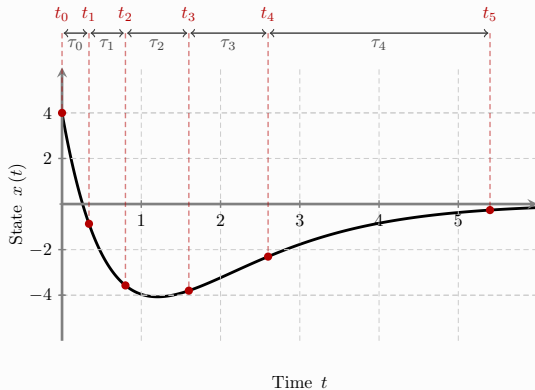# Optimal Control

Martina Maggio

Saarland University
Department of Computer Science

October 30, 2025

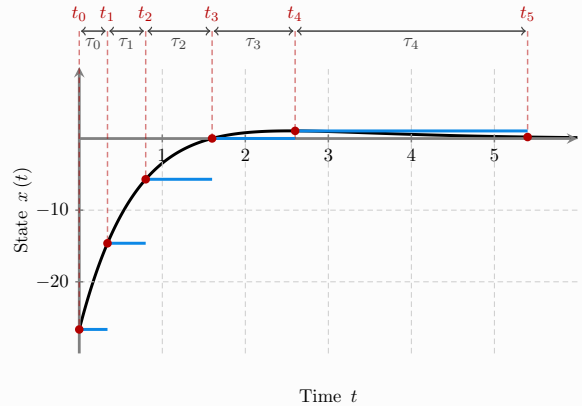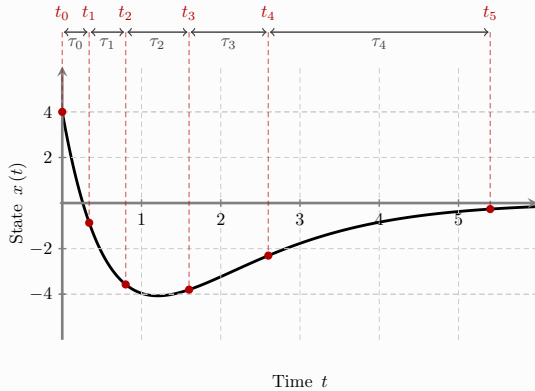# SAMPLING

Sampling

► $\{ t_0,\ t_1,\ \dots,\ ,t_N \}$: Sampling Pattern

► t_k : Sampling Instants

► $\tau_k = t_{k+1} - t_k$: Interarrival Times

Sampling

The Problem

Given

$$\begin{cases} \dot{x}(t) & = & A\,x(t) + B\,u(t) \\ x(0) & = & x_0 \end{cases}$$

- ▶ We restrict $u(t)$ to be in the class of piecewise constant functions and denote it as $\overline{u}(t)$
- ▶ Finite horizon $[0, T]$
- ▶ Number N of control updates

find

- ▶ Sampling instants $0 = t_1 < \ldots < t_N = T$
- ▶ Piecewise constant $\overline{u}(t) = u_k, \quad \forall t \in [t_{k-1}, t_k)$

that minimizes the performance index

- ▶ $\mathcal{J}(\overline{u}) = \int_0^T \left(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t) \right) dt + x'(T)\,S\,x(T)$

Quadratic Performance Index

$$\mathcal{J}(\overline{u}) = \int_0^T \left(x'(t) \underbrace{Q}_{\geq} x(t) + \overline{u}'(t) \underbrace{R}_{>} \overline{u}'(t) \right) dt + x'(T) \underbrace{S}_{\geq} x(T)$$

$$\text{s.t.} \begin{cases} \dot{x}(t) &=& A x(t) + B u(t) \\ x(0) &=& x_0 \end{cases}$$

> ► $x(t) \in \mathbb{R}^n$: system state
> ► $u(t) \in \mathbb{R}^m$: system input
> ► A, B, Q, R, S : constant matrices

Outline

- ▶ Computation of optimal control $u(t)$ for continuous-time systems
- ▶ Discretization Process for given $\{t_0, \ldots, t_N\}$
    1. System Discretization
    2. Computation of optimal control $(u_k)_{k \in \{0, \ldots, N\}}$ for the discrete-time system
- ▶ Sampling Density and Sampling Method Cost

