Chapter 1

Naive Set Theory

Instead of forming a rigorous, axiomatic basis for sets, we will simply take an informal approach to sets guided by our intuition. Ultimately, our introduction to real analysis does not fiddle with the fine details of set theory, so it's safe to take a naive approach.

1.1 Sets

Definition 1.1.1 ▸ Set

A ***set*** is a collection of distinct objects.

For example, $\mathbb{N} := \{1, 2, 3 \dots\}$ is the set of all natural numbers, and $\mathbb{Z} := \{\dots, 1, 2, 3, \dots\}$ is the set of all integers. It's conventional to use capital letters to denote sets and use lowercase letters to denote elements of sets. Throughout this chapter, we will use A and B to represent arbitrary sets.

Definition 1.1.2 ▸ Membership, $\in$

We write $a \in A$ to mean "a is in A".

Definition 1.1.3 ▸ Subset, $\subseteq$

A is a ***subset*** of B if everything in A is also in B.

$$A \subseteq B \iff \forall(x \in A)(x \in B)$$

Definition 1.1.4 ▸ Set Equality, =

A ***equals*** B if A is a subset of B and B is a subset of A.

$$A = B \iff (A \subseteq B \land B \subseteq A)$$

Definition 1.1.5 ▸ Proper Subset, ⊊

A is a ***proper subset*** of B if A is a subset of B but B is not a subset of A.

$$A \subsetneq B \iff (A \subseteq B \land B \nsubseteq A)$$

In other words, A is a proper subset of B if everything in A is also in B, but B has something that A does not.

Among mathematical texts, the generic subset symbol $\subset$ has no standardized definition. Some use it to represent subset or equal; others use it to represent proper subset. We will simply not use $\subset$ to avoid any ambiguity.

Definition 1.1.6 ▸ Empty Set (∅)

The ***empty set*** is the set that contains no elements.

$$\varnothing \coloneqq \{\}$$

As convention, we assume that $\varnothing$ is a subset of every set, including itself.

Technique 1.1.7 ▸ Proving a Subset Relation

To prove that $A \subseteq B$:

1. Let x be an arbitrary element of A.
2. Show that $x \in B$.

To prove that $A \nsubseteq B$, choose a specific $x \in A$ and show $x \notin B$.

Example 1.1.8 ▸ Proving Simple Subset Relation

Suppose that $A \subseteq B$ and $B \subseteq C$. Prove that $A \subseteq C$.

Proof. Let $x \in A$ be arbitrary. Since $A \subseteq B$, then $x \in B$. Similarly, since $B \subseteq C$, then $x \in C$. Therefore, $A \subseteq C$. ▢

Definition 1.1.9 ▸ Union

The ***union*** of two sets is the set of all things that are in one or the other set.

$$A \cup B \coloneqq \{x : x \in A \vee x \in B\}$$

Definition 1.1.10 ▸ Intersection

The ***intersection*** of two sets is the set of all things that are in both sets.

$$A \cap B \coloneqq \{x : x \in A \wedge x \in B\}$$

More generally, we can apply union and intersection to an arbitrary number of sets, finite or infinite. We use a notation similar to summation using $\sum$. Let Λ be an indexing set, and for each $\lambda \in \Lambda$, let A_λ be a set.

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$$
$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\}$$

Example 1.1.11 ▸ Indexed Sets

For $n \in \mathbb{N}$, let $A_n = \left[\frac{1}{n}, 1\right] = \left\{x \in \mathbb{R} : \frac{1}{n} \leq x \leq 1\right\}$. Prove that:

(a) $\bigcup_{n=1}^{\infty} = (0, 1]$

(b) $\bigcap_{n=1}^{\infty} = \{1\}$

Proof of (a). Suppose $x \in \bigcup_{n=1}^{\infty} A_n$. Then there exists $n \in \mathbb{N}$ such that $x \in A_n = \left[\frac{1}{n}, 1\right]$. That is, $0 < \frac{1}{n} \leq x \leq 1$. Therefore, $x \in (0, 1]$.

