Two-Player Games

Who is this man?

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Does Garry Kasparov have a winning strategy?

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2. the player Q can choose a boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

The player Q wins if after n steps the chosen values make the formula G false.

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The existence of a winning strategy can be expressed by a quantified boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \dots \exists p_n \forall q_n G$.

Quantified Boolean Formulas

Propositional formula:

- Every boolean variable is a formula.
- $\top$ and $\bot$ are formulas.
- If $F_1, \ldots, F_n$ are formulas, where $n \geq 2$, then $(F_1 \wedge \ldots \wedge F_n)$ and $(F_1 \vee \ldots \vee F_n)$ are formulas.
- If F is a formula, then $\neg F$ is a formula.
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Quantified boolean formulas:

- If p is a boolean variable and F is a formula, then $\forall p F$ and $\exists p F$ are formulas.

Quantifiers

- $\forall$ is called the universal quantifier.
- $\exists$ is called the existential quantifier.
- Read $\forall pF$ as "for all p, F".
- Read $\exists pF$ as "there exists p such that F" or "for some p, F".

Connective	Name	Precedence
$\forall$	for all	5
$\exists$	exists	5
$\neg$	negation	5
$\wedge$	conjunction	4
$\vee$	disjunction	3
$\rightarrow$	implication	2
$\leftrightarrow$	equivalence	1

New Notation

Define

$$I_p^b(q) \stackrel{\text{def}}{=} \begin{cases} I(q), & \text{if } p \neq q; \\ b, & \text{if } p = q. \end{cases}$$

Example: let $I = \{p \mapsto 1, q \mapsto 0, r \mapsto 1\}$. Then

$$\begin{array}{lcl} I_q^1 & = & \{p \mapsto 1, q \mapsto 1, r \mapsto 1\} \\ I_q^0 & = & \{p \mapsto 1, q \mapsto 0, r \mapsto 1\} \quad = \quad I \\ I_p^0 & = & \{p \mapsto 0, q \mapsto 0, r \mapsto 1\} \end{array}$$

Semantics

1. $I(\top) = 1$ and $I(\bot) = 0$.
2. $I(F_1 \wedge \ldots \wedge F_n) = 1$ if and only if $I(F_i) = 1$ for all i.
3. $I(F_1 \vee \ldots \vee F_n) = 1$ if and only if $I(F_i) = 1$ for some i.
4. $I(\neg F) = 1$ if and only if $I(F) = 0$.
5. $I(F \rightarrow G) = 1$ if and only if $I(F) = 0$ or $I(G) = 1$.
6. $I(F \leftrightarrow G) = 1$ if and only if $I(F) = I(G)$.
7. $I(\forall p F) = 1$ if and only if $I_p^0(F) = 1$ and $I_p^1(F) = 1$.
8. $I(\exists p F) = 1$ if and only if $I_p^0(F) = 1$ or $I_p^1(F) = 1$.

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7. $I(\forall pF) = 1$ if and only if $I_p^0(F) = 1$ and $I_p^1(F) = 1$.
8. $I(\exists pF) = 1$ if and only if $I_p^0(F) = 1$ or $I_p^1(F) = 1$.

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\begin{tikzcd}[column sep=small]
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Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $*$ to denote "any" boolean value.

$$I_{**} \models \forall p \exists q (p \leftrightarrow q)$$

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The variables p and q are bound by quantifiers $\forall p$ and $\exists q$, so the value of the formula does not depend on these variables.

Subformula

Propositional formulas:

- The formulas $F_1, \ldots, F_n$ are the immediate subformulas of the formulas $F_1 \wedge \ldots \wedge F_n$ and $F_1 \vee \ldots \vee F_n$.
- The formulas F is the immediate subformula of the formula $\neg F$.
- The formulas F_1, F_2 are the immediate subformulas of the formulas $F_1 \rightarrow F_2$ and $F_1 \leftrightarrow F_2$.
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Quantified boolean formulas:

- The formula F_1 is the immediate subformula of the formulas $\forall p F_1$ and $\exists p F_1$.

Positions and Polarity

Let $F|_\pi = G$.
Propositional formulas:

- If G has the form $G_1 \wedge \ldots \wedge G_n$ or $G_1 \vee \ldots \vee G_n$, then for all $i \in \{1, \ldots, n\}$ the position $\pi.i$ is a position in F and $pol(F, \pi.i) \stackrel{\text{def}}{=} pol(F, \pi)$.
- If G has the form $\neg G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} -pol(F, \pi)$.
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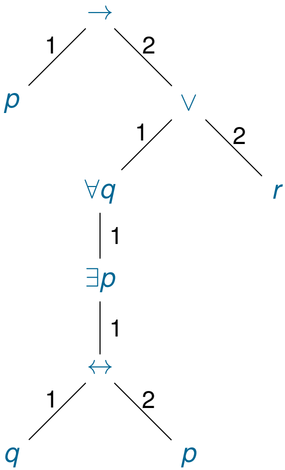
Example



$$p \to \forall q \exists p (q \leftrightarrow p) \vee r$$

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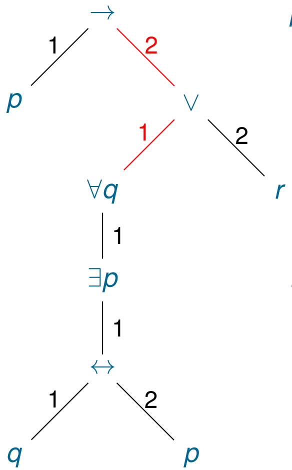

