
These notes suck

Calculus III

UT Knoxville, Spring 2023, MATH 341

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March 4, 2023

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Three-Dimensional Space

In past math classes, we have been used to dealing in \mathbb{R}^2 where we work with two degrees of freedom: x and y . Now, we will be working in \mathbb{R}^3 with three degrees of freedom: x , y , and z .

1.1 Points

Definition 1.1.1 ► Point

A **point** in \mathbb{R}^n space is an n -tuple that specifies a location in that space.

$$p = (p_1, \dots, p_n) \in \mathbb{R}^n$$

Definition 1.1.2 ► Distance

Given two points $a, b \in \mathbb{R}^n$, the **distance** between the two points is defined as:

$$d(a, b) := \sqrt{(b_1 - a_1)^2 + \dots + (b_n - a_n)^2}$$

Example 1.1.3 ► Distance Between Points

Find the distance between $p_1 = (-1, -1, 4)$ and $p_2 = (-1, 4, -1)$.

$$\begin{aligned} d(p_1, p_2) &= \sqrt{(-1 - (-1))^2 + (4 - (-1))^2 + (-1 - 1)^2} \\ &= \sqrt{0^2 + 5^2 + (-5)^2} \\ &= \sqrt{50} \end{aligned}$$

Definition 1.1.4 ► Sphere

Given a point $c = (h, k, l) \in \mathbb{R}^3$, a **sphere** is the set of all points $(x, y, z) \in \mathbb{R}^3$ that are a distance r from the point $c = (h, k, l)$.

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Note that all the points of the sphere are equidistant to the center of the sphere. This means the sphere is really a hollow shell.

Example 1.1.5 ► Circle

Show that the following quadratic equation represents a circle by rewriting it in standard form. Find the center $c = (h, k)$ and the radius r .

$$x^2 + y^2 + x = 0$$

To solve this, we will have to complete the square:

$$\begin{aligned} x^2 + x + y^2 &= 0 \\ \implies x^2 + x + \frac{1}{4} + y^2 &= \frac{1}{4} \\ \implies \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4} \end{aligned}$$

1.2 Vectors

Definition 1.2.1 ► Vector

A **vector** is a mathematical object that contains multiple objects of the same type.

$$\vec{v} = \langle v_1, \dots, v_n \rangle \in \mathbb{R}^n$$

As customary in most mathematics textbooks, we will always denote vectors using the little arrow thing. In the context of three-dimensional space, we will only be working with vectors with three components. In addition, we will think of vectors as having a magnitude and direction.

Definition 1.2.2 ► Scalar Multiplication

Given a vector \vec{v} and scalar k , we define **scalar multiplication** as:

$$k \cdot \vec{v} := \langle kv_1, \dots, kv_n \rangle$$

Note that scalar multiplication is associative, commutative, and distributive.

- $a(b\vec{v}) = b(a(\vec{v})) = (ab)\vec{v}$

- $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$
- $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$

Definition 1.2.3 ► Norm

A vector's **norm** is its magnitude or length.

$$\|\vec{v}\| := \sqrt{v_1^2 + \cdots + v_n^2}$$

Definition 1.2.4 ► Unit Vector

A **unit vector** is a vector whose magnitude is 1.

We will introduce shorthand notation for the three standard unit vectors:

- $\hat{i} := \langle 1, 0, 0 \rangle$
- $\hat{j} := \langle 0, 1, 0 \rangle$
- $\hat{k} := \langle 0, 0, 1 \rangle$

These three vectors form the **standard basis** for \mathbb{R}^3 . That is, we can express any vector in \mathbb{R}^3 as a linear combination of $\hat{i}, \hat{j}, \hat{k}$.

