Theorem 0.0.1 ▸ $\mathbb{N}$ is not Bounded Above

Proof. Suppose for contradiction $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is not empty, then $\mathbb{N}$ has a supremum in $\mathbb{R}$. Let $s := \sup \mathbb{N} \in \mathbb{R}$. Then $n \leq s$ for all $n \in \mathbb{N}$. By the Peano axioms, n has a successor $n + 1 \in \mathbb{N}$, so $n + 1 \leq s$ for all $n \in \mathbb{N}$. Therefore, $n \leq s - 1$ for all $n \in \mathbb{N}$. This contradicts s being the least upper bound for $\mathbb{N}$. ▢

Theorem 0.0.2 ▸ Archimedean Principle

Suppose $x, y \in \mathbb{R}$ where $x > 0$. Then, there exists $n \in \mathbb{N}$ such that $nx > y$.

Intuition: This is basically an extension of the fact that $\mathbb{N}$ is not bounded above.

Proof. Since y/x is not an upper bound for $\mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $n > y/x$. Since $x > 0$, then $nx > y$. ▢

Theorem 0.0.3 ▸ Density of $\mathbb{Q}$ in $\mathbb{R}$

Suppose $x, y \in \mathbb{R}$ where $x < y$. Then there exists $r \in Q$ such that $x < r < y$.

Intuition: Given any two different real numbers, there's some rational number between them.

Proof. We will consider three cases:

1. If $x \geq 0$, then $0 \leq x < y$. Since $y - x > 0$, then by the Archimedean Principle, there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$. We want to show there is a natural number between nx and ny. Let $A := \{k \in \mathbb{N} : k > nx\}$. Since $\mathbb{N}$ isn't bounded above, then A is not empty. By the **??**, A has a minimum. Let $m := \min A$. Then $m > nx$, and $m - 1 \leq nx$. Thus, $m \leq nx + 1$, so:

$$nx < m \leq nx + 1 < ny$$

   Dividing through by n yields $x < m/n < y$. Note that $m, n \in \mathbb{N} \subseteq \mathbb{Z}$, so $m/n \in \mathbb{Q}$.

▢