Calculus III

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Contents

1 Three-Dimensional Space

In past math classes, we have been used to dealing in $\mathbb{R}^2$ where we work with two degrees of freedom: x and y. Now, we will be working in $\mathbb{R}^3$ with three degrees of freedom: x, y, and z.

1.1 Points

Definition 1.1.1 ▸ Point

A ***point*** in $\mathbb{R}^n$ space is an n-tuple that specifies a location in that space.

$$p = (p_1, \dots, p_n) \in \mathbb{R}^n$$

Definition 1.1.2 ▸ Distance

Given two points $a, b \in \mathbb{R}^n$, the ***distance*** between the two points is defined as:

$$d(a, b) := \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$$

Example 1.1.1 ▸ Distance Between Points

Find the distance between $p_1 = (-1, -1, 4)$ and $p_2 = (-1, 4, -1)$.

$$\begin{aligned} d(p_1, p_2) &= \sqrt{(-1 - (-1))^2 + (4 - (-1))^2 + (-1 - 1)^2} \\ &= \sqrt{0^2 + 5^2 + (-5)^2} \\ &= \sqrt{50} \end{aligned}$$

Definition 1.1.3 ▸ Sphere

Given a point $c = (h, k, l) \in \mathbb{R}^3$, a ***sphere*** is the set of all points $(x, y, z) \in \mathbb{R}^3$ that are a distance r from the point $c = (h, k, l)$.

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Note that all the points of the sphere are equidistant to the center of the sphere. This means the sphere is really a hollow shell.

Example 1.1.2 ▸ Circle

Show that the following quadratic equation represents a circle by rewriting it in standard form. Find the center $c = (h, k)$ and the radius r.

$$x^2 + y^2 + x = 0$$

To solve this, we will have to complete the square:

$$\begin{aligned} x^2 + x + y^2 &= 0 \\ \implies x^2 + x + \frac{1}{4} + y^2 &= \frac{1}{4} \\ \implies \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4} \end{aligned}$$

1.2 Vectors

Definition 1.2.1 ▸ Vector

A ***vector*** is a mathematical object that contains multiple objects of the same type.

$$\vec{v} = \langle v_1, \ldots, v_n \rangle \in \mathbb{R}^n$$

As customary in most mathematics textbooks, we will always denote vectors using the little arrow thing. In the context of three-dimensional space, we will only be working with vectors with three components. In addition, we will think of vectors as having a magnitude and direction.

Definition 1.2.2 ▸ Scalar Multiplication

Given a vector $\vec{v}$ and scalar k, we define ***scalar multiplication*** as:

$$k \cdot \vec{v} \coloneqq \langle kv_1, \ldots, kv_n \rangle$$

Note that scalar multiplication is associative, commutative, and distributive.

- $a(b\vec{v}) = b(a(\vec{v})) = (ab)\vec{v}$

- $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$
- $k(\vec{v} + \vec{w}) = kv + k\vec{w}$

Definition 1.2.3 ▶ Norm

A vector's ***norm*** is its magnitude or length.

$$\|v\| := \sqrt{v_1^2 + \cdots + v_n^2}$$

Definition 1.2.4 ▶ Unit Vector

A ***unit vector*** is a vector whose magnitude is 1.

We will introduce shorthand notation for the three standard unit vectors:

- $\hat{\imath} := \langle 1, 0, 0 \rangle$
- $\hat{\jmath} := \langle 0, 1, 0 \rangle$
- $\hat{k} := \langle 0, 0, 1 \rangle$

These three vectors form the ***standard basis*** for $\mathbb{R}^3$. That is, we can express any vector in $\mathbb{R}^3$ as a linear combination of $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Technique 1.2.1 ▶ Finding a Unit Vector from a Given Vector

Given a vector $\vec{v} = \langle x, y, z \rangle \in \mathbb{R}^3$, we can find the ***unit vector*** $\vec{u}$ with the same direction by:

$$\vec{u} = \frac{\vec{v}}{\|v\|} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|}, \frac{z}{\|v\|} \right\rangle$$

Definition 1.2.5 ▶ Dot Product

Given two vectors $\vec{a}$ and $\vec{b}$ whose cardinality are both n, we define the ***dot product*** of $\vec{a}$ and $\vec{b}$ as:

$$\vec{a} \cdot \vec{b} := a_1 b_1 + \cdots + a_n b_n$$

Like scalar multiplication, dot product is also associative, commutative, and distributive.

