Calculus III

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Contents

1 Introduction

Much of our focus will be on Stoke's Theorem.

2 Three-Dimensional Space

In past math classes, we have been used to dealing in $\mathbb{R}^2$ where we work with two degrees of freedom: x and y. Now, we will be working in $\mathbb{R}^3$ with three degrees of freedom: x, y, and z.

2.1 Points

Definition 2.1.1 ▶ Point

A ***point*** in $\mathbb{R}^n$ space is an n-tuple that specifies a location in that space.

$$p = (p_1, \dots, p_n) \in \mathbb{R}^n$$

Definition 2.1.2 ▶ Distance

Given two points $a, b \in \mathbb{R}^n$, the ***distance*** between the two points is defined as:

$$d(a, b) \coloneqq \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$$

Example 2.1.1 ▶ Distance Between Points

Find the distance between $p_1 = (-1, -1, 4)$ and $p_2 = (-1, 4, -1)$.

$$\begin{aligned} d(p_1, p_2) &= \sqrt{(-1 - (-1))^2 + (4 - (-1))^2 + (-1 - 1)^2} \\ &= \sqrt{0^2 + 5^2 + (-5)^2} \\ &= \sqrt{50} \end{aligned}$$

Definition 2.1.3 ▶ Sphere

Given a point $c = (h, k, l) \in \mathbb{R}^3$, a ***sphere*** is the set of all points $(x, y, z) \in \mathbb{R}^3$ that are a distance r from the point $c = (h, k, l)$.

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Note that all the points of the sphere are equidistant to the center of the sphere. This means the sphere is really a hollow shell.

Example 2.1.2 ▶ Circle

Show that the following quadratic equation represents a circle by rewriting it in standard form. Find the center $c = (h, k)$ and the radius r.

$$x^2 + y^2 + x = 0$$

To solve this, we will have to complete the square:

$$\begin{aligned} x^2 + x + y^2 &= 0 \\ \implies x^2 + x + \frac{1}{4} + y^2 &= \frac{1}{4} \\ \implies \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4} \end{aligned}$$

Definition 2.1.4 ▶ Cylinder

Given a planar curve c, the surface in $\mathbb{R}^3$ defined by all parallel lines crossing the curve c is called a ***cylinder***.

Similarly, the set of all points (x, y, z) such that x and y satisfy $(x - h)^2 + (y - k)^2 = r^2$ forms a circular cylinder. Note that our broad definition of cylinder does not require the cylinder to be circular.

Regarding the quadrants:

$$\begin{aligned} Q1 &= \left\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\right\} \\ Q4 &= \left\{(x, y) \in \mathbb{R}^2 : x > 0, y < 0\right\} \end{aligned}$$

2.2 Vectors

Definition 2.2.1 ▶ Vector

A ***vector*** is a mathematical object that contains multiple objects of the same type.

$$\vec{v} = \langle v_1, \ldots, v_n \rangle \in \mathbb{R}^n$$

As customary in most mathematics textbooks, we will always denote vectors using the little arrow thing. In the context of three-dimensional space, we will only be working with vectors with three components. In addition, we will think of vectors as having a magnitude and direction.

Definition 2.2.2 ▶ Scalar Multiplication

Given a vector $\vec{v}$ and scalar k, we define ***scalar multiplication*** as:

$$k \cdot \vec{v} \coloneqq \langle kv_1, \ldots, kv_n \rangle$$

Note that scalar multiplication is associative, commutative, and distributive.

- $a(b\vec{v}) = b(a(\vec{v})) = (ab)\vec{v}$
- $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$
- $k(\vec{v} + \vec{w}) = kv + k\vec{w}$

Definition 2.2.3 ▶ Norm

A vector's ***norm*** is its magnitude or length.

$$\|v\| \coloneqq \sqrt{v_1^2 + \cdots + v_n^2}$$

Definition 2.2.4 ▶ Unit Vector

A ***unit vector*** is a vector whose magnitude is 1.

We will introduce shorthand notation for the three standard unit vectors:

- $\hat{\imath} \coloneqq \langle 1, 0, 0 \rangle$
- $\hat{\jmath} \coloneqq \langle 0, 1, 0 \rangle$
- $\hat{k} \coloneqq \langle 0, 0, 1 \rangle$

These three vectors form the ***standard basis*** for $\mathbb{R}^3$. That is, we can express any vector in $\mathbb{R}^3$ as a linear combination of $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Technique 2.2.1 ▶ Finding a Unit Vector from a Given Vector

Given a vector $\vec{v} = \langle x, y, z \rangle \in \mathbb{R}^3$, we can find the ***unit vector*** $\vec{u}$ with the same direction by:

$$\vec{u} = \frac{\vec{v}}{\|v\|} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|}, \frac{z}{\|v\|} \right\rangle$$

Definition 2.2.5 ▶ Dot Product

Given two vectors $\vec{a}$ and $\vec{b}$ whose cardinality are both n, we define the ***dot product*** of $\vec{a}$ and $\vec{b}$ as:

$$\vec{a} \cdot \vec{b} \coloneqq a_1 b_1 + \cdots + a_n b_n$$

Like scalar multiplication, dot product is also associative, commutative, and distributive.

