Random Walks Script

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1 Introduciton

Hi all, welcome to my talk on random walks! It's a topic which I came across while I was attempting to do a random computing lab which I thought was pretty interesting, so I hope you guys don't find it too dull! !!!SLIDE!!!

2 Definition of random walks

So i suppose lets get right into it! I'll quickly define the type of random walk we're talking about so what I'm about to say makes sense. It's basically just a sequence of locations, where each new position is found by starting at the old one, and taking a step to one of its neighbours, where each possible step is independently and identically distributed from the standard d-dimensional unit vectors. !!!SLIDE!!!

3 Markov Chains

I'll quickly generalise this a bit. If any of you have seen Markov chains before, this might sound similar – the proposed random walk is just a simplified example of the more fundamental Markov chain, which consists of three things and one defining property; !!!SLIDE!!!

- First, State space - basically means the set of states which can be occupied. These can be anything – the positions in snakes and ladders, !!!SLIDE!!!, !!!SLIDE!!! types of cheesegraters, or in the d-dimensional simple symmetric random walk case, the **'hypercube lattice'** which sounds a lot cooler than it is... in 2D, it looks like.... This! !!!SLIDE!!!... just the integer points in a cartesian axis. !!!SLIDE!!!
- Transition probabilities are the probabilities of transitioning from some starting state to any of the states which can be reached from that position. !!!SLIDE!!! For 1D, you might have this. You could move from A to B with probability 1/2, or to C with 1/3, but it's also possible to not transition; here with probability 1/6. !!!SLIDE!!!

  In the n-dimensional case, and applied to our considered random walk, we should have $\mathbb{P}(X_i = \hat{\mathbf{e}}_j) = \mathbb{P}(X_i = \hat{\mathbf{e}}_j) = \frac{1}{2d}$, with $j = 1, ..., d$. i.e., for a dimension d, there are $2d$ possible steps with each having equal chance of selection. !!!SLIDE!!! They look like these lovely figures in two and three D. !!!SLIDE!!!
- Naturally there's also an Initial distribution - In a markov chain, you can choose (with certainty or with each state having an assigned initial probability) in which state the chain starts. !!!SLIDE!!! For our random walk, we'll start at the origin with certainty. !!!SLIDE!!!

The defining property of a Markov chain is that it has no memory (memorylessness)...– when you're at a state, the transition probabilities are fully independent of the route with which you got to that state. This will be important later. !!!SLIDE!!!

4 The question.

The main focus on my talk is the question will a walk return to the origin with certainty? If so, it's called recurrent, and there exists an n such that the position at that step is back where the walker started. i.e., $\mathbb{P}(S_n = 0) = 1$ –the green one in the figure. If its not recurrent, it's transient – there is a chance that the walk never returns to the origin, like the red one? !!!SLIDE!!!

To answer this question, clearly we need to work out the return probability. !!!SLIDE!!!

5 The brunt

Let's introduce some letters to make stuff easier. Call V the number of times the walker returns to the origin, assuming that n tends to infinity, i.e., the walker has been walking for an infinite amount of 'time'. !!!SLIDE!!! For a recurrent walk, if we're certain to reach the origin once, by memorylessness we require the probability that V equals infinity is 1, or equivalently the expected value of V is infinity. !!!SLIDE!!! For a transcient walk, of course the expected value of V must be finite. !!!SLIDE!!! Well use this later, but for the mean time let's work out the expected value of V.

Let's also introduce an indicator variable, J, which is 1 if, at step n, the walker is at the start point, and 0 if they're anywhere else. With that, V must just be that quantity summed over all steps, i.e., from $n = 1$ to infinity. Now the expected value of V is now easy to calculate, as its got the be the sum over all steps of the probability of the walker being at the origin for that step, which shouldn't be too hard to work out!

6 Calculating $\mathbb{P}$

For our simple state space, any walk which starts and ends at the same place must have an even number of steps–the steps 'up' must be the steps 'down', and there will be the same number 'left' and 'right'.

Clearly therefore the probability of getting back to the origin on an odd step is zero, so we can ignore all those contributions.