Suppose $x \in (0,1]$. Then $x > 0$, so there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. Then $\frac{1}{n_0} \leq x \leq 1$, so $x \in A_{n_0}$. Therefore, $x \in \bigcup_{n=1}^{\infty} A_n$. ▢

Proof of (b). Suppose $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_1 = \{1\}$.
Suppose $x \in \{1\}$. Then $x = 1 \in \left[\frac{1}{n}, 1\right]$ for all $n \in \mathbb{N}$. Therefore, $x \in \bigcap_{n=1}^{\infty} A_n$. ▢

Definition 1.1.12 ▸ Set Minus

The ***set difference*** of two sets is the set of all things that are in first set but not the second set.

$$A \setminus B = \{x \in A : x \notin B\}$$

Definition 1.1.13 ▸ Complement

Let X be a set called the ***universal set***. The ***complement*** of A in X is defined as $X \setminus A$.

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

Theorem 1.1.14 ▸ De Morgan's Laws for Sets

Suppose X is a set, and for any subset S of X, let $S^c = X \setminus S$. Suppose that $A_\lambda \subseteq X$ for every λ belonging to some index set Λ. Prove that:

(a) $\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$;
(b) $\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c$.

Proof of (a). First, let $a \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c$. Then, $a \in X \setminus \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)$, so $a \in X$ but $a \notin \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)$. Thus, $a \notin A_\lambda$ for any $\lambda \in \Lambda$, so $a \in X \setminus A_\lambda$ for all $\lambda \in \Lambda$. In other words, $a \in \bigcap_{\lambda \in \Lambda} A_\lambda^c$.
Next, let $a \in \bigcap_{\lambda \in \Lambda} A_\lambda^c$. Then $a \in A_\lambda^c$ for all $\lambda \in \Lambda$, so $a \in X$ but $a \notin A_\lambda$ for all $\lambda \in \Lambda$. That is, $a \notin \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)$. In other words, $a \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c$. ▢

Proof of (b). First, let $a \in \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^c$. Then, $a \in X \setminus \bigcap_{\lambda \in \Lambda} A_\lambda$, so $a \in X$ but $a \notin \bigcap_{\lambda \in \Lambda} A_\lambda$. That is, $a \notin A_\lambda$ for some $\lambda \in \Lambda$. Thus, $a \in X \setminus A_\lambda$ for some $\lambda \in \Lambda$. Therefore, $a \in \bigcup_{\lambda \in \Lambda} A_\lambda^c$.
Next, let $a \in \bigcup_{\lambda \in \Lambda} A_\lambda^c$. Then $a \in A_\lambda^c$ for some $\lambda \in \Lambda$, so $a \in X$ but $a \notin A_\lambda$ for some

$\lambda \in \Lambda$. That is, $a \notin \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. Therefore, $a \in \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c$. ▢

1.2 Functions

We generally think of functions as a "map" or "rule" that assigns numbers to other numbers. For example, $f(x) = 2x$ maps $1 \mapsto 2$, $2 \mapsto 4$, etc. More formally, we define functions in terms of sets.

Definition 1.2.1 ▸ Cartesian Product

Let X and Y be sets. The ***Cartesian product*** of X and Y is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$.

$$X \times Y := \{(x, y) : x \in X \wedge y \in Y\}$$

Definition 1.2.2 ▸ Relation

Let X and Y be sets. A ***relation*** between X and Y is a subset of the Cartesian product $X \times Y$.

Definition 1.2.3 ▸ Function

Let X and Y be sets. A ***function*** from X to Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ where $(x, y) \in f$.

