The Optimal Sampling Pattern for Linear Control Systems

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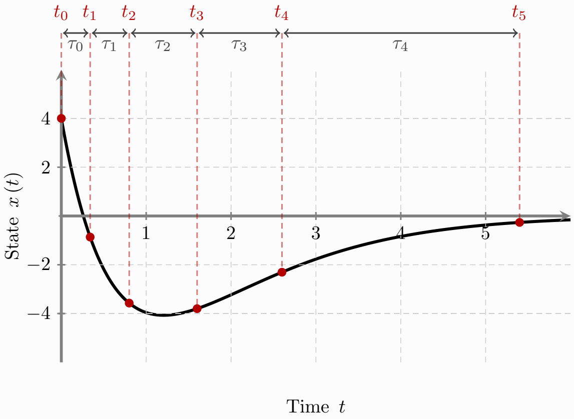
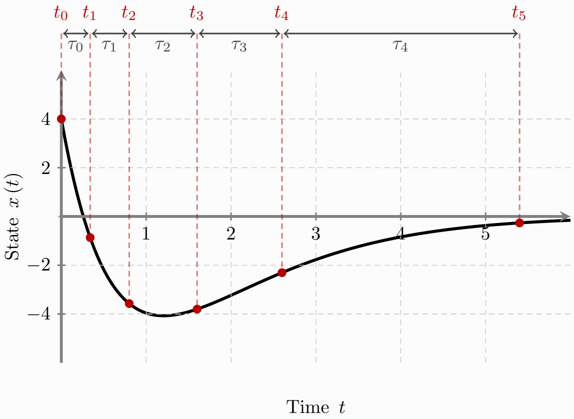
Sampling



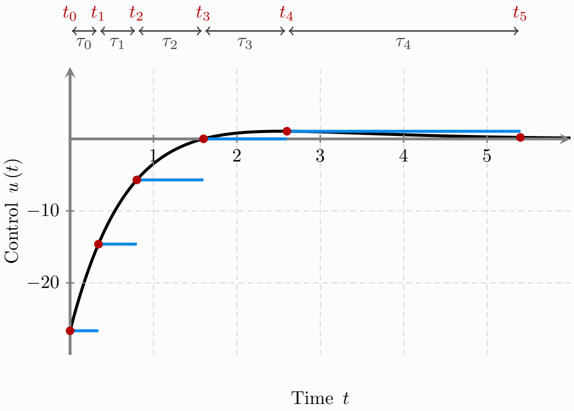
Sampling

- $\{t_0, t_1, \ldots, t_N\}$: Sampling pattern
- t_k: Sampling instants
- $\tau_k = t_{k+1} - t_k$: Interarrival times

Sampling



t_0 t_1 t_2 t_3 t_4 t_5
τ_0 τ_1 τ_2 τ_3 τ_4
State $x(t)$
4
2
−2
−4
1 2 3 4 5
Time t



t_0 t_1 t_2 t_3 t_4 t_5
τ_0 τ_1 τ_2 τ_3 τ_4
Control $u(t)$
−10
−20
1 2 3 4 5
Time t

QUESTIONS

Question

left plot jumps with \only. how to fix?

The Problem

Given

$$\begin{cases} \dot{x}(t) & = & A\,x\,(t) + B\,u\,(t) \\ x\,(0) & = & x_0 \end{cases}$$

- We restrict $u\,(t)$ to be in the class of piecewise constant functions and denote it as $\overline{u}\,(t)$
- Finite horizon $[0, T]$
- Number N of control updates

find

- Sampling instants $0 = t_1 < \ldots < t_N = T$
- Piecewise constant $\overline{u}\,(t) = u_k, \quad \forall t \in [t_{k-1}, t_k)$

that minimizes the performance index

- $\mathcal{J}\,(\overline{u}) = \int_0^T \big(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t)\big)\,dt + x'(T)\,S\,x(T)$

Quadratic Performance Index

$$\mathcal{J}(\overline{u}) = \int_0^T \left(x'(t) \underbrace{Q}_{\succeq} x(t) + \overline{u}'(t) \underbrace{R}_{\succ} \overline{u}'(t)\right) dt + x'(T) \underbrace{S}_{\succeq} x(T)$$

$$\text{s.t.} \begin{cases} \dot{x}(t) & = & A\,x(t) + B\,u(t) \\ x(0) & = & x_0 \end{cases}$$

- $x(t) \in \mathbb{R}^n$: system state
- $u(t) \in \mathbb{R}^m$: system input
- A, B, Q, R, S: constant matrices

Outline

- Computation of optimal control $u(t)$ for continuous-time systems
- Discretization Process for given $\{t_0, \ldots, t_N\}$
  1. System Discretization
  2. Computation of optimal control $(u_k)_{k\in\{0,\ldots,N\}}$ for the discrete-time system

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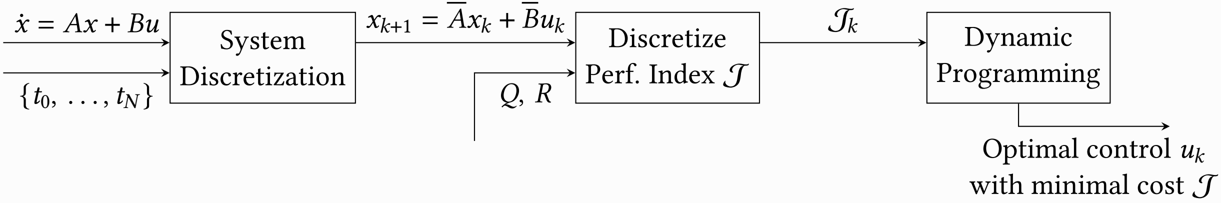

Outline

- Computation of optimal control $u(t)$ for continuous-time systems
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  1. System Discretization
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- Sampling Density and Sampling Method Cost
- Sampling Methods for finding $\{t_0, \ldots, t_N\}$
  1. Periodic sampling
  2. Lebesgue sampling
  3. Quantization-based sampling
- Results

Optimal Control: Continuous-Time Systems

$$\text{minimize}_u \int_0^T \Big(x'(t)\, Q\, x(t) + u'(t)\, R\, u(t)\Big)\, dt + x(T)'\, S\, x(T)$$

Riccati equation for continuous-time systems

$$\begin{cases} \dot{K}(t) & = & K(t)\, B R^{-1} B' K(t) - A' K(t) - K(t)\, A - Q, \\ K(T) & = & S \end{cases}$$

which gives us the optimal control $u(t) = -R^{-1} B' K(t)\, x(t)$

with achieved cost

$$\mathcal{J}_\infty = x_0' K(0)\, x_0$$

Optimal Control: Discrete-Time Systems

Given the time points $\{t_0, \ldots, t_N\}$, with interarrivals τ_k, $k \in \{0, \ldots, N\}$, the discrete-time state space equations are:

$$\begin{cases} x_{k+1} &= \overline{A}_k \, x_k + \overline{B}_k \\ x\,(0) &= x_0 \end{cases}$$

