MS&E 228: Inference with Modern Non-Linear Prediction

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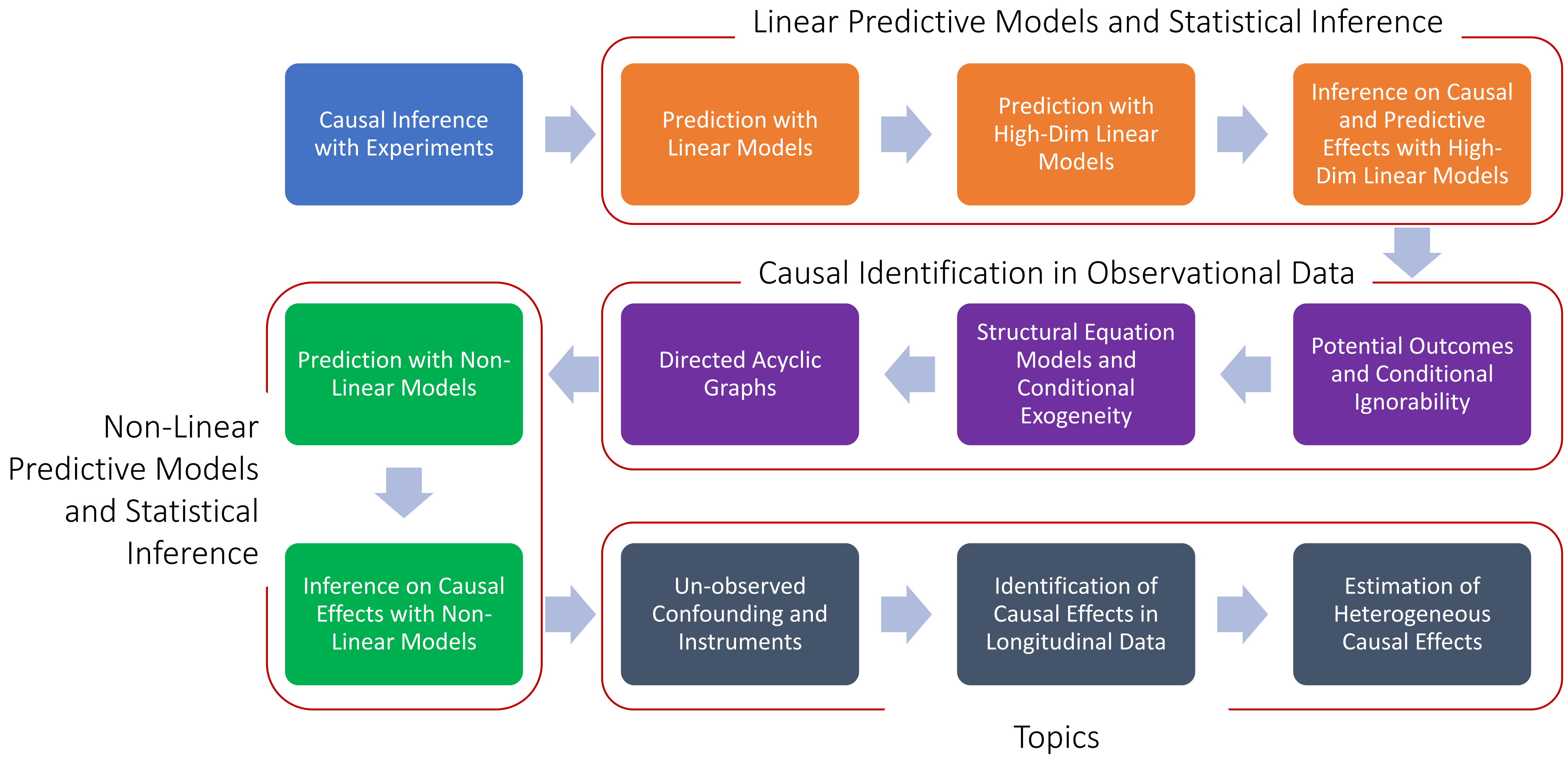
Linear Predictive Models and Statistical Inference
Causal Inference with Experiments
Prediction with Linear Models
Prediction with High-Dim Linear Models
Inference on Causal and Predictive Effects with High-Dim Linear Models
Causal Identification in Observational Data
Potential Outcomes and Conditional Ignorability
Structural Equation Models and Conditional Exogeneity
Directed Acyclic Graphs
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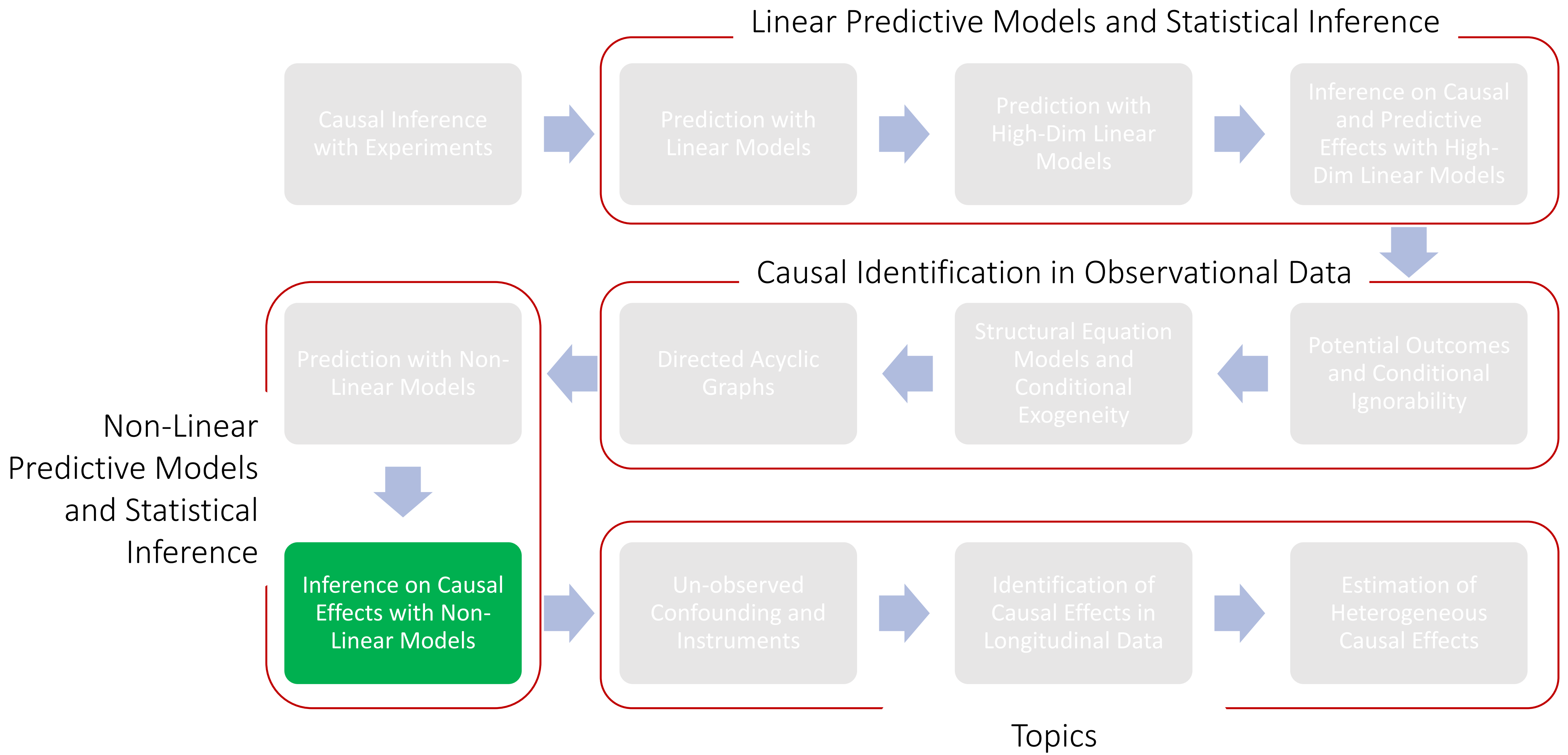
Un-observed Confounding and Instruments
Identification of Causal Effects in Longitudinal Data
Estimation of Heterogeneous Causal Effects
Topics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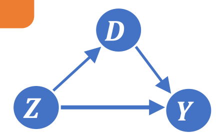
Recap of Last Lecture