TODO

Slide animation with the plot

TODO

Update Results bullet point

- ▶ Computation of optimal control $u(t)$ for continuous-time systems
- ▶ Discretization Process for given $\{t_0, \ldots, t_N\}$
  1. System Discretization
  2. Computation of optimal control $\{u_1, \ldots, u_N\}$ for the discrete-time system
- ▶ Sampling Density and Sampling Method Cost
- ▶ Sampling Methods for finding $\{t_0, \ldots, t_N\}$
  1. Periodic sampling
  2. Lebesgue sampling
  3. Quantization-based sampling
- ▶ Results

$$\text{minimize}_u \int_0^T \Big(x'(t)\,Q\,x(t) + u'(t)\,R\,u(t)\Big)\,dt + x(T)'\,S\,x(T)$$

Riccati equation for continuous-time systems

$$\begin{cases} \dot{K}(t) &= K(t)\,B\,R^{-1}\,B'\,K(t) - A'K(t) - K(t)\,A - Q, \\ K(T) &= S \end{cases}$$

which gives us the optimal control $u(t) = -R^{-1}B'\,K(t)\,x(t)$

with achieved cost

$$\mathcal{J}_\infty = x_0'\,K(0)\,x_0$$

Optimal Control: Discrete-Time Systems

Given the time points $\{t_0, \ldots, t_N\}$, with interarrivals τ_k, $k \in \{0, \ldots, N\}$, the discrete-time state space equations are:

$$\begin{cases} x_{k+1} & = & \overline{A}_k \, x_k + \overline{B}_k \\ x\left(0\right) & = & x_0 \end{cases}$$

▶ $\overline{A}_k = \Phi(\tau_k)$, $\Phi(\tau) = e^{A\tau_k}$

▶ $\overline{B}_k = \Gamma(\tau_k)$, $\Gamma(\tau) = \int_0^\tau e^{A(\tau - t)} \, dt \, B$

Comment

Integration limits? + a note about a variable change?

Performance Index for Discrete-Time Systems

$$\int_0^T \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t) \right) dt + x'(T)\, S\, x(T)$$

$$= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t) \right) dt + x'(T)\, S\, x(T)$$

$$= \sum_{k=0}^{N-1} \left[\int_{t_k}^{t_{k+1}} \left(\Phi\, x_k + \Gamma\, u_k \right)'\, Q\, \left(\Phi\, x_k + \Gamma\, u_k \right) dt + \int_{t_k}^{t_{k+1}} u_k'\, R\, u_k \right] + x'(T)\, S\, x(T)$$

$$= x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q\, \Phi\, dt \right)}_{\overline{Q}} x_k + u_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma' Q \Gamma\, dt + \underbrace{\int_{t_k}^{t_{k+1}} R\, dt}_{\tau_k R} \right)}_{\overline{R}} u_k + 2 x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q\, \Gamma\, dt \right)}_{\overline{P}} u_k + x'(T)\, S\, x(T)$$

Performance Index for Discrete-Time Systems

$$\int_0^T \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t) \right) dt + x'(T)\, S\, x(T)$$

$$= x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi'\, Q\, \Phi\, dt \right)}_{\overline{Q}} x_k + u_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma'\, Q\, \Gamma\, dt + \underbrace{\int_{t_k}^{t_{k+1}} R\, dt}_{\tau_k\, R} \right)}_{\overline{R}} u_k + 2 x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi'\, Q\, \Gamma\, dt \right)}_{\overline{P}} u_k + x'(T)\, S\, x(T)$$

$$\mathcal{J}(\overline{u}) = \sum_{k=0}^{N-1} \left(x_k'\, \overline{Q}\, x_k + u_k'\, \overline{R}\, u_k + 2 x_k'\, \overline{P}\, u_k \right) + x_N'\, S\, x_N$$

Question

Different brackets for integrals due to underbrace.

