Calculus and Vectors, Grade 12

University Preparation   MCV4U

This course builds on students' previous experience with functions and their developing understanding of rates of change. Students will solve problems involving geometric and algebraic representations of vectors and representations of lines and planes in three-dimensional space; broaden their understanding of rates of change to include the derivatives of polynomial, sinusoidal, exponential, rational, and radical functions; and apply these concepts and skills to the modelling of real-world relationships. Students will also refine their use of the mathematical processes necessary for success in senior mathematics. This course is intended for students who choose to pursue careers in fields such as science, engineering, economics, and some areas of business, including those students who will be required to take a university-level calculus, linear algebra, or physics course.

Note: The new Advanced Functions course (MHF4U) must be taken prior to or concurrently with Calculus and Vectors (MCV4U).

MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

Problem Solving

- develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;

Reasoning and Proving

- develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;

Reflecting

- demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);

Selecting Tools and Computational Strategies

- select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;

Connecting

- make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);

Representing

- create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;

Communicating

- communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

A. RATE OF CHANGE

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of rate of change by making connections between average rate of change over an interval and instantaneous rate of change at a point, using the slopes of secants and tangents and the concept of the limit;

2. graph the derivatives of polynomial, sinusoidal, and exponential functions, and make connections between the numeric, graphical, and algebraic representations of a function and its derivative;

3. verify graphically and algebraically the rules for determining derivatives; apply these rules to determine the derivatives of polynomial, sinusoidal, exponential, rational, and radical functions, and simple combinations of functions; and solve related problems.

SPECIFIC EXPECTATIONS

1. Investigating Instantaneous Rate of Change at a Point

By the end of this course, students will:

1.1 describe examples of real-world applications of rates of change, represented in a variety of ways (e.g., in words, numerically, graphically, algebraically)

1.2 describe connections between the average rate of change of a function that is smooth (i.e., continuous with no corners) over an interval and the slope of the corresponding secant, and between the instantaneous rate of change of a smooth function at a point and the slope of the tangent at that point

Sample problem: Given the graph of $f(x)$ shown below, explain why the instantaneous rate of change of the function cannot be determined at point P.
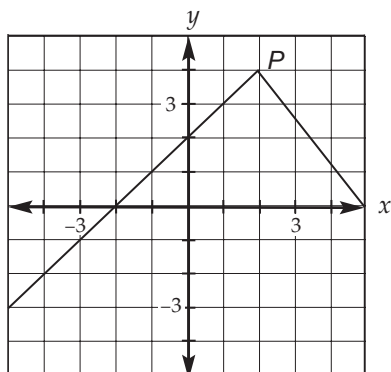
1.3 make connections, with or without graphing technology, between an approximate value of the instantaneous rate of change at a given point on the graph of a smooth function and average rates of change over intervals containing the point (i.e., by using secants through the given point on a smooth curve to approach the tangent at that point, and determining the slopes of the approaching secants to approximate the slope of the tangent)

1.4 recognize, through investigation with or without technology, graphical and numerical examples of limits, and explain the reasoning involved (e.g., the value of a function approaching an asymptote, the value of the ratio of successive terms in the Fibonacci sequence)

Sample problem: Use appropriate technology to investigate the limiting value of the terms in the sequence $\left(1 + \frac{1}{1}\right)^1, \left(1 + \frac{1}{2}\right)^2, \left(1 + \frac{1}{3}\right)^3,$ $\left(1 + \frac{1}{4}\right)^4,$ …, and the limiting value of the series $4 \times 1 - 4 \times \frac{1}{3} + 4 \times \frac{1}{5} - 4 \times \frac{1}{7} + 4 \times \frac{1}{9} - \dots .$

1.5 make connections, for a function that is smooth over the interval $a \le x \le a + h$, between the average rate of change of the function over this interval and the value of the expression $\frac{f(a + h) - f(a)}{h}$, and between the instantaneous rate of change of the function at $x = a$ and the value of the limit $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

Sample problem: What does the limit

$$\lim_{h \to 0} \frac{f(4 + h) - f(4)}{h} = 8 \text{ indicate about the}$$

graph of the function $f(x) = x^2$? The graph of a general function $y = f(x)$?

