# MS&E 228: Unobserved Confounding and Instruments

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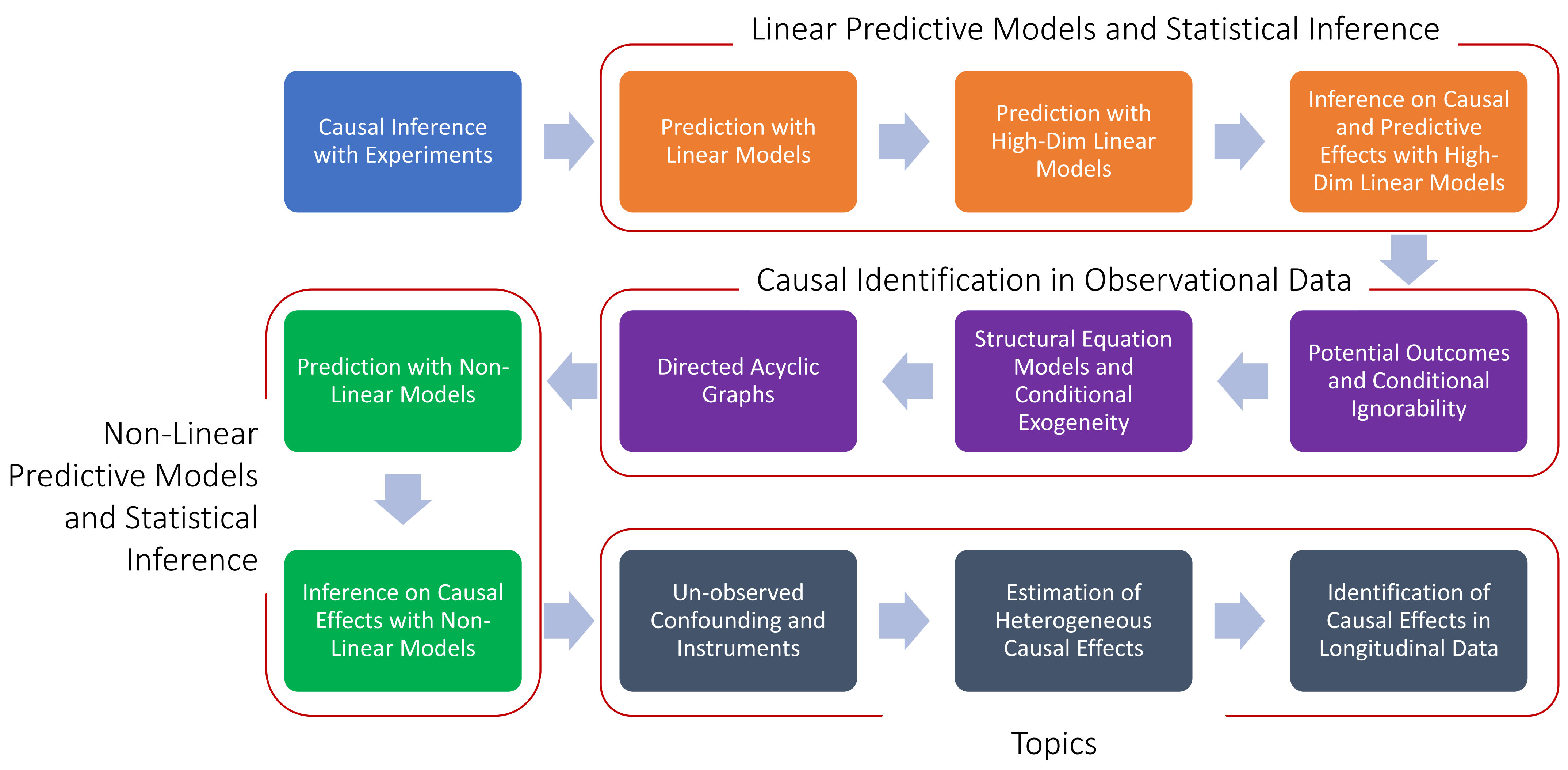
Linear Predictive Models and Statistical Inference
Causal Inference with Experiments
Prediction with Linear Models
Prediction with High-Dim Linear Models
Inference on Causal and Predictive Effects with High-Dim Linear Models
Causal Identification in Observational Data
Prediction with Non-Linear Models