Technique 1.2.5 ► Finding a Unit Vector from a Given Vector

Given a vector $\vec{v} = \langle x, y, z \rangle \in \mathbb{R}^3$, we can find the **unit vector** \vec{u} with the same direction by:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{x}{\|\vec{v}\|}, \frac{y}{\|\vec{v}\|}, \frac{z}{\|\vec{v}\|} \right\rangle$$

Definition 1.2.6 ► Dot Product

Given two vectors \vec{a} and \vec{b} whose cardinality are both n , we define the **dot product** of \vec{a} and \vec{b} as:

$$\vec{a} \cdot \vec{b} := a_1b_1 + \cdots + a_nb_n$$

Like scalar multiplication, dot product is also associative, commutative, and distributive.

Theorem 1.2.7 ► Angle Between Vectors

If \vec{a} and \vec{b} are vectors and θ is the angle between \vec{a} and \vec{b} , then:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cdot \cos(\theta)$$

Proof. TODO: finish proof

**Definition 1.2.8 ► Parallel, Perpendicular**

- Two vectors are **parallel** if the angle between the vectors is 0 deg.
- Two vectors are **perpendicular** if the angle between the vectors is 90 deg.

Definition 1.2.9 ► Vector Orthogonality

We say two vectors are **orthogonal** if their dot product is 0.

In other words, \vec{a} and \vec{b} are **orthogonal** if $\vec{a} \cdot \vec{b} = 0$.

Given a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$, we have:

$$\frac{\vec{a}}{\|\vec{a}\|} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

where:

- $\alpha = \cos^{-1} \left(\frac{a_1}{\|\vec{a}\|} \right)$ (angle between \vec{a} and the x -axis)
- $\beta = \cos^{-1} \left(\frac{a_2}{\|\vec{a}\|} \right)$ (angle between \vec{a} and the y -axis)
- $\gamma = \cos^{-1} \left(\frac{a_3}{\|\vec{a}\|} \right)$ (angle between \vec{a} and the z -axis)

Definition 1.2.10 ► Work

If F is a force moving a particle from a point P to a point Q , the **work** performed by the force is given by:

$$W = \vec{F} \cdot \vec{PQ}$$

Example 1.2.11 ► Finding Work

Find the work done by a force $\vec{F} = \langle 3, 4, 5 \rangle$ in moving an object from $p = (2, 1, 0)$ to $q = (4, 6, 2)$.

First, we find \vec{pq} as such:

$$\begin{aligned}\vec{pq} &= \langle 4 - 2, 6 - 1, 2 - 0 \rangle \\ &= \langle 2, 5, 2 \rangle\end{aligned}$$

Then, we can find work:

$$\begin{aligned}W &= \vec{F} \cdot \vec{PQ} \\ &= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 \\ &= 36\end{aligned}$$

1.3 Gradient

Definition 1.3.1 ► Gradient

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. The **gradient** of f is a function $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by:

$$\nabla f(x_1, \dots, x_n) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

Example 1.3.2 ► Gradient

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function defined by $f(T, L, \rho) = \frac{1}{2L} \sqrt{\frac{T}{\rho}}$

The gradient of $f(T, L, P)$ is denoted

$$\begin{aligned}\nabla f(T, L, \rho) &= \left\langle \frac{\partial f}{\partial T}, \frac{\partial f}{\partial L}, \frac{\partial f}{\partial \rho} \right\rangle \\ &= \left\langle \frac{1}{4L\sqrt{T\rho}}, -\frac{1}{2L^2} \sqrt{\frac{T}{\rho}}, -\frac{1}{4L} \sqrt{\frac{T}{\rho^3}} \right\rangle\end{aligned}$$

We can then calculate gradient as such:

$$\begin{aligned}\nabla f(2, 1, 1) &= \left\langle \frac{1}{4(1)\sqrt{(2)(1)}}, -\frac{1}{2(1)}\sqrt{\frac{2}{1}}, -\frac{1}{(4)(1)}\sqrt{\frac{2}{1}} \right\rangle \\ &= \left\langle \frac{1}{4\sqrt{2}}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4} \right\rangle\end{aligned}$$

Definition 1.3.3 ► Directional Derivative

The **directional derivative** of $f(x, y, z)$ in the direction of \vec{a} is defined as:

$$\nabla f(x, y, z) \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

Example 1.3.4 ► Directional Derivative

If $f(x, y, z) = xy^2z^5$, find the directional derivative of $f(x, y, z)$ at the point $(1, 0, -2)$ in the direction of the unit vector $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$, $\vec{a} = \langle 1, 2, -2 \rangle$.