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$$\begin{cases} x_{k+1} & = & \overline{A}_k\, x_k + \overline{B}_k \\ x\,(0) & = & x_0 \end{cases}$$

$\overline{A}_k = \Phi(\tau_k)$	$\Phi(\tau) = e^{A\tau}$	State Matrix
$\overline{B}_k = \Gamma(\tau_k)$	$\Gamma(\tau) = \int_0^{\tau} e^{A(\tau - t)}\, dt\, B$	Input Matrix

Performance Index for Discrete-Time Systems

$$\int_0^T \left(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t)\right)dt + x'(T)\,S\,x(T)$$

$$= \sum_{k=0}^{N-1}\left[\int_{t_k}^{t_{k+1}} \left(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t)\right)dt + x'(T)\,S\,x(T)\right]$$

$$= \sum_{k=0}^{N-1}\left[\int_{t_k}^{t_{k+1}} (\Phi\,x_k + \Gamma\,u_k)'\;Q\;(\Phi\,x_k + \Gamma\,u_k)\,dt + \int_{t_k}^{t_{k+1}} u_k'\,R\,u_k\right] + x'(T)\,S\,x(T)$$

$$= \sum_{k=0}^{N-1}\left[x_k'\underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi'\,Q\,\Phi\,dt\right)}_{\overline{Q}} x_k + u_k'\underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma'\,Q\Gamma\,dt + \underbrace{\int_{t_k}^{t_{k+1}} R dt}_{\tau_k R}\right)}_{\overline{R}} u_k + 2x_k'\underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi'\,Q\,\Gamma\,dt\right)}_{\overline{P}} u_k\right] + x'(T)\,S\,x(T)$$

Performance Index for Discrete-Time Systems

$$\int_0^T \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t)\right) dt + x'(T)\, S\, x(T)$$

$$= \sum_{k=0}^{N-1} \left[x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q \Phi \, dt \right)}_{\overline{Q}} x_k + u_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma' Q \Gamma \, dt + \underbrace{\int_{t_k}^{t_{k+1}} R dt}_{\tau_k R} \right)}_{\overline{R}} u_k + 2 x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi' Q \Gamma \, dt \right)}_{\overline{P}} u_k \right] + x'(T)\, S\, x(T)$$

$$\mathcal{J}(\overline{u}) = \sum_{k=0}^{N-1} \left(x_k' \, \overline{Q}\, x_k + u_k' \, \overline{R}\, u_k + 2 x_k' \, \overline{P}\, u_k \right) + x_N' \, S\, x_N$$

COMMENTS

Comment

- cannot use \pause within eqarray (so that equations on the first "Performance Index for Discrete-Time Systems" slide appear one by one). any alternatives, without deconstruction into separate equations? Additional comment in code!
- and the first line of equations shouldnt jump!

Dynamic Programming: Bellman Equation

Bellman Equation

For $k \in \{0, \ldots, N\}$, we define the function $\mathcal{J}_k : \mathbb{R}^n \to \mathbb{R}$, which gives the minimal cost achievable from stage k onward, given the state x_k as

$$\begin{cases} \mathcal{J}_k(x_k) = \min_u \Big[\underbrace{x_k' \overline{Q}_k x_k}_{\text{State penalty}} + \underbrace{u' \overline{R}_k u}_{\text{Control penalty}} + \underbrace{2x_k' \overline{P}_k u}_{\text{Cross term}} + \mathcal{J}_{k+1}(x_{k+1}) \Big] & \text{for } k \in \{0, \ldots, N-1\} \\ \mathcal{J}_N(x) = x' S x. \end{cases}$$

Intuition: define a *backward recursive function* that gives the minimal achievable cost from the current state onward.

Dynamic Programming: Bellman Equation

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$$x_{k+1} = \overline{A}_k x_k + \overline{B}_k u_k$$

Dynamic Programming: Quadratic Form

$\mathcal{J}_k(x)$ is a quadratic form of x

Unknown at this point!

$$\mathcal{J}_k(x) = x' \overline{K}_k \, x$$

$$\Downarrow$$

$$\mathcal{J}_{k+1}(\overline{A}_k x + \overline{B}_k u) = (\overline{A}_k x + \overline{B}_k u)' \, \overline{K}_{k+1} \, (\overline{A}_k x + \overline{B}_k u)$$

$$\Downarrow$$

Plug into **Bellman equation**

$$\mathcal{J}_k(x) = \; x' \underbrace{(\overline{Q}_k + \overline{A}_k' \, \overline{K}_{k+1} \overline{A}_k)}_{\hat{Q}_k} x + u' \underbrace{(R_k + \overline{B}_k' \overline{K}_{k+1} \overline{B}_k)}_{\hat{R}_k} u \; + 2x' \underbrace{(\overline{P}_k + \overline{A}_k' \, \overline{K}_{k+1} \, \overline{B}_k)}_{\hat{S}_k} u$$

Dynamic Programming: Quadratic Form

$\mathcal{J}_k(x)$ is a quadratic form of x

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$$\mathcal{J}_k(x) = x' \overline{K}_k \, x$$

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$$\Downarrow$$

Plug into **Bellman equation**

$$\boxed{\mathcal{J}_k(x) = x' \, \hat{Q}_k \, x + u' \, \hat{R}_k \, u + 2x' \, \hat{S}_k \, u}$$

Dynamic Programming: Minimization

We aim to **minimize** the cost:

$$\min_u \Big(x' \, \hat{Q}_k \, x + u' \, \hat{R}_k \, u + 2x' \, \hat{S}_k \, u \Big)$$

Optimality condition:

$$\frac{\partial}{\partial u}(\,\cdot\,) = 0 \quad \Longrightarrow \quad \overline{R}_k \, u + \overline{S}_k' x = 0$$

$$\Downarrow \qquad \text{Solving for } u$$

$$u_k^* = -\,\hat{R}_k^{-1} \, \hat{S}_k' \, x_k$$

Dynamic Programming: Riccati Solution

We plug u_k^* into the **Bellman equation:**

$$\mathcal{J}_k(x) = x' \left(\hat{Q}_k - \hat{B}_k \, \hat{R}_k^{-1} \, \hat{B}_k' \right) x$$