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Structural Equation Models and Conditional Exogeneity
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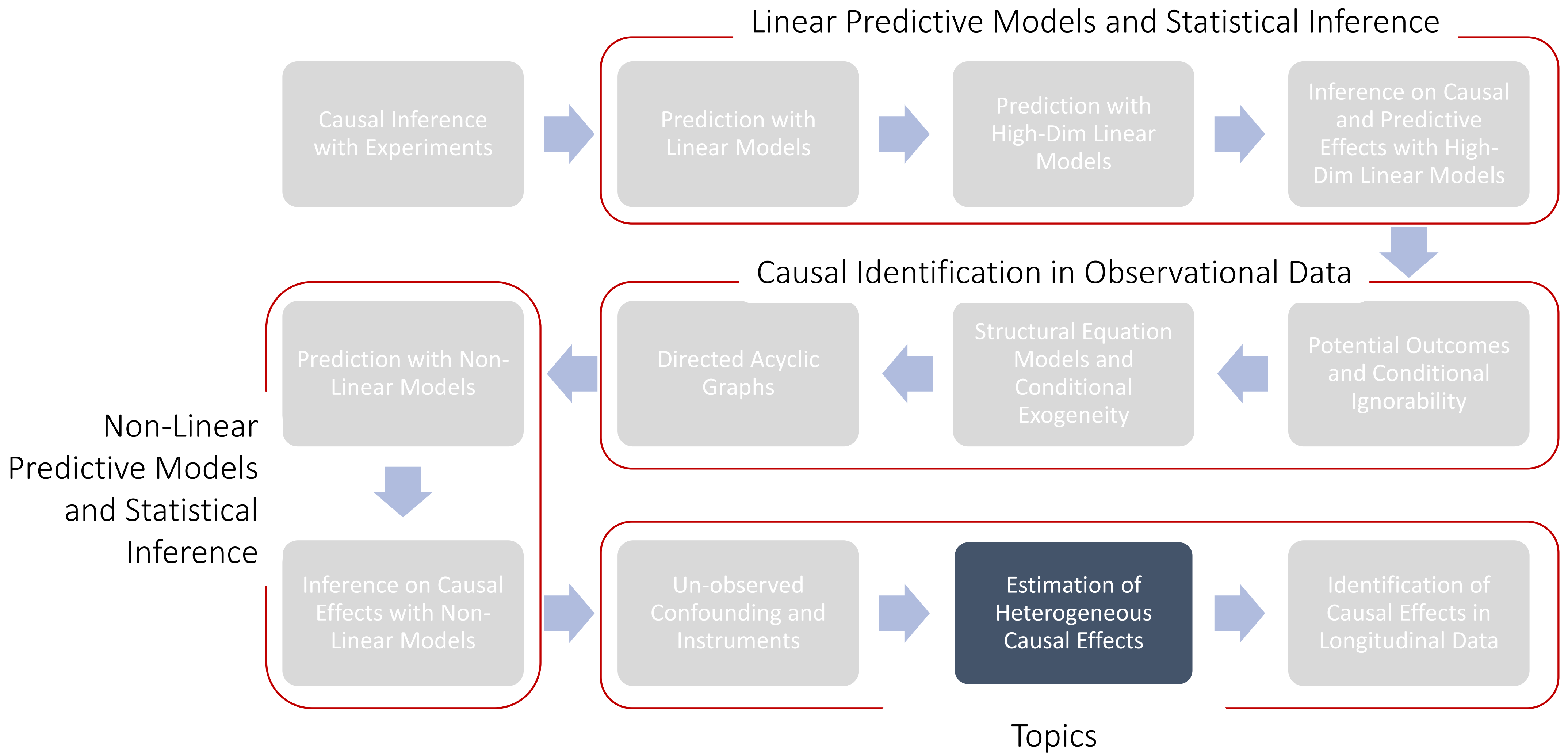
Un-observed Confounding and Instruments
Estimation of Heterogeneous Causal Effects
Identification of Causal Effects in Longitudinal Data
Topics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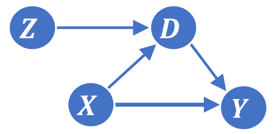
# Goals for Today