Causal Inference Pipeline

Theory

DAGs/ASEMs



a_d
$=$
$E\left[E[Y|D = d, Z]\right]$

Construct estimate $\hat{g}(d, z)$ of predictive model
$E[\,Y \mid D = d, Z = z\,]$
Estimate mean counterfactual
$\hat{a}_d = E_n[\hat{g}(d, Z)]$

$a_d \in \hat{a}_d \pm 2\hat{\sigma}$

Practice

Data Collection → Domain Assumption Elicitation → Identification → Estimation (training) → Validation (testing)

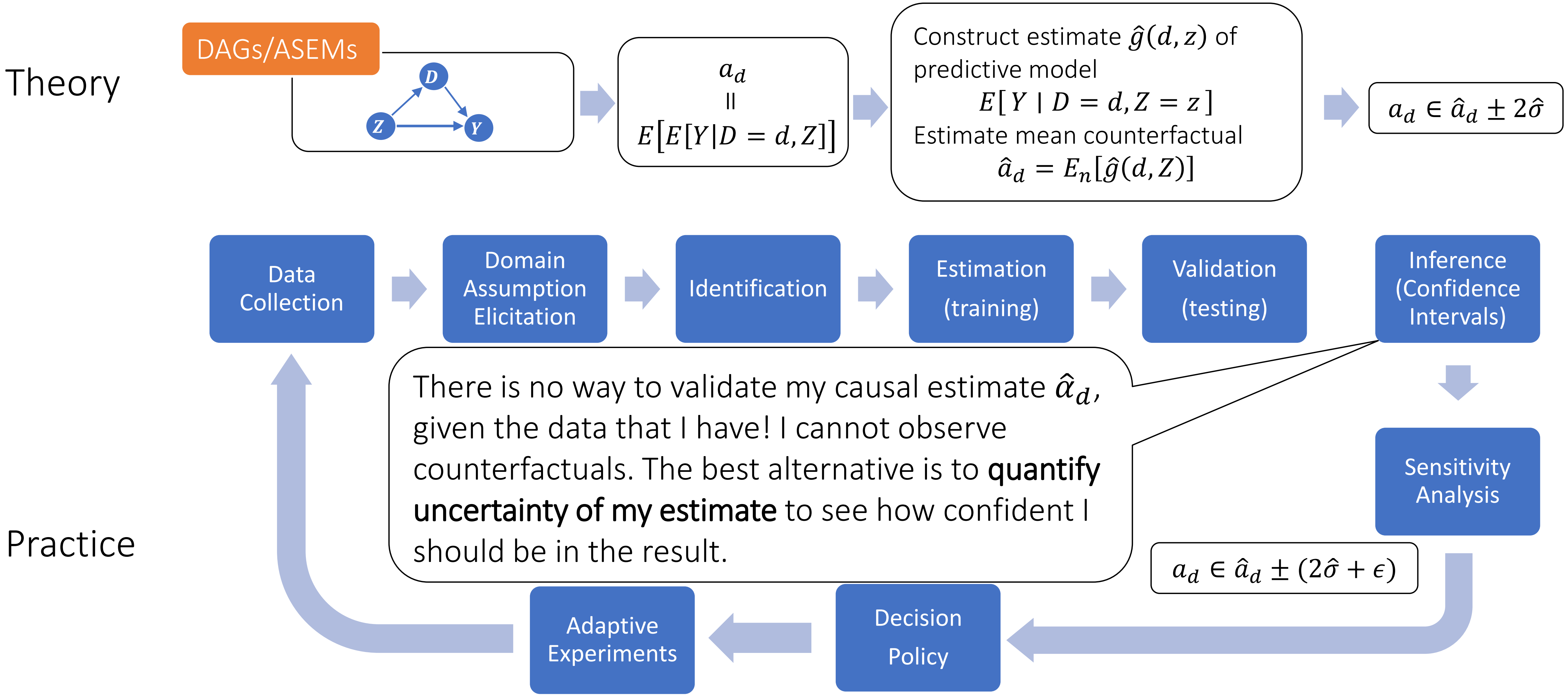
Inference (Confidence Intervals) → Sensitivity Analysis

$a_d \in \hat{a}_d \pm (2\hat{\sigma} + \epsilon)$

→ Decision Policy → Adaptive Experiments → Data Collection

So I need to estimate this **bigger predictive model, predicting the outcome from the treatment $\boldsymbol{D}$ and the controls $\boldsymbol{S}$**, in some manner. Then average the predictive values over S fixing $d = 1$.

