COMP26120: Algorithms and Imperative Programming
Lecture 6: Introducing Complexity

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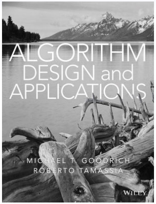
- You need this book: 

- Make sure you use the up-to-date edition. It is available on the course materials page:

  `http://studentnet.cs.manchester.ac.uk/ugt/2015/COMP26120/syllabus/`

- Read Ch. 1 (pp. 1–50).
- Pay particular attention to:
  - Pseudocode
  - Big-O notation and its relatives
  - The mathematical basics.
- Also read pp. pp. 689–690 and 695–696.

Outline

Getting started: two ways of computing variance

Big-O notation

Some details: What is an operation, and how big is a number?

Example: powers in modular arithmetic

Euclid's algorithm for finding highest common factors

- Let us begin with a simple example.
- Suppose we have a collection of numbers $x_1, \ldots, x_n$, and want to compute the *variance*, defined by the formula:

$$\sigma^2 = \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (x_i - x_j)^2 = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2.$$

- We could just do it:

$\mathtt{var1}(x_1, \ldots, x_n)$
  $s := 0$
  $\mathtt{for}\ i\ \mathtt{from}\ 1\ \mathtt{to}\ n-1$
    $\mathtt{for}\ j\ \mathtt{from}\ i+1\ \mathtt{to}\ n$
      $s := s + (x_i - x_j)^2$
  $\mathtt{return}\ s/n^2$
$\mathtt{end}$

- To see why this wouldn't be a good idea, let's count how much work is done.

```
var1(x_1, ..., x_n)
    s := 0
    for i from 1 to n-1
        for j from i+1 to n
            s := s + (x_i - x_j)^2
    return s/n^2
end
```

- We do $\sum_{i=1}^{n-1}(n-i) = \sum_{i=1}^{n-1} i = \frac{1}{2}(n-1)n$ executions of the line $s := s + (x_i - x_j)^2$ plus one final squaring and division—about $\frac{3}{2}(n-1)n + 2$ operations.

- But suppose you notice that the variance of $x_1, \ldots, x_n$ is actually the mean squared distance from the mean, μ. Noting that $\mu = \sum_{i=1}^{n} x_i/n$:

$$\sigma^2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (x_i - x_j)^2/n^2 = \sum_{i=1}^{n} (x_i - \mu)^2/n.$$

Then the following algorithm will then work:

var2($x_1, \ldots, x_n$)
  $m := 0$
  for i from 1 to n
    $m := m + x_i$
  $m := m/n$ % m now holds the mean
  $s := 0$
  for i from 1 to n
    $s := s + (x_i - m)^2$
  return s/n
end

- Now let's see how much work was done again:

```
var2(x_1, ..., x_n)
    m := 0
    for i from 1 to n
        m := m + x_i
    m := m/n                    % m now holds the mean
    s := 0
    for i from 1 to n
        s := s + (x_i - m)^2
    return s/n
end
```

- Here we do n additions in the first loop, and n subtractions, squarings and additions in the second loop, plus one division after each loop, making $4n + 2$ operations, much less (for large n) than $\frac{3}{2}(n-1)n + 2$.

- Observe
  - Algorithms are given in pseudocode.
  - The correctness of the algorithm `var2` needs to be established. Specifically, we have to prove that

$$\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{i=i+1}^{n}(x_i - x_j)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{i=1}^{n}x_i\right)^2.$$

  - We could quibble endlessly about *exactly* how many operations are involved in these algorithms, but we'd rather not ...
  - Such quibbles are irrelevant, because `var2` is clearly superior to `var1`.
- This lecture is about how to articulate these ideas.

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- When comparing growth-rates of functions, it is often useful to ignore
  - small values
  - linear factors
- That is, we are interested in how functions behave in the long run, and up to a linear factor.
- This is essentially to enable us to abstract away from relatively trivial implementation details.

- The main device used for this is big-O notation. If $f : \mathbb{N} \to \mathbb{N}$ is a function, then $O(f)$ denotes the set of functions:

  $$\{g : \mathbb{N} \to \mathbb{N} \mid \exists n_0 \in \mathbb{N} \text{ and } c \in \mathbb{R}^+ \text{ s.t. } \forall n > n_0,\ g(n) \leq c \cdot f(n)\}.$$

- Thus, $O(f)$ denotes a *set* of functions.

