Galerkin Discretizations of the Boltzmann Equation

Alex Alekseenko,
California State University Northridge

Collaborators: Prof. A. Wood, Dr. S. Gimelshein.
Students: T. Nguyen, J. Limbacher, A. Grandilli

Support Acknowledged: NSF DMS-1620497, AFIT/AFOSR MOU, AFIT COE

CSUN, January 28, 2019

Mathematical Background of Galerkin Approximation

A vector space over a field F is a set V whose elements called vectors that is closed with respect to two operations:

- **Associativity of addition:** $+ : V \times V \to V$, that is, for each $\mathbf{v}, \mathbf{w} \in V$ the vector $\mathbf{v} + \mathbf{w}$ is defined and belongs to V;
- **multiplication by a scalar:** $\cdot : R \times V \to V$, that is, for any $a \in F$ and $\mathbf{v} \in V$ the element $a\mathbf{v}$ is defined and belongs to V.

Also, the operations $+$ and $\cdot$ satisfy the following axioms:

- **Associativity of addition:** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- **Commutativity of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- **Identity element of addition:** $\exists \mathbf{0} \in V$, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- **Compatibility of scalar multiplication:** $a(b\mathbf{u}) = (ab)\mathbf{u}$
- **Identity of scalar multiplication:** $1\mathbf{u} = \mathbf{u}$ where 1 is the multiplicative identity of F
- **Distributivity, vector addition:** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- **Distributivity, field addition:** $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Examples of Vector Spaces, $\mathbb{R}^n$

Finitely dimensional: $\mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n)$.

Linear Independence: A set of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is linearly independent if the only possibility to have

$$a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n = 0$$

is all $a_1, \ldots, a_n$ are equal to zero.

Basis of the Vector Space: A set of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ that is linearly independent and any other vector $\mathbf{u}$ can be expressed as

$$\mathbf{u} = a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n$$

for some scalars $a_1, \ldots, a_n \in \mathbb{R}$.

Standard basis in $\mathbb{R}^n$:

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, 0, \ldots, n) \end{aligned}$$

Vector Norms

Norm of a Vector. Let V be a real linear space (e.g., $\mathbb{R}^n$). It is normed if there is a function $\|\cdot\| : V \to \mathbb{R}$, which we call a norm, satisfying all of the following:

- $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;
- $\|\alpha x\| = |\alpha| \, \|x\|$, for any $\alpha \in \mathbb{R}$;
- $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

Some norms frequently used to measure vectors of $\mathbb{R}^n$:

$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{1/p}, \quad 1 \leq p \leq \infty, \quad \text{in particular} \quad \|x\|_2 = \sqrt{\sum_i |x_i|^2}$$

and **infinity norm**

$$\|x\|_\infty = max_i |x_i|$$

Dot Product

Inner Product. Let V be a real vector space. Function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is an inner product if all of the following apply :

1. $\langle x, y\rangle = \langle y, x\rangle$,
2. $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$,
3. $\langle ax, y\rangle = a\langle x, y\rangle$ real scalar a,
4. $\langle x, x\rangle > 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

Example. Inner product on $\mathbb{R}^n$,

$$\langle x, y\rangle = y^T x = \sum_i x_i y_i.$$

Orthogonal Vectors. x and y are orthogonal if $\langle x, y\rangle = 0$.

Norms Induced by the Inner Product. An inner product is often used to define the subordinate norm:

$$\|x\| = \sqrt{\langle x, y\rangle}$$

Ortogonal Projection.

Direct Sum of Spaces. Let V and W be subspaces of $\mathbb{R}^n$. The direct sum of V and W, denotes by $V \oplus W$ is the set of all vectors $\mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Orthogonality of Spaces. Let V and W be subspaces of $\mathbb{R}^n$. We say that $V \perp W$ if $\mathbf{v} \perp \mathbf{w}$ for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Orthogonal Projections. Let V be a subspace of $\mathbb{R}^n$, then there is a unique subspace $V^\perp$ of $\mathbb{R}^n$ such that $V \perp V^\perp$ and $\mathbb{R}^n = V \oplus V^\perp$, that is, for any vector $\mathbf{u} \in \mathbb{R}^n$ there exists $\mathbf{v} \in V$ and $\mathbf{w} \in V^\perp$, such that

$$\mathbf{u} = \mathbf{v} + \mathbf{w} \quad \text{where} \quad \mathbf{v} \perp \mathbf{w}$$

The space $V^\perp$ and vectors $\mathbf{v}$ and $\mathbf{w}$ are unique.

$\mathbf{v}$ is called the orthogonal projection of $\mathbf{u}$ onto V
$\mathbf{w}$ is the orthogonal compliment.

If $\|w\|$ is small, then $\mathbf{v}$ is an approximation of $\mathbf{u}$ and $\mathbf{w}$ is the error of the approximation.

Vector Spaces of Functions.