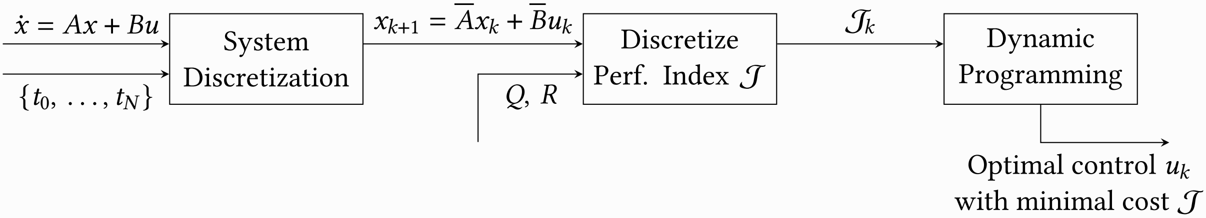
$$\Downarrow \qquad \text{Recall} \quad \mathcal{J}_k(x) = x' \overline{K}_k x$$

$$\overline{K}_k = \hat{Q}_k - \hat{S}_k \, \hat{R}_k^{-1} \, \hat{S}_k'$$

Minimal cost equals

$$\mathcal{J} = x_0' \, \overline{K}_0 \, x_0$$

Outline


$\dot{x} = Ax + Bu$
$\{t_0, \ldots, t_N\}$
System Discretization
$x_{k+1} = \overline{A}x_k + \overline{B}u_k$
Q, R
Discretize Perf. Index $\mathcal{J}$
$\mathcal{J}_k$
Dynamic Programming
Optimal control u_k with minimal cost $\mathcal{J}$

Outline

$\dot{x} = Ax + Bu$

$\{t_0, \ldots, t_N\}$

System Discretization

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Q, R

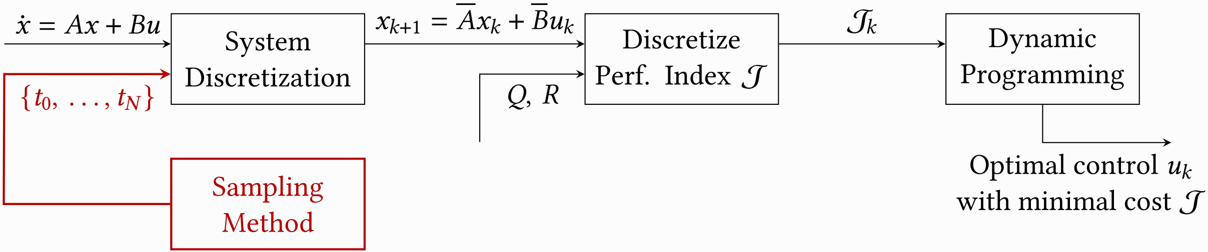
Discretize Perf. Index $\mathcal{J}$

$\mathcal{J}_k$

Dynamic Programming

Optimal control u_k with minimal cost $\mathcal{J}$

Outline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$\{t_0, \ldots, t_N\}$
Sampling Method
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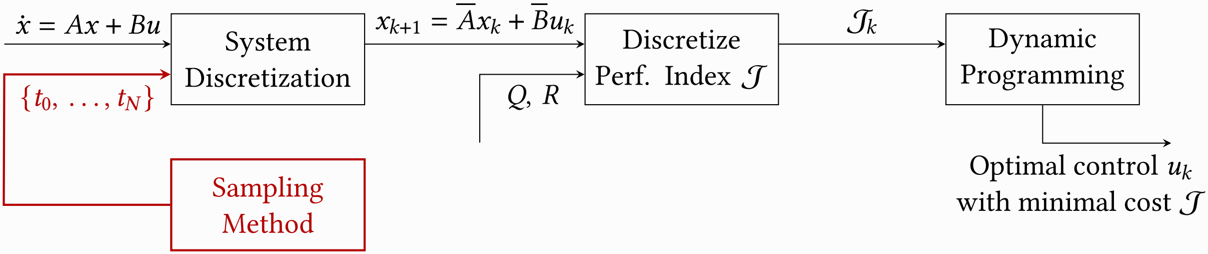
Q, R
Discretize Perf. Index $\mathcal{J}$
$\mathcal{J}_k$
Dynamic Programming
Optimal control u_k with minimal cost $\mathcal{J}$

Outline



- Sampling Density and Sampling Method Cost
- Sampling Methods for finding $\{t_0, \ldots, t_N\}$
  1. Periodic sampling
  2. Lebesgue sampling
  3. Quantization-based sampling

Comparing Sampling Methods: Sampling Density

Sampling Density

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *sampling density* $\sigma_{N,m} : [0, T] \to \mathbb{R}^+$ of any sampling method m is defined as

$$\sigma_{N,m}(t) = \frac{1}{N\,\tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

- Sampling density is normalized

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Asymptotic Sampling Density

To remove the dependency on N, we define the *asymptotic sampling density* as $\sigma_m : [0, T] \longrightarrow \mathbb{R}^+$ as the limit

$$\sigma_m(t) = \lim_{N\to\infty} \sigma_{N,m}(t)$$

Comparing Sampling Methods: Sampling Density

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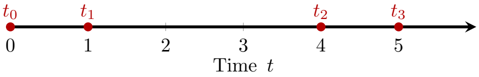
Example ($T = 5$, $N = 4$)



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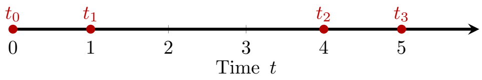
Example ($T = 5$, $N = 4$)



- $\sigma_4\,(t) = \frac{1}{4 \cdot 1} = \frac{1}{4}, \quad \forall t \in [0, 1)$
- $\sigma_4\,(t) = \frac{1}{4 \cdot 3} = \frac{1}{12}, \quad \forall t \in [1, 4)$

QUESTIONS

Question

Is it okay that the derivation for sampling density being normalized is shown afterwards or should it also be shown directly? Right now it also stays for the following slides, should the item for the normalized disappear afterwards?

Comparing Sampling Methods: Normalized Cost

Normalized Cost

Given x_0, A, B, Q, R, and S, and interval length T, and a number of samples N, the *normalized cost* of any sampling method m is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method m with N samples, and $\mathcal{J}_\infty$ is the minimal cost of the continuous-time system.

Comparing Sampling Methods: Normalized Cost

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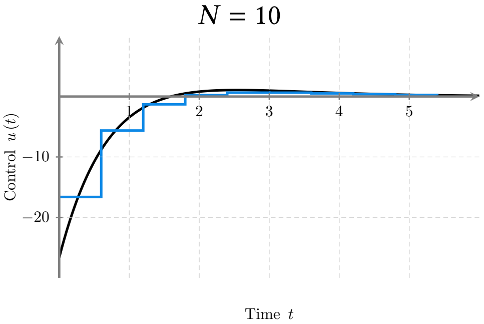
Asymptotic Normalized Cost

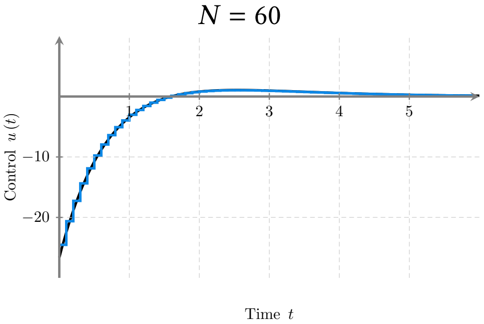
To remove the dependency on N, we define the *asymptotic normalized cost* as the limit

$$c_m = \lim_{N \to \infty} c_{N,m}$$

Comparing Sampling Methods: Normalized Cost

Example (Normalized Cost for Periodic Sampling)