Theorem 1.2.1 ▶ Angle Between Vectors

If $\vec{a}$ and $\vec{b}$ are vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cdot \cos(\theta)$$

Proof. TODO: finish proof ☐

Definition 1.2.6 ▶ Parallel, Perpendicular

- Two vectors are ***parallel*** if the angle between the vectors is 0 deg.
- Two vectors are ***perpendicular*** if the angle between the vectors is 90 deg.

Definition 1.2.7 ▶ Orthogonal

$\vec{a}$ and $\vec{b}$ are ***orthogonal*** if $\vec{a} \cdot \vec{b} = 0$.

Given a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$, we have:

$$\frac{\vec{a}}{\|a\|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

where:

- $\alpha = \cos^{-1}\left(\frac{a_1}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the x-axis)
- $\beta = \cos^{-1}\left(\frac{a_2}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the y-axis)
- $\beta = \cos^{-1}\left(\frac{a_3}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the z-axis)

Definition 1.2.8 ▶ Work

If F is a force moving a particle from a point P to a point Q, the ***work*** performed by the force is given by:

$$W = \vec{F} \cdot \overrightarrow{PQ}$$

Example 1.2.1 ▶ Finding Work

Find the work done by a force $\vec{F} = \langle 3, 4, 5 \rangle$ in moving an object from $p = (2, 1, 0)$ to $q = (4, 6, 2)$.

First, we find $\overrightarrow{pq}$ as such:

$$\begin{aligned}\overrightarrow{pq} &= \langle 4-2, 6-1, 2-0\rangle \\ &= \langle 2, 5, 2\rangle\end{aligned}$$

Then, we can find work:

$$\begin{aligned}W &= \vec{F}\cdot\overrightarrow{PQ} \\ &= \langle 3,4,5\rangle\cdot\langle 2,5,2\rangle \\ &= 6+20+10 \\ &= 36\end{aligned}$$

1.3 Gradient

Definition 1.3.1 ▸ Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The ***gradient*** of f is a function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\nabla f(x_1, \dots x_n) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right\rangle$$

Example 1.3.1 ▸ Gradient

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $f(T, L, \rho) = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

The gradient of $f(T, L, P)$ is denoted

$$\begin{aligned}\nabla f(T, L, \rho) &= \left\langle \frac{\partial f}{\partial T}, \frac{\partial f}{\partial L}, \frac{\partial f}{\partial \rho}\right\rangle \\ &= \left\langle \frac{1}{4L\sqrt{T\rho}}, -\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}, -\frac{1}{4L}\sqrt{\frac{T}{\rho^3}}\right\rangle\end{aligned}$$

We can then calculate gradient as such:

$$\begin{aligned}\nabla f(2,1,1) &= \left\langle \frac{1}{4(1)\sqrt{(2)(1)}}, -\frac{1}{2(1)}\sqrt{\frac{2}{1}}, -\frac{1}{(4)(1)}\sqrt{\frac{2}{1}} \right\rangle \\ &= \left\langle \frac{1}{4\sqrt{2}}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4} \right\rangle\end{aligned}$$

Definition 1.3.2 ▸ Directional Derivative

The ***directional derivative*** of $f(x,y,z)$ in the direction of $\vec{a}$ is defined as:

$$\nabla f(x,y,z) \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

Example 1.3.2 ▸ Directional Derivative

If $f(x,y,z) = xy^2z^5$, find the directional derivative of $f(x,y,z)$ at the point $(1,0,-2)$ in the direction of the unit vector $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$, $\vec{a} = \langle 1,2,-2\rangle$.

For this, we calculate $\nabla f(1,0,-1)$, then calculate the dot product of $\nabla f(1,0,-1)$ with the unit vector $\vec{u} = \langle 1/3, 2/3, -2/3\rangle$. Thus, the directional derivative of $f(x,y,z)$ at $(1,0,-1)$ denoted by $Df(1,0,-1)$ in the direction of $\vec{u}$ is:

$$\begin{aligned}Df(1,0,-1) &= \nabla f(1,2,-2) \cdot \vec{u} \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \vec{u} \\ &= \langle 0,0,0\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \\ &= 0\end{aligned}$$

1.4 Projecting Vectors

Projecting a vector onto another vector

Definition 1.4.1 ▸ Scalar Projection

Given $\vec{a}$ and $\vec{b}$, the ***scalar projection*** of $\vec{b}$ onto $\vec{a}$ is the norm of the vector projection of $\vec{b}$ onto $\vec{a}$.

