# The Optimal Sampling Pattern for Linear Control Systems

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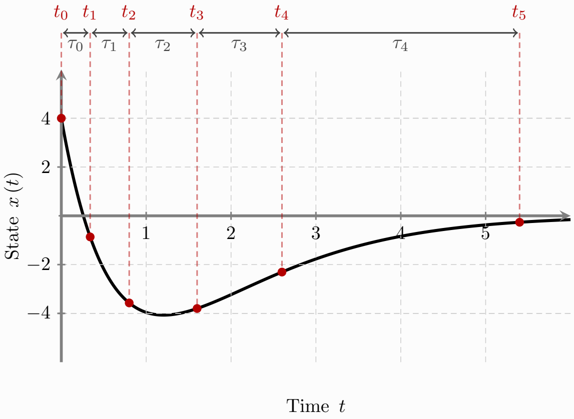
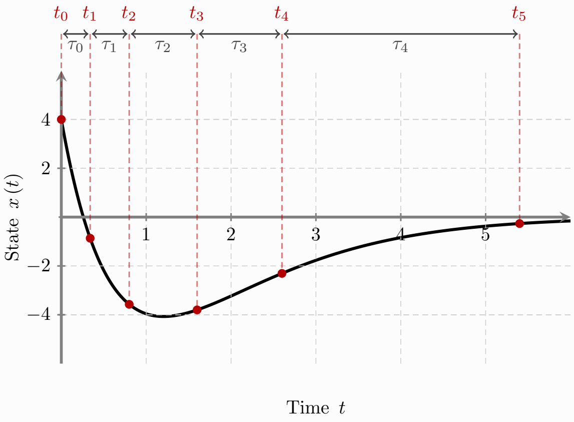
# Sampling



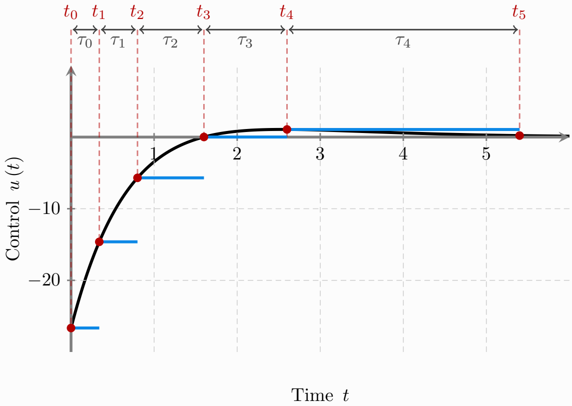

## Sampling

- $\{t_0, t_1, \ldots, t_N\}$: Sampling pattern
- $t_k$: Sampling instants
- $\tau_k = t_{k+1} - t_k$: Interarrival times

# Sampling



t0 t1 t2 t3 t4 t5
τ0 τ1 τ2 τ3 τ4
4
2
−2
−4
1 2 3 4 5
State x(t)
Time t

t0 t1 t2 t3 t4 t5
τ0 τ1 τ2 τ3 τ4
−10
−20
1 2 3 4 5
Control u(t)
Time t

## The Problem

Given

$$\begin{cases} \dot{x}(t) & = & A\,x(t) + B\,u(t) \\ x(0) & = & x_0 \end{cases}$$

- We restrict $u(t)$ to be in the class of piecewise constant functions and denote it as $\overline{u}(t)$
- Finite horizon $[0, T]$
- Number $N$ of control updates

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that minimizes the performance index

- $\mathcal{J}(\overline{u}) = \int_0^T \big(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t)\big)\,dt + x'(T)\,S\,x(T)$

# Quadratic Performance Index

$$\mathcal{J}(\overline{u}) = \int_0^T \left(x'(t) \underbrace{Q}_{\succeq} x(t) + \overline{u}'(t) \underbrace{R}_{\succ} \overline{u}'(t)\right) dt + x'(T) \underbrace{S}_{\succeq} x(T)$$

$$\text{s.t.} \begin{cases} \dot{x}(t) & = & A\,x(t) + B\,u(t) \\ x(0) & = & x_0 \end{cases}$$

- $x(t) \in \mathbb{R}^n$: system state
- $u(t) \in \mathbb{R}^m$: system input
- $A$, $B$, $Q$, $R$, $S$: constant matrices

# Outline

- Computation of optimal control $u(t)$ for continuous-time systems
- Discretization Process for given $\{t_0, \ldots, t_N\}$
    1. System Discretization
    2. Computation of optimal control $(u_k)_{k \in \{0, \ldots, N\}}$ for the discrete-time system

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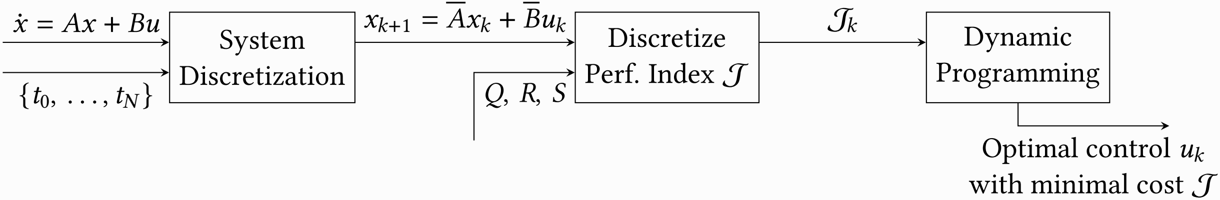

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- Sampling Density and Sampling Method Cost
- Sampling Methods for finding $\{t_0, \ldots, t_N\}$
    1. Periodic sampling
    2. Lebesgue sampling
    3. Quantization-based sampling
- Results

# Optimal Control: Continuous-Time Systems

$$\text{minimize}_u \int_0^T \Big(x'(t)\, Q\, x(t) + u'(t)\, R\, u(t)\Big)\, dt + x(T)'\, S\, x(T)$$

Riccati equation for continuous-time systems

$$\begin{cases} \dot{K}(t) &= K(t)\, B R^{-1} B' K(t) - A' K(t) - K(t) A - Q, \\ K(T) &= S \end{cases}$$

which gives us the optimal control $u(t) = -R^{-1} B' K(t)\, x(t)$

with achieved cost

$$\mathcal{J}_\infty = x_0' K(0)\, x_0$$

# Optimal Control: Discrete-Time Systems

Given the time points $\{t_0, \ldots, t_N\}$, with interarrivals $\tau_k$, $k \in \{0, \ldots, N-1\}$, the discrete-time state space equations are:

$$\begin{cases} x_{k+1} & = & \overline{A}_k \, x_k + \overline{B}_k \, u_k \\ x\,(0) & = & x_0 \end{cases}$$

