21111 SAT competition

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Deadline: 20th of November

Implement any of DPLL, GSAT, Walk SAT . . . or any other!

Data Structures for Large Propositional Formulas

Suppose that large propositional formulas are reused over and over again. For example, we may

- Build a conjunction of several formulas;
- Negate a formula;
- Check if two formulas are equivalent . . .

One needs data structures which

- give compact representation of formulas, or the boolean functions represented by the formulas;
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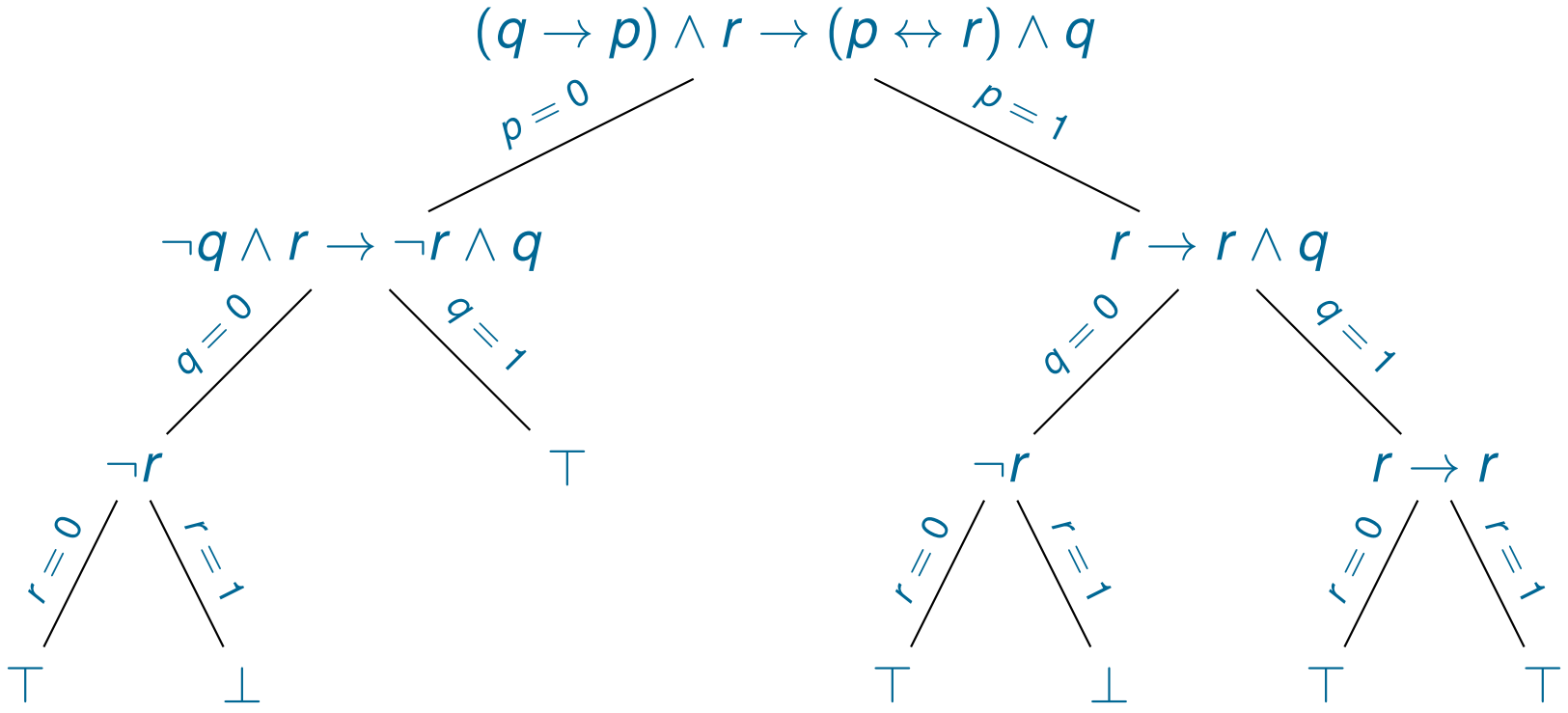
Splitting Tree



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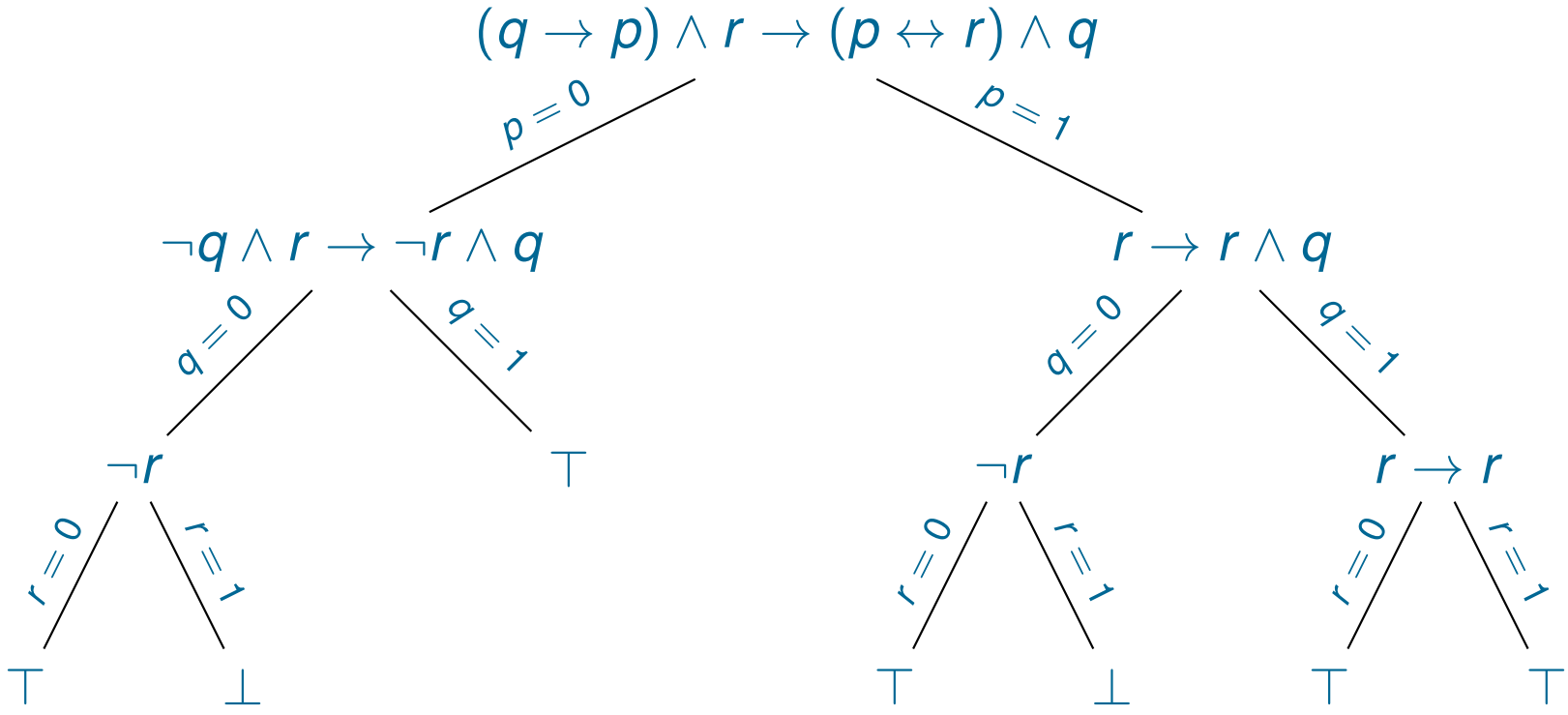

Let us “forget” about formulas in the tree

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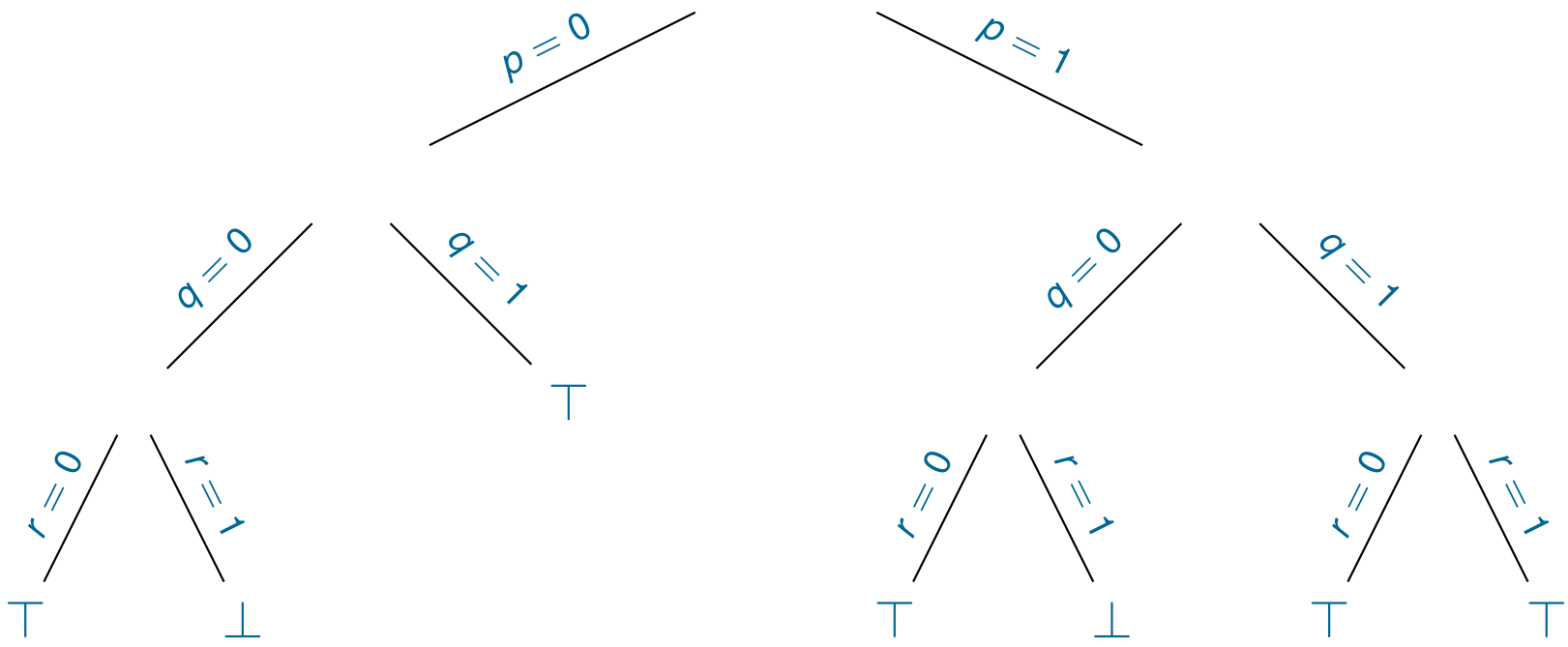

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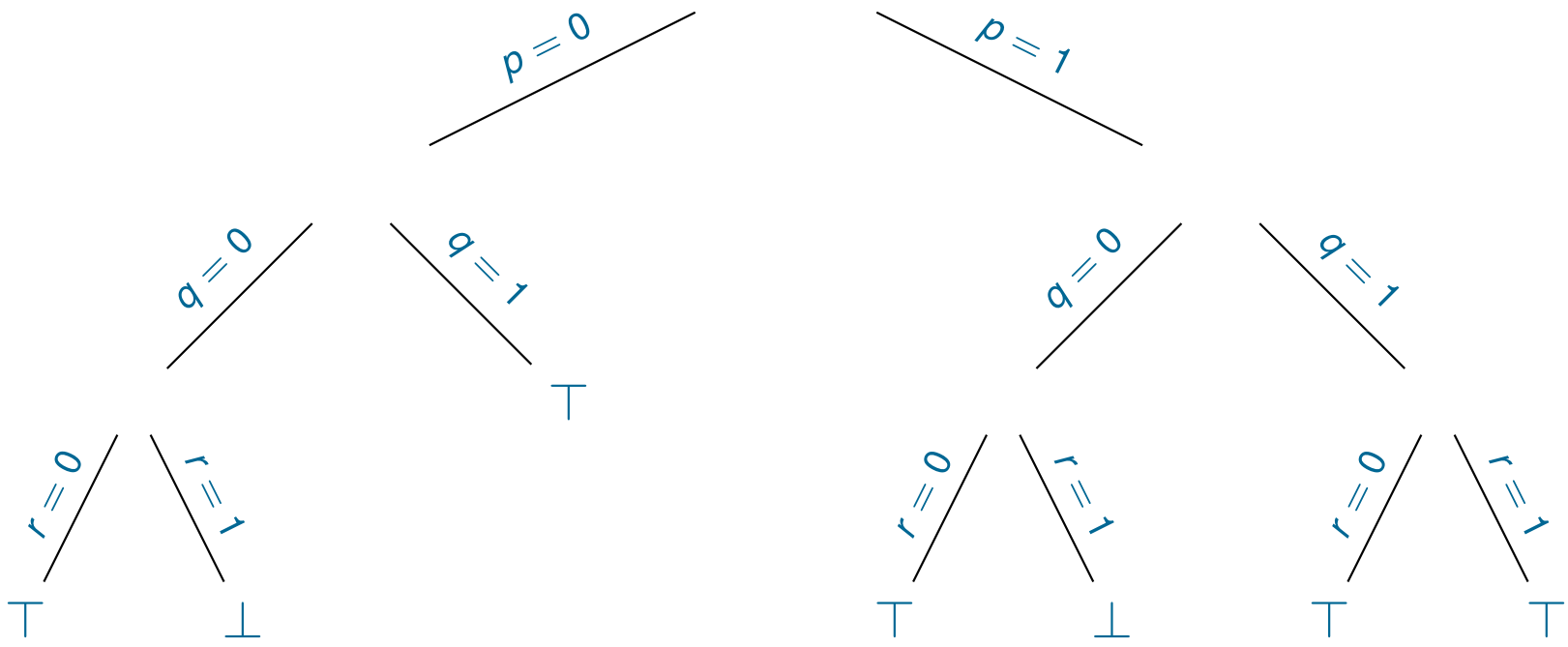
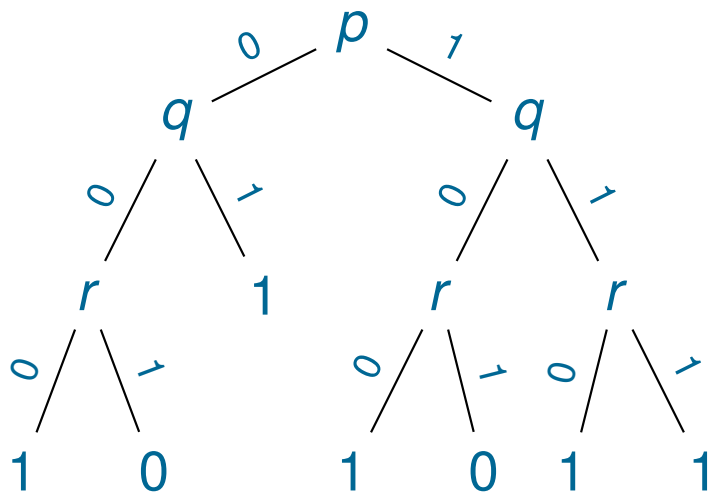


We do not see the syntax of the formula but the semantics is preserved: the tree encodes all models of the formula. Any formula having the same tree with “forgotten” formulas has the same models.

Binary Decision Tree



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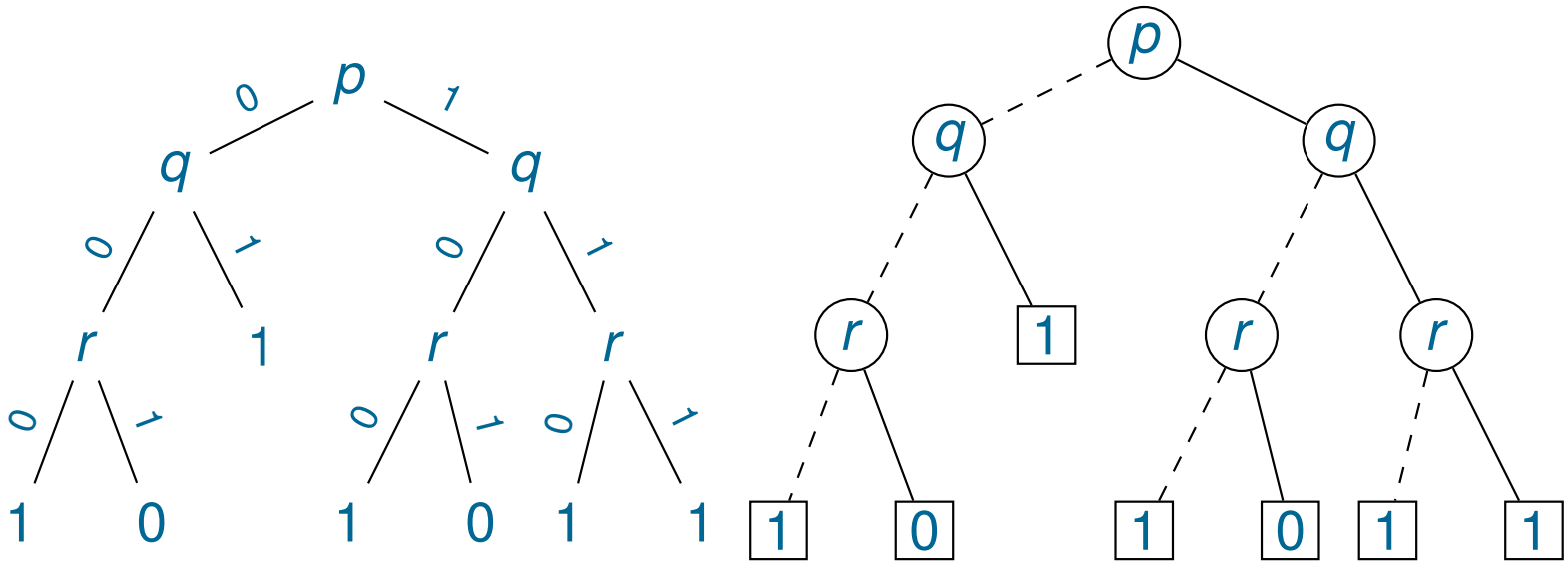

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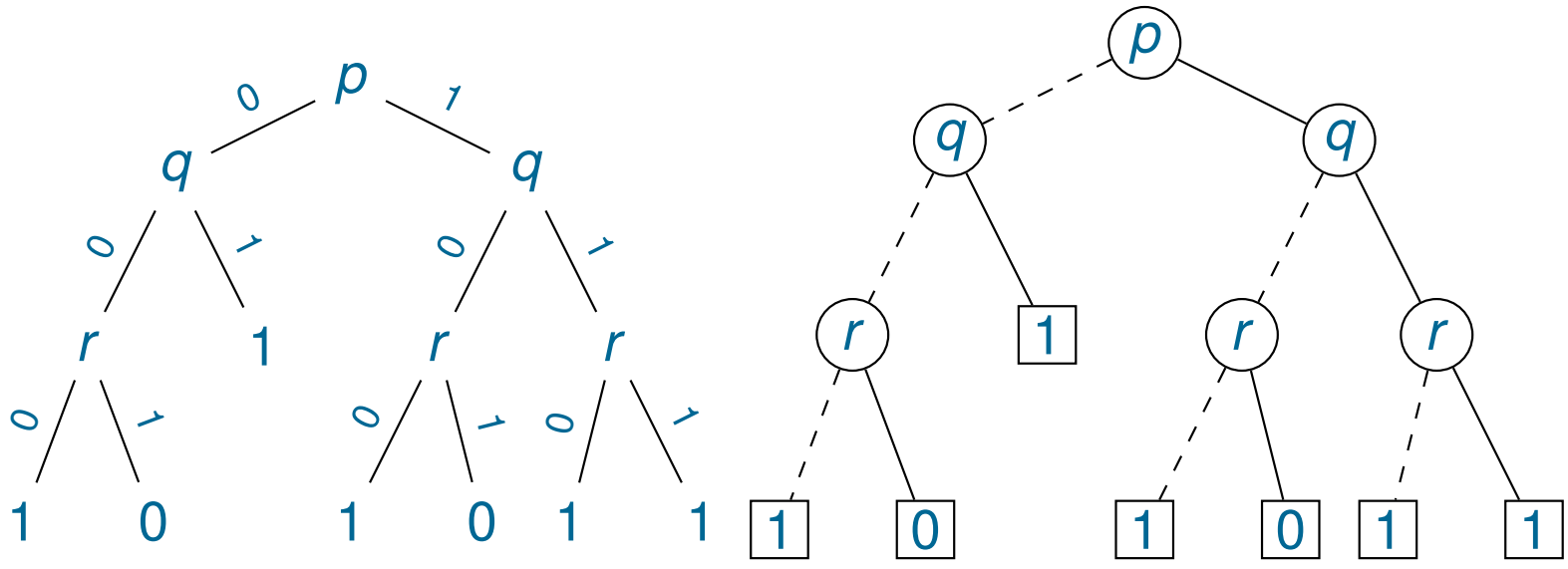

A circled node, such as (p), denotes the decision on the variable of this node.