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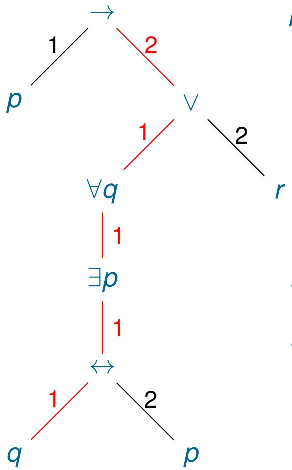

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$F|_{2.1.1.1.1} = q$

Free and bound occurrences of variables

Let p be a boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall p G$ or $\exists p G$ for some G.

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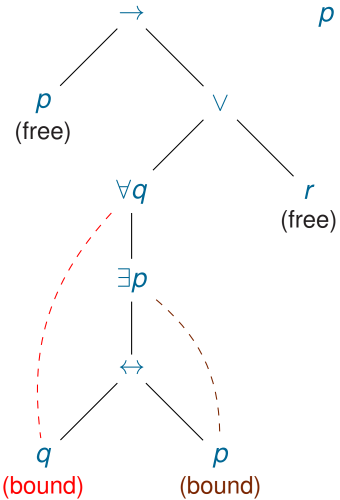
- Closed formula: formula with no free variables.

Example: Free and Bound Variables



$p \to \forall q \exists p (q \leftrightarrow p) \lor r$

Only Free Variables Matter

The truth value of a formula depends only on the truth values of free variables of the formula:

Lemma

Let for all free variables p of a formula F we have $I_1(p) = I_2(p)$. Then $I_1 \models F$ if and only if $I_2 \models F$.

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Theorem

Let F be a closed formula and I_1, I_2 be interpretations. Then $I_1 \models F$ if and only if $I_2 \models F$.

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Validity and satisfiability can be expressed through truth:

Lemma

Let F be a formula with free variables $p_1, \ldots, p_n$.

- *F is satisfiable if and only if the formula $\exists p_1 \ldots \exists p_n F$ is satisfiable (true, valid).*
- *F is valid if and only if the formula $\forall p_1 \ldots \forall p_n F$ is valid (true, satisfiable).*

Substitutions for propositional formulas

Substitution: $(F)^G_p$: denotes the formula obtained from F by replacing all occurrences of the variable p by G.

Example:

$$((p \vee s) \wedge (q \rightarrow p))^{(l \wedge s)}_p =$$
$$(((l \wedge s) \vee s) \wedge (q \rightarrow (l \wedge s)))$$

Properties: If we apply any substitution to a valid formula then we also obtain a valid formula.

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Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g. $(\forall p p \to q) \wedge (q \vee (q \to p))$

Renaming bound variables

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Renaming bound variables in F:
Let $F[\ \exists\!\!\!\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
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Notation: $\exists\!\!\!\forall$: any of $\exists$, $\forall$ and $\wedge\!\!\!\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:
Let $F[\ \exists\!\!\!\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
2. Replace all free occurrences of p in G (note: not in F!) by q obtaining G'.
3. So we obtain the $F[\ \exists\!\!\!\forall qG'\ ]$ as the result.

Lemma

$F[\ \exists\!\!\!\forall pG\ ] \equiv F[\ \exists\!\!\!\forall qG'\ ]$

Example:
$\exists q(\ \forall p((p \rightarrow q) \wedge p)\ ) \vee p$.

Then we can rename p into r obtaining
$\exists q(\ \forall r((r \rightarrow q) \wedge r)\ ) \vee p$.

Rectified formulas

Rectified formula F:

- no variable appears both free and bound in F;
- for every variable p, the formula F contains at most one occurrence of quantifiers $\exists\forall p$.

Any formula can be transformed into a rectified formula by renaming bound variables.

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Rectification: Example

$$p \to \exists p(p \land \forall p(p \lor r \to \neg p)) \Rightarrow$$

$$p \to \exists p(p \land \forall p_1(p_1 \lor r \to \neg p_1)) \Rightarrow$$

$$p \to \exists p_2(p_2 \land \forall p_1(p_1 \lor r \to \neg p_1))$$

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$$p \to \exists p(p \land \boxed{\forall p(p \lor r \to \neg p)}\,) \Rightarrow$$

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Rectification: Example

$$p \rightarrow \exists p(p \wedge \boxed{\forall p(p \vee r \rightarrow \neg p)}) \Rightarrow$$

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Another problem

$\exists q(\neg p \leftrightarrow q)$: there exists a truth value equivalent to $\neg p$. This formula is valid.