More formally, a ***function*** $f : X \to Y$ is a subset of $X \times Y$ satisfying:

1. $\forall (x \in X)\, [\exists (y \in Y)((x, y) \in f)]$
2. $(x, y_1), (x, y_2) \in f \implies y_1 = y_2$

Given a function $f : X \to Y$, we call X the ***domain*** of f and Y the ***codomain*** of f. Given $x \in X$, we write $f(x)$ to denote the unique element of Y such that $(x, y) \in f$.

$$f(x) = y \iff (x, y) \in f$$

Definition 1.2.4 ▸ Function Image

Let $f : X \to Y$ be a function and $A \subseteq X$. The ***image*** of A under f is the set containing all possible function outputs from all inputs in A.

$$f[A] \coloneqq \{f(a) : a \in A\}$$

Given $f : X \to Y$, we call $f[X]$ the ***range*** of f.

Example 1.2.5 ▸ Function Images

Suppose $f : X \to Y$ is a function, and $A_\lambda \subseteq X$ for each $\lambda \in \Lambda$. Then:

(a) $f\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] = \bigcup_{\lambda\in\Lambda} f\left[A_\lambda\right]$

(b) $f\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] \subseteq \bigcap_{\lambda\in\Lambda} f\left[A_\lambda\right]$

In this example, we will only prove the "forward" direction. That is, we want to show that $f\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] \subseteq \bigcup_{\lambda\in\Lambda} f\left[A_\lambda\right]$.

Proof of (a). Let $y \in f\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right]$. By definition of Function Image, there exists $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$ such that $y = f(x)$. Thus, there exists $\lambda_0 \in \Lambda$ such that $x \in \lambda_0$. That is, $y \in f\left[A_{\lambda_0}\right]$. Therefore, $y \in \bigcup_{\lambda\in\Lambda} f\left[A_\lambda\right]$. ▢

Definition 1.2.6 ▸ Function Inverse Image

Let $f : X \to Y$ be a function and $B \subseteq Y$. The ***inverse image*** of B under f is the set containing all possible function inputs whose output is in B.

$$f^{-1}[B] \coloneqq \{x \in X : f(x) \in B\}$$

Note the following logical equivalence:

$$x \in f^{-1}[B] \iff f(x) \in B$$

Example 1.2.7 ▶ Function Inverse Images

Suppose $f : X \to Y$ is a function, and $B_\lambda \subseteq Y$ for each $\lambda \in \Lambda$. Then:

$$f^{-1}\left[\bigcup_{\lambda \in \Lambda} B_\lambda\right] = \bigcup_{\lambda \in \Lambda} f^{-1}[B_\lambda]$$

Again, we will only prove the "forward direction".

Proof. Let $x \in f^{-1}\left[\bigcup_{\lambda\in\Lambda} B_\lambda\right]$. Then, $f(x) \in \bigcup_{\lambda\in\Lambda} B_\lambda$. That is, $f(x) \in B_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Thus, $x \in f^{-1}[B_{\lambda_0}]$, so $x \in \bigcup_{\lambda\in\Lambda} f^{-1}[B_\lambda]$. ▢

1.3 Injectivity and Surjectivity

Definition 1.3.1 ▶ Injective, One-to-one

A function $f : X \to Y$ is ***injective*** or ***one-to-one*** if no two inputs in X have the same output in Y.

$$\forall(x_1, x_2 \in X)\,[x_1 \neq x_2 \implies f(x_1) \neq f(x_2)]$$

We can also think of injectivity as, "if two inputs have the same output, then the two inputs must be the same". It's really just the contrapositive of our initial definition, which we know must be logically equivalent.

$$\forall(x_1, x_2 \in x)\,[f(x_1) = f(x_2) \implies x_1 = x_2]$$

For example, the function $f(x) = x^2$ is not injective, because $f(-1) = 1$ and $f(1) = 1$. We have two distinct inputs that map to the same output.

Technique 1.3.2 ▶ Proving a Function is Injective

To prove a function $f : X \to Y$ is injective:

1. Let $x_1, x_2 \in X$ where $f(x_1) = f(x_2)$.
2. Reason that $x_1 = x_2$.

Example 1.3.3 ▸ Proving Injectivity

$f(x) = -3x - 7$ is injective.

