Exercise 2. Problem 1

Consider the following puzzle.

Isaac and Albert were excitedly describing the result of the Third Annual International Science Fair Extravaganza in Sweden. There were three contestants, Louis, Rene, and Johannes. Isaac reported that Louis won the fair, while Rene came in second. Albert, on the other hand, reported that Johannes won the fair, while Louis came in second.

In fact, neither Isaac nor Albert had given a correct report of the results of the science fair. Each of them had given one true statement and one false statement. What was the actual placing of the three contestants?

Represent this puzzle as a set P of propositional formulas using the following variables L_1-J_3 denoting a person and his placing:

person/placing	1	2	3
Louis	L_1	L_2	L_3
Rene	R_1	R_2	R_3
Johannes	J_1	J_2	J_3

Find a model of P. Note that this model should give you the solution of the puzzle.

Solution

First, we should represent that exacty one of the three contestants came in first (respectively, second and third). This gives us the following 12 clauses:

$$
\begin{array}{llll}
L_1 \vee R_1 \vee J_1 & \neg L_1 \vee \neg R_1 & \neg L_1 \vee \neg J_1 & \neg R_1 \vee \neg J_1 \\
L_2 \vee R_2 \vee J_2 & \neg L_2 \vee \neg R_2 & \neg L_2 \vee \neg J_2 & \neg R_2 \vee \neg J_2 \\
L_3 \vee R_3 \vee J_3 & \neg L_3 \vee \neg R_3 & \neg L_3 \vee \neg J_3 & \neg R_3 \vee \neg J_3
\end{array}
$$

Solution

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$$\begin{array}{llll} L_1 \vee R_1 \vee J_1 & \neg L_1 \vee \neg R_1 & \neg L_1 \vee \neg J_1 & \neg R_1 \vee \neg J_1 \\ L_2 \vee R_2 \vee J_2 & \neg L_2 \vee \neg R_2 & \neg L_2 \vee \neg J_2 & \neg R_2 \vee \neg J_2 \\ L_3 \vee R_3 \vee J_3 & \neg L_3 \vee \neg R_3 & \neg L_3 \vee \neg J_3 & \neg R_3 \vee \neg J_3 \end{array}$$

Likewise, we should represent that each of the contestants cannot take more than one place. This gives us the following 12 clauses:

$$\begin{array}{llll} L_1 \vee L_2 \vee L_3 & \neg L_1 \vee \neg L_2 & \neg L_1 \vee \neg L_3 & \neg L_2 \vee \neg L_3 \\ R_1 \vee R_2 \vee R_3 & \neg R_1 \vee \neg R_2 & \neg R_1 \vee \neg R_3 & \neg R_2 \vee \neg R_3 \\ J_1 \vee J_2 \vee J_3 & \neg J_1 \vee \neg J_2 & \neg J_1 \vee \neg J_3 & \neg J_2 \vee \neg J_3 \end{array}$$

Note that Isaac's report can be expressed by $L_1 \wedge R_2$, while Albert's report by $J_1 \wedge L_2$. We can express that that exactly one statement in each of these reports is true using the following clauses

$$\begin{array}{l} \neg L_1 \vee \neg R_2 \\ L_1 \vee R_2 \\ \neg J_1 \vee \neg L_2 \\ J_1 \vee L_2 \end{array}$$

This set of clauses has a model (in fact, a single model):

$$\left\{ \begin{array}{rrr} L_1 \mapsto 0, & L_2 \mapsto 0, & L_3 \mapsto 1, \\ R_1 \mapsto 0, & R_2 \mapsto 1, & R_3 \mapsto 0, \\ J_1 \mapsto 1, & J_2 \mapsto 0, & J_3 \mapsto 0 \end{array} \right\}$$

That is, Johannes won and Rene came second.

Exercise 2. Problem 2

Show that the formulas $\neg q \vee \neg r \rightarrow \neg p$ and $p \rightarrow \perp$ are not equivalent by finding an interpretation in which they have different truth values.

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Show that the formulas $\neg q \vee \neg r \rightarrow \neg p$ and $p \rightarrow \bot$ are not equivalent by finding an interpretation in which they have different truth values.

Solution

Take the interpretation

$$I \stackrel{\text{def}}{=} \{p \mapsto 1, q \mapsto 1, r \mapsto 1\}.$$

Then we have

$$\begin{array}{lll} I & \models & \neg q \vee \neg r \rightarrow \neg p \\ I & \not\models & p \rightarrow \bot \end{array}$$

Therefore, these formulas are not equivalent.