- To see why this is useful, consider the sets of functions

$$O(n) = \{g : \mathbb{N} \to \mathbb{N} \mid \exists n_0 \in \mathbb{N},\ c \in \mathbb{R}^+ \text{ s.t. } \forall n > n_0,\ g(n) \leq cn\}$$
$$O(n^2) = \{g : \mathbb{N} \to \mathbb{N} \mid \exists n_0 \in \mathbb{N},\ c \in \mathbb{R}^+ \text{ s.t. } \forall n > n_0,\ g(n) \leq cn^2\}.$$

- The following should now be obvious:
  - The function $g_2(n) = 4n + 2$ is in $O(n)$.
  - The function $g_1(n) = \frac{3}{2}(n-1)n + 2$ is in $O(n^2)$.
  - The function $g_1(n)$ is not in $O(n)$.
- Notice, of course, that $O(n) \subsetneq O(n^2)$.

- So now we can express succinctly the difference between running times of our algorithms `var1` and `var2`:
  - The running time of `var1` is in $O(n^2)$ (but not in $O(n)$);
  - The running time of `var2` is in $O(n)$.
- Often, we forget that $O(f)$ is technically a set of functions, and say:
  - The running time of `var1` is $O(n^2)$ (or: is order n^2);
  - The running time of `var2` is $O(n)$ (or: is order n).

But this is really just a manner of speaking.

- Of course, you can have $O(f)$ for any $f : \mathbb{N} \to \mathbb{N}$:
  - $O(\log n)$
  - $O(\log^2 n)$
  - $O(\sqrt{n})$
  - $O(n)$, $O(n^2)$, $O(n^3)$, ...
  - $O(2^n)$, $O(2^{n^2})$, ...
  - $O(2^{2^n})$, $O(2^{2^{2^n}})$, ...

- To think about:
    - Make sure you understand why $f(n) \leq g(n)$ for all n implies $O(f) \subseteq O(g)$.
    - Why do you not hear people talking about $O(6n + 7)$?
    - Give a succinct but accurate characterization of $O(1)$ in plain English.

Outline

- We said that the time-complexity of `var2` is in $O(n)$, but what, exactly, does this mean?
- Answer: to say that an algorithm A runs in time g means the following.

  Given an input of size n, the number of operations executed by A is bounded above by $g(n)$.

- This raises two important issues:
  - What is an operation?
  - How do we measure the size of the input?

- Deciding what to count as an operation is a bit of a black art. It depends on what you want your analysis for.
- For most practical applications, it is okay to take the following as operations:
  - arithmetic operations (e.g. +, *, /, %) on all the basic number types)
  - assignments (e.g. `a:= b`, `a[i] = t`, `t = a[i]`)
  - basic tests (e.g. `a = b`, `a` $\geq$ `b`
  - Boolean operations (e.g. `&`, `!`, `||`).
- Things like allocating memory, managing loops are often ignored—again, this may depend on the application.

- Note that, for some applications, this accounting régime might be misleading.
- Imagine, for example, an cryptographic algorithm requiring to perform arithmetic on numbers hundreds of digits long.
- In this case, we would probably want to count the number of logical operations involved.
- For example, to multiply numbers with p bits and q bits, we require in general about pq logical operations.
- There is a formal model of computation, the Turing Machine, which specifies precisely what counts as a basic operation.
- But in this course, we shall not use the Turing machine model.

- The question of how to measure the size of the input is rather trickier.
- Officially, the input to an algorithm is a string.
- Often, that string represents a number, or a sequence of numbers, but it is still a string.
- What is the size of the following inputs?
    - The cat sat on the mat
    - 1
    - 13
    - 445
    - 65535
- The size of a positive integer n (in canonical decimal representation) is $\lfloor \log_{10} n \rfloor + 1$, not: n.

Outline

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Example: powers in modular arithmetic

Euclid's algorithm for finding highest common factors

- Recall the definition of $m \mod k$, for k an integer greater than 1:

$$17 \mod 6 = 5$$
$$14 \mod 2 = 0$$
$$117 \mod 10 = 7$$

- When performing arithmetic mod k, we can stay within the numbers $0, \ldots, k-1$:

$$17 + 5564 \mod 10 = 7 + 4 \mod 10 = 11 \mod 10 = 1$$
$$17 \cdot 5564 \mod 10 = 7 \cdot 4 \mod 10 = 28 \mod 10 = 8$$
$$5564^{17} \mod 10 = 4^{17} \mod 10 = 17179869184 \mod 10 = 4$$

- Modular arithmetic is important in cryptography.

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- Modular arithmetic is important in cryptography.