Proof. Suppose $f(x_1) = f(x_2)$. Then $-3x_1 + 7 = -3x_2 + 7$, so $-3x_1 = -3x_2$. Thus, $x_1 = x_2$, so f is injective. □

Example 1.3.4 ▸ Disproving Injectivity

Prove that $f(x) = x^2$ is not injective.

Proof. $f(-1) = 1$ and $f(1) = 1$, but $-1 \neq 1$. Thus, f is not injective. □

Definition 1.3.5 ▸ Surjective, Onto

A function $f : X \to Y$ is ***surjective*** or ***onto*** if everything in Y has a corresponding input in X.

$$\forall(y \in Y)\,[\exists(x \in X)(f(x) = y)]$$

Note that $f : X \to f[X]$ is **always** surjective.

Technique 1.3.6 ▸ Proving a Function is Surjective

To prove a function $f : X \to Y$ is surjective:

1. Let $y \in Y$ be arbitrary.
2. "Undo" the function f to obtain $x \in X$ where $f(x) = y$.

Example 1.3.7 ▸ Proving Surjectivity

Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -3x + 7$ is surjective.

Proof. Let $y \in Y$ be arbitrary. Let $x \coloneqq \frac{y-7}{-3}$. Then $x \in \mathbb{R}$, and:

$$\begin{aligned} f(x) &= -3\left(\frac{y-7}{-3}\right) + 7 \\ &= (y-7) + 7 \\ &= y \end{aligned}$$

Therefore, f is surjective. $\square$

Definition 1.3.8 ▸ Bijective

A function $f : X \to Y$ is ***bijective*** if it is both injective and surjective.

Definition 1.3.9 ▸ Function Composition

Let $f : X \to Y$ and $g : Y \to Z$ be functions. The ***composition*** of f and g is a function $g \circ f : X \to Z$ defined by:

$$(g \circ f)(x) \coloneqq g(f(x))$$

Theorem 1.3.10 ▸ Composition Preserves Injectivity and Surjectivity

Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.

(a) If f and g are injective, then $g \circ f$ is injective.
(b) If f and g are surjective, then $g \circ f$ is surjective.
(c) If f and g are bijective, then $g \circ f$ is bijective.

Proof of (a). Let $x_1, x_2 \in X$. Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then, $g(f(x_1)) = g(f(x_2))$. Because g is injective, we have $f(x_1) = f(x_2)$. Because f is injective, we have $x_1 = x_2$. Therefore, $g \circ f$ is injective. $\square$

Proof of (b). Let $z \in Z$. Because g is surjective, there exists an element $y \in Y$ such that $g(y) = z$. Because f is surjective, there exists an element $x \in X$ such that $f(x) = y$. Thus, $(g \circ f)(x) = g(f(x)) = g(y) = z$. Therefore, $g \circ f$ is surjective. $\square$

Proof of (c). We know that from (a) and (b) composition preserves injectivity and surjectivity. Thus, composition must also preserve bijectivity. $\square$

Definition 1.3.11 ▸ Inverse Function

Let $f : X \to Y$ be a bijection. The ***inverse function*** of f is a function $f^{-1} : Y \to X$ defined by:

$$f^{-1} \coloneqq \{(y, x) \in Y \times X : (x, y) \in f\}$$

The notation for inverse functions conflicts with the notation for inverse images. A key distinction to make it that only bijections can have an inverse function, but we can apply the inverse image to any function. Thus, given a bijection $f : X \to Y$, we know $f^{-1}(f(x)) = x$ for all $x \in X$, and $f(f^{-1}(y)) = y$ for all $y \in Y$.

Example 1.3.12

Let $f : X \to Y$ and $g : Y \to X$ be functions such that $(g \circ f) = x$ for all $x \in X$, and $(f \circ g)(y) = y$ for all $y \in Y$. $f^{-1} = g$.

Proof. todo: finish proof $\square$