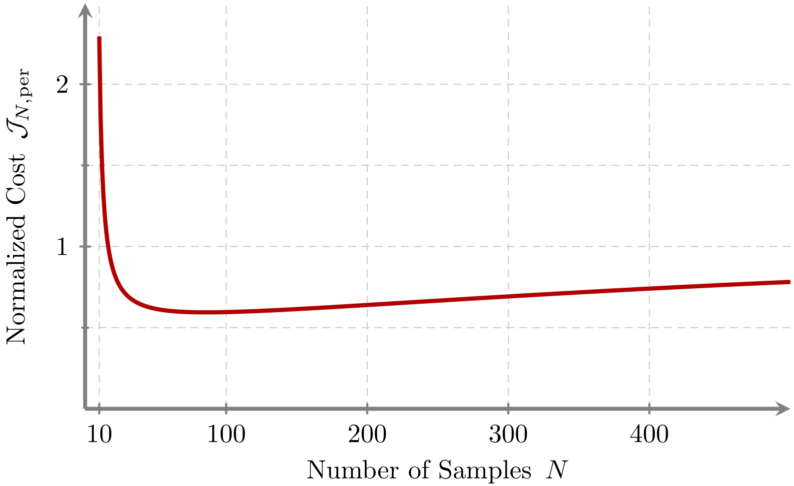




- $\mathcal{J}_\infty = 383.2, \ \mathcal{J}_{10,\,\mathrm{per}} = 699.7$
- $c_{10,\,\mathrm{per}} = \frac{10^2}{6^2} \cdot \frac{699.7-383.2}{383.2} = 2.3$
- $\mathcal{J}_\infty = 383.2, \ \mathcal{J}_{60,\,\mathrm{per}} = 385.5$
- $c_{60,\,\mathrm{per}} = \frac{60^2}{6^2} \cdot \frac{385.5-383.2}{383.2} = 0.6$

Comparing Sampling Methods: Normalized Cost

Example (Normalized Cost for Periodic Sampling)



QUESTIONS

Question

- Is it okay that we started at $N = 10$ and also the way we denoted it in the graph, or is the x axis too chaotic?
- Plot good? do x labels 10 and 60 look weird?

Periodic Sampling

We divide the interval $[0, T]$ into N parts of equal size

$$
\begin{aligned}
\tau_k &= \tau = \frac{T}{N}, && k \in \{0, \ldots, N-1\}, \\
t_k &= k\,\tau = k\,\frac{T}{N}, && k \in \{0, \ldots, N\}
\end{aligned}
$$

Periodic Sampling

We divide the interval $[0, T]$ into N parts of equal size

$$\tau_k = \tau = \frac{T}{N}, \qquad k \in \{0, \dots, N-1\},$$

$$t_k = k\,\tau = k\,\frac{T}{N}, \quad k \in \{0, \dots, N\}$$

For $N \in \mathbb{N}$, we get the constant sampling density

$$\sigma_{per,N}(t) = \frac{1}{N \cdot \tau_k} = \frac{1}{N} \cdot \frac{N}{T} = \frac{1}{T}$$

Periodic Sampling: Optimal Control

For sampling period τ, the solution $\overline{K}(\tau)$ of the discrete-time Riccati equation can be determined analytically as

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2) \quad [1]$$

where K_∞ is the solution of the Riccati equation in continuous-time, and X is the solution of a Lyapunov equation[1]

[1] Melzer, Stuart M., and Benjamin C. Kuo. "Sampling period sensitivity of the optimal sampled data linear regulator." Automatica 7.3 (1971): 367-370.

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. . . informally, optimal controller of the discrete-time can be expressed as the continuous-time solution K_∞ plus a correction term that is proportional to the square of the sampling period τ.

[1] Melzer, Stuart M., and Benjamin C. Kuo. "Sampling period sensitivity of the optimal sampled data linear regulator." Automatica 7.3 (1971): 367-370.

QUESTIONS

Question

Should there be a [1] in the equation, or only after Lyapunov equation?

Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\text{per}} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,\text{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$c_{\text{per}} = \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0' \,\overline{K}(\tau)\, x_0 - x_0' \,K_\infty\, x_0}{x_0' \,K_\infty\, x_0}$$

Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\mathrm{per}} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,\mathrm{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$\begin{aligned} c_{\mathrm{per}} &= \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0' \,\overline{K}(\tau)\, x_0 - x_0' \,K_\infty\, x_0}{x_0' \,K_\infty\, x_0} \\ &= \lim_{N\to\infty} \frac{N^2}{T^2} \frac{x_0' \left(K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)\right) x_0 - x_0' \,K_\infty\, x_0}{x_0' \,K_\infty\, x_0} \end{aligned}$$

Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\text{per}} = \frac{N^2}{T^2}\frac{\mathcal{J}_{N,\text{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$\begin{aligned}
c_{\text{per}} &= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\,\overline{K}(\tau)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\left(K_\infty + X\cdot\frac{\tau^2}{2} + o(\tau^2)\right)x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\left(K_\infty + X\cdot\frac{T^2}{2N^2} + o\left(\frac{T^2}{2N^2}\right)\right)x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0}
\end{aligned}$$

Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\mathrm{per}} = \frac{N^2}{T^2}\frac{\mathcal{J}_{N,\mathrm{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}\,(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$\begin{aligned}
c_{\mathrm{per}} &= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\,\overline{K}(\tau)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\left(K_\infty + X\cdot\frac{\tau^2}{2} + o(\tau^2)\right)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\left(K_\infty + X\cdot\frac{T^2}{2N^2} + o\left(\frac{T^2}{2N^2}\right)\right)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\,\frac{XT^2}{2N^2}\,x_0 + x_0'\,o\left(N^{-2}\right)\,x_0}{x_0'\,K_\infty\,x_0}
\end{aligned}$$

Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\mathrm{per}} = \frac{N^2}{T^2}\frac{\mathcal{J}_{N,\mathrm{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$\begin{aligned}
c_{\mathrm{per}} &= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\,\overline{K}(\tau)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\left(K_\infty + X\cdot\frac{\tau^2}{2} + o(\tau^2)\right)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\left(K_\infty + X\cdot\frac{T^2}{2N^2} + o\left(\frac{T^2}{2N^2}\right)\right)\,x_0 - x_0'\,K_\infty\,x_0}{x_0'\,K_\infty\,x_0} \\
&= \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0'\,\frac{XT^2}{2N^2}\,x_0 + x_0'\,o\left(N^{-2}\right)\,x_0}{x_0'\,K_\infty\,x_0} = \frac{x_0'\,X\,x_0}{2\,x_0'\,K_\infty\,x_0}
\end{aligned}$$

