The Planck Length

For thousands of years, science has pushed limits regarding what we know. As humans, we're born with senses which allow us to explore the world in which we live. Though this is nice, they also place a limit on what we're able to directly experience. In my talk, I'll be answering the question 'is there a fundamental limit to the size an object or a region of space can be? If so, how do we calculate it.'

In an attempt to answer this question, let's consider a pair of electrons separated by distance r. We'll ask the question 'what would happen if you squeeze a pair of electrons into a smaller and smaller volume of space?' i.e. r decreases. So what are the forces acting on the electrons? There's the gravitational force of attraction, which can be conveniently calculated using Newtons universal law of gravitation, $F_G = \frac{Gm_1m_2}{r^2}$ which, in our case can be simplified to $\frac{Gm^2}{r^2}$ since the mass of both electrons Is the same and constant. Interestingly, this follows an inverse square relationship, which will be interesting later. The other force acting on the electrons is the electrostatic force of repulsion which will attempt to push the two electrons apart. This can be quantified using Coulomb's law: $F_E = \frac{Q_1Q_2}{4\pi\varepsilon_0 r^2}$. Again, this can be simplified since the charge of the two electrons is the same. For reference, this also follows an inverse square relationship, and epsilon naught is the permittivity of free space – it has a numerical value so isn't particularly relevant, but if you're wondering, it's a measure of how polarised charged particles become in the presence of an electric field. At this stage, it would be useful to know which of these two is larger.

To do this, we can take both of them and consider them as a ratio. Dividing the electrostatic by the gravitational, we find that the electrostatic attraction is 4 times ten to the 42 times larger than the gravitational force. That's surprising because it was the gravitational force which was noticed first [because atoms are neutral since negative electrons and positive protons cancel each other out, so we don't notice the really strong electrostatic forces inside of the atom from the outside since they are being perfectly neutralised inside]. Also, because both followed inverse square relationships, the ratio of forces is independent of the separation of the two electrons.

Unfortunately, however, thing's aren't as simple as that. We need to consider quantum mechanics. One thing we learn from quantum mechanics is that at very small distances, classical laws of physics start to misbehave. Specifically, Heisenberg's uncertainty principle says that as the electrons are pushed into a smaller and smaller region of space, there's an inherent quantum 'jitter' in the movement of the two electrons. This jittery motion of the electrons in turn increases the energy of the electrons. At some point, this energy of the jittery electrons is comparable to the mass energy required to create a new electron! Pretty weirdly, Einstein showed us that energy can be converted into matter and vice versa. This energy can be easily calculated using the nice equation $E = mc^2$. By combining Einstein's energy mass relationship and the uncertainty principle, we're able to calculate the distance at which this happens. As we know from the uncertainty principle, the uncertainty in the momentum of the electrons must be greater than or equal to $\frac{\hbar}{2\Delta x}$. For our simple purposes, we can approximate Δx as r. Multiplying both sides by the speed of light, we work out $\Delta E \geq \frac{\hbar c}{2r}$. By equating our two expressions for ΔE, we get the following expression for r. putting in numbers, we get a number in the order of ten to the minus thirteen meters. Contextually, this is smaller than an atom but larger than a nucleus.

Naturally, the next question to ask is what happens as we push them closer? As they are pushed closer, the quantum mechanical uncertainty means the mass of the system increases, therefore the force due to gravity must also increase! You can ask the question at what point does the force of electrostatic repulsion equal coolthe force due to gravity? Well it's around 10 to the minus thirty four meters. [to do this, use Fe/Fg and work out m. use equation $r \leq \frac{\hbar}{mc}$ to calculate r.]

So again, what happens if we push them closer? Well, first we ought to look at black holes. Consider a small mass, little m, located at the edge of a larger mass big M. let them be separated by a distance of r. naturally there is a gravitational force of attraction between the two masses as shown on the slides. In order to pull the small mass away from the large mass, you must do work against the force of gravity. Imagine you move the small mass a distance of dr, then the work required to do this, dW can be writen as Fdr. What if you want to work out the work done required to move the small mass away from the large mass to a distance of infinity? Well to do this, you can integrate the expression between the limits r and infinity. You need to use calculus to do this because the magnitude of the force changes with distance. So, integrating and substituting in the limits, you get the expression $W = \frac{GMm}{r}$. We can consider work done to be equal to the gain in potential energy. Now consider that you throw the mass small m away from the large mass with a speed v. you can ask the question 'what velocity will I need to throw small m such that it escapes the gravitational attraction of big m?' Since the energy required is equal to the gain in potential energy, we can also equate this to the kinetic energy of the little mass and rearrange for v. therefore we find that $v = \sqrt{\frac{2GM}{r}}$. This is referred to as the escape velocity of an object. For example, the escape velocity of earth is around eleven kilometres per second.

The obvious question to ask now is 'what if the escape velocity is greater than the speed of light?'. In that case, not even light could escape the object [hence it would appear black..]. pretty much, an object which is so dense that not even light can escape it is called a black hole. nevertheless, to see the condition for this to happen, we can let v equal c. rearranging the expression for r, we find that r equals $\frac{2GM}{c^2}$. This is known as the Schwarzschild radius. It basically tells us how small we'd need to squeeze an object of mass m to form a black hole! Now you can see why we're squeezing together our two electrons.

So going back to our two electrons, if we keep on squeezing them into a smaller and smaller volume of space, and by extension increase the amount of mass contained within that area, then at some point we will cross a threshold where a black hole will form. To calculate at what distance this will happen, we can calculate at what distance will happen – all we need to do is use all of our results so far. We've seen already that using quantum mechanics and special relativity, $r \sim \frac{\hbar}{mc}$. We also just saw how the Schwarzschild radius is related to the gravitational mass, i.e. $r \sim \frac{Gm}{c^2}$. By rearranging for m and equating, we see that this distance occurs when $r = \sqrt{\frac{G\hbar}{c^3}}$. Substituting in the numbers, we get a value of 1.616×10^{-35} meters. In other words, if we squeeze our electrons into a volume of space approximately equal to ten to the minus thirty five meters, then a black hole will form. We're in a situation where we're trying to find out what's happening in this region of space at higher and higher energies, but by doing so, the uncertainty principle is pushing so much mass into the region we're looking at such that it collapses to form a black hole, and at this point, no information can be got out. Pretty much, this means that there's no way to look at nature on distances smaller than this – its known as the Planck length, and is pretty much the fundamental limit to space. You can also calculate the amount of mass which is squeezed into this very small volume – it ends up being around $2.21 \times 10^{-8} kg$. This is known as the Planck mass. Interestingly, all of this can be worked out using unit consistency – if you work out the units for h bar, c and G, you can do some maths and work out all of Plank's things – i.e. Planck time, mass, length etc. I can go through it later, I guess, but not at the moment... probably ran out of time.

Nevertheless, thanks very much for listening and I hope you found it interesting!