Theorem 2.2.1 ▶ Angle Between Vectors

If $\vec{a}$ and $\vec{b}$ are vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cdot \cos(\theta)$$

Proof. TODO: finish proof ▢

Definition 2.2.6 ▶ Parallel, Perpendicular

- Two vectors are ***parallel*** if the angle between the vectors is 0 deg.
- Two vectors are ***perpendicular*** if the angle between the vectors is 90 deg.

Definition 2.2.7 ▶ Orthogonal

$\vec{a}$ and $\vec{b}$ are ***orthogonal*** if $\vec{a} \cdot \vec{b} = 0$.

Given a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$, we have:

$$\frac{\vec{a}}{\|a\|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

where:

- $\alpha = \cos^{-1}\left(\frac{a_1}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the x-axis)
- $\beta = \cos^{-1}\left(\frac{a_2}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the y-axis)
- $\beta = \cos^{-1}\left(\frac{a_3}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the z-axis)

Definition 2.2.8 ▶ Work

If F is a force moving a particle from a point p to a point q, the ***work*** performed by the force is given by:

$$W = \vec{F} \cdot \overrightarrow{PQ}$$

Example 2.2.1 ▶ Finding Work

Find the work done by a force $\vec{F} = \langle 3, 4, 5 \rangle$ in moving an object from $p = (2, 1, 0)$ to $q = (4, 6, 2)$.

First, we find $\overrightarrow{pq}$ as such:

$$\begin{aligned} \overrightarrow{pq} &= \langle 4 - 2, 6 - 1, 2 - 0 \rangle \\ &= \langle 2, 5, 2 \rangle \end{aligned}$$

Then, we can find work:

$$\begin{aligned} W &= \vec{F} \cdot \overrightarrow{PQ} \\ &= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 \\ &= 36 \end{aligned}$$

2.3 Gradient

Definition 2.3.1 ▶ Gradient

Example 2.3.1 ▶ Gradient

$f(T, L, \rho) = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

The gradient of $f(T, L, P)$ is denoted

$$\begin{aligned}\nabla f(T, L, \rho) &= \left\langle \frac{\partial f}{\partial T}, \frac{\partial f}{\partial L}, \frac{\partial f}{\partial \rho} \right\rangle \\ &= \left\langle \frac{1}{4L\sqrt{T\rho}}, -\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}, -\frac{1}{4L}\sqrt{\frac{T}{\rho^3}} \right\rangle\end{aligned}$$

We can then calculate gradient as such:

$$\begin{aligned}\nabla f(2,1,1) &= \left\langle \frac{1}{4(1)\sqrt{(2)(1)}}, -\frac{1}{2(1)}\sqrt{\frac{2}{1}}, -\frac{1}{(4)(1)}\sqrt{\frac{2}{1}} \right\rangle \\ &= \left\langle \frac{1}{4\sqrt{2}}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4} \right\rangle\end{aligned}$$

Definition 2.3.2 ▶ Directional Derivative

The ***directional derivative*** of $f(x, y, z)$ in the direction of $\vec{u}$ is defined as:

$$\nabla f(x, y, z) \cdot \vec{u}$$

The dot product of the gradient $\nabla f(x, y, z)$ with a unit vector $\frac{\vec{a}}{\|\vec{a}\|}$

Example 2.3.2 ▶ Directional Derivative

If $f(x, y, z) = xy^2z^5$, find the directional derivative of $f(x, y, z)$ at the point $(1, 0, -2)$ in the direction of the unit vector $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$, $\vec{a} = \langle 1, 2, -2 \rangle$.

For this, we calculate $\nabla f(1, 0, -1)$, then calculate the dot product of $\nabla f(1, 0, -1)$ with the unit vector $\vec{u} = \langle 1/3, 2/3, -2/3 \rangle$. Thus, the directional derivative of $f(x, y, z)$ at $(1, 0, -1)$ denoted by $Df(1, 0, -1)$ in the direction of $\vec{u}$ is:

$$\begin{aligned}Df(1,0,-1) &= \nabla f(1,2,-2) \cdot \vec{u} \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \vec{u} \\ &= \langle 0, 0, 0 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \\ &= 0\end{aligned}$$

2.4 Projecting Vectors

Definition 2.4.1 ▶ Vector Projection

Given $\vec{a}$ and $\vec{b}$ that are non-zero vectors, the ***vector projection*** of $\vec{b}$ over the vector $\vec{a}$ is denoted by:

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