

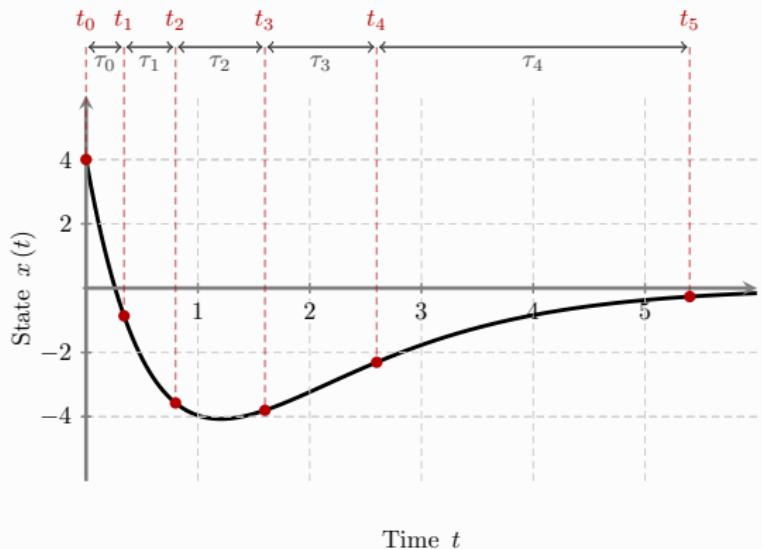
Optimal Control

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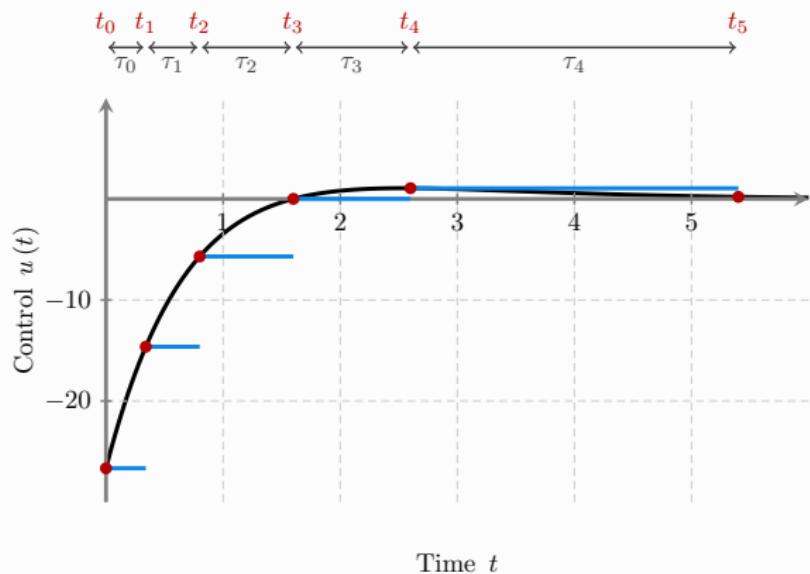
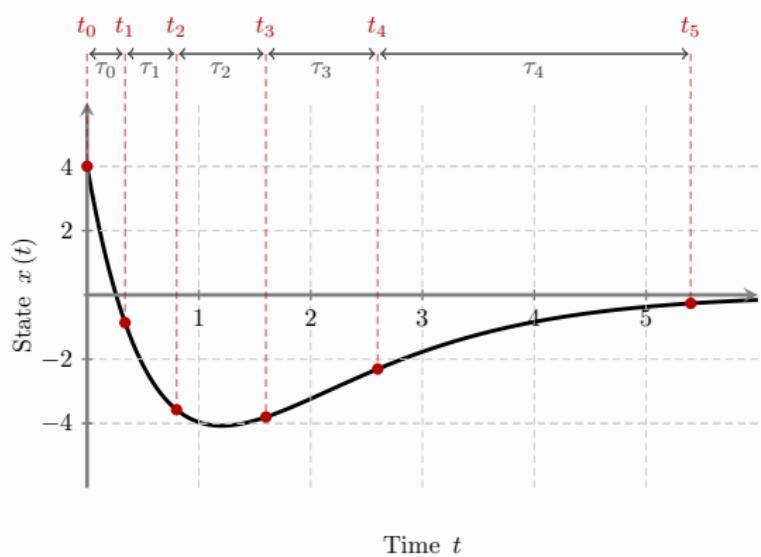
SAMPLING



Sampling

- ▶ $\{t_0, t_1, \dots, t_N\}$: Sampling pattern
- ▶ t_k : Sampling instants
- ▶ $\tau_k = t_{k+1} - t_k$: Interarrival times

SAMPLING



QUESTIONS

Question

left plot jumps with \only. how to fix?

