Introduction to Abstract Algebra

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Contents

1 Introduction

TODO: Pentagon rotation and mirroring example

1.1 Relations

Definition 1.1.1 ▸ Relation

Let A and B be sets.

- A ***relation*** from A to B is a subset of the Cartesian product $A \times B$.
- A ***relation*** on A is a subset of the Cartesian product $A \times A$.

Given a relation ρ, we denote $(a, b) \in \rho$ as $a \mathrel{\rho} b$. If $(a, b) \notin \rho$, we write $a \not\rho\, b$.

Definition 1.1.2 ▸ Reflexive, symmetric, transitive, equivalence relation

Let ρ be a relation on a set A.

- ρ is ***reflexive*** if, for any $a \in A$, $a \mathrel{\rho} a$.
- ρ is ***symmetric*** if $a \mathrel{\rho} b$ implies $b \mathrel{\rho} a$.
- ρ is ***transitive*** if, whenever $a \mathrel{\rho} b$ and $b \mathrel{\rho} c$, we have $a \mathrel{\rho} c$.

If ρ satisfies all three properties, it is called an ***equivalence relation***. We often use $\sim$ to denote an equivalence relation.

Definition 1.1.3 ▸ Equivalence class

Let $\sim$ be an equivalence relation on a set A, and let $a \in A$. The ***equivalence class*** of a is a set defined as:

$$[a] \coloneqq \{b \in A : a \sim b\}$$

1.2 Functions

Definition 1.2.1 ▶ Function

Let X and Y be sets. A ***function*** from X to Y is a relation f from X to Y such that, for each $x \in X$, there exists exactly one $y \in Y$ where $x\ f\ y$. We write $f : X \to Y$ to mean f is a function from X to Y, and we write $f(x) = y$ to mean $x\ f\ y$.

Definition 1.2.2 ▶ Injective, surjective, bijective

Let $f : X \to Y$ be a function.

- f is ***injective*** if, for all x_1 and x_2 where $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.
- f is ***surjective*** if, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
- f is ***bijective*** if it is both injective and surjective.

Definition 1.2.3 ▶ Permutation

A ***permutation*** of a set A is a function from A to A.

Definition 1.2.4 ▶ Binary operation

A ***binary operation*** on a set A is a function from $A \times A$ to A.

Wowzers

2

The Integers and Modular Arithmetic

Theorem 2.0.1 ▶ Well Ordering Axiom

If S is a nonempty subset of $\mathbb{N}$, then S has a minimum value.

Theorem 2.0.2 ▶ Principle of Mathematical Induction

For each $n \in \mathbb{N}$, let $P(n)$ denote a statement. Suppose that:

1. $P(1)$ is true, and
2. for each $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

2.1 Divisibility

Theorem 2.1.1 ▶ Division Algorithm

TODO: division algorithm

Definition 2.1.2 ▶ Divides

Let $a, b \in \mathbb{Z}$. We say a ***divides*** b if there exists an integer k such that $b = ka$. We write $a \mid b$ to mean a divides b.

Definition 2.1.3 ▶ Greatest common divisor (GCD)

Let $a, b \in \mathbb{Z}$ where at least one is non-zero. The ***greatest common divisor (GCD)*** of a and b is the largest positive integer g such that $g \mid a$ and $g \mid b$. We write $\gcd(a, b)$ or simply (a, b) to denote the greatest common divisor of a and b.

Definition 2.1.4 ▶ Relatively prime, coprime

Let $a, b \in \mathbb{Z}$, where at least one is non-zero. We say a and b are ***relatively prime*** (or ***coprime***) if $\gcd(a, b) = 1$.

Theorem 2.1.5

Let $a, b \in \mathbb{Z}$, where at least one is non-zero. Then there exist $u, v \in \mathbb{Z}$ where $\gcd(a, b) = au + bv$. Moreover, $\gcd(a, b)$ is the smallest possible number of all values of u and v.

Theorem 2.1.6 ▶ Euclidean Algorithm

TODO

2.2 Prime Factorization

Definition 2.2.1 ▶ Prime, composite

A natural number $p > 1$ is ***prime*** if its only positive divisors are 1 and p itself. Otherwise, p is ***composite***.

Theorem 2.2.2 ▶ Euclid's Lemma

Let $p \in \mathbb{N}$ where $p > 1$. p is prime if and only if, for any integers a and b where $p \mid ab$, then $p \mid a$ or $p \mid b$.