Periodic Sampling: Example

$$c_{\text{per}} = \frac{x_0' \, X \, x_0}{2 \, x_0' \, K_\infty \, x_0}$$

Example

For a first-order system ($n = 1$), wlog $B = R = 1$, we obtain

$$K_\infty = A + \sqrt{A^2 + Q},$$

$$X = \frac{1}{12}(K_\infty - A)K_\infty^2$$

which gives us the asymptotic normalized cost

$$c_{\text{per}} = \frac{1}{24} A\sqrt{A^2 + Q} + A^2 + Q.$$

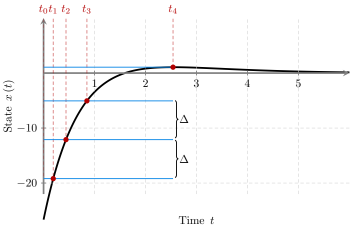
Deterministic Lebesgue Sampling

- **Intuition:** Sample more frequently where the control changes faster
- Sample whenever the optimal u changes by a fixed threshold Δ, so after any sampling instant t_k, the next t_{k+1} is determined s.t.

$$\| u\,(t_{t+1}) - u(t_k) \| = \Delta$$

where u is the optimal continuous-time input

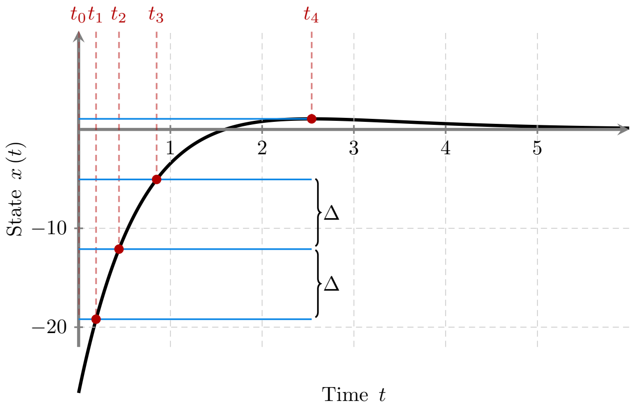
Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

$$\| \, u\,(t_{t+1}) - u(t_k) \, \| = \Delta$$

Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

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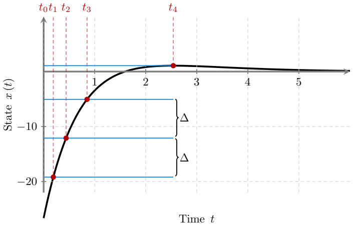
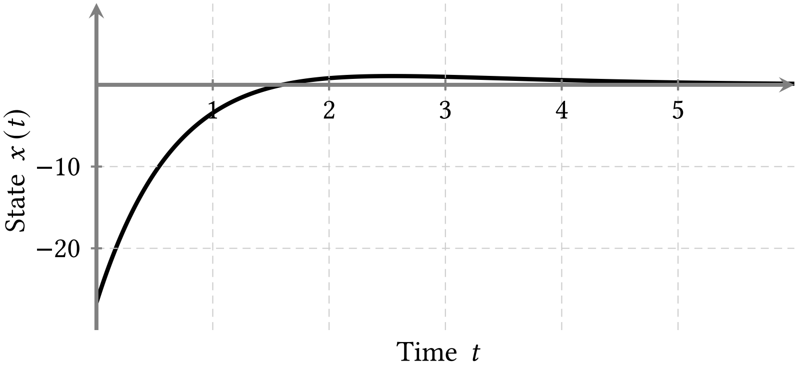


Deterministic Lebesgue Sampling

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

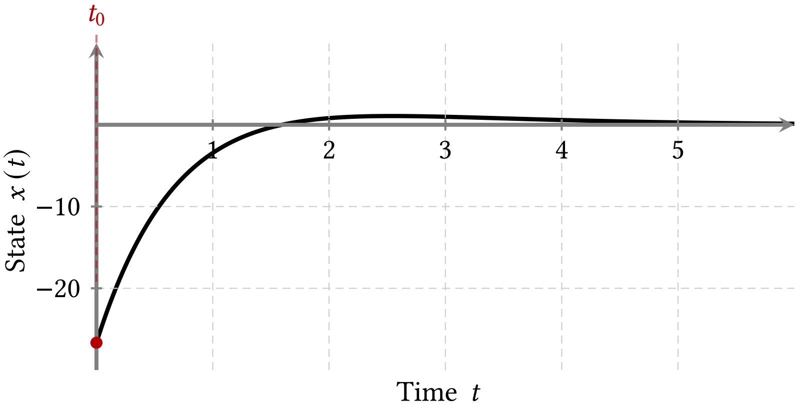
Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

$$\| u\,(t_{t+1}) - u(t_k)\, \| = \Delta$$

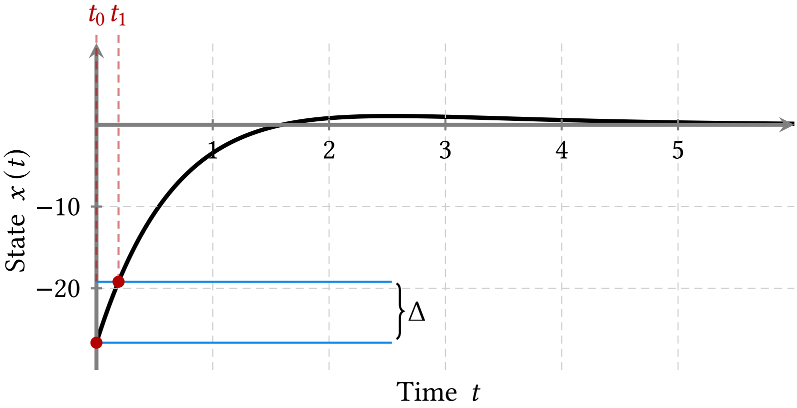
Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

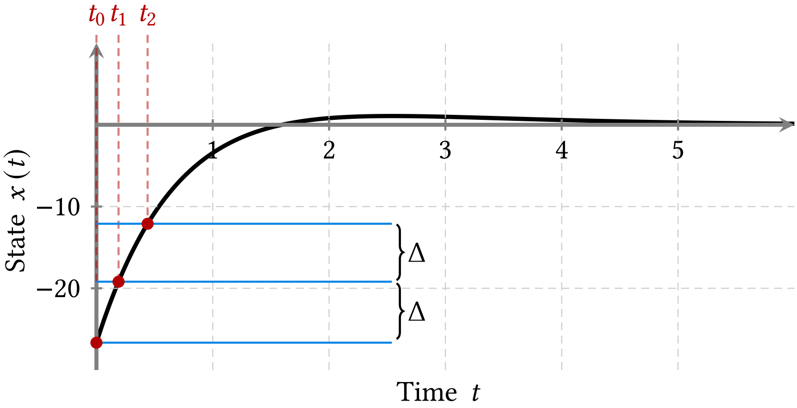
Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