$$\text{comp}_{\vec{a}}\,\vec{b} := \frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|}$$

Definition 1.4.2 ▸ Vector Projection

Given $\vec{a}$ and $\vec{b}$ that are non-zero vectors, the ***vector projection*** of $\vec{b}$ onto the vector $\vec{a}$ is defined by:

$$\text{proj}_{\vec{a}}\,\vec{b} := \text{comp}_{\vec{a}}\,\vec{b}\frac{\vec{a}}{\|\vec{a}\|}$$

1.5 Cross Product

Definition 1.5.1 ▸ Cross Product

Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, the ***cross product*** of $\vec{a}$ and $\vec{b}$ is a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b} := \vec{c} \quad \text{where} \quad \vec{a}\cdot\vec{c} = 0 \quad \textbf{and} \quad \vec{b}\cdot\vec{c} = 0$$

The cross product is exclusive to vectors in three dimensions.

Technique 1.5.1 ▸ Calculating Cross Product

Let $\vec{a} := \langle a_1, a_2, a_3\rangle$ and $\vec{b} := \langle b_1, b_2, b_3\rangle$ To find $\vec{a}\times\vec{b}$, we:

1. Create a matrix as such:

$$\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

2. Find the determinant of the matrix by cofactor expansion on the first row.

$$\begin{aligned}\vec{a} \times \vec{b} &= \hat{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{\imath}(a_2 b_3 - b_2 a_3) - \hat{\jmath}(a_1 b_3 - b_1 a_3) + \hat{k}(a_1 b_2 - b_1 a_2) \\ &= \langle a_2 b_3 - b_2 a_3,\ -a_1 b_3 + b_1 a_3,\ a_1 b_2 - b_1 a_2 \rangle\end{aligned}$$

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- If $r \in \mathbb{R}$, then $(r\vec{a}) \times \vec{b} = \vec{a} \times (r\vec{b}) = r(\vec{a} \times \vec{b})$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot c$
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Theorem 1.5.1

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors in $\mathbb{R}^3$ and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$$

Technique 1.5.2 ▶ Using Cross Product to Calculate Torque

If $\vec{F}$ is a force applied to an object with position vector $\vec{r}$, then the torque $\vec{\mathrm{T}}$ produced by $\vec{F}$ is given by:

$$\vec{\mathrm{T}} \coloneqq \vec{r} \times \vec{F}$$

1.6 Cylinders and Quadratic Surfaces

Definition 1.6.1 ▶ Planar Curve

A ***planar curve*** is any curve that lies on a single plane.

Definition 1.6.2 ▸ Cylinder

Given a planar curve c, the surface in $\mathbb{R}^3$ defined by all parallel lines crossing the curve c is called a ***cylinder***.

Note that our broad definition of cylinder does not require the cylinder to be circular, nor does it require it to be straight. For example, we could have a planar curve defined by $x^2 + y^2 = 1$ and create a circular cylinder with radius 1. We could also have a planar curve defined by $y = x^2$ and create a ***parabolic cylinder***.

Example 1.6.1

Consider the curve $x^2 + y^2 = z^2$. For every z_0 at $x = y = 0$, we have a point, say $p \coloneqq (0, 0, 0)$yh

Definition 1.6.3 ▸ Cone

Definition 1.6.4 ▸ Conic Surface

A ***conic surface*** is a surface that is attained by taking a cross-section of a cone.

There are four types of conic surfaces:

1. The cross-section parallel to the xy-plane is a ***circle***.
2. The cross-section slightly angled from the xy-plane is a ***ellipse***.
3. The cross-section parallel to a generating line is a ***parabola***.
4. The cross-section parallel to the z axis is a ***hyperbola***.

Definition 1.6.5 ▸ Quadratic Surface

A ***quadratic surface*** in $\mathbb{R}^3$ is the set of points whose coordinates satisfy a quadratic polynomial in the variables x, y, z.

For example, the standard equation for a sphere is a quadratic surface.

Definition 1.6.6 ▸ Ellipsoid

An ***ellipsoid*** is a quadratic surface in $\mathbb{R}^3$ defined by the equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

The cross-sections of an ellipsoid with each coordinate plane (xy-plane, xz-plane, yz-plane) is just an ellipse. In other words:

- If we set $z = l$, we get an ellipse in the xy-plane
- If we set $y = k$, we get an ellipse in the xz-plane
- If we set $x = h$, we get an ellipse in the yz-plane

We can also have ***hyperboloids***:

- Type 1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Type 2: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

For type 2 hyperboloids, its cross-section with the xy and xz plane is a hyperbola. Note that there is no cross-section with the yz plane. This is because the equation $-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ has no solution in the real numbers.