Theorem 2.2.3 ▶ Fundamental Theorem of Arithmetic

For every natural number a greater than 1, there exists a unique set of primes $\{p_1, \dots, p_n\}$ such that $a = p_1 \cdots p_n$.

2.3 Properties of Integers

2.4 Modular Arithmetic

Definition 2.4.1 ▶ Modular congruency

Let $n \in \mathbb{N}$ where $n > 1$, and let $a, b \in \mathbb{Z}$. We say a is ***congruent*** to b ***modulo*** n if $n \mid (a-b)$ (that is, if a and b have the same remainder when divided by n). We write $a \equiv b \pmod{n}$ to mean a is congruent to b modulo n.

Theorem 2.4.2

Let $n \in \mathbb{N}$ where $n > 1$. Then $a \equiv b \pmod{n}$ is an equivalence relation.

The equivalence classes of $a \equiv b \pmod{n}$ are conventionally written as:

$$[0], [1], \dots, [n-1]$$

These are called the ***congruence classes modulo*** n, where:

$$\mathbb{Z}_n := \{[0], [1], \dots, [n-1]\}$$

On $\mathbb{Z}_n$, we define addition modulo n and multiplication modulo n as:

$$[a] + [b] = [a+b]$$
$$[a] \cdot [b] = [ab]$$

For example, in $\mathbb{Z}_7$, we have $[5] + [6] = [4]$. We will often shorten this as $5 + 6 = 4$ when the context is clear.

Theorem 2.4.3

Addition modulo n and multiplication modulo n are well-defined.

Proof. Fix $n \in \mathbb{N}$ where $n > 1$. Suppose $a_1 \equiv a_2 \pmod{n}$ and $b_1 \equiv b_2 \pmod{n}$. To prove addition modulo n is well-defined, we need to verify the following equality:

$$[a_1] + [b_1] = [a_2] + [b_2]$$

Note that:

$$(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$$

Since $n \mid (a_1 - a_2)$ and $n \mid (b_1 - b_2)$, we have $n \mid [(a_1 + b_1) - (a_2 + b_2)]$, so addition is well-defined.
To prove multiplication is well-defined, we need to verify the following equality:

$$[a_1][b_1] = [a_2][b_2]$$

Note that:

$$a_1b_1 - a_2b_2 = a_1b_1 - a_1b_2 + a_1b_2 - a_2b_2 = a_1(b_1 - b_2) + (a_1 - a_2)b_2$$

So multiplication modulo n is also well-defined ▢

These operations follow similar properties as traditional integer addition and multiplication. Addition in $\mathbb{Z}_n$ is closed, associative, commutative, and has additive identity $[0]$ and additive inverse $[-a]$ for any $a \in \mathbb{Z}_n$.

Multiplication in $\mathbb{Z}_n$ is closed, associative, commutative, distributive, and has multiplicative identity $[1]$. However, not every $\mathbb{Z}_n$ has a multiplicative inverse for all elements.

Example 2.4.4 ▸ Multiplicative inverse in $\mathbb{Z}_n$

In $\mathbb{Z}_6$, does $ab = 0$ mean that $a = 0$ or $b = 0$? Not necessarily: $a = 3$ and $b = 2$ is a counterexample.

In $\mathbb{Z}_7$, does $ab = 0$ mean $a = 0$ or $b = 0$? For any $a \in \mathbb{Z}_7$ where $a \neq 0$, note that $\gcd(a, 7) = 1$. Thus, there exist $u, v \in \mathbb{Z}$ where $au + 7v = 1$. Rearranging, we get $7v = 1 - au$, so $7 \mid (au - 1)$. That means $[a][u] = [1]$, so u is the multiplicative inverse of a. Since our choice of a was arbitrary, then every element in $\mathbb{Z}_7$ has a multiplicative inverse.

Example 2.4.5

In $\mathbb{Z}_5$, what is 4^{91}?

$$\begin{aligned} 4^1 &= 4 \\ 4^2 &= 1 \\ 4^3 &= 4 \\ 4^4 &= 1 \\ &\vdots \\ 4^{91} &= 4 \end{aligned}$$

$3^1 = 3, 3^2 = 4, 3^2 = 2, 3^4 = 1$, so $3^{91} = (3^4)^{22} \cdot 3^3 = 2$.