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| | | |
|---|---|---|
| $\overline{A}_k = \Phi(\tau_k)$ | $\Phi(\tau) = e^{A\tau}$ | State Matrix |
| $\overline{B}_k = \Gamma(\tau_k)$ | $\Gamma(\tau) = \int_0^{\tau} e^{A(\tau - t)}\, dt\, B$ | Input Matrix |

# Performance Index for Discrete-Time Systems

$$\int_0^T \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t)\right) dt + x'(T)\, S\, x(T)$$

$$= \sum_{k=0}^{N-1} \left[ \int_{t_k}^{t_{k+1}} \left(x'(t)\, Q\, x(t) + \overline{u}'(t)\, R\, \overline{u}(t)\right) dt + x'(T)\, S\, x(T)\right]$$

# Performance Index for Discrete-Time Systems

$$
\begin{aligned}
&\int_0^T \left(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t)\right)dt + x'(T)\,S\,x(T) \\
&= \sum_{k=0}^{N-1}\left[\int_{t_k}^{t_{k+1}} \left(x'(t)\,Q\,x(t) + \overline{u}'(t)\,R\,\overline{u}(t)\right)dt + x'(T)\,S\,x(T)\right] \\
&= \sum_{k=0}^{N-1}\left[\int_{t_k}^{t_{k+1}} (\Phi\,x_k + \Gamma\,u_k)'\,Q\,(\Phi\,x_k + \Gamma\,u_k)\,dt + \int_{t_k}^{t_{k+1}} u_k'\,R\,u_k\right] + x'(T)\,S\,x(T)
\end{aligned}
$$

# Performance Index for Discrete-Time Systems

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$$= \sum_{k=0}^{N-1}\left[\int_{t_k}^{t_{k+1}} (\Phi\,x_k + \Gamma\,u_k)'\,Q\,(\Phi\,x_k + \Gamma\,u_k)\,dt + \int_{t_k}^{t_{k+1}} u_k'\,R\,u_k\right] + x'(T)\,S\,x(T)$$

$$= \sum_{k=0}^{N-1}\left[x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi'\,Q\,\Phi\,dt\right)}_{\overline{Q}} x_k + u_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Gamma'\,Q\Gamma\,dt + \underbrace{\int_{t_k}^{t_{k+1}} R dt}_{\tau_k R}\right)}_{\overline{R}} u_k + 2x_k' \underbrace{\left(\int_{t_k}^{t_{k+1}} \Phi'\,Q\,\Gamma\,dt\right)}_{\overline{P}} u_k\right] + x'(T)\,S\,x(T)$$

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$$\mathcal{J}(\overline{u}) = \sum_{k=0}^{N-1}\left(x_k'\,\overline{Q}\,x_k + u_k'\,\overline{R}\,u_k + 2x_k'\,\overline{P}\,u_k\right) + x_N'\,S\,x_N$$

## Bellman Equation

For $k \in \{0, \ldots, N\}$, we define the function $\mathcal{J}_k : \mathbb{R}^n \to \mathbb{R}$, which gives the minimal cost achievable from stage $k$ onward, given the state $x_k$ as

$$\begin{cases} \mathcal{J}_k(x_k) = \min_u \Big[ \underbrace{x_k' \overline{Q}_k x_k}_{\text{State penalty}} + \underbrace{u' \overline{R}_k u}_{\text{Control penalty}} + \underbrace{2x_k' \overline{P}_k u}_{\text{Cross term}} + \mathcal{J}_{k+1}(x_{k+1}) \Big] & \text{for } k \in \{0, \ldots, N-1\} \\ \mathcal{J}_N(x) = x' S x. \end{cases}$$

**Intuition:** define a *backward recursive function* that gives the minimal achievable cost from the current state onward.

# Dynamic Programming: Quadratic Form

$\mathcal{J}_k(x)$ is a quadratic form of $x$

*Unknown at this point!*

$$\mathcal{J}_k(x) = x' \overline{K}_k \, x$$

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$$\Downarrow$$

Recall $x_{k+1} = \overline{A}_k x_k + \overline{B}_k u_k$

$$\mathcal{J}_{k+1}(\overline{A}_k x_k + \overline{B}_k u_k) = (\overline{A}_k x_k + \overline{B}_k u_k)' \, \overline{K}_{k+1} \, (\overline{A}_k x_k + \overline{B}_k u_k)$$

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$$\Downarrow$$

Plug into **Bellman equation**

$$\mathcal{J}_k(x) = x_k' \underbrace{(\overline{Q}_k + \overline{A}_k' \, \overline{K}_{k+1} \overline{A}_k)}_{\hat{Q}_k} x_k + u_k' \underbrace{(\overline{R}_k + \overline{B}_k' \overline{K}_{k+1} \overline{B}_k)}_{\hat{R}_k} u_k + 2 x_k' \underbrace{(\overline{P}_k + \overline{A}_k' \, \overline{K}_{k+1} \, \overline{B}_k)}_{\hat{B}_k} u_k$$

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$$\Downarrow$$

Plug into **Bellman equation**

$$\boxed{\mathcal{J}_k(x) = x_k' \, \hat{Q}_k \, x_k + u_k' \, \hat{R}_k \, u_k + 2x_k' \, \hat{B}_k \, u_k}$$

# Dynamic Programming: Minimization

We aim to **minimize** the cost:

$$\min_u \left( x' \, \hat{Q}_k \, x + u' \, \hat{R}_k \, u + 2x' \, \hat{B}_k \, u \right)$$

**Optimality** condition:

$$\frac{\partial}{\partial u}(\,\cdot\,) = 0 \qquad \Longrightarrow \qquad \hat{R}_k \, u + \hat{B}_k' x = 0$$

$$\Downarrow \qquad \text{Solving for } u$$

$$u_k^* = -\,\hat{R}_k^{-1} \, \hat{B}_k' \, x_k$$

# Dynamic Programming: Riccati Solution

We plug $u_k^*$ into the **Bellman equation:**

$$\mathcal{J}_k(x) = x' \left( \hat{Q}_k - \hat{B}_k \, \hat{R}_k^{-1} \, \hat{B}_k' \right) x$$

