Ramsey model with Epstein-Zin Preferences

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September 2, 2024

We employ recursive preferences in the form of Epstein-Zin utility function. We use the utility function introduced in Cai and Lontzek (2019):

$$U_t = \left[(1-\beta) \frac{(C_t/L_t)^{1-1/\psi}}{1-1/\psi} L_t + \beta \mathbb{E}_t \left[U_{t+1}^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}} \tag{1}$$

In the paper Cai and Lontzek (2019) assumed $\psi = 1.5 > 1.$ to be greater than unity. In case of $\psi < 1$ the form of the utility function should be modified:

$$U_t = -\left[-(1-\beta) \frac{(C_t/L_t)^{1-1/\psi}}{1-1/\psi} L_t + \beta \mathbb{E}_t \left[-U_{t+1}^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}} \tag{2}$$

Then we define

$$V_t^{1-1/\psi} := \frac{1-1/\psi}{1-\beta} U_t^{1-1/\psi} \tag{3}$$

From Eq. (3) we can infer:

$$U_t = \left(V_t^{1-1/\psi} \frac{1-\beta}{1-1/\psi} \right)^{\frac{1}{1-1/\psi}} \tag{4}$$

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We transform Eq. (1) by substituting Eq. (4):

$$
\begin{aligned}
U_t^{1-1/\psi} &= (1-\beta)\frac{(C_t/L_t)^{1-1/\psi}}{1-1/\psi}L_t + \beta\mathbb{E}_t\left[U_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \\
\frac{1-1/\psi}{1-\beta}U_t^{1-1/\psi} &= (C_t/L_t)^{1-1/\psi}L_t + \frac{1-1/\psi}{1-\beta}\beta\mathbb{E}_t\left[U_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \\
V_t^{1-1/\psi} &= (C_t/L_t)^{1-1/\psi}L_t + \frac{1-1/\psi}{1-\beta}\beta\mathbb{E}_t\left[\left(\frac{1-\beta}{1-1/\psi}\left(V_{t+1}^{1-1/\psi}\right)^{\frac{1}{1-1/\psi}}\right)^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \\
V_t^{1-1/\psi} &= (C_t/L_t)^{1-1/\psi}L_t + \beta\mathbb{E}_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}}
\end{aligned}
$$

Then we formulate the complete optimization problem:

$$
V_t(K_t)^{1-1/\psi} = \max_{K_{t+1},C_t}\left\{\left(\frac{C_t}{L_t}\right)^{1-1/\psi}L_t + e^{-\rho}\mathbb{E}_t\left[V_{t+1}(K_{t+1})^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}}\right\} \tag{5}
$$

$$
\text{s.t.}\quad K_t^{\alpha}(\zeta_t A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t - K_{t+1} = 0 \quad (\lambda_t) \tag{6}
$$

$$
(1-\delta)K_t + I_t - K_{t+1} = 0 \tag{7}
$$

$$
\log(\zeta_{t+1}) = \log(\zeta_t) + \frac{\chi_t}{1-\alpha} + \frac{\varrho}{1-\alpha}\omega_{\zeta,t+1}, \quad \omega_{\zeta,t+1} \sim \mathcal{N}(0,1) \tag{8}
$$

$$
\chi_{t+1} = r\chi_t + \varsigma\omega_{\chi,t+1}, \quad \omega_{\chi,t+1} \sim \mathcal{N}(0,1) \tag{9}
$$

We normalize the variables with respect to the deterministic TFP and labor evolution:

$$
c_t := \frac{C_t}{A_t L_t}, k_t := \frac{K_t}{A_t L_t}, i_t := \frac{I_t}{A_t L_t}, k_{t+1} := \frac{K_{t+1}}{A_{t+1}L_{t+1}} \tag{10}
$$

This normalization implies the following changes:

- In the objective function we replace $v_t = \frac{V_t}{A_t L_t^{\frac{1}{1-1/\psi}}}$:

$$
\begin{aligned}
V_t^{1-1/\psi} &= \max_{K_{t+1},C_t} \left\{ \left(\frac{C_t}{A_t L_t}\right)^{1-1/\psi} A_t^{1-1/\psi} L_t + e^{-\rho}\mathbb{E}_t \left[V_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \right\} \\
\left(\frac{V_t}{A_t L_t^{\frac{1}{1-1/\psi}}}\right)^{1-1/\psi} &= \max_{k_{t+1},c_t} \left\{ c_t^{1-1/\psi} + \frac{e^{-\rho}}{A_t^{1-1/\psi} L_t} \mathbb{E}_t \left[\left(\frac{V_{t+1}}{A_{t+1} L_{t+1}^{\frac{1}{1-1/\psi}}}\right)^{1-\gamma} \left(A_{t+1} L_{t+1}^{\frac{1}{1-1/\psi}}\right)^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right\} \\
v_t^{1-1/\psi} &= \max_{k_{t+1},c_t} \left\{ c_t^{1-1/\psi} + \frac{\exp(-\rho)}{A_t^{1-1/\psi} L_t} \left(\exp(g_t^A) A_t\right)^{1-1/\psi} \exp(g_t^L) L_t \mathbb{E}_t \left[v_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \right\} \\
v_t^{1-1/\psi} &= \max_{k_{t+1},c_t} \left\{ c_t^{1-1/\psi} + \exp\left(-\rho + g_t^A(1-1/\psi) + g_t^L\right) \mathbb{E}_t \left[v_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \right\} \\
v_t^{1-1/\psi} &= \max_{k_{t+1},c_t} \left\{ c_t^{1-1/\psi} + \beta_t \mathbb{E}_t \left[v_{t+1}^{1-\gamma}\right]^{\frac{1-1/\psi}{1-\gamma}} \right\} \qquad (11)
\end{aligned}
$$