Example 2.4.6

Find b satisfying:

$$\begin{aligned} b &\equiv 3 \pmod{5} \\ b &\equiv 4 \pmod{11} \\ b &\equiv 6 \pmod{14} \end{aligned}$$

Note that 5 and 11 are relatively prime, so there exist $u, v \in \mathbb{Z}$ where $5u + 11v = 1$. In this case, we can take $u = -2$ and $v = 1$. Note that:

$$\begin{aligned} 5(-2)4 + 11(1)3 &\equiv 3 \pmod{5} \\ 5(-2)4 + 11(1)3 &\equiv 4 \pmod{11} \end{aligned}$$

More generally, we can take $b = -7 + 55k$ for any $k \in \mathbb{Z}$.

Alternatively, we can let:

$$\begin{aligned} d_1 &:= 11 \cdot 14 = 154 \\ d_2 &:= 5 \cdot 14 = 70 \\ d_3 &:= 5 \cdot 11 = 55 \end{aligned}$$

Note that $\gcd(5, 154) = 1$, so:

$$5(31) + 154(-1) = 1 \implies 5 \cdot 31 \equiv 1 \pmod{5}$$

$$11(-19) + 70(3) = 1 \implies 70 \cdot 3 \equiv 1 \pmod{11}$$

$$14(4) + 55(-1) = 1 \implies 55(-1) \equiv 1 \pmod{14}$$

Let $b := 154(-1)(3) + 70(3)4 + 55(-1)6$. Then:

$$b \pmod{5} = 154(-1)(3) = 3$$

$$b \pmod{11} = 4$$

$$b \pmod{14} = 6$$

Theorem 2.4.7 ▸ Chinese Remainder Theorem

Let $n_1, \ldots, n_k$ be positive integers, all greater than 1, where any two different n_i and n_j are relatively prime. If $a_1, \ldots, a_n \in \mathbb{Z}$, we can find $b \in \mathbb{Z}$ satisfying $b \equiv a_i \pmod{n_i}$ for all $1 \leq i \leq k$. Moreover, if $c \equiv a_i \pmod{n_i}$, then $b \equiv c \pmod{n_1 n_2 \cdots n_k}$.

3 Introduction to Groups

3.1 The Basics

Definition 3.1.1 ▶ Group

A ***group*** is a set G together with a binary operation $*$ satisfying for any $a, b, c \in G$:

- ***closure*** under $*$, meaning $a * b \in G$;
- ***associativity*** under $*$, meaning $(a * b) * c = a * (b * c)$;
- existence of an ***identity element*** $e \in G$ satisfying $e * a = a * e$; and
- existence of an ***inverse*** for a, say $a^{-1} \in G$ where $a * a^{-1} = a^{-1} * a = e$.

A group is ***abelian*** if it is commutative under $*$, meaning $a * b = b * a$ for any $a, b \in G$.

Some examples of groups include $\mathbb{Z}$ under addition, $\mathbb{Z}_n$ where $n \geq 2$ under addition, and D_{10} under $\circ$, the dihedral group of the regular pentagon, often called D_5. (TODO: pentagon example)

Theorem 3.1.2 ▶ Uniqueness of identities and inverses

Let G be a group.

1. The identity of G is unique (that is, there is only one identity element in G).
2. For any $a \in G$, its inverse a^{-1} is unique.

Proof of 1. Let e and f be identity elements in G Then $ef = e$ because f is an identity, and $ef = f$ because e is an identity. Thus, $e = f$. ▢

Proof of 2. Let b and c be inverses of a. Then $bac = (ba)c = ec = c$, and $bac = b(ac) = be = b$. Thus, $b = c$. ▢

Theorem 3.1.3 ▶ Cancellation

Let G be a group, and let $a, b, c \in G$. If $ab = ac$ or $ba = ca$, then $b = c$.

Proof sketch. If $ab = ac$, then $a^{-1}(ab) = a^{-1}(ac)$ … so $b = c$. ▢

Theorem 3.1.4

Let G be a group, and let $a, b \in G$. Then there is a unique $c \in G$ satisfying $ac = b$, and there is a unique $d \in G$ satisfying $da = b$.

Intuition: $c = a^{-1}b$ and $d = ba^1$.

Definition 3.1.5 ▶ Permutation

A ***permutation*** on a set A is an injective function $\sigma : A \to A$, written as:

$$\sigma \coloneqq \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

to mean $\sigma(1) = a, \sigma(2) = b, \sigma(3) = c$.