1.6 compare, through investigation, the calculation of instantaneous rates of change at a point $(a, f(a))$ for polynomial functions [e.g., $f(x) = x^2$, $f(x) = x^3$], with and without simplifying the expression $\dfrac{f(a + h) - f(a)}{h}$ before substituting values of h that approach zero [e.g., for $f(x) = x^2$ at $x = 3$, by determining

$$\frac{f(3 + 1) - f(3)}{1} = 7, \quad \frac{f(3 + 0.1) - f(3)}{0.1} = 6.1,$$

$$\frac{f(3 + 0.01) - f(3)}{0.01} = 6.01, \text{ and}$$

$$\frac{f(3 + 0.001) - f(3)}{0.001} = 6.001, \text{ and}$$

by first simplifying $\dfrac{f(3 + h) - f(3)}{h}$ as

$\dfrac{(3 + h)^2 - 3^2}{h} = 6 + h$ and then substituting

the same values of h to give the same results]

2. Investigating the Concept of the Derivative Function

By the end of this course, students will:

2.1 determine numerically and graphically the intervals over which the instantaneous rate of change is positive, negative, or zero for a function that is smooth over these intervals (e.g., by using graphing technology to examine the table of values and the slopes of tangents for a function whose equation is given; by examining a given graph), and describe the behaviour of the instantaneous rate of change at and between local maxima and minima

Sample problem: Given a smooth function for which the slope of the tangent is always positive, explain how you know that the function is increasing. Give an example of such a function.

2.2 generate, through investigation using technology, a table of values showing the instantaneous rate of change of a polynomial function, $f(x)$, for various values of x (e.g., construct a tangent to the function, measure its slope, and create a slider or animation to move the point of tangency), graph the ordered pairs, recognize that the graph represents a function called the derivative, $f'(x)$ or $\dfrac{dy}{dx}$, and make connections between

the graphs of $f(x)$ and $f'(x)$ or y and $\dfrac{dy}{dx}$ [e.g., when $f(x)$ is linear, $f'(x)$ is constant; when $f(x)$ is quadratic, $f'(x)$ is linear; when $f(x)$ is cubic, $f'(x)$ is quadratic]

Sample problem: Investigate, using patterning strategies and graphing technology, relationships between the equation of a polynomial function of degree no higher than 3 and the equation of its derivative.

2.3 determine the derivatives of polynomial functions by simplifying the algebraic expression $\dfrac{f(x + h) - f(x)}{h}$ and then taking the limit of the simplified expression as h approaches zero [i.e., determining $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$]

2.4 determine, through investigation using technology, the graph of the derivative $f'(x)$ or $\dfrac{dy}{dx}$ of a given sinusoidal function [i.e., $f(x) = \sin x$, $f(x) = \cos x$] (e.g., by generating a table of values showing the instantaneous rate of change of the function for various values of x and graphing the ordered pairs; by using dynamic geometry software to verify graphically that when $f(x) = \sin x$, $f'(x) = \cos x$, and when $f(x) = \cos x$, $f'(x) = -\sin x$; by using a motion sensor to compare the displacement and velocity of a pendulum)

2.5 determine, through investigation using technology, the graph of the derivative $f'(x)$ or $\dfrac{dy}{dx}$ of a given exponential function [i.e., $f(x) = a^x$ ($a > 0$, $a \neq 1$)] [e.g., by generating a table of values showing the instantaneous rate of change of the function for various values of x and graphing the ordered pairs; by using dynamic geometry software to verify that when $f(x) = a^x$, $f'(x) = kf(x)$], and make connections between the graphs of $f(x)$ and $f'(x)$ or y and $\dfrac{dy}{dx}$ [e.g., $f(x)$ and $f'(x)$ are both exponential; the ratio $\dfrac{f'(x)}{f(x)}$ is constant, or $f'(x) = kf(x)$; $f'(x)$ is a vertical stretch from the x-axis of $f(x)$]

Sample problem: Graph, with technology, $f(x) = a^x$ ($a > 0$, $a \neq 1$) and $f'(x)$ on the same set of axes for various values of a (e.g., 1.7, 2.0, 2.3, 3.0, 3.5). For each value of a, investigate the ratio $\dfrac{f'(x)}{f(x)}$ for various values of x, and explain how you can use this ratio to determine the slopes of tangents to $f(x)$.

