Quantum Field Theory: What is a Particle?

Physics.

Disclaimer.

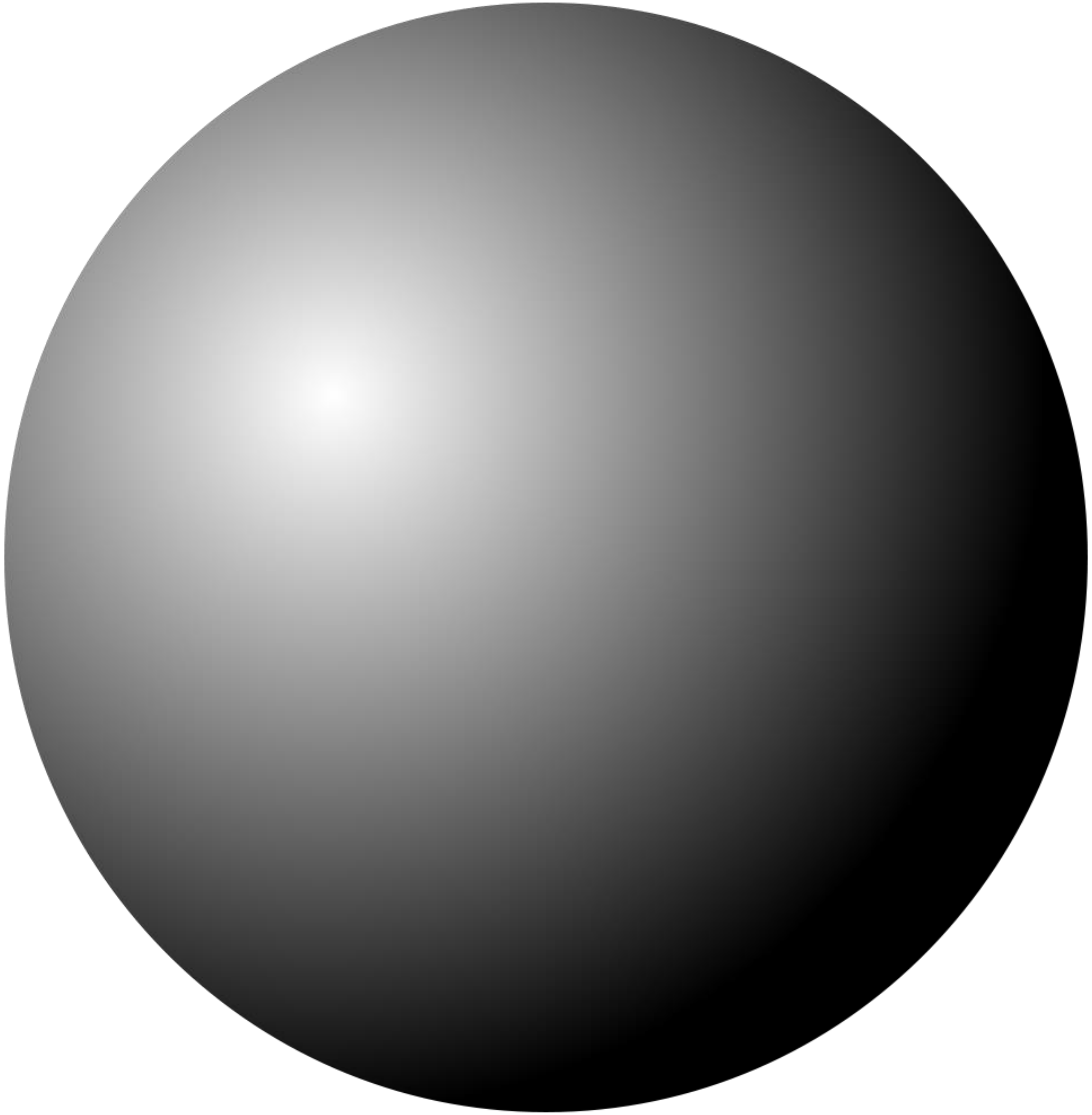
- Calling this talk ‘Quantum Field Theory’ is maybe a bit extreme. **No *real* QFT will really be discussed**—it’s well beyond the scope, but we will take inspiration from it, and right at the end take a Giant leap to hop to the QFT answer.
- That said, the talk contains a lot of *very* useful techniques, and stuff which I’ll hopefully learn again in 1st year of university!
- Enjoy!

The plan of action.