- Heterogeneous Treatment Effects
- Statement of the problem
- A basic solution

# Causal Inference Pipeline

## Theory

DAGs/ASEMs




$$\alpha = \frac{E[\tilde{Y}\tilde{Z}]}{E[\tilde{D}\tilde{Z}]}$$

Estimate approximate residuals
$\check{Y}, \check{Z}, \breve{D}$ using cross-fitting
Solve moment restriction

$$E_n[(\check{Y} - \hat{a}\breve{D})\check{Z}] = 0$$

$$a \in \hat{a} \pm 2\hat{\sigma}$$

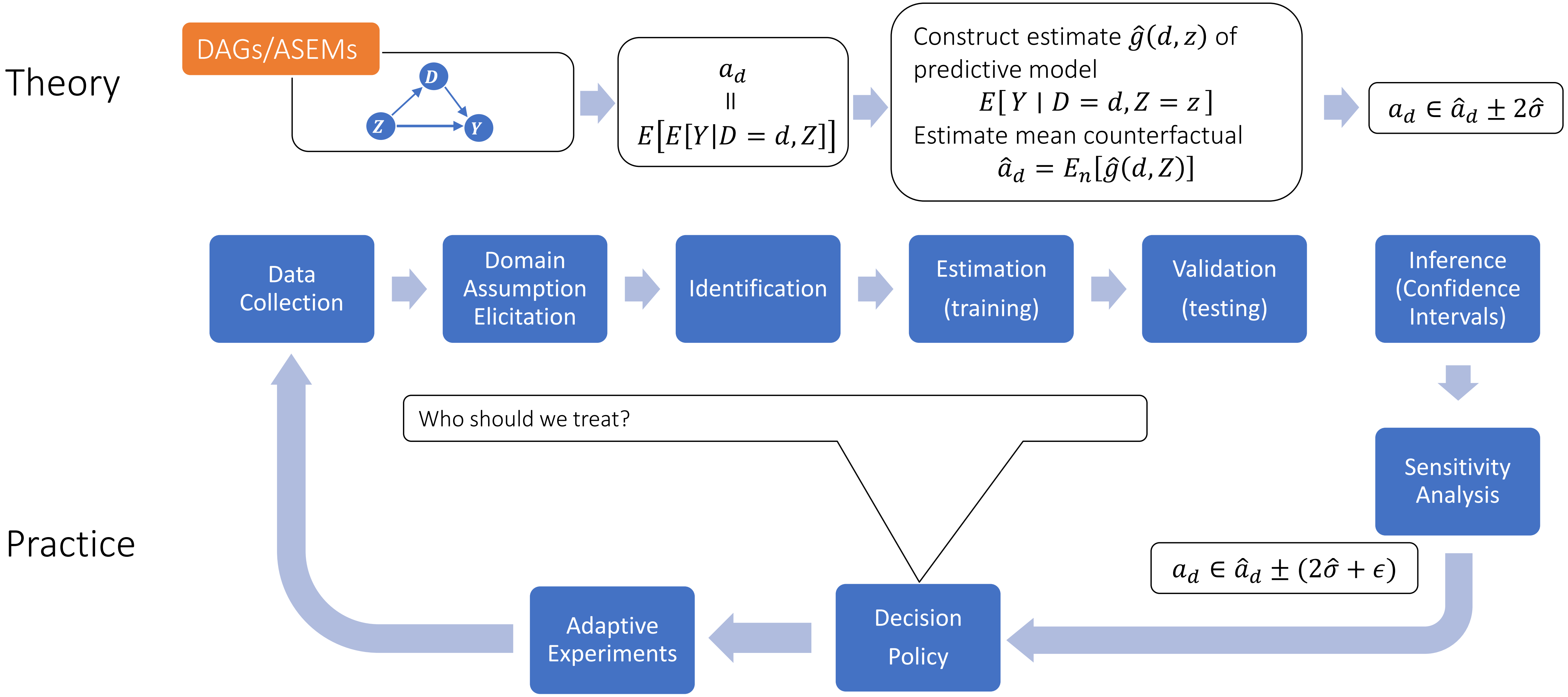
## Practice

Data Collection → Domain Assumption Elicitation → Identification → Estimation (training) → Validation (testing) → Inference (Confidence Intervals) → Sensitivity Analysis → Decision Policy → Adaptive Experiments → Data Collection

**Instrumental Variable:** any random variable Z that affects the treatment (log-price) D but does not affect the outcome (log-demand) Y other than through the treatment

$$a \in \hat{a} \pm (2\hat{\sigma} + \epsilon)$$

# Causal Inference Pipeline



# Conditional Average Treatment Effects (CATE)

aka Heterogeneous Treatment Effects

# Problem with Average Treatment Effect

- So far, we mostly focused on understanding average treatment effects
$$\theta = E[Y(1) - Y(0)]$$

- This quantity is not informative of who to treat
- At best we can use it to make a uniform decision for the population
$$\text{treat everyone if } \theta > 0 \text{ and don't treat otherwise}$$

- Such uniform policies can lead to severe adverse effects
- Such uniform analyses can lead us to miss on “responder subgroups”

# Personalized (Refined) Policies

- To understand who to treat, we need to learn how effect varies
- Conditional Average Treatment Effect

$$\theta(x) = E[\,Y(1) - Y(0) \mid X = x\,]$$

- Allows us to understand differences (heterogeneities) in the response to treatment for different parts of the population
- We can deploy more refined “personalized” policies
- For every person that comes, we observe an $\mathrm{X} = x$ and decide

$$\text{treat if } \theta(x) > 0 \text{ else don't treat}$$

# The intrinsic hardness of CATE

- The CATE quantity is not just a parameter
- It is a whole function…