2.6 determine, through investigation using technology, the exponential function $f(x) = a^x$ $(a > 0, a \neq 1)$ for which $f'(x) = f(x)$ (e.g., by using graphing technology to create a slider that varies the value of a in order to determine the exponential function whose graph is the same as the graph of its derivative), identify the number e to be the value of a for which $f'(x) = f(x)$ [i.e., given $f(x) = e^x$, $f'(x) = e^x$], and recognize that for the exponential function $f(x) = e^x$ the slope of the tangent at any point on the function is equal to the value of the function at that point

Sample problem: Use graphing technology to determine an approximate value of e by graphing $f(x) = a^x$ $(a > 0, a \neq 1)$ for various values of a, comparing the slope of the tangent at a point with the value of the function at that point, and identifying the value of a for which they are equal.

2.7 recognize that the natural logarithmic function $f(x) = \log_e x$, also written as $f(x) = \ln x$, is the inverse of the exponential function $f(x) = e^x$, and make connections between $f(x) = \ln x$ and $f(x) = e^x$ [e.g., $f(x) = \ln x$ reverses what $f(x) = e^x$ does; their graphs are reflections of each other in the line $y = x$; the composition of the two functions, $e^{\ln x}$ or $\ln e^x$, maps x onto itself, that is, $e^{\ln x} = x$ and $\ln e^x = x$]

2.8 verify, using technology (e.g., calculator, graphing technology), that the derivative of the exponential function $f(x) = a^x$ is $f'(x) = a^x \ln a$ for various values of a [e.g., verifying numerically for $f(x) = 2^x$ that $f'(x) = 2^x \ln 2$ by using a calculator to show that $\lim\limits_{h \to 0} \dfrac{(2^h - 1)}{h}$ is $\ln 2$ or by graphing $f(x) = 2^x$, determining the value of the slope and the value of the function for specific x-values, and comparing the ratio $\dfrac{f'(x)}{f(x)}$ with $\ln 2$]

Sample problem: Given $f(x) = e^x$, verify numerically with technology using $\lim\limits_{h \to 0} \dfrac{(e^{x+h} - e^x)}{h}$ that $f'(x) = f(x) \ln e$.

3. Investigating the Properties of Derivatives

By the end of this course, students will:

3.1 verify the power rule for functions of the form $f(x) = x^n$, where n is a natural number [e.g., by determining the equations of the derivatives of the functions $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, and $f(x) = x^4$ algebraically using $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ and graphically using slopes of tangents]

3.2 verify the constant, constant multiple, sum, and difference rules graphically and numerically [e.g., by using the function $g(x) = kf(x)$ and comparing the graphs of $g'(x)$ and $kf'(x)$; by using a table of values to verify that $f'(x) + g'(x) = (f + g)'(x)$, given $f(x) = x$ and $g(x) = 3x$], and read and interpret proofs involving $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ of the constant, constant multiple, sum, and difference rules (student reproduction of the development of the general case is not required)

Sample problem: The amounts of water flowing into two barrels are represented by the functions $f(t)$ and $g(t)$. Explain what $f'(t)$, $g'(t)$, $f'(t) + g'(t)$, and $(f + g)'(t)$ represent. Explain how you can use this context to verify the sum rule, $f'(t) + g'(t) = (f + g)'(t)$.

3.3 determine algebraically the derivatives of polynomial functions, and use these derivatives to determine the instantaneous rate of change at a point and to determine point(s) at which a given rate of change occurs

Sample problem: Determine algebraically the derivative of $f(x) = 2x^3 + 3x^2$ and the point(s) at which the slope of the tangent is 36.

3.4 verify that the power rule applies to functions of the form $f(x) = x^n$, where n is a rational number [e.g., by comparing values of the slopes of tangents to the function $f(x) = x^{\frac{1}{2}}$ with values of the derivative function determined using the power rule], and verify

algebraically the chain rule using monomial functions [e.g., by determining the same derivative for $f(x) = (5x^3)^{\frac{1}{3}}$ by using the chain rule and by differentiating the simplified form, $f(x) = 5^{\frac{1}{3}}x$] and the product rule using polynomial functions [e.g., by determining the same derivative for $f(x) = (3x + 2)(2x^2 - 1)$ by using the product rule and by differentiating the expanded form $f(x) = 6x^3 + 4x^2 - 3x - 2$]

Sample problem: Verify the chain rule by using the product rule to look for patterns in the derivatives of $f(x) = x^2 + 1$, $f(x) = (x^2 + 1)^2$, $f(x) = (x^2 + 1)^3$, and $f(x) = (x^2 + 1)^4$.