For this, we calculate $\nabla f(1, 0, -1)$, then calculate the dot product of $\nabla f(1, 0, -1)$ with the unit vector $\vec{u} = \langle 1/3, 2/3, -2/3 \rangle$. Thus, the directional derivative of $f(x, y, z)$ at $(1, 0, -1)$ denoted by $Df(1, 0, -1)$ in the direction of \vec{u} is:

$$\begin{aligned}Df(1, 0, -1) &= \nabla f(1, 2, -2) \cdot \vec{u} \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \vec{u} \\ &= \langle 0, 0, 0 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \\ &= 0\end{aligned}$$

1.4 Projecting Vectors

Projecting a vector onto another vector

Definition 1.4.1 ► Scalar Projection

Given \vec{a} and \vec{b} , the **scalar projection** of \vec{b} onto \vec{a} is the norm of the vector projection of \vec{b} onto \vec{a} .

$$\text{comp}_{\vec{a}} \vec{b} := \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|}$$

Definition 1.4.2 ► Vector Projection

Given \vec{a} and \vec{b} that are non-zero vectors, the **vector projection** of \vec{b} onto the vector \vec{a} is defined by:

$$\text{proj}_{\vec{a}} \vec{b} := \text{comp}_{\vec{a}} \vec{b} \frac{\vec{a}}{\|\vec{a}\|}$$

1.5 Cross Product

Definition 1.5.1 ► Cross Product

Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, the **cross product** of \vec{a} and \vec{b} is a vector that is orthogonal to both \vec{a} and \vec{b} .

$$\vec{a} \times \vec{b} := \vec{c} \quad \text{where} \quad \vec{a} \cdot \vec{c} = 0 \quad \text{and} \quad \vec{b} \cdot \vec{c} = 0$$

The cross product is exclusive to vectors in three dimensions.

Technique 1.5.2 ► Calculating Cross Product

Let $\vec{a} := \langle a_1, a_2, a_3 \rangle$ and $\vec{b} := \langle b_1, b_2, b_3 \rangle$ To find $\vec{a} \times \vec{b}$, we:

1. Create a matrix as such:

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

2. Find the determinant of the matrix by cofactor expansion on the first row.

$$\begin{aligned}\vec{a} \times \vec{b} &= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{i}(a_2b_3 - b_2a_3) - \hat{j}(a_1b_3 - b_1a_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \langle a_2b_3 - b_2a_3, -a_1b_3 + b_1a_3, a_1b_2 - b_1a_2 \rangle\end{aligned}$$

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- If $r \in \mathbb{R}$, then $(r\vec{a}) \times \vec{b} = \vec{a} \times (r\vec{b}) = r(\vec{a} \times \vec{b})$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Theorem 1.5.3

If \vec{a} and \vec{b} are two non-zero vectors in \mathbb{R}^3 and θ is the angle between \vec{a} and \vec{b} , then:

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin \theta$$

Technique 1.5.4 ► Using Cross Product to Calculate Torque

If \vec{F} is a force applied to an object with position vector \vec{r} , then the torque \vec{T} produced by \vec{F} is given by:

$$\vec{T} := \vec{r} \times \vec{F}$$

1.6 Cylinders and Quadratic Surfaces

Definition 1.6.1 ► Planar Curve

A *planar curve* is any curve that lies on a single plane.

Definition 1.6.2 ► Cylinder

Given a planar curve c , the surface in \mathbb{R}^3 defined by all parallel lines crossing the curve c is called a **cylinder**.