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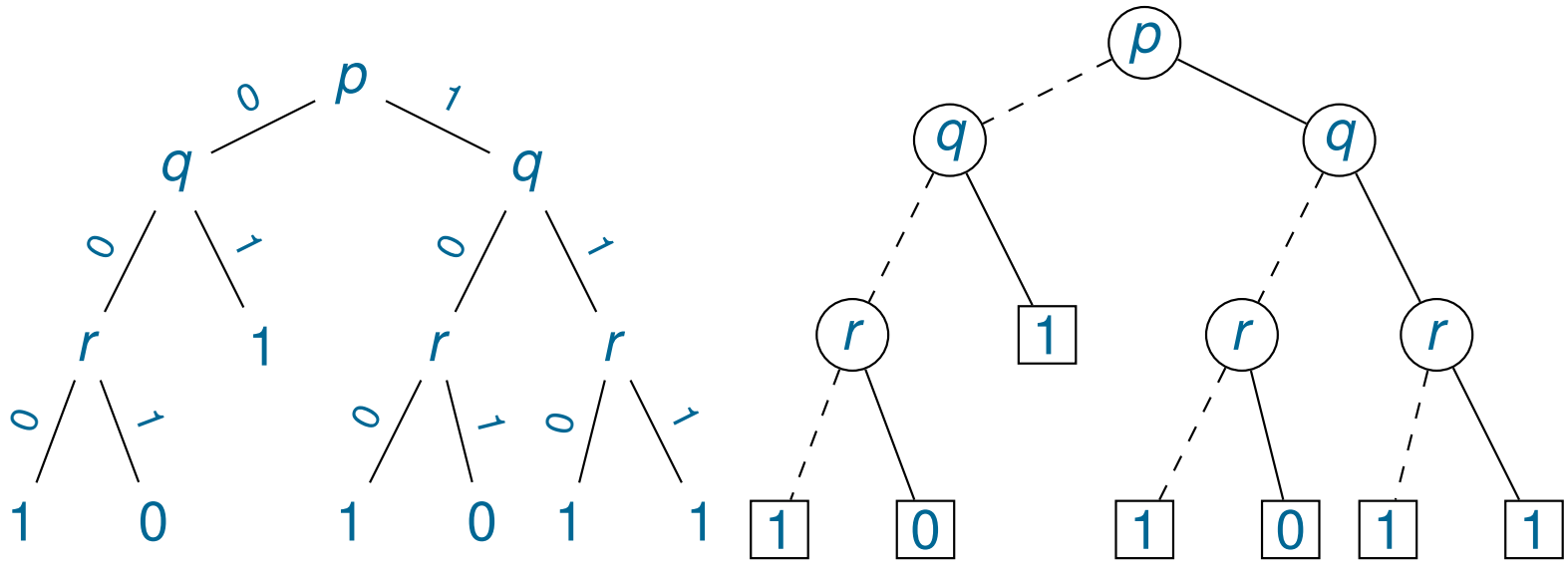

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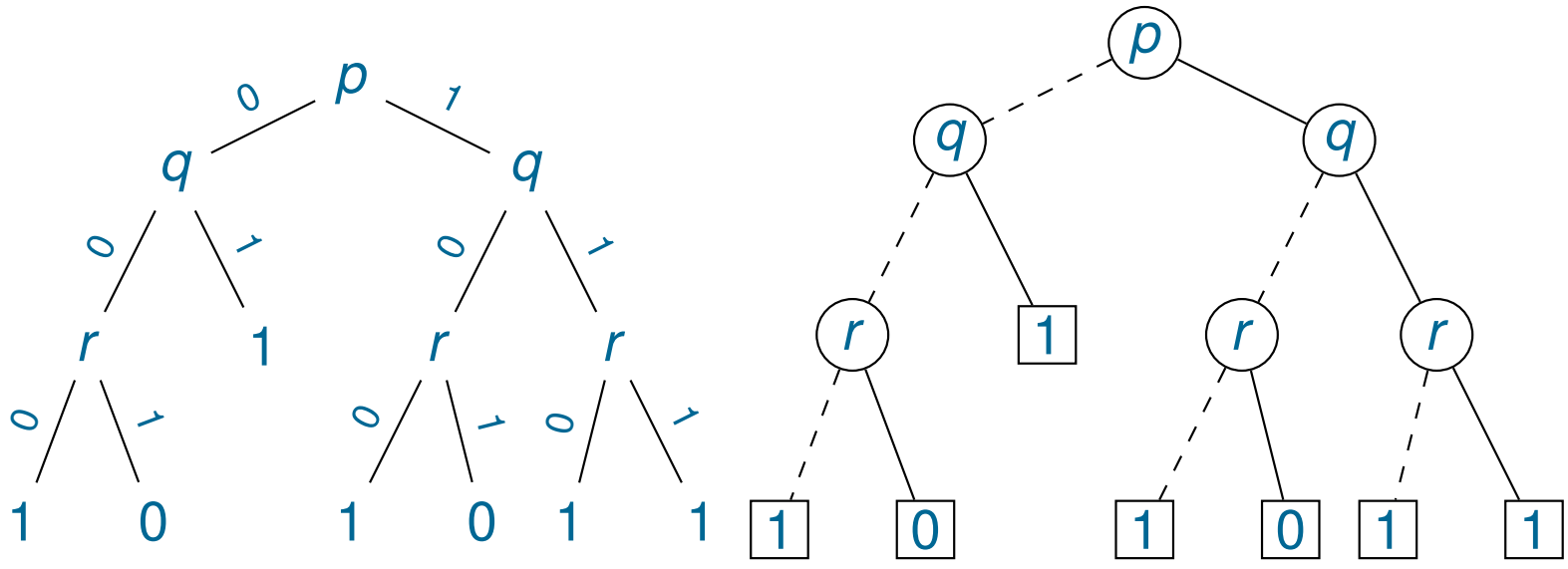

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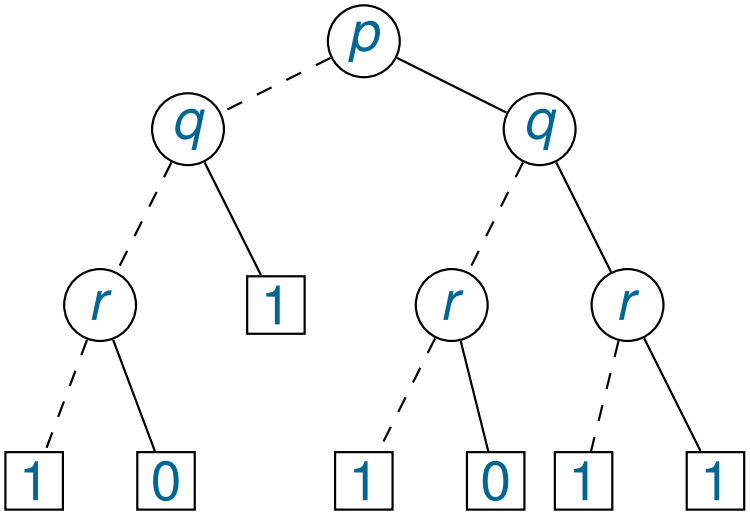
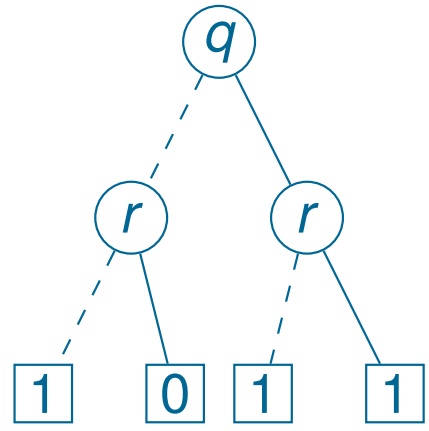
Solid lines correspond to choices when the variable is true, dashed lines to the choices when the variable is false.

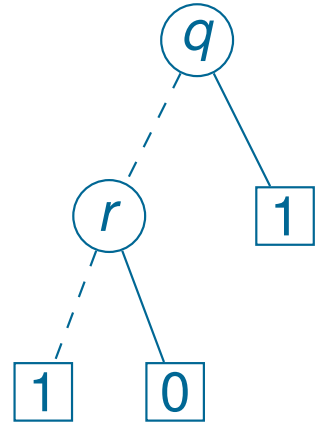
Tests correspond to “if-then-else”



if *p* *then*



else



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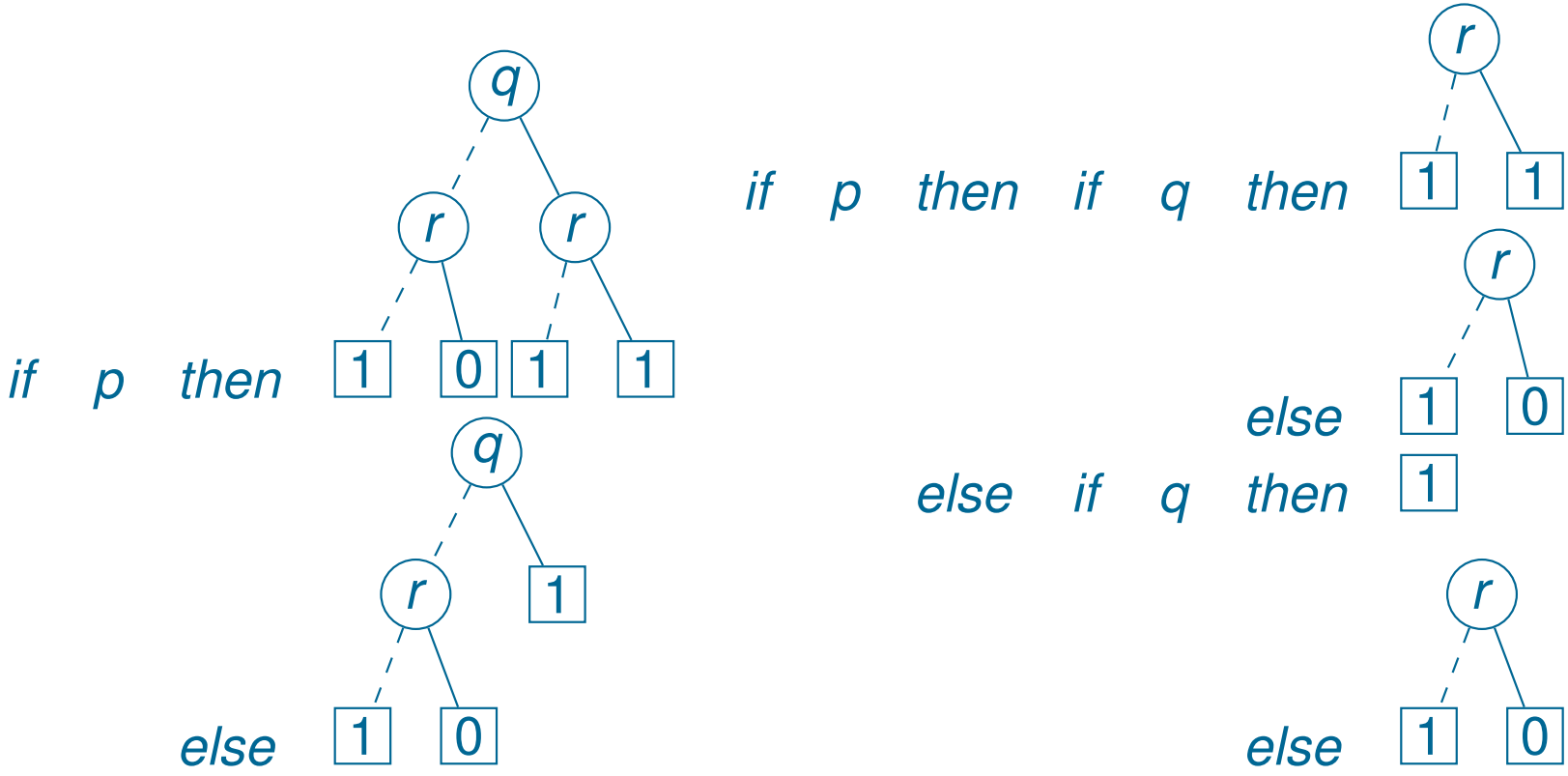

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$$
\begin{array}{llllllllll}
\mathit{if} & p & \mathit{then} & \mathit{if} & q & \mathit{then} & \mathit{if} & r & \mathit{then} & \top \\
 & & & & & & & & \mathit{else} & \top \\
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If-Then-Else Normal Form

- The only connectives are if-then-else, $\top$ and $\bot$;
- All if-formulas A in *if* A *then* B *else* C are atomic.

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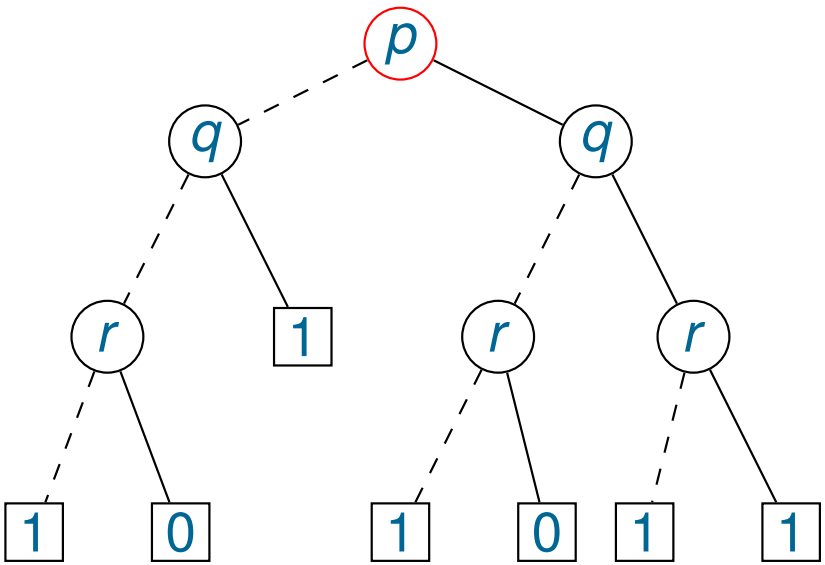

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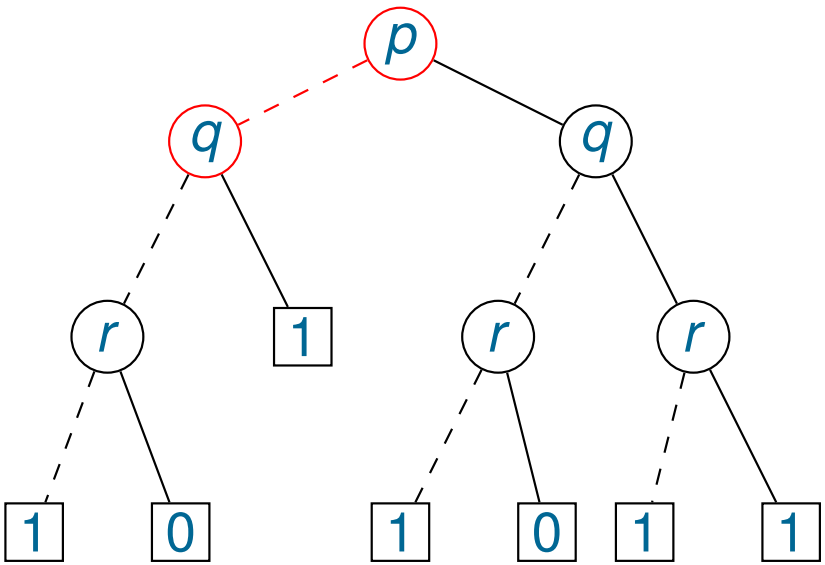

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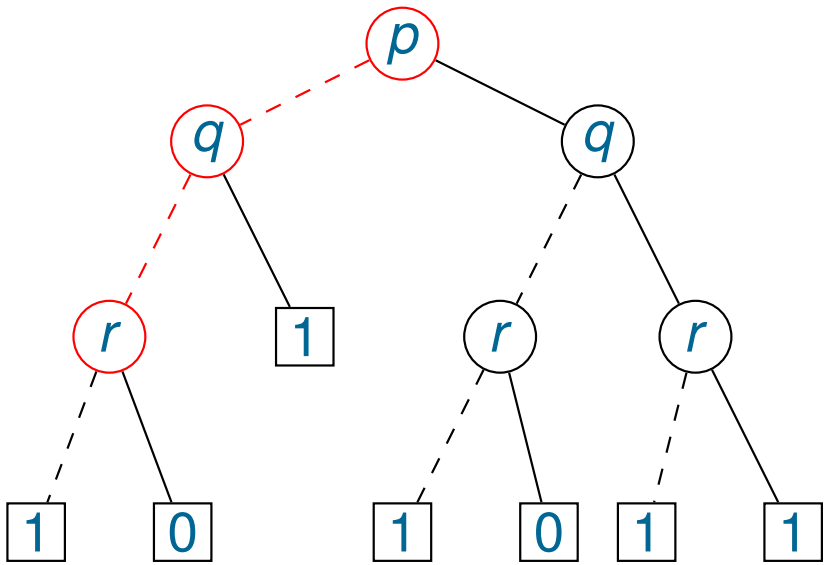

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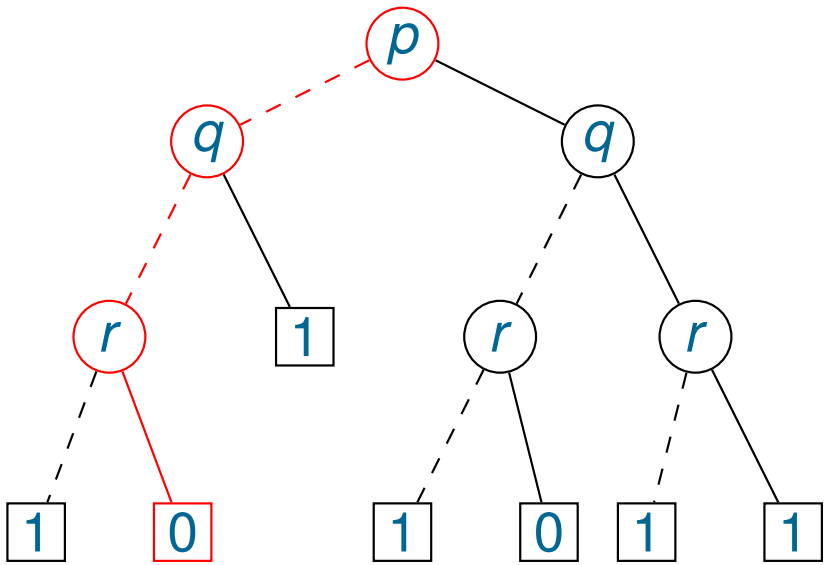

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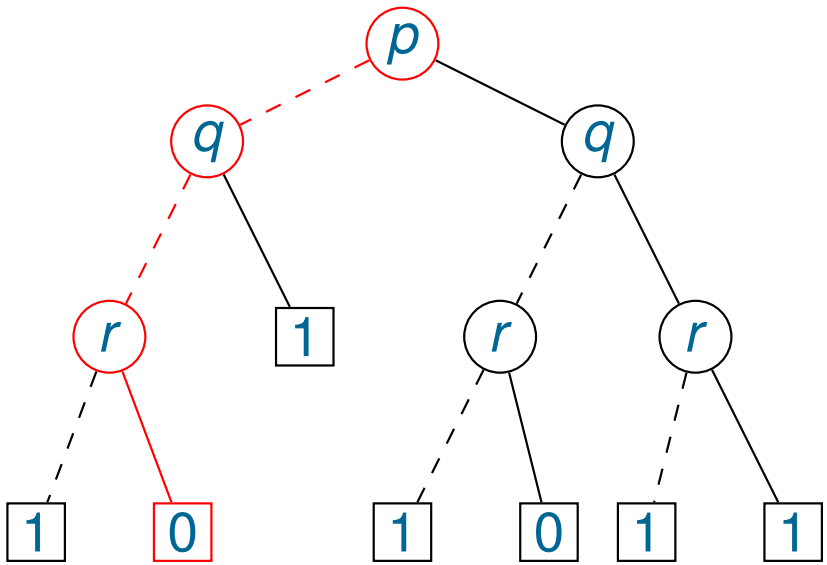

So any formula with this decision tree is false in this interpretation.

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Are binary decision trees compact?

Algorithm for Building Binary Decision Trees

procedure $bdt(A)$
input: propositional formula A
output: a binary decision tree
parameters: function $select\_variable$
begin
  $A := simplify(A)$
  if $A = \bot$ **then return** $\boxed{0}$
  if $A = \top$ **then return** $\boxed{1}$
  $p := select\_variable(A)$
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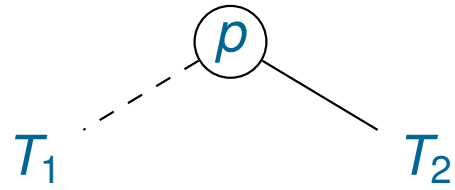
Here $tree(T_1, p, T_2)$ builds the tree



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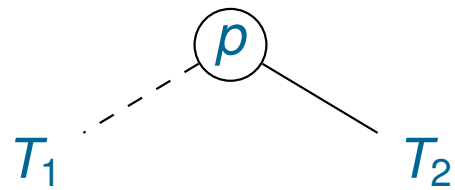
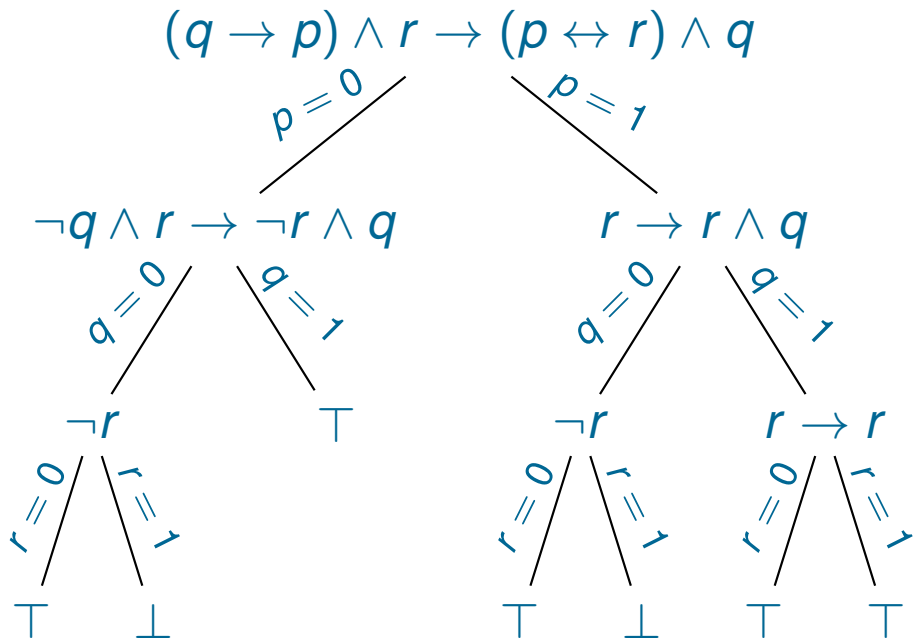


Note resemblance to the splitting algorithm!