Since these are functions, we can compose two or more permutations.

Permutation example, composition example

The set of permutations, under function composition, is a ***group***.

- Closed
- Associativity
- Existence of an identity element e where $e \circ \sigma = \sigma \circ e$ for all σ. In this case, e is simply the identity function.
- Existence of an inverse for each σ. That is, for any σ, there exists τ where $\sigma \circ \tau = \tau \circ \sigma = e$.

Definition 3.1.6 ▶ Symmetric group (S_n)

The set of permutations on 3 elements under function composition is called S_3, the ***symmetric group*** on 3 elements.

Let $n \geq 2$. Let $U(n)$ denote the set of all $a \in \mathbb{Z}_n$ where $\gcd(a, n) = 1$, under the multiplication modulo n.

Definition 3.1.7 ▶ Direct product

Let G be a group with operation $*$, and let H be a group with operation $\cdot$. On the Cartesian product $G \times H$, define the operation $\diamond$ by:

$$(g_1, h_1) \diamond (g_2, h_2) := (g_1 * g_2, h_1 \cdot h_2)$$

for all $g_i \in G, h_i \in H$. We call this the ***direct product*** of G and H.

Theorem 3.1.8 ▸ Direct product is always a group

The direct product of any two groups is itself a group.

Example 3.1.9 ▸ Simple direct product

Consider the direct product $\mathbb{Z}_3 \times S_3$.

- How many elements are in the direct product?
- What is the identity element?
- What is the inverse of $\left(2, \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)\right)$?

- There are 3 elements in $\mathbb{Z}_3$ and 6 elements in S_3, so there are a total of 18 elements in the direct product.
- The identity element is $\left(0, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}\right)$
- The inverse of $\left(2, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right)$ is $\left(1, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right)$

matrix size

3.2 Order

Integer powers

In groups under an addition operation such as $\mathbb{Z}_{15}$, we write $7 \cdot 2$ instead of 2^7 to avoid ambiguity with the notation for integer powers.

Theorem 3.2.1 ▸ Properties of power

1. $a^m a^n = a^{m+n}$
2. $(a^m)^n = a^{mn}$
3. $a^{-n} = (a^{-1})^n = (a^n)^{-1}$

Definition 3.2.2 ▸ Order

Let G be a group under operation $\cdot$.

- The ***order*** of G (denoted $|G|$) is the number of elements in G. G is ***finite*** if its order is finite; otherwise, it's an ***infinite*** group.
- The ***order*** of an element $a \in G$ (denoted $|a|$) is the smallest positive integer where:

$$\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}} = e \quad \text{(the identity element of } G\text{)}$$

If such an n exists, a has ***finite order***; otherwise, a has ***infinite order***.

In any group, the identity element is the only element that has order 1.

Example 3.2.3 ▸ Order of common groups

- $|Z| = \infty$
- $|Z_{15}| = 15$
- $|D_{10}| = 10$
- $|S_5| = 5!$
- $|D_6 \times S_4| = 6 \cdot 4!$

Example 3.2.4 ▸ Order of elements in common groups

- Order of $2 \in Z_4$ is 2 because $2 + 2 = 0 = e$
- Order of $3 \in U(8)$ is 3 because $3^2 = 1 = e$
- Order of $\sigma \coloneqq \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right) \in S_3$ is 3 because $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)^3 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)$

Theorem 3.2.5 ▸ Properties of order

Let G be a group, and let $a \in G$.

1. If a has infinite order, then $a^i = a^j$ if and only if $i = j$.
2. If a has order $n \in \mathbb{Z}^+$, then $a^i = a^j$ if and only if $n \mid (i - j)$.

Proof sketch. Consider i and j where $a^{i-j} = e$.

1. If a has infinite order, then $i - j = 0$.
2. If a has finite order, write $i - j = nq + r$ for $0 \le r < n$ (by the division algorithm TODO: REF). Then:

$$a^{i-j} = (a^n)^q a^r = e$$

So $a^r = e$. But $r < n$, and n is the smallest positive integer satisfying $a^n = e$. Thus, $r = 0$.

$\square$

Corollary 3.2.6

Let G be a group, and let $a \in G$ where $|a| = n \in \mathbb{Z}^+$. Then $a^i = e$ if and only if $n \mid i$.

Example 3.2.7

Show that ab and ba have the same order.

