Chapter 1

Sequences and Convergence

Definition 1.0.1 ▶ Sequence

A ***sequence*** is an ordered list of real numbers.

$$s = (s_1, s_2, s_3, s_4, \dots)$$

Formally, a ***sequence*** is a function $s : \mathbb{N} \to \mathbb{R}$. We write s_n to denote $s(n)$.

We can define a sequence using an expression, like $s_n \coloneqq n^2$. Then $s = (1, 4, 9, 16, \dots)$. Also, we can informally define a sequence in terms of its elements, like $s = (3, 1, 4, 1, 5, 9, \dots)$. We could just have a random sequence like $s \coloneqq (12.3, e^2, 1 - \pi, 10000. \dots)$.

Let's consider how we can formalize the definitions of limits and convergence. Consider the sequence $s_n \coloneqq 1/n$, then $(s_n) = (1, 1/2, 1/3, 1/4, \dots)$. We have an intuitive idea that, as n gets bigger, then $1/n$ gets closer to 0. We can say that this sequence "converges" to 0.

Now consider the sequence $s \coloneqq (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \dots)$. Does this sequence converge? This really depends on our definition of convergence. We might define this as, "s_n gets close to l as n gets large". It certainly matches our intuition, but what exactly does "close to l" mean? Maybe we could say, "$|s_n - l|$ gets small as n gets large". More precisely, this might be "for all $\epsilon > 0$, $|s_n - l| < \epsilon$ when n is large". That "n is large" is still imprecise. Fixing that part, we get the formal definition for convergence:

Definition 1.0.2 ▶ Convergence

Let $s \coloneqq (s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and let $l \in \mathbb{R}$. We say s_n ***converges*** to l if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - l| < \epsilon$ for all $n > N$.

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(n > N)(|s_n - l| < \epsilon)$$

Like in the approximation property, we use ϵ to denote some arbitrarily tiny value that's really really close to 0, but not actually 0. We can also write $\lim_{n\to\infty} s_n = l$ or $s_n \to l$ to mean s_n converges to l.

Technique 1.0.3 ▶ Proving Convergence

To prove that a sequence s converges to l, we carry out the following steps:

1. As some scratch work, solve the inequality $|s_n - l| < \epsilon$ for n.
2. In the formal proof, let $\epsilon >$, and let N be greater than the solved thing. Let $n > N$, then work towards $|s_n - l| < \epsilon$.

Make this explanation better

Example 1.0.4 ▶ $1/n$ converges to 0

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Intuition: Since we're proving something for all $\epsilon > 0$, let's start by choosing some arbitrary $\epsilon > 0$. Next, we need to choose some $N \in \mathbb{N}$ where $|s_n - l| < \epsilon$ for all $n > N$. Thus:

$$\begin{aligned}
|s_n - l| &< \epsilon \\
\left|\frac{1}{n} - 0\right| &< \epsilon \\
\frac{1}{n} &< \epsilon \\
n &> \frac{1}{\epsilon}
\end{aligned}$$

So we choose $N > \frac{1}{\epsilon}$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ where $N > 1/\epsilon$. If $n > N > 1/\epsilon$, then $1/n < \epsilon$. Thus:

$$|s_n - l| = |1/n - 0| = 1/n < \epsilon$$

Therefore, s converges to 0. $\square$

Example 1.0.5

Prove that $\lim_{n\to\infty} \frac{2n+3}{3n+7} = \frac{2}{3}$.

Intuition: This time, we want to choose some $N \in \mathbb{N}$ such that $|s_n - l| < \epsilon$. Thus:

$$\begin{aligned}
\left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| &< \epsilon \\
\left|\frac{6n+9-6n-14}{9n+21}\right| &< \epsilon \\
\frac{5}{9n+21} &< \epsilon \\
\frac{5}{\epsilon} &< 9n+21 \\
\frac{1}{9}\left(\frac{5}{\epsilon} - 21\right) &< n
\end{aligned}$$

Thus, we want to choose $N > {}^1/_9\,({}^5/_\epsilon - 21)$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > {}^1/_9\,({}^5/_\epsilon - 21)$. If $n > N > {}^1/_9\,({}^5/_\epsilon - 21)$, then:

$$\begin{aligned}
9n &> {}^5/_\epsilon - 21 \\
9n &> \frac{5}{\epsilon} - 21 \\
9n + 21 &> \frac{5}{\epsilon} \\
\frac{5}{9n+21} &< \epsilon
\end{aligned}$$

Thus:

$$\begin{aligned}
|s_n - l| &= \left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| \\
&= \left|\frac{6n+9-6n-14}{9n+21}\right| \\
&= \frac{5}{9n+21} \\
&< \epsilon
\end{aligned}$$

$\square$

The above proof chooses a sort of "optimal" or "best possible" N. We could have thrown away the 21 in the denominator, and the inequality we're aiming for will still be the same.

Alternate proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{5}{9\epsilon}$. If $n > N > \frac{5}{9\epsilon}$, then $\frac{5}{9n} < \epsilon$, so $\frac{5}{9n+21} < \frac{5}{9n} < \epsilon$. Then:

$$|s_n - l| = \left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| = \frac{5}{9n+21} < \epsilon$$

$\square$

Example 1.0.6

Prove that $\lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}$.