Example



$(q \to p) \wedge r \to (p \leftrightarrow r) \wedge q$

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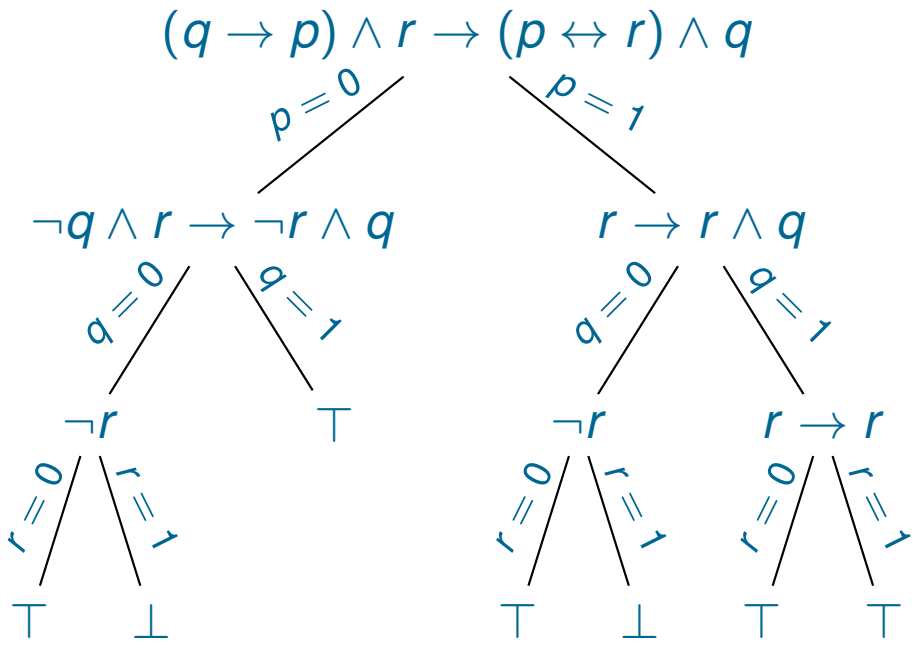
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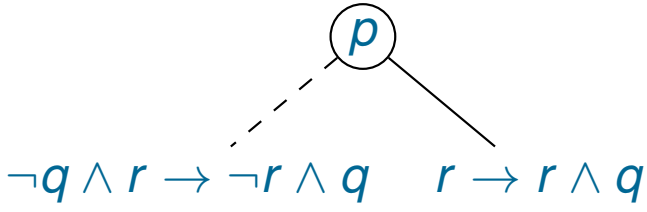
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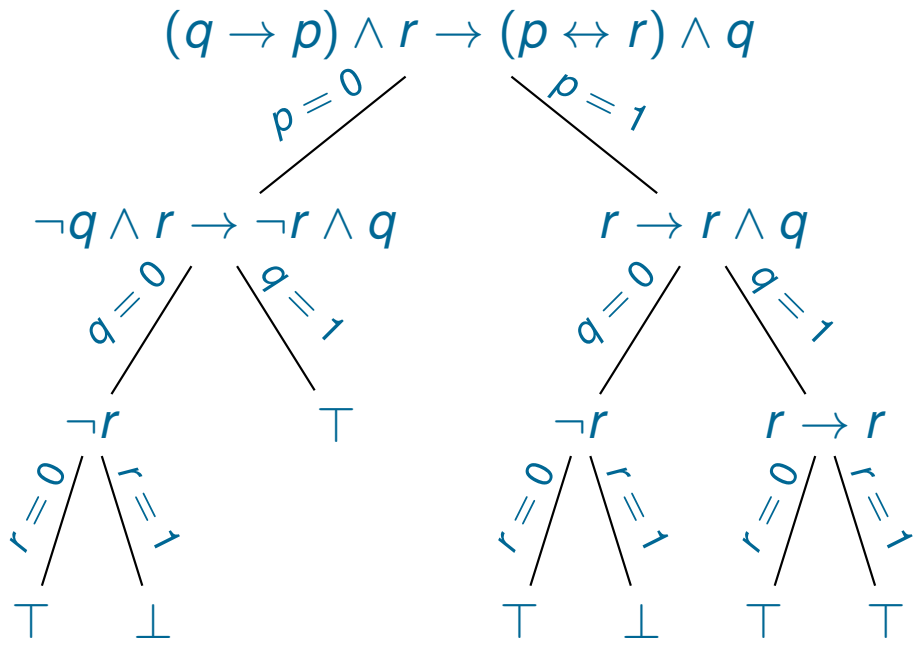
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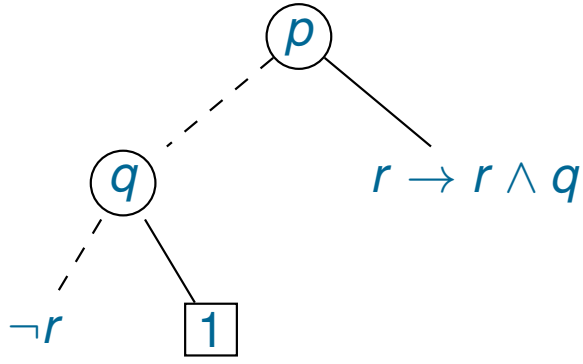
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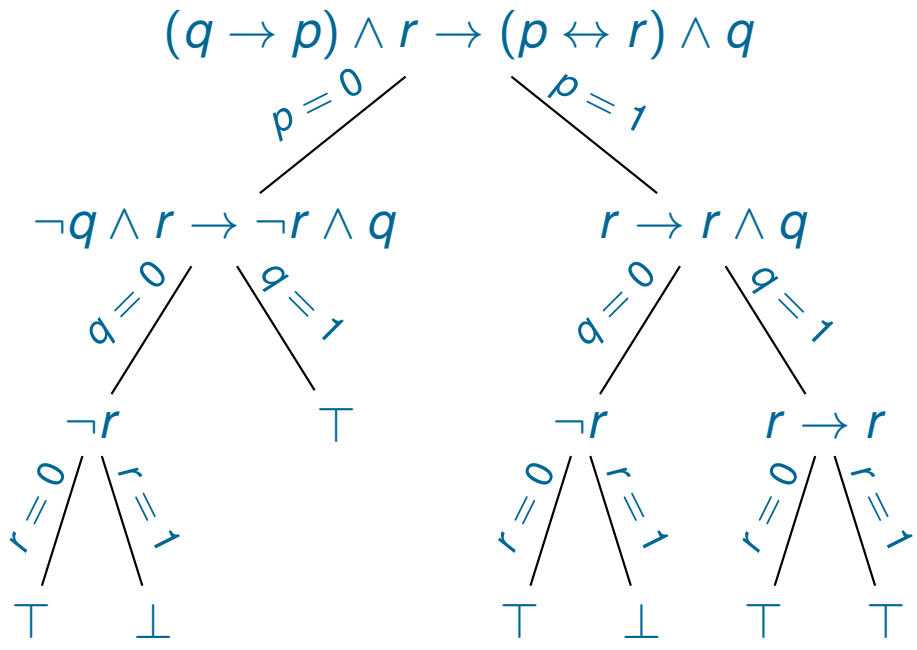
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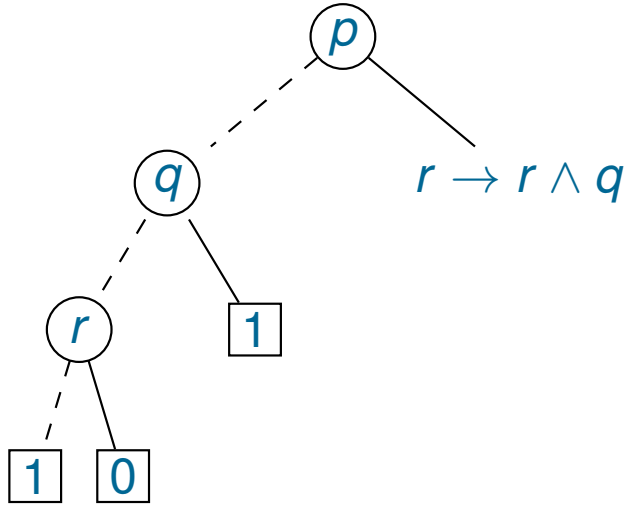
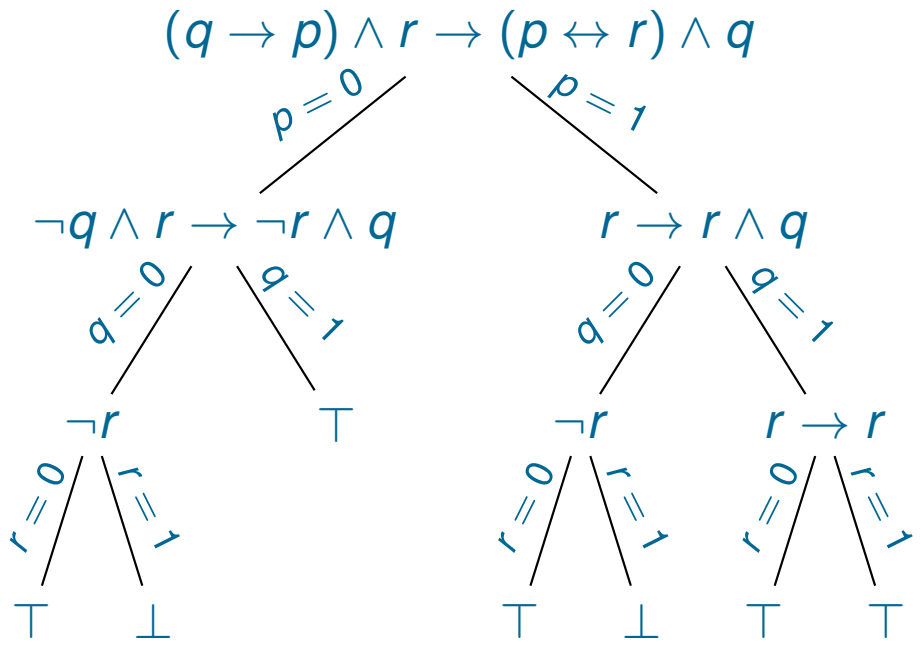
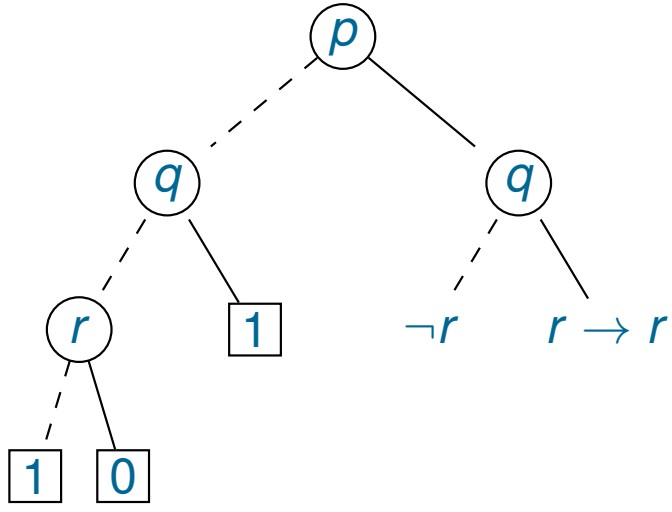
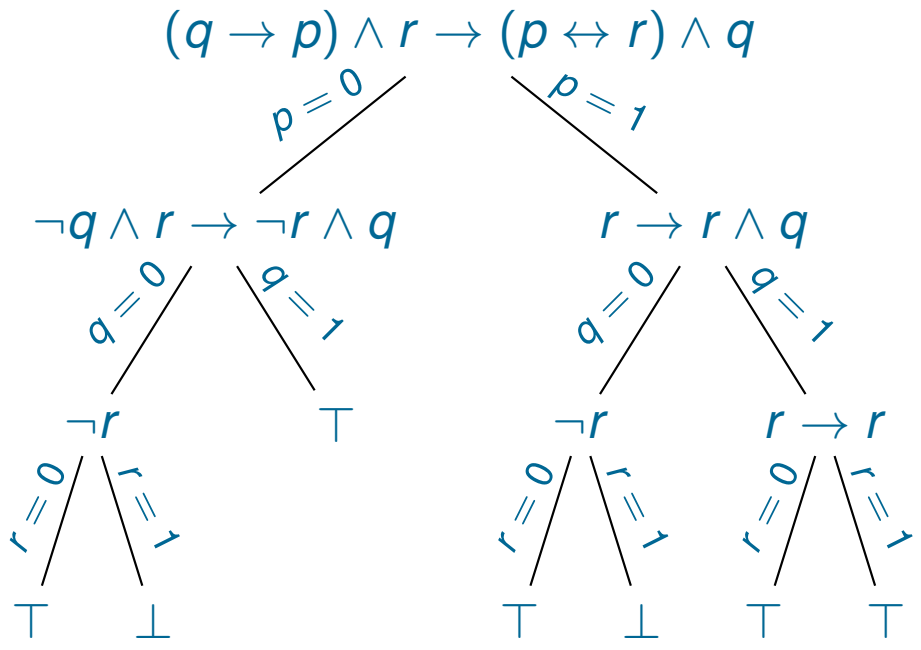

p
q
$r \to r \wedge q$
r
1
1
0

Example


$(q \to p) \wedge r \to (p \leftrightarrow r) \wedge q$
$p = 0$
$p = 1$
$\neg q \wedge r \to \neg r \wedge q$
$r \to r \wedge q$
$q = 0$
$q = 1$
$q = 0$
$q = 1$
$\neg r$
$\top$
$\neg r$
$r \to r$
$r = 0$
$r = 1$
$r = 0$
$r = 1$
$r = 0$
$r = 1$
$\top$
$\bot$
$\top$
$\bot$
$\top$
$\top$


p
q
q
r
1
$\neg r$
$r \to r$
1
0

Example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$p = 1$
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$r \to r \wedge q$
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$q = 1$
$\neg r$
$\top$
$\neg r$
$r \to r$
$r = 0$
$r = 1$
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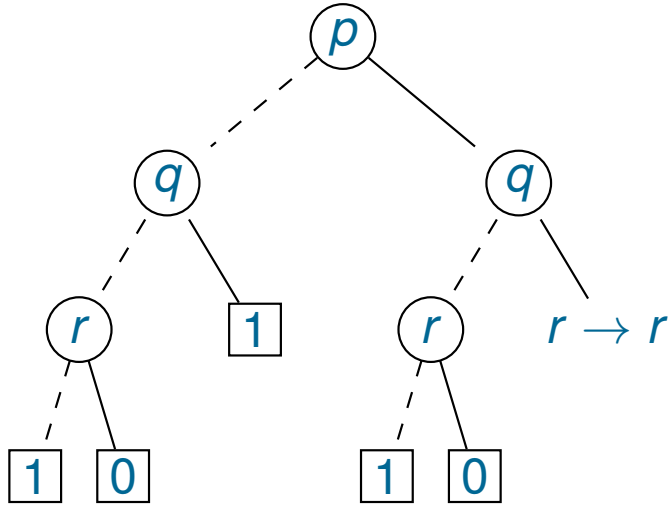
p
q
q
r
1
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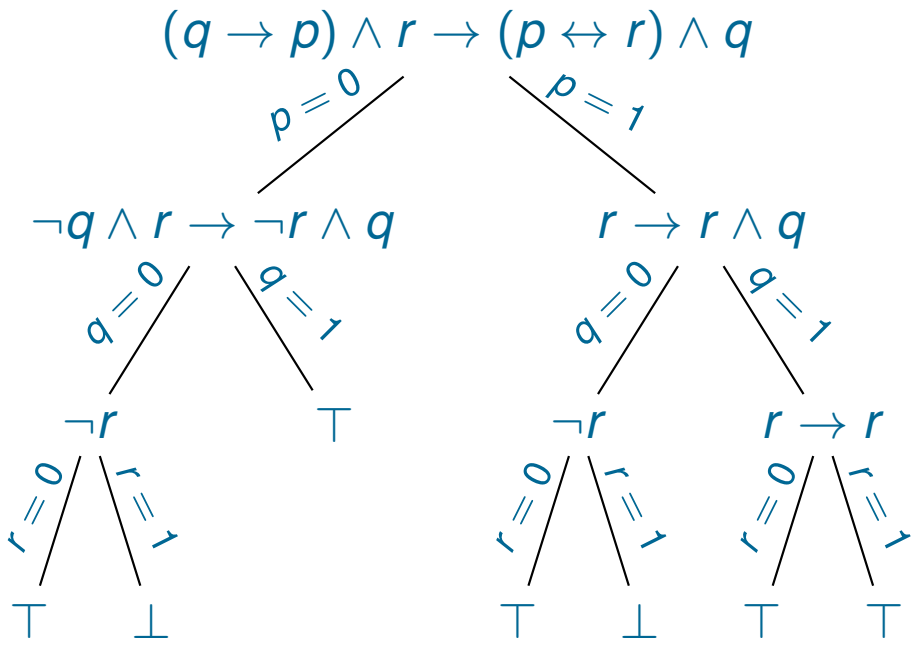
$r \to r$
1
0
1
0

Example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$p = 0$
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$q = 0$
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$r \to r$
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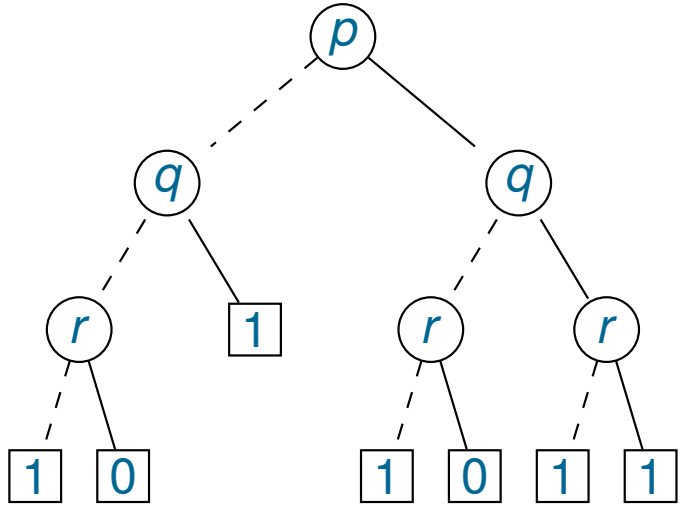
p
q
q
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r
r
1
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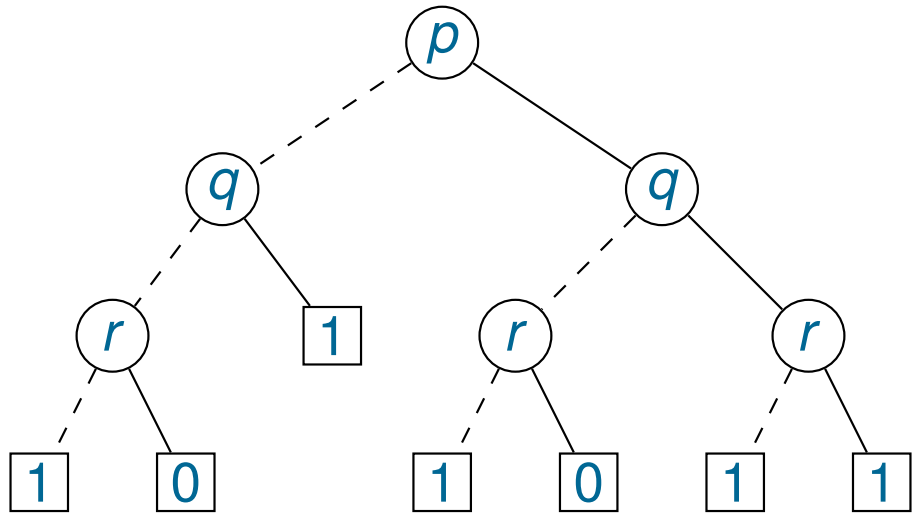
0
1
1

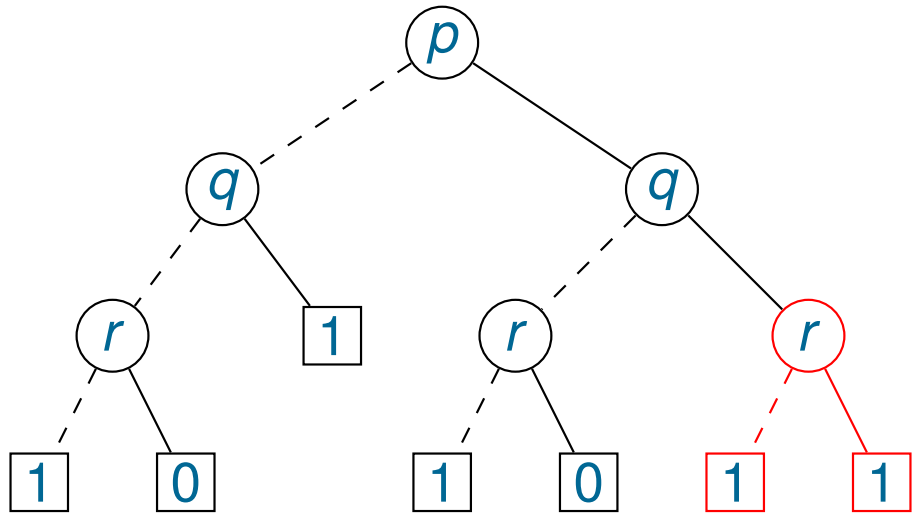
Are BDTs compact?

Are binary decision trees compact?



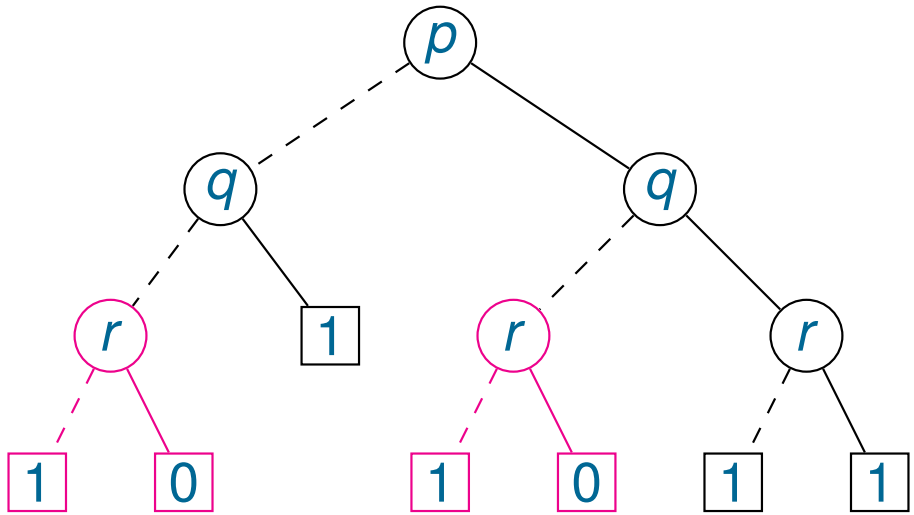
Are BDTs compact?

Are binary decision trees compact?
No: they may contain redundant tests:



Are BDTs compact?

Are binary decision trees compact?
No: they may contain redundant tests:
No: they may contain isomorphic subtrees:

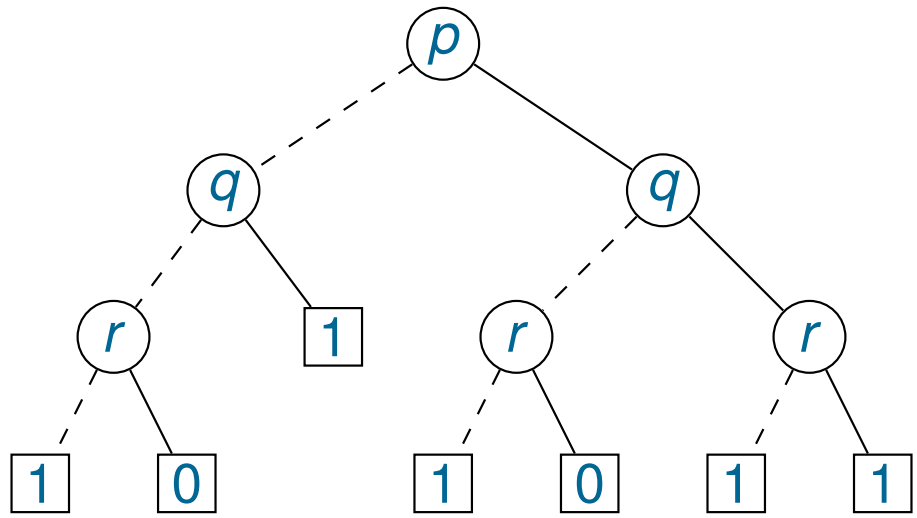


Binary Decision Diagrams

A binary decision diagrams or simply a BDD is a dag (directed acyclic graph) built like a binary decision tree but containing

- no redundant tests; and
- no isomorphic subtrees.

Transforming Binary Decision Tree into a BDD



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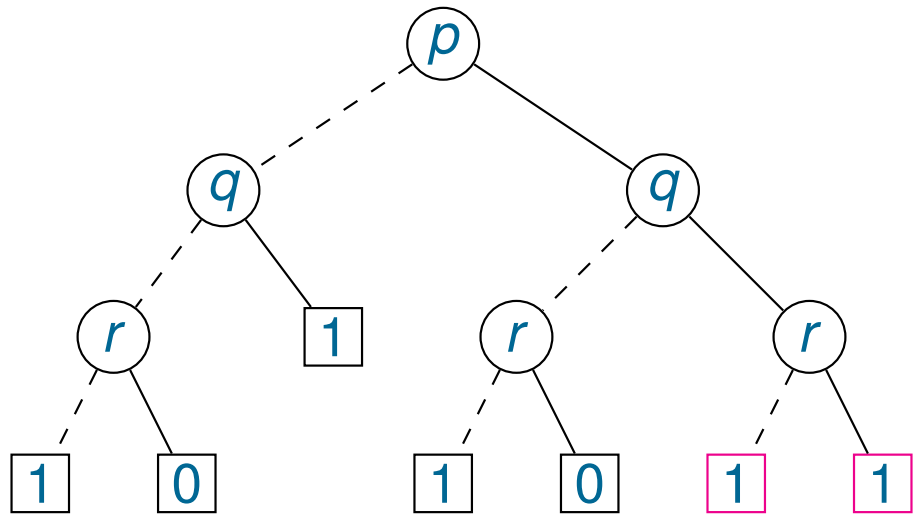

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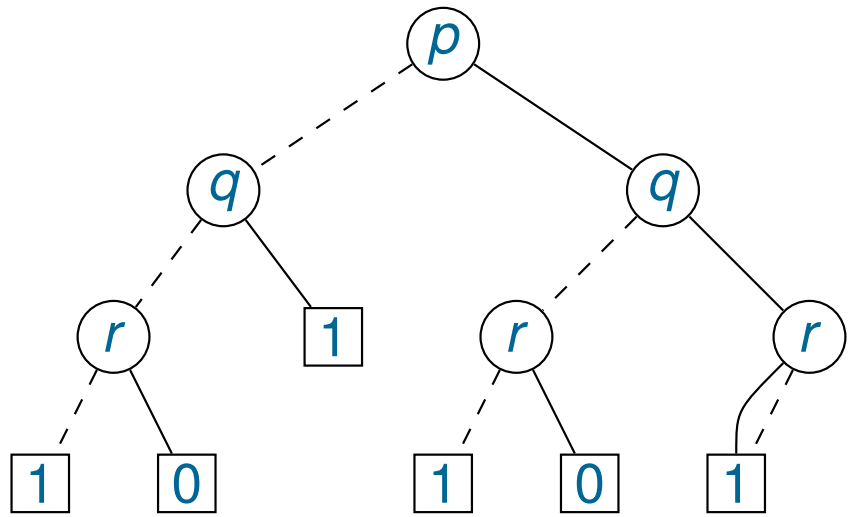

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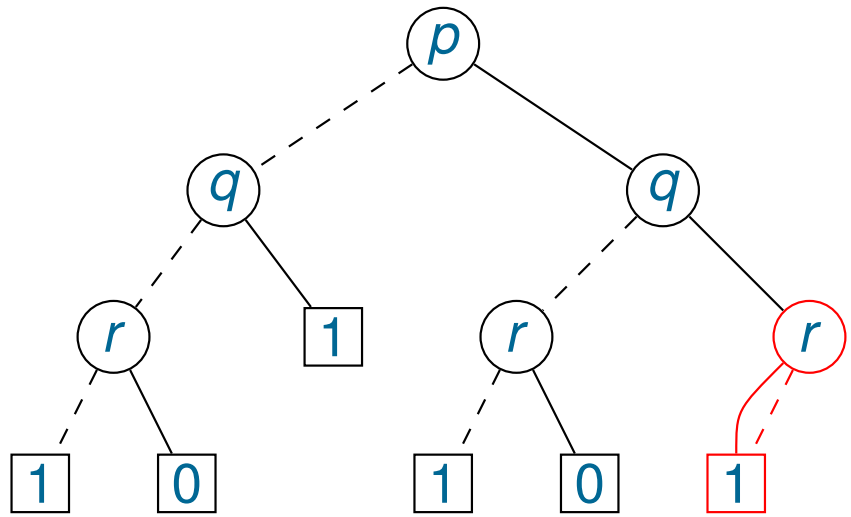