- Modular arithmetic is particularly nice when the modulus is a prime number, p.
- If $1 \leq a < p$, then there is a unique number b such that

$$a \cdot b = b \cdot a = 1 \mod p.$$

  In that case we call b the inverse of a (modulo p) and write $b = a^{-1}$.
- For example,

$$3 \cdot 5 = 5 \cdot 3 = 1 \mod 7,$$

  so 3 and 5 are inverses modulo 7.

- Here is an algorithm to compute $a^b \mod k$.
(Note that we may as well assume that $a < k$.)

```
pow1(a, b, k)
    s := 1
    for i from 1 to b
        s := s · a mod k
    return s
end
```

- The number of operations performed here is clearly $O(b)$.
- Therefore the time complexity is . . . .

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$$
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\qquad \texttt{for } i \texttt{ from 1 to } b \\
\qquad\qquad s := s \cdot a \mod k \\
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\texttt{end}
\end{array}
$$

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\quad\quad \texttt{for } i \texttt{ from 1 to } b \\
\quad\quad\quad\quad s := s \cdot a \mod k \\
\quad\quad \texttt{return } s \\
\texttt{end}
\end{array}
$$

- The number of operations performed here is clearly $O(b)$.
- Therefore the time complexity is $O(2^n)$—i.e. exponential.
- That's right, exponential, not linear: the size of the input b is $\log b$. (Note that a and k don't really matter here.)
- Reminder: $2^{\log_2 n} = n$.

- Here is a better algorithm to compute $a^b \mod k$.

$$
\begin{array}{l}
\texttt{pow2}(a, b, k) \\
\quad d := a,\ e := b,\ s := 1 \\
\quad \texttt{until}\ \ e = 0 \\
\quad\quad \texttt{if}\ e \text{ is odd} \\
\quad\quad\quad s := s \cdot d \mod k \\
\quad\quad d := d^2 \mod k \\
\quad\quad e := \lfloor e/2 \rfloor \\
\quad \texttt{return}\ s \\
\texttt{end}
\end{array}
$$

- The number of operations performed here is proportional to the number of times $d = b$ can be halved before reaching 0, i.e. at most $\lceil \log_2 b \rceil$. Thus, this algorithm has running time in $O(n)$, i.e. linear in the size n of the input b. (Again, a and k don't really matter here.)

- To see how this number works, think of b in terms of its binary representation $b = b_{n-1}, \ldots, b_0$, i.e.

$$b = \sum_{h=0}^{n-1} b_h \cdot 2^h,$$

so that

$$a^b = \prod_{h=0}^{n-1} a^{(b_h \cdot 2^h)}$$

And of course

$$a^{(b_h \cdot 2^h)} = \begin{cases} 1 & \text{if } b_h = 0 \\ a^{(2^h)} & \text{if } b_h = 1. \end{cases}$$

But the variable e holds a^{2^h} on entry to the hth iteration on the loop (counting from $h = 0$ to $h = n - 1$).

- Compute $7^{11} \mod 10$. (N.B. 7^{11} is actually 1977326743.)

```
pow2(a, b, k)
    d := a, e := b, s := 1
    until e = 0
        if e is odd
            s := s · d mod k
        d := d^2 mod k
        e := ⌊e/2⌋
    return s
end
```

Before loop:
$s \leftarrow 1$
$d \leftarrow 7$
$e \leftarrow 11$ (= Binary 1011)

Round 1:
$s \leftarrow 7$ (e is odd and $1 \cdot 7 = 7 \mod 10$)
$d \leftarrow 9$ ($7^2 = 9 \mod 10$)
$e \leftarrow 5$

Round 2:
$s \leftarrow 3$ (e is odd and $7 \cdot 9 = 3 \mod 10$)
$d \leftarrow 1$ ($7^4 = 9^2 = 1 \mod 10$)
$e \leftarrow 2$

Round 3:
$s \leftarrow 3$ (e is even)
$d \leftarrow 1$ ($7^8 = 1^2 = 1 \mod 10$)
$e \leftarrow 1$

Round 4:
$s \leftarrow 3$ (e is odd and $3 \cdot 1 = 1 \mod 10$)
$d \leftarrow 1$ ($7^{16} = 1^2 = 1 \mod 10$)
$e \leftarrow 0$

- At some point d became 1. Do you see an optimization?