Space of Functions. Consider the set of all functions defined on an interval $[a, b]$. Let $f(x), g(x) : [a, b] \to \mathbb{R}$ and let $m \in \mathbb{R}$. It is straightforward to define

$$(f+g)(x) = f(x)_+ g(x), \quad \text{and} \quad (mf)(x) = mf(x).$$

It is straightforward to check that with these definitions, functions on $[a, b]$ make a vector field over $\mathbb{R}$.

Example. $C[a, b]$ is the space of all continuous functions, $C^n[a, b]$ is the space of all functions whose n^{th} derivative is continuous.

Question. In what sense two functions are orthogonal?

Example. Consider set of all functions that are square-integrable on $[a, b]$ (in sense of Lebesgue), that is all $f(x) : [a, b] \to \mathbb{R}$

$$\int_a^b f^2(x)\, dx \quad \text{is bounded}$$

These functions form a vector space, $L^2[a, b]$.

Inner Product for Spaces of Functions.

Inner Product. Let $f(x), g(x) : [a, b] \to \mathbb{R}$. Define

$$\langle f(x), g(x) \rangle = \int_a^b f(x) g(x) \, dx.$$

This definition is well-posed for $L^2[a, b]$.

Orthogonality. $f(x) \in L^2[a, b]$ is orthogonal to $g(x) \in L^2[a, b]$ if and only if

$$\langle f(x), g(x) \rangle = \int_a^b f(x) g(x) \, dx = 0.$$

Norm. Let $f(x) \in L^2[a, b]$, define

$$\|f(x)\|_2 = \sqrt{\int_a^b f^2(x) \, dx}.$$

Orthogonal Projections. Let V be a subspace of $L^2[a, b]$, then there is a unique subspace $V^\perp$ of $L^2[a, b]$ such that $L^2[a, b] = V \oplus V^\perp$, that is, for any function $f(x) \in L^2[a, b]$ there exists $g(x) \in V$ and $h(x) \in V^\perp$, such that $f(x) = g(x) + h(x)$. The space $V^\perp$ and functions $g(x)$ and $h(x)$ are unique.

Dimensionality of $L^2[a,b]$. Dimensionality of $L^2[a,b]$ is infinite. Notice that polynomials $p(x) = c_n x^n + \ldots + c_1 x + c_0$ also form a space P

It is straightforward to check that space $P[a,b]$ of polynomials on interval $[a,b]$ is a subset of $L^2[a,b]$.

$P[a,b]$ is spanned by $1, x, x^2, \ldots, x^n, \ldots$. (countably many basis vectors). Monomials x^i are linearly independent. This implies that $L^2[a,b]$ has infinitely many dimensions.

Bases for $L^2[a,b]$. One can show that x^i form a basis for $L^2[a,b]$. Also, other polynomial families can be used, e.g., Legendre polynomials, Chebyshev polynomials.

Fourier basis on $[-1,1]$,

$$\phi_n = \exp(\imath \pi n x)$$

or use

$$\sin(\pi n x) \quad \text{and} \quad \cos(\pi n x).$$

Galerkin Approximation of Functions.

Galerkin Solution Space. Let $\phi_1(x), \ldots, \phi_n(x)$ be linearly independent functions from $L^2[a,b]$. We consider the vector space $V = \mathrm{Span}(\phi_1(x), \ldots, \phi_n(x))$, which is the set of all linear combinations of $\phi_i(x)$.

V is called the Galerkin solution space, $\phi_i(x)$ are called basis functions.

Let $f \in L^2[a,b]$. Let us consider the orthogonal projection of $f(x)$ onto V:

$$f(x) = g(x) + h(x), \quad g(x) \in V, \quad h(x) \perp V.$$

Since $V = \mathrm{Span}(\phi_1(x), \ldots, \phi_n(x))$, we can represent

$$g(x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

That is,

$$f(x) = \sum_{i=1}^{n} a_i \phi_i(x) + h(x), \quad h(x) \perp V.$$

Galerkin Approximation of Functions.

$$f(x) = \sum_{i=1}^{n} a_i \phi_i(x) + h(x), \quad h(x) \perp V.$$

Multiplying both sides by $\phi_j(x)$ and integrating over $[a, b]$, we have

$$\int_a^b f(x)\phi_j(x)\,dx = \int_a^b \Big(\sum_{i=1}^{n} a_i \phi_i(x)\Big)\phi_j(x)\,dx + \int_a^b h(x)\phi_j(x)\,dx, \quad h(x) \perp V.$$

The last integral is zero, since $h(x) \perp V$. Rearranging terms,

$$\int_a^b f(x)\phi_j(x)\,dx = \sum_{i=1}^{n} a_i \int_a^b \phi_i(x)\phi_j(x)\,dx$$

Define vector $f_j = \int_a^b f(x)\phi_j(x)\,dx$ and matrix $M_{ji} = \int_a^b \phi_i(x)\phi_j(x)\,dx$. The above equation is a linear system:

$$f = Ma$$

M is called the Mass matrix of the basis $\phi_i(x)$. If $\phi_i(x)$ are linearly independent, then M is invertible and

$$a = M^{-1} f$$