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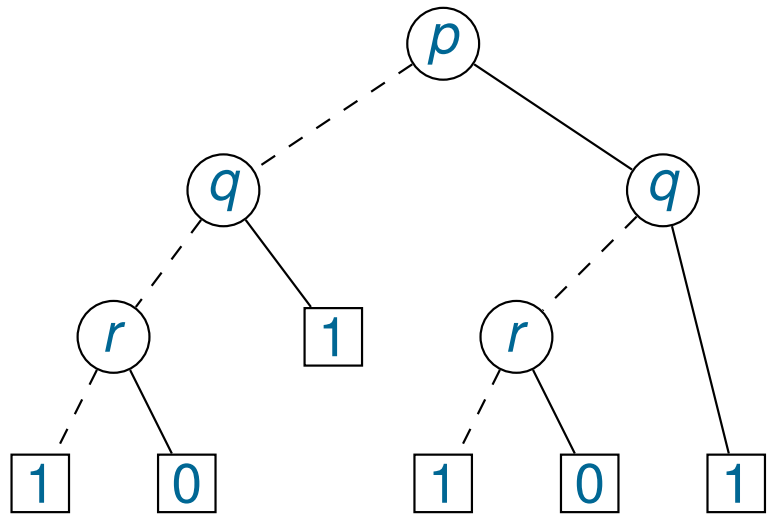

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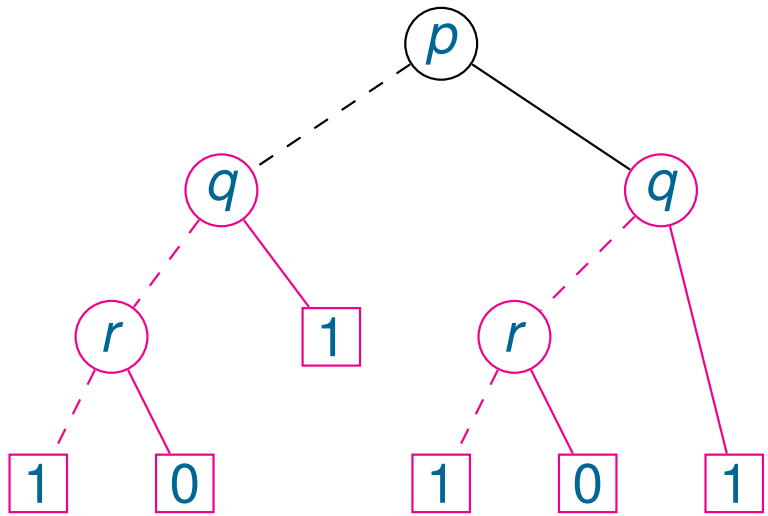

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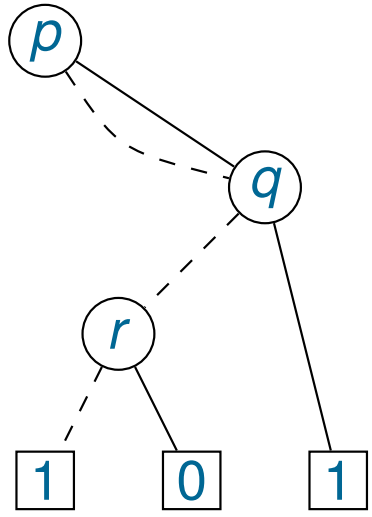

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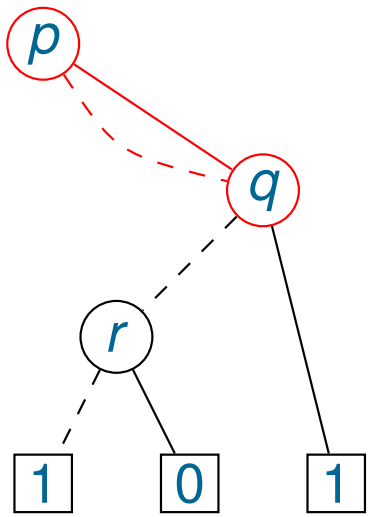

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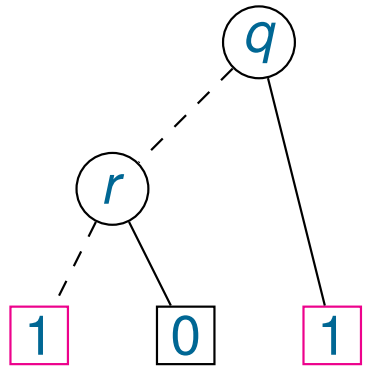

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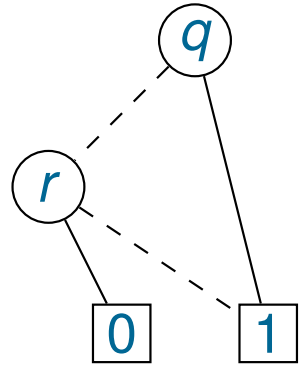

Properties

What is the complexity of satisfiability, validity and equivalence checking for BDDs?

- Satisfiability checking can be done in constant time;
- Validity checking can be done in constant time;
- Equivalence checking
- Some boolean operations, e.g., conjunction

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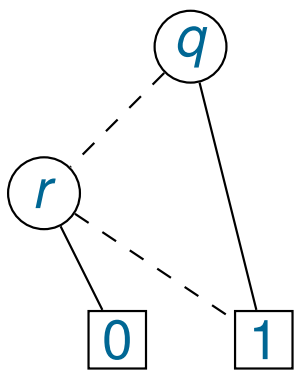

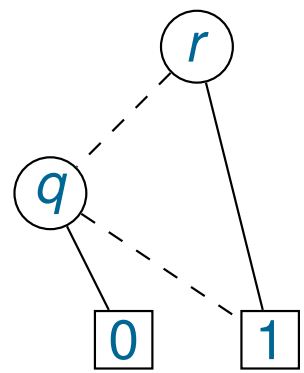

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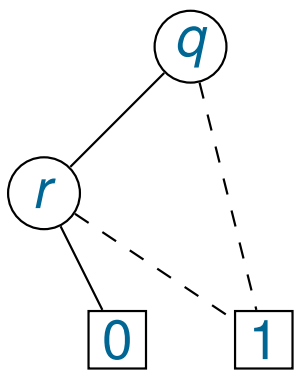

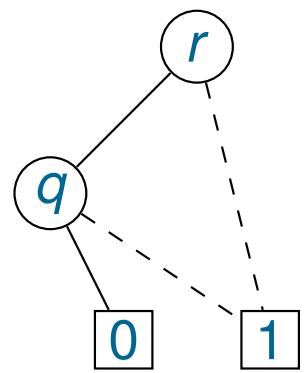

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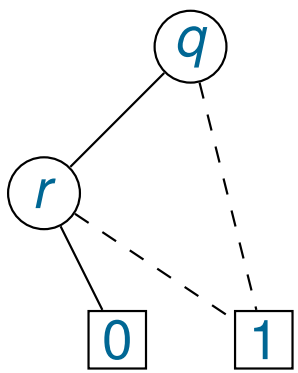

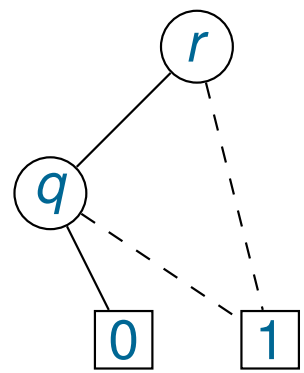

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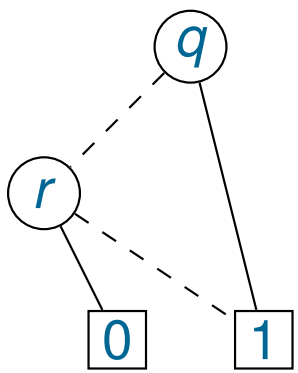
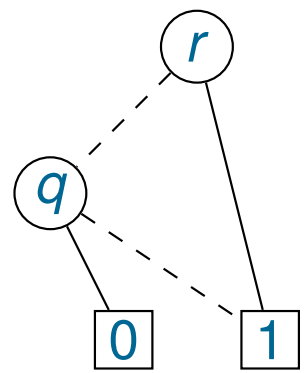


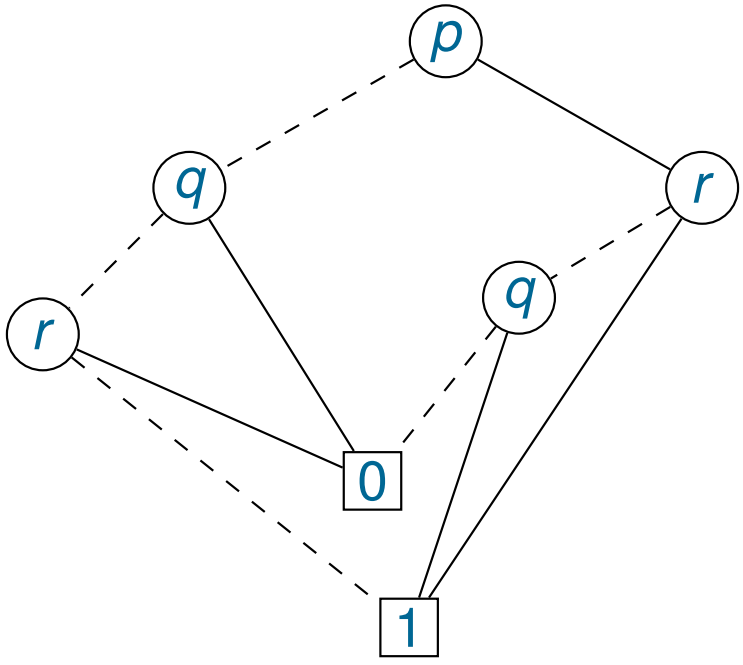


OBDDs

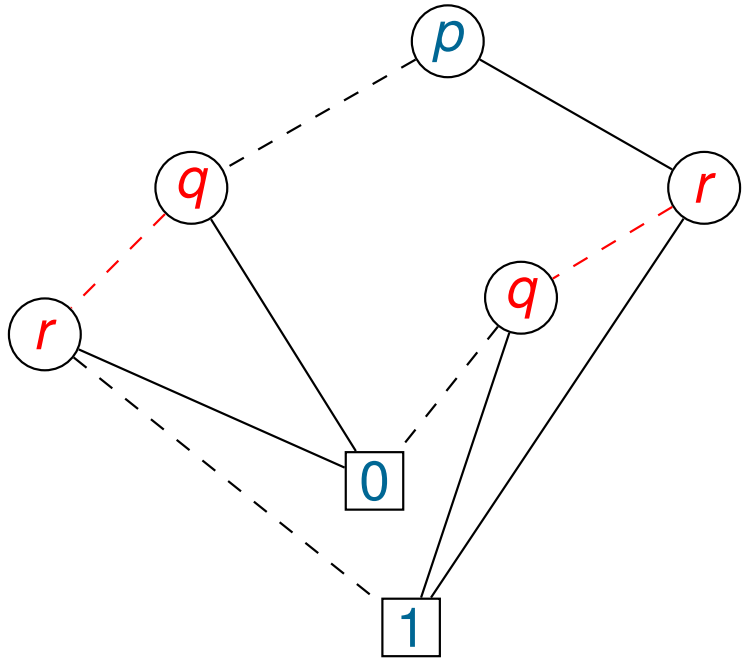
Ordered Binary Decision Diagrams (OBDDs)

Donald Knuth regarding OBDDs:
“one of the only really fundamental data structures that came out in the last twenty-five years”

BDDs

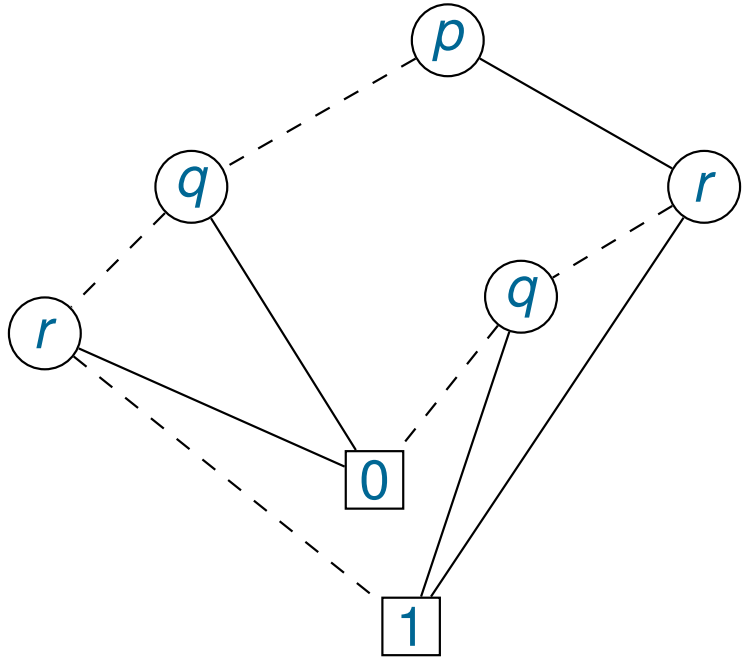


BDDs



Problem: on different branches of the dag tests are made in a different order.

BDDs



Problem: on different branches of the dag tests are made in a different order.

OBDDs:

- introduce an order $>$ on variables;
- make tests in this order.

OBDDs Properties

- Every Bolean function has a unique representation as OBDD!
  In other words OBDDs are canonical for Boolean functions.
- Validity checking can be done in constant time;
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- Equivalence checking can be done in constant time;
- Boolean operations e.g., conjunction, are easy to implement.

Integrating a node in a dag

All OBDD algorithms will use the same procedure for integrating a node in a dag.

procedure *integrate*(n_1, p, n_2, D)
parameters: global dag D
input: nodes n_1, n_2 in D representing formulas F_1, F_2, variable p
output: node n in (modified) D representing *if* p *then* F_1 *else* F_2

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begin
  if $n_1 = n_2$ **then return** n_1;

end

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begin
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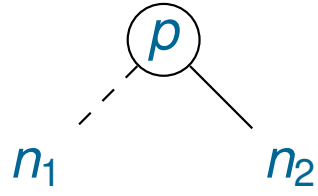
  if D contains a node n having the form



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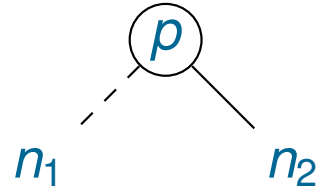
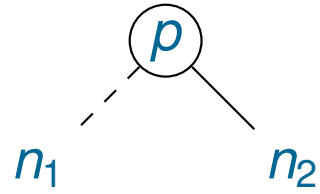


 then return n;
 else add to D a new node n of the form



 return n
end

Building OBDDs

procedure *obdd*(F)
input: propositional formula F
parameters: global dag D
output: a node n in (modified) D which represents F
begin
  $F := simplify(F)$
  if $F = \bot$ **then return** $\boxed{0}$
  if $F = \top$ **then return** $\boxed{1}$
  $p := max\_variable(F)$
  $n_1 := obdd(F_p^{\bot})$
  $n_2 := obdd(F_p^{\top})$
  return $integrate(n_1, p, n_2, D)$
end

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procedure *obdd*(F)
input: propositional formula F
parameters: global dag D
output: a node n in (modified) D which represents F
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  $F := simplify(F)$
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- Redundancy elimination is done by the procedure *integrate*.

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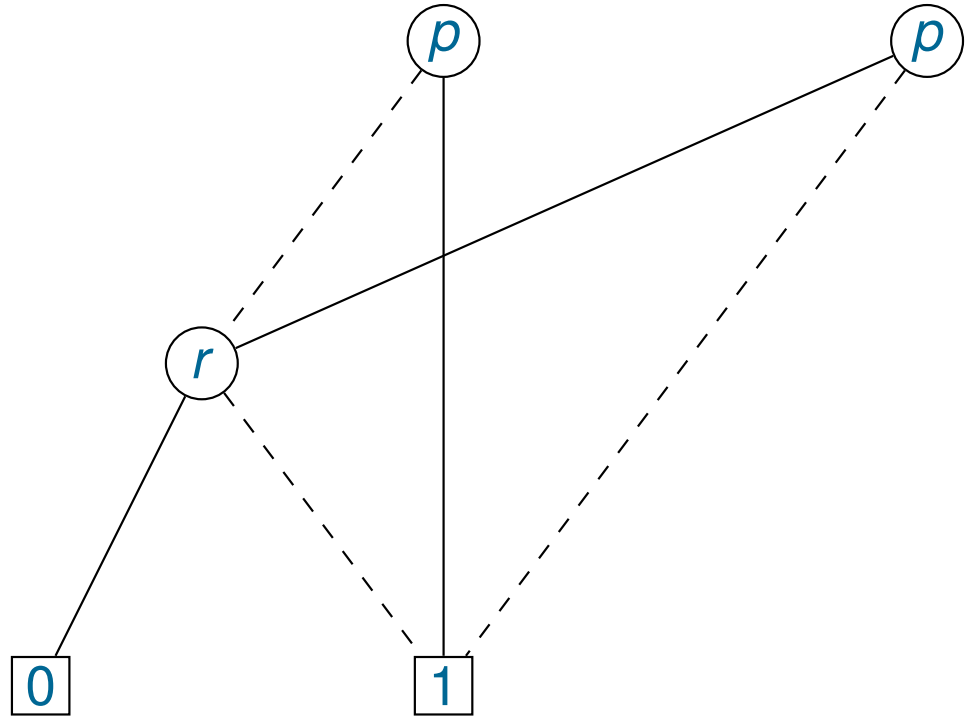
$obdd((q \rightarrow p) \wedge r \rightarrow (p \leftrightarrow r) \wedge q)$



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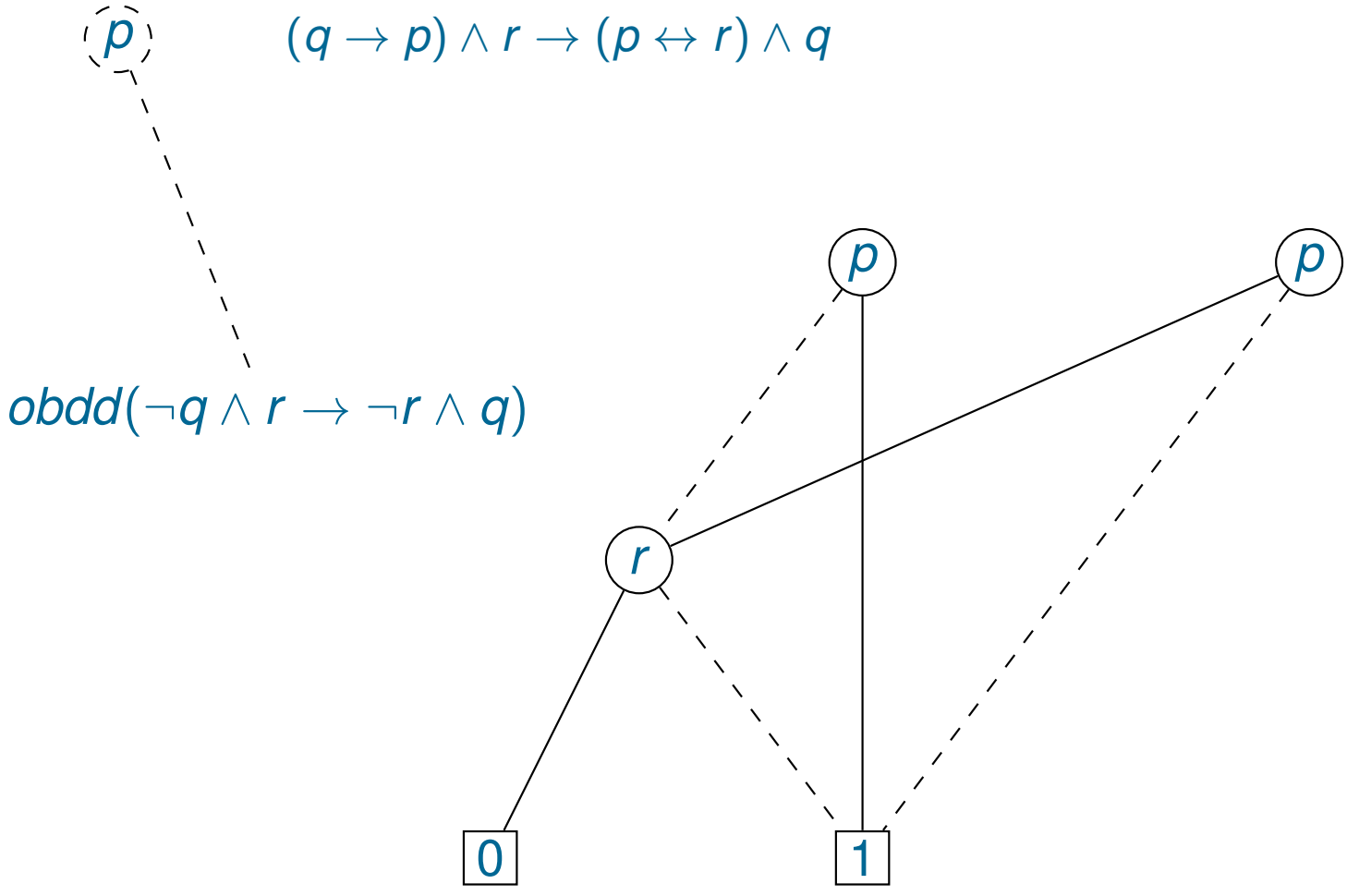

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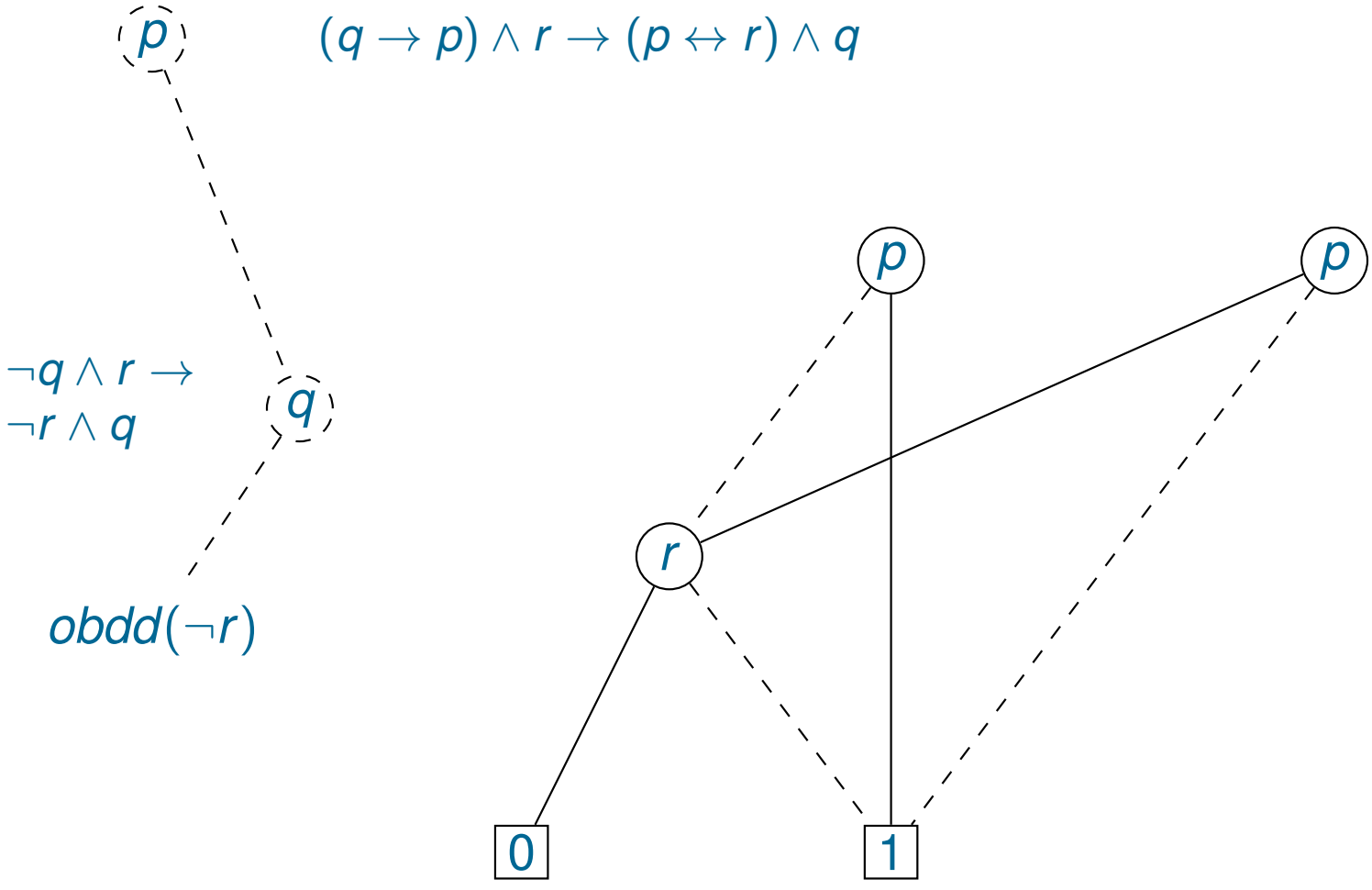