Note that our broad definition of cylinder does not require the cylinder to be circular, nor does it require it to be straight. For example, we could have a planar curve defined by $x^2 + y^2 = 1$ and create a circular cylinder with radius 1. We could also have a planar curve defined by $y = x^2$ and create a **parabolic cylinder**.

Example 1.6.3

Consider the curve $x^2 + y^2 = z^2$. For every z_0 at $x = y = 0$, we have a point, say $p := (0, 0, 0)$ yh

Definition 1.6.4 ► Cone**Definition 1.6.5 ► Conic Surface**

A **conic surface** is a surface that is attained by taking a cross-section of a cone.

There are four types of conic surfaces:

1. The cross-section parallel to the xy -plane is a **circle**.
2. The cross-section slightly angled from the xy -plane is a **ellipse**.
3. The cross-section parallel to a generating line is a **parabola**.
4. The cross-section parallel to the z axis is a **hyperbola**.

Definition 1.6.6 ► Quadratic Surface

A **quadratic surface** in \mathbb{R}^3 is the set of points whose coordinates satisfy a quadratic polynomial in the variables x, y, z .

For example, the standard equation for a sphere is a quadratic surface.

Definition 1.6.7 ► Ellipsoid

An **ellipsoid** is a quadratic surface in \mathbb{R}^3 defined by the equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

The cross-sections of an ellipsoid with each coordinate plane (xy -plane, xz -plane, yz -plane) is just an ellipse. In other words:

- If we set $z = l$, we get an ellipse in the xy -plane
- If we set $y = k$, we get an ellipse in the xz -plane
- If we set $x = h$, we get an ellipse in the yz -plane

We can also have **hyperboloids**:

- Type 1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Type 2: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

For type 2 hyperboloids, its cross-section with the xy and xz plane is a hyperbola. Note that there is no cross-section with the yz plane. This is because the equation $-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ has no solution in the real numbers.

The standard form equation for a parabola is:

$$z - l = \pm c [(x - h)^2 + (y - k)^2]$$

TODO: more quadratic surfaces here

Vector Functions

Definition 2.0.1 ► Vector Function

A **vector function** $\vec{r}(t)$ is a function that maps each $t \in \mathbb{R}$ to a corresponding vector in \mathbb{R}^n . In other words:

$$\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^n$$

For example, we can have $\vec{r}(t) = \langle 2 - 3t, 5 - 7t, t \rangle$. Then $\vec{r}(t)$ is a vector function that parameterizes a line that passes through $(2, 5, 0)$ and with vector direction $\vec{v} = \langle -3, -7, 1 \rangle$.

Example 2.0.2 ► Circle Vector Function

Let $\vec{r}_1(t) = \langle \cos t, \sin t \rangle$ where $0 \leq t < 2\pi$. This function's graph is a circle of radius 1. We can think of this as $\cos^2 t + \sin^2 t = 1$.

Similarly, consider $\vec{r}_2(t) = \langle 5 \cos t, 8 \sin t \rangle$. Thus, $x/5 = \cos t$ and $y/8 = \sin t$. Using the equation $\cos^2 t + \sin^2 t = 1$, we now have $(x/5)^2 + (y/8)^2 = 1$. It's an ellipse.

If a vector function is given by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then the domain of $\vec{r}(t)$ is the intersection of the domains of f, g, h . This is denoted as $\text{Dom}(\vec{r}(t))$.

2.1 Limits and Continuity

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector function, we say that:

$$\lim_{t \rightarrow a} \vec{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

Recall L'Hopital's rule:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

Only applies if the left-hand side is an indeterminate form (i.e. a fraction whose denominator is 0).

Definition 2.1.1 ▶ Continuity

We say a function $f : X \rightarrow Y$ is **continuous** at some value a if:

1. $a \in X$,
2. $\lim_{x \rightarrow a} f(x)$ exists, and
3. $f(a) = \lim_{x \rightarrow a} f(x)$

Similarly, a vector function is continuous if it satisfies the above conditions.