- Raising positive numbers to various powers modulo k produces 1 more often than you think.
- This is of special interest when k is some prime number, p.
- For example, set $k = p = 7$. In the following, all calculations are performed modulo 7.

$$
\begin{array}{llllll}
1^1 = 1 & & & & & \\
2^1 = 2 & 2^2 = 4 & 2^3 = 1 & & & \\
3^1 = 3 & 3^2 = 2 & 3^3 = 6 & 3^4 = 4 & 3^5 = 5 & 3^6 = 1 \\
4^1 = 4 & 4^2 = 2 & 4^3 = 1 & & & \\
5^1 = 5 & 5^2 = 4 & 5^3 = 6 & 5^4 = 2 & 5^5 = 3 & 5^6 = 1
\end{array}
$$

- Notice that, for $1 \leq a < p$, the smallest k such that $a^k = 1 \mod p$ divides $p - 1$ (this is always true for p prime).
- Hence a^{p-1} is always 1 (this is Fermat's little theorem).
- However, there is always some number a such that the various powers g^i cover the whole of $\{1, 2, \ldots, p-1\}$ (g is a primitive root modulo p).

- Let p be a prime, and consider the equation

$$a^x = y \mod p.$$

- If a is a primitive root modulo p, then, for every y $(1 \leq y < p)$, such an x $(1 \leq y < p)$ exists.
- In that case, the number x is called the discrete logarithm of y with base a, modulo p.
- Thus, the discrete logarithm is an inverse of exponentiation.
- We have seen that, for fixed a and p, computing

$$y = a^x \mod p$$

  for a given x is very fast. However, no such fast algorithm is known for recovering x from y.
- That is: modular exponentiation may be an example of a one-way function—easy to compute, hard to invert.

- Such one-way functions can be used for cryptography.
- Fix a prime, p a primitive root g modulo p.
- Choose a private key: x $(1 \leq x < p - 1)$.
- Broadcast the public key: (p, g, y), where $y = g^x$.
- Suppose someone wants to send you a message M (assume M is an integer $1 \leq M < p$).
- He picks k relatively prime to $p - 1$, sets

$$a \leftarrow g^k \mod p$$
$$b \rightarrow My^k \mod p$$

  and sends the ciphertext $C = (a, b)$.
- To decode $C = (a, b)$, you set

$$M' \leftarrow b/(a^x) \mod p.$$

- To see that you get the proper message:

$$\begin{aligned} M' = b/(a^x) \mod p &= My^k(a^x)^{-1} \mod p \\ &= M(g^{xk})(g^{xk})^{-1} \mod p \\ &= M \end{aligned}$$

- To see that this is secure, notice that to encode, one needs the public key $y = g^x$, but to decode, one needs the private key x (which only you have).
- In other words, to break the code, an enemy needs to be able to find the discrete logarithm of y to the base g, modulo p.
- The existence of one-way functions is equivalent to the conjecture $P \neq NP$.

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Some details: What is an operation, and how big is a number?

Example: powers in modular arithmetic

Euclid's algorithm for finding highest common factors

- Suppose you want to compute the highest common factor (hcf) of two non-negative integers a and b.
- Note that the hcf is sometimes called the *greatest common divisor* (*gcd*).
- Assume $a \geq b$. A little thought shows that, letting $r = a \bmod b$, we have, for some q

$$a = qb + r$$
$$r = qb - a$$

so that the common factors of a and b are the same as the common factors of b and r. Hence:

$$\mathrm{hcf}(a, b) = \mathrm{hcf}(b, r).$$

- This gives us the following very elegant algorithm for computing highest common factors.

```
hcf(a, b)          (Assume 0 ≠ a ≥ b)
    if b = 0
        return a
    else
        r := a mod b
        return hcf(b, r)
end
```

- This is so simple, it hurts.

- How long does `hcf(a, b)` take to run?

  Well, let $a_1, a_2, \ldots, a_\ell$ be the first arguments of successive calls to `hcf` in the computation of `hcf(a, b)`. (Thus, $a = a_1$.)

  Certainly $a_1 > a_2 > \cdots > a_\ell$, so the algorithm definitely terminates.

  Assuming $\ell > 2$, consider h in the range $1 \leq h \leq \ell - 2$. If $a_{h+1} \leq a_h/2$, then $a_{h+2} < a_h/2$. On the other hand, if $a_{h+1} > a_h/2$, then $a_{h+2} = a_h \mod a_{h+1} < a_h/2$.

  Either way, $a_{h+2} < a_h/2$. So the number of iterations is at most $\max(2, 2\lceil \log_2 a \rceil)$. That is, the algorithm is linear in the size of the input. (Actually, the algorithm performs slightly better than this.)