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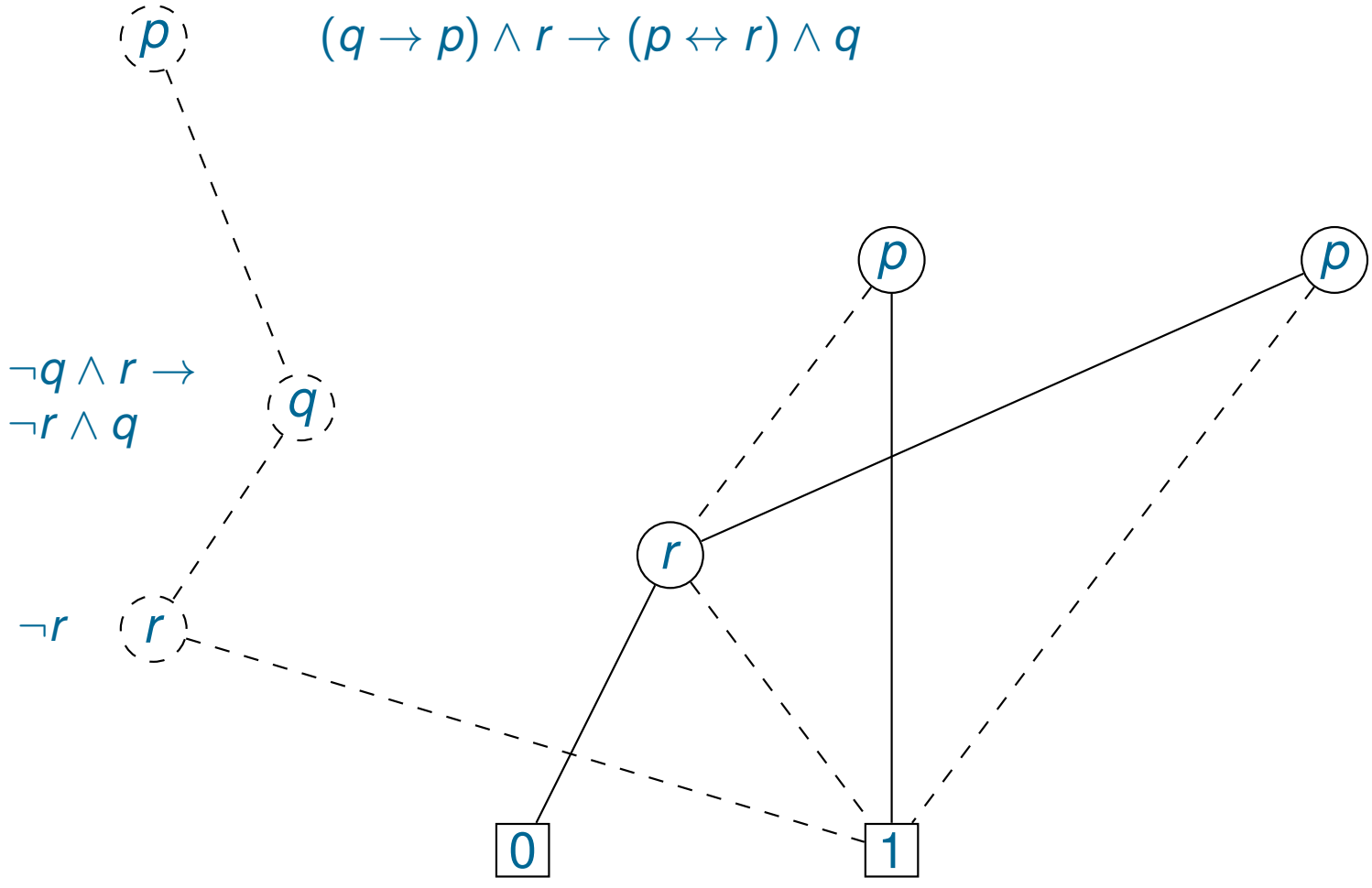

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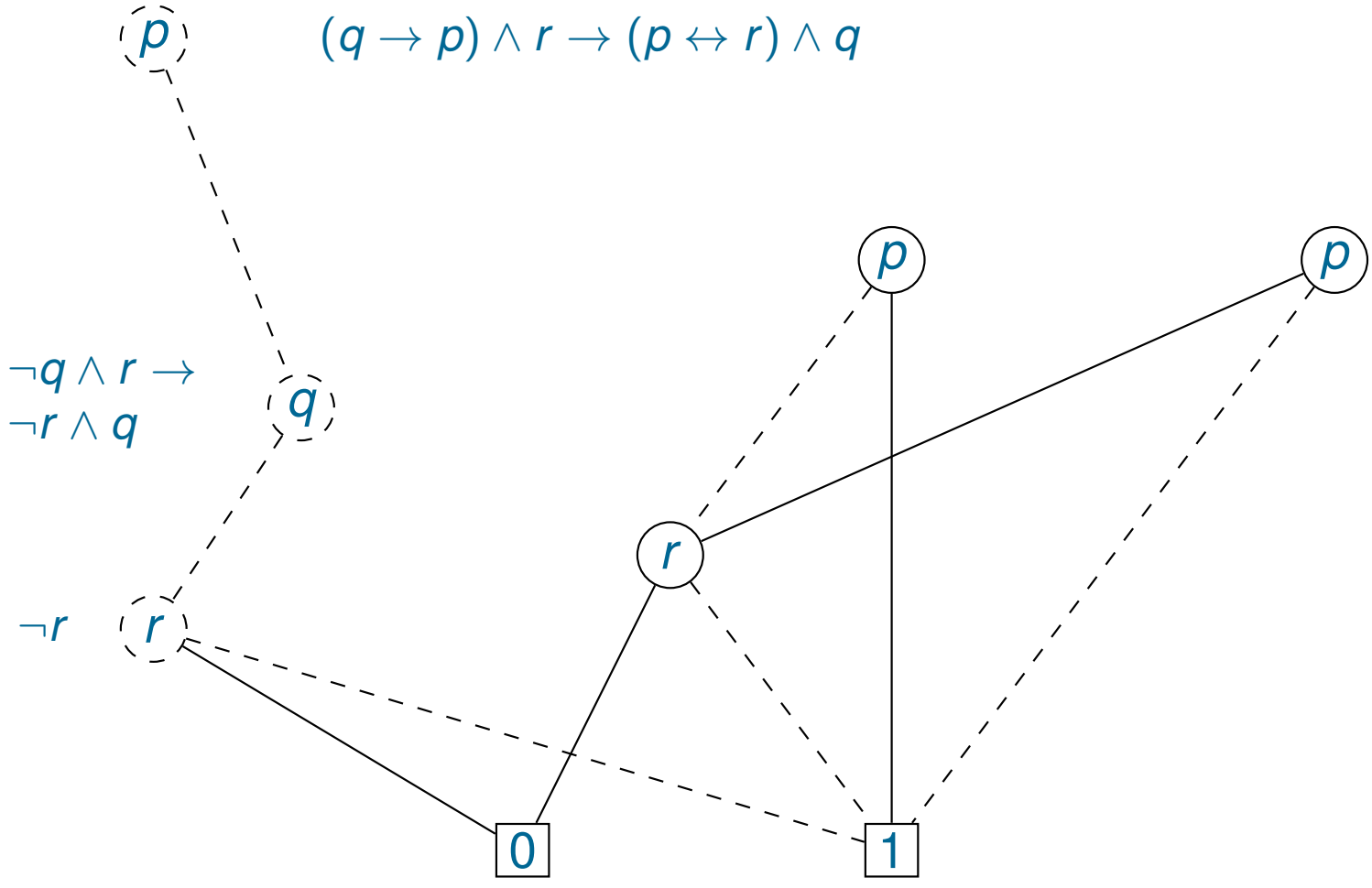

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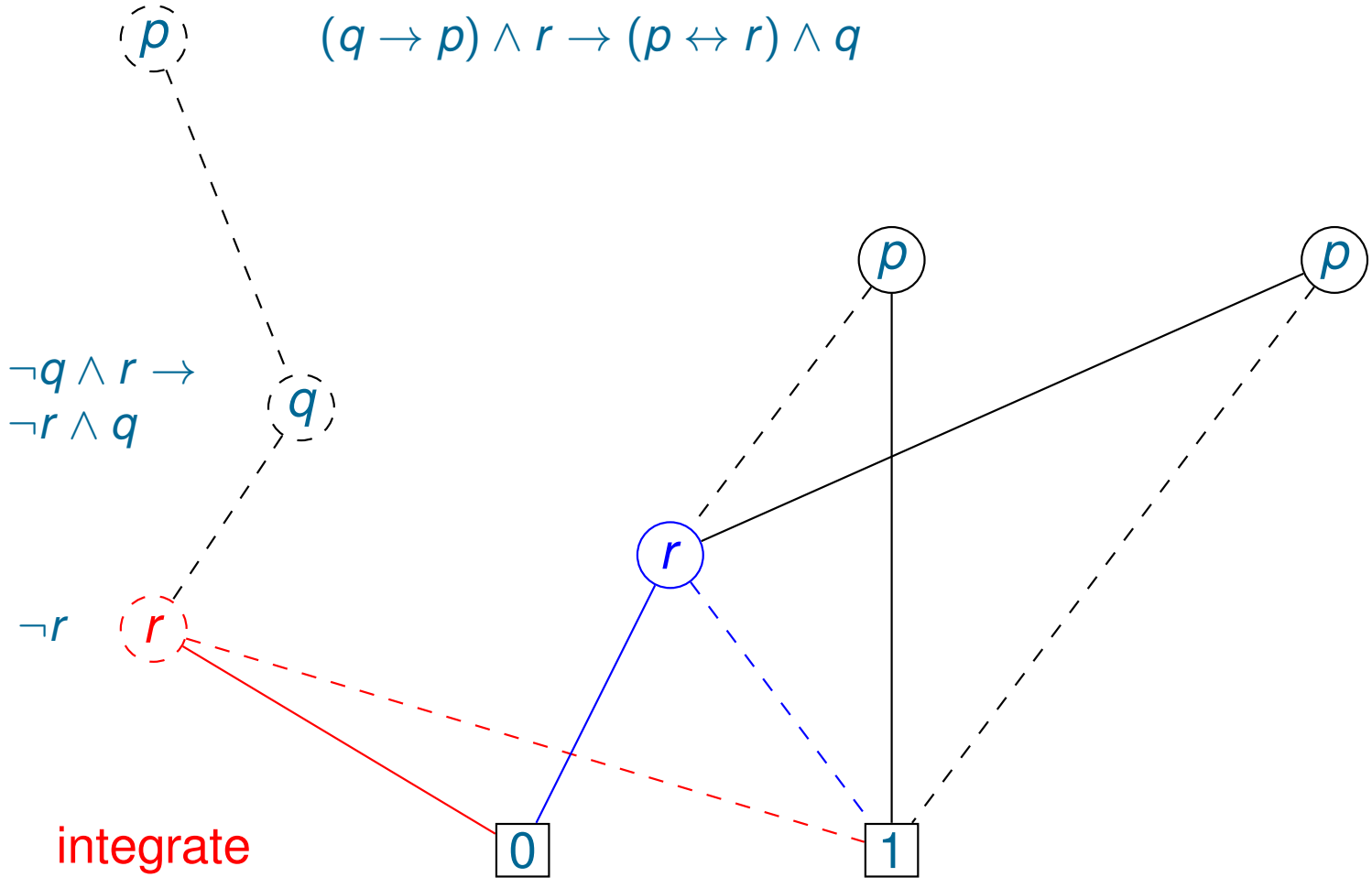

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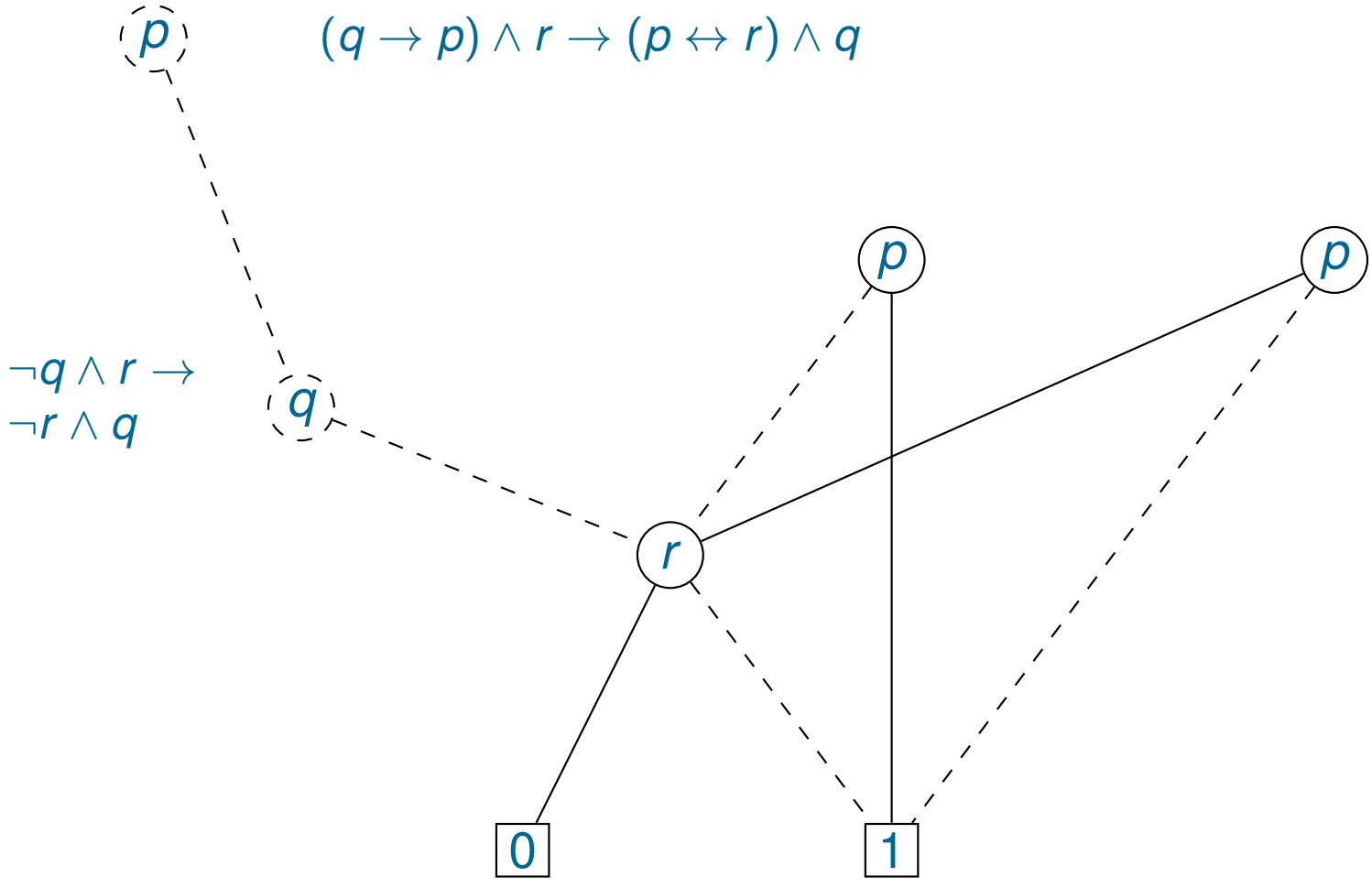

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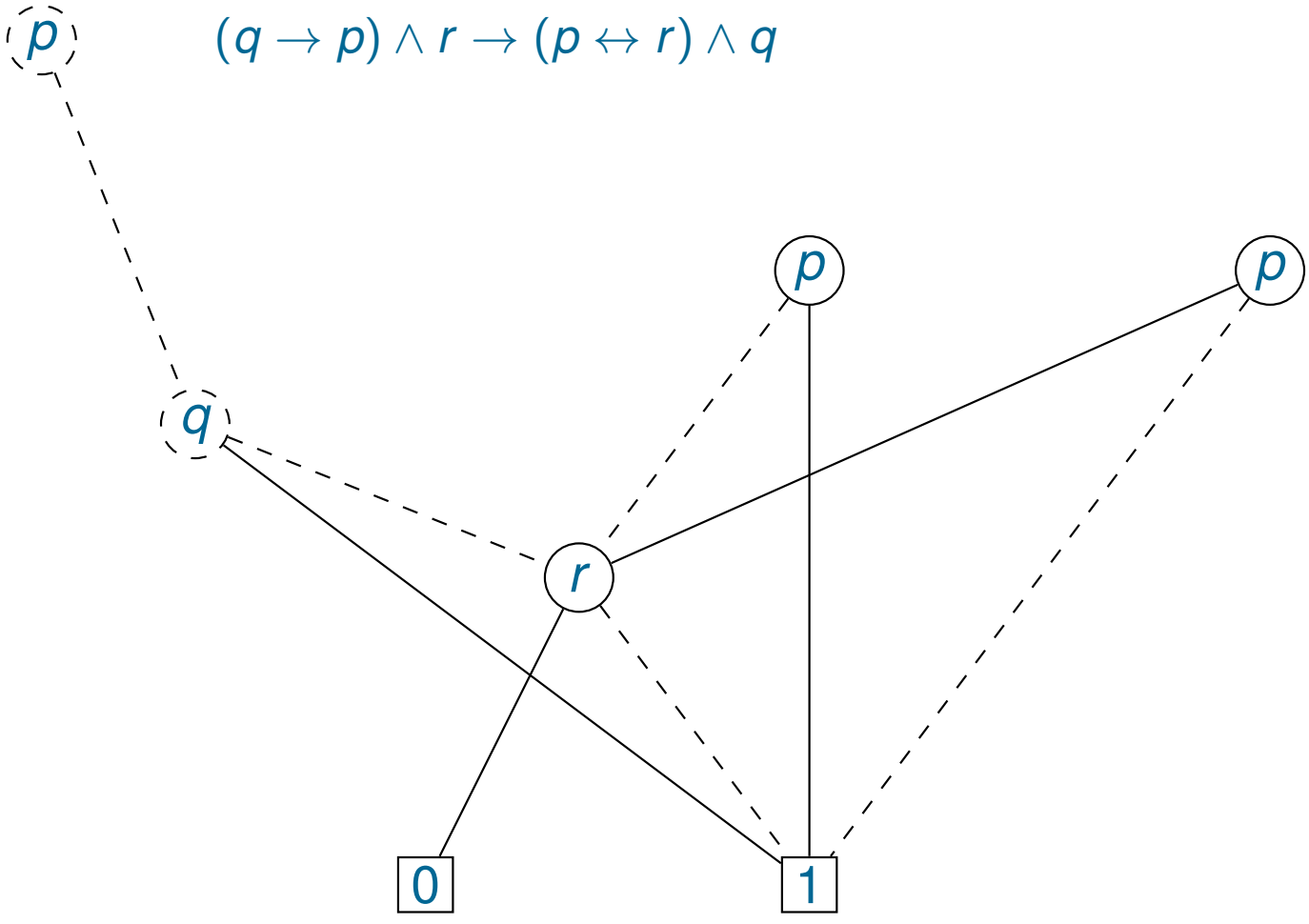

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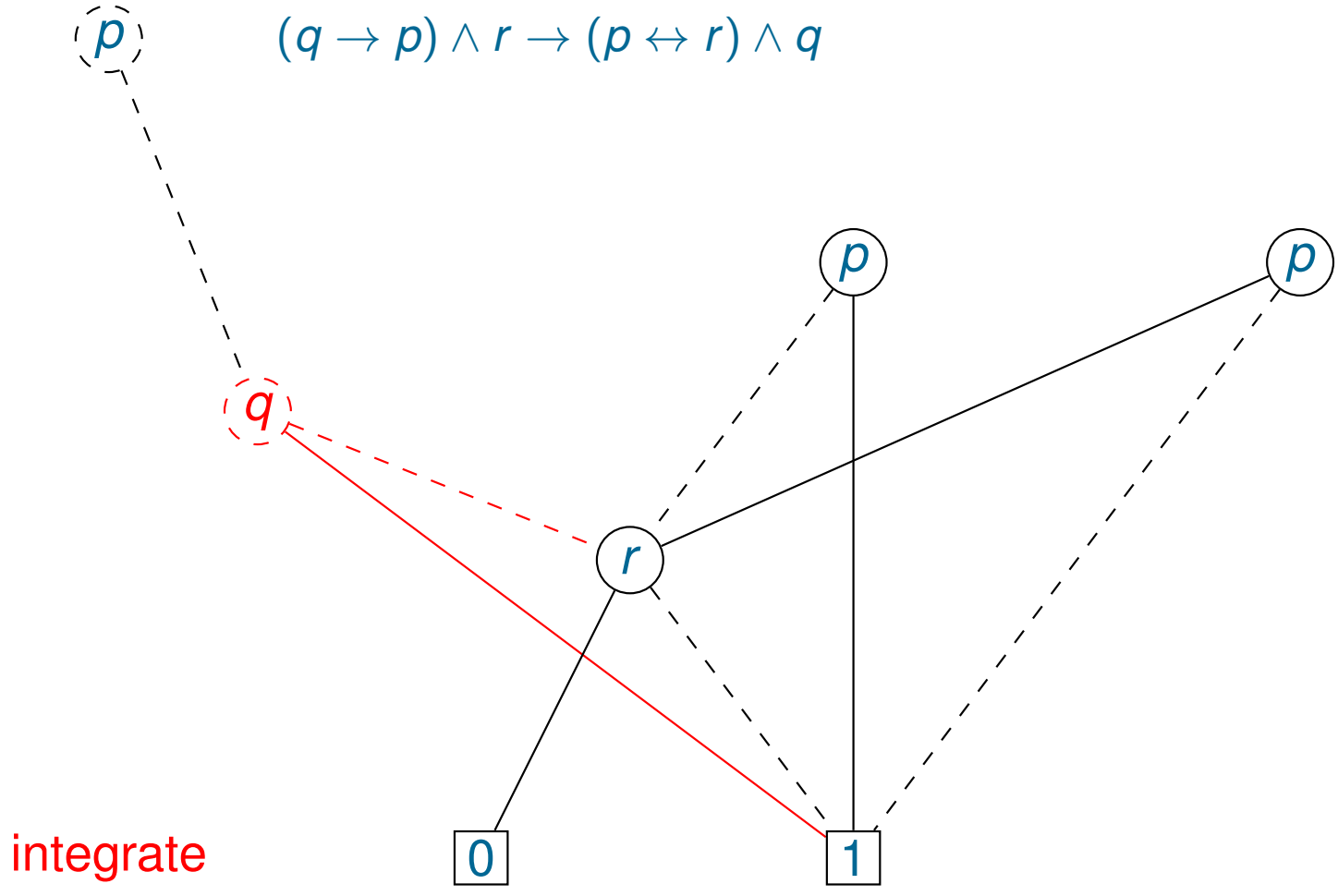