- Learning such conditional expectation functions is inherently harder than learning parameters
- For instance: we might never have seen in our data other samples with the exact same $x$
- Such quantities are known as statistically “irregular” quantities
- We have seen such quantities when were solving the best prediction rule $E[Y|X]$

# The intrinsic hardness of CATE

- Estimating CATE at least as hard as estimating the best prediction rule
- Inherently harder than estimating an “average”
- So far for our target causal quantities we wanted fast estimation rates and confidence intervals
- We were only ok with “decent” estimation rates for the auxiliary (nuisance) predictive models that entered our analysis

- We might want to relax our goals…

# Different Approaches to Relaxing our Goals

- Goal 1: Maybe estimate a simpler projection (e.g. analogue of BLP)
- Goal 2: Confidence intervals for predictions of this simple projection
- Goal 3: Simultaneous confidence bands for predictions of this simple projection
- Goal 4: Estimation error rate for the true CATE
- Goal 5: Confidence intervals for the prediction of a CATE model
- Goal 6: Simultaneous confidence bands for joint predictions of CATE model

Linear Doubly Robust Learner

**Meta-learner** approaches: S-Learner, T-Learner, X-Learner, R-Learner, DR-Learner
**Neural Network** approaches: TARNet, CFR
**Random Forest** approaches: BART

**Modified (honest) ML** methods: Generalized Random Forest, Orthogonal Random Forest, Sub-sampled Nearest Neighbor Regression

**??** (only classical non-parametric statistic results on confidence bands of non-parametric functions)

Policy Learning

- Goal 7: Go after optimal simple treatment policies; give me a policy with value close to the best
- Goal 8: Inference on value of candidate treatment policies
- Goal 9: Inference on value of optimal policy
- Goal 10: Identify responder or heterogeneous sub-groups; policies with statistical significance;

Doubly Robust Policy Evaluation

Doubly Robust Policy Learning

# Different Approaches to Relaxing our Goals

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Doubly Robust Policy Evaluation