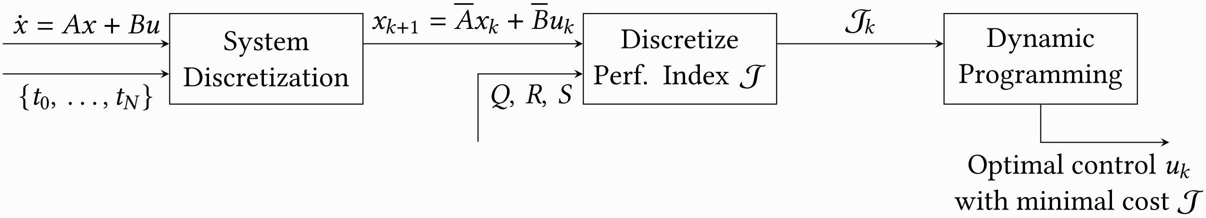
$$\Downarrow \qquad \text{Recall } \; \mathcal{J}_k(x) = x' \overline{K}_k x$$

$$\overline{K}_k = \hat{Q}_k - \hat{B}_k \, \hat{R}_k^{-1} \, \hat{B}_k'$$

Minimal cost equals

$$\mathcal{J} = x_0' \, \overline{K}_0 \, x_0$$

# Outline



$\dot{x} = Ax + Bu$
$\{t_0, \ldots, t_N\}$
System Discretization
$x_{k+1} = \overline{A}x_k + \overline{B}u_k$
$Q, R, S$
Discretize Perf. Index $\mathcal{J}$
$\mathcal{J}_k$
Dynamic Programming
Optimal control $u_k$ with minimal cost $\mathcal{J}$

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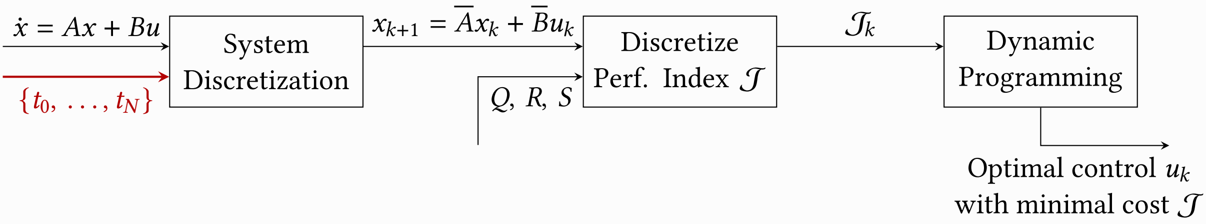
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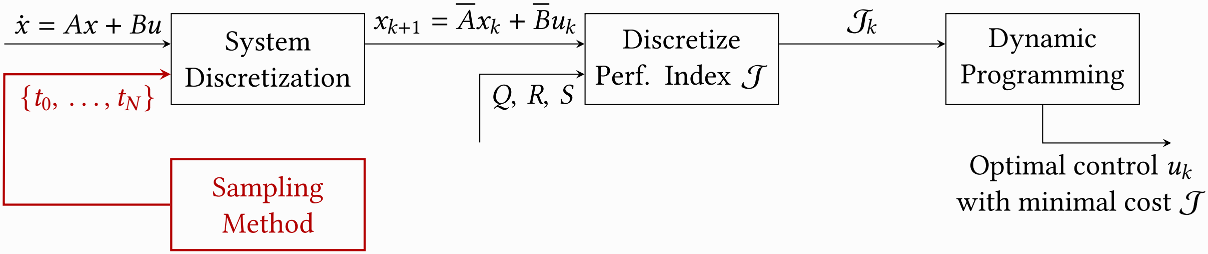

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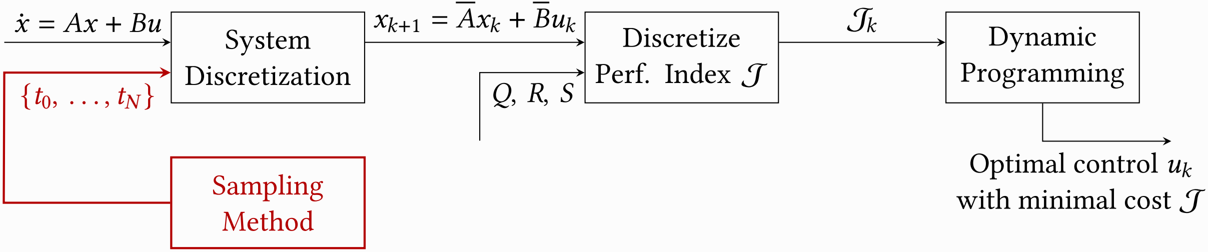
Dynamic Programming
$\{t_0, \dots, t_N\}$
$Q, R, S$
Sampling Method
Optimal control $u_k$ with minimal cost $\mathcal{J}$

# OUTLINE

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  1. Periodic sampling
  2. Lebesgue sampling
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# Comparing Sampling Methods: Sampling Density

## Sampling Density

Given a sampling sequence $t_0 = 0,\ t_1,\ \ldots,\ t_N = T$, we define the *sampling density* $\sigma_{N,m} : [0, T] \longrightarrow \mathbb{R}^+$ of any sampling method $m$ as

$$\sigma_{N,m}(t) = \frac{1}{N\,\tau_k} \quad \forall t \in [t_k, t_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

- Sampling density is normalized

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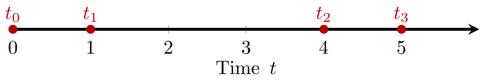
## Example ($T = 5$, $N = 4$)



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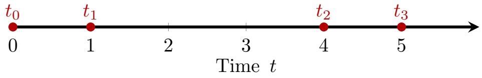


- $\sigma_4\,(t) = \frac{1}{4 \cdot 1} = \frac{1}{4}, \quad \forall t \in [0, 1)$
- $\sigma_4\,(t) = \frac{1}{4 \cdot 3} = \frac{1}{12}, \quad \forall t \in [1, 4)$

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## Asymptotic Sampling Density

To remove the dependency on $N$, we define the *asymptotic sampling density* as $\sigma_m : [0, T] \longrightarrow \mathbb{R}^+$ as the limit

$$\sigma_m(t) = \lim_{N\to\infty} \sigma_{N,m}(t)$$

# Comparing Sampling Methods: Normalized Cost

## Normalized Cost

Given an interval length $T$, and a number of samples $N$, the *normalized cost* $c_{N,m}$ of any sampling method $m$ is defined as

$$c_{N,m} = \frac{N^2}{T^2} \frac{\mathcal{J}_{N,m} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

where $\mathcal{J}_{N,m}$ is the minimal cost of the sampling method $m$ with $N$ samples, and $\mathcal{J}_\infty$ is the minimal cost of the continuous-time system.