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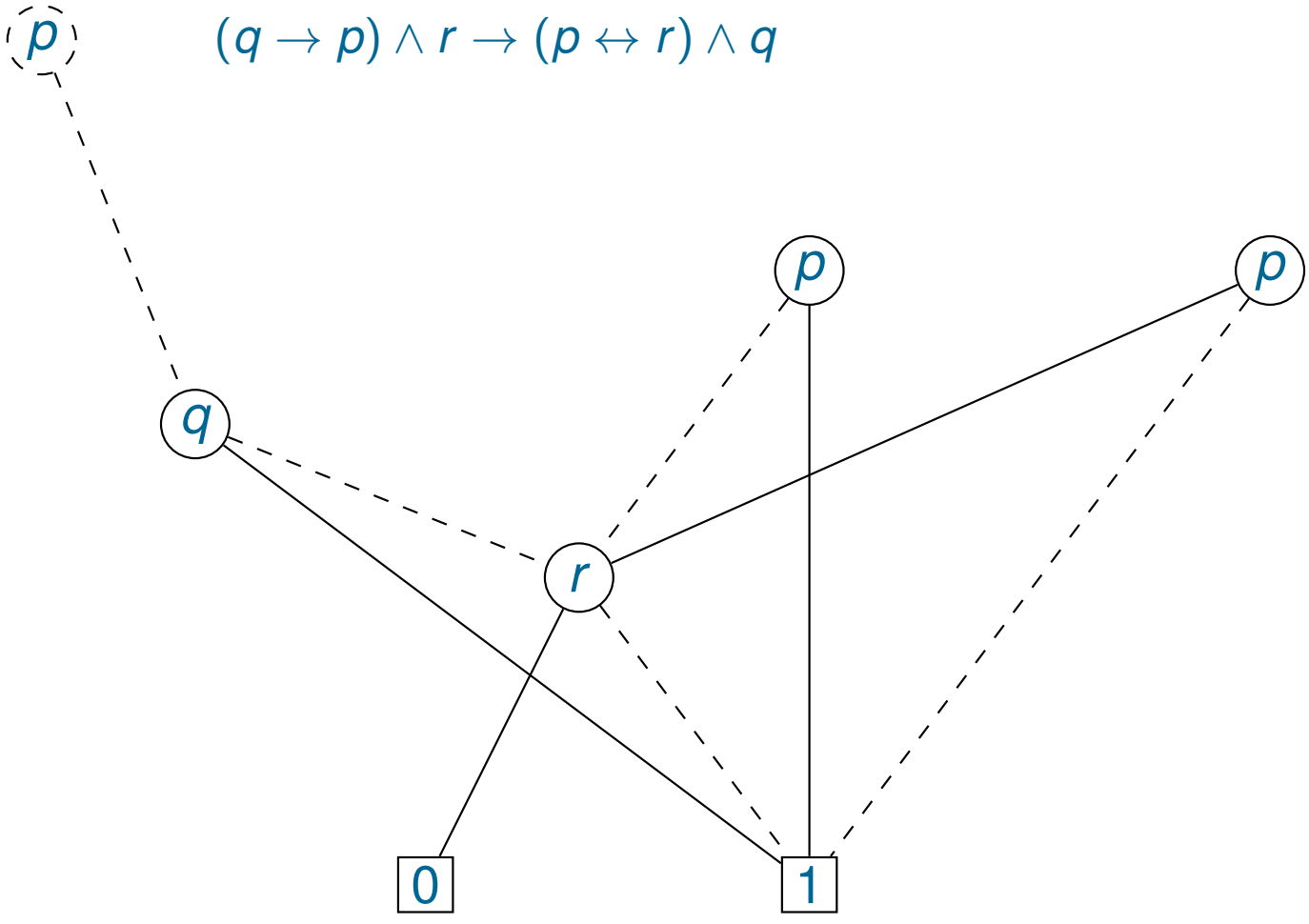

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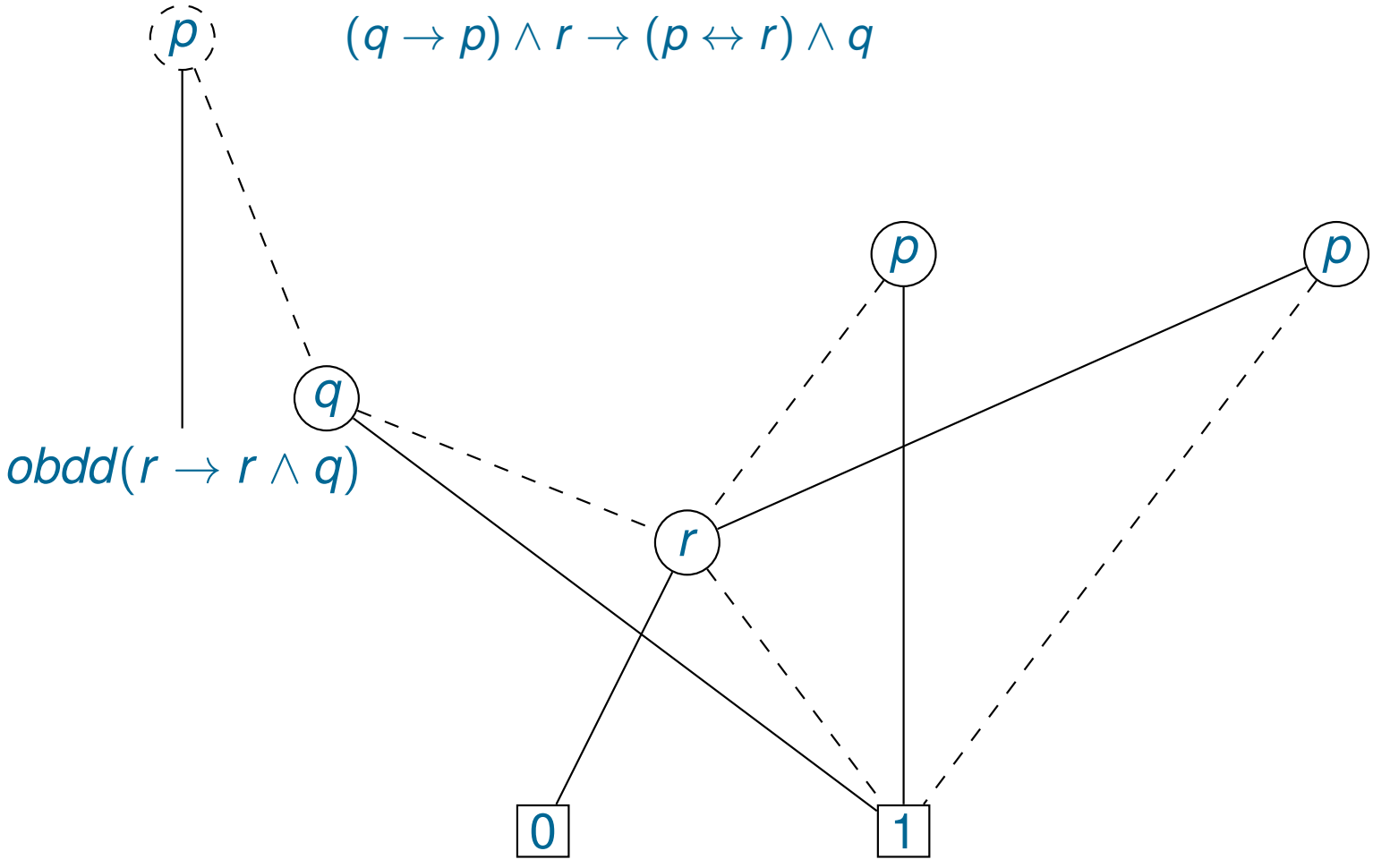

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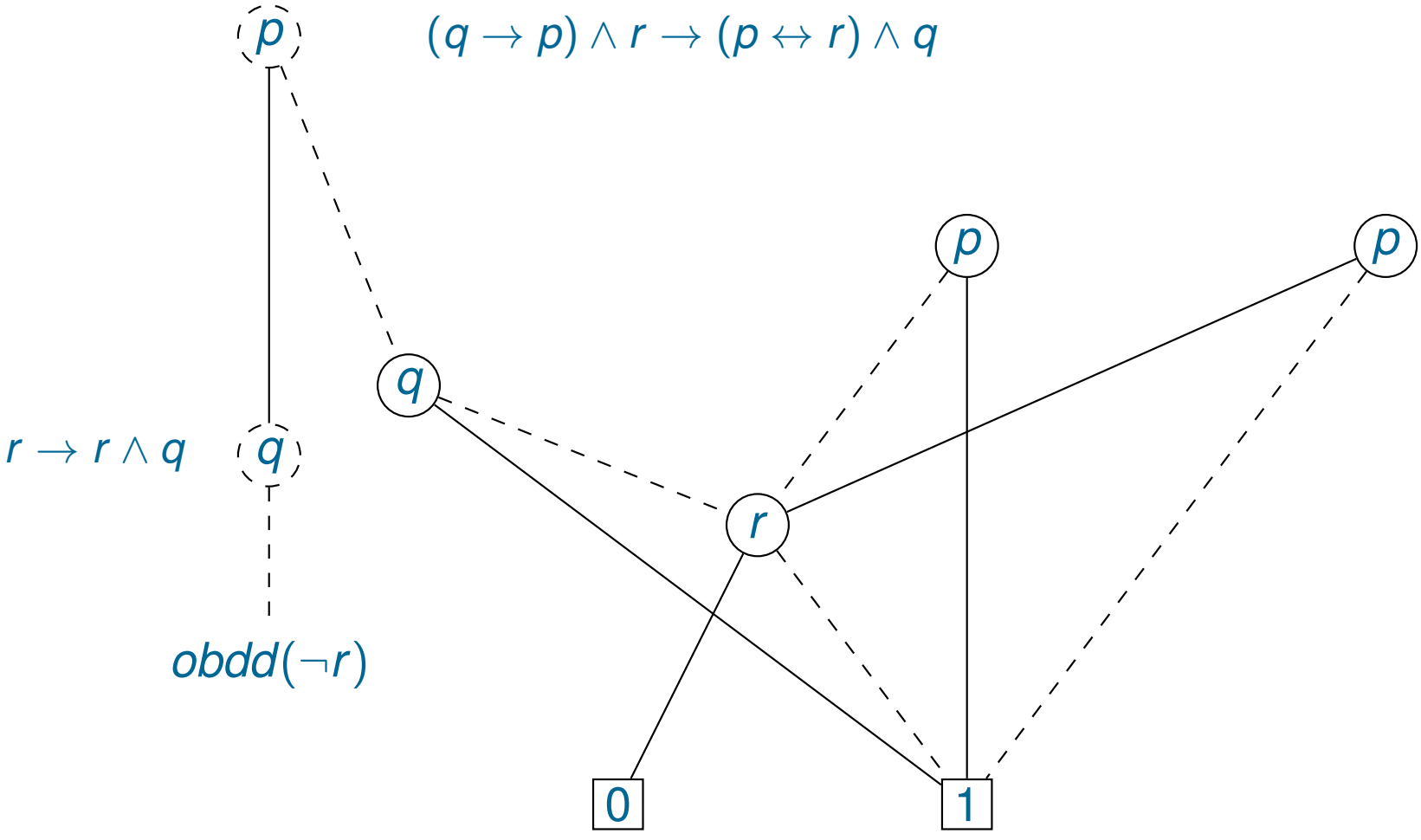

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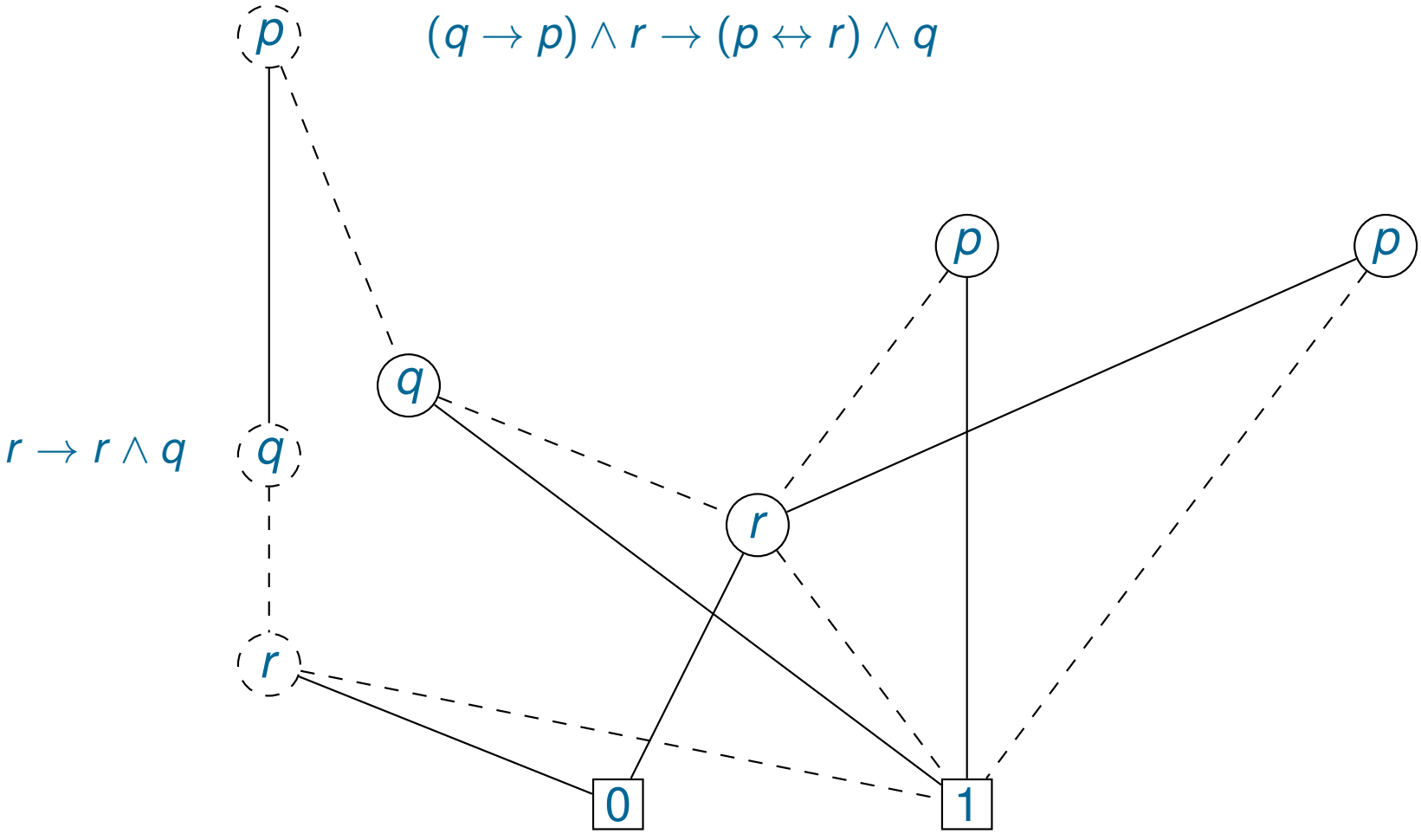

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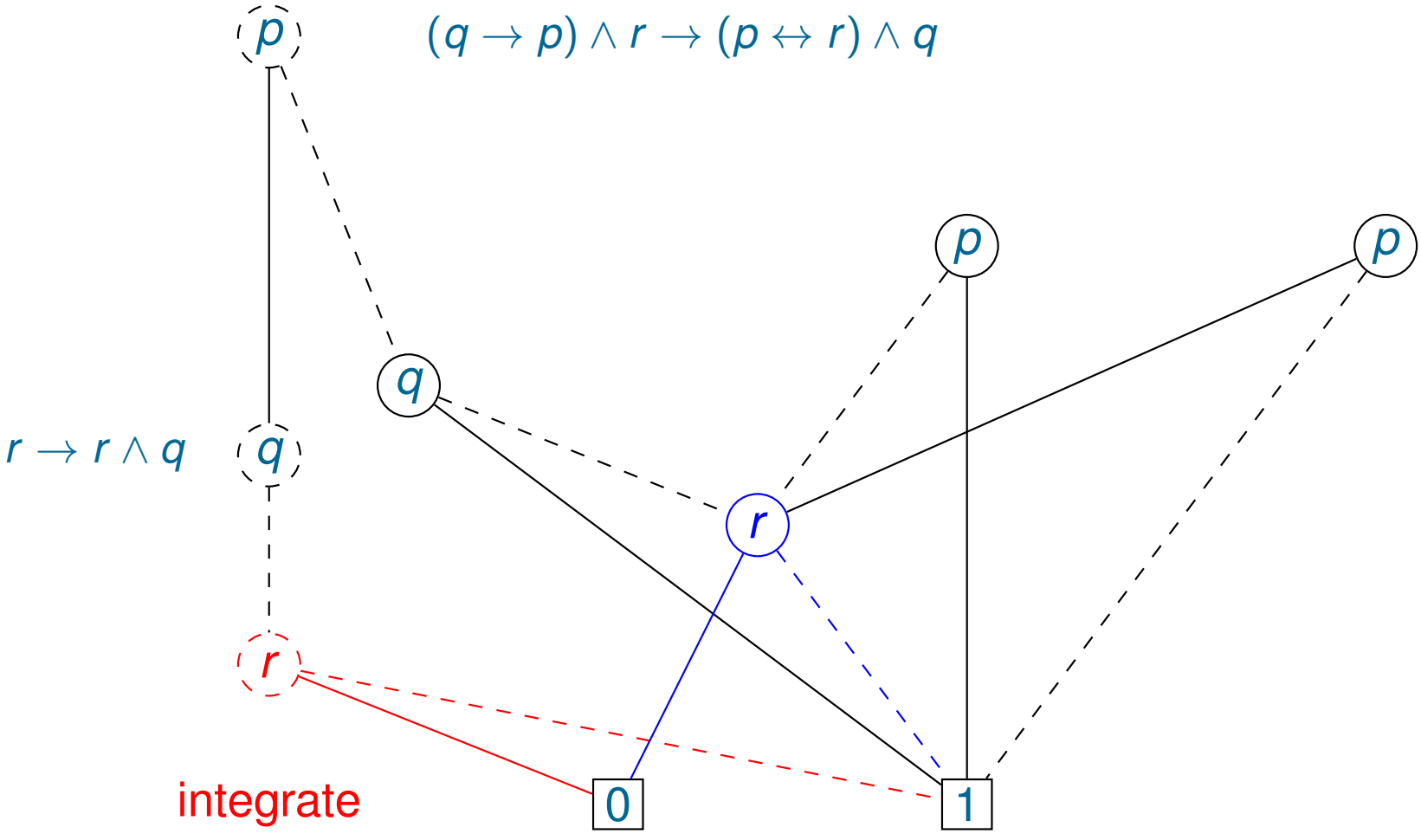

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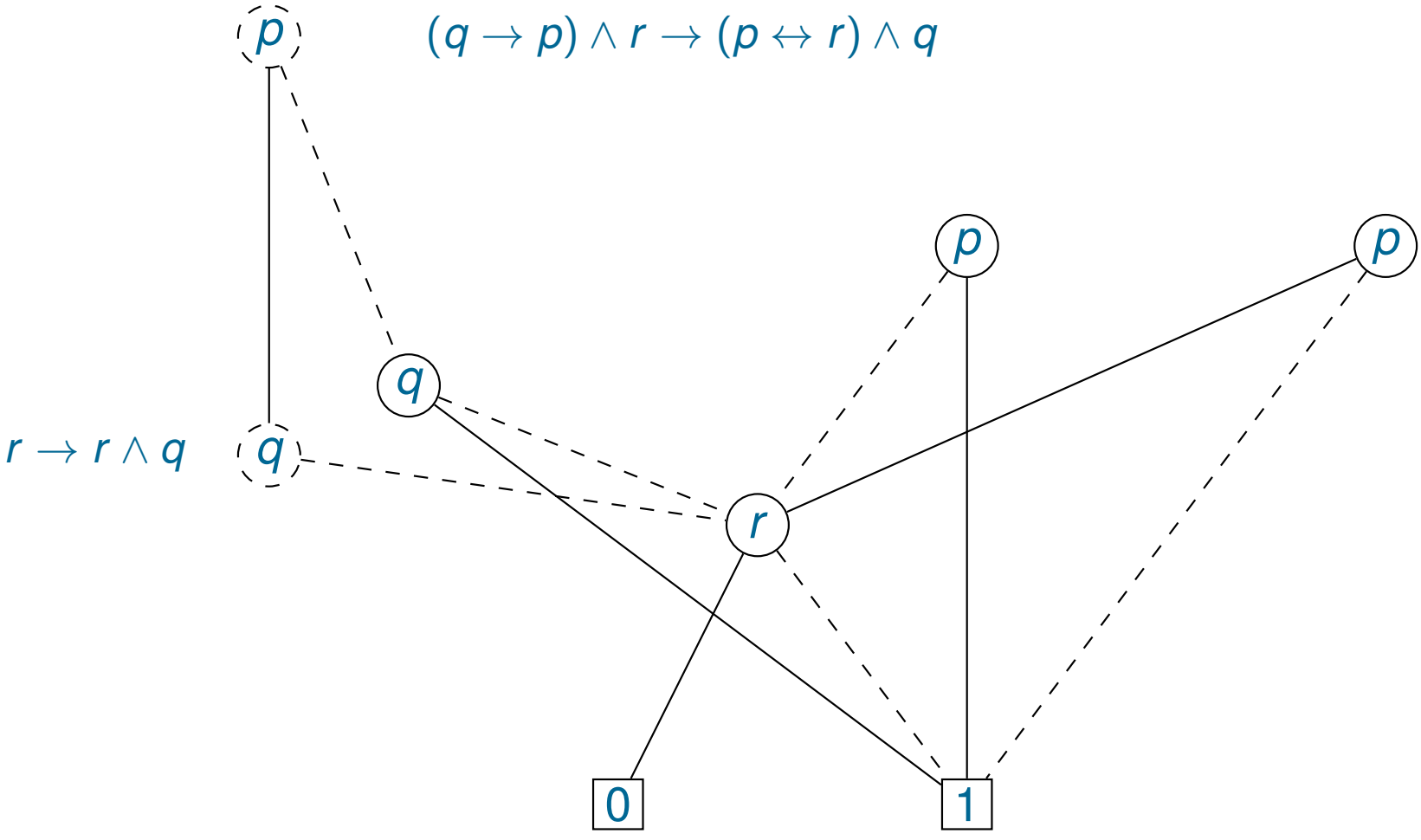

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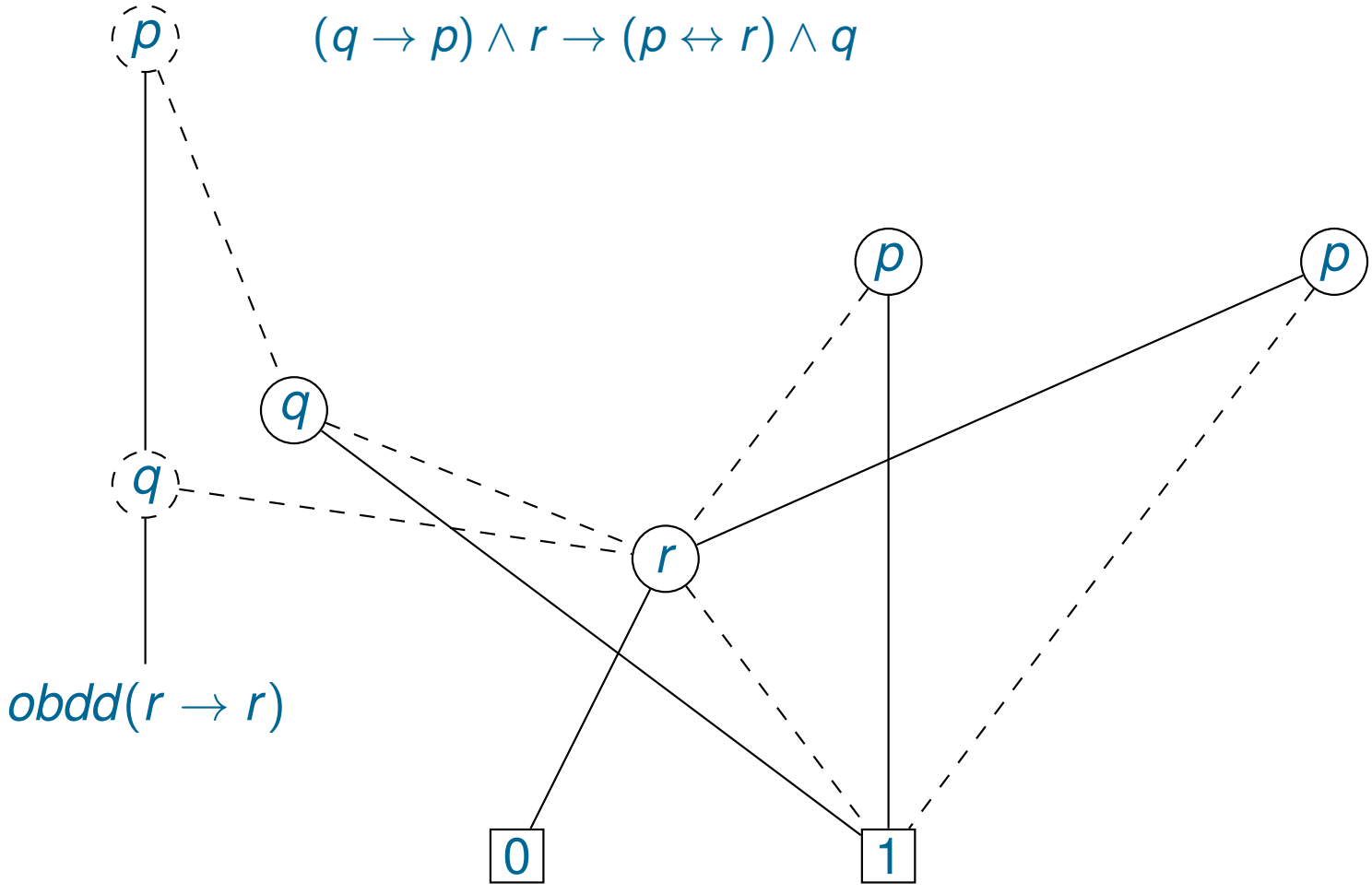