The standard form equation for a parabola is:

$$z - l = \pm c\left[(x-l)^2 + (y-k)^2\right]$$

TODO: more quadratic surfaces here

2

Vector Functions

Definition 2.0.1 ▶ Vector Function

A ***vector function*** $\vec{r}(t)$ is a function that maps each $t \in \mathbb{R}$ to a corresponding vector in $\mathbb{R}^n$. In other words:

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

For example, we can have $\vec{r}(t) = \langle 2-3t, 5-7t, t\rangle$. Then $\vec{r}(t)$ is a vector function that parameterizes a line that passes through $(2, 5, 0)$ and with vector direction $\vec{v} = \langle -3, -7, 1\rangle$.

Example 2.0.1 ▶ Circle Vector Function

Let $\vec{r}_1(t) = \langle \cos t, \sin t\rangle$ where $0 \leq t < 2\pi$. This function's graph is a circle of radius 1. We can think of this as $\cos^2 t + \sin^2 t = 1$.

Similarly, consider $\vec{r}_2(t) = \langle 5\cos t, 8\sin t\rangle$. Thus, $x/5 = \cos t$ and $y/8 = \sin t$. Using the equation $cos^2 t + sin^2 t = 1$, we now have $(x/5)^2 + (y/8)^2 = 1$. It's an ellipse.

If a vector function is given by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then the domain of $\vec{r}(t)$ is the intersection of the domains of f, g, h. This is denoted as $\mathrm{Dom}(\vec{r}(t))$.

2.1 Limits and Continuity

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a vector function, we say that:

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

Recall L'Hopital's rule:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Only applies if the left-hand side is an indeterminate form (i.e. a fraction whose denominator is 0).

Definition 2.1.1 ▸ Continuity

We say a function $f : X \to Y$ is ***continuous*** at some value a if:

1. $a \in X$,
2. $\lim_{x\to a} f(x)$ exists, and
3. $f(a) = \lim_{x\to a} f(x)$

Similarly, a vector function is continuous if it satisfies the above conditions.

Recall that for a function $f : \mathbb{R} \to \mathbb{R}$, the derivative at some value x can be generalized by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Similarly, the derivative of a vector function at some value x can be generalized as:

$$\vec{r}(t) = \lim_{h\to 0} \frac{\vec{r}(x+h) - \vec{r}(x)}{h}$$

Thus, if $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, we have:

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Theorem 2.1.1 ▸ Properties of the differentials of vector functions

Let $\vec{u}(t) = \langle f_1(t), g_1(t), h_1(t)\rangle$ and $\vec{v}(t) = \langle f_2(t), g_2(t), h_2(t)\rangle$ where $f_1, g_1, h_1, f_2, g_2, h_2$ are differentiable. Let c be a scalar. Then:

- $\frac{d}{dt}\vec{u}(t) = \langle {f_1}'(t), {g_1}'(t), {h_1}'(t)\rangle$
- $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$
- $\frac{d}{dt}c\vec{v}(t) = c\frac{d}{dt}\vec{v}(t)$
- $\frac{d}{dt}f(t)\vec{v}(t) = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$
- $\frac{d}{dt}\vec{u}(t)\cdot\vec{v}(t) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$
- $\frac{d}{dt}\vec{u}(t)\times\vec{v}(t) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$ (note that order here is sensitive)
- $\frac{d}{dt}\vec{u}(f(t)) = \vec{u}'(f(t))f'(t)$

If we consider the curve C formed by all the terminal points of the graph of a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we call $\vec{r}(t)$ a ***parameterization*** of the curve C, and we call $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ the ***displacement vector***.

Example 2.1.1

Find the parametric equations of the tangent line to the helix with parameterization $\vec{r}(t) \coloneqq \langle 2\cos t, \sin t, t\rangle$ at the point $(0, 1, \pi/2)$

$$\vec{r}'\left(\frac{\pi}{2}\right) = \left\langle -2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1\right\rangle$$

Then the parameterized line is:

$$\left\langle 0, 1, \frac{\pi}{2}\right\rangle + t\langle -2, 0, 1\rangle$$

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