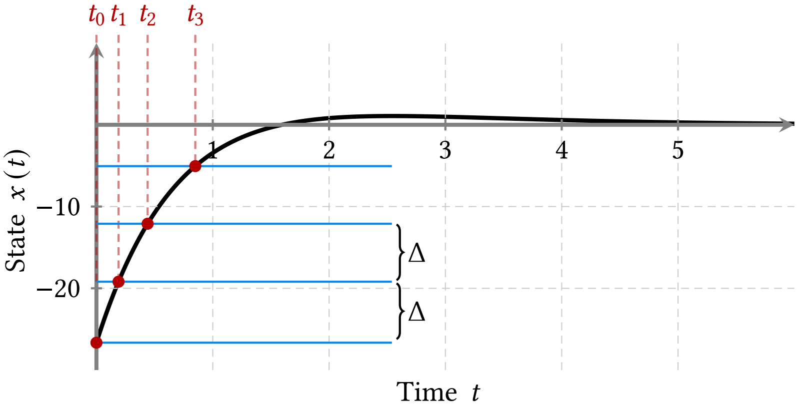
Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

$$\| u\,(t_{t+1}) - u(t_k)\, \| = \Delta$$

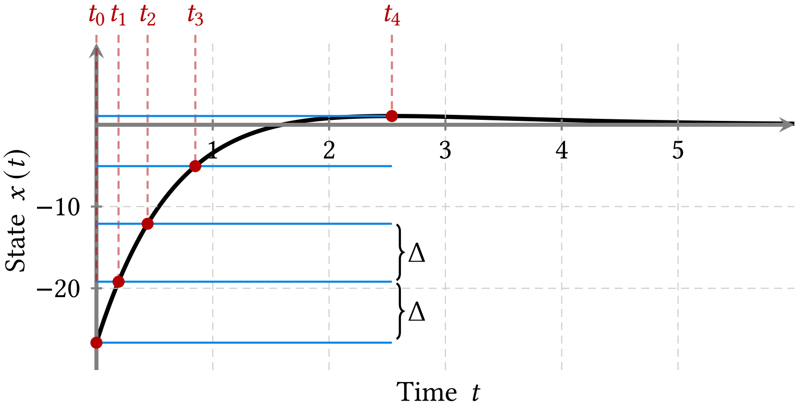
Example ($\Delta = 7$)



Deterministic Lebesgue Sampling

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

Example ($\Delta = 7$)



QUESTIONS

Question

To make the elements appear step by step in slide 22, I did the tikzpicture directly in the frame instead of including the plot as pdf. Slide 21 is still the pdf version. I noted that they actually look different, which one is the prefered one, as so far we always include pdfs. Should we change the other plots too, to have the tikzpicture directly in the frame?

Deterministic Lebesgue Sampling: Deriving the Relation for N

$$\| u(t_{t+1}) - u(t_k) \| = \Delta$$

In the case of a scalar input ($m = 1$) and a given number N of sampling instances in $[0, T]$, the sampling instants t_k satisfy

$$\int_{t_k}^{t_{k+1}} |\dot{u}(t)| \, dt = |u(t_{k+1}) - u(t_k)| = \Delta$$

Deterministic Lebesgue Sampling: Deriving the Relation for N

$$\| \, u\,(t_{t+1}) - u(t_k) \, \| = \Delta$$

In the case of a scalar input ($m = 1$) and a given number N of sampling instances in $[0, T]$, the sampling instants t_k satisfy

$$\begin{aligned} \int_{t_k}^{t_{k+1}} |\, \dot{u}\,(t)\,|\; dt \quad &= \quad |\, u\,(t_{k+1}) - u\,(t_k)\,| \; = \; \Delta \\ &= \quad \frac{1}{N} \cdot \underbrace{\int_0^T |\, \dot{u}\,(s)\,|\; ds}_{N \cdot \Delta} \end{aligned}$$

QUESTIONS

Question

- For the last two reminder boxes with the equation $\| \, u\,(t_{t+1}) - u(t_k) \, \| = \Delta$, should we write it with the absolute value instead, so $| \, u\,(t_{t+1}) - u(t_k) \, | = \Delta$, as both of them are cases for scalar input?
- Wanted one more transition in between for slide 22, where the last equation appears without the underbrace first and then with it. Didnt manage to do so, with only the equations changed their places and uncover didnt work :(

Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\boxed{\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt = \frac{1}{N} \cdot \int_0^T |\,\dot{u}\,(s)\,|\,ds}$$

$$\sigma_{\text{dls}}\,(t) = \lim_{N \to \infty} \frac{1}{N \cdot \tau_k} = \lim_{N \to \infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}(t)\,|\,dt}{\int_0^T |\,\dot{u}(s)\,|\,ds} \cdot \frac{1}{\tau_k}$$

Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\boxed{\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt = \frac{1}{N}\cdot\int_0^T |\,\dot{u}\,(s)\,|\,ds}$$

$$\begin{aligned}
\sigma_{\mathrm{dls}}\,(t) \quad &= \quad \lim_{N\to\infty}\frac{1}{N\cdot\tau_k} = \lim_{N\to\infty}\frac{\int_{t_k}^{t_{k+1}}|\,\dot{u}(t)\,|\,dt}{\int_0^T |\,\dot{u}(s)\,|\,ds}\cdot\frac{1}{\tau_k} \\
&= \quad \lim_{N\to\infty}\frac{\int_{t_k}^{t_{k+1}}|\,\dot{u}\,(t)\,|\,dt}{\tau_k}\cdot\frac{1}{\int_0^T |\,\dot{u}\,(s)\,|\,ds}
\end{aligned}$$

Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\boxed{\int_{t_k}^{t_{k+1}} |\, \dot{u}\,(t)\,|\, dt = \frac{1}{N} \cdot \int_0^T |\, \dot{u}\,(s)\,|\, ds}$$

$$\begin{aligned}
\sigma_{\mathrm{dls}}\,(t) &= \lim_{N\to\infty} \frac{1}{N \cdot \tau_k} = \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\, \dot{u}(t)\,|\, dt}{\int_0^T |\, \dot{u}(s)\,|\, ds} \cdot \frac{1}{\tau_k} \\
&= \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\, \dot{u}\,(t)\,|\, dt}{\tau_k} \cdot \frac{1}{\int_0^T |\, \dot{u}\,(s)\,|\, ds} \\
&= \lim_{N\to\infty} \frac{|\, u\,(t_{k+1}) - u\,(t_k)\,|}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T |\, \dot{u}\,(s)\,|\, ds}
\end{aligned}$$

Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\boxed{\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt = \frac{1}{N} \cdot \int_0^T |\,\dot{u}\,(s)\,|\,ds}$$

$$\begin{aligned}
\sigma_{\text{dls}}\,(t) &= \lim_{N\to\infty} \frac{1}{N \cdot \tau_k} = \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}(t)\,|\,dt}{\int_0^T |\,\dot{u}(s)\,|\,ds} \cdot \frac{1}{\tau_k} \\
&= \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt}{\tau_k} \cdot \frac{1}{\int_0^T |\,\dot{u}\,(s)\,|\,ds} \\
&= \lim_{N\to\infty} \frac{|\,u\,(t_{k+1}) - u\,(t_k)\,|}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T |\,\dot{u}\,(s)\,|\,ds} \\
&= \frac{|\,\dot{u}\,(t)\,|}{\int_0^T |\,\dot{u}\,(s)\,|\,ds}
\end{aligned}$$

QUESTIONS

Question

should we use \displaystile for the integrals in fractions? If we do, integral sign looks better, but the expression explodes a bit.

