Calculus III

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1 Introduction

Much of our focus will be on Stoke's Theorem.

2 Three-Dimensional Space

In past math classes, we have been used to dealing in $\mathbb{R}^2$ where we work with two degrees of freedom: x and y. Now, we will be working in $\mathbb{R}^3$ with three degrees of freedom: x, y, and z.

2.1 Points

Definition 2.1.1 ▶ Point

A ***point*** in $\mathbb{R}^n$ space is an n-tuple that specifies a location in that space.

$$p = (p_1, \dots, p_n) \in \mathbb{R}^n$$

Definition 2.1.2 ▶ Distance

Given two points $a, b \in \mathbb{R}^n$, the ***distance*** between the two points is defined as:

$$d(a, b) \coloneqq \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$$

Example 2.1.1 ▶ Distance Between Points

Find the distance between $p_1 = (-1, -1, 4)$ and $p_2 = (-1, 4, -1)$.

$$\begin{aligned} d(p_1, p_2) &= \sqrt{(-1 - (-1))^2 + (4 - (-1))^2 + (-1 - 1)^2} \\ &= \sqrt{0^2 + 5^2 + (-5)^2} \\ &= \sqrt{50} \end{aligned}$$

Definition 2.1.3 ▶ Sphere

Given a point $c = (h, k, l) \in \mathbb{R}^3$, a ***sphere*** is the set of all points $(x, y, z) \in \mathbb{R}^3$ that are a distance r from the point $c = (h, k, l)$.

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Note that all the points of the sphere are equidistant to the center of the sphere. This means the sphere is really a hollow shell.

Example 2.1.2 ▶ Circle

Show that the following quadratic equation represents a circle by rewriting it in standard form. Find the center $c = (h, k)$ and the radius r.

$$x^2 + y^2 + x = 0$$

To solve this, we will have to complete the square:

$$\begin{aligned} x^2 + x + y^2 &= 0 \\ \implies x^2 + x + \frac{1}{4} + y^2 &= \frac{1}{4} \\ \implies \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4} \end{aligned}$$

2.2 Vectors

Definition 2.2.1 ▶ Vector

A ***vector*** is a mathematical object that contains multiple objects of the same type.

$$\vec{v} = \langle v_1, \dots, v_n \rangle \in \mathbb{R}^n$$

As customary in most mathematics textbooks, we will always denote vectors using the little arrow thing. In the context of three-dimensional space, we will only be working with vectors with three components. In addition, we will think of vectors as having a magnitude and direction.

Definition 2.2.2 ▶ Scalar Multiplication

Given a vector $\vec{v}$ and scalar k, we define ***scalar multiplication*** as:

$$k \cdot \vec{v} \coloneqq \langle kv_1, \dots, kv_n \rangle$$

Note that scalar multiplication is associative, commutative, and distributive.

- $a(b\vec{v}) = b(a(\vec{v})) = (ab)\vec{v}$
- $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$

- $k(\vec{v} + \vec{w}) = kv + k\vec{w}$

Definition 2.2.3 ▶ Norm

A vector's ***norm*** is its magnitude or length.

$$\|v\| := \sqrt{v_1^2 + \cdots + v_n^2}$$

Definition 2.2.4 ▶ Unit Vector

A ***unit vector*** is a vector whose magnitude is 1.

We will introduce shorthand notation for the three standard unit vectors:

- $\hat{\imath} := \langle 1, 0, 0 \rangle$
- $\hat{\jmath} := \langle 0, 1, 0 \rangle$
- $\hat{k} := \langle 0, 0, 1 \rangle$

These three vectors form the ***standard basis*** for $\mathbb{R}^3$. That is, we can express any vector in $\mathbb{R}^3$ as a linear combination of $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Technique 2.2.1 ▶ Finding a Unit Vector from a Given Vector

Given a vector $\vec{v} = \langle x, y, z \rangle \in \mathbb{R}^3$, we can find the ***unit vector*** $\vec{u}$ with the same direction by:

$$\vec{u} = \frac{\vec{v}}{\|v\|} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|}, \frac{z}{\|v\|} \right\rangle$$

Definition 2.2.5 ▶ Dot Product

Given two vectors $\vec{a}$ and $\vec{b}$ whose cardinality are both n, we define the ***dot product*** of $\vec{a}$ and $\vec{b}$ as:

$$\vec{a} \cdot \vec{b} := a_1 b_1 + \cdots + a_n b_n$$

Like scalar multiplication, dot product is also associative, commutative, and distributive.