Bellman Equation

For $k \in \{\, 0, \ldots, N \,\}$, we define the function $\mathcal{J}_k : \mathbb{R}^n \to \mathbb{R}$, which gives the minimal cost achievable from stage k onward, given the state x_k as

$$
\begin{cases}
\mathcal{J}_k\,(x_k) & = & \min_u \left[\underbrace{x_k'\,\overline{Q}_k\,x_k}_{\substack{\text{State}\\\text{penalty}}} + \underbrace{u'\,\overline{R}_k\,u}_{\substack{\text{Control}\\\text{penalty}}} + \underbrace{2x_k'\,\overline{P}_k\,u}_{\text{Cross term}} + \mathcal{J}_{k+1}(x_{k+1}) \right] & \text{for } k \in \{\, 0, \ldots, N-1 \,\} \\[2em]
\mathcal{J}_N\,(x) & = & x'S\,x &
\end{cases}
$$

Intuition: define a *backward recursive function* that gives the minimal achievable cost from the current state onward.

Bellman Equation

For $k \in \{0, \ldots, N\}$, we define the function $\mathcal{J}_k : \mathbb{R}^n \to \mathbb{R}$, which gives the minimal cost achievable from stage k onward, given the state x_k as

$$
\begin{cases}
\mathcal{J}_k(x_k) = \min_u \Big[\underbrace{x_k' \overline{Q}_k x_k}_{\substack{\text{State} \\ \text{penalty}}} + \underbrace{u' \overline{R}_k u}_{\substack{\text{Control} \\ \text{penalty}}} + \underbrace{2x_k' \overline{P}_k u}_{\text{Cross term}} + \mathcal{J}_{k+1}(x_{k+1}) \Big] & \text{for } k \in \{0, \ldots, N-1\} \\
\\
\mathcal{J}_N(x) = x'Sx
\end{cases}
$$

$$
x_{k+1} = \overline{A}_k x_k + \overline{B}_k u_k
$$

DYNAMIC PROGRAMMING: QUADRATIC FORM

$\mathcal{J}_k(x)$ is a quadratic form of x

Unknown at this point!

$$\mathcal{J}_k(x) = x' \overline{K}_k x$$

$$\Downarrow$$

$$\mathcal{J}_{k+1}(\overline{A}_k x + \overline{B}_k u) = (\overline{A}_k x + \overline{B}_k u)' \, \overline{K}_{k+1} \, (\overline{A}_k x + \overline{B}_k u)$$

$\Downarrow$ Plug into **Bellman equation**

$$\mathcal{J}_k(x) = x' \underbrace{(\overline{Q}_k + \overline{A}'_k \overline{K}_{k+1} \overline{A}_k)}_{\hat{Q}_k} x + u' \underbrace{(R_k + \overline{B}'_k \overline{K}_{k+1} \overline{B}_k)}_{\hat{R}_k} u + 2x' \underbrace{(\overline{P}_k + \overline{A}'_k \overline{K}_{k+1} \overline{B}_k)}_{\hat{S}_k} u$$

NOTES

Question

can't move the whole thing to the left

Question

a frame for the last eq? or copy it to the next slide

Dynamic Programming: Minimization

We aim to **minimize** the cost:

$$\min_u \left(x' \, \hat{Q}_k \, x + u' \, \hat{R}_k \, u + 2x' \, \hat{S}_k \, u \right)$$

Optimality condition:

$$\frac{\partial}{\partial u}(\,\cdot\,) = 0 \quad \implies \quad \overline{R}_k \, u + \overline{S}'_k x = 0$$

$$\Downarrow \quad \text{Solving for } u$$

$$u_k^* = -\,\overline{R}_k^{-1} \, \overline{S}'_k \, x_k$$

We plug u_k^* into the **Bellman equation:**

$$\mathcal{J}_k(x) = x'\left(\hat{Q}_k - \hat{B}_k\,\hat{R}_k^{-1}\,\hat{B}_k'\right)x$$

$$\Downarrow \qquad \text{Recall } \mathcal{J}_k(x) = x'K_k x$$

$$\overline{K}_k = \hat{Q}_k - \hat{S}_k\,\hat{R}_k^{-1}\,\hat{S}_k'$$

Minimal cost equals

$$\mathcal{J} = x_0'\,\overline{K}_0\,x_0$$

Sampling Density

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *sampling density* $\sigma_{N,m} : [0, T] \to \mathbb{R}^+$ of any sampling method m is defined as
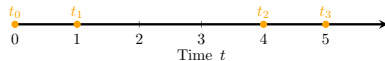
$$\sigma_{N,m}(t) = \frac{1}{N\,\tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

