Numerical Analysis

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1 Introduction

Numerical analysis studies algorithms that use numerical approximation, focused on practical use by computers rather than theoretical use by mathematicians. It covers:

- computation, instabilities and rounding,
- solution of nonlinear equations and systems,
- interpolation and approximation by polynomials,
- numerical differentiation and integration, and
- solution of initial and boundary value problems for ordinary differential equations.

To prime ourselves for numerical analysis, let's review some key concepts from previous math courses:

1.1 Calculus Review

Concepts from calculus play a key role in numerical techniques. Let's begin by reviewing some familiar theorems.

Theorem 1.1.1 ▸ Extreme Value Theorem

(Simplified) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value and an absolute minimum value at some values x_0 and x_1 in $[a, b]$.

(More formal) Suppose K is a nonempty and compact subset of $\mathbb{R}$, and suppose $f : K \to \mathbb{R}$ is continuous. Then:

(a) f is bounded on K (that is, $f[K]$ is bounded),
(b) there exists $x_0 \in K$ such that $f(x_0) = \sup(f[K])$, and
(c) there exists $x_1 \in K$ such that $f(x_1) = \inf(f[K])$.

Theorem 1.1.2 ▸ Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$. Then for any function value y between the minimum and maximum function values from $[a, b]$, there exists some x in

$[a, b]$ where $f(x) = y$.

A notable application of the Intermediate Value Theorem is figuring out whether a polynomial function attains zero (i.e. whether a polynomial has a root). For example, let's consider the following polynomial:

$$f(x) = 2x^3 - 3x - 7$$

Using the Intermediate Value Theorem, all we need to do is find a value x_0 where $f(x_0) < 0$, and a value x_1 where $f(x_1) > 0$. Since polynomial functions are continuous, we can use the IVT to ensure that the polynomial has a root. For the above example, we can consider $f(-1) = -2$ and $f(0) = 3$. By the IVT, there must exist some $x \in [-1, 0]$ where $f(x) = 0$.

Moving on, let's recall the following two theorems related to derivatives:

Theorem 1.1.3 ▶ Rolle's Theorem

Let $a, b \in \mathbb{R}$ where $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = 0$ and $f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 1.1.4 ▶ Mean Value Theorem

Let $a, b \in \mathbb{R}$ where $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Intuitively, the Mean Value Theorem states the following: given any two points on a differentiable function's curve, there exists some x value where the tangent line at $f(x)$ is the same slope as the slope between those two points.

One of the most important theorems in analysis is the following:

Theorem 1.1.5 ▶ Taylor's Theorem

Let n be a non-negative integer. Let $f : (a, b) \to \mathbb{R}$ be a function where $f^{(n+1)}(x)$ exists for any $x \in (a, b)$. Then, for any $c, x \in (a, b)$, there exists some number ζ between c and x such that:

$$f(x) = p_n + R_n(\zeta)$$

Here, $p_n(x)$ represents the Taylor polynomial of f:

$$p_n(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

And, $R_n(\zeta)$ represents the remainder (i.e. how far off the Taylor polynomial was to approximating the actual answer):

$$R_n(\zeta) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

The Taylor expansion is well-liked in numerical analysis since it can turn an incomputable function such as sine or cosine to a computable one (albeit, only an approximation). The above theorem gives us a formula for determining the inaccuracy of certain Taylor polynomial approximations of functions.

We use $|R_n(\zeta)|$ as the error term for how different the approximation is from the actual answer. In practical application, we often do not know the value for ζ. However, since the theorem states ζ is between c and x, it is easy to attain and upper and lower bound for ζ.

A Taylor polynomial's accuracy is good near the initial point c, but deteriorates farther from initial point. It's a consequence of the fact that all information about the approximated function f is coming from a single point of that function.

On the other hand, interpolation is not so accurate at any specific point, but it's often more accurate across a range of points.

1.2 Sequences

Many algorithms generate sequences. For example, consider Newton's method for calculating $\sqrt{3}$:

$$\begin{cases} x_{k+1} = \frac{x_k}{2} + \frac{3}{2x_k}, & k = 0, 1, 2, \ldots \\ x_0 = 2 \end{cases}$$

In this course, we will use 0 to index the first element of a sequence.

This method works as this sequence (x_k) converges to $\sqrt{3}$. We can notate this as:

$$\lim_{k\to\infty} x_k = \sqrt{3}$$

Intuitively, this means that the numbers of the sequences gets closer to the limit $\sqrt{3}$ as k gets bigger. We formalize this idea in the following definition:

Definition 1.2.1 ▶ Convergence

We say a sequence x_n converges if: for any $\epsilon > 0$, there exists $K \in \mathbb{N}_0$ such that, for all $k \geq K$, $|x_k - L| < \epsilon$.

In practical applications, we want the sequence to converge quickly. The "speed" of convergence matters, especially in numerical analysis. For example, consider the following two sequences:

$$x_n = \frac{n}{n+1} \quad \text{and} \quad y_n = 1 - {}^1\!/_{n^2}$$

Both of these sequences converge to 1. However, it might be intuitive to think that (y_n) converges "faster" than (x_n). The problem is: how do prove that (y_n) converges "faster" than (x_n)? And how do we quantify this? We may look at the progress achieved at each iteration of the sequence. More specifically, we can calculate:

$$\frac{|x_{n+1} - L|}{|x_n - L|}$$

where L is the limit of this sequence. The lower this expression evaluates, the "faster" or "better" the sequence.

Definition 1.2.2 ▶ Sublinear, linear, superlinear

Let x_n be a sequence that converges to L, and suppose that $\lim_{n\to\infty} \frac{|x_{n+1}-L|}{|x_n-L|} = \lambda$ where $0 \leq \lambda \leq 1$.

- If $\lambda = 1$, then (x_n) is ***sublinear***.
- If $0 < \lambda < 1$, then (x_n) is ***linear***.
- If $\lambda = 0$, then (x_n) is ***superlinear***.

Although all three of these types are convergent, it is clear that a lower value of λ yields a more easily-computable sequence.

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