Calvin and the Magic Wand Puzzle

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1 Problem

Calvin has to cross several signals when he walks from his home to school. Each of these signals operate independently. They alternate every 80 seconds between green light and red light. At each signal, there is a counter display that tells him how long it will be before the current signal light changes. Calvin has a magic wand which lets him turn a signal from red to green instantaneously. However, this wand comes with limited battery life, so he can use it only for a specified number of times.

(a) If the total number of signals is 2 and Calvin can use his magic wand only once, then what is the expected waiting time at the signals when Calvin optimally walks from his home to school?

(b) What if the number of signals is 3 and Calvin can use his magic wand only once?

(c) Can you write a script (in any programming language of your choice) that takes as inputs the number of signals and the number of times Calvin can use his magic wand, and outputs the expected waiting time?

2 Formulation and particular solutions

2.1 Formulation of the problem

First, let us define the different random variables that will be useful to solve this problem. We enumerate the signals as s_i, for $i = 1, 2, \ldots, N$. With the same index, the remaining time before a change of state for each signal is noted t_i. We call a_i the action taken by Calvin at signal i. Finally, the number of charges for the wand Calvin has left is noted c.

- $s_i \overset{i.i.d.}{\sim} B(1/2)$ the state of the signal number i. s_i is a Bernoulli random variable with parameter $1/2$. Since Calvin does not wait when signal is "green", it is more interesting to define $s_i = 1$ when signal is "red" and $s_i = 0$ when signal is "green".

- $t_i \overset{i.i.d.}{\sim} U(0, 80)$ the remaining time before signal i changes of state is a random variable uniformly distributed on the interval $[0, 80)$.

- a_i has to be equal to 1 if Calvin uses the wand at signal i and 0 otherwise. Our goal is to find the optimal time for Calvin's travel so it translates into finding the optimal limit l above which it is optimal for him to use the wound. Therefore we construct a_i in the following way

$$a_i = \begin{cases} 1, & \text{if } s_i = 1,\ t_i > l,\ c > 0, \\ 0, & \text{if } s_i = 0, \end{cases}$$

  where $l \in [0, 80)$. So a_i can be re-written as an indicator function $a_i = \mathbb{1}_{\{s_i t_i > l\}}$.

2.2 Particular solutions (a) and (b)

2.2.1 2 signals and 1 charge

Let us call the total time of the travel T_c^N in the general case. In this question it will the be T_1^2. Calvin arrives at the first signal, if he uses the wound there ($a_1 = 1$) then he will wait whatever time is remaining at the second signal, which is zero if it is "green" and t_2 if it is "red". If he chooses not to use the wand at the first signal, he will use it at the second one as soon as there is time left to wait there (i.e. $s_2 = 1$ and $t_2 > 0$). Therefore we can write

$$T_1^2 = a_1 s_2 t_2 + a_2 s_1 t_1 \Rightarrow \mathbb{E}[T_1^2] = \mathbb{E}[a_1 s_2 t_2] + \mathbb{E}[a_2 s_1 t_1].$$

The problem is now to find $\mathbb{E}[T_1^2]$ as a function of l and to minimise it with respect to this argument. Remember that all s_i and all t_i are mutually independent.

$$\begin{aligned}
\mathbb{E}[a_1 s_2 t_2] &= \mathbb{E}[\mathbb{1}_{\{s_1 t_1 > l\}} s_2 t_2] \\
&= \mathbb{E}[\mathbb{1}_{\{s_1 t_1 > l\}}]\mathbb{E}[s_2 t_2] \\
&= \mathbb{P}(s_1 t_1 > l)\mathbb{E}[s_2]\mathbb{E}[t_2] \\
&= \mathbb{P}(s_1 = 1, t_1 > l)\mathbb{E}[s_2]\mathbb{E}[t_2] \\
&= \mathbb{P}(s_1 = 1)\mathbb{P}(t_1 > l)\mathbb{E}[s_2]\mathbb{E}[t_2] \\
&= \frac{1}{2}\left(1 - \frac{l}{80}\right)\frac{80}{2}\frac{1}{2} \\
&= 10 - \frac{l}{8}.
\end{aligned}$$

Now for the second expectation, since the actions a_1 and a_2 sum up to 1 (we have one charge) we write $a_2 = 1 - a_1$ but this means that we loose the independence with the two other terms, i.e.