Causal Inference Pipeline



Goals for Today

- Methods for Confidence Intervals for ATE with non-linear models
- General Neyman Orthogonality Framework (Double/Debiased ML)
- Methods for Confidence Intervals for ATE in a partially-linear model
- Sample-splitting and cross-fitting

- Proof sketch of main theorem*

The Example Problem

Identification under Conditional Ignorability

- Once we condition on enough variables X that affect treatment assignment, remnant variation in D is exogenous (as-if trial)

$$Y^{(d)} \perp D \mid X \quad \text{(conditional ignorability)}$$

- Why useful:

$$\begin{aligned} E[\,Y \mid D = d, X\,] &= E\left[\,Y^{(D)} \mid D = d, X\,\right] \\ &= E\left[\,Y^{(d)} \mid D = d, X\,\right] = E\left[\,Y^{(d)} \mid X\,\right] \end{aligned}$$

- Average treatment effect is "identified" as (g-formula):

$$\begin{aligned} \theta_0 = E\left[Y^{(1)} - Y^{(0)}\right] &= E\left[E\left[\,Y^{(1)} - Y^{(0)} \mid X\,\right]\right] \\ &= E\left[E[Y|D = 1, X] - E[\,Y \mid D = 0, X\,]\right] \end{aligned}$$

Let’s take it to data

- We observe n samples $Z_1, \ldots, Z_n$ where $Z_i = (X_i, D_i, Y_i)$

- Want to estimate average effect θ_0, which satisfies:

$$\theta_0 = E[g_0(1, X) - g_0(0, X)]$$

- Where:

$$g_0(D, X) \coloneqq E[\, Y \mid D, X \,]$$

- We want to be able to use ML to learn regression function g_0!

A General Estimation Framework

Semi-Parametric Moment Restrictions

- Observe samples $Z_1, \ldots, Z_n$ i.i.d. from data distribution D
- Distribution D satisfies vector of moment restrictions
$$M(\theta_0, g_0) \coloneqq E_{Z\sim D}[m(Z; \theta_0, g_0)] = 0$$
- $\theta_0 \in R^d$ finite dimensional target parameter of interest
- $g_0 \in G$ potentially infinite dimensional (an un-known function) we don't care (nuisance)
- g_0 is un-known and needs to be estimated from data

ATE under Conditional Exogeneity

- We observe n samples $Z_1, \ldots, Z_n$ where $Z_i = (X_i, D_i, Y_i)$
- Want to estimate average effect θ_0, which satisfies:

$$M(\theta_0, g_0) \coloneqq E[g_0(1, X) - g_0(0, X) - \theta_0] = 0$$

- Where:

$$g_0(D, X) \coloneqq E[\, Y \mid D, X \,]$$

- We want to be able to use ML to learn regression function g!

Given n samples we want to produce estimate $\widehat{\theta}$

What do we want from $\hat{\theta}$?

- Consistency

$$\hat{\theta} \rightarrow_p \theta_0$$

What do we want from $\hat{\theta}$?

- Finite sample parametric rate:

$$\|\hat{\theta} - \theta_0\| = O_p\left(\sqrt{d/n}\right)$$

What do we want from $\hat{\theta}$?

- Asymptotic normality

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) \to_d N\left(0, \sigma^2\right)$$

- Construction of confidence intervals:

with prob. $\approx$ 95%: $\theta_0 \in \left[\hat{\theta} - 1.96\hat{\sigma}, \hat{\theta} + 1.96\hat{\sigma}\right]$.

- Calculation of p-value for zero effect

What do we want from $\hat{\theta}$?

- Asymptotic linearity

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi_0(Z_i) + o_p(1)$$

- Consistency of bootstrap confidence intervals

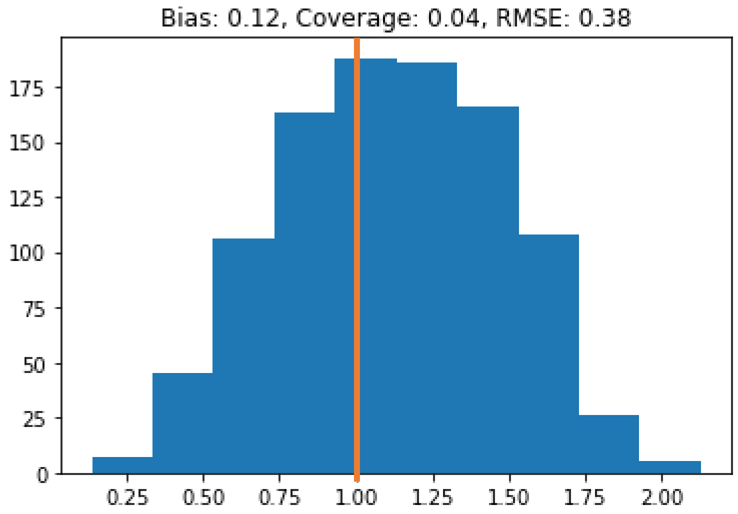
Natural Estimation Algorithm

- Estimate $\hat{g}$ of g_0 from data
- Return solution $\hat{\theta}$ to empirical plug-in moment equation:

$$M_n(\hat{\theta}, \hat{g}) \coloneqq \frac{1}{n} \sum_{i=1}^{n} m(Z_i; \hat{\theta}, \hat{g}) = 0$$

Natural Algorithm Gone Wrong

```
def est(X, D, y): # direct non-orthogonal estimator of average effect
    est = RandomForestRegressor(min_samples_leaf=20)
    est.fit(np.hstack([D.reshape(-1, 1), X]), y)
    ones = np.hstack([np.ones((X.shape[0], 1)), X])
    zeros = np.hstack([np.zeros((X.shape[0], 1)), X])
    preds = est.predict(ones) - est.predict(zeros)
    return np.mean(preds), np.std(preds)/np.sqrt(X.shape[0])
```



Natural Estimation Algorithm (Draft 2)

- Split the data in half
- On first half, estimate $\hat{g}$ of g_0
- On second half, solution $\hat{\theta}$ to empirical plug-in moment equation:

$$M_n(\hat{\theta}, \hat{g}) \coloneqq \frac{1}{n_2} \sum_{i \in S_2} m(Z_i; \hat{\theta}, \hat{g}) = 0$$