Suppose $(ab)^n = e$. Then:

$$\begin{aligned}(ba)^n &= \underbrace{baba \cdots ba}_{n \text{ times}} \\ &= b(ab)^{n-1}a\end{aligned}$$

So $(ba)^n b = b(ab)^{n-1}ab = b(ab)^n = b$. Thus, $(ba)^n = e$. Thus, $n \mid |ba|$, or $|ab| \mid |ba|$.

3.3 Cyclic Groups

Definition 3.3.1 ▸ Cyclic

A group G is ***cyclic*** if there exists $a \in G$ where, for any $b \in G$:

$$b = a^n \quad \text{for some } n \in \mathbb{Z}$$

In other words, G is cyclic if there exists $a \in G$ where any element of G is a power of a. In this context, we say a is a ***generator*** of G and write $G = \langle a \rangle$, where:

$$\langle a \rangle := \{a^k : k \in \mathbb{Z}\}$$

For example, $\mathbb{Z}$ under addition is a cyclic group. For any $n \in \mathbb{Z}$, we have:

$$1 \cdot n = n$$

Note here that $1 \cdot n$ reflects the idea of integer powers under addition. We apply the group operation of addition n-times. For example:

$$5 = 1^5 = 1 \cdot 51 + 1 + 1 + 1 + 1$$
$$-2 = 1^{-2} = 1 \cdot (-2) = -(1 + 1)$$

When dealing with additive operations, we usually omit the exponent notation and simply write the multiplicative expression. Note also that $\mathbb{Z}$ can be generated by -1. Thus, the generator of a cyclic group is not guaranteed to be unique.

Another example, in $\mathbb{Z}_{12}$, we have:

$$\langle 1 \rangle = \{0, 1, 2, \dots, 10, 11\}$$

$$\langle 4 \rangle = \{0, 4, 8\}$$

In fact, this $\langle 4 \rangle$ is itself a group under addition modulo 12.

Theorem 3.3.2 ▸ Every cyclic group is abelian

Let G be a group. If G is cyclic, then it is abelian.

3.4 Subgroups

Definition 3.4.1 ▸ Subgroup

Let G be a group under an operation $*$. Then a subset $H \subseteq G$ is considered a ***subgroup*** of G if H itself also a group under $*$. H is called a ***proper subgroup*** of G if $H \subsetneq G$.

Trivially, every group is a subgroup of itself. Also, $\{e\}$ is a subgroup of every group. More substantially, $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$, and $\mathbb{Q}$ is a subgroup of $\mathbb{R}$. This is sometimes written as $\mathbb{Z} \leq \mathbb{Q}$, and $\mathbb{Q} \leq \mathbb{R}$.

Theorem 3.4.2 ▸ Conditions for subgroup

Let G be a group under operation $*$, and let $H \subseteq G$. Then H is a subgroup of G if and only if:

1. $e \in H$ (the subset contains the identity);
2. for any $a, b \in H$, $a * b \in H$ (the subset is closed under $*$); and
3. for any $a \in H$, $a^{-1} \in H$ (the subset contains all inverses).

Example 3.4.3 ▸ Determining $3\mathbb{Z}$ is a subgroup of $\mathbb{Z}$

Consider the following set:

$$3\mathbb{Z} := \{3x : x \in \mathbb{Z}\}$$

We have:

1. $0 \in 3\mathbb{Z}$
2. For any $a, b \in 3\mathbb{Z}$, $a = 3m$ and $b = 3n$ for some $m, n \in \mathbb{Z}$. Thus, $a+b = 3m+3n = 3(m+n) \in 3\mathbb{Z}$.
3. For any $a \in 3\mathbb{Z}$, $a^{-1} = -a = 3(-m)$.