Doubly Robust Policy Learning

# Best Linear Projection of CATE

# Identification by Conditioning

- Under conditional ignorability

$$Y(1), Y(0) \perp\!\!\!\perp D \mid Z$$

- CATE can be identified by conditioning

$$\alpha(Z) \coloneqq E[Y(1) - Y(0)|Z] = E[Y|D = 1, Z] - E[Y|D = 0, Z] = \pi(Z)$$

- If we want a CATE on some subset of variables $X$

$$\theta(X) = E[\,\alpha(Z) \mid X\,] = E[\pi(Z) \mid X]$$

# Identification with Propensity Scores

- Under conditional ignorability
$$Y(1), Y(0) \perp\!\!\!\perp D \mid Z$$
- CATE can be identified by propensity scores

$$\alpha(Z) \coloneqq E[Y(1) - Y(0)|Z] = E[Y\, H(D,Z)|Z] = \pi(Z)$$
$$H(D,Z) = \frac{D}{\Pr(D=1|Z)} - \frac{1-D}{1-\Pr(D=1|Z)}$$

- If we want a CATE on some subset of variables $X$
$$\theta(X) = E[\,\alpha(Z) \mid X\,] = E[\pi(Z) \mid X]$$

# Doubly Robust Identification

- Under conditional ignorability

$$Y(1), Y(0) \perp\!\!\!\perp D \mid Z$$

- CATE can be identified by combination of conditioning and propensity scores

$$a(Z) \coloneqq E\big[\, g(1,Z) - g(0,Z) - H(D,Z)\left(Y - g(D,Z)\right) \mid Z \,\big] = \pi(Z)$$

$$H(D,Z) = \frac{D}{p(Z)} - \frac{1-D}{1-p(Z)}$$

$$g(D,Z) \coloneqq E[Y|D,Z], \qquad p(Z) \coloneqq \Pr(D=1|Z)$$

- If we want a CATE on some subset of variables $X$

$$\theta(X) = E[\pi(Z) \mid X] = E\big[\, g(1,Z) - g(0,Z) - H(D,Z)\left(Y - g(D,Z)\right) \mid X \,\big]$$

# From Identification to Estimation

- If we knew the propensity or regression, we have a random variable
$$Y_{DR}(g,p) \coloneqq g(1,Z) - g(0,Z) - H(D,Z)\left(Y - g(D,Z)\right)$$

- Such that what we are looking for is the CEF
$$\theta(X) \coloneqq E[Y_{DR}(g,p)|X]$$

- In the non-linear prediction section, we saw that this is the solution to the Best Prediction rule problem!

# **Blast from the Past:** Best Prediction Rule

- Given $n$ samples $(Z_1, Y_1), \dots, (Z_n, Y_n)$ drawn iid from a distribution $D$
- Want an estimate $\hat{g}$ that approximates the Best Prediction

$$g \coloneqq \operatorname*{argmin}_{\tilde{g}} E\left[\left(Y - \tilde{g}(Z)\right)^2\right]$$

- Best Prediction rule is Conditional Expectation Function (CEF)

$$g(Z) = E[Y|Z]$$

- We want our estimate $\tilde{g}$ to be close to $g$ in RMSE

$$\|\hat{g} - g\| = \sqrt{E_Z\left(\hat{g}(S) - g(Z)\right)^2} \rightarrow 0, \qquad \text{as } n \rightarrow \infty$$

# **Blast from the Past:** Linear CEF

- If CEF is assumed linear with respect to known engineered features
$$E[\, Y \mid Z \,] = \beta' \psi(Z)$$

- Then the Best Prediction rule (CEF) coincides with the Best Linear Prediction rule (BLP)

- We can use OLS if $\psi(Z)$ is low-dimensional (p≪n) or the multitude of approaches we learned if $\psi(Z)$ is high-dimensional (Lasso, ElasticNet, Ridge, Lava)