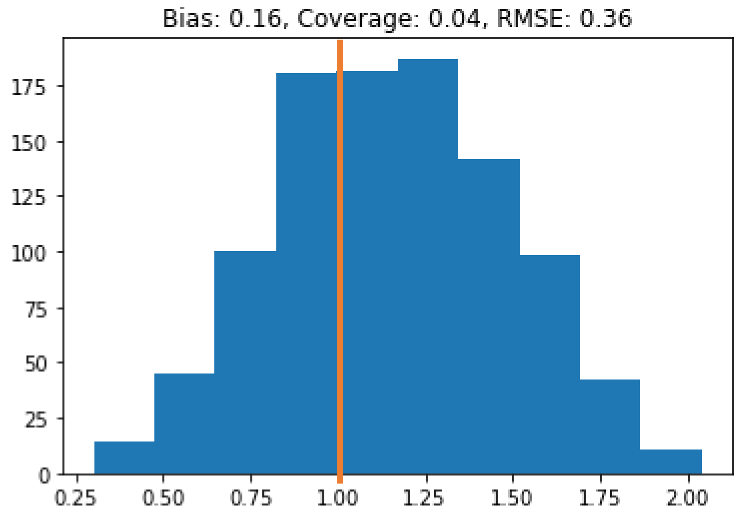
Natural Estimation Algorithm (Draft 3)

- Split data in K parts, $S_1, \dots, S_K$
- For each part k, estimate $\hat{g}_k$ using data from all parts except S_k
- Return solution $\hat{\theta}$ to cross-fitted empirical moment equation:

$$\frac{1}{n}\sum_{k=1}^{K}\sum_{i\in S_k} m\left(Z_i;\hat{\theta},\hat{g}_k\right) = 0$$

Natural Algorithm (Draft 3) Gone Wrong

```
def est2(X, D, y): # direct non-orthogonal estimator with sample splitting
    effects = np.zeros(X.shape[0])
    for train, test in KFold(n_splits=3).split(X):
        est = RandomForestRegressor(min_samples_leaf=20)
        est.fit(np.hstack([D[train].reshape(-1, 1), X[train]]), y[train])
        ones = np.hstack([np.ones((X[test].shape[0], 1)), X[test]])
        zeros = np.hstack([np.zeros((X[test].shape[0], 1)), X[test]])
        effects[test] = est.predict(ones) - est.predict(zeros)
    return np.mean(effects), np.std(effects)/np.sqrt(X.shape[0])
```

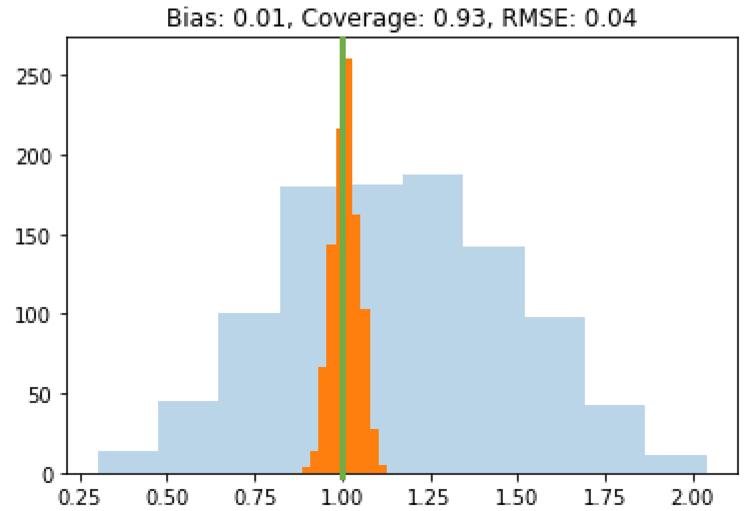


When is estimate $\hat{\theta}$ $\sqrt{n}$-asymptotically normal?

Natural Algorithm (Draft 3) Gone Right

[redacted]

TO BE UNCOVERED!



When is estimate $\hat{\theta}$ $\sqrt{n}$-asymptotically normal?
We need to change the moment we use

Debiasing Intuition

ATE under Conditional Exogeneity

- We observe n samples $Z_1, \dots, Z_n$ where $Z_i = (X_i, D_i, Y_i)$
- Want to estimate average effect θ_0, which satisfies:

$$M(\theta_0, g_0) \coloneqq E[g_0(1, X) - g_0(0, X) - \theta_0] = 0$$

- Where:

$$g_0(D, X) \coloneqq E[\, Y \mid D, X \,]$$

- The moment $M(\theta, g)$ is sensitive to variations in g
- Any bias or error in g propagates to bias or error in moment and $\hat{\theta}$
- Can we add a correction that corrects the biases of $\hat{g}$

Better Moment for ATE

- Add a "debiasing" correction:

$$\tilde{M}(\theta, g, a) = M(\theta, g) + E\big[a(D, X)\,\big(Y\ - g(D, X)\big)\big]$$

- What is a_0? Should be such that

$$E[a_0(D, X)\ g(D, X)] = E[g(1, X) - g(0, X)]$$

- If this holds then if g is very wrong but a is correct:

$$\tilde{M}(\theta, g) = E[a(D, X)Y] = E\big[a(D, X)E[Y \mid D, X\,]\big]$$
$$= E[a(D, X)g(D, X)] = E[g(1, X) - g(0, X)]$$

Inverse Propensity Weighting (IPW)

- The following works: inverse propensity scoring
$$a_0(D,X) = \frac{D}{\Pr[D=1|X]} - \frac{1-D}{\Pr[D=0|X]}$$
- Sketch:
$$\begin{aligned} E\left[\frac{D}{\Pr[D=1|X]}g(D,X)\right] &= E\left[\frac{D}{\Pr[D=1|X]}g(1,X)\right] \\ &= E\left[\frac{E[D|X]}{\Pr[D=1|X]}g(1,X)\right] \\ &= E[g(1,X)] \end{aligned}$$

New Moment is Insensitive

$$\widetilde{M}(\theta, g, a) = M(\theta, g) + E\left[a(D, X)\left(Y - g(D, X)\right)\right]$$

- Take derivative with respect to g at θ_0, g_0, a_0 in any direction $\nu \in G$

$$\frac{\partial}{\partial t}\widetilde{M}(\theta_0, g_0 + t\,\nu, a_0)\Big|_{t=0} = E[\nu(1, X) - \nu(0, X)] + E[a(D, X)\,\nu(D, X)]$$
$$= 0$$

- Take derivative with respect to a at θ_0, g_0, a_0 in any direction $\nu \in A$

$$\frac{\partial}{\partial t}\widetilde{M}(\theta_0, g_0, a_0 + t\nu)\Big|_{t=0} = E\left[\nu(D, X)\left(Y - g_0(D, X)\right)\right]$$
$$= 0$$

Neyman Orthogonality

Formal Definition

- Moment $M(\theta, g)$ is Neyman orthogonal if for any $\nu \in G - g_0$:

$$D_g M(\theta_0, g_0)[\nu] \coloneqq \frac{\partial}{\partial t} M(\theta_0, g_0 + t\,\nu)\Big|_{t=0} = 0$$

