# Chapter 1

# Vector Calculus

## 1.1 Vector Fields

**Definition 1.1.1 ▸ Vector Field**

Let $D \subseteq \mathbb{R}^n$. A ***vector field*** on $\mathbb{R}^n$ is a function $\vec{F}$ that assigns every point $p \in D$ to an $n$-dimensional vector in $\mathbb{R}^n$.

For example, a vector field in $\mathbb{R}^3$ may have the form $\vec{F} := \langle P, Q, R \rangle$, where $P$, $Q$, and $R$ are functions of three variables. That is:

$$\vec{F}\langle P, Q, R \rangle = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

We say a vector field in $\mathbb{R}^3$ is ***conservative*** if there exists a function $f(x, y, z)$ satisfying:

$$\nabla f(x, y, z) = \vec{F}(x, y, z)$$

That is:

$$\left\langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \right\rangle = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

Thus:

$$\begin{aligned}
f_x(x,y,z) = P(x,y,z) &\implies \int P(x,y,z)\,dx = f(x,y,z) + K(y,z) \\
f_y(x,y,z) = Q(x,y,z) &\implies \int Q(x,y,z)\,dy = f(x,y,z) + K(x,z) \\
f_z(x,y,z) = R(x,y,z) &\implies \int R(x,y,z)\,dx = f(x,y,z) + K(x,y)
\end{aligned}$$

**Definition 1.1.2 ▸ Force Field**