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## Asymptotic Normalized Cost

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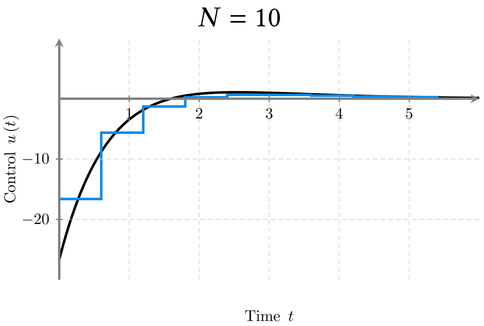
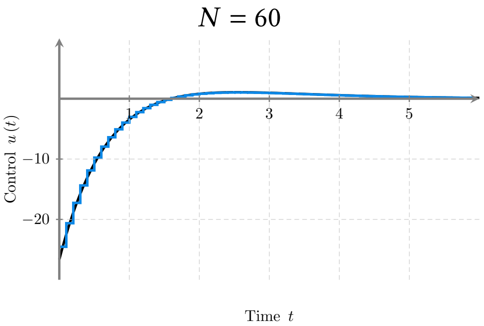
$$c_m = \lim_{N \to \infty} c_{N,m}$$

# Comparing Sampling Methods: Normalized Cost

## Example (Normalized Cost for Periodic Sampling)




- $\mathcal{J}_\infty = 383.2, \quad \mathcal{J}_{10,\,\mathrm{per}} = 699.7$
- $c_{10,\,\mathrm{per}} = \frac{10^2}{6^2} \cdot \frac{699.7 - 383.2}{383.2} = 2.3$

- $\mathcal{J}_\infty = 383.2, \quad \mathcal{J}_{60,\,\mathrm{per}} = 385.5$
- $c_{60,\,\mathrm{per}} = \frac{60^2}{6^2} \cdot \frac{385.5 - 383.2}{383.2} = 0.6$

# Comparing Sampling Methods: Normalized Cost

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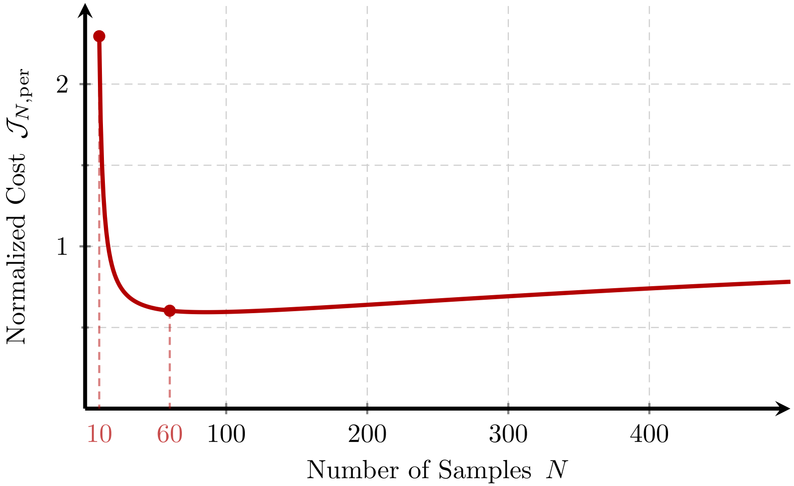

# Periodic Sampling

We divide the interval $[0, T]$ into $N$ parts of equal size

$$\tau_k = \tau = \frac{T}{N}, \qquad k \in \{0, \ldots, N-1\},$$
$$t_k = k\,\tau = k\,\frac{T}{N}, \quad k \in \{0, \ldots, N\}$$

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$$t_k = k\,\tau = k\,\frac{T}{N}, \quad k \in \{0, \ldots, N\}$$

For $N \in \mathbb{N}$, we get the constant sampling density

$$\sigma_{N,\,\mathrm{per}}(t) = \frac{1}{N \cdot \tau_k} = \frac{1}{N} \cdot \frac{N}{T} = \frac{1}{T}$$

# Periodic Sampling: Optimal Control

For sampling period $\tau$, the solution $\overline{K}(\tau)$ of the discrete-time Riccati equation can be determined analytically as

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

where $K_\infty$ is the solution of the Riccati equation in continuous-time, and $X$ is the solution of a Lyapunov equation[1]

[1] Melzer, Stuart M., and Benjamin C. Kuo. "Sampling period sensitivity of the optimal sampled data linear regulator." Automatica 7.3 (1971): 367-370.

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*... informally, the optimal controller of the discrete-time can be expressed as the continuous-time solution $K_\infty$ plus a correction term that is proportional to the square of the sampling period $\tau$.*

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# Periodic Sampling: Asymptotic Normalized Cost

$$c_{N,\mathrm{per}} = \frac{N^2}{T^2}\frac{\mathcal{J}_{N,\mathrm{per}} - \mathcal{J}_\infty}{\mathcal{J}_\infty}$$

$$\overline{K}(\tau) = K_\infty + X \cdot \frac{\tau^2}{2} + o(\tau^2)$$

$$c_{\mathrm{per}} = \lim_{N\to\infty} \frac{N^2}{T^2}\frac{x_0' \, \overline{K}(\tau)\, x_0 - x_0' \, K_\infty \, x_0}{x_0' \, K_\infty \, x_0}$$

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&= \lim_{N\to\infty} \frac{N^2}{T^2}\,\frac{x_0'\,X\,\frac{T^2}{2N^2}\,x_0 + x_0'\,o\left(N^{-2}\right)\,x_0}{x_0'\,K_\infty\,x_0}
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&= \lim_{N\to\infty} \frac{N^2}{T^2}\,\frac{x_0'\,X\frac{T^2}{2N^2}\,x_0 + x_0'\,o\left(N^{-2}\right)\,x_0}{x_0'\,K_\infty\,x_0} = \frac{x_0'\,X\,x_0}{2\,x_0'\,K_\infty\,x_0}
\end{aligned}$$

# Periodic Sampling: Example

$$c_{\text{per}} = \frac{x_0' \, X \, x_0}{2 \, x_0' \, K_\infty \, x_0}$$

## Example

For a first-order system ($n = 1$), wlog $B = R = 1$, we obtain

$$\begin{aligned} K_\infty &= A + \sqrt{A^2 + Q}, \\ X &= \frac{1}{12}(K_\infty - A)K_\infty^2 \end{aligned}$$

which gives us the asymptotic normalized cost

$$c_{\text{per}} = \frac{1}{24} A \sqrt{A^2 + Q} + A^2 + Q.$$

# Deterministic Lebesgue Sampling

- **Intuition:** Sample more frequently where the control changes faster
- Sample whenever the optimal $u$ changes by a fixed threshold $\Delta$, so after any sampling instant $t_k$, the next $t_{k+1}$ is determined s.t.

$$\| u(t_{k+1}) - u(t_k) \| = \Delta$$

where $u$ is the optimal continuous-time input

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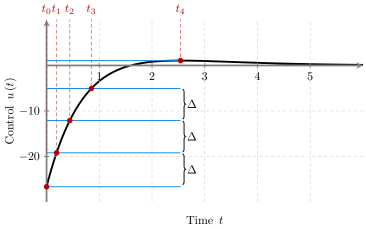
## Example ($\Delta = 7$)