$$\begin{aligned}\mathbb{E}[a_2 s_1 t_1] &= \mathbb{E}[(1-a_1)s_1 t_1] = \mathbb{E}[\mathbb{1}_{\{s_1 t_1 < l\}} s_1 t_1] \\ &= \mathbb{E}[\mathbb{1}_{\{s_1 t_1 < l\}} s_1 t_1 \mathbb{1}_{\{s_1=0\}}] + \mathbb{E}[\mathbb{1}_{\{s_1 t_1 < l\}} s_1 t_1 \mathbb{1}_{\{s_1=1\}}] \\ &= 0 + \mathbb{E}[\mathbb{1}_{\{t_1 < l\}} t_1]\mathbb{P}(s_1 = 1) \\ &= \mathbb{E}[\mathbb{1}_{\{t_1 < l\}} t_1] \cdot 1/2.\end{aligned}$$

Now, the density of $t := \mathbb{1}_{\{t_1 < l\}} t_1$ is

$$f_t(x) := f_{t_1}(x) \cdot \mathbb{1}_{[0,l)}(x) = 1/80 \cdot \mathbb{1}_{[0,80)}(x) \cdot \mathbb{1}_{[0,l)}(x) = \mathbb{1}_{[0,l)}(x)/80.$$

Therefore the expectation is

$$\mathbb{E}[t] = \int_{\mathbb{R}} f_t(x)dx = \int_{\mathbb{R}} \mathbb{1}_{[0,l)}(x)x/80dx = \frac{1}{80}\int_0^l xdx = \frac{1}{80}\left[\frac{1}{2}x^2\right]_0^l = \frac{l^2}{160}.$$

Finally we have

$$F(l;2,1) := \mathbb{E}[T_1^2] = \frac{l^2}{320} - \frac{l}{8} + 10,$$

where $F(l;2,1)$ is the expected time for 2 signals and 1 wand charge with a limit time to action of l. We then want to minimise this equation in l so we derive once, equalise to zero and solve.

$$\frac{\partial F(l;2,1)}{\partial l} = 0 \quad \Leftrightarrow \quad \frac{2l^*}{320} - \frac{1}{8} = 0 \quad \Leftrightarrow \quad l^* = 20.$$

This is a unique global minimum since the initial function is a twice-continuously differentiable convex function.

So the answer to the first question is : the optimal waiting time for Calvin when he has to go through 2 signals with 1 wand charge is **8.75 seconds** with an optimal limit to action time of **20 seconds**. We then write

$$F(l^*;2,1) = \min_{l\in[0,80)} \mathbb{E}[T_1^2] \Leftrightarrow F(20;2,1) = 8.75.$$

2.3 3 signals and 1 charge

Now we have 3 signals but only 1 charge for our wand. If we use it on the first signal and we wait at second plus third. But if we don't use it at the first signal, we wait at least at the first signal and then we have to decide whether to use it at the second or the third signal, i.e. we recover the case from the previous question. To decide the optimal waiting time, we will then need two limit to action times l_1 and l_2. We can write that our total waiting time in this case is

$$\begin{aligned}T_1^3 &= a_1(s_2 t_2 + s_3 t_3) + (1-a_1)(s_1 t_1 + a_2 s_3 t_3 + (1-a_2)s_2 t_2) \\ &= \mathbb{1}_{\{s_1 t_1 > l_1\}}(s_2 t_2 + s_3 t_3) + \mathbb{1}_{\{s_1 t_1 < l_1\}}(s_1 t_1 + \mathbb{1}_{\{s_2 t_2 > l_2\}} s_3 t_3 + \mathbb{1}_{\{s_2 t_2 < l_2\}} s_2 t_2) \\ &= \mathbb{1}_{\{s_1 t_1 > l_1\}}(s_2 t_2 + s_3 t_3) + \mathbb{1}_{\{s_1 t_1 < l_1\}}(s_1 t_1 + T_1^2).\end{aligned}$$

Taking the expectation keeping in mind that all variables are mutually independent we get

$$\begin{aligned}
\mathbb{E}[T_1^3] &= \mathbb{E}[a_1(s_2t_2 + s_3t_3)] + \mathbb{E}[(1-a_1)(s_1t_1 + T_1^2)] \\
&= \mathbb{E}[a_1]\mathbb{E}[s_2]\mathbb{E}[t_2] + \mathbb{E}[a_1]\mathbb{E}[s_3]\mathbb{E}[t_3] + \mathbb{E}[(1-a_1)s_1t_1] + \mathbb{E}[1-a_1]\mathbb{E}[T_1^2] \\
&= 2\mathbb{E}[a_1]\mathbb{E}[s_2]\mathbb{E}[t_2] + \mathbb{E}[(1-a_1)s_1t_1] + (1-\mathbb{E}[a_1])\mathbb{E}[T_1^2] \\
&= 2\left(\mathbb{P}(s_1t_1 > l_1)\frac{1}{2}\frac{80}{2}\right) + \frac{l_1^2}{320} + (1-\mathbb{P}(s_1t_1 > l_1))\left(\frac{l_2^2}{320} - \frac{l_2}{8} + 10\right) \\
&= 2\left(\frac{1}{2}\left(1-\frac{l_1}{80}\right)\frac{1}{2}\frac{80}{2}\right) + \frac{l_1^2}{320} + \left(1-\frac{1}{2}\left(1-\frac{l_1}{80}\right)\right)\left(\frac{l_2^2}{320} - \frac{l_2}{8} + 10\right) \\
&= 20 - \frac{l_1}{4} + \frac{l_1^2}{320} + \left(\frac{1}{2} + \frac{l_1}{160}\right)\left(\frac{l_2^2}{320} - \frac{l_2}{8} + 10\right) \\
&= F(l_1, l_2; 3, 1).
\end{aligned}$$