THE PROBLEM

Given

$$\begin{cases} \dot{x}(t) = A x(t) + B u(t) \\ x(0) = x_0 \end{cases}$$

- ▶ We restrict $u(t)$ to be in the class of piecewise constant functions and denote it as $\bar{u}(t)$
- ▶ Finite horizon $[0, T]$
- ▶ Number N of control updates

find

- ▶ Sampling instants $0 = t_1 < \dots < t_N = T$
- ▶ Piecewise constant $\bar{u}(t) = u_k, \quad \forall t \in [t_{k-1}, t_k)$

that minimizes the performance index

- ▶ $\mathcal{J}(\bar{u}) = \int_0^T (x'(t) Q x(t) + \bar{u}'(t) R \bar{u}(t)) dt + x'(T) S x(T)$

COMMENTS

Comment

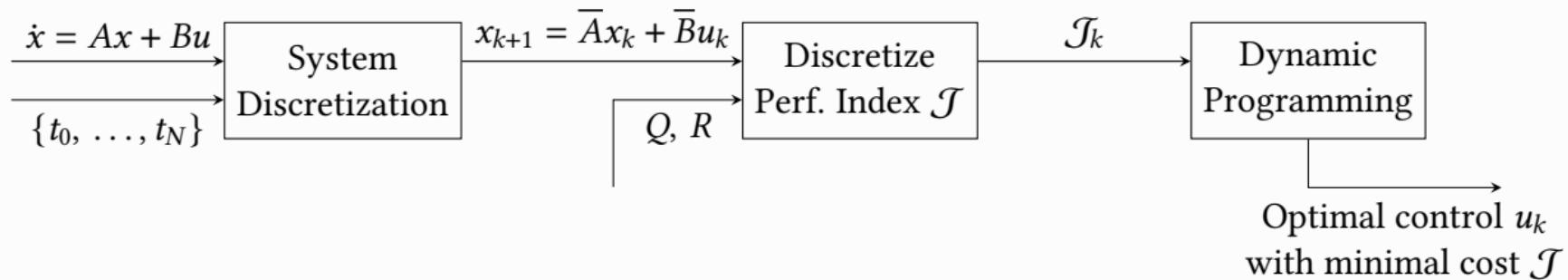
"find" looks weird?

QUADRATIC PERFORMANCE INDEX

$$\begin{aligned}\mathcal{J}(\bar{u}) &= \int_0^T \left(x'(t) \underbrace{\underline{Q}}_{\succeq} x(t) + \bar{u}'(t) \underbrace{\underline{R}}_{\succ} \bar{u}'(t) \right) dt + x'(T) \underbrace{\underline{S}}_{\succeq} x(T) \\ \text{s.t. } &\begin{cases} \dot{x}(t) &= A x(t) + B u(t) \\ x(0) &= x_0 \end{cases}\end{aligned}$$

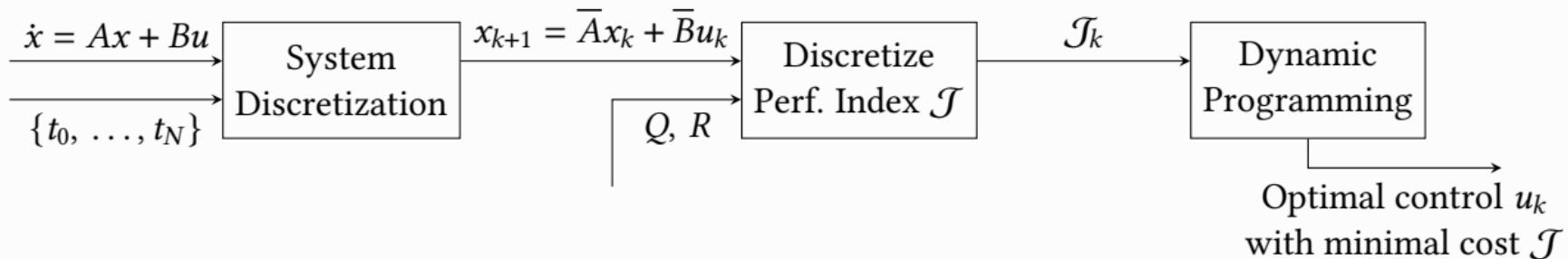
- ▶ $x(t) \in \mathbb{R}^n$: system state
- ▶ $u(t) \in \mathbb{R}^m$: system input
- ▶ A, B, Q, R, S : constant matrices