3.5 solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions [e.g., by expressing $f(x) = \dfrac{x^2 + 1}{x - 1}$ as the product $f(x) = (x^2 + 1)(x - 1)^{-1}$], radical functions [e.g., by expressing $f(x) = \sqrt{x^2 + 5}$ as the power $f(x) = (x^2 + 5)^{\frac{1}{2}}$], and other simple combinations of functions [e.g., $f(x) = x \sin x$, $f(x) = \dfrac{\sin x}{\cos x}$]*

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*The emphasis of this expectation is on the application of the derivative rules and not on the simplification of resulting complex algebraic expressions.

B. DERIVATIVES AND THEIR APPLICATIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. make connections, graphically and algebraically, between the key features of a function and its first and second derivatives, and use the connections in curve sketching;

2. solve problems, including optimization problems, that require the use of the concepts and procedures associated with the derivative, including problems arising from real-world applications and involving the development of mathematical models.

SPECIFIC EXPECTATIONS

1. Connecting Graphs and Equations of Functions and Their Derivatives

By the end of this course, students will:

1.1 sketch the graph of a derivative function, given the graph of a function that is continuous over an interval, and recognize points of inflection of the given function (i.e., points at which the concavity changes)

Sample problem: Investigate the effect on the graph of the derivative of applying vertical and horizontal translations to the graph of a given function.

1.2 recognize the second derivative as the rate of change of the rate of change (i.e., the rate of change of the slope of the tangent), and sketch the graphs of the first and second derivatives, given the graph of a smooth function

1.3 determine algebraically the equation of the second derivative $f''(x)$ of a polynomial or simple rational function $f(x)$, and make connections, through investigation using technology, between the key features of the graph of the function (e.g., increasing/decreasing intervals, local maxima and

minima, points of inflection, intervals of concavity) and corresponding features of the graphs of its first and second derivatives (e.g., for an increasing interval of the function, the first derivative is positive; for a point of inflection of the function, the slopes of tangents change their behaviour from increasing to decreasing or from decreasing to increasing, the first derivative has a maximum or minimum, and the second derivative is zero)

Sample problem: Investigate, using graphing technology, connections between key properties, such as increasing/decreasing intervals, local maxima and minima, points of inflection, and intervals of concavity, of the functions $f(x) = 4x + 1$, $f(x) = x^2 + 3x - 10$, $f(x) = x^3 + 2x^2 - 3x$, and $f(x) = x^4 + 4x^3 - 3x^2 - 18x$ and the graphs of their first and second derivatives.

1.4 describe key features of a polynomial function, given information about its first and/or second derivatives (e.g., the graph of a derivative, the sign of a derivative over specific intervals, the x-intercepts of a derivative), sketch two or more possible graphs of the function that are consistent with the given information, and explain why an infinite number of graphs is possible

Sample problem: The following is the graph of the function $g(x)$.
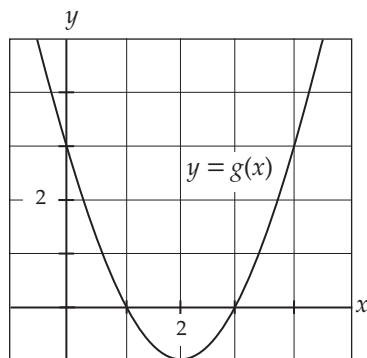
If $g(x)$ is the derivative of $f(x)$, and $f(0) = 0$, sketch the graph of $f(x)$. If you are now given the function equation $g(x) = (x − 1)(x − 3)$, determine the equation of $f''(x)$ and describe some features of the equation of $f(x)$. How would $f(x)$ change graphically and algebraically if $f(0) = 2$?

1.5 sketch the graph of a polynomial function, given its equation, by using a variety of strategies (e.g., using the sign of the first derivative; using the sign of the second derivative; identifying even or odd functions) to determine its key features (e.g., increasing/decreasing intervals, intercepts, local maxima and minima, points of inflection, intervals of concavity), and verify using technology

2. Solving Problems Using Mathematical Models and Derivatives

By the end of this course, students will:

2.1 make connections between the concept of motion (i.e., displacement, velocity, acceleration) and the concept of the derivative in a variety of ways (e.g., verbally, numerically, graphically, algebraically)

Sample problem: Generate a displacement–time graph by walking in front of a motion sensor connected to a graphing calculator. Use your knowledge of derivatives to sketch the velocity–time and acceleration–time graphs. Verify the sketches by displaying the graphs on the graphing calculator.

