# Exercise 3. Problem 1

Check, using splitting, whether the formula $\neg((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$ is satisfiable. Split on the variable $p$ first.

# Solution

The splitting tree is shown below (the simplification steps are omitted). Since all leaves contain $\bot$, the formula is unsatisfiable.

$$
\begin{tikzcd}
 & \neg((p \to (q \to r)) \to ((p \to q) \to (p \to r))) \arrow[dl, "p=0"'] \arrow[dr, "p=1"] & & & \\
\bot & & \neg((q \to r) \to (q \to r)) \arrow[dl, "q=0"'] \arrow[dr, "q=1"] & & \\
 & \bot & & \neg(r \to r) \arrow[dl, "r=0"'] \arrow[dr, "r=1"] & \\
 & & \bot & & \bot
\end{tikzcd}
$$

# Exercise 3. Problem 2

Apply the standard CNF transformation algorithm to the following formula:

$$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) \qquad \Rightarrow$$
$$(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p)$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$(p \to q) \leftrightarrow (\neg q \to \neg p) \qquad \Rightarrow$$
$$(\neg p \lor q) \leftrightarrow (\neg q \to \neg p)$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) &
\end{array}
$$

## Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \to (q \vee \neg p)) \wedge ((q \vee \neg p) \to (\neg p \vee q)) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \to (q \vee \neg p)) \wedge ((q \vee \neg p) \to (\neg p \vee q)) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$\begin{array}{lr}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) &
\end{array}$$

## Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \to (q \vee \neg p)) \wedge ((q \vee \neg p) \to (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \to (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land \neg\neg p) \lor \neg p \lor q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \to (q \vee \neg p)) \wedge ((q \vee \neg p) \to (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \to (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge p) \vee \neg p \vee q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land \neg\neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land p) \lor \neg p \lor q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) \quad \Rightarrow$$
$$(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) \quad \Rightarrow$$
$$(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) \quad \Rightarrow$$
$$(\neg p \vee q) \leftrightarrow (q \vee \neg p) \quad \Rightarrow$$
$$((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) \quad \Rightarrow$$
$$((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) \quad \Rightarrow$$
$$(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) \quad \Rightarrow$$
$$(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) \quad \Rightarrow$$
$$(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge p) \vee \neg p \vee q) \quad \Rightarrow$$
$$(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q)$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) & \Rightarrow \\
((\neg\neg p \wedge \neg q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land \neg\neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) & \Rightarrow \\
((\neg\neg p \land \neg q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land \neg\neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) & \Rightarrow \\
((\neg\neg p \land \neg q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) & \Rightarrow \\
((p \land \neg q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) & \Rightarrow \\
((\neg\neg p \wedge \neg q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) & \Rightarrow \\
((p \wedge \neg q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) &
\end{array}
$$

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{ll}
(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg q \rightarrow \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (\neg\neg q \vee \neg p) & \Rightarrow \\
(\neg p \vee q) \leftrightarrow (q \vee \neg p) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow (\neg p \vee q)) & \Rightarrow \\
((\neg p \vee q) \rightarrow (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg(q \vee \neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge \neg\neg p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge ((\neg q \wedge p) \vee \neg p \vee q) & \Rightarrow \\
(\neg(\neg p \vee q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) & \Rightarrow \\
((\neg\neg p \wedge \neg q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) & \Rightarrow \\
((p \wedge \neg q) \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) & \Rightarrow \\
(p \vee q \vee \neg p) \wedge (\neg q \vee q \vee \neg p) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q) &
\end{array}
$$

The last formula is in CNF.

# Solution

The transformation steps are given below. At each step we show in red the subformula to which the transformation is applied.

$$
\begin{array}{lr}
(p \to q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg q \to \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (\neg\neg q \lor \neg p) & \Rightarrow \\
(\neg p \lor q) \leftrightarrow (q \lor \neg p) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land ((q \lor \neg p) \to (\neg p \lor q)) & \Rightarrow \\
((\neg p \lor q) \to (q \lor \neg p)) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg(q \lor \neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land \neg\neg p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land ((\neg q \land p) \lor \neg p \lor q) & \Rightarrow \\
(\neg(\neg p \lor q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) & \Rightarrow \\
((\neg\neg p \land \neg q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) & \Rightarrow \\
((p \land \neg q) \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) & \Rightarrow \\
(p \lor q \lor \neg p) \land (\neg q \lor q \lor \neg p) \land (\neg q \lor \neg p \lor q) \land (p \lor \neg p \lor q) &
\end{array}
$$

The last formula is in CNF.
One could also simplify the formulas by removing tautological clauses from the CNF.

# Exercise 3. Problem 3

Find a model of the formula $((\neg p \to q) \to p) \to \neg p$ using only the pure atom rule.

# Exercise 3. Problem 3

Find a model of the formula $((\neg p \to q) \to p) \to \neg p$ using only the pure atom rule.

**Solution**

One can not that the only occurrence of $q$ is positive, therefore $q$ can be replaced by $\top$ so that we obtain an equi-satisfiable formula

$$((\neg p \to \top) \to p) \to \neg p.$$

# Exercise 3. Problem 3

Find a model of the formula $((\neg p \rightarrow q) \rightarrow p) \rightarrow \neg p$ using only the pure atom rule.

**Solution**

One can not that the only occurrence of $q$ is positive, therefore $q$ can be replaced by $\top$ so that we obtain an equi-satisfiable formula

$$((\neg p \rightarrow \top) \rightarrow p) \rightarrow \neg p.$$

This formula can be simplified to

$$p \rightarrow \neg p.$$

# Exercise 3. Problem 3

Find a model of the formula $((\neg p \to q) \to p) \to \neg p$ using only the pure atom rule.

**Solution**

One can not that the only occurrence of $q$ is positive, therefore $q$ can be replaced by $\top$ so that we obtain an equi-satisfiable formula

$$((\neg p \to \top) \to p) \to \neg p.$$

This formula can be simplified to

$$p \to \neg p.$$

Now both occurrences of $p$ are negative, hence $p$ can be replaced by $\bot$ obtaining

$$\bot \to \neg \bot.$$

which can be simplified to $\top$.

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**Solution**

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$$((\neg p \to \top) \to p) \to \neg p.$$

This formula can be simplified to

$$p \to \neg p.$$

Now both occurrences of $p$ are negative, hence $p$ can be replaced by $\perp$ obtaining

$$\perp \to \neg \perp.$$

which can be simplified to $\top$.

Therefore, the original formula is satiafiable and $\{p \mapsto 0, q \mapsto 1\}$ is a model of this formula.