Theorem 2.2.1 ▸ Angle Between Vectors

If $\vec{a}$ and $\vec{b}$ are vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cdot \cos(\theta)$$

Proof. TODO: finish proof □

Definition 2.2.6 ▸ Parallel, Perpendicular

- Two vectors are ***parallel*** if the angle between the vectors is 0 deg.
- Two vectors are ***perpendicular*** if the angle between the vectors is 90 deg.

Definition 2.2.7 ▸ Orthogonal

$\vec{a}$ and $\vec{b}$ are ***orthogonal*** if $\vec{a} \cdot \vec{b} = 0$.

Given a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$, we have:

$$\frac{\vec{a}}{\|a\|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

where:

- $\alpha = \cos^{-1}\left(\frac{a_1}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the x-axis)
- $\beta = \cos^{-1}\left(\frac{a_2}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the y-axis)
- $\beta = \cos^{-1}\left(\frac{a_3}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the z-axis)

Definition 2.2.8 ▸ Work

If F is a force moving a particle from a point P to a point Q, the ***work*** performed by the force is given by:

$$W = \vec{F} \cdot \overrightarrow{PQ}$$

Example 2.2.1 ▸ Finding Work

Find the work done by a force $\vec{F} = \langle 3, 4, 5 \rangle$ in moving an object from $p = (2, 1, 0)$ to $q = (4, 6, 2)$.

First, we find $\overrightarrow{pq}$ as such:

$$\begin{aligned}\overrightarrow{pq} &= \langle 4-2, 6-1, 2-0\rangle \\ &= \langle 2, 5, 2\rangle\end{aligned}$$

Then, we can find work:

$$\begin{aligned}W &= \vec{F}\cdot\overrightarrow{PQ} \\ &= \langle 3,4,5\rangle\cdot\langle 2,5,2\rangle \\ &= 6+20+10 \\ &= 36\end{aligned}$$

2.3 Gradient

Definition 2.3.1 ▶ Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The ***gradient*** of f is a function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\nabla f(x_1, \dots x_n) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right\rangle$$

Example 2.3.1 ▶ Gradient

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $f(T, L, \rho) = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

The gradient of $f(T, L, P)$ is denoted

$$\begin{aligned}\nabla f(T, L, \rho) &= \left\langle \frac{\partial f}{\partial T}, \frac{\partial f}{\partial L}, \frac{\partial f}{\partial \rho}\right\rangle \\ &= \left\langle \frac{1}{4L\sqrt{T\rho}}, -\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}, -\frac{1}{4L}\sqrt{\frac{T}{\rho^3}}\right\rangle\end{aligned}$$

We can then calculate gradient as such:

$$\begin{aligned}\nabla f(2,1,1) &= \left\langle \frac{1}{4(1)\sqrt{(2)(1)}}, -\frac{1}{2(1)}\sqrt{\frac{2}{1}}, -\frac{1}{(4)(1)}\sqrt{\frac{2}{1}} \right\rangle \\ &= \left\langle \frac{1}{4\sqrt{2}}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4} \right\rangle\end{aligned}$$

Definition 2.3.2 ▶ Directional Derivative

The ***directional derivative*** of $f(x, y, z)$ in the direction of $\vec{a}$ is defined as:

$$\nabla f(x, y, z) \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

Example 2.3.2 ▶ Directional Derivative

If $f(x, y, z) = xy^2z^5$, find the directional derivative of $f(x, y, z)$ at the point $(1, 0, -2)$ in the direction of the unit vector $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$, $\vec{a} = \langle 1, 2, -2 \rangle$.

For this, we calculate $\nabla f(1, 0, -1)$, then calculate the dot product of $\nabla f(1, 0, -1)$ with the unit vector $\vec{u} = \langle 1/3, 2/3, -2/3 \rangle$. Thus, the directional derivative of $f(x, y, z)$ at $(1, 0, -1)$ denoted by $Df(1, 0, -1)$ in the direction of $\vec{u}$ is:

$$\begin{aligned}Df(1, 0, -1) &= \nabla f(1, 2, -2) \cdot \vec{u} \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \vec{u} \\ &= \langle 0, 0, 0 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \\ &= 0\end{aligned}$$

2.4 Projecting Vectors

Projecting a vector onto another vector

Definition 2.4.1 ▶ Scalar Projection

Given $\vec{a}$ and $\vec{b}$, the ***scalar projection*** of $\vec{b}$ onto $\vec{a}$ is the norm of the vector projection of $\vec{b}$ onto $\vec{a}$.

$$\operatorname{comp}_{\vec{a}} \vec{b} := \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|}$$