Sample-Splitting Estimation Algorithm

- Split the data in half
- On first half, estimate $\hat{g}$ of g_0
- On second half, solution $\hat{\theta}$ to empirical plug-in moment equation:

$$M_n(\hat{\theta}, \hat{g}) \coloneqq \frac{1}{n_2} \sum_{i \in S_2} m(Z_i; \hat{\theta}, \hat{g}) = 0$$

Main Theorem

- If moment is Neyman orthogonal and RMSE of $\hat{g}$ is $o_p\left(n^{-1/4}\right)*$

$$\sqrt{n}\,\left(\hat{\theta}-\theta_0\right) \rightarrow N\left(0, A^{-1}\Sigma\left(\mathrm{A}^{-1}\right)^{\top}\right)$$

- $A = \nabla_\theta M(\theta_0, g_0)$ and $\Sigma = E\left[m(Z;\theta_0, g_0)\, m(Z;\theta_0, g_0)^{\top}\right]$

*plug regularity conditions

Continuous Treatments under Partial Linearity

Partially Linear Model

- Relevant in many applications: dose-response curve in healthcare, effect of price on demand, return-on-investment
- Assume conditional exogeneity
$$Y^{(d)} \perp D \mid X$$
- Assume partially linear response
$$g_0(D, X) = E[\, Y \mid D, X \,] = \theta_0 D + f_0(X)$$
- Parameter of interest θ_0 is constant marginal effect of treatment

Partially Linear Model

- By definition of CEF we have the decomposition
$$Y = g_0(D, X) + \epsilon = \theta_0 D + f_0(X) + \epsilon, \qquad E[\epsilon | D, X] = 0$$

- By definition of g_0, for any function $q(D, X)$
$$E\left[\left(Y - \theta_0 D - f_0(X)\right) q(D, X)\right] = E[E[Y - g_0(D, X) \mid D, X]\, q(D, X)] = 0$$

- Direct non-orthogonal method, estimate $\hat{f}$ and solve:
$$E\left[\left(Y - \theta D - \hat{f}(X)\right) D\right] = 0$$

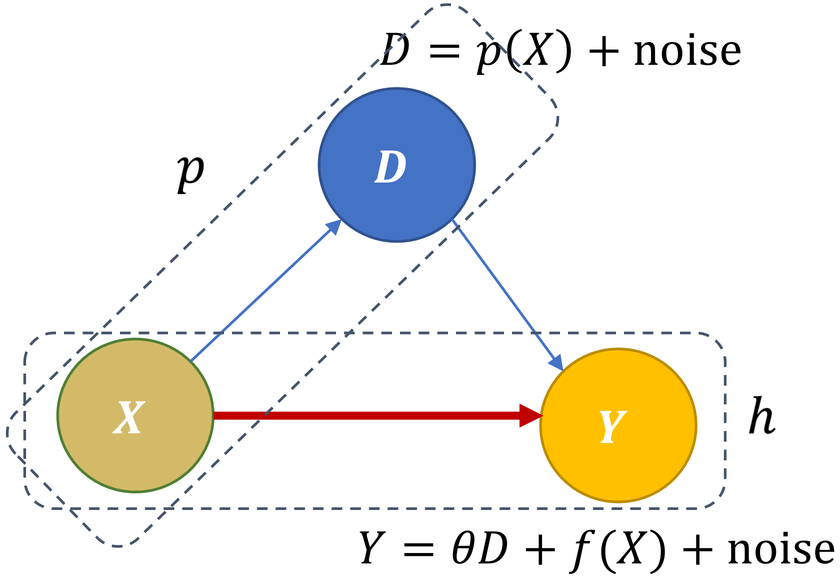
Generalization of FWL Theorem

- Let's define a slight variant of residualization
$$\tilde{V} = V - E[V|X]$$
- Generalization of FWL theorem to partially linear models
$$\tilde{Y} = \theta_0 \widetilde{D} + \epsilon, \qquad E\left[\epsilon \middle| \widetilde{D}\right] = 0$$
- Let's consider the residual outcome

$$\begin{aligned}
\tilde{Y} &= Y - E[Y|X] \\
&= \theta_0 D + f_0(X) + \epsilon - E[\theta_0 D + f_0(X) + \epsilon|X] \\
&= \theta_0 D + f_0(X) + \epsilon - \theta_0 E[D|X] - f_0(X) \\
&= \theta_0 (D - E[D|X]) + \epsilon
\end{aligned}$$

Orthogonal Method: Double ML

- Double ML. Split samples in half
  - Regress $Y \sim X$ with ML on first half, to get estimate $\hat{h}(S)$ of $E[Y|X]$
  - Regress $D \sim X$ with ML on first half, to get estimate $\hat{p}(S)$ of $E[D|X]$



Orthogonal Method: Double ML

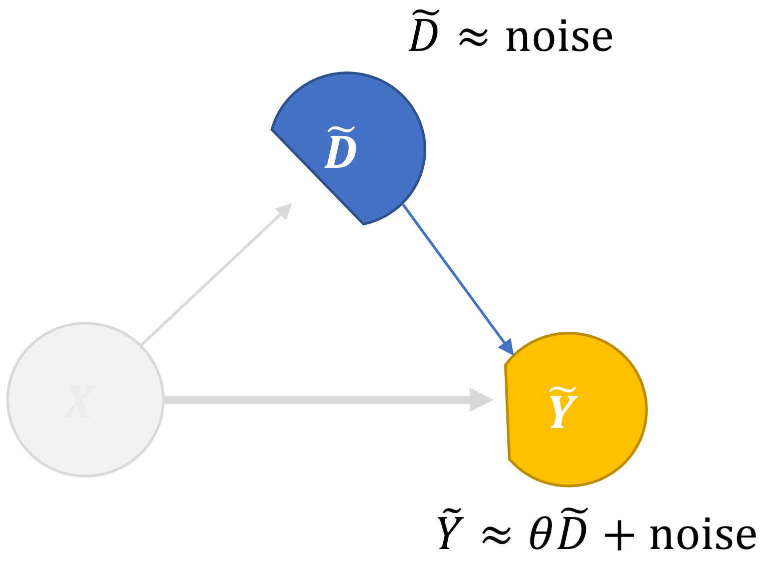
- Double ML. Split samples in half
  - Regress $Y \sim X$ with ML on first half, to get estimate $\hat{h}(S)$ of $E[Y|X]$
  - Regress $D \sim X$ with ML on first half, to get estimate $\hat{p}(S)$ of $E[D|X]$
  - Construct residuals on other half, $\widetilde{D} \coloneqq D - \hat{p}(X)$ and $\tilde{Y} \coloneqq Y - \hat{h}(X)$
  - Run OLS on residuals: $\tilde{Y} \sim \widetilde{D}$ to get $\hat{\theta}$

- OLS equivalent to solving moment condition:

$$E\left[\left(\tilde{Y} - \theta\widetilde{D}\right)\widetilde{D}\right] = 0$$

- Orthogonal Moment condition:

$$M(\theta, h, p) = E\left[\left(Y - h(X) - \theta\left(D - p(X)\right)\right)\ \left(D - p(X)\right)\right]$$



Practical Variants of Sample-Splitting

Cross-fitting and semi-cross-fitting

Cross-fitting

- Sample splitting is statistically lossy

- Only half of the data are used for the final parameter estimation
- Can we utilize all the data?