Substitute p by q.

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Another restriction

Suppose we want to substitute $(F)_p^G$.
Then we require: no free variable in G become bound in $(F)_p^G$.

In previous example $\exists q(\neg p \leftrightarrow q)$:
Substitute p by q. ($\exists q(\neg q \leftrightarrow q)$ does not satisfy above)

Uniform solution – renaming of bound variables
$\exists q(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$
Now we can substitute p by q obtaining $\exists r(\neg q \leftrightarrow r)$

From now on we always assume that:

- formulas are rectified.
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Equivalent replacement

Lemma

Let I be an interpretation and $I \models F_1 \leftrightarrow F_2$. Then $I \models G[F_1] \leftrightarrow G[F_2]$.

Theorem (Equivalent Replacement)

Let $F_1 \equiv F_2$. Then $G[F_1] \equiv G[F_2]$.

Prenex form

- Quantifier-free formula: no quantifiers (that is, propositional).
- Prenex formula has the form $\exists\forall_1 p_1 \ldots \exists\forall_n p_n G$, where G is quantifier-free.
- Outermost prefix of $\exists\forall_1 p_1 \ldots \exists\forall_n p_n G$: the longest subsequence $\exists\forall_1 p_1 \ldots \exists\forall_k p_k$ of $\exists\forall_1 p_1 \ldots \exists\forall_n p_n$ such that $\exists\forall_1 = \ldots = \exists\forall_k$.
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Prenexing rules

We assume that the formula rectified before application of

Prenexing rules:

$$\exists\!\!\!\forall p F_1 \wedge\!\!\!\!\vee \ldots \wedge\!\!\!\!\vee F_n \;\Rightarrow\; \exists\!\!\!\forall p (F_1 \wedge\!\!\!\!\vee \ldots \wedge\!\!\!\!\vee F_n)$$

$$F_1 \leftrightarrow F_2 \;\Rightarrow\; (F_1 \to F_2) \wedge (F_2 \to F_1)$$

$$\forall p F_1 \to F_2 \;\Rightarrow\; \exists p (F_1 \to F_2) \qquad \exists p F_1 \to F_2 \;\Rightarrow\; \forall p (F_1 \to F_2)$$

$$F_1 \to \forall p F_2 \;\Rightarrow\; \forall p (F_1 \to F_2) \qquad F_1 \to \exists p F_2 \;\Rightarrow\; \exists p (F_1 \to F_2)$$

$$\neg \forall p F \;\Rightarrow\; \exists p \neg F \qquad \neg \exists p F \;\Rightarrow\; \forall p \neg F$$

Warning: Sound only when the formula is rectified!

Some useful equivalences: $\neg\forall p\, F \equiv \exists p\, \neg F$ and $\neg\exists p\, F \equiv \forall p\, \neg F$

Prenexing. Example I

$$\exists q(q \to p) \to \neg\forall r(r \to p) \lor p \Rightarrow$$
$$\forall q((q \to p) \to \neg\forall r(r \to p) \lor p) \Rightarrow$$
$$\forall q((q \to p) \to \exists r\neg(r \to p) \lor p) \Rightarrow$$
$$\forall q((q \to p) \to \exists r(\neg(r \to p) \lor p)) \Rightarrow$$
$$\forall q\exists r((q \to p) \to \neg(r \to p) \lor p).$$

Prenexing. Example II

$$
\begin{array}{l}
\exists q(q \to p) \to \neg\forall r(r \to p) \vee p \Rightarrow \\
\exists q(q \to p) \to \exists r\neg(r \to p) \vee p \Rightarrow \\
\exists q(q \to p) \to \exists r(\neg(r \to p) \vee p) \Rightarrow \\
\exists r(\exists q(q \to p) \to \neg(r \to p) \vee p) \Rightarrow \\
\exists r\forall q((q \to p) \to \neg(r \to p) \vee p).
\end{array}
$$

Summary

- quantified boolean formulas (QBF): $\exists x\, \forall y \exists z\; F$
- syntax, semantics
- evaluating QBF formula on an interpretations: and-or trees
- positions/polarity
- bound/free occurrences of variables
- for closed formulas: validity and satisfiability coincide;
  for open formulas we can express satisfiability/validity using $\exists/\forall$ quantifiers respectively.
- rectified formulas: i) no variable occurs free and bound, ii) every variable is quantified at most once.
- rectification: rename bound variables
- prenex normal form: all quantifiers are on the left-hand-side
- prenexing: rectify + apply prenexing rules