2.2 make connections between the graphical or algebraic representations of derivatives and real-world applications (e.g., population and rates of population change, prices and inflation rates, volume and rates of flow, height and growth rates)

Sample problem: Given a graph of prices over time, identify the periods of inflation and deflation, and the time at which the maximum rate of inflation occurred. Explain how derivatives helped solve the problem.

2.3 solve problems, using the derivative, that involve instantaneous rates of change, including problems arising from real-world applications (e.g., population growth, radioactive decay, temperature changes, hours of daylight, heights of tides), given the equation of a function*

Sample problem: The size of a population of butterflies is given by the function
$$P(t) = \frac{6000}{1 + 49(0.6)^t}$$ where t is the time in days. Determine the rate of growth in the population after 5 days using the derivative, and verify graphically using technology.

2.4 solve optimization problems involving polynomial, simple rational, and exponential functions drawn from a variety of applications, including those arising from real-world situations

Sample problem: The number of bus riders from the suburbs to downtown per day is represented by $1200(1.15)^{−x}$, where x is the fare in dollars. What fare will maximize the total revenue?

2.5 solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results

Sample problem: A bird is foraging for berries. If it stays too long in any one patch it will be spending valuable foraging time looking for the hidden berries, but when it leaves it will have to spend time finding another patch. A model for the net amount of food energy in

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*The emphasis of this expectation is on the application of the derivative rules and not on the simplification of resulting complex algebraic expressions.

joules the bird gets if it spends t minutes in a patch is $E = \dfrac{3000t}{t + 4}$. Suppose the bird takes 2 min on average to find each new patch, and spends negligible energy doing so. How long should the bird spend in a patch to maximize its average rate of energy gain over the time spent flying to a patch and foraging in it? Use and compare numeric, graphical, and algebraic strategies to solve this problem.

C. GEOMETRY AND ALGEBRA OF VECTORS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of vectors in two-space and three-space by representing them algebraically and geometrically and by recognizing their applications;

2. perform operations on vectors in two-space and three-space, and use the properties of these operations to solve problems, including those arising from real-world applications;

3. distinguish between the geometric representations of a single linear equation or a system of two linear equations in two-space and three-space, and determine different geometric configurations of lines and planes in three-space;

4. represent lines and planes using scalar, vector, and parametric equations, and solve problems involving distances and intersections.

SPECIFIC EXPECTATIONS

1. Representing Vectors Geometrically and Algebraically

By the end of this course, students will:

1.1 recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors (e.g., displacement, forces involved in structural design, simple animation of computer graphics, velocity determined using GPS)

Sample problem: Position is represented using vectors. Explain why knowing that someone is 69 km from Lindsay, Ontario, is not sufficient to identify their exact position.

1.2 represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways (e.g., 320°; N 40° W), and algebraically (e.g., using Cartesian coordinates; using polar coordinates), and recognize vectors with the same magnitude and direction but different positions as equal vectors

1.3 determine, using trigonometric relationships [e.g., $x = r\cos\theta$, $y = r\sin\theta$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ or $\tan^{-1}\left(\frac{y}{x}\right) + 180°$, $r = \sqrt{x^2 + y^2}$], the Cartesian representation of a vector in two-space given as a directed line segment, or the representation as a directed line segment of a vector in two-space given in Cartesian form [e.g., representing the vector (8, 6) as a directed line segment]

Sample problem: Represent the vector with a magnitude of 8 and a direction of 30° anti-clockwise to the positive *x*-axis in Cartesian form.

1.4 recognize that points and vectors in three-space can both be represented using Cartesian coordinates, and determine the distance between two points and the magnitude of a vector using their Cartesian representations

2. Operating With Vectors

By the end of this course, students will:

2.1 perform the operations of addition, subtraction, and scalar multiplication on vectors represented as directed line segments in two-space, and on vectors represented in Cartesian form in two-space and three-space

2.2 determine, through investigation with and without technology, some properties (e.g., commutative, associative, and distributive properties) of the operations of addition, subtraction, and scalar multiplication of vectors