- *Cross-fitting:* analogous to cross-validation
- Use the second half to train g and predict on first half
- Then calculate parameter using all the data

Cross-fitting Estimation Algorithm

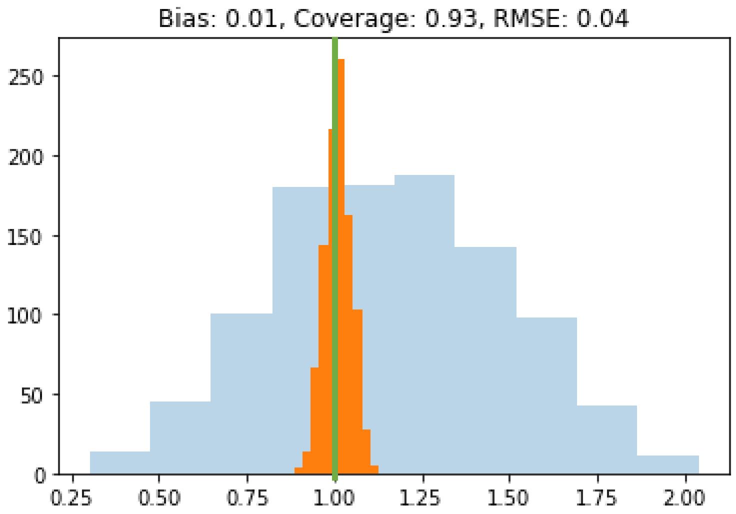
- Split the data in half
- On first half, estimate $\hat{g}_1$ of g_0 and predict on second half
- On second half, estimate $\hat{g}_2$ of g_0 and predict on first half
- On all data, solution $\hat{\theta}$ to empirical plug-in moment equation:

$$M_n(\hat{\theta}, \hat{g}) \coloneqq \frac{1}{n} \sum_{i \in S_1} m(Z_i; \hat{\theta}, \hat{g}_2) + \frac{1}{n} \sum_{i \in S_2} m(Z_i; \hat{\theta}, \hat{g}_1) = 0$$

- In practice do this with $K \approx 3 \text{ to } 5$ folds: for each fold k train on all other folds and predict on fold k

Natural Algorithm (Draft 3) Gone Right

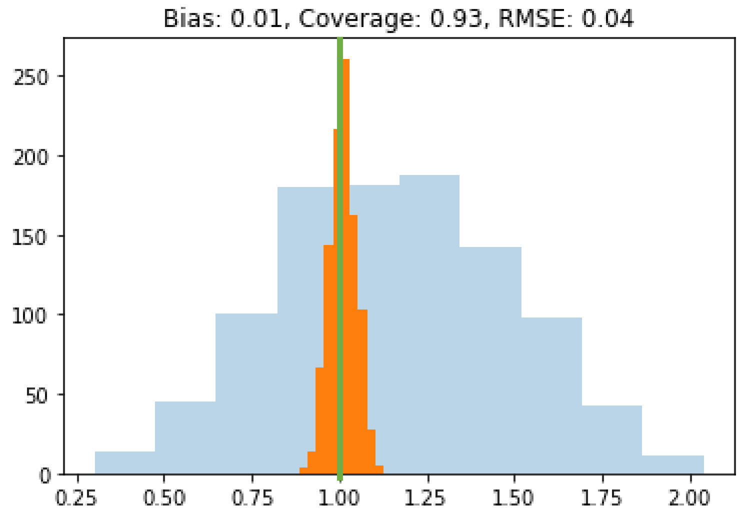
```
def dml2(X, D, y): # orthogonal dml with sample-splitting
    est_y = RandomForestRegressor(min_samples_leaf=20)
    yres = y - cross_val_predict(est_y, X, y, cv=3)
    est_t = RandomForestRegressor(min_samples_leaf=20)
    Dres = D - cross_val_predict(est_t, X, D, cv=3)
    theta = np.mean(yres * Dres) / np.mean(Dres**2)
    var = np.mean((Dres**2) * (yres - theta*Dres)**2) / np.mean(Dres**2)
    stderr = np.sqrt(var / X.shape[0])
    return theta, stderr
```



Natural Algorithm (Draft 3) Gone Right

```
from econml.dml import LinearDML

dml = LinearDML(model_y=RandomForestRegressor(min_samples_leaf=20),
                model_t=RandomForestRegressor(min_samples_leaf=20))
est.fit(y, D, W=X).effect_inference()
```



Stacking and Model Selection

- If we want to choose among many models or perform stacking, we can just use a stacked or automl model in place of each ML model

Stacking ML Models

```
def dml2(X, D, y): # orthogonal dml with sample-splitting
    est_y = StackingRegressor([rf, nnet, gbf, lasso])
    yres = y - cross_val_predict(est_y, X, y, cv=3)
    est_t = StackingRegressor([rf, nnet, gbf, lasso])
    Dres = D - cross_val_predict(est_t, X, D, cv=3)
    theta = np.mean(yres * Dres) / np.mean(Dres**2)
    var = np.mean((Dres**2) * (yres - theta*Dres)**2) / np.mean(Dres**2)
    stderr = np.sqrt(var / X.shape[0])
    return theta, stderr
```

AutoML Models

```python
from flaml import AutoML

def dml2(X, D, y): # orthogonal dml with sample-splitting
    est_y = AutoML()
    yres = y - cross_val_predict(est_y, X, y, cv=3)
    est_t = AutoML()
    Dres = D - cross_val_predict(est_t, X, D, cv=3)
    theta = np.mean(yres * Dres) / np.mean(Dres**2)
    var = np.mean((Dres**2) * (yres - theta*Dres)**2) / np.mean(Dres**2)
    stderr = np.sqrt(var / X.shape[0])
    return theta, stderr
```

Stacking and Model Selection

- If we want to choose among many models or perform stacking, we can just use a stacked or automl model in place of each ML model
- Model selection or stacking done many times within each training fold
- Computationally expensive and statistically lossy

- Can we use all the data to at least select among models?