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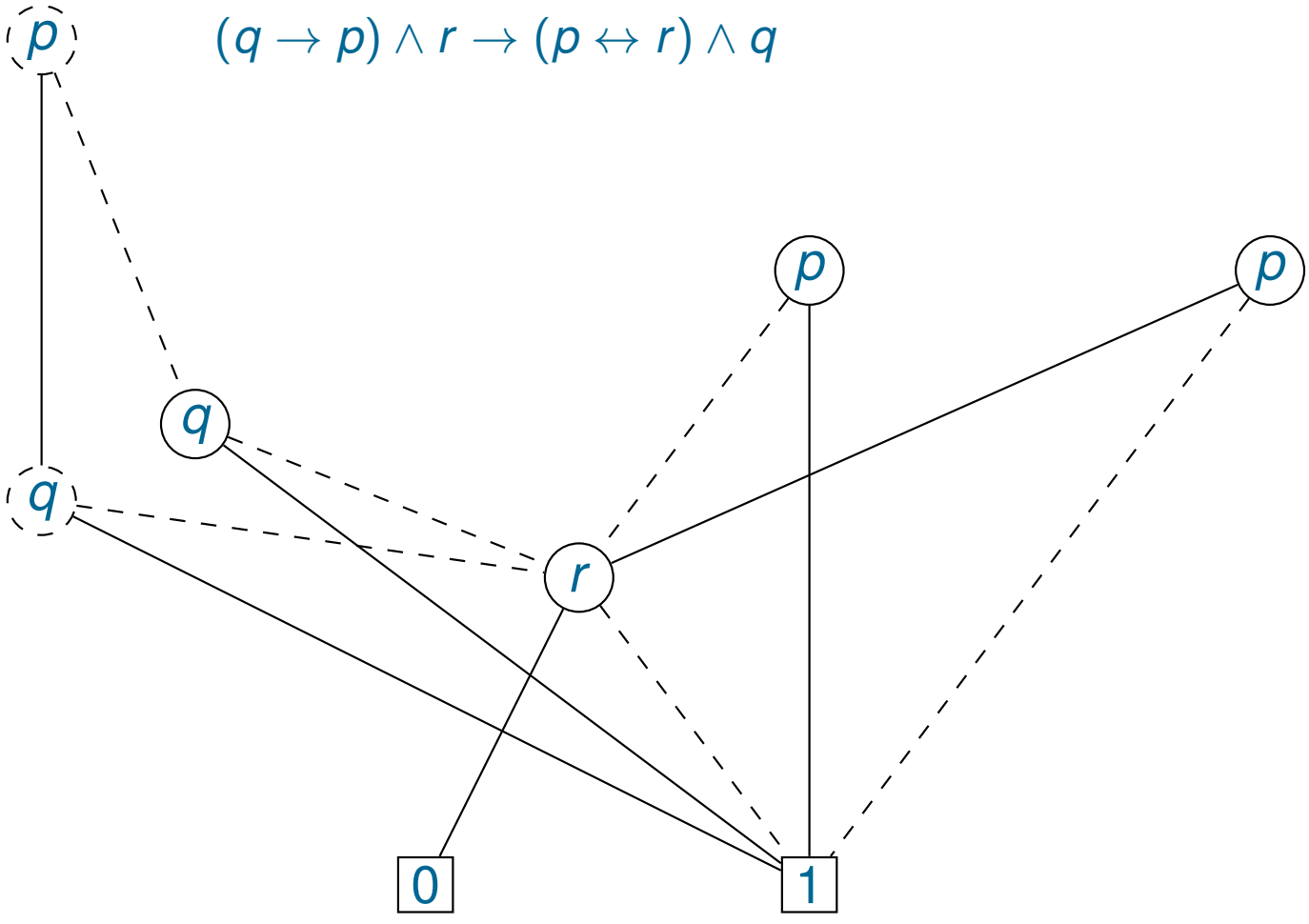

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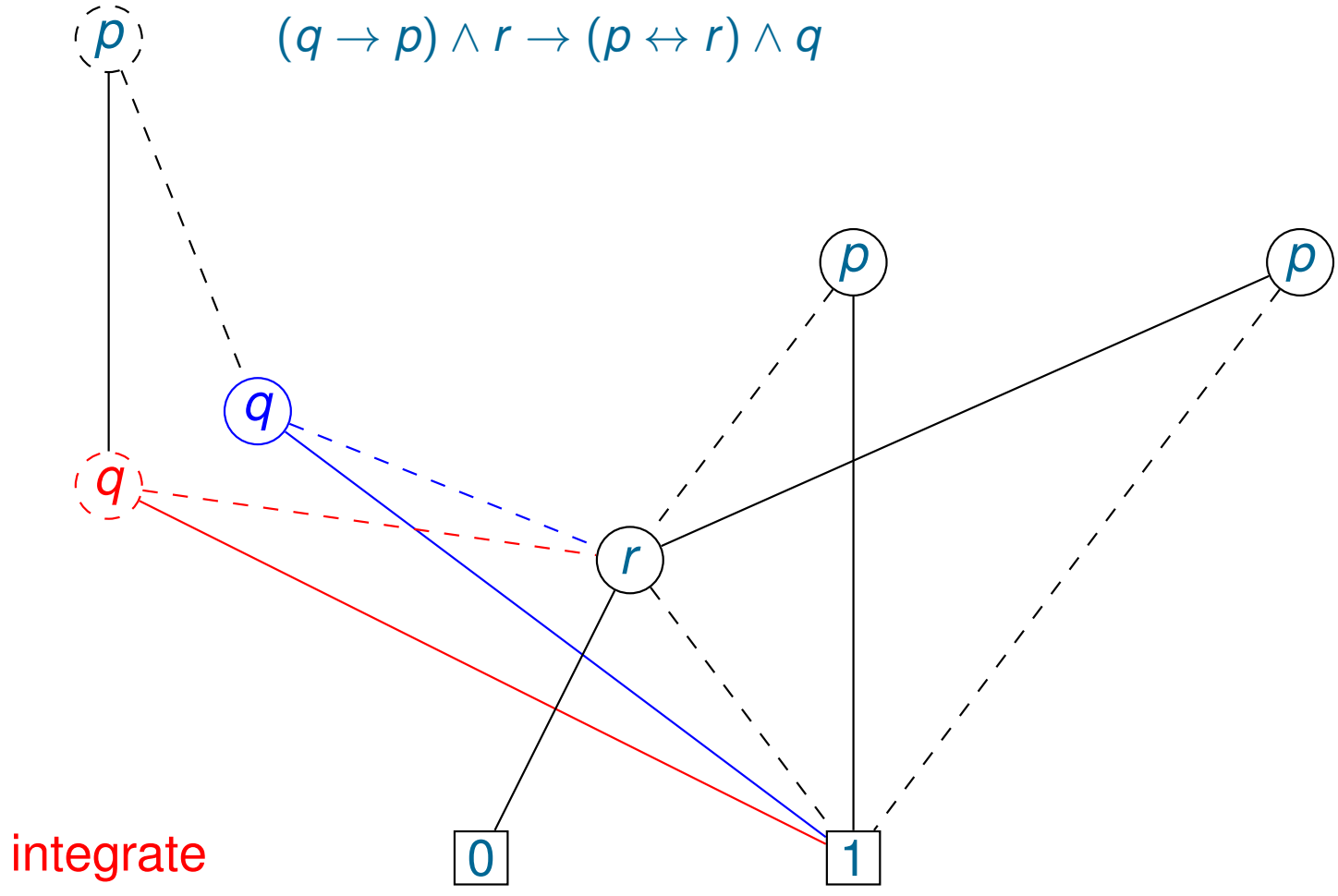

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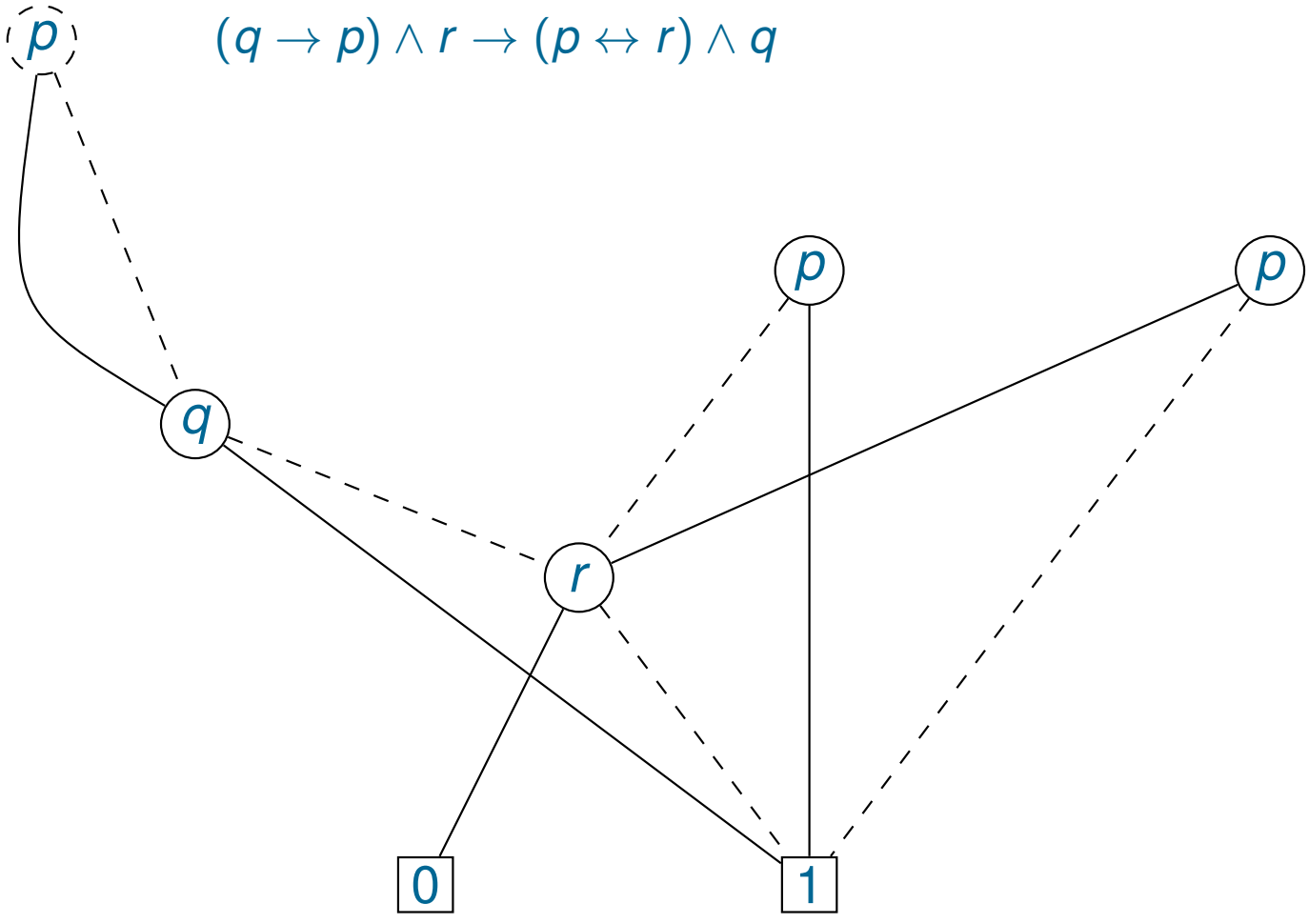

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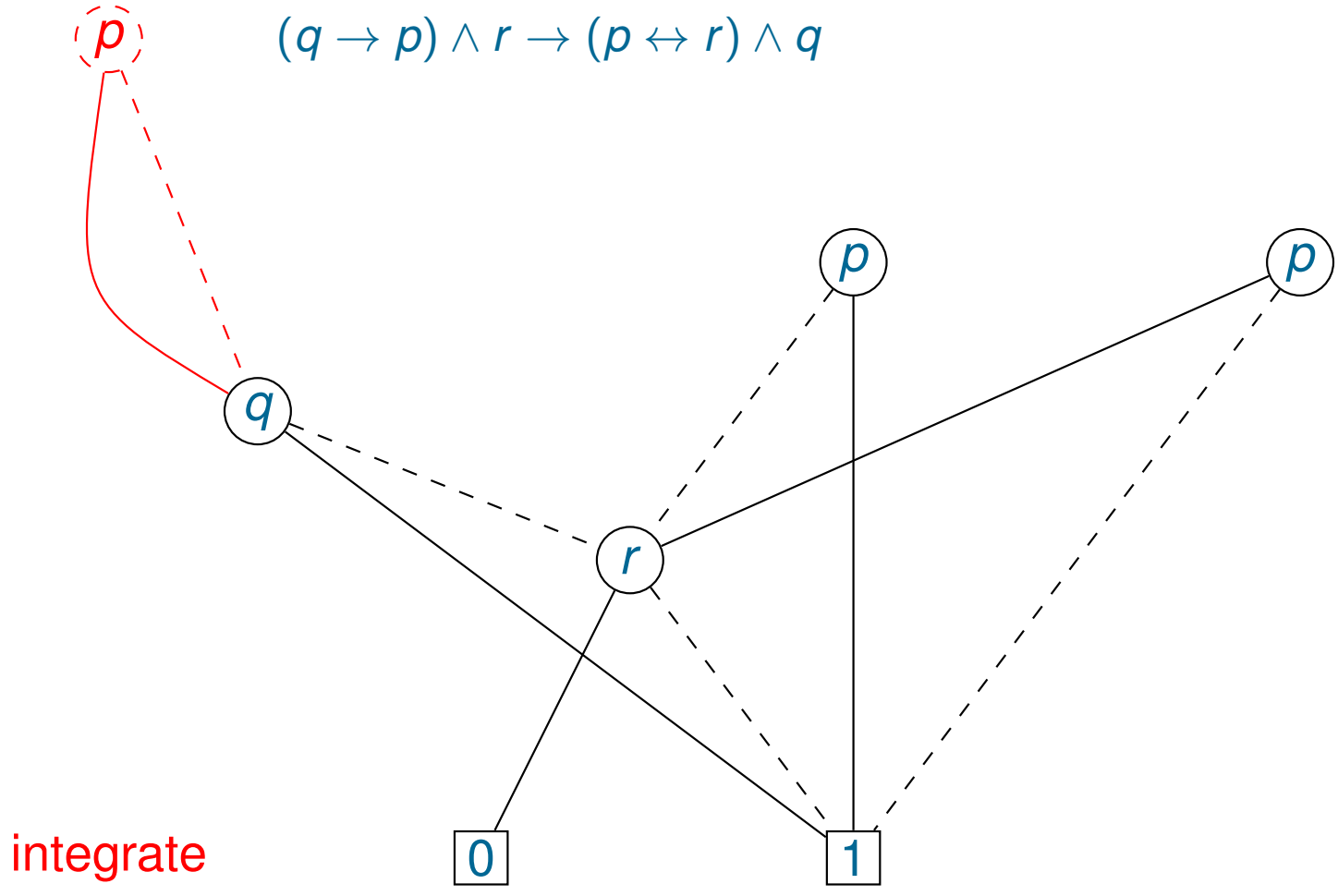

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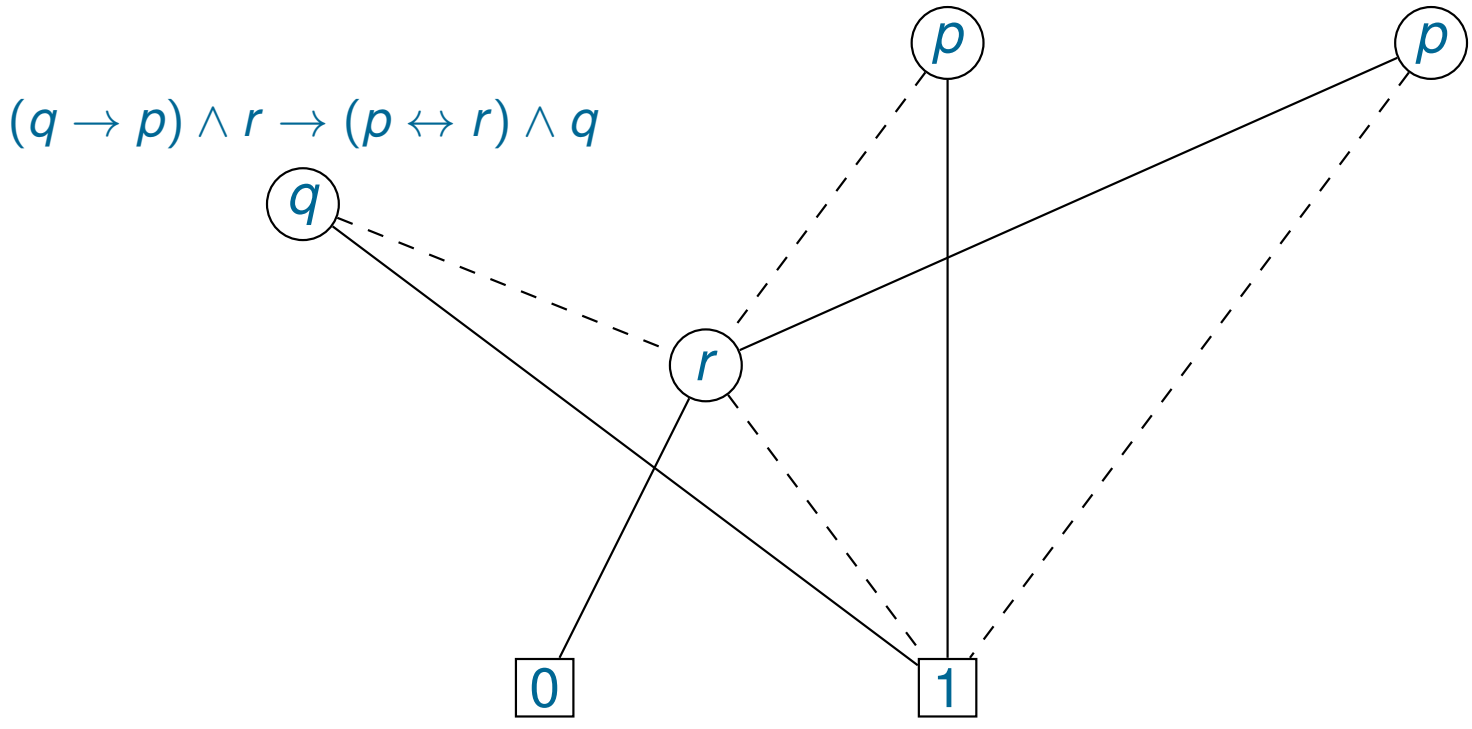

We return the new node rooted at q.

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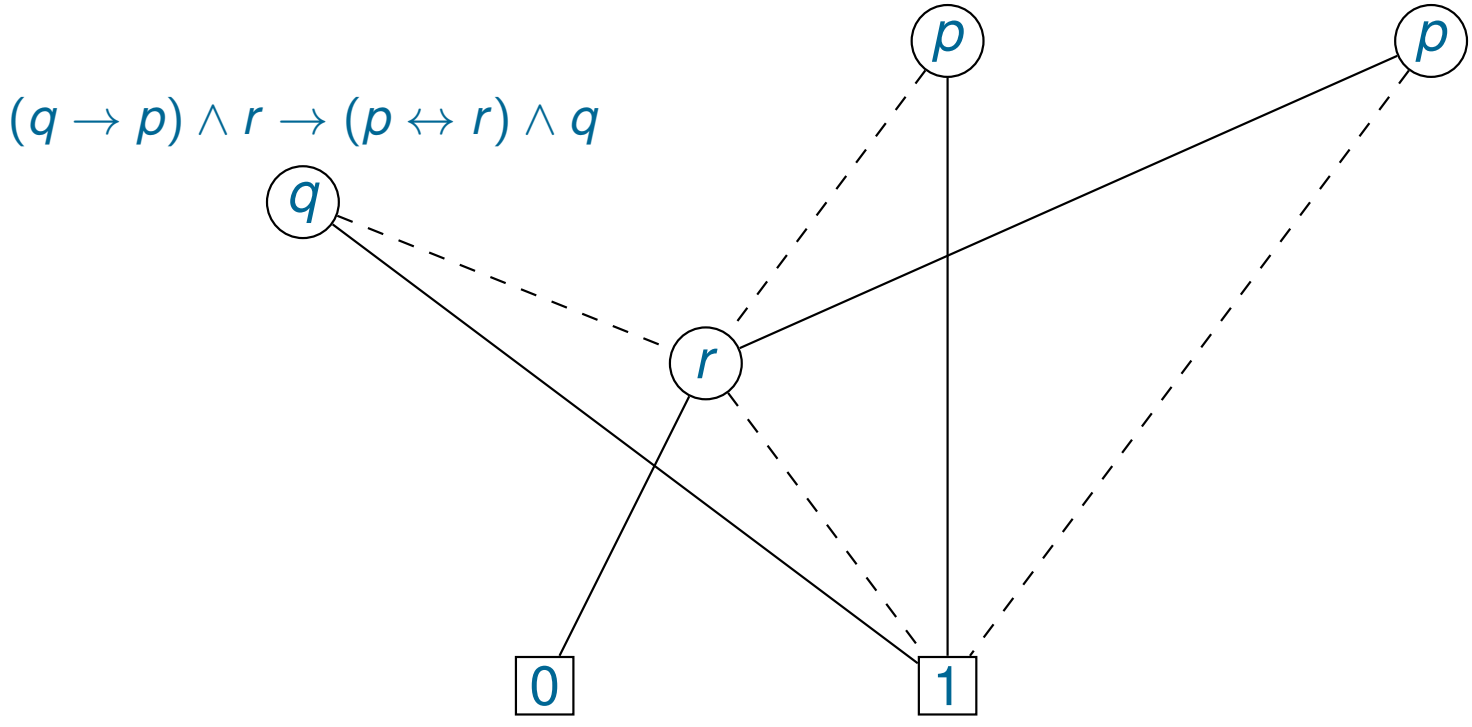

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Note that the application of this procedure modified the global dag.

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- We assume a global dag that contains all OBDDs so that all isomorphic subdags are shared.
- Use the following fundamental property of if-then-else: if-then-else commutes with functions.

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- Consider the expression in Java

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- Can we simplify it?
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&\quad \textit{if } p \textit{ then } l_n \textit{ else } r_n\ ) =\\
&\textit{if } p \textit{ then } f(l_1, \ldots, l_n) \textit{ else } f(r_1, \ldots, r_n)
\end{aligned}
$$

Proof? By case analysis on p.