Optimal Sampling

Discrete-Time Optimal Cost

For a fixed sampling pattern $\{\tau_0, \ldots, \tau_{N-1}\}$, the optimal cost is

$$\mathcal{J} = x_0^\top \bar{K}_0 x_0,$$

where $\bar{K}_k$ satisfies the Riccati recursion

$$\bar{K}_k = r(\tau_k, \bar{K}_{k+1}), \bar{K}_N = S.$$

$\bar{K}_k$ depends only on the current and future sampling intervals $\{\tau_k, \tau_{k+1}, \ldots, \tau_{N-1}\}$. Hence,

$$h < k \implies \frac{\partial \bar{K}_k}{\partial \tau_h} = 0.$$

Optimal Sampling: Necessary Optimality Condition

KKT Optimality Condition

The sampling intervals satisfy

$$\sum_{k=0}^{N-1} \tau_k = T.$$

At the optimal sampling pattern, the gradient of the cost must be proportional to $[1,\ 1,\ \ldots,\ 1]$ which implies that

$$\frac{\partial \mathcal{J}}{\partial \tau_h} = \frac{\partial \mathcal{J}}{\partial \tau_{h+1}}, \qquad h = 0, \ldots, N-2.$$

Riccati-Based Condition

Using the chain rule on the Riccati recursion, this condition can be rewritten as

$$x_0^\top \left[\frac{\partial r}{\partial \tau}(\tau_h,\ \bar{K}_{h+1}) - \frac{\partial r}{\partial \bar{K}}(\tau_h,\ \bar{K}_{h+1})\ \frac{\partial r}{\partial \tau}(\tau_{h+1},\ \bar{K}_{h+2}) \right] = 0$$

Optimal Sampling: First Order Systems

Discrete-Time Riccati Recursion

In discrete time, the Riccati recursion has the form:

$$\bar{K}_k = r(\tau_k, \bar{K}_{k+1})$$

For a first-order system ($n = 1$), the recurrence function r becomes a scalar rational function of τ and $\bar{K}$.

$$r(\tau, \bar{K}) = \frac{\bar{Q}_k\bar{R}_k - \bar{P}_k^2 + (\bar{A}_k^2\bar{R}_k - 2\bar{A}_k\bar{B}_k\bar{P}_k + \bar{B}_k^2\bar{Q}_k)\bar{K}}{\bar{R}_k + \bar{B}_k^2\bar{K}}.$$

With partial derivatives of $r(\tau, \bar{K}) : \frac{\partial r}{\partial \tau}, \frac{\partial r}{\partial \bar{K}}$

Necessary Condition for Optimal Sampling

General Optimality Condition

The general necessary condition for optimality is

$$\frac{\partial \mathcal{J}}{\partial \tau_h} = 0.$$

Using the Riccati recursion and the chain rule, this condition becomes

$$\frac{\partial r}{\partial \tau}\big(\tau_h,\ r(\tau_{h+1}, \bar{K}_{h+2})\big) - \frac{\partial r}{\partial \bar{K}}\big(\tau_h,\ r(\tau_{h+1}, \bar{K}_{h+2})\big) \cdot \frac{\partial r}{\partial \tau}\big(\tau_{h+1},\ \bar{K}_{h+2}\big) = 0$$

Quantization based Sampling

Quantization-based sampling approximates a continuous input $u(t)$ by a piecewise-constant function $\bar{u}(t)$ with N values, aiming to minimize the approximation error. This method provides a near-minimal cost.

The quantization error is defined as

$$E_{\text{qnt}} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} |u(t) - u_k|^2 \, dt.$$

- Unknowns: constants u_k and instants t_k
- Both values and sampling instants are optimized

Quantization: Optimality Conditions

Differentiating E_{qnt} with respect to u_k yields

$$u_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u(t)\, dt.$$

Differentiating E_{qnt} with respect to t_k gives

$$|u_{k-1} - u(t_k)|^2 = |u_k - u(t_k)|^2$$

- This method is appplicable for any linear system with dimension of input space $m = 1$ and any dimension n of the state space.
- Sampling instants are chosen so that the approximation error is balanced across intervals.

Asymptotic Quantization

As $N \longrightarrow \infty$, $\tau_k = 0$:

- Sampling intervals shrink
- Piecewise-constant approximation becomes dense
- Sampling pattern converges to a sampling density

For a scalar input ($m = 1$), known results from optimal quantization imply

$$\sigma(t) \propto |\dot{u}(t)|^{\frac{p}{p+1}}.$$

Choosing $p = 2$ (norm = 2), we obtain the quantization density

$$\sigma_{\text{qnt}}(t) = \frac{|\dot{u}(t)|^{2/3}}{\int_0^T |\dot{u}(s)|^{2/3} ds}$$

NOTES

Question

Explanation for denominator s and should eqn.28 be added?(sampling instants)

Lemma 6

For the first-order systems, the asymptotic quantization density σ_{qnt} coincides with the optimal sampling density σ_{opt}.

Assumptions (for simplification):

- $B = 1$, $R = 1$
- $Q = 0$, $A > 0$
- $S = K_\infty = A + \sqrt{A^2 + Q}$

These assumptions enables us to have an expression for optimal continuous-time input u: $u(t) = -x_0(A + \sqrt{A^2 + Q})e^{-\sqrt{A^2+Q}\,t}$.

Asymptotic Optimal Sampling Density

For small sampling intervals:

- $A_k R_k \neq B_k P_k$
- $\tau_h \longrightarrow \tau_{h+1}$ as both tend to zero

We obtain the expansion

$$\tau_h = \alpha \tau_{h+1} + \beta \tau_{h+1}^2 + o(\tau_{h+1}^2)$$

Equating constant terms yields

$$\alpha = 1, \qquad \beta = -\frac{2}{3} \frac{(Q + A^2)\bar{K}_{h+2}}{Q + A\bar{K}_{h+2}}.$$

Asymptotic Optimal Sampling Density

The asymptotic sampling density of the optimal pattern $\sigma_{\text{opt}}(t)$ satisfies the differential equation

$$\dot{\sigma}_{\text{opt}}(t) = -\frac{2}{3}\frac{(Q + A^2)\,K(t)}{Q + AK(t)}\,\sigma_{\text{opt}}(t),$$

where $K(t)$ is the solution of the continuous-time Riccati equation.
When $S = K_\infty = A + \sqrt{A^2 + Q}$

$$\dot{\sigma}_{\text{opt}}(t) = -\frac{2}{3}\sqrt{Q + A^2}\,\sigma_{\text{opt}}(t),$$

results in

$$\sigma_{\text{opt}}(t) = c\,e^{-\frac{2}{3}\sqrt{Q+A^2}\,t}, \qquad \int_0^T \sigma_{\text{opt}}(t)\,dt = 1.$$

Key Result

Using the expression of the optimal continuous-time input,

$$\sigma_{\text{opt}}(t) \;\propto\; |\dot{u}(t)|^{2/3}.$$

NOTES

Question

Layout issue in the Key result block

Lemma 7: Asymptotic Normalized Cost

Lemma 7

For a sampling method m_α with asymptotic density

$$\sigma_{m_\alpha}(t) = \frac{\alpha(S-A)}{1-e^{-\alpha(S-A)T}} e^{-\alpha(S-A)t} \propto |\dot{u}(t)|^\alpha,$$

the asymptotic normalized cost is

$$c_{m_\alpha} = \frac{S}{12(S-A)T^2} \frac{1-e^{-2(1-\alpha)(S-A)T}}{2(1-\alpha)} \frac{1-e^{-\alpha(S-A)T}}{\alpha}$$

NOTES

Question

Are backup slides enough for lemma 6 and lemma 7?