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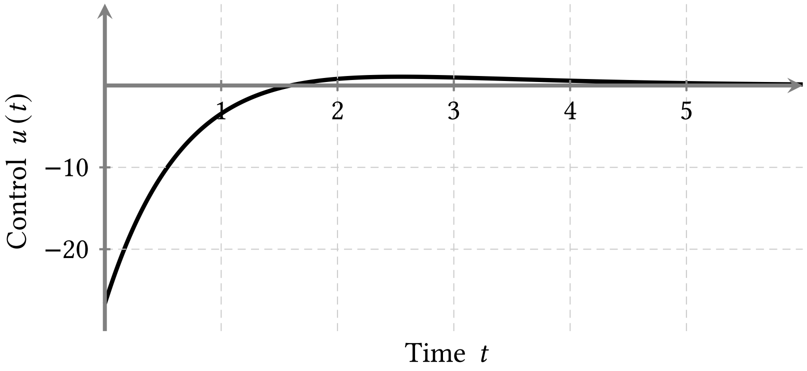

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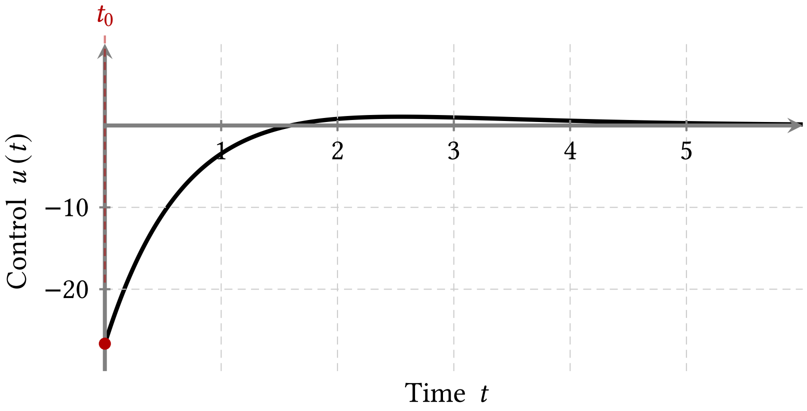

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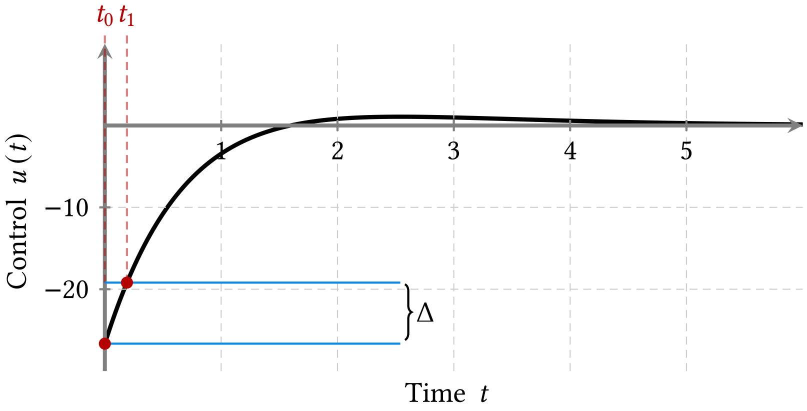

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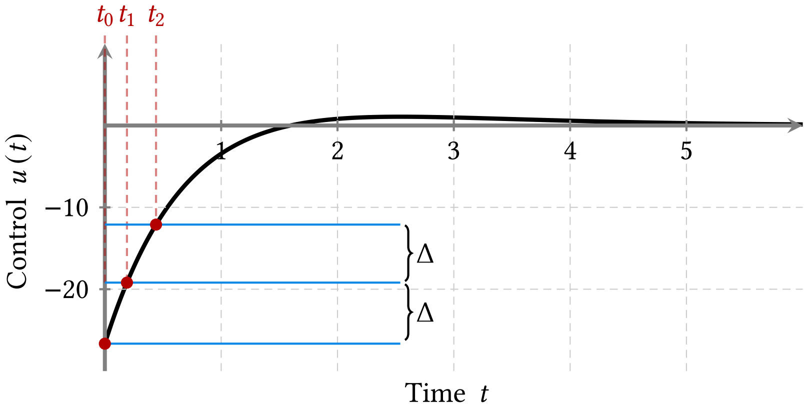

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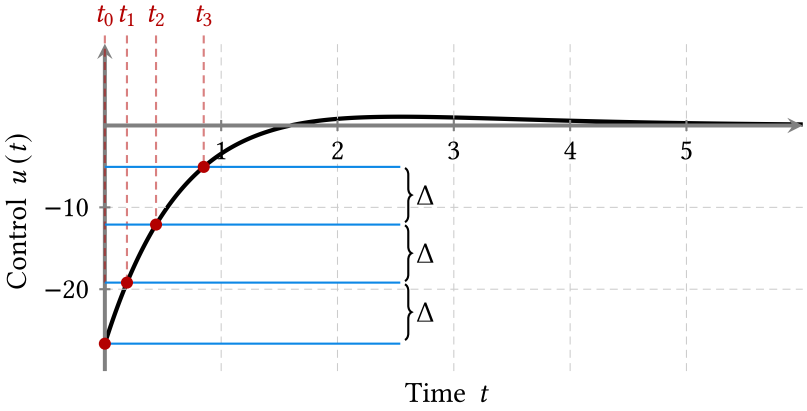

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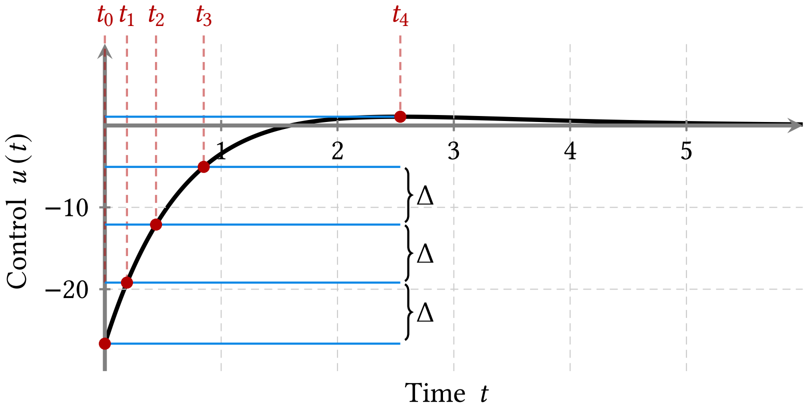