where we define the TFP and labor-adjusted discount rate as:

$$\beta_t = \exp\left(-\rho + \left(1 - \frac{1}{\psi}\right) g_t^A + g_t^L\right). \qquad (12)$$

- The budget constraint becomes:

$$
\begin{aligned}
&\left(\frac{K_t}{A_t L_t}\right)^{\alpha} \frac{(A_t L_t)^{\alpha}}{A_t L_t} (\zeta_t A_t L_t)^{1-\alpha} + (1-\delta)\frac{K_t}{A_t L_t} - \frac{C_t}{A_t L_t} - \frac{K_{t+1}}{A_t L_t} = 0 \\
&k_t^{\alpha} \zeta_t^{1-\alpha} + (1-\delta) k_t - c_t - \exp\left(g_t^A + g_t^L\right) \frac{K_{t+1}}{A_{t+1} L_{t+1}} = 0 \\
&k_t^{\alpha} \zeta_t^{1-\alpha} + (1-\delta) k_t - c_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0 \qquad (13)
\end{aligned}
$$

- Law of motion for capital:

$$(1-\delta) k_t + i_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0 \qquad (14)$$

We present a normalized problem:

$$v_t^{1-1/\psi}(k_t) = \max_{k_{t+1}, c_t} \left\{ c_t^{1-1/\psi} + \beta_t \mathbb{E}_t \left[v_{t+1}(k_{t+1})^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right\} \tag{15}$$

$$\text{s.t.} \quad k_t^{\alpha} \zeta_t^{1-\alpha} + (1-\delta) k_t - c_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0 \quad (\lambda_t) \tag{16}$$

$$(1-\delta) k_t + i_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0 \tag{17}$$

$$\log(\zeta_{t+1}) = \log(\zeta_t) + \frac{\chi_t}{1-\alpha} + \frac{\varrho}{1-\alpha} \omega_{\zeta,t+1}, \quad \omega_{\zeta,t+1} \sim \mathcal{N}(0,1) \tag{18}$$

$$\chi_{t+1} = r\chi_t + \varsigma \omega_{\chi,t+1}, \quad \omega_{\chi,t+1} \sim \mathcal{N}(0,1) \tag{19}$$

Then we formulate a set of equilibrium conditions, which consists of conditions implied by the envelope theorem, first-order conditions, KKT condition and value function equality:

- Envelope theorem:

$$\left(1 - \frac{1}{\psi}\right) v_t^{-1/\psi} v_{k,t} = \lambda_t \left(\zeta_t^{1-\alpha} \alpha k_t^{\alpha-1} + (1-\delta) \right) \tag{20}$$

- FOCs:

$$\left(1 - \frac{1}{\psi}\right) \beta_t \mathbb{E}_t \left[v_{t+1}^{1-\gamma} \right]^{\frac{\gamma-1/\psi}{1-\gamma}} \mathbb{E}_t \left[v_{t+1}^{-\gamma} v_{k,t+1} \right] - \lambda_t \exp\left(g_t^A + g_t^L\right) = 0 \tag{21}$$

$$\left(1 - \frac{1}{\psi}\right) c_t^{-\frac{1}{\psi}} - \lambda_t = 0 \tag{22}$$

$$\zeta_t^{1-\alpha} k_t^{\alpha} + (1-\delta) k_t - c_t - \exp\left(g_t^A + g_t^L\right) k_{t+1} = 0. \tag{23}$$

- Value function equality holds at optimality:

$$\tilde{v}^*(k_t)^{1-1/\psi} = c_t^{*1-1/\psi} + \beta_t \mathbb{E}_t \left[\tilde{v}^*(k_{t+1})^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}}. \tag{24}$$

References

Yongyang Cai and Thomas S. Lontzek. The Social Cost of Carbon with Economic and Climate Risks. *Journal of Political Economy*, 127(6):2684–2734, dec 2019. ISSN 0022-3808. doi: 10.1086/701890. [1]