Results

- Optimal sampling for linear control systems depends on the system dynamics and cost.
- For first-order systems, the optimal sampling pattern can be characterized by an **asymptotic sampling density**.
- The optimal density admits the closed-form structure

$$\sigma_{\mathrm{opt}}(t) \propto |\dot{u}(t)|^{2/3}.$$

- This density coincides with that obtained from **quantization-based sampling** of the optimal continuous-time input.
- Optimal and quantization-based sampling achieve a **lower asymptotic cost** than periodic sampling.

NOTES

Question

Is result content fine or should I add more explanation?

COMMENTS

Comment

From now on, these slides are our thoughts on important quantization slides. It must be checked with Sree's slides!

Quantization of Continuous-Time Control

Quantization Problem

Given a continuous-time optimal control signal $u : x \longrightarrow \mathbb{R}^m$, we approximate it by a *piecewise-constant* control $\overline{u}$ with N updates, by minimizing an L^p approximation error:

$$\begin{cases} \min\limits_{\{\tau_k,\, u_k\}} \displaystyle\int_\Omega \underbrace{\|u(x) - \overline{u}(x)\|_p}_{\substack{\text{Approximation}\\ \text{error}}} \, dx \\ \text{s.t.} \quad \overline{u}(x) = u_k, \quad x \in [\tau_k, \tau_{k+1}), \quad k = 0, \ldots, N-1 \end{cases}$$

Idea: replace the continuous optimal solution by a piecewise-constant one that is *closest in the* $L^p(\Omega)$ *sense.*

QUESTIONS

Question

- is $\min_{\{\tau_k, u_k\}}$ valid? or should be $\{\tau_1, \ldots, \tau_k\}$ instead?
- Ω undefined is fine? Should the integration limits be $\int_0^T$ or $\int_{0^m}^{T^m}$ instead?
- in the paper, they integrate over Ω and w.r.t. dx. Should it be wrt t and between $(0, T)$ instead?

Quantization Problem: Example

Example (Input space dimension $m = 1$)

$$\min_u \int_\Omega \left|u(x) - \overline{u}(x)\right|^2 dx$$

$$\Downarrow$$

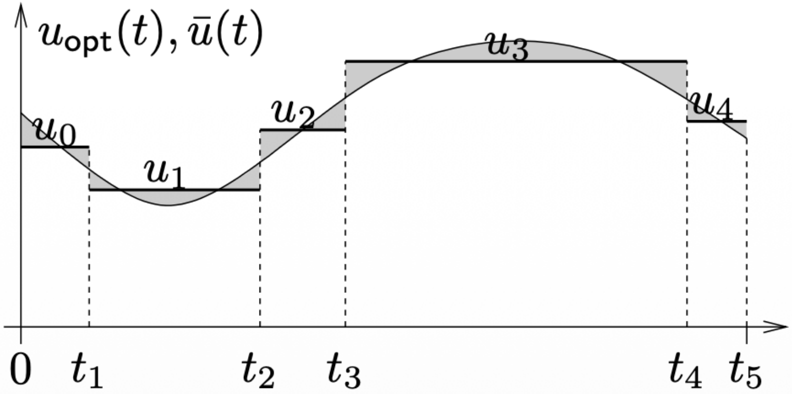
$$\sum_{k=0}^{N-1} \int_{\tau_k}^{\tau_{k+1}} \left|u(x) - u_k\right|^2 dx \longleftarrow \text{partition } x$$

$$\Downarrow \longleftarrow \textit{Optimality condition}$$

$$\boxed{\left|u_{k-1} - u(t_k)\right|^2 = \left|u_k - u(t_k)\right|^2}$$

Quantization: Intuition

$$|u_{k-1} - u(t_k)|^2 = |u_k - u(t_k)|^2$$



Quantization-Based Sampling: Algorithm ($m = 1$, $p = 2$)

Iterative Procedure

1. **Initialization** Set

$$t_k^{(0)} = \frac{k}{N} T, \quad k = 0, \ldots, N.$$

2. **Centroid update** (fix $\{t_k^{(r)}\}$) For each interval $k = 0, \ldots, N-1$, compute

$$\bar{u}_k^{(r+1)} = \frac{1}{t_{k+1}^{(r)} - t_k^{(r)}} \int_{t_k^{(r)}}^{t_{k+1}^{(r)}} u(t)\, dt.$$

3. **Boundary update** (fix $\{u_k^{(r+1)}\}$) For each $k = 1, \ldots, N-1$, update $t_k^{(r+1)}$ as the solution of

$$u(t_k) = \frac{\bar{u}_{k-1}^{(r+1)} + \bar{u}_k^{(r+1)}}{2}, \qquad t_{k-1}^{(r+1)} < t_k < t_{k+1}^{(r)}.$$

4. **Repeat** steps 2–3 until convergence.

Quantization-Based Sampling: Algorithm ($m = 1$, $p = 2$)

Iterative Procedure

1. **Initialization** Set

$$t_k^{(0)} = \frac{k}{N} T, \quad k = 0, \dots, N.$$

2. **Centroid update** (fix $\{t_k^{(r)}\}$) For each interval $k = 0, \dots, N-1$, compute

$$\overline{u}_k^{(r+1)} = \frac{1}{t_{k+1}^{(r)} - t_k^{(r)}} \int_{t_k^{(r)}}^{t_{k+1}^{(r)}} u(t)\, dt.$$

3. **Boundary update** (fix $\{u_k^{(r+1)}\}$) For each $k = 1, \dots, N-1$, update $t_k^{(r+1)}$ as the solution of

$$u(t_k) = \frac{\overline{u}_{k-1}^{(r+1)} + \overline{u}_k^{(r+1)}}{2}, \qquad t_{k-1}^{(r+1)} < t_k < t_{k+1}^{(r)}.$$

4. **Repeat** steps 2–3 until convergence.

... centroids minimize error inside intervals, boundaries balance errors between neighboring intervals.

QUESTIONS

Question

- does the algorithm actually make sense? it was not in the paper explicitly, I think.
- Is the algorithm too small on the second slide?
- We cannot use a screenshot and need a proper plot