# Deterministic Lebesgue Sampling: Deriving the Relation for $N$

$$\| u(t_{k+1}) - u(t_k) \| = \Delta$$

In the case of a scalar input ($m = 1$) and a given number $N$ of sampling instances in $[0, T]$, the sampling instants $t_k$ satisfy

$$\int_{t_k}^{t_{k+1}} |\dot{u}(t)| \, dt = |u(t_{k+1}) - u(t_k)| = \Delta$$

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$$\begin{aligned}\int_{t_k}^{t_{k+1}} |\dot{u}(t)| \, dt &= |u(t_{k+1}) - u(t_k)| = \Delta \\ &= \frac{1}{N} \cdot \int_0^T |\dot{u}(s)| \, ds\end{aligned}$$

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In the case of a scalar input ($m = 1$) and a given number $N$ of sampling instances in $[0, T]$, the sampling instants $t_k$ satisfy

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} |\dot{u}(t)| \, dt &= |u(t_{k+1}) - u(t_k)| = \Delta \\
&= \frac{1}{N} \cdot \underbrace{\int_0^T |\dot{u}(s)| \, ds}_{N \cdot \Delta}
\end{aligned}$$

# Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt = \frac{1}{N} \cdot \int_0^T |\,\dot{u}\,(s)\,|\,ds$$

$$\sigma_{\mathrm{dls}}\,(t) \quad = \quad \lim_{N\to\infty} \frac{1}{N \cdot \tau_k} \;=\; \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}(t)\,|\,dt}{\int_0^T |\,\dot{u}(s)\,|\,ds} \cdot \frac{1}{\tau_k}$$

# Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt = \frac{1}{N} \cdot \int_0^T |\,\dot{u}\,(s)\,|\,ds$$

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&= \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt}{\tau_k} \cdot \frac{1}{\int_0^T |\,\dot{u}\,(s)\,|\,ds}
\end{aligned}$$

# Deterministic Lebesgue Sampling: Asymptotic Sampling Density

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&= \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}\,(t)\,|\,dt}{\tau_k} \cdot \frac{1}{\int_0^T |\,\dot{u}\,(s)\,|\,ds} \\
&= \lim_{N\to\infty} \frac{|\,u\,(t_{k+1}) - u\,(t_k)\,|}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T |\,\dot{u}\,(s)\,|\,ds}
\end{aligned}$$

# Deterministic Lebesgue Sampling: Asymptotic Sampling Density

$$\int_{t_k}^{t_{k+1}} |\,\dot{u}(t)\,|\,dt = \frac{1}{N} \cdot \int_0^T |\,\dot{u}(s)\,|\,ds$$

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\sigma_{\text{dls}}(t) &= \lim_{N\to\infty} \frac{1}{N \cdot \tau_k} = \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}(t)\,|\,dt}{\int_0^T |\,\dot{u}(s)\,|\,ds} \cdot \frac{1}{\tau_k} \\
&= \lim_{N\to\infty} \frac{\int_{t_k}^{t_{k+1}} |\,\dot{u}(t)\,|\,dt}{\tau_k} \cdot \frac{1}{\int_0^T |\,\dot{u}(s)\,|\,ds} \\
&= \lim_{N\to\infty} \frac{|\,u(t_{k+1}) - u(t_k)\,|}{t_{k+1} - t_k} \cdot \frac{1}{\int_0^T |\,\dot{u}(s)\,|\,ds} \\
&= \frac{|\,\dot{u}(t)\,|}{\int_0^T |\,\dot{u}(s)\,|\,ds}
\end{aligned}$$

## Discrete-Time Optimal Cost

For a fixed sampling pattern $\{\tau_0, \ldots, \tau_{N-1}\}$, the optimal cost is

$$\mathcal{J} = x_0^\top \overline{K}_0 x_0,$$

where $\overline{K}_k$ satisfies the Riccati recursion

$$\overline{K}_k = r(\tau_k, \overline{K}_{k+1}), \overline{K}_N = S.$$

$\overline{K}_k$ depends only on the current and future sampling intervals $\{\tau_k, \tau_{k+1}, \ldots, \tau_{N-1}\}$. Hence,

$$h < k \implies \frac{\partial \overline{K}_k}{\partial \tau_h} = 0.$$

# Optimal Sampling: KKT Optimality Condition

The sampling intervals satisfy

$$\sum_{k=0}^{N-1} \tau_k = T.$$

At the optimal sampling pattern, the gradient of the cost must be proportional to $[1, 1, \ldots, 1]$ which implies that

$$\frac{\partial \mathcal{J}}{\partial \tau_h} = \frac{\partial \mathcal{J}}{\partial \tau_{h+1}}, \qquad h = 0, \ldots, N-2.$$

Since $\mathcal{J} = x_0^\top \bar{K}_0 x_0$, the above condition is equivalent to

$$x_0^\top \frac{\partial \bar{K}_0}{\partial \tau_h} = x_0^\top \frac{\partial \bar{K}_0}{\partial \tau_{h+1}},$$

which, after expanding the Riccati recursion, leads to the optimality condition

## Riccati-Based Condition

Using the chain rule on the Riccati recursion, this condition can be rewritten as

$$x_0' \prod_{i=0}^{h-1} \left[ \frac{\partial r}{\partial \overline{K}}(\tau_i, \overline{K}_{i+1}) \right] \left[ \frac{\partial r}{\partial \tau}(\tau_h, \overline{K}_{h+1}) - \frac{\partial r}{\partial \overline{K}}(\tau_h, \overline{K}_{h+1}) \frac{\partial r}{\partial \tau}(\tau_{h+1}, \overline{K}_{h+2}) \right] = 0$$

## Discrete-Time Riccati Recursion

In discrete time, the Riccati recursion has the form:

$$\overline{K}_k = r(\tau_k, \overline{K}_{k+1})$$

For a first-order system ($n = 1$), the recurrence function $r$ becomes a scalar rational function of $\tau$ and $\overline{K}$.

$$r(\tau, \overline{K}) = \frac{\overline{Q}_k \overline{R}_k - \overline{P}_k^2 + (\overline{A}_k^2 \overline{R}_k - 2\overline{A}_k \overline{B}_k \overline{P}_k + \overline{B}_k^2 \overline{Q}_k)\overline{K}}{\overline{R}_k + \overline{B}_k^2 \overline{K}}.$$

With partial derivatives of $r(\tau, \overline{K}) : \frac{\partial r}{\partial \tau}, \frac{\partial r}{\partial \overline{K}}$

# Necessary Condition for Optimal Sampling

## General Optimality Condition

The general necessary condition for optimality is

$$\frac{\partial \mathcal{J}}{\partial \tau_h} = 0.$$

Using the Riccati recursion and the chain rule, this condition becomes

$$\frac{\partial r}{\partial \tau}\big(\tau_h,\ r(\tau_{h+1}, \overline{K}_{h+2})\big) - \frac{\partial r}{\partial \overline{K}}\big(\tau_h,\ r(\tau_{h+1}, \overline{K}_{h+2})\big) \cdot \frac{\partial r}{\partial \tau}\big(\tau_{h+1},\ \overline{K}_{h+2}\big) = 0$$

## Quantization based Sampling

Quantization-based sampling approximates a continuous input $u(t)$ by a piecewise-constant function $\overline{u}(t)$ with $N$ values, aiming to minimize the approximation error. This method provides a near-minimal cost.

