Optimal Control

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The Problem

Given

- $\begin{cases} \dot{x}(t) & = & A\,x\,(t) + B\,u\,(t) \\ x\,(0) & = & x_0 \end{cases}$
- We restrict $u\,(t)$ to be in the class of piecewise constant functions and denote it as $\overline{u}\,(t)$
- Finite horizon $[0, T]$
- Number N of control updates

find

- Sampling instants $0 = t_1 < \ldots < t_N = T$
- Piecewise constant $\overline{u}\,(t) = u_k \quad \forall t \in [t_{k-1}, t_k)$

that minimizes the performance index

- $\mathcal{J}\,(\overline{u}) = \int_0^T \left(x'\,(t)\,Q\,x\,(t) + \overline{u}'\,(t)\,R\,\overline{u}\,(t)\right) dt + x'\,(T)\,S\,x\,(T)$

NOTES

Question

How can we separate the system more, but keep the same spacing for other bulletpoints?

Question

For all quantifier in the items

Quadratic Performance Index

$$\mathcal{J}(\overline{u}) = \int_0^T \left(x'(t) \underbrace{Q}_{\succeq} x(t) + \overline{u}'(t) \underbrace{R}_{\succ} \overline{u}'(t)\right) dt + x'(T) \underbrace{S}_{\succeq} x(T)$$

$$\text{s.t.} \begin{cases} \dot{x}(t) & = & A\,x(t) + B\,u(t) \\ x(0) & = & x_0 \end{cases}$$

- $x(t) \in \mathbb{R}^n$: system state
- $u(t) \in \mathbb{R}^m$: system input
- A, B, Q, R, S : constant matrices

Outline

- Computation of optimal control $u(t)$ for continuous-time systems
- Discretization Process for given $\{t_0, \ldots, t_N\}$
  1. System Discretization
  2. Computation of optimal control $\{u_1, \ldots, u_N\}$ for the discrete-time system
- Sampling Density and Sampling Method Cost

TODO

Slide animation with the plot

TODO

Update Results bullet point

Outline

- Computation of optimal control $u(t)$ for continuous-time systems
- Discretization Process for given $\{t_0, \ldots, t_N\}$
  1. System Discretization
  2. Computation of optimal control $\{u_1, \ldots, u_N\}$ for the discrete-time system
- Sampling Density and Sampling Method Cost
- Sampling Methods for finding $\{t_0, \ldots, t_N\}$
  1. Periodic sampling
  2. Lebesgue sampling
  3. Quantization-based sampling
- Results

NOTES

Question

In the def, u_k or a set of us?

Optimal Control: Continuous-Time Systems

$$\text{minimize}_u \int_0^T \Big(x'(t)\,Q\,x(t) + u'(t)\,R\,u(t)\Big)\,dt + x(T)'\,S\,x(T)$$

Riccati equation for continuous-time systems

$$\begin{cases} \dot{K}(t) & = & K(t)\,B\,R^{-1}\,B'\,K(t) - A'K(t) - K(t)\,A - Q, \\ K(T) & = & S \end{cases}$$

which gives us the optimal control $u(t)$

$$u(t) = -R^{-1}B'\,K(t)\,x(t)$$

with achieved cost

$$\mathcal{J}_\infty = x_0'\,K(0)\,x_0$$

NOTES

Question

K or $K(t)$?

Optimal Control: Discrete-Time Systems

Given the time points $\{t_0, \ldots, t_N\}$, with interarrivals τ_k, $k \in \{0, \ldots, N\}$, the discrete-time state space equations are:

$$\begin{cases} x_{k+1} & = & \overline{A}_k \, x_k + \overline{B}_k \\ x(0) & = & x_0 \end{cases}$$

- $\overline{A}_k = \Phi(\tau_k), \qquad \Phi(\tau) = e^{A\tau_k}$
- $\overline{B}_k = \Gamma(\tau_k), \qquad \Gamma(\tau) = \int_0^{\tau} e^{A(\tau-t)} \, dt \, B$

NOTES

Question

A bar over A is weird.

Comment

Integration limits? + a note about a variable change?