▶ Temporal distrubtion of sampling instants

▶ Sampling density is normalized

Example

Consider $T = 5$ units, $N = 4$ samples, and the following sampling instants

▶ $\sigma_4\,(t) = \frac{1}{4 \cdot 1} = \frac{1}{4}, \quad \forall t \in [0, 1)$

▶ $\sigma_4\,(t) = \frac{1}{4 \cdot 3} = \frac{1}{12}, \quad \forall t \in [1, 4)$

Sampling Density

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *sampling density* $\sigma_{N,m} : [0, T] \to \mathbb{R}^+$ of any sampling method m is defined as

$$\sigma_{N,m}(t) = \frac{1}{N\,\tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

► Temporal distrubtion of sampling instants
► Sampling density is normalized

Asymptotic Sampling Density

To remove the dependency on N, we define the *asymptotic sampling density* as $\sigma_m : [0, T] \to \mathbb{R}^+$ as the limit

$$\sigma_m(t) = \lim_{N \to \infty} \sigma_{N,m}(t)$$

Sampling Density

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *sampling density* $\sigma_{N,m} : [0, T] \to \mathbb{R}^+$ of any sampling method m is defined as

$$\sigma_{N,m}(t) = \frac{1}{N \, \tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

▶ Temporal distrubtion of sampling instants
▶ Sampling density is normalized

Asymptotic Sampling Density

To remove the dependency on N, we define the *asymptotic sampling density* as $\sigma_m : [0, T] \to \mathbb{R}^+$ as the limit

$$\sigma_m(t) = \lim_{N \to \infty} \sigma_{N,m}(t)$$

Comparing Sampling Methods: Normalized Cost

Normalized Cost

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *normalized cost* of any sampling method m is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method m with N samples, and $\mathcal{J}_\infty$ is the minimal cost of the continuous-time system.

Example

TODO

Comparing Sampling Methods: Normalized Cost

Normalized Cost

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *normalized cost* of any sampling method m is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method m with N samples, and $\mathcal{J}_\infty$ is the minimal cost of the continuous-time system.

Asymptotic Normalized Cost

To remove the dependency on N, we define the *asymptotic normalized cost* as the limit

$$c_m = \lim_{N \to \infty} c_{N,m}$$

Periodic Sampling

We divide the interval $[0, T]$ into N parts of equal size

$$\tau_k = \tau = \frac{T}{N}, \quad k \in \{0, \ldots, N-1\}$$

$$t_k = k \cdot \frac{T}{N}, \quad k \in \{0, \ldots, N\}$$

For $N \in \mathbb{N}$, we get the constant sampling density

$$\sigma_{per,N}(t) = \frac{1}{N \cdot t_k} = \frac{1}{N} \cdot \frac{N}{T} = \frac{1}{T}$$

Periodic Sampling: Optimal Control

For sampling period τ, the solution $\overline{K}(\tau)$ of the discrete-time Riccati equation can be determined analytically as

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

where K_∞ is the solution of the continuous-time Riccati equation, and X is the solution of a particular equation.

> *... informally, optimal controller of the discrete-time can be expressed as the continuous-time solution K_∞ plus a correction term that is proportional to the square of the sampling period τ.*

NOTES

Question

How do we do a reference? just as a bibliography entry?

Question

phrasing of the X intro? maybe some more precise reference. Feels out of blue.