# From Identification to Estimation

- If we knew the propensity or regression, we have a random variable

$$Y_{DR}(g,p) \coloneqq g(1,Z) - g(0,Z) - H(D,Z)\left(Y - g(D,Z)\right)$$

- Such that what we are looking for is the CEF

$$\theta(X) \coloneqq E[Y_{DR}(g,p)|X]$$

- We can reduce CATE estimation to a Best Prediction rule problem!

$$\theta \coloneqq \operatorname*{argmin}_{g} E\left[\left(Y_{DR}(g,p) - g(X)\right)^2\right]$$

- ML techniques can be used to solve this problem and provide RMSE rates

$$\sqrt{E\left[\left(\theta(X) - \hat{\theta}(X)\right)^2\right]} \approx 0$$

# Doubly Robust Learning

[Foster, Syrgkanis, '19 Orthogonal Statistical Learning]

- Split your data in half
  - Train ML model $\hat{g}$ for $g_0(D,Z) \triangleq E[Y|D,Z]$ on the first, predict on the second and calculate regression estimate of each potential outcome

$$\tilde{Y}_i^{(d)} = \hat{g}(d, Z_i)$$

    and vice versa

  - Train ML classification model $\hat{p}_d$ for $p_d(Z) \triangleq Pr[D = d\,|Z]$ on the first, predict on the second, calculate propensity $\hat{p}_{d,i} = \Pr[D = d|Z_i]$ and vice versa
- Calculate doubly robust values:

$$\tilde{Y}_{i,DR}^{(d)} = \tilde{Y}_i^{(d)} + \frac{\left(Y_i - \tilde{Y}_i^{(D_i)}\right) 1\{D_i = d\}}{\hat{p}_{d,i}}$$

- Any ML algorithm to solve the regression:

$$\tilde{Y}_{i,DR}^{(1)} - \tilde{Y}_{i,DR}^{(0)} \quad \sim \quad X$$

# **Blast from the Past:** Best Linear Prediction (BLP) Problem

- The BLP minimizes the MSE

$$\min_{b \in \mathbb{R}^p} E\left[\left(Y - b'\psi(X)\right)^2\right]$$

- Since by the variance decomposition

$$E\left[\left(Y - b'\psi(X)\right)^2\right] = E\left[(Y - E[Y|X])^2\right] + E\left[\left(E[Y|X] - b'\psi(X)\right)^2\right]$$

- First part does not depend on $b$. The BLP minimizes

$$\min_{b \in \mathbb{R}^p} E\left[\left(E[Y|X] - b'\psi(X)\right)^2\right]$$

- The BLP is the **best linear approximation of the CEF**

# From Identification to Estimation

- If we knew the propensity or regression, we have a random variable
$$Y_{DR}(g,p) \coloneqq g(1,Z) - g(0,Z) - H(D,Z)\left(Y - g(D,Z)\right)$$
- Such that what we are looking for is the CEF
$$\theta(X) \coloneqq E[Y_{DR}(g,p)|X]$$
- Estimate best linear approximation to the CATE via the BLP problem:
$$\beta \coloneqq \underset{b}{\operatorname{argmin}}\, E\left[\left(Y_{DR}(g,p) - b'\psi(X)\right)^2\right]$$

$$\theta_{BLP}(X) = \beta'\psi(X)$$

# Normal Equations

- Equivalently, the solution to the normal equations

$$E\left[\left(Y_{DR}(g,p) - \beta'\psi(X)\right)\psi(X)\right] = 0$$

- Falls into the moment equation framework with nuisance components
- Nuisance components are $g, p$ and target parameter is $\beta$
- Moment is Neyman orthogonal with respect to $g, p$ (why?)
- Local insensitivity (orthogonality) holds even conditional on $X$

$$\lim_{\epsilon \to 0} \frac{E\left[\, Y_{DR}\left(g + \epsilon\, \nu_g, p + \epsilon\, \nu_p\right) \mid X \,\right] - E[\, Y_{DR}(g,p) \mid X \,]}{\epsilon} = 0$$