Performance Index for Discrete-Time Systems

$$
\begin{aligned}
&\int_0^T \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t)\right) dt + x'(T)\, S\, x(T) \\
&= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t)\right) dt + x'(T)\, S\, x(T) \\
&= \sum_{k=0}^{N-1} \left[\int_{t_k}^{t_{k+1}} (\Phi\, x_k + \Gamma\, u_k)'\; Q\; (\Phi\, x_k + \Gamma\, u_k)\, dt + \int_{t_k}^{t_{k+1}} u_k'\, R\, u_k\right] + x'(T)\, S\, x(T) \\
&= x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q \Phi\, dt\right)}_{\overline{Q}} x_k + u_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma' Q \Gamma\, dt + \underbrace{\int_{t_k}^{t_{k+1}} R\, dt}_{\tau_k R}\right)}_{\overline{R}} u_k + 2x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q \Gamma\, dt\right)}_{\overline{P}} u_k + x'(T)\, S\, x(T)
\end{aligned}
$$

$$
\mathcal{J}(\overline{u}) = \sum_{k=0}^{N-1} \left(x_k'\, \overline{Q}\, x_k + u_k'\, \overline{R}\, u_k + 2x_k'\, \overline{P}\, u_k\right) + x_N'\, S\, x_N
$$

NOTES

TODO

add more of \pause in equations

Question

Different brackets for integrals due to underbrace.

Dynamic Programming: Bellman Equation

Idea: We define a *recursive cost-to-go function* that minimizes the remaining cost given the current state.

$$\mathcal{J}_k(x_k) = \min_u \Big[\underbrace{x_k^\top Q_k x_k}_{\text{State penalty}} + \underbrace{u^\top R_k u}_{\text{Control penalty}} + \underbrace{2x_k^\top S_k u}_{\text{Cross term}} + \mathcal{J}_{k+1}(x_{k+1}) \Big]$$

$$\mathcal{J}_N(x) = x^\top S x$$

$$x_{k+1} = A_k x_k + B_k u_k$$

Comment

This slide not reviewed yet! Just a test. DP slides to be added later!

Comparing Sampling Methods: Sampling Density

Sampling Density

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *sampling density* $\sigma_{N,m} : [0, T] \rightarrow \mathbb{R}^+$ of any sampling method m is defined as

$$\sigma_{N,m}(t) = \frac{1}{N\,\tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

- Temporal distrubtion of sampling instants
- Sampling density is normalized

Example

Consider $T = 5$ units, $N = 4$ samples, and the following sampling instants

- $\sigma_4\,(t) = \frac{1}{4 \cdot 1} = \frac{1}{4}, \quad \forall t \in [0, 1)$
- $\sigma_4\,(t) = \frac{1}{4 \cdot 3} = \frac{1}{12}, \quad \forall t \in [1, 4)$

Comparing Sampling Methods: Sampling Density

Sampling Density

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *sampling density* $\sigma_{N,m} : [0, T] \longrightarrow \mathbb{R}^+$ of any sampling method m is defined as

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- Temporal distrubtion of sampling instants
- Sampling density is normalized

Asymptotic Sampling Density

To remove the dependency on N, we define the *asymptotic sampling density* as $\sigma_m : [0, T] \longrightarrow \mathbb{R}^+$ as the limit

$$\sigma_m(t) = \lim_{N \to \infty} \sigma_{N,m}(t)$$

Comparing Sampling Methods: Normalized Cost

Normalized Cost

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *normalized cost* of any sampling method m is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method m with N samples, and $\mathcal{J}_\infty$ is the minimal cost of the continuous-time system.

Example

TODO

Comparing Sampling Methods: Normalized Cost

Normalized Cost

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *normalized cost* of any sampling method m is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method m with N samples, and $\mathcal{J}_\infty$ is the minimal cost of the continuous-time system.

Asymptotic Normalized Cost

To remove the dependency on N, we define the *asymptotic normalized cost* as the limit

$$c_m = \lim_{N \to \infty} c_{N,m}$$

Periodic Sampling

We divide the interval $[0, T]$ into N parts of equal size

$$\tau_k = \tau = \frac{T}{N}, \quad k \in \{0, \ldots, N-1\}$$
$$t_k = k \cdot \frac{T}{N}, \quad k \in \{0, \ldots, N\}$$

For $N \in \mathbb{N}$, we get the constant sampling density

$$\sigma_{per,N}(t) = \frac{1}{N \cdot t_k} = \frac{1}{N} \cdot \frac{N}{T} = \frac{1}{T}$$

Periodic Sampling: Optimal Control

For sampling period τ, the solution $\overline{K}(\tau)$ of the discrete-time Riccati equation can be determined analytically as

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

where K_∞ is the solution of the continuous-time Riccati equation, and X is the solution of a particular equation.