PICTURE



OUTLINE

- ▶ Computation of optimal control $u(t)$ for continuous-time systems
- ▶ Discretization Process for given $\{t_0, \dots, t_N\}$
 1. System Discretization
 2. Computation of optimal control $(u_k)_{k \in \{0, \dots, N\}}$ for the discrete-time system



OUTLINE

- ▶ Computation of optimal control $u(t)$ for continuous-time systems
- ▶ Discretization Process for given $\{t_0, \dots, t_N\}$
 1. System Discretization
 2. Computation of optimal control $(u_k)_{k \in \{0, \dots, N\}}$ for the discrete-time system
- ▶ Sampling Density and Sampling Method Cost
- ▶ Sampling Methods for finding $\{t_0, \dots, t_N\}$
 1. Periodic sampling
 2. Lebesgue sampling
 3. Quantization-based sampling
- ▶ Results

OPTIMAL CONTROL: CONTINUOUS-TIME SYSTEMS

$$\text{minimize}_u \int_0^T \left(x'(t) Q x(t) + u'(t) R u(t) \right) dt + x(T)' S x(T)$$

Riccati equation for continuous-time systems

$$\begin{cases} \dot{K}(t) = K(t) B R^{-1} B' K(t) - A' K(t) - K(t) A - Q, \\ K(T) = S \end{cases}$$

which gives us the optimal control $u(t) = -R^{-1}B'K(t)x(t)$

with achieved cost

$$\mathcal{J}_\infty = x_0' K(0) x_0$$



OPTIMAL CONTROL: DISCRETE-TIME SYSTEMS

Given the time points $\{t_0, \dots, t_N\}$, with interarrivals $\tau_k, k \in \{0, \dots, N\}$, the discrete-time state space equations are:

$$\begin{cases} x_{k+1} &= \bar{A}_k x_k + \bar{B}_k \\ x(0) &= x_0 \end{cases}$$

$$\bar{A}_k = \Phi(\tau_k)$$

$$\Phi(\tau) = e^{A\tau}$$

State Matrix

$$\bar{B}_k = \Gamma(\tau_k)$$

$$\Gamma(\tau) = \int_0^\tau e^{A(\tau-t)} dt B$$

Input Matrix

PERFORMANCE INDEX FOR DISCRETE-TIME SYSTEMS

$$\int_0^T (x'(t) Q x(t) + \bar{u}'(t) R \bar{u}(t)) dt + x'(T) S x(T)$$

$$= \sum_{k=0}^{N-1} \text{Bigg} [\int_{t_k}^{t_{k+1}} (x'(t) Q x(t) + \bar{u}'(t) R \bar{u}(t)) dt + x'(T) S x(T) \text{Bigg}]$$

$$= \sum_{k=0}^{N-1} \left[\int_{t_k}^{t_{k+1}} (\Phi x_k + \Gamma u_k)' Q (\Phi x_k + \Gamma u_k) dt + \int_{t_k}^{t_{k+1}} u'_k R u_k \right] + x'(T) S x(T)$$

$$= \sum_{k=0}^{N-1} \left[\underbrace{x'_k \left(\int_{t_k}^{t_{k+1}} \Phi' Q \Phi dt \right)}_{\bar{Q}} x_k + \underbrace{u'_k \left(\int_{t_k}^{t_{k+1}} \Gamma' Q \Gamma dt + \underbrace{\int_{t_k}^{t_{k+1}} R dt}_{\tau_k R} \right)}_{\bar{R}} u_k + \underbrace{2x'_k \left(\int_{t_k}^{t_{k+1}} \Phi' Q \Gamma dt \right)}_{\bar{P}} u_k \right] + x'(T) S x(T)$$