Now all is left is to work out he probability of return on an even step number. Let's consider the 2D case, and fix the total number of steps to be 16. One path that the walker might take is shown on the screen, and it's not hard to calculate the probability that this one occurs - the probability of going right on the first step is 1/4, and then up on the next is also 1/4 since due to the Markov property the steps are independent, so we can say the probability of this path is $\left(\frac{1}{4}\right)^{16}$.

But that's not the only path. This dashed blue one is another. Note how intersecting twice is not an issue, since we're at this stage only considering what happens on the 16th step, and since it is at the origin then, it's a valid path. Any other intersections are accounted for at a different value of n. Regardless, logically you'd expect there to be 16! paths, and there are!

It makes sense that you'd expect the total probability of returning to the origin in $2n$ steps to be the probability of a route returning (which is the same for each route) times the number of routes, so maybe you'd expect that the probability is $(16)!/4^{2n}$, but it turns out by doing that we're over counting the number of paths – some are degenerate.

If you consider a path that has i steps 'left', it must also have i steps 'right, and equivalently $n - i$ in the up and down directions. We're can take one of the i steps 'left', and swap it with another one of them providing us with the same route, but one which we had previously considered to be distinct. Clearly, this will be solved by dividing by the number of permutations of swaps we can make; i factorial, and the same applies for all the other directions. With that done, and summing over i from 0 to n (since there can be either 0 steps left for a path going only up then down, or n for a path only travelling left or right), you find the probability of returning to the origin to be the equation on the bottom of this slide. Analysing this with Stirling's formula or something similar, you'll find it scales as 1 over n – you might see where this is going...

Having done that, it's not hard for 3D. The only thing that has changed is the probability of taking a step in some direction – it's now a sixth instead of a forth. Of course there will still be $2n$! possible walks, and we also use the same logic to account for degenerate walks, assume i walks left, but j up, and $n - i - j$ walks out – the degree of freedom has increased by one. Again, we can just sum over i and j to account for all the walks. When the dust settles on this, you find the probability of return on the $2n$th step as this expression. For this one, i'll tell you it scales with $\frac{1}{n^{3/2}}$.

We're interested in the expectation value of V, which earlier we said was just the sum of all the $2n$ step return probabilities. For the 2D case, we worked out these scale as $1/n$, and as we all know the harmonic series diverges! So the expected number of returns is infinite, and the origin is certain to be intersected an infinite number of times (which is.. obviously.. greater than one times...)– i.e., a 2D random walk is guaranteed to return to the origin. It's recurrent. For the 3D case, with the series scales as $1/n^{3/2}$ (which converges, as the zeta function converges for all $s > 1$), we're not guaranteed an infinite number of returns, and by the earlier analysis, the walk must be transient!

That's basically what the whole talk was about! We've shown the maybe somewhat surprising result a 2D walker gets home, in 3D they may get lost. Some of the actual probabilities were worked out by George Polya, and it turns out that the probability of return is surprisingly small – only a 34 percent chance that in 3D you return to the origin! Of course, higher dimensional results are also available but maybe less intuitive.. Note how, maybe unsurprisingly, they decrease as the dimension number increases. This can be thought of intuitively as there being more paths that take you away from the starting point than back towards it, so the return probability decreases. It's less intuitive why the cut off is between 2 and 3 dimensions, but if you'd like to find out more feel free to ask me after the talk

7 Intuition (eech)

8 Application

What we just talked about may have sounded pretty abstract and not very helpful, but random walks (and more generally markov chains) have quite important applications. A very small number of examples are;

- financial economics – used to work out waht to do in stock markets or something like that (i don't really understand those ones i've not yet sold my soul to the finance devil)
- model cascades of neurons firing in the brain
- one one of the interesting physics applications is brownian motion, which can be considered a continues time 'wiener process', viewed as a simple version of a 3D random walk – I made a quick simulation to show this.

9 Thanks

So yeah! Thanks everyone for watching, and i really hope you enjoyed/found it interesting or informative... If you did (or if you didn't) please take pity on me and don't ask horrid questions!

References