Recall that for a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the derivative at some value x can be generalized by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Similarly, the derivative of a vector function at some value x can be generalized as:

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Thus, if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, we have:

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Theorem 2.1.2 ▶ Properties of the differentials of vector functions

Let $\vec{u}(t) = \langle f_1(t), g_1(t), h_1(t) \rangle$ and $\vec{v}(t) = \langle f_2(t), g_2(t), h_2(t) \rangle$ where $f_1, g_1, h_1, f_2, g_2, h_2$ are differentiable. Let c be a scalar. Then:

- $\frac{d}{dt} \vec{u}(t) = \langle f_1'(t), g_1'(t), h_1'(t) \rangle$
- $\frac{d}{dt} (\vec{u}(t) + \vec{v}(t)) = \frac{d}{dt} \vec{u}(t) + \frac{d}{dt} \vec{v}(t)$
- $\frac{d}{dt} c\vec{v}(t) = c \frac{d}{dt} \vec{v}(t)$
- $\frac{d}{dt} f(t)\vec{v}(t) = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$
- $\frac{d}{dt} \vec{u}(t) \cdot \vec{v}(t) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt} \vec{u}(t) \times \vec{v}(t) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ (note that order here is sensitive)
- $\frac{d}{dt} \vec{u}(f(t)) = \vec{u}'(f(t))f'(t)$

If we consider the curve C formed by all the terminal points of the graph of a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we call $\vec{r}(t)$ a **parameterization** of the curve C , and we call $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ the **displacement vector**.

Example 2.1.3

Find the parametric equations of the tangent line to the helix with parameterization $\vec{r}(t) := \langle 2 \cos t, \sin t, t \rangle$ at the point $(0, 1, \pi/2)$

$$\vec{r}'\left(\frac{\pi}{2}\right) = \left\langle -2 \sin \frac{\pi}{2}, \cos \frac{\pi}{2}, 1 \right\rangle$$

Then the parameterized line is:

$$\left\langle 0, 1, \frac{\pi}{2} \right\rangle + t \langle -2, 0, 1 \rangle$$

2.2 Integrability

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $t \in \mathbb{R}$, and $x(t)$, $y(t)$, and $z(t)$ are integrable on the interval $[a, b]$, then:

$$\begin{aligned} \int_a^b \vec{r}(t) dt &= \int_a^b \langle x(t), y(t), z(t) \rangle dt \\ &= \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle \end{aligned}$$

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $t \in [a, b]$, and if \vec{r} is a parameterization of a curve C , then $\vec{v}(t) = \vec{r}'(t)$ is the **velocity** vector, and $\vec{a}(t) = \vec{r}''(t)$ is the **acceleration vector**. Also, $\|\vec{r}'(t)\|$ is the **speed**, and $\|\vec{r}''(t)\|$ is the **acceleration**.

Definition 2.2.1 ► Unit Tangent Vector

Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, the **unit tangent vector** is a unit vector that is tangent to the curve defined by $\vec{r}(t)$.

$$T(t) := \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Definition 2.2.2 ► Normal Vector

Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, the **normal vector** is a unit vector that

finish definition

$$N(t) := \frac{\vec{r}''(t)}{\|\vec{r}''(t)\|}$$

Definition 2.2.3 ► Osculating Plane, Binormal Vector

At each point t , $T(t)$ and $N(t)$ determine a plane called the **osculating plane**. An equation of the osculating plane to C at $P_0 = \vec{r}(t)$ has a normal direction $B(t) := T(t) \times N(t)$ called the **binormal vector** to the curve C at p_0 . The plane determined by $N(t)$ and $B(t)$ is called the **normal plane** to the curve C at terminal point of $\vec{r}(t) = p_0$, and it has normal direction given by $T(t)$.