To find the minimum in l_1, l_2, again we derive it and find the global minimum as $F(l_1, l_2; 3, 1)$ is a convex function in space. We solve

$$\vec{\nabla}F(l_1, l_2; 3, 1) = 0 \Leftrightarrow \begin{cases} \frac{\partial F}{\partial l_1} = -\frac{1}{4} + \frac{l_1}{160} + \frac{1}{160}\left(\frac{l_2^2}{320} - \frac{l_2}{8} + 10\right) = 0 \\ \frac{\partial F}{\partial l_2} = \left(\frac{1}{2} + \frac{l_1}{160}\right)\left(\frac{l_2}{160} - \frac{1}{8}\right) = 0 \end{cases} \Leftrightarrow \begin{cases} l_1^* = 31.25 \\ l_2^* = 20. \end{cases}$$

Finally, the result for this question is Calvin optimally waits for approximately $F(l_1^*, l_2^*; 3, 1) =$ **21.32 seconds**.

3 General case

In the general case, we need to find a stable recurrence relationship to compute T_c^N as a combination of the previous waiting times. We already saw two cases

$$\begin{cases} T_1^2 = a_1s_2t_2 + a_2s_1t_1, \text{ and} \\ T_1^3 = a_1(s_2t_2 + s_3t_3) + (1-a_1)(s_1t_1 + T_1^2). \end{cases}$$

We need a general way of writing down T_c^N so let us write the few first terms of the recurrence to see if we can extract pattern.

3.1 Recurrence

Let's take T_1^2 first. Note that $T_n^N = 0$ for all $n \in \mathbb{N}$ and $n \geq N$, i.e. if we have as many charges as there are traffic lights, there is no point in waiting. When Calvin has 1 charge for 2 signals, either he uses it at the first one (waits s_2t_2) and end up in a T_0^1 situation or he doesn't use it at the first signal (waits s_1t_1) and ends up in a T_1^1 situation at the second signal. We can then write

$$T_1^2 = a_1T_0^1 + (1-a_1)(s_1t_1 + T_1^1).$$

Now let's consider T_1^3. Either Calvin uses the wand at first signal and ends up in a T_0^2 situation (waits $s_2t_2 + s_3t_3$), or he doesn't use it at first signal (waits at least s_1t_1) and ends up in a T_1^2 situation. We then have

$$T_1^3 = a_1T_0^2 + (1-a_1)(s_1t_1 + T_1^2).$$

But that isn't enough. Let's see what happens for T_2^3 for example. If we use a charge on the first signal we are left with a T_1^2 situation. but if we don't, we wait at the first signal and then we're left with a T_2^2 situation, which is zero in terms of waiting time but we still write it to exhibit the pattern. We now have

$$\begin{cases} T_1^2 = a_1 T_0^1 + (1-a_1)(s_1t_1 + T_1^1), \\ T_1^3 = a_1 T_0^2 + (1-a_1)(s_1t_1 + T_1^2), \\ T_2^3 = a_1 T_1^2 + (1-a_1)(s_1t_1 + T_2^2). \end{cases}$$

We are starting to see a pattern of the form

$$T_c^N = a_1 T_{c-1}^{N-1} + (1-a_1)(s_1t_1 + T_c^{N-1}).$$

This recurrence is intuitive, for example, let's continue with T_c^4.

- If Calvin has 1 wand charge and 4 signals to go through, either he uses the charge at first signal and ends up in a T_0^3 situation, or he doesn't and ends up in a T_1^3 situation, i.e.

$$T_1^4 = a_1 T_0^3 + (1-a_1)(s_1t_1 + T_1^3).$$

- If Calvin has 2 wand charges and 4 signals to go through, either he uses one charge at first signal and ends up in a T_1^3 situation, or he doesn't and ends up in a T_2^3 situation, i.e.

$$T_2^4 = a_1 T_1^3 + (1-a_1)(s_1t_1 + T_2^3).$$

- If Calvin has 3 wand charges and 4 signals to go through, either he uses one charge at first signal and ends up in a T_2^3 situation, or he doesn't and ends up in a T_3^3 situation, i.e.

$$T_2^4 = a_1 T_1^3 + (1-a_1)(s_1t_1 + T_3^3).$$

- and so on ...