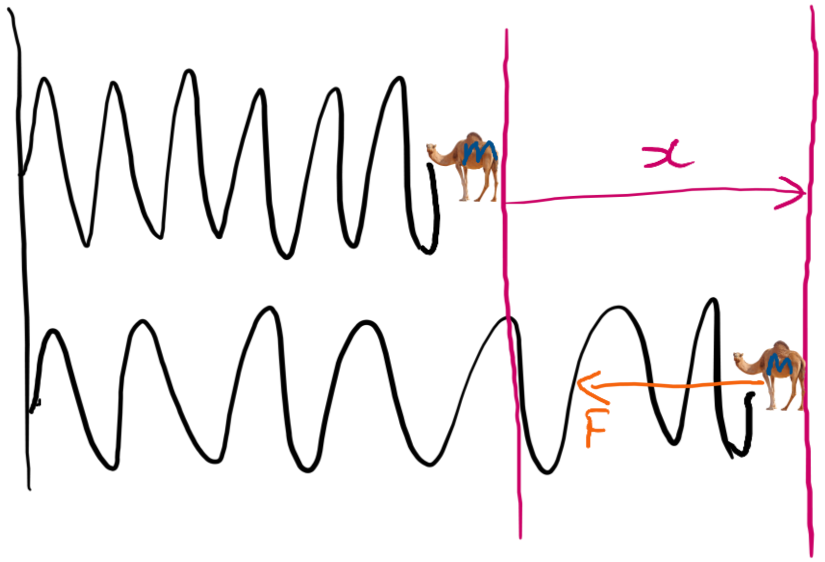
- How can you describe vibrations and waves?
- How can you describe quantum vibrations and waves?
- What’s a field? How about a quantum field?
- What is a particle? (according to QFT, at least)

What I’ll cover…

- Greeks
- Hooke
- Simple harmonic motion
- Energy etc.
- Quantum effects!
- Zero-point energy
- Waves, the wave equation and a modified wave equation
- The energy of a wave
- Quantum waves
- Fields, relativistic and non relativistic
- What a particle is…

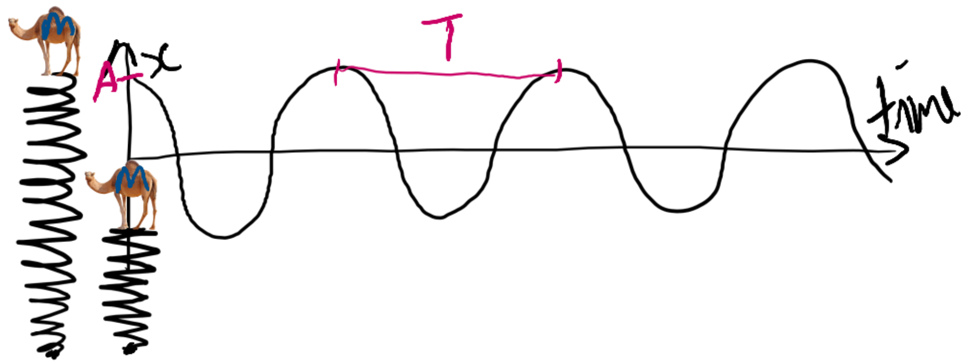


Hooke.



- Hooke's law:
  - $F \propto -x$
  - $F = -kx$
  - $ma = -kx \Rightarrow a = -\frac{k}{m}x$
    - Since acceleration is proportional and in the oposite direction to displacement, SHM ($a \propto -x$)
- From Hooke's law:
  - $a(t) = -\frac{k}{m}x(t)$
  - $a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$
  - Combining these, we get $\frac{d^2x}{dt^2} = -\frac{k}{m}x$

Special Harmonic Motion



- $x(t) \propto \cos(t)$
- In more detail:
  - $x(t) = A\cos(\omega t)$
    - Where $\omega = \frac{2\pi}{T} = 2\pi f$ as
  - $x(t) = A\cos(2\pi f t)$
- Checking:
  - $\frac{dx}{dt} = \frac{d}{dt}(A\cos(\omega t)) = -A\omega\sin(\omega t)$ - the velocity of the SHM oscillator.
  - $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(-A\omega\sin(\omega t)) = -\omega^2 A\cos(\omega t) = -\omega^2 x$
  - Therefore, $-\omega^2 x = -\frac{k}{m}x \Rightarrow \omega = \sqrt{\frac{k}{m}}$

Frequancy and Time Period

- Know: $\omega = \sqrt{\frac{k}{m}}$, but we also know $\omega = \frac{2\pi}{T} = 2\pi f$
- $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and $\mathrm{T} = 2\pi\sqrt{\frac{m}{k}}$.