Intuition: Here, we have to be careful about throwing away terms.

$$\begin{aligned} |s_n - l| &< \epsilon \\ \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| &< \epsilon \\ \left|\frac{6n+9-6n+14}{9n-21}\right| &< \epsilon \\ \frac{23}{|9n-21|} &< \epsilon \end{aligned}$$

We want $9n - 21 > 0$, so we must have $n \geq 3$. We can apply this restriction on n to get rid of the absolute value:

$$\begin{aligned} \frac{23}{9n-21} &< \epsilon \\ \frac{23}{\epsilon} &< 9n - 21 \\ \frac{1}{9}\left(\frac{23}{\epsilon} + 21\right) &< n \end{aligned}$$

Thus, we want to choose some $N > \frac{1}{9}\left(\frac{23}{\epsilon} + 21\right)$ and $N \geq 3$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{1}{9}\left(\frac{23}{\epsilon} + 21\right)$. Then $N > \frac{21}{9}$, and since N is a natural number, then $N \geq 3$. Let $n \in \mathbb{N}$ where $n > N$. Then:

$$9n > \frac{23}{\epsilon} + 21$$
$$9n - 21 > \frac{23}{\epsilon}$$
$$\epsilon > \frac{23}{9n - 21}$$

Thus:

$$|s_n - l| = \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \left|\frac{23}{9n-21}\right| = \frac{23}{9n-21} < \epsilon$$

□

Definition 1.0.7 ▸ Divergence

A sequence ***diverges*** if it does not converge.

$$\exists(\epsilon > 0)\forall(N \in \mathbb{N})\exists(n > N)(|s_n - l| \geq \epsilon)$$

Example 1.0.8 ▸ Diverging Sequence

Prove that $s = (1, 0, 1, 0, 0, 1, 0, 0, 0, \ldots)$ does not converge to 0.

Proof. Let $\epsilon = 1/2$. Then for all $N \in \mathbb{N}$, there exists $n > N$ such that $s_n = 1$. Then:

$$|s_n - 0| = |1 - 0| > \epsilon$$

Therefore, s does not converge. □

1.1 Properties of Limits

A sequence can only converge to one value, not more. That is, if a sequence has a limit, then that limit is unique.

Lemma 1.1.1

Let $x \in \mathbb{R}$. If $x < \epsilon$ for all $\epsilon > 0$, then $x \leq 0$.

Proof. We proceed by contraposition. Suppose $x > 0$. Let $\epsilon \coloneqq x/2 > 0$. Then $x \geq \epsilon = x/2$. ▢

Theorem 1.1.2 ▸ Uniqueness of Limits

Let s_n be a sequence of real numbers. If s_n converges to l_1 and converges to l_2, then $l_1 = l_2$.

Proof. Let $\epsilon > 0$. Since s_n converges to l_1, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - l_1| < \epsilon/2$ for all $n > N_1$. Similarly, since s_n converges to l_2, then there exists $N_2 \in \mathbb{N}$ such that $|s_n - l_2| < \epsilon/2$ for all $n > N_2$.
Let $n \in \mathbb{N}$ where $n > N_1$ and $n > N_2$. Then:

$$|l_1 - l_2| = |l_1 - s_n + s_n - l_2| \leq \underbrace{|l_1 - s_n| + |s_n - l_2|}_{??} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $|l_1 - l_2| < \epsilon$ for all $\epsilon > 0$. Thus, by Lemma 1.1.1, $|l1 - l2| \leq 0$. However, we know that $|l1 - l2| \geq 0$ since it's an absolute value. Thus, we have $|l1 - l2| = 0$, so $l1 = l2$. ▢

Definitions of bounds for sequences, show that convergent implies boundedness

Theorem 1.1.3

Suppose (s_n) and (t_n) are sequences of real numbers, and $s, t \in \mathbb{R}$ such that s_n converges to s and t_n converges to t. Then:

1. cs_n converges to s.
2. $s_n + t_n$ converges to $s + t$.
3. $s_n t_n$ converges to st
4. If $t_n \neq 0$, then for all n and $t \neq 0$, $\frac{s_n}{t_N}$ converges to $\frac{s}{t}$.

Proof of 1. Let $\epsilon > 0$. Since (s_n) converges to s, then there exists $N \in \mathbb{N}$ such that $|s_n - s| < \frac{\epsilon}{1+|c|}$ for all $n > N$. Then, for all $n > N$, we have:

$$|cs_n - cs| = |c(s_n - s)| = |c||s_n - s| < |c|\frac{\epsilon}{1 + |c|} = \frac{|c|}{1 + |c|}\epsilon < \epsilon$$

▢

Proof of 2. Let $\epsilon > 0$. Since (s_n) converges to s, then there exists $N_1 \in \mathbb{N}$ such that

$|s_n - s| < \epsilon/2$ for all $n > N$. Similarly, since t_n converges to t, then there exists $N_2 \in \mathbb{N}$ such that $|t_n - t| < \epsilon/2$. Let $N \in \mathbb{N}$ where $N \geq N_1$ and $N \geq N_2$. Then:

$$|(s_n + t_n) - (s + t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

That is, $s_n + t_n$ converges to $s + t$. ▢

Proof of 3. Let $\epsilon > 0$. Since s_n converges to s, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2(|t|+1)$ for all $n > N$. Also, (s_n) converges, so (s_n) is bounded. That is, there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Since t_n converges to t, there exists $N_2 \in \mathbb{N}$ such that $|t_n - t| < \frac{\epsilon}{2(M+1)}$ for all $n > N$. Let $N \in \mathbb{N}$ such that $N \geq N_1$ and $N \geq N_2$. If $n > N$, then:

$$\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&= |s_n(t_n - t) + (s_n - s)t| \\
&\leq |s_n(t_n - t)| + |(s_n - s)t| \\
&= |s_n||t_n - t| + |s_n - s||t| \\
&< M\frac{\epsilon}{2(1+M)} + \frac{\epsilon}{2(1+|t|)}|t| \\
&= \frac{M}{1+M}\frac{\epsilon}{2} + \frac{\epsilon}{2}\frac{abst}{1+|t|} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

▢

Explain new notation