Thus, we can confirm that $3\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

Note that in the above example, we can also write:

$$3\mathbb{Z} := \langle 3 \rangle = \{3x : x \in \mathbb{Z}\}$$

Definition 3.4.4 ▸ Cyclic subgroup

Let G be a group, and let $a \in G$. The ***cyclic subgroup*** generated by a is defined as:

$$\langle a \rangle := \{a^n : n \in \mathbb{Z}\}$$

For example, in $\mathbb{Z}_{12}$, we have:

$$\begin{aligned}
\langle 0 \rangle &= \{0\} \\
\langle 1 \rangle &= \mathbb{Z}_{12} \\
\langle 2 \rangle &= \{0, 2, 4, 6, 8, 10\} \\
\langle 3 \rangle &= \{0, 3, 6, 9\} \\
\langle 4 \rangle &= \{0, 4, 8\} \\
\langle 5 \rangle &= \{0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7\} = \mathbb{Z}_{12} \\
\langle 6 \rangle &= \{0, 6\} \\
\langle 7 \rangle &= \ldots = \mathbb{Z}_{12} \\
\langle 8 \rangle &= \{0, 8, 4\} \\
\langle 9 \rangle &= \{0, 9, 6, 3\} \\
\langle 10 \rangle &= \{0, 10, 8, 6, 4, 2\} = \langle 2 \rangle \\
\langle 11 \rangle &= \ldots = \mathbb{Z}_{12}
\end{aligned}$$

From this, it seems that numbers relatively prime with 12 can generate the entirety of $\mathbb{Z}_{12}$. In fact, if $|a| = n$, then $|a^i| = \frac{n}{\gcd(n,i)}$.

Check this fact!!!

Theorem 3.4.5 ▸ Cyclic subgroups are groups

Let G be a group, and let $a \in G$. $\langle a \rangle$ is a subgroup.

Proof sketch. We simply check the three conditions.

1. $e = a^0$.
2. $a^m a^n = a^{m+n}$
3. if $a^m \in \langle a \rangle$, then $a^{-m} \in \langle a \rangle$.

Thus, $\langle G \rangle$ is a subgroup of G. ▢

Theorem 3.4.6 ▸ Conditions for subgroup relaxed

Let G be a group, and let $H \subseteq G$. Then H is a subgroup G if and only if:

1. $e \in H$, and
2. $ab^{-1} \in H$ for any $a, b \in H$.

Proof sketch. Let $a \in H$. $e \in H$ by (1), so $1 \cdot a^{-1} \in H$.
Let $a, b \in H$. $b^{-1} \in H$ by the first statement, so then $a(b^{-1})^{-1} \in H$, so $ab \in H$. ▢

Theorem 3.4.7 ▸ Conditions for finite subgroup

Let G be a group, and let H be a **finite** subset of G. Then H is a subgroup of G if and only if:

1. $e \in H$, and
2. $ab \in H$ for any $a, b \in H$.

Intuition: This theorem is saying that if we take a finite subset of G, then these two conditions alone imply the existence of inverses, and vice versa. For any $a \in H$, we have:

$$\langle a \rangle = \{e, a, a^2, a^3, \ldots\} \subseteq H$$

Since H is finite, then these a's must "wrap around" back to e. For example, we might have $a^5 = a^{17}$, which implies that $e = a^{12} = a(a^{11})$. Thus, the inverse of a is a^{11}.

Crucially, this theorem does not apply for infinite subsets/subgroups.

Definition 3.4.8 ▸ Center

Let G be a group. The ***center*** of G is defined as:

$$Z(G) := \{z \in G : az = za \text{ for all } a \in G\}$$

If G is abelian, then $Z(G) = G$.

TODO: dihedral groups, diagram thing

3.5 Cyclic Groups

Definition 3.5.1 ▸ Euler phi-function

The ***Euler phi-function*** is a function $\phi : \mathbb{N} \to \mathbb{N}$ where $\phi(n)$ is the number of integers $1 \leq i \leq n$ where $\gcd(i, n) = 1$.

For example, to calculate $\phi(10)$, we can look at all the integers 1 through 10 and see if they are relatively prime to 10. From doing this, we see that only 1, 3, 7, and 9 are relatively prime to 10. Thus, $\phi(10) = 4$.

Much much more stuff

3.6 Cosets and Lagrange's Theorem

Definition 3.6.1 ▸ Modular congruency (groups)

Let G be a group, and let H be a subgroup of G. For any $a, b \in G$, we say a is ***congruent*** to b ***modulo*** if $a^{-1}b \in H$. That is:

$$a \equiv b \pmod{H} \iff a^{-1}b \in H$$

Theorem 3.6.2

For any group H, congruence modulo H is an equivalence relation.