PERFORMANCE INDEX FOR DISCRETE-TIME SYSTEMS

$$\begin{aligned} & \int_0^T (x'(t) Q x(t) + \bar{u}'(t) R \bar{u}(t)) dt + x'(T) S x(T) \\ &= \sum_{k=0}^{N-1} \left[\underbrace{x'_k \left(\int_{t_k}^{t_{k+1}} \Phi' Q \Phi dt \right)}_{\bar{Q}} x_k + u'_k \underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma' Q \Gamma dt + \underbrace{\int_{t_k}^{t_{k+1}} R dt}_{\tau_k R} \right)}_{\bar{R}} u_k + 2x'_k \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q \Gamma dt \right)}_{\bar{P}} u_k \right] + x'(T) S x(T) \end{aligned}$$

$$\mathcal{J}(\bar{u}) = \sum_{k=0}^{N-1} \left(x'_k \bar{Q} x_k + u'_k \bar{R} u_k + 2x'_k \bar{P} u_k \right) + x'_N S x_N$$

COMMENTS

Comment

- ▶ Different brackets for integrals due to underbrace.
- ▶ Γ and Φ without args? Double check!

DYNAMIC PROGRAMMING: BELLMAN EQUATION

Bellman Equation

For $k \in \{0, \dots, N\}$, we define the function $\mathcal{J}_k : \mathbb{R}^n \rightarrow \mathbb{R}$, which gives the minimal cost achievable from stage k onward, given the state x_k as

$$\left\{ \begin{array}{lcl} \mathcal{J}_k(x_k) & = & \min_u \left[\underbrace{x'_k \bar{Q}_k x_k}_{\text{State penalty}} + \underbrace{u' \bar{R}_k u}_{\text{Control penalty}} + \underbrace{2x'_k \bar{P}_k u}_{\text{Cross term}} + \mathcal{J}_{k+1}(x_{k+1}) \right] \quad \text{for } k \in \{0, \dots, N-1\} \\ \\ \mathcal{J}_N(x) & = & x' S x \end{array} \right. .$$

Intuition: define a *backward recursive function* that gives the minimal achievable cost from the current state onward.

DYNAMIC PROGRAMMING: BELLMAN EQUATION

Bellman Equation

For $k \in \{0, \dots, N\}$, we define the function $\mathcal{J}_k : \mathbb{R}^n \rightarrow \mathbb{R}$, which gives the minimal cost achievable from stage k onward, given the state x_k as

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$$x_{k+1} = \bar{A}_k x_k + \bar{B}_k u_k$$

QUESTIONS

Question

same issue: the first block moves

DYNAMIC PROGRAMMING: QUADRATIC FORM

$\mathcal{J}_k(x)$ is a quadratic form of x

$$\mathcal{J}_k(x) = x' \bar{K}_k x$$

Unknown at this point!



$$\mathcal{J}_{k+1}(\bar{A}_k x + \bar{B}_k u) = (\bar{A}_k x + \bar{B}_k u)' \bar{K}_{k+1} (\bar{A}_k x + \bar{B}_k u)$$



Plug into **Bellman equation**

$$\mathcal{J}_k(x) = \underbrace{x' (\bar{Q}_k + \bar{A}'_k \bar{K}_{k+1} \bar{A}_k)}_{\hat{Q}_k} x + \underbrace{u' (R_k + \bar{B}'_k \bar{K}_{k+1} \bar{B}_k)}_{\hat{R}_k} u + \underbrace{2x' (\bar{P}_k + \bar{A}'_k \bar{K}_{k+1} \bar{B}_k) u}_{\hat{S}_k}$$

DYNAMIC PROGRAMMING: QUADRATIC FORM

$\mathcal{J}_k(x)$ is a quadratic form of x

$$\rightarrow \mathcal{J}_k(x) = x' \bar{K}_k x$$

Unknown at this point!



$$\mathcal{J}_{k+1}(\bar{A}_k x + \bar{B}_k u) = (\bar{A}_k x + \bar{B}_k u)' \bar{K}_{k+1} (\bar{A}_k x + \bar{B}_k u)$$



Plug into Bellman equation

$$\mathcal{J}_k(x) = x' \hat{Q}_k x + u' \hat{R}_k u + 2x' \hat{S}_k u$$

DYNAMIC PROGRAMMING: MINIMIZATION

We aim to **minimize** the cost:

$$\min_u \left(x' \hat{Q}_k x + u' \hat{R}_k u + 2x' \hat{S}_k u \right)$$

Optimality condition:

$$\frac{\partial}{\partial u} (\cdot) = 0 \implies \bar{R}_k u + \bar{S}'_k x = 0$$