This is what this formula teaches us: to apply f to n OBDDs rooted at p,

- apply f to the subdags corresponding to $p = 0$ (the subdags under the dashed edge), obtaining a dag D_0;
- apply f to the subtrees corresponding to $p = 1$ (the subdags under the solid edge), obtaining a dag D_1;

Fundamental property of if-then-else

In fact, if-then-else commutes with any function.

$$
\begin{aligned}
&f(\ \textit{if } p \textit{ then } l_1 \textit{ else } r_1,\\
&\quad \ldots,\\
&\quad \textit{if } p \textit{ then } l_n \textit{ else } r_n\ ) =\\
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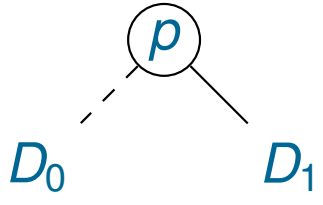
- apply f to the subdags corresponding to $p = 0$ (the subdags under the dashed edge), obtaining a dag D_0;
- apply f to the subtrees corresponding to $p = 1$ (the subdags under the solid edge), obtaining a dag D_1;
- Build and return the dag of the form

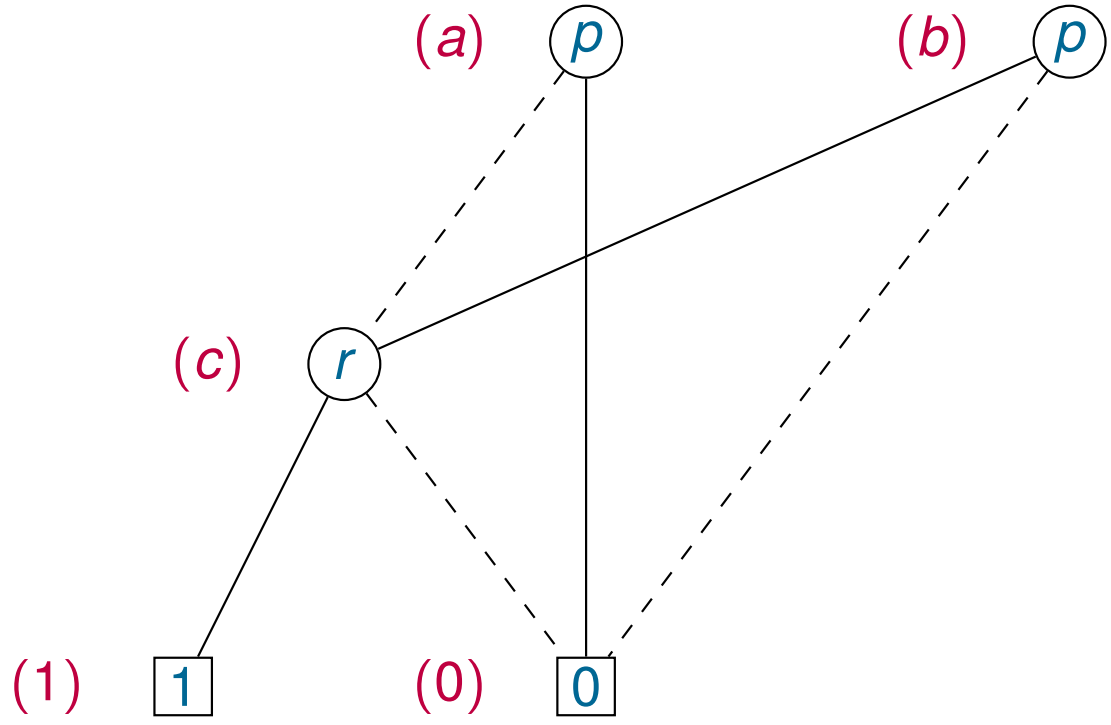


Disjunction

procedure *disjunction*$(n_1, \ldots, n_m)$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $F_1 \vee \ldots \vee F_m$ in (modified) D
begin
  if some n_i is $\boxed{1}$ **then return** $\boxed{1}$
  if $m = 1$ **then return** n_1
  if some n_i is $\boxed{0}$ **then**
    return *disjunction*$(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$
  p := *max_variable*$(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then (l_i, r_i) := $(neg(n_i), pos(n_i))$
      else (l_i, r_i) := (n_i, n_i)
  k_1 := *disjunction*$(l_1, \ldots, l_m)$
  k_2 := *disjunction*$(r_1, \ldots, r_m)$
  return *integrate*(k_1, p, k_2, D)
end

Example: Disjunction $p > r$

(a) represents $(\neg p \wedge r)$; (b) represents $(p \wedge r)$



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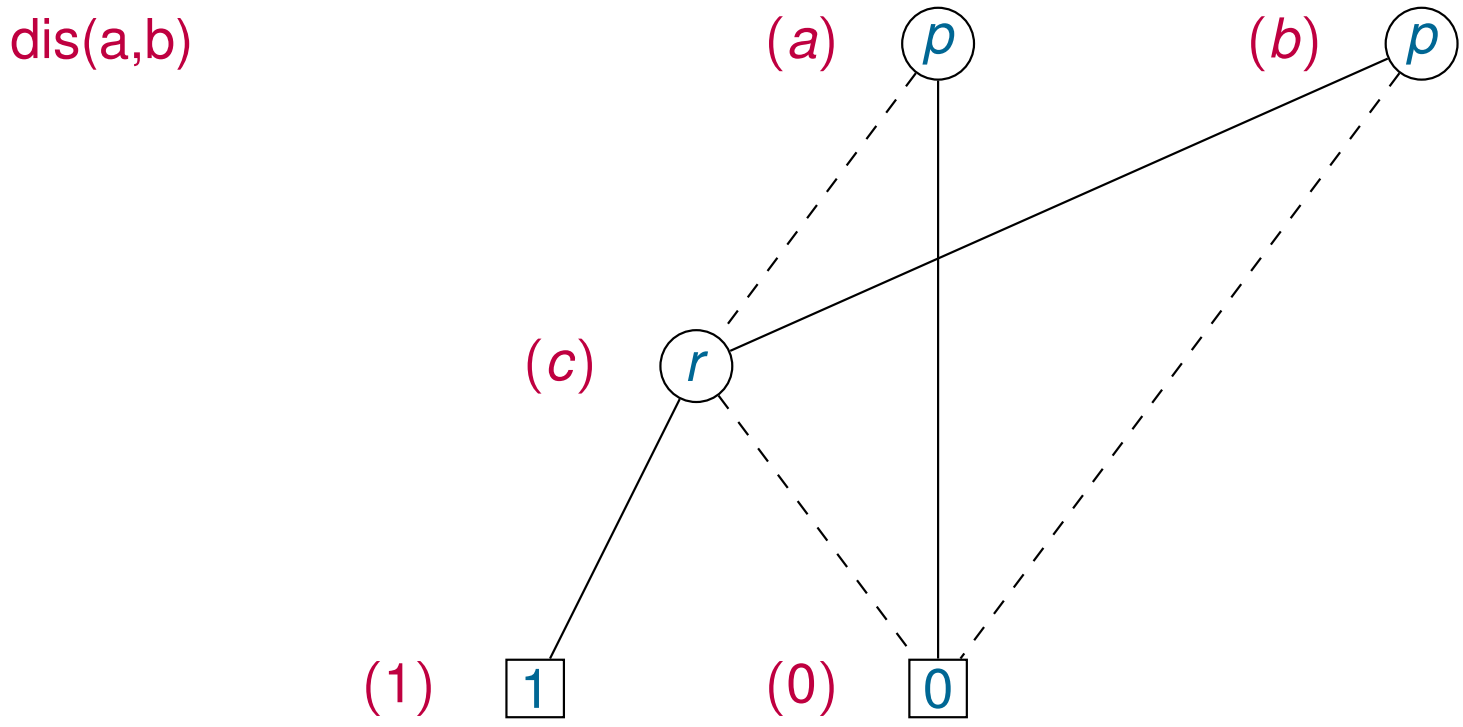

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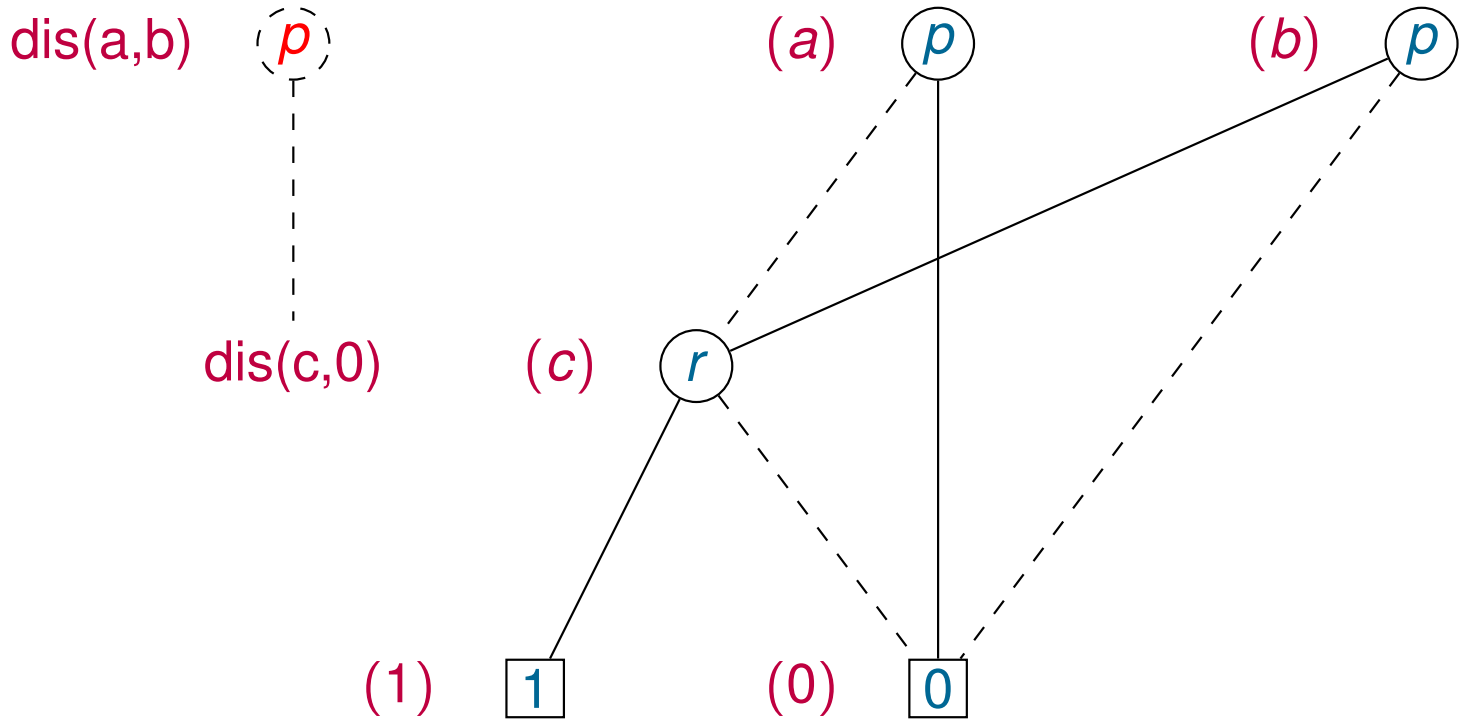

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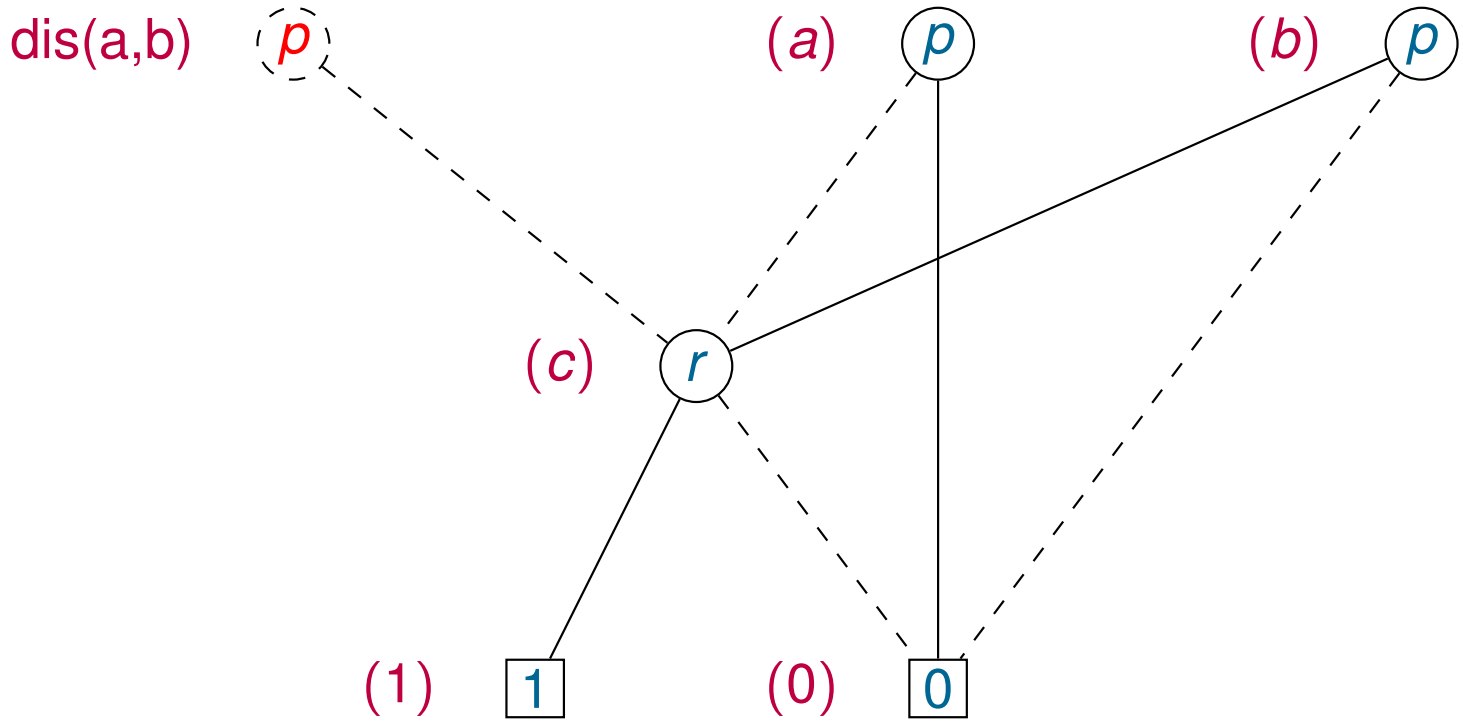

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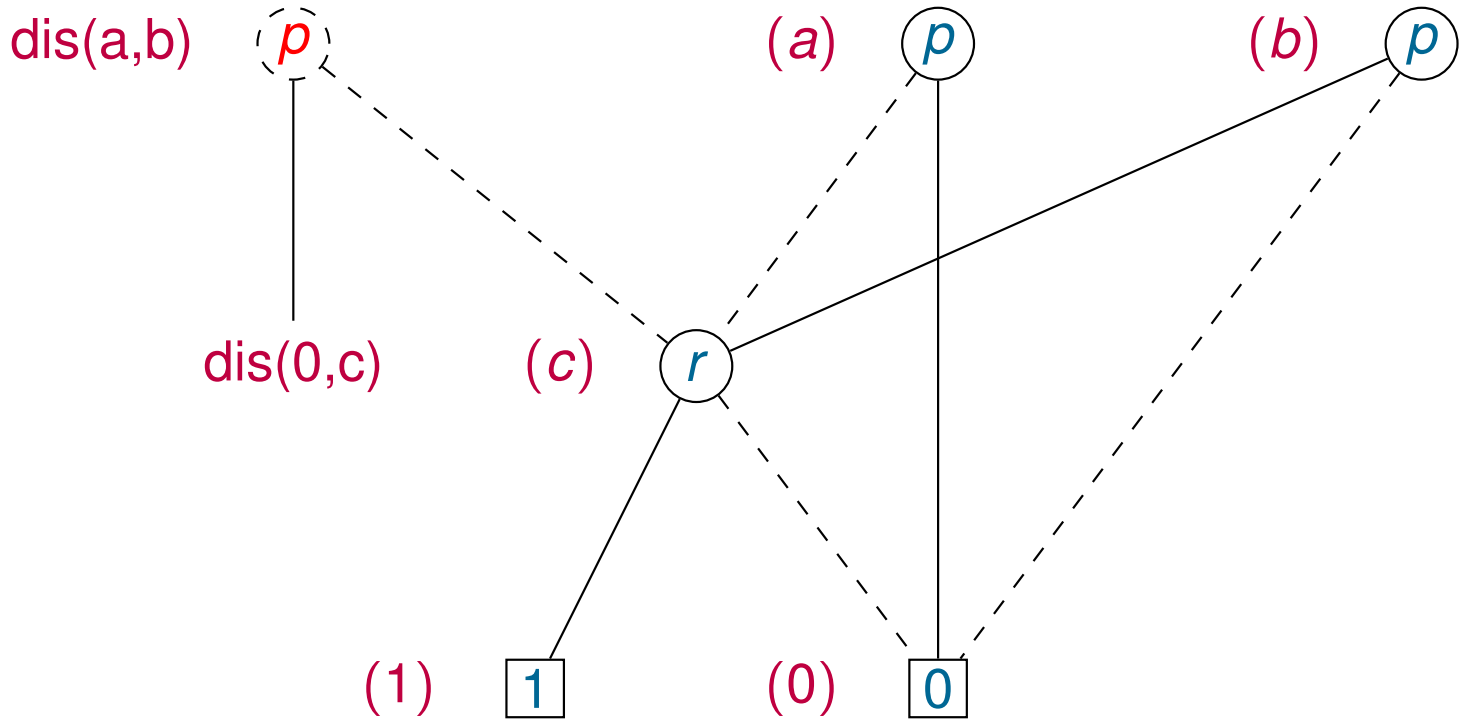

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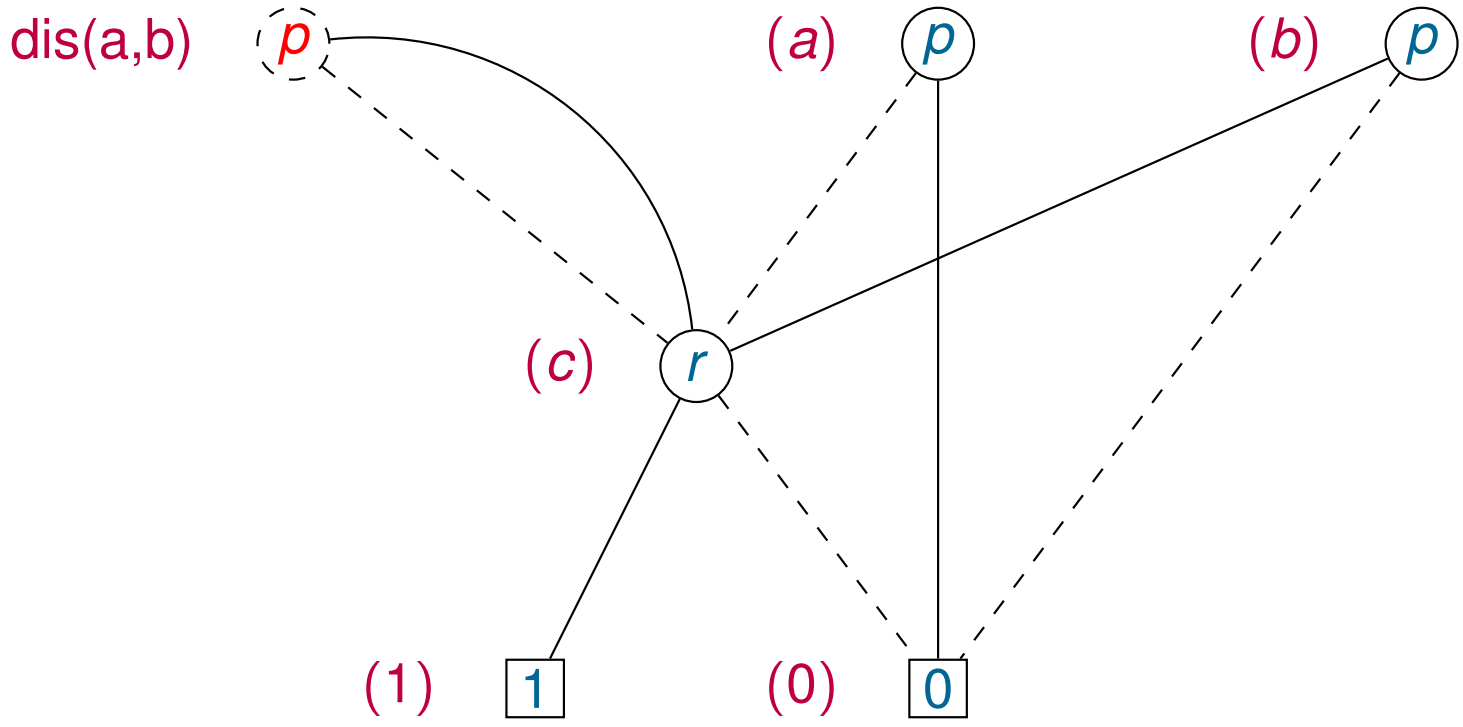

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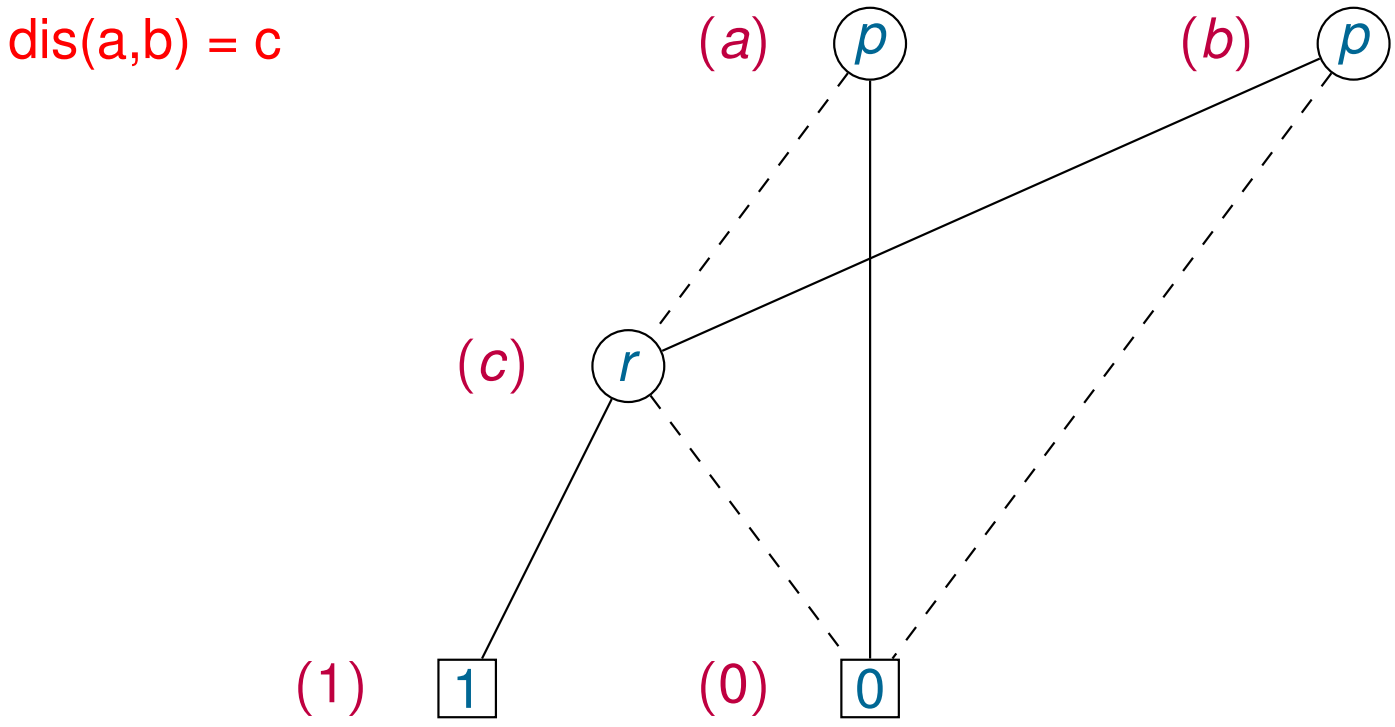

Disjunction

procedure *disjunction*$(n_1, \ldots, n_m)$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $F_1 \vee \ldots \vee F_m$ in (modified) D
begin
  if some n_i is $\boxed{1}$ **then return** $\boxed{1}$
  if $m = 1$ **then return** n_1
  if some n_i is $\boxed{0}$ **then**
    return *disjunction*$(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$
  p := *max_variable*$(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then (l_i, r_i) := $(neg(n_i), pos(n_i))$
      else (l_i, r_i) := (n_i, n_i)
  k_1 := *disjunction*$(l_1, \ldots, l_m)$
  k_2 := *disjunction*$(r_1, \ldots, r_m)$
  return *integrate*(k_1, p, k_2, D)
end

Disjunction

procedure *disjunction*$(n_1, \ldots, n_m)$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $F_1 \vee \ldots \vee F_m$ in (modified) D
begin
  if some n_i is 1 **then return** 1
  if $m = 1$ **then return** n_1
  if some n_i is 0 **then**
    return *disjunction*$(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$
  p := *max_variable*$(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then (l_i, r_i) := $(neg(n_i), pos(n_i))$
      else (l_i, r_i) := (n_i, n_i)
  k_1 := *disjunction*$(l_1, \ldots, l_m)$
  k_2 := *disjunction*$(r_1, \ldots, r_m)$
  return *integrate*(k_1, p, k_2, D)
end

$F \vee \top \equiv \top$
$F \vee \bot \equiv F$

Conjunction

procedure $conjunction(n_1, \ldots, n_m)$
parameters: global dag D
input: nodes $n_1, \ldots, n_m$ representing $F_1, \ldots, F_m$ in D
output: a node n representing $F_1 \wedge \ldots \wedge F_m$ in (modified) D
begin
  if some n_i is $\boxed{0}$ **then return** $\boxed{0}$
  if $m = 1$ **then return** n_1
  if some n_i is $\boxed{1}$ **then**
    return $conjunction(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$
  $p := max\_variable(n_1, \ldots, n_m)$
  forall $i = 1 \ldots m$
    if n_i is labelled by p
      then $(l_i, r_i) := (neg(n_i), pos(n_i))$
      else $(l_i, r_i) := (n_i, n_i)$
  $k_1 := conjunction(l_1, \ldots, l_m)$
  $k_2 := conjunction(r_1, \ldots, r_m)$
  return $integrate(k_1, p, k_2, D)$
end

$$F \wedge \bot \equiv \bot$$
$$F \wedge \top \equiv F$$

Summary

Binary decision trees, BDDs, OBDDs.

OBDDs:

- canonical representation of Boolean functions;
- can transform any formula to equivalent OBDD representation. But in some cases the size of the OBDD can grow exponentially.
- constant time checks for main logical problems: satisfiability, validity, equivalence;
- Boolean operations e.g. conjunction, disjunction, etc. are easy to implement.