Exercise 2. Problem 3

Evaluate the formula $\neg q \lor \neg r \to \neg p$ in the interpretation $\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$ using (i) the method using the table of subformulas; (ii) the rewriting-based algorithm.

Solution Using the Table of Subformulas

- Build the table of subfomulas

formula	value
$\neg q \lor \neg r \to \neg p$	
$\neg p$	
p	
$\neg q \lor \neg r$	
$\neg q$	
q	
$\neg r$	
r	

Solution Using the Table of Subformulas

- Build the table of subfomulas
- Evaluate atomic subformulas using the values from the interpretation

formula	value
$\neg q \lor \neg r \to \neg p$	
$\neg p$	
p	1
$\neg q \lor \neg r$	
$\neg q$	
q	0
$\neg r$	
r	1

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

Solution Using the Table of Subformulas

- Build the table of subfomulas
- Evaluate atomic subformulas using the values from the interpretation
- Evaluate all subformulas

formula	value
$\neg q \lor \neg r \rightarrow \neg p$	
$\neg p$	
p	1
$\neg q \lor \neg r$	
$\neg q$	
q	0
$\neg r$	0
r	1

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

Solution Using the Table of Subformulas

- Build the table of subfomulas
- Evaluate atomic subformulas using the values from the interpretation
- Evaluate all subformulas

formula	value
$\neg q \lor \neg r \rightarrow \neg p$	
$\neg p$	
p	1
$\neg q \lor \neg r$	
$\neg q$	1
q	0
$\neg r$	0
r	1

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

Solution Using the Table of Subformulas

- Build the table of subfomulas
- Evaluate atomic subformulas using the values from the interpretation
- Evaluate all subformulas

formula	value
$\neg q \vee \neg r \rightarrow \neg p$	
$\neg p$	
p	1
$\neg q \vee \neg r$	1
$\neg q$	1
q	0
$\neg r$	0
r	1

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

Solution Using the Table of Subformulas

- Build the table of subfomulas
- Evaluate atomic subformulas using the values from the interpretation
- Evaluate all subformulas

formula	value
$\neg q \lor \neg r \to \neg p$	
$\neg p$	0
p	1
$\neg q \lor \neg r$	1
$\neg q$	1
q	0
$\neg r$	0
r	1

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

Solution Using the Table of Subformulas

- Build the table of subfomulas
- Evaluate atomic subformulas using the values from the interpretation
- Evaluate all subformulas
- The formula is false.

formula	value
$\neg q \vee \neg r \rightarrow \neg p$	0
$\neg p$	0
p	1
$\neg q \vee \neg r$	1
$\neg q$	1
q	0
$\neg r$	0
r	1

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

Solution Using the Rewriting-Based Algorithm

First, using the values of the variables in the interpretation:

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$$

we replace the formula $\neg q \lor \neg r \to \neg p$ by $\neg\bot \lor \neg\top \to \neg\top$.

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Then it can be rewritten as follows:

$$\neg\bot \vee \neg\top \to \neg\top$$

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Then it can be rewritten as follows:

$$\begin{array}{ll} \neg\bot \lor \neg\top \to \neg\top & \Rightarrow \\ \top \lor \neg\top \to \neg\top & \end{array}$$

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\neg \bot \vee \neg \top \to \neg \top & \Rightarrow \\
\top \vee \neg \top \to \neg \top & \Rightarrow \\
\top \to \neg \top &
\end{array}$$

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\top \vee \neg\top \to \neg\top & \Rightarrow \\
\top \to \neg\top & \Rightarrow \\
\neg\top &
\end{array}$$

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\top \rightarrow \neg\top & \Rightarrow \\
\neg\top &
\end{array}$$

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\neg\bot \vee \neg\top \rightarrow \neg\top & \Rightarrow \\
\top \vee \neg\top \rightarrow \neg\top & \Rightarrow \\
\top \rightarrow \neg\top & \Rightarrow \\
\neg\top & \Rightarrow \\
\bot &
\end{array}$$

Solution Using the Rewriting-Based Algorithm

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\neg\bot \vee \neg\top \rightarrow \neg\top & \Rightarrow \\
\top \vee \neg\top \rightarrow \neg\top & \Rightarrow \\
\top \rightarrow \neg\top & \Rightarrow \\
\neg\top & \Rightarrow \\
\bot &
\end{array}$$

Thus, the formula is false.