↓ Solving for u

$$u_k^* = -\hat{R}_k^{-1} \hat{S}'_k x_k$$



DYNAMIC PROGRAMMING: RICCATI SOLUTION

We plug u_k^* into the **Bellman equation**:

$$\mathcal{J}_k(x) = x' \left(\hat{Q}_k - \hat{B}_k \hat{R}_k^{-1} \hat{B}'_k \right) x$$



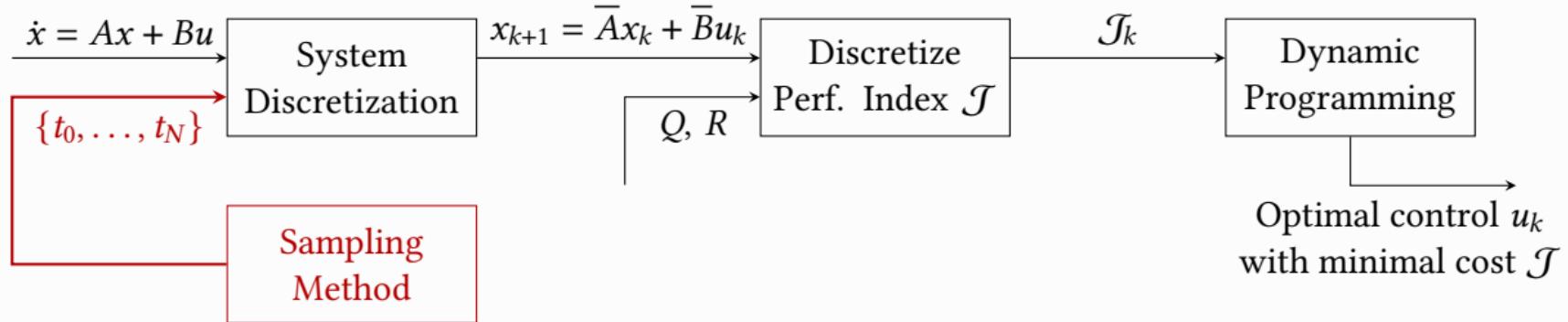
Recall $\mathcal{J}_k(x) = x' \bar{K}_k x$

$$\bar{K}_k = \hat{Q}_k - \hat{S}_k \hat{R}_k^{-1} \hat{S}'_k$$

Minimal cost equals

$$\mathcal{J} = x'_0 \bar{K}_0 x_0$$

OUTLINE



- ▶ Sampling Density and Sampling Method Cost
- ▶ Sampling Methods for finding $\{t_0, \dots, t_N\}$
 1. Periodic sampling
 2. Lebesgue sampling
 3. Quantization-based sampling

COMPARING SAMPLING METHODS: SAMPLING DENSITY

Sampling Density

Given x_0, A, B, Q, R , and S , and interval length T , and a number of samples N , the *sampling density* $\sigma_{N,m} : [0, T] \rightarrow \mathbb{R}^+$ of any sampling method m is defined as

$$\sigma_{N,m}(t) = \frac{1}{N \tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \dots, N - 1\}$$

- ▶ Temporal distribution of sampling instants
- ▶ Sampling density is normalized

Asymptotic Sampling Density

To remove the dependency on N , we define the *asymptotic sampling density* as $\sigma_m : [0, T] \rightarrow \mathbb{R}^+$ as the limit

$$\sigma_m(t) = \lim_{N \rightarrow \infty} \sigma_{N,m}(t)$$

COMPARING SAMPLING METHODS: SAMPLING DENSITY

Sampling Density

Given x_0, A, B, Q, R , and S , and interval length T , and a number of samples N , the *sampling density* $\sigma_{N,m} : [0, T] \rightarrow \mathbb{R}^+$ of any sampling method m is defined as

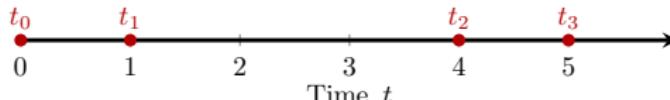
$$\sigma_{N,m}(t) = \frac{1}{N \tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \dots, N-1\}$$

- ▶ Temporal distribution of sampling instants
- ▶ Sampling density is normalized

Example ($T = 5, N = 4$)

$$\blacktriangleright \sigma_4(t) = \frac{1}{4 \cdot 1} = \frac{1}{4}, \quad \forall t \in [0, 1)$$

$$\blacktriangleright \sigma_4(t) = \frac{1}{4 \cdot 3} = \frac{1}{12}, \quad \forall t \in [1, 4)$$



COMPARING SAMPLING METHODS: NORMALIZED COST

Normalized Cost

Given x_0, A, B, Q, R , and S , and interval length T , and a number of samples N , the *normalized cost* of any sampling method m is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method m with N samples, and \mathcal{J}_∞ is the minimal cost of the continuous-time system.