2.3 Arc Length and Curvature

If a curve in the xy -plane is parameterized by $x = f(t)$ and $y = g(t)$ for $t \in [a, b]$, and f and g are differentiable on $[a, b]$, then the total arc length of the curve is given by:

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt$$

Similarly, if C is a curve in \mathbb{R}^3 parameterized by a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $t \in [a, b]$. Then the total arc length of the curve is given by:

Arc Length of a Curve in \mathbb{R}^3

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt$$

Definition 2.3.1 ► Curvature, Osculating Circle

The **curvature** of a curve at some point is how much it curves at that point. More specifically, it's the inverse of the radius of the **osculating circle**, tangent to the point that most closely “matches” the curve. This circle is called the **osculation circle**, and it lies in the osculating plane.

Formally, if C is a curve parameterized by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $t \in [a, b]$, and f, g, h are differentiable twice on $[a, b]$, then the **curvature** of C is given by:

$$k(t) := \left\| \frac{d}{ds} \vec{T}(t) \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

A lower curvature at some point actually means that it curves more.

Theorem 2.3.2

The curvature given by a vector function $\vec{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^3$ is given by:

$$k(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

If C is a planar curve given by $y = f(x)$, then $k(t) = \frac{\|f''(t)\|}{(1+[f'(t)]^2)^{3/2}}$

Definition 2.3.3 ► Torsion

The **torsion** of a curve C at a point $\vec{r}(t)$ (the endpoint of $\vec{r}(t)$) is a measure of how much C departs from the osculating plane.

$$\tau := \frac{d}{ds} \vec{B}(t) \cdot \vec{N}(t)$$

Theorem 2.3.4

If C is a “smooth” curve parameterized by $\vec{r}(t)$, then the torsion is given by:

$$\tau := \frac{[\vec{r}'(t) \times \vec{r}''(t)] \cdot \vec{r}'''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|}$$

Tangential component of \vec{a} is given by:

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|}$$

2.4 Projectile Motion

Suppose we launch some projectile from an initial position $(0, 0)$, initial angle θ , and initial velocity \vec{v}_0 . Disregarding air resistance, the only force acting on this projectile is gravity. Thus, the force vector is given by $\vec{F} = \langle 0, -g, 0 \rangle$ where g is a gravitational constant. Then, the velocity at any time t is given by:

$$\vec{v} = \vec{v}_0 + \vec{F}t$$

Partial Derivatives

3.1 Functions of Several Variables

We will only be concerned with functions in the real numbers.

Definition 3.1.1 ► Function of n Variables

A **function of n variables** is a function $f : X \rightarrow Y$ where $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}$. In other words, it maps ordered pairs of real numbers $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ to some real number $y \in \mathbb{R}$. In this context, we call x_1, \dots, x_n **independent variables** and call y a **dependent variable**.

We will mostly be concerned with functions of two variables, mapping ordered pairs of real numbers to a unique real number. These functions exist in a three-dimensional space, and are thus relatively easy to visualize.

Definition 3.1.2 ► Graph

Let f be a function of two variables with domain D . The **graph** of f is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $(x, y) \in D$ and $z = f(x, y)$

The graph of a function gives us a really nice visualization of three-dimensional functions. Take for instance a function in two variables defined as $f(x, y) = 6 - 3x - 2y$.

finish
example

Definition 3.1.3 ► Level Curve

Let f be a function of two variables. A **level curve** of f is a curve with equation $f(x, y) = k$ for a fixed k in the range of f .

Formally, let $f : A \rightarrow B$ be a function where $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}$. Let $z \in f[A]$ (the image of A under f). A **level curve** is the set of all (x, y, z) such that $f(x, y) = z$.

We call a collection of level curves a **contour map**. Contour maps are most descriptive when the level curves are equally spaced; that is, we choose equally spaced z values.

3.2 Limits and Continuity

Definition 3.2.1 ► Limits for functions of two variables

Let f be a function of two variables with domain D . We say the **limit** as f approaches (a, b) is L if, for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $(x, y) \in D$, if $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

$$\forall(\epsilon > 0)\exists(\delta > 0)\forall((x, y) \in D)\left(0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \implies |f(x, y) - L| < \epsilon\right)$$

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