Energies

- $E_T = E_K + E_P = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$
  - $= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$
  - $= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t) + \frac{1}{2}kA^2\cos^2(\omega t)$ and since $k = m\omega^2$,
  - $= \frac{1}{2}m\omega^2 A^2[\sin^2(\omega t) + \cos^2(\omega t)]$
  - $= \frac{1}{2}m\omega^2 A^2 = 2\pi^2 f^2 A^2 m$ i.e. $E_T \propto f^2 A^2$

Introducing quantum!

- Planck's quantum energy:
  - $E_n = nhf$
- SHM oscillator + quantum energy:
  - $nhf = 2\pi^2 f^2 A^2 m$
  - $A^2 = \frac{nhf}{2\pi^2 f^2 m} \Rightarrow A = \frac{1}{2\pi}\sqrt{\frac{2nh}{mf}}$
- For 100g mass oscillating at a frequency of 1Hz;
  - $\Delta A = 1.8 \times 10^{-17} m$ – pretty small.

Zero point energy.

- Energy of classical oscillator.
  - $E_T = E_K + E_P = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2$
- Heisenberg's uncertainty principle:
  - $\Delta x \Delta P \geq \frac{\hbar}{2}, \hbar = \frac{h}{2\pi}$
- Uncertainty in the energy:
  - $\Delta E \geq \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2\Delta x^2$
  - $\frac{d\Delta E}{d(\Delta x)} = -\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2\Delta x = 0 \Rightarrow \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$

More zero point energy..

- $\Delta E \geq \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}m\omega^2\Delta x^2$, $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$
- Substituting and simplifying:
  - $\Delta E \geq \frac{1}{2}hf \Rightarrow E_0 = \frac{1}{2}hf$
- Therefore…
  - $E_n = nhf$ changes to $E_n = \left(n + \frac{1}{2}\right)hf$ which you may see writen as $(n + \frac{1}{2})\hbar\omega$

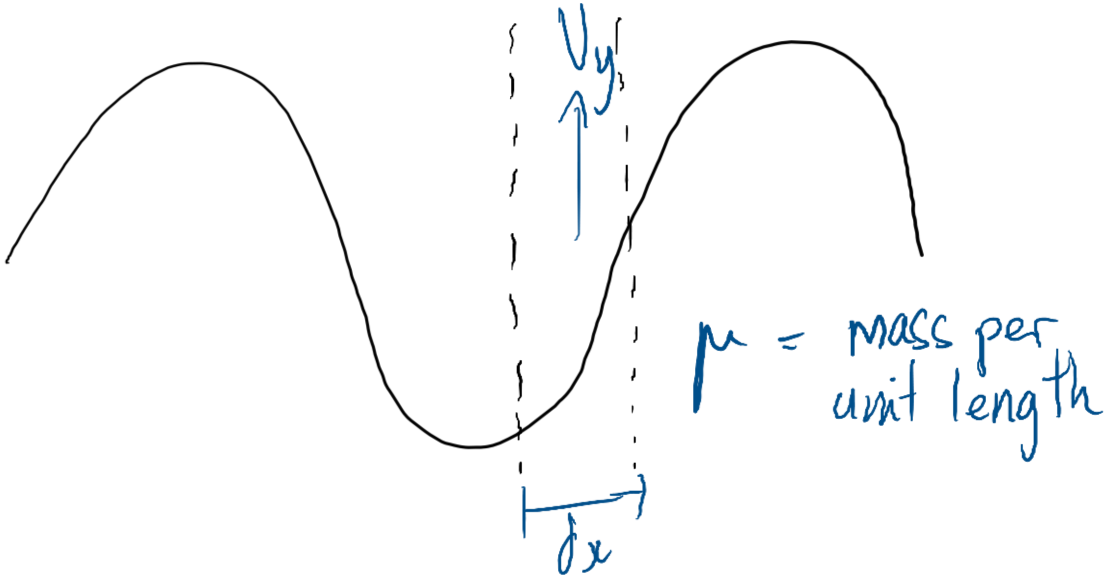
Waves and the wave equation. (only in one dimension)

- Waves?
  - $y(x,t) = A\cos(kx - \omega t)$, at $x = 0$, $y(0,t) = A\cos(\omega t)$
- Wave equation.
  - $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$
  - From displacement function:
    - $\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t)$
    - $\frac{\partial^2 y}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t)$ and by combining,
    - $-\omega^2 y = -v^2 k^2 y \quad \Rightarrow \omega^2 = v^2 k^2 \quad \Rightarrow v = \frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = f\lambda$