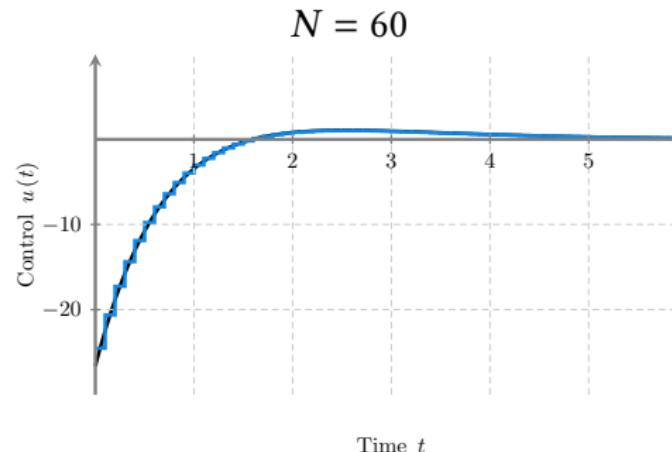
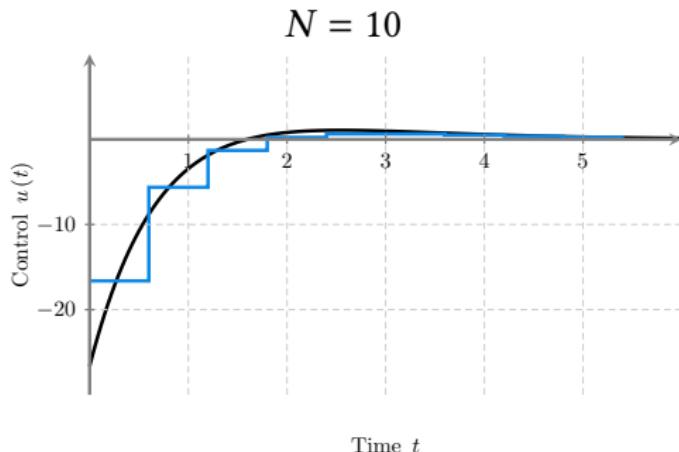
Asymptotic Normalized Cost

To remove the dependency on N , we define the *asymptotic normalized cost* as the limit

$$c_m = \lim_{N \rightarrow \infty} c_{N,m}$$

COMPARING SAMPLING METHODS: NORMALIZED COST

Example (Normalized Cost for Periodic Sampling)



► $\mathcal{J}_\infty = 383.2, \mathcal{J}_{10,\text{per}} = 699.7$

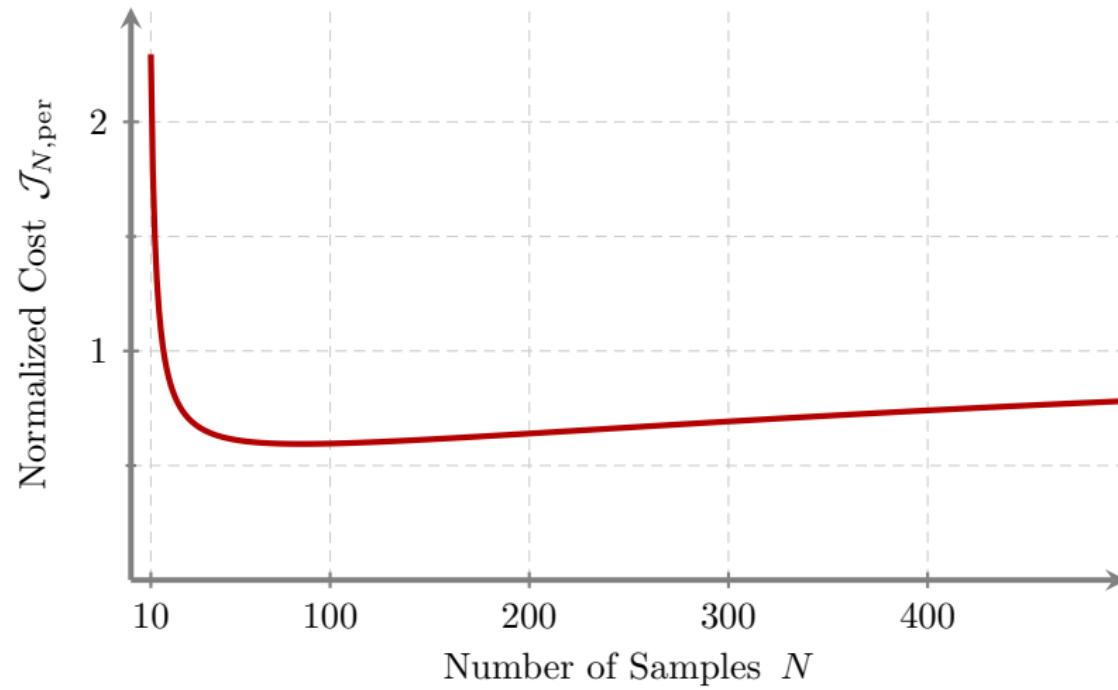
► $c_{10,\text{per}} = \frac{10^2}{6^2} \frac{699.7 - 383.2}{383.2} = 2.3$