So the recurrence relation seems reasonable. Note that we are interested in the expectation of T_c^N which is

$$\begin{aligned} \mathbb{E}[T_c^N] &= \mathbb{E}[a_1 T_{c-1}^{N-1}] + \mathbb{E}[(1-a_1)(s_1t_1 + T_c^{N-1})] \\ &= \mathbb{E}[\mathbb{1}_{\{s_1t_1 > l_1\}} T_{c-1}^{N-1}] + \mathbb{E}[\mathbb{1}_{\{s_1t_1 < l_1\}}(s_1t_1 + T_c^{N-1})] \\ &= \mathbb{E}[\mathbb{1}_{\{s_1t_1 > l_1\}}]\mathbb{E}[T_{c-1}^{N-1}] + \mathbb{E}[\mathbb{1}_{\{s_1t_1 < l_1\}} s_1t_1] + \mathbb{E}[\mathbb{1}_{\{s_1t_1 < l_1\}}]\mathbb{E}[T_c^{N-1}] \\ &= \mathbb{P}(s_1t_1 > l_1)\mathbb{E}[T_{c-1}^{N-1}] + \mathbb{E}[\mathbb{1}_{\{s_1t_1 < l_1\}} s_1t_1] + \mathbb{P}(s_1t_1 < l_1)\mathbb{E}[T_c^{N-1}]. \end{aligned}$$

Also remember that these expectations are themselves functions of different limit-to-action levels we called $l_1, l_2, \dots$ Note that there are always one level less than the number of traffic lights Calvin has to go through so we can write $\mathbb{E}[T_c^N] = F(l_1, l_2, \dots, l_{N-1}; N, c)$. Therefore

$$\begin{aligned} F(l_1, \dots, l_{N-1}; N, c) &= \mathbb{P}(s_1t_1 > l_1)\mathbb{E}[T_{c-1}^{N-1}] + \mathbb{E}[\mathbb{1}_{\{s_1t_1 < l_1\}} s_1t_1] + \mathbb{P}(s_1t_1 < l_1)\mathbb{E}[T_c^{N-1}] \\ &= \left(\frac{1}{2} - \frac{l_1}{160}\right) F(l_2, \dots, l_{N-1}; N-1, c-1) \\ &\qquad + \frac{l_1^2}{320} + \left(\frac{1}{2} + \frac{l_1}{160}\right) F(l_2, \dots, l_{N-1}; N-1, c). \end{aligned}$$

3.2 First Algorithmic Solution

Based on the recurrence

$$\mathbb{E}[T_c^N] = \left(\frac{1}{2} - \frac{l_1}{160}\right)\mathbb{E}[T_{c-1}^{N-1}] + \frac{l_1^2}{320} + \left(\frac{1}{2} + \frac{l_1}{160}\right)\mathbb{E}[T_c^{N-1}],$$

we now have to come up with the algorithm that will compute the optimal waiting time for any case. Since this is a recurring problem, we will code a recurring solution. As in every recursive algorithm, we need a stopping condition and here we have several. If $N = 0$ or $c \geq N$ then the expected time is null. But if $c = 0$ and $N \geq 1$ then

$$\mathbb{E}[T_0^N] = \mathbb{E}\left[\sum_{i=1}^{N} s_i t_i\right] \stackrel{i.i.d}{=} N\mathbb{E}[s_1]\mathbb{E}[t_1] = N\frac{1}{2}\frac{80}{2} = 20N.$$

```
# Optimal time recurring function
F = function(N,c)
{
  # check for stopping conditions first
  if(N == 0 || c >= N) return(0)
  if(c == 0) return(N*20)

  # recursive part
  part1 = F(N - 1, c - 1)
  part2 = F(N - 1, c)
  obj = function(x)
  {
    (0.5 - x/160)*part1 + x^2/320 + (0.5 + x/160)*part2
  }

  # finding the minimum
  res = optimize(obj, c(0,80))
  print(paste('For',N,'signals␣and',c,'charges,
␣␣␣␣Calvin␣decides␣at␣level',res$minimum,'and␣waits',res$objective))
  return(res$objective)
}
```

Even though this solution gives the correct answer, it is quite slow as the numbers at work become larger. In fact it is pretty easy to see that we will have duplicate intermediary steps. We need to optimise our running time, this one being exponential.

3.3 Dynamic Programming

It is clear that we have to use a dynamic programming approach here to be able to compute results in a realistic time. Indeed we can see that if we don't "remember" steps we already computed, the program will end up computing the same ones repeatedly which will be very inefficient. We then use the "memoization" principle which does exactly this, avoid useless calculations we already did.

4 Conclusion

A Appendix

A.1 code