Modified wave equation

- The modified wave equation:
  - $\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = -(2\pi\chi)^2 y$
  - $\Rightarrow \omega^2 - k^2 v^2 = 4\pi^2\chi^2, \;\; \omega = 2\pi f, \; k = \frac{2\pi}{\lambda}$
    - $\Rightarrow 4\pi^2 f^2 - v^2 \frac{4\pi^2}{\lambda^2} = 4\pi^2\chi^2$
  - $\Rightarrow f^2 - \frac{v^2}{\lambda^2} = \chi^2$
  - $\Rightarrow f = \sqrt{\frac{v^2}{\lambda^2} + \chi^2} \rightarrow f \geq \chi \rightarrow \boldsymbol{\chi = f_{min}}$
- Now we can consider our new modified wave equation:
  - $\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = -(2\pi f_{min})^2 y$

The energy of a wave.



- Kinetic energy of wave:
  - $dE_K = \frac{1}{2} dm v_y^2, \quad dm = \mu dx,\ v_y = \frac{dy}{dt}$
  - $dE_K = \frac{1}{2} \mu A^2 \omega^2 \sin^2(kx - \omega t) dx$
- Potential energy of wave:
  - $dE_P = Fx = T(ds - dx), \quad ds^2 = dx^2 + dy^2$
  - $dE_P = T(\sqrt{dx^2 + dy^2} - dx) = Tdx\left(\sqrt{1 - \left(\frac{dy}{dx}\right)^2} - 1\right)$
    - $\sqrt{1 - \left(\frac{dy}{dx}\right)^2} \sim 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \cdots$

The energy of a wave continued

- $dE_P = \frac{1}{2} T \left(\frac{dy}{dx}\right)^2 dx, \quad y(x,t) = Acos(kx - \omega t)$
- $dE_p = \frac{1}{2} T k^2 A^2 \sin^2(kx - \omega t)\, dx\,, \quad v = \frac{\omega}{k} = \sqrt{\frac{T}{\mu}}$

- Total energy in one section of a wave with width dx:
  - $dE = dE_K + dE_P = \mu \omega^2 A^2 \sin^2(kx - \omega t)\, \mathrm{d}x$
- Total energy in one wavelength of wave:
  - $E_\lambda = \mu \omega^2 A^2 \int_0^\lambda \sin^2(kx - \omega t)\, dx = \mu \omega^2 A^2 \int_0^\lambda \frac{1}{2}\big(1 - \cos\big(2(kx - \omega t)\big)\big) dx$
  - $E_\lambda = \frac{1}{2} \mu \omega^2 A^2 \lambda = 2\pi^2 f^2 A^2 \mu, \quad E_\lambda \propto f^2 A^2$

Quantum waves (oooh spooky)

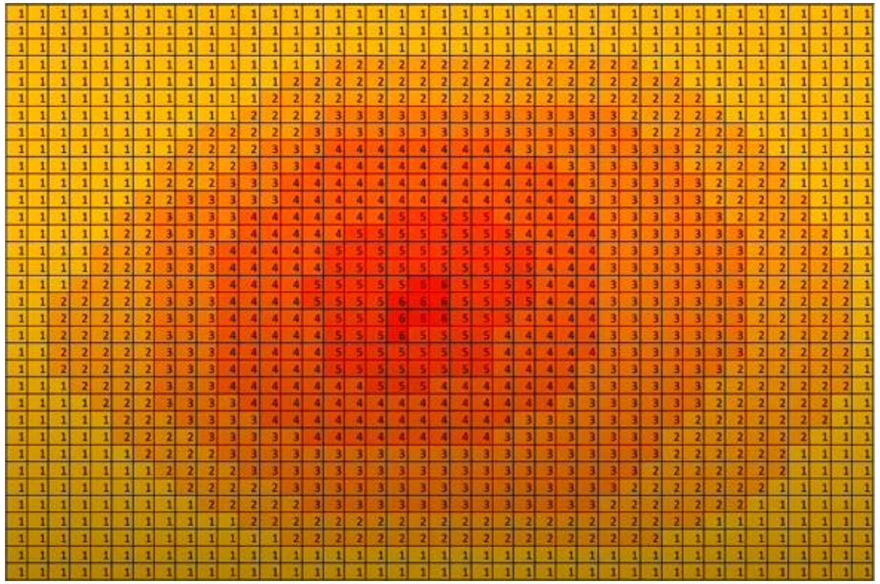
- From earlier…
  - $E_n = \left(n + \frac{1}{2}\right) hf$
- The energy contained in a region of length L:
  - $E_L = E_\lambda \left(\frac{L}{\lambda}\right) = 2\pi^2 f^2 A^2 \mu \left(\frac{L}{\lambda}\right) \approx nhf$
  - Rearranging for A,
    - $A^2 \approx \frac{nhf\lambda}{2\pi^2 f^2 L\mu} \quad \Rightarrow \mathrm{A} \approx \frac{1}{2\pi}\sqrt{\frac{2nh\lambda}{fL\mu}} \to A \propto \sqrt{\frac{nh}{f}} \to A \propto \sqrt{n}$