Semi-Cross-fitting Estimation Algorithm

- Split the data in half *(in practice K folds)*
- On first half, estimate $\hat{g}_1^{(1)}, \dots, \hat{g}_1^{(\mathrm{L})}$ of g_0 and predict on second half
- On second half, estimate $\hat{g}_2^{(1)}, \dots, \hat{g}_2^{(\mathrm{L})}$ of g_0 and predict on first half
- Choose the model $\ell \in \{1, \dots, L\}$ that optimizes out-of-sample RMSE
- On all data, solution $\hat{\theta}$ to empirical plug-in moment equation:

$$M_n(\hat{\theta}, \hat{g}) := \frac{1}{n} \sum_{i \in S_1} m\left(Z_i; \hat{\theta}, \hat{g}_2^{(\ell)}\right) + \frac{1}{n} \sum_{i \in S_2} m\left(Z_i; \hat{\theta}, \hat{g}_1^{(\ell)}\right) = 0$$

Semi-Crossfitting

```
def dml2(X, D, y): # orthogonal dml with semi-crossfitting
    # cross val predict with many models
    est_y = [rf, gbf, lasso]
    yres = np.array([y - cross_val_predict(est, X, y, cv=3) for est in est_y])
    est_d = [rf, gbf, lasso]
    Dres = np.array([D - cross_val_predict(est, X, D, cv=3) for est in est_d])
    # select models with best out of fold performance
    best_y = np.argmin(np.mean(yres**2, axis=1))
    best_d = np.argmin(np.mean(Dres**2, axis=1))
    yres = yres[best_y]
    Dres = Dres[best_d]
    # go with their corresponding residuals
    theta = np.mean(yres * Dres) / np.mean(Dres**2)
    var = np.mean((Dres**2) * (yres - theta*Dres)**2) / np.mean(Dres**2)
    stderr = np.sqrt(var / X.shape[0])
    return theta, stderr
```

Semi-Crossfitting

- If the number of models L is small, then “spillover” is ok and approach still works. For practical purposes L should be thought as constant.
- Under further regularity, provably asymptotic normality holds if

$$\sqrt{\log(L)} = \mathrm{o}\left(\mathrm{n}^{1/4}\right)$$

Semi-Cross-fitting with Stacking

- Split the data in half *(in practice K folds)*
- On first half, estimate $\hat{g}_1^{(1)}, \dots, \hat{g}_1^{(\mathrm{L})}$ of g_0 and predict on second half
- On second half, estimate $\hat{g}_2^{(1)}, \dots, \hat{g}_2^{(\mathrm{L})}$ of g_0 and predict on first half
- Construct weights $\alpha_1, \dots, \alpha_\ell$ on the models using all the data (stacking)

- On all data, solution $\hat{\theta}$ to empirical plug-in moment equation:

$$M_n(\hat{\theta}, \hat{g}) := \frac{1}{n} \sum_{i \in S_1} m\left(Z_i; \hat{\theta}, \hat{g}_2^{(\ell)}\right) + \frac{1}{n} \sum_{i \in S_2} m\left(Z_i; \hat{\theta}, \hat{g}_1^{(\ell)}\right) = 0$$

Semi-Crossfitting with Stacking

```python
def dml2(X, D, y): # orthogonal dml with semi-crossfitting and stacking
    # cross val predict with many models
    est_y = [rf, gbf, lasso]
    ypreds = np.array([cross_val_predict(est, X, y, cv=3) for est in est_y]).T
    est_d = [rf, gbf, lasso]
    Dpreds = np.array([cross_val_predict(est, X, D, cv=3) for est in est_d]).T
    # calculate stacked residuals by finding optimal coefficients
    # and weigthing out-of-sample predictions by these coefficients
    yres = y - LinearRegression().fit(ypreds, y).predict(ypreds)
    Dres = D - LinearRegression().fit(Dpreds, D).predict(Dpreds)
    # go with the stacked residuals
    theta = np.mean(yres * Dres) / np.mean(Dres**2)
    var = np.mean((Dres**2) * (yres - theta*Dres)**2) / np.mean(Dres**2)
    stderr = np.sqrt(var / X.shape[0])
    return theta, stderr
```

Semi-Crossfitting

- If the number of models L is small, then "spillover" is ok and approach still works. For practical purposes L should be thought as constant.
- Under further regularity, provably asymptotic normality holds if

$$\sqrt{L} = \mathrm{o}\left(\mathrm{n}^{1/4}\right)$$

Equivalent view of cross-fitting with stacking (lens of FWL theorem)

- Construct out of fold predictions based on many ML models
- Use these predictions as engineered features X in a simple OLS regression on D, X
- Use the coefficient and standard error of D from this final OLS

Proving the Main Theorem

Linear in θ Moments

- We will restrict attention to a broad class that simplifies proof

- Moment is linear in target parameter

$$m(Z; \theta, g) = a(Z; g)'\theta + \nu(Z; g)$$

- Expected moment also linear in θ

$$M(\theta, g) = A(g)'\theta + V(g)$$

Proof Ingredients: Linear in θ Moments

- Since $M_n(\hat{\theta}, \hat{g}) = 0$ we expect by concentration and sample splitting $M(\hat{\theta}, \hat{g}) \approx n^{-1/2}$
- Since $M(\theta_0, g_0) = 0$ we expect by Neyman orthogonality $M(\theta_0, \hat{g}) \approx RMSE(\hat{g})^2 = o(n^{-1/2})$
- Since moment is linear in θ: $A(\hat{g})\,(\hat{\theta} - \theta_0) = M(\hat{\theta}, \hat{g}) - M(\theta_0, \hat{g})$
- Since A is Lipschitz and $\hat{g} \to g_0$: $A(g_0)\,(\hat{\theta} - \theta_0) = M(\hat{\theta}, \hat{g}) - M(\theta_0, \hat{g}) + \mathrm{o_p}(\|\hat{\theta} - \theta_0\|)$
- Since $A(g_0)$ is invertible: $\|\hat{\theta} - \theta_0\| = \mathrm{O}(\|M(\hat{\theta}, \hat{g})\| + \|M(\theta_0, \hat{g})\|) = O_p(n^{-1/2})$