► $\mathcal{J}_\infty = 383.2, \mathcal{J}_{60,\text{per}} = 385.5$

► $c_{60,\text{per}} = \frac{60^2}{6^2} \frac{385.5 - 383.2}{383.2} = 0.6$

COMPARING SAMPLING METHODS: NORMALIZED COST

Example (Normalized Cost for Periodic Sampling)



PERIODIC SAMPLING

We divide the interval $[0, T]$ into N parts of equal size

$$\tau_k = \tau = \frac{T}{N}, \quad k \in \{0, \dots, N - 1\}$$

$$t_k = k \cdot \frac{T}{N}, \quad k \in \{0, \dots, N\}$$

For $N \in \mathbb{N}$, we get the constant sampling density

$$\sigma_{per,N}(t) = \frac{1}{N \cdot t_k} = \frac{1}{N} \cdot \frac{N}{T} = \frac{1}{T}$$

PERIODIC SAMPLING: OPTIMAL CONTROL

For sampling period τ , the solution $\overline{K}(\tau)$ of the discrete-time Riccati equation can be determined analytically as

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2) \quad [1]$$

where K_∞ is the solution of the continuous-time Riccati equation, and X is the solution of a Lyapunov equation¹

... informally, optimal controller of the discrete-time can be expressed as the continuous-time solution K_∞ plus a correction term that is proportional to the square of the sampling period τ .

¹ Melzer, Stuart M., and Benjamin C. Kuo. "Sampling period sensitivity of the optimal sampled data linear regulator." *Automatica* 7.3 (1971): 367-370.

QUESTIONS

Question

Should there be a [1] in the equation, or only after Lyapunov equation?

PERIODIC SAMPLING: ASYMPTOTIC NORMALIZED COST

$$c_{N,\text{per}} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,\text{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$\begin{aligned}
 c_{\text{per}} &= \lim_{N \rightarrow \infty} \frac{N^2}{T^2} \frac{x_0' \overline{K}(\tau) x_0 - x_0' K_\infty x_0}{x_0' K_\infty x_0} \\
 &= \lim_{N \rightarrow \infty} \frac{N^2}{T^2} \frac{x_0' \left(K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2) \right) x_0 - x_0' K_\infty x_0}{x_0' K_\infty x_0} \\
 &= \lim_{N \rightarrow \infty} \frac{N^2}{T^2} \frac{x_0' \left(K_\infty + X \cdot \frac{T^2}{2N^2} + o\left(\frac{T^2}{2N^2}\right) \right) x_0 - x_0' K_\infty x_0}{x_0' K_\infty x_0} \\
 &= \lim_{N \rightarrow \infty} \frac{N^2}{T^2} \frac{x_0' \frac{XT^2}{2N^2} x_0 + x_0' o(N^{-2}) x_0}{x_0' K_\infty x_0} \\
 &= \frac{x_0' X x_0}{2 x_0' K_\infty x_0}
 \end{aligned}$$