# Main Theorem (linear moments)

- If moments are linear
$$m(Z;\theta,g) = \nu(Z;g) - a(Z;g)\theta$$
- Estimate is closed form:
$$\hat{\theta} = \hat{J}^{-1}E_n[\nu(Z;g)], \qquad \hat{J} = E_n[a(Z;g)]$$
- Then the estimate $\hat{\theta}$ is *asymptotically linear*
$$\sqrt{n}\big(\hat{\theta} - \theta_0\big) \approx \sqrt{n}\, E_n[\phi_0(Z)], \qquad \phi_0(Z) = -J_0^{-1}\, m(Z;\theta_0,g_0), \qquad J_0 \coloneqq E[a(Z;\ g_0)]$$
- Consequently, it is *asymptotically normal*
$$\sqrt{n}\,\big(\hat{\theta} - \theta_0\big) \sim_a N(0,V), \qquad V \coloneqq E[\phi_0(Z)\phi_0(Z)']$$
- *Confidence intervals* for any projection based on estimate of variance are asymptotically valid
$$\ell'\theta \in \left[\ell'\hat{\theta} \pm c\sqrt{\frac{\ell'\hat{V}\ell}{n}}\right], \qquad \hat{\mathrm{V}} = \mathrm{Var_n}\left(\hat{\phi}(Z)\right), \qquad \hat{\phi}(Z) \coloneqq -\hat{J}^{-1}m\big(Z;\hat{\theta},\hat{g}\big), \qquad \hat{J} = E_n[a(Z;\hat{g})]$$

# Main Theorem (linear moments)

- If moments are linear

$$m(Z;\beta,g,p) = Y_{DR}(g,p)\psi(X) - \psi(X)\psi(X)'\theta$$

- Estimate is closed form:

$$\hat{\theta} = \hat{J}^{-1} E_n[Y_{DR}(g,p)\psi(X)], \qquad \hat{J} = E_n[\psi(X)\psi(X)']$$

- Then the estimate $\hat{\beta}$ is *asymptotically linear*

$$\sqrt{n}\big(\hat{\beta} - \beta_0\big) \approx \sqrt{n}\, E_n[\phi_0(Z)], \qquad \phi_0(Z) = -J_0^{-1}\, m(Z;\beta_0,g_0,p_0), \qquad J_0 \coloneqq E[\psi(X)\psi(X)']$$

- Consequently, it is *asymptotically normal*

$$\sqrt{n}\,\big(\hat{\beta} - \beta_0\big) \sim_a N(0,V), \qquad V \coloneqq E[\phi_0(Z)\phi_0(Z)']$$

- *Confidence intervals* for any projection based on estimate of variance are asymptotically valid

$$x'\beta \in \left[x'\hat{\beta} \pm c\sqrt{\frac{x'\hat{V}x}{n}}\right], \qquad \hat{\mathrm{V}} = \mathrm{Var_n}\left(\hat{\phi}(Z)\right), \qquad \hat{\phi}(Z) \coloneqq -\hat{J}^{-1} m\big(Z;\hat{\theta},\hat{g}\big), \qquad \hat{J} = E_n[\psi(X)\psi(X)']$$

# Confidence Bands

- Since $\hat{\beta}$ are asymptotically linear, predictions are asymptotically linear
- Then the estimate $\hat{\beta}$ is *asymptotically linear*

$$\sqrt{n}\big(\hat{\theta}_{BLP}(x) - \theta_{BLP}(x)\big) = \sqrt{n}\big(x'\hat{\beta} - x'\beta_0\big) \approx \sqrt{n}\, E_n[x'\phi_0(Z)]$$

- Holds jointly for all $x \in X$ (as long as $|X|$ not growing exponential in $n$)

$$\max_{x \in X} \left| \sqrt{n}\left(\hat{\theta}_{BLP}(x) - \theta_{BLP}(x)\right) - \sqrt{n}\, E_n[x'\phi_0(Z)] \right| \approx 0$$