- More fine-grained analysis of $M(\hat{\theta}, \hat{g})$ term, shows: $\sqrt{n}M(\hat{\theta}, \hat{g}) \to \mathrm{N}(0, \mathrm{V})$

Proof of Main Theorem

- Since moment $M(\theta, g)$ is linear in θ and $M_n(\hat{\theta}, \hat{g}) = 0$ and $M(\theta_0, g_0) = 0$

$$
\begin{aligned}
A(\hat{g})\left(\hat{\theta} - \theta_0\right) &= M\left(\hat{\theta}, \hat{g}\right) - M(\theta_0, \hat{g}) \\
&= M\left(\hat{\theta}, \hat{g}\right) - M_n\left(\hat{\theta}, \hat{g}\right) + M(\theta_0, g_0) - M(\theta_0, \hat{g})
\end{aligned}
$$

- Since $\mathrm{RMSE}(\hat{g}) = \|\hat{g} - g_0\| = o_p(1)$

$$A(g_0)\left(\hat{\theta} - \theta_0\right) = A(\hat{g})\left(\hat{\theta} - \theta_0\right) + \mathrm{o_p}\left(\|\hat{\theta} - \theta_0\|\right)$$

- Thus

$$A(g_0)\left(\hat{\theta} - \theta_0\right) = \underbrace{M\left(\hat{\theta}, \hat{g}\right) - M_n\left(\hat{\theta}, \hat{g}\right)}_{\to N(0,V)} + \underbrace{M(\theta_0, g_0) - M(\theta_0, \hat{g})}_{= o_p\left(n^{-1/2}\right)} + \mathrm{o_p}\left(\|\hat{\theta} - \theta_0\|\right)$$

$\to N(0, V)$
via CLT
+ sample-splitting
+ concentration
+ $\hat{g} \to g_0$

$= o_p\left(n^{-1/2}\right)$
via orthogonality
+ $\|\hat{g} - g_0\| = o_p\left(n^{-1/4}\right)$

Proof of Main Theorem: Orthogonality

- By Neyman orthogonality and bounded second derivative of $M(\theta_0, g)$ w.r.t. g
$$M(\theta_0, g_0) - M(\theta_0, \hat{g}) = D_g M(\theta_0, g_0)[g_0 - g] + O\left(\|\hat{g} - g_0\|^2\right) = o_p\left(n^{-1/2}\right)$$

- Thus
$$A(g_0)\left(\hat{\theta} - \theta_0\right) = \underbrace{M\left(\hat{\theta}, \hat{g}\right) - M_n\left(\hat{\theta}, \hat{g}\right)}_{G_n(\hat{\theta}, \hat{g})} + \mathrm{o_p}\left(n^{-1/2} + \left\|\hat{\theta} - \theta_0\right\|\right)$$

Proof of Main Theorem: Sample-Splitting (1)

- Let $G_n(\theta, g) = M(\theta, g) - M_n(\theta, g)$
$$G_n(\hat{\theta}, \hat{g}) = G_n(\theta_0, g_0) + G_n(\hat{\theta}, \hat{g}) - G_n(\theta_0, \hat{g}) + G_n(\theta_0, \hat{g}) - G_n(\theta_0, g_0)$$

- Linearity of moment + (sample-splitting and concentration $\Rightarrow$ $\|A(\hat{g}) - A_n(\hat{g})\| = o_p(1)$):
$$G_n(\hat{\theta}, \hat{g}) - G_n(\theta_0, \hat{g}) = \left(A(\hat{g}) - A_n(\hat{g})\right)\left(\hat{\theta} - \theta_0\right) = o_p\left(\left\|\hat{\theta} - \theta_0\right\|\right)$$

- Thus
$$A(g_0)\left(\hat{\theta} - \theta_0\right) = G_n(\theta_0, g_0) + G_n(\theta_0, \hat{g}) - G_n(\theta_0, g_0) + \mathrm{o_p}\left(n^{-1/2} + \left\|\hat{\theta} - \theta_0\right\|\right)$$

Proof of Main Theorem: Sample-Splitting (2)

- Note for $X_i = m(Z_i; \theta_0, \hat{g}) - m(Z_i; \theta_0, \hat{g}) - E[m(Z_i; \theta_0, \hat{g}) - m(Z_i; \theta_0, \hat{g})]$

$$G_n(\theta_0, \hat{g}) - G_n(\theta_0, g_0) = \frac{1}{n_2} \sum_{i \in S_2} X_i$$

- By sample splitting, X_i are i.i.d. with $E[X_i] = 0$. By variance decomposition (concentration)

$$\left\| \frac{1}{n_2} \sum_{i \in S_2} X_i \right\|_{L_2} \leq \sqrt{\frac{E\left[X_i^2\right]}{n}}$$

- Thus

$$\|G_n(\theta_0, \hat{g}) - G_n(\theta_0, g_0)\|_{L_2} \leq \frac{1}{\sqrt{n}} \sqrt{E\left[\left(m(Z_i; \theta_0, \hat{g}) - m(Z_i; \theta_0, \hat{g})\right)^2\right]} = O\left(\frac{\|\hat{g} - g_0\|}{\sqrt{n}}\right) = o_p\left(n^{-1/2}\right)$$

Concluding

- So far

$$A(g_0)\left(\hat{\theta}-\theta_0\right) = G_n(\theta_0, g_0) + \mathrm{o_p}\left(n^{-1/2} + \left\|\hat{\theta}-\theta_0\right\|\right)$$

- Since $A(g_0)$ is invertible and $G_n(\theta_0, g_0) = O_p\left(n^{-1/2}\right)$ by concentration

$$\left\|\hat{\theta}-\theta_0\right\| = O_p\left(n^{-1/2}\right)$$

- Thus, we have asymptotic linearity

$$\sqrt{n}\left(\hat{\theta}-\theta_0\right) = \sqrt{n}A(g_0)^{-1}G_n(\theta_0, g_0) + \mathrm{o_p}(1) = \frac{1}{\sqrt{n_2}}\sum_{i\in S_2} A(g_0)^{-1} m(Z_i; \theta_0, g_0) + o_p(1)$$

- By CLT we get the theorem

Main Theorem

- If moment is Neyman orthogonal and RMSE of $\hat{g}$ is $o_p\left(n^{-1/4}\right)*$

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) \to N\left(0, A^{-1}\Sigma\left(\mathrm{A}^{-1}\right)^{\top}\right)$$

- $A = \nabla_\theta M(\theta_0, g_0)$ and $\Sigma = E\left[m(Z;\theta_0, g_0)\, m(Z;\theta_0, g_0)^{\top}\right]$

*plus regularity conditions