The quantization error is defined as

$$E_{\text{qnt}} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} |u(t) - u_k|^2 \, dt.$$

- Unknowns: constants $u_k$ and instants $t_k$
- Both values and sampling instants are optimized

# Quantization: Optimality Conditions (m=1)

Differentiating $E_{\text{qnt}}$ with respect to $u_k$ yields

$$u_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u(t)\, dt.$$

Differentiating $E_{\text{qnt}}$ with respect to $t_k$ gives

$$|u_{k-1} - u(t_k)|^2 = |u_k - u(t_k)|^2$$

- This method is appplicable for any linear system with dimension of input space $m = 1$ and any dimension $n$ of the state space.
- Sampling instants are chosen so that the approximation error is balanced across intervals.

# Quantization-Based Sampling: Optimality Condition

$$u_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u(t)\, dt$$

$$\begin{aligned}
E_{\text{qnt}} &= \int_0^T |u(t)|^2\, dt - \sum_{k=0}^{N-1} (t_{k+1} - t_k)\, |u_k|^2 \\
&= -|u_k|^2 + 2u'(t_k)u_k + |u_{k-1}|^2 - 2u'(t_k)u_{k-1} \\
&= |u_{k-1} - u(t_k)|^2 - |u_k - u(t_k)|^2 \\
&\implies |u_{k-1} - u(t_k)|^2 - |u_k - u(t_k)|^2
\end{aligned}$$

# Asymptotic Quantization

As $N \longrightarrow \infty$, $\tau_k = 0$:

- Sampling intervals shrink
- Piecewise-constant approximation becomes dense

For a scalar input ($m = 1$), known results from optimal quantization imply

$$\sigma(t) \propto |\dot{u}(t)|^{\frac{p}{p+1}}.$$

Choosing $p = 2$, we obtain the quantization density

$$\sigma_{\text{qnt}}(t) = \frac{|\dot{u}(t)|^{2/3}}{\int_0^T |\dot{u}(s)|^{2/3} ds}$$

# Asymptotic Quantization

$$\frac{f(y)^{\frac{m}{m+p}}}{\int f(y)^{\frac{m}{m+p}}\,dy}$$

$$m = 1 \qquad \Longrightarrow \qquad f(y) = \frac{1}{|\dot u|}\,u^{-1}(y)$$

$$\frac{|\dot u|^{-\frac{1}{1+p}}\,u^{-1}(y)}{\int |\dot u|^{-\frac{1}{1+p}}\,u^{-1}(y)\,dy}$$

$$\sigma_{\text{qnt}}(t) = \frac{|\dot u(t)|^{\frac{p}{1+p}}}{\int_0^T |\dot u(t)|^{\frac{p}{1+p}}\,dt}$$

$$p = 2 \qquad \Longrightarrow \qquad \sigma_{\text{qnt}}(t) = \frac{|\dot u(t)|^{\frac{2}{3}}}{\int_0^T |\dot u(s)|^{\frac{2}{3}}\,ds}$$

$$\int_{t_k}^{t_{k+1}} |\dot u(t)|^{\frac{2}{3}}\,dt = \frac{1}{N}\int_0^T |\dot u(t)|^{\frac{2}{3}}\,dt$$

# Lemma 6

For the first-order systems, the asymptotic quantization density $\sigma_{\text{qnt}}$ coincides with the optimal sampling density $\sigma_{\text{opt}}$.

Assumptions (for simplification):

- $B = 1, R = 1$
- $Q = 0, A > 0$
- $S = K_\infty = A + \sqrt{A^2 + Q}$

These assumptions enables us to have an expression for optimal continuous-time input u: $u(t) = -x_0(A + \sqrt{A^2 + Q})e^{-\sqrt{A^2+Q}\,t}$.

# Asymptotic Optimal Sampling Density

For small sampling intervals:

- $A_k R_k \neq B_k P_k$
- $\tau_h \longrightarrow \tau_{h+1}$ as both tend to zero

We obtain the expansion

$$\tau_h = \alpha \tau_{h+1} + \beta \tau_{h+1}^2 + o(\tau_{h+1}^2)$$

Equating constant terms yields

$$\alpha = 1, \qquad \beta = -\frac{2}{3} \frac{(Q + A^2)\overline{K}_{h+2}}{Q + A\overline{K}_{h+2}}.$$

# Asymptotic Optimal Sampling Density

The asymptotic sampling density of the optimal pattern $\sigma_{\text{opt}}(t)$ satisfies the differential equation

$$\dot{\sigma}_{\text{opt}}(t) = -\frac{2}{3}\frac{(Q + A^2)\,K(t)}{Q + AK(t)}\,\sigma_{\text{opt}}(t),$$

where $K(t)$ is the solution of the continuous-time Riccati equation.

When $S = K_\infty = A + \sqrt{A^2 + Q}$

$$\dot{\sigma}_{\text{opt}}(t) = -\frac{2}{3}\sqrt{Q + A^2}\,\sigma_{\text{opt}}(t)$$

$$\frac{\dot{\sigma}_{\text{opt}}(t)}{\sigma_{\text{opt}}(t)} = -\frac{2}{3}\sqrt{Q + A^2}$$

$$\int \frac{\dot{\sigma}_{\text{opt}}(t)}{\sigma_{\text{opt}}(t)}\,dt = -\frac{2}{3}\sqrt{Q + A^2}\,t$$

$$\ln \sigma_{\text{opt}}(t) = -\frac{2}{3}\sqrt{Q + A^2}\,t + \ln c$$

$$\sigma_{\text{opt}}(t) = c\,e^{-\frac{2}{3}\sqrt{Q+A^2}\,t}$$

# Lemma 7: Asymptotic Normalized Cost

## Lemma 7

For a sampling method $m_\alpha$ with asymptotic density

$$\sigma_{m_\alpha}(t) = \frac{\alpha(K_\infty - A)}{1 - e^{-\alpha(S-A)T}} e^{-\alpha(K_\infty - A)t} \propto |\dot{u}(t)|^\alpha,$$

the asymptotic normalized cost is

$$c_{m_\alpha} = \frac{S}{12(K_\infty - A)T^2} \frac{1 - e^{-2(1-\alpha)(K_\infty - A)T}}{2(1-\alpha)} \frac{1 - e^{-\alpha(K_\infty - A)T}}{\alpha}$$