PERIODIC SAMPLING: EXAMPLE

$$c_{\text{per}} = \frac{x_0' X x_0}{2 x_0' K_\infty x_0}$$

Example

For a first-order system ($n = 1$), wlog $B = R = 1$, we obtain

$$\begin{aligned} X &= \frac{1}{12}(K_\infty - A)K_\infty^2, \\ K_\infty &= A + \sqrt{A^2 + Q}. \end{aligned}$$

which gives us the asymptotic normalized cost

$$c_{\text{per}} = \frac{1}{24}A\sqrt{A^2 + Q} + A^2 + Q.$$

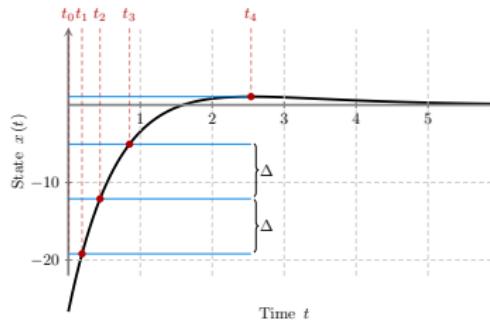
DETERMINISTIC LEBESGUE SAMPLING

- ▶ **Intuition:** Sample more frequently where the control changes faster
- ▶ Sample whenever the optimal u changes by a fixed threshold Δ , so after any sampling instant t_k , the next t_{k+1} is determined s.t.

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

where u is the optimal continuous-time input

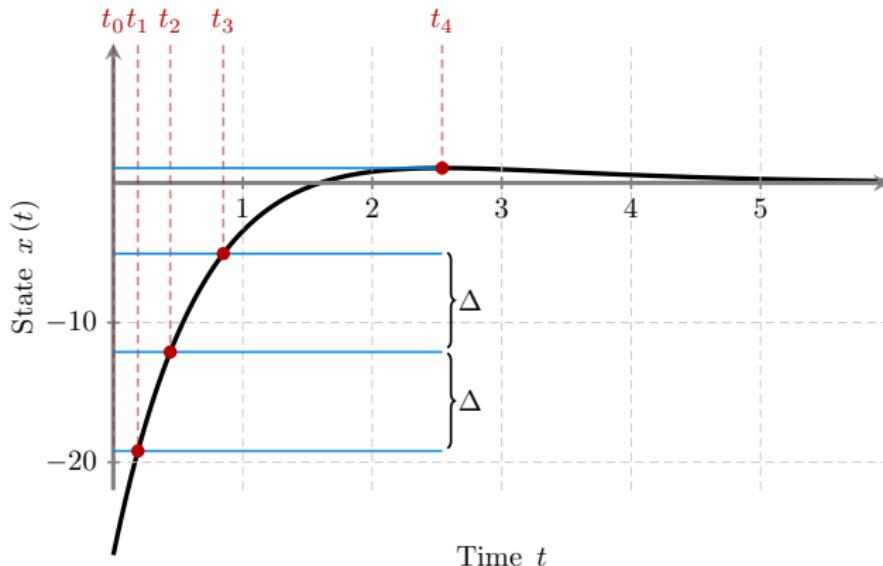
Example ($\Delta = 7$)



DETERMINISTIC LEBESGUE SAMPLING

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

Example ($\Delta = 7$)



DETERMINISTIC LEBESGUE SAMPLING

$$\| u(t_{k+1}) - u(t_k) \| = \Delta$$

In the case of a scalar input ($m = 1$) and a given number N of sampling instances in $[0, T]$, the sampling instants t_k satisfy

$$\begin{aligned} \int_{t_k}^{t_{k+1}} |\dot{u}(t)| dt &= |u(t_{k+1}) - u(t_k)| = \Delta \\ &= \frac{1}{N} \cdot \underbrace{\int_0^T |\dot{u}(t)| dt}_{N \cdot \Delta} \end{aligned}$$

DETERMINISTIC LEBESGUE SAMPLING: SAMPLING DENSITY

$$\int_{t_k}^{t_{k+1}} |\dot{u}(t)| dt = \frac{1}{N} \cdot \int_0^T |\dot{u}(t)| dt$$

$$\begin{aligned}\sigma_{\text{dls}}(t) &= \frac{1}{N \cdot \tau_k} \\ &= \frac{\int_{t_k}^{t_{k+1}} |\dot{u}(t)| dt}{\int_0^T |\dot{u}(t)| dt} \cdot \frac{1}{\tau_k} \\ &= \frac{\int_{t_k}^{t_{k+1}} |\dot{u}(t)| dt}{\tau_k} \cdot \frac{1}{\int_0^T |\dot{u}(t)| dt} \\ &= \frac{|u(t_{k+1}) - u(t_k)|}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T |\dot{u}(t)| dt}\end{aligned}$$

QUESTIONS

Question

Last equation missing