- High-dimensional CLT theorems also imply that jointly:

$$\left\{\sqrt{n}\left(\hat{\theta}_{BLP}(x) - \theta_{BLP}(x)\right)\right\}_{x \in X} \sim_a N(0, V), \qquad V_{x_1 x_2} = E[x_1'\phi_0(Z)\phi_0(Z)x_2]$$

# Confidence Bands

- Similar to inference on many coefficients
- Now the many predictions take the role of the many coefficients
- Confidence band: construct intervals

$$CI(x) := \left[\hat{\theta}(x) \pm c\sqrt{\hat{V}_{xx}/n}\right]$$

- Such that

$$\Pr\big(\forall x:\ \theta(x) \in CI(x)\big) \to 1 - \alpha$$

# Confidence Bands

- Confidence band: construct intervals

$$CI(x) := \left[\hat{\theta}(x) \pm c\sqrt{\frac{\hat{V}_{xx}}{n}}\right], \qquad \Pr\big(\forall x:\ \theta(x) \in CI(x)\big) \to 1 - \alpha$$

- Note that

$$\Pr\big(\forall x:\ \theta(x) \in CI(x)\big) = \Pr\left(\max_{x \in X}\left|\frac{\sqrt{n}\left(\theta(x) - \hat{\theta}(x)\right)}{\sqrt{\hat{V}_{xx}}}\right| \le c\right)$$

- By Gaussian approximation, for $D = \operatorname{diag}(V)$

$$\Pr\left(\max_{x \in X}\left|\frac{\sqrt{n}\left(\theta(x) - \hat{\theta}(x)\right)}{\sqrt{\hat{V}_{xx}}}\right| \le c\right) \approx \Pr\left(\left\|N\left(0, D^{-1/2} V D^{-1/2}\right)\right\|_{\infty} \le c\right)$$

By Gaussian approximation, choose $c$ as the $1-\alpha$ quantile of the maximum entry in a gaussian vector drawn with covariance $D^{-1/2}VD^{-1/2}$

$$\mathrm{D} \coloneqq \mathrm{diag(V)} = \begin{bmatrix} V_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & V_{mm} \end{bmatrix}$$

For 95% confidence band, c slightly larger than 1.96

# Computationally Friendlier Version: Multiplier Bootstrap

- By asymptotic linearity we know that:

$$\frac{\sqrt{n}\left(\theta(x) - \hat{\theta}(x)\right)}{\sqrt{\hat{V}_{xx}}} \approx \sqrt{n}\, E_n\left[\frac{x'\phi_0(Z)}{\sqrt{V_{xx}}}\right]$$

- For every sample $i = 1 \dots n$, draw an independent Gaussian $\epsilon_i \sim N(0,1)$ and consider the variable

$$Q(x; \epsilon_1, \dots, \epsilon_n) \coloneqq \sqrt{n}\, E_n\left[\frac{x'\phi_0(Z)}{\sqrt{V_{xx}}}\epsilon\right] = \frac{1}{\sqrt{n}}\sum_i \frac{x'\phi_0(Z)}{\sqrt{V_{xx}}}\epsilon_i$$

- The vector of random variables $\left(Q(x_1), \dots, Q\left(x_{|X|}\right)\right) \sim_a N\left(0, D^{-1/2} V D^{-1/2}\right)$
- Approximately the same holds for $\left(\hat{Q}(x_1), \dots, \hat{Q}\left(x_{|X|}\right)\right)$ with $\hat{Q}(x; \epsilon_1, \dots, \epsilon_n) = \frac{1}{\sqrt{n}}\sum_i \frac{x'\hat{\phi}(Z)}{\sqrt{\hat{V}_{xx}}}\epsilon_i$

- **Repeat process $\boldsymbol{B}$ times:** each repetition $b$ draw vector $\epsilon_1^{(b)}, \dots, \epsilon_n^{(b)}$ and calculate maximum over $x$

$$Z^{(b)} \coloneqq \max_{x \in X}\left|\hat{Q}(x; \epsilon_1, \dots, \epsilon_n)\right|$$

- Set $c$ to be the $1 - \alpha$ quantile of $Z^{(b)}$ over the $B$ repetitions