# Results

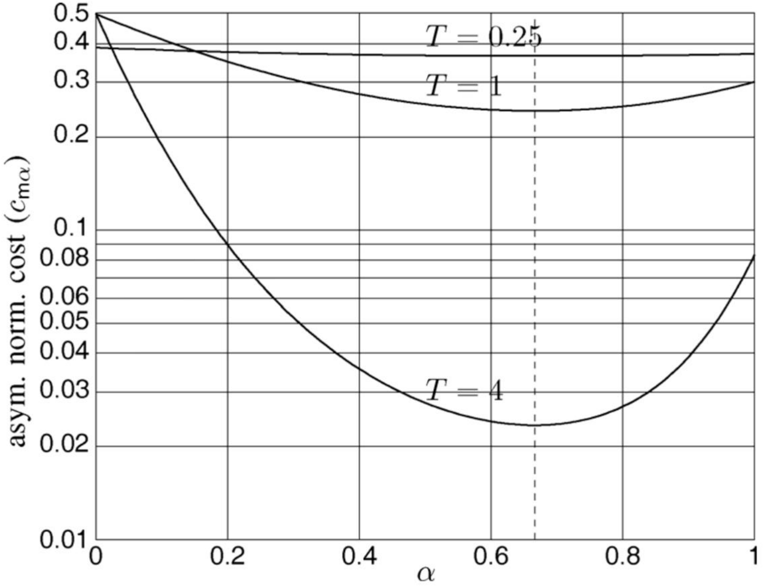
- Optimal sampling for linear control systems depends on the system dynamics and cost.
- For first-order systems, the optimal sampling pattern can be characterized by an **asymptotic sampling density**.
- The optimal density admits the closed-form structure

$$\sigma_{\text{opt}}(t) \propto |\dot{u}(t)|^{2/3}$$

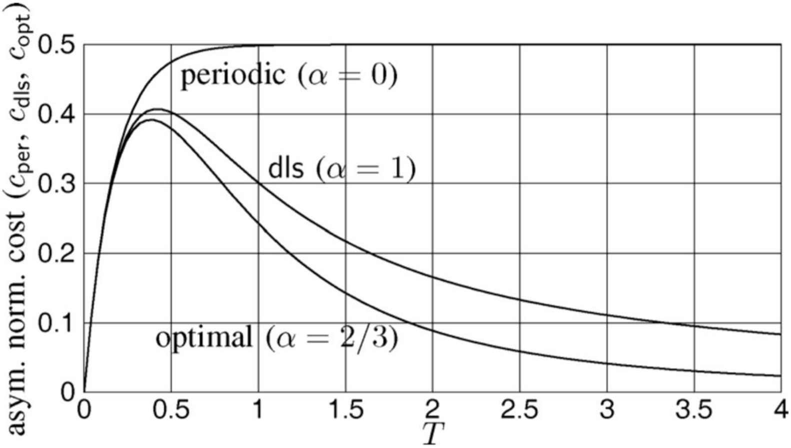
- This density coincides with that obtained from **quantization-based sampling** of the optimal continuous-time input.
- Optimal and quantization-based sampling achieve a **lower asymptotic cost** than periodic sampling.

# Backup Slides

# Quantization: Normalized Costs

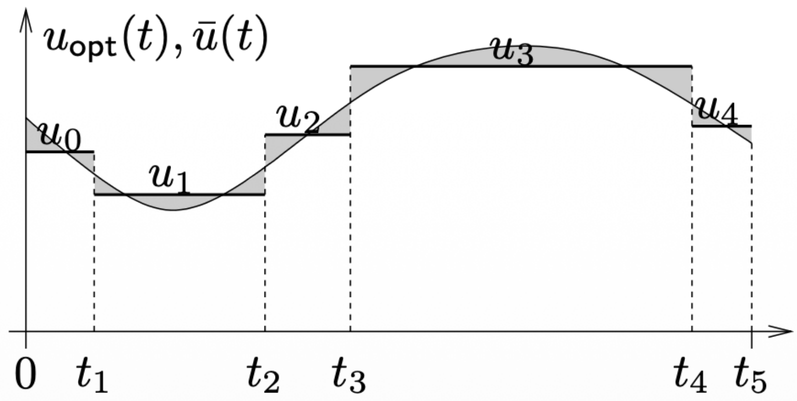


# Quantization: Comparison of Sampling Methods



# Quantization: Intuition

$$|u_{k-1} - u(t_k)|^2 = |u_k - u(t_k)|^2$$



# Quantization of Continuous-Time Control

## Quantization Problem

Given a continuous-time optimal control signal $u : x \to \mathbb{R}^m$, we approximate it by a *piecewise-constant* control $\overline{u}$ with $N$ updates, by minimizing an $L^p$ approximation error:

$$\begin{cases} \min\limits_{\{\tau_k, u_k\}} \displaystyle\int_\Omega \underbrace{\|u(x) - \overline{u}(x)\|_p}_{\text{Approximation error}} \, dx \\ \text{s.t.} \quad \overline{u}(x) = u_k, \quad x \in [\tau_k, \tau_{k+1}), \quad k = 0, \dots, N-1 \end{cases}$$

**Idea:** replace the continuous optimal solution by a piecewise-constant one that is *closest in the $L^p(\Omega)$ sense.*

# Quantization Problem: Example

**Example (Input space dimension $m = 1$)**

$$\min_u \int_\Omega \left| u(x) - \overline{u}(x) \right|^2 dx$$

$$\Downarrow$$

$$\sum_{k=0}^{N-1} \int_{\tau_k}^{\tau_{k+1}} \left| u(x) - u_k \right|^2 dx \longleftarrow \text{partition } x$$

$$\Downarrow \longleftarrow \textit{Optimality condition}$$

$$\boxed{\left| u_{k-1} - u(t_k) \right|^2 = \left| u_k - u(t_k) \right|^2}$$

# Quantization-Based Sampling: Algorithm ($m = 1$, $p = 2$)

## Iterative Procedure

1. **Initialization** Set
$$t_k^{(0)} = \frac{k}{N} T, \quad k = 0, \dots, N.$$

2. **Centroid update** (fix $\{t_k^{(r)}\}$) For each interval $k = 0, \dots, N-1$, compute
$$\overline{u}_k^{(r+1)} = \frac{1}{t_{k+1}^{(r)} - t_k^{(r)}} \int_{t_k^{(r)}}^{t_{k+1}^{(r)}} u(t)\, dt.$$

3. **Boundary update** (fix $\{u_k^{(r+1)}\}$) For each $k = 1, \dots, N-1$, update $t_k^{(r+1)}$ as the solution of
$$u(t_k) = \frac{\overline{u}_{k-1}^{(r+1)} + \overline{u}_k^{(r+1)}}{2}, \qquad t_{k-1}^{(r+1)} < t_k < t_{k+1}^{(r)}.$$

4. **Repeat** steps 2–3 until convergence.