Proof. If $a \equiv b \pmod{H}$, then $a^{-1}b \in H$. Thus, $a^{-1}b = h$ for some $h \in H$. Also, $b = ah \in aH$, and clearly $a \in aH$. If $b = ah$ for some $h, \in H$, then $a^{-1}b = h \in H$. Thus, $a \equiv b \pmod{H}$. □

As with any equivalence relation, we have equivalence classes defined below:

Definition 3.6.3 ▸ Left coset, right coset

Let H be a subgroup of G. For any $g \in G$, the ***left cosets*** of H in G are sets defined as:

$$gH := \{gh : h \in H\}$$

Similarly, we define ***right cosets*** of H in G as:

$$Hg := \{hg : h \in H\}$$

Note: If the group operation is addition, we write $g + H$ instead of gH.

Theorem 3.6.4 ▸ Cosets partition a group

Let H be a group, and let H be a subgroup of G. Then the left cosets of H in G partition G.

1. Each $a \in G$ is in exactly one left coset, aH; and
2. if $a, b \in G$, either $aH = bH$ or $aH \cap bH = \varnothing$.

Example 3.6.5 ▸ Left cosets and partitioning

Consider the group $U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$ with $H = \langle 3 \rangle$. Then we have the following left cosets of H in $U(16)$:

1. $1H = H$
2. $3H = \{3, 9, 11, 1\} = H$
3. $5H = \{5 \cdot 1, 5 \cdot 3, 5 \cdot 9, 5 \cdot 11\} = \{5, 15, 13, 7\}$
4. $7H = \{7, 5, 15, 13\}$
5. $9H = \{9, 11, 1, 3\} = H$
6. $11H = \{11, 1, 3, 9\} = H$
7. $13H = \{13, 7, 5, 15\}$
8. $15H = \{15, 13, 7, 5\}$

From this, there are only two distinct equivalence classes (and thus, only two left cosets): $\{1, 3, 9, 11\}$ and $\{5, 7, 13, 15\}$. These two left cosets partition $U(16)$.

Theorem 3.6.6 ▸ Lagrange's Theorem

Let G be a group, and let H be a subgroup of G. Then $|H|$ divides $|G|$.

Definition 3.6.7 ▸ Index

Let H be a subgroup of G. The index of H in G, written $[G : H]$, is the number of left cosets of H in G.

Corollary 3.6.8

If G is a group and $a \in G$, then $|a|$ divides $|G|$.

Corollary 3.6.9

Every group of prime order is cyclic.

Proof. $\square$

Example 3.6.10

Let G be a group having subgroups H and K, where $|H| = 20$ and $|K| = 63$. Show that $H \cap K = \{e\}$.

4

Factor groups and Homomorphisms

4.1 Normal Subgroups

Definition 4.1.1 ▶ Normal subgroup

Let N be a subgroup of G. We say N is a ***normal subgroup*** of G if $aN = Na$ for all $a \in G$. That is:

$$N \trianglelefteq G \iff \forall(a \in G)(aN = Na)$$

Intuitively, to be a normal subgroup means some symmetry between left and right cosets. For example, $\{e\}$ and G are normal subgroups of G. Also, any subgroup of an abelian group G is normal.

In D_8, $\langle R_{90} \rangle$ is normal, even though $F_1 R_{90} \neq R_{90} F_1$. Also, $\langle R_{180} \rangle$ is normal, and $\langle R_{180} \rangle = Z(D_8)$ (the center of D_8).

Theorem 4.1.2

Let G be a group. Any subgroup of index 2 is normal.

Theorem 4.1.3

Let H be a subgroup of G, and let $a \in G$. Then:

$$a^{-1}Ha = \{a^{-1}ha : h \in H\}$$

$a^{-1}Ha$ is a subgroup and $|a^{-1}Ha| = |H|$.

Proof. ▢

Theorem 4.1.4

Let H be a subgroup of G. The following are equivalent:

1. H is normal in G.

2. $a^{-1}ha \in H$ for all $h \in H$ and $a \in G$.
3. $a^{-1}Ha \subseteq H$ for all $a \in G$.
4. $a^{-1}Ha = H$ for all $a \in G$.

Definition 4.1.5 ▶ Product of subgroups

Let H and K be subgroups of G. We define the ***product of subgroups*** H and K as:

$$HK \coloneqq \{hk : h \in H, k \in K\}$$

4.2 Factor Groups

Definition 4.2.1 ▶ Factor groups

Let N be a normal subgroup of G. Then the ***factor group*** G/N is the set of all left cosets gN for all $g \in G$. For any $aN, bN \in G/N$, we define the group operation to be $(aN)(bN) = (ab)N$.

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