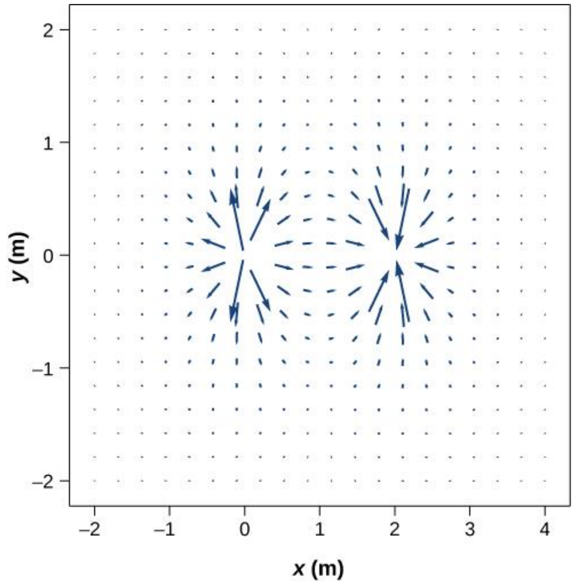
Summary of normal/quantum waves.

- Unmodified wave equation:
  - $\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$
  - $f = \frac{v}{\lambda}$
  - $E = hf$
- Modified wave equation:
  - $\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = -(2\pi\chi)^2 y$
  - $f = \sqrt{\frac{v^2}{\lambda^2} + f_{min}^2} \geq f_{min}$
  - $E_{min} = hf_{min}$

Fields?

- A function of space and time where each point has a related equation of motion which describes how this function changes with time.
  - Scalar Fields
  - Vector Fields





Relativistic Fields

- Combining equations of electricity and magnetism:
  - $\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \frac{\partial^2 \vec{E}}{\partial x^2} = 0$ – look familiar?
  - $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3.0 \times 10^8\, ms^{-1}$
- Two (main) kinds of relativistic fields: note: $\Psi(x,t) =$ relativistic field
  - $\frac{\partial^2 \Psi}{\partial t^2} - c^2 \frac{\partial^2 \Psi}{\partial^2 x} = 0$
  - $\frac{\partial^2 \Psi}{\partial t^2} - c^2 \frac{\partial^2 \Psi}{\partial^2 x} = -(2\pi f_{min})^2 \Psi$
    - Constraint equation: $f = \sqrt{\frac{c^2}{\lambda^2} + f_{min}^2} \quad \Rightarrow f^2 = \frac{c^2}{\lambda^2} + f_{min}^2$

Relativistic quantum fields

- Mass(y?) particles: $f^2 = \frac{c^2}{\lambda^2} + f_{min}^2$
  - $\Rightarrow h^2 f^2 = \left(\frac{hc}{\lambda}\right)^2 + h^2 f_{min}^2$
  - $E^2 = \left(\frac{hc}{\lambda}\right)^2 + (hf_{min})^2$ - similar to energy momentum relationship? ($E^2 = p^2c^2 + (mc^2)^2$)
    1. $pc = \frac{hc}{\lambda}$ - de-Broglie relationship
    2. $mc^2 = hf_{min}$ - $m = \frac{hf_{min}}{c^2}$ - mass of a particle
- Massless particles:
  - As $f_{min} = 0, m = 0$
  - $E^2 = \left(\frac{hc}{\lambda}\right)^2 \rightarrow E = \frac{hc}{\lambda}$
  - $E^2 = p^2c^2 \rightarrow E = pc$

After much, much maths… what’s a particle?

- A particle is the smallest possible vibration (quantum) of a relativistic quantum field.

Thanks for listening, and hopefully if I’ve timed this right there won’t be much time for questions.