PERIODIC SAMPLING: ASYMPTOTIC NORMALIZED COST

$$c_{N,\text{per}} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,\text{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$
\begin{aligned}
c_{\text{per}} &= \lim_{N \to \infty} \frac{N^2}{T^2} \frac{x_0' \, \overline{K}(\tau) \, x_0 - x_0' \, K_\infty \, x_0}{x_0' \, K_\infty \, x_0} \\
&= \lim_{N \to \infty} \frac{N^2}{T^2} \frac{x_0' \left(K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2) \right) x_0 - x_0' \, K_\infty \, x_0}{x_0' \, K_\infty \, x_0} \\
&= \lim_{N \to \infty} \frac{N^2}{T^2} \frac{x_0' \left(K_\infty + X \cdot \frac{T^2}{2N^2} + o\left(\frac{T^2}{2N^2} \right) \right) x_0 - x_0' \, K_\infty \, x_0}{x_0' \, K_\infty \, x_0} \\
&= \lim_{N \to \infty} \frac{N^2}{T^2} \frac{x_0' \frac{XT^2}{2N^2} x_0 + x_0' \, o\left(N^{-2} \right) x_0}{x_0' \, K_\infty \, x_0} \\
&= \frac{x_0' \, X \, x_0}{2 \, x_0' \, K_\infty \, x_0}
\end{aligned}
$$

NOTES

> **Question**
>
> The boxes are not boxing :(

$$c_{\text{per}} = \frac{x_0' \, X \, x_0}{2 \, x_0' \, K_\infty \, x_0}$$

Example

For a first-order system ($n = 1$), wlog $B = R = 1$, we obtain

$$X = \frac{1}{12}(K_\infty - A)K_\infty^2,$$

$$K_\infty = A + \sqrt{A^2 + Q}.$$

which gives us the asymptotic normalized cost

$$c_{\text{per}} = \frac{1}{24}A\sqrt{A^2 + Q} + A^2 + Q.$$

> **Question**
>
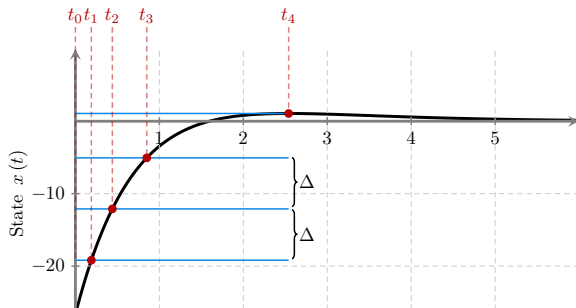> should we do italics every time we use a defined instance? Like *asymptotic normalized cost.*

- **Intuition:** Sample more frequently where the control changes faster
- Sample whenever the optimal u changes by a fixed threshold Δ, so after any sampling instant t_k, the next t_{k+1} is determined s.t.

$$\| u\left(t_{t+1}\right) - u(t_k) \| = \Delta$$

where u is the optimal continuous-time input

Example ($\Delta = 7$)

Deterministic Lebesgue Sampling

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

In the case of a scalar input ($m = 1$) and a given number N of sampling instance in $[0, T]$, we compute the sampling instants t_k

$$
\begin{aligned}
\int_{t_k}^{t_{k+1}} | \dot{u}(t) | \, dt &= | u(t_{k+1}) - u(t_k) | \\
&= \Delta \\
&= \frac{1}{N} \cdot \underbrace{\int_0^T | \dot{u}(t) | \, dt}_{N \cdot \Delta}
\end{aligned}
$$

Deterministic Lebesgue Sampling: Sampling Density

$$\int_{t_k}^{t_{k+1}} | \dot{u}(t) | \, dt = \frac{1}{N} \cdot \int_0^T | \dot{u}(t) | \, dt$$

$$
\begin{aligned}
\sigma_{\text{dls}}(t) &= \frac{1}{N \cdot \tau_k} \\
&= \frac{\int_{t_k}^{t_{k+1}} | \dot{u}(t) | \, dt}{\int_0^T | \dot{u}(t) | \, dt} \cdot \frac{1}{\tau_k} \\
&= \frac{\int_{t_k}^{t_{k+1}} | \dot{u}(t) | \, dt}{\tau_k} \cdot \frac{1}{\int_0^T | \dot{u}(t) | \, dt} \\
&= \frac{| u(t_{k+1}) - u(t_k) |}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T | \dot{u}(t) | \, dt}
\end{aligned}
$$

- Sampling intervals vary depending on $| \dot{u}(t) |$
- No closed formula for the asymptotic normalized cost c_{dls}

Question

integrals are inlined, even with \displaystyle