Definition 2.4.2 ▶ Vector Projection

Given $\vec{a}$ and $\vec{b}$ that are non-zero vectors, the ***vector projection*** of $\vec{b}$ onto the vector $\vec{a}$ is defined by:

$$\operatorname{proj}_{\vec{a}} \vec{b} := \operatorname{comp}_{\vec{a}} \vec{b} \frac{\vec{a}}{\|\vec{a}\|}$$

2.5 Cross Product

Definition 2.5.1 ▶ Cross Product

Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, the ***cross product*** of $\vec{a}$ and $\vec{b}$ is a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} := \vec{c} \quad \text{where} \quad \vec{a} \cdot \vec{c} = 0 \quad \textbf{and} \quad \vec{b} \cdot \vec{c} = 0$$

The cross product is exclusive to vectors in three dimensions.

Technique 2.5.1 ▶ Calculating Cross Product

Let $\vec{a} := \langle a_1, a_2, a_3 \rangle$ and $\vec{b} := \langle b_1, b_2, b_3 \rangle$ To find $\vec{a} \times \vec{b}$, we:

1. Create a matrix as such:

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

2. Find the determinant of the matrix by cofactor expansion on the first row.

$$\begin{aligned}\vec{a}\times\vec{b} &= \hat{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{\imath}(a_2b_3 - b_2a_3) - \hat{\jmath}(a_1b_3 - b_1a_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \langle a_2b_3 - b_2a_3,\ -a_1b_3 + b_1a_3,\ a_1b_2 - b_1a_2\rangle\end{aligned}$$

- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
- $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$
- If $r\in\mathbb{R}$, then $(r\vec{a})\times\vec{b} = \vec{a}\times(r\vec{b}) = r(\vec{a}\times\vec{b})$
- $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot c$
- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

Theorem 2.5.1

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors in $\mathbb{R}^3$ and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta$$

2.6 Torque

Definition 2.6.1 ▶ Torque

If $\vec{F}$ is a force applied to an object with position vector $\vec{r}$, then the torque $\vec{\mathrm{T}}$ produced by $\vec{F}$ is given by:

$$\vec{\mathrm{T}} := \vec{r}\times\vec{F}$$

2.7 Cylinders and Quadratic Surfaces

Definition 2.7.1 ▸ Planar Curve

A ***planar curve*** is any curve that lies on a single plane.

Definition 2.7.2 ▸ Cylinder

Given a planar curve c, the surface in $\mathbb{R}^3$ defined by all parallel lines crossing the curve c is called a ***cylinder***.

Note that our broad definition of cylinder does not require the cylinder to be circular, nor does it require it to be straight. For example, we could have a planar curve defined by $x^2 + y^2 = 1$ and create a circular cylinder with radius 1. We could also have a planar curve defined by $y = x^2$ and create a ***parabolic cylinder***.

Example 2.7.1

Consider the curve $x^2 + y^2 = z^2$. For every z_0 at $x = y = 0$, we have a point, say $p \coloneqq (0, 0, 0)$yh

Definition 2.7.3 ▸ Cone

Definition 2.7.4 ▸ Conic Surface

A ***conic surface*** is a surface that is attained by taking a cross-section of a cone.

There are four types of conic surfaces:

1. The cross-section parallel to the xy-plane is a ***circle***.
2. The cross-section slightly angled from the xy-plane is a ***ellipse***.
3. The cross-section parallel to a generating line is a ***parabola***.
4. The cross-section parallel to the z axis is a ***hyperbola***.

Definition 2.7.5 ▸ Quadratic Surface

A ***quadratic surface*** in $\mathbb{R}^3$ is the set of points whose coordinates satisfy a quadratic polynomial in the variables x, y, z.

For example, the standard equation for a sphere is a quadratic surface.

Definition 2.7.6 ▸ Ellipsoid

An ***ellipsoid*** is a shape in $\mathbb{R}^3$ defined by the equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

The cross-sections of an ellipsoid with each coordinate plane (xy-plane, xz-plane, yz-plane) is just an ellipse. In other words:

- If we set $z = l$, we get an ellipse in the xy-plane
- If we set $y = k$, we get an ellipse in the xz-plane
- If we set $x = h$, we get an ellipse in the yz-plane

We can also have ***hyperboloids***:

- Type 1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Type 2: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

For type 2 hyperboloids, its cross-section with the xy and xz plane is a hyperbola. Note that there is no cross-section with the yz plane. This is because the equation $-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ has no solution in the real numbers.

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