... informally, optimal controller of the discrete-time can be expressed as the continuous-time solution K_∞ plus a correction term that is proportional to the square of the sampling period τ.

NOTES

Question

How do we do a reference? just as a bibliography entry?

Question

phrasing of the X intro? maybe some more precise reference. Feels out of blue.

Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\text{per}} = \frac{N^2}{T^2}\frac{\mathcal{J}_{N,\text{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$\begin{aligned}
c_{\text{per}} &= \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0'\,\overline{K}(\tau)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0'\left(K_\infty + X\cdot\frac{\tau^2}{2} + o(\tau^2)\right)x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0'\left(K_\infty + X\cdot\frac{T^2}{2N^2} + o\left(\frac{T^2}{2N^2}\right)\right)x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0'\,\frac{XT^2}{2N^2}\,x_0 + x_0'\,o\left(N^{-2}\right)x_0}{x_0'\,K_\infty\,x_0} \\
&= \frac{x_0'\,X\,x_0}{2\,x_0'\,K_\infty\,x_0}
\end{aligned}$$

NOTES

Question

The boxes are not boxing :(

Periodic Sampling: Example

$$c_{\text{per}} = \frac{x_0' \, X \, x_0}{2 \, x_0' \, K_\infty \, x_0}$$

Example

For a first-order system ($n = 1$), wlog $B = R = 1$, we obtain

$$X = \frac{1}{12}(K_\infty - A)K_\infty^2,$$
$$K_\infty = A + \sqrt{A^2 + Q}.$$

which gives us the asymptotic normalized cost

$$c_{\text{per}} = \frac{1}{24} A \sqrt{A^2 + Q} + A^2 + Q.$$

NOTES

Question

should we do italics every time we use a defined instance? Like *asymptotic normalized cost.*

Deterministic Lebesgue Sampling

- **Intuition:** Sample more frequently where the optimal control changes faster
- Sample whenever the optimal u changes by a fixed threshold Δ, so after any sampling instant t_k, the next t_{k+1} is determined s.t.

$$\| u\left(t_{t+1}\right) - u(t_k) \| = \Delta$$

where u is the optimal continuous-time input

Example

TODO

Deterministic Lebesgue Sampling

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

In the case of a scalar input ($m = 1$) and a given number N of sampling instance in $[0, T]$, we compute the sampling instants t_k

$$\begin{aligned} \int_{t_k}^{t_{k+1}} |\dot{u}(t)|\, dt &= |u(t_{k+1}) - u(t_k)| \\ &= \Delta \\ &= \frac{1}{N} \cdot \underbrace{\int_0^T |\dot{u}(t)|\, dt}_{N \cdot \Delta} \end{aligned}$$

Deterministic Lebesgue Sampling: Sampling Density

$$\int_{t_k}^{t_{k+1}} |\dot{u}(t)|\, dt = \frac{1}{N} \cdot \int_0^T |\dot{u}(t)|\, dt$$

$$\begin{aligned}
\sigma_{\text{dls}}(t) &= \frac{1}{N \cdot \tau_k} \\
&= \frac{\int_{t_k}^{t_{k+1}} |\dot{u}(t)|\, dt}{\int_0^T |\dot{u}(t)|\, dt} \cdot \frac{1}{\tau_k} \\
&= \frac{\int_{t_k}^{t_{k+1}} |\dot{u}(t)|\, dt}{\tau_k} \cdot \frac{1}{\int_0^T |\dot{u}(t)|\, dt} \\
&= \frac{|u(t_{k+1}) - u(t_k)|}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T |\dot{u}(t)|\, dt}
\end{aligned}$$

- Sampling intervals vary depending on $|\dot{u}(t)|$
- No closed formula for the asymptotic normalized cost c_{dls}

NOTES

Question

integrals are inlined, even with \displaystyle