Chapter 1

Sequences and Convergence

Definition 1.0.1 ▸ Sequence

A ***sequence*** is an infinite ordered list of real numbers.

$$s = (s_1, s_2, s_3, s_4, \dots)$$

Formally, a ***sequence*** is a function $s : \mathbb{N} \to \mathbb{R}$. We write s_n to denote $s(n)$.

We can define a sequence using an expression like, for every $n \in \mathbb{N}$, let $s_n := n^2$. Then $s = (1, 4, 9, 16, \dots)$. Also, we can informally define a sequence in terms of its elements, like $s := (3, 1, 4, 1, 5, 9, \dots)$. We could just have a random sequence like $s := (12.3, e^2, 1 - \pi, 10000, \dots)$.

Let's consider how we can formalize the definitions of limits and convergence. Consider the sequence $s_n := 1/n$, so $(s_n) = (1, 1/2, 1/3, 1/4, \dots)$. We have an intuitive idea that, as n gets bigger, then $1/n$ gets closer to 0. We can say that this sequence "converges" to 0.

Now consider the sequence $s := (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \dots)$. Does this sequence converge? This really depends on our definition of convergence. We might define this as, "s_n gets close to l as n gets large". It certainly matches our intuition, but what exactly does "close to l" mean? Maybe we could say, "$|s_n - l|$ gets small as n gets large". More precisely, this might be "for all $\epsilon > 0$, $|s_n - l| < \epsilon$ when n is large". That "n is large" is still imprecise. Fixing that part, we get the formal definition for convergence:

Definition 1.0.2 ▸ Convergence

A sequence of real numbers s_n ***converges*** to $l \in \mathbb{R}$ if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ where $n > N$, $|s_n - l| < \epsilon$.

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(n \in \mathbb{N})(n > N \implies |s_n - l| < \epsilon)$$

Like in the approximation property, we may think of ϵ as some tiny positive value that's really really close to 0, but not actually 0. So perhaps more intuitively, we can think of the definition of convergence as, no matter how small we make ϵ, there is some point in the sequence where every sequence point afterwards is within ϵ distance of our limit. We can also write $\lim_{n\to\infty} s_n = l$ or $s_n \to l$ to mean s_n converges to l.

Technique 1.0.3 ▸ Proving Convergence (Epsilon-Delta Proof)

To prove that a sequence s converges to l, we carry out the following steps:

1. As some scratch work, solve the inequality $|s_n - l| < \epsilon$ for n.
2. In the formal proof, let $\epsilon > 0$, and let N be greater than the solved thing. Let $n > N$, then work towards $|s_n - l| < \epsilon$.

Make this explanation better

Example 1.0.4 ▸ $1/n$ converges to 0

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Intuition: Since we're proving something for all $\epsilon > 0$, let's start by choosing some arbitrary $\epsilon > 0$. Next, we need to choose some $N \in \mathbb{N}$ where $|s_n - l| < \epsilon$ for all $n > N$. Thus:

$$\begin{aligned} |s_n - l| &< \epsilon \\ \left|\frac{1}{n} - 0\right| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ n &> \frac{1}{\epsilon} \end{aligned}$$

So we choose $N > \frac{1}{\epsilon}$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ where $N > 1/\epsilon$. If $n > N > 1/\epsilon$, then $1/n < \epsilon$. Thus:

$$|s_n - l| = |1/n - 0| = 1/n < \epsilon$$

Therefore, s converges to 0. ▢

Example 1.0.5

Prove that $\lim_{n\to\infty} \frac{2n+3}{3n+7} = \frac{2}{3}$.

Intuition: This time, we want to choose some $N \in \mathbb{N}$ such that $|s_n - l| < \epsilon$. Thus:

$$\begin{aligned}
\left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| &< \epsilon \\
\left|\frac{6n+9-6n-14}{9n+21}\right| &< \epsilon \\
\frac{5}{9n+21} &< \epsilon \\
\frac{5}{\epsilon} &< 9n+21 \\
\frac{1}{9}\left(\frac{5}{\epsilon} - 21\right) &< n
\end{aligned}$$

Thus, we want to choose $N > 1/9\,(5/\epsilon - 21)$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > 1/9\,(5/\epsilon - 21)$. If $n > N > 1/9\,(5/\epsilon - 21)$, then:

$$\begin{aligned}
9n &> 5/\epsilon - 21 \\
9n &> \frac{5}{\epsilon} - 21 \\
9n + 21 &> \frac{5}{\epsilon} \\
\frac{5}{9n+21} &< \epsilon
\end{aligned}$$

Thus:

$$\begin{aligned}
|s_n - l| &= \left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| \\
&= \left|\frac{6n+9-6n-14}{9n+21}\right| \\
&= \frac{5}{9n+21} \\
&< \epsilon
\end{aligned}$$

□

The above proof chooses a sort of "optimal" or "best possible" N. We could have thrown away the 21 in the denominator, and the inequality we're aiming for will still be the same.

Alternate proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{5}{9\epsilon}$. If $n > N > \frac{5}{9\epsilon}$, then $\frac{5}{9n} < \epsilon$, so $\frac{5}{9n+21} < \frac{5}{9n} < \epsilon$. Then:

$$|s_n - l| = \left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| = \frac{5}{9n+21} < \epsilon$$

□

Example 1.0.6

Prove that $\lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}$.

Intuition: Here, we have to be careful about throwing away terms.

$$\begin{aligned}
|s_n - l| &< \epsilon \\
\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| &< \epsilon \\
\left|\frac{6n+9-6n+14}{9n-21}\right| &< \epsilon \\
\frac{23}{|9n-21|} &< \epsilon
\end{aligned}$$

We want $9n - 21 > 0$, so we must have $n \geq 3$. We can apply this restriction on n to get rid of the absolute value:

$$\begin{aligned}\frac{23}{9n-21} &< \epsilon \\ \frac{23}{\epsilon} &< 9n-21 \\ \frac{1}{9}\left(\frac{23}{\epsilon}+21\right) &< n\end{aligned}$$

Thus, we want to choose some $N > \frac{1}{9}\left(\frac{23}{\epsilon}+21\right)$ and $N \geq 3$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{1}{9}\left(\frac{23}{\epsilon}+21\right)$. Then $N > \frac{21}{9}$, and since N is a natural number, then $N \geq 3$. Let $n \in \mathbb{N}$ where $n > N$. Then:

$$\begin{aligned}9n &> \frac{23}{\epsilon}+21 \\ 9n-21 &> \frac{23}{\epsilon} \\ \epsilon &> \frac{23}{9n-21}\end{aligned}$$

Thus:

$$|s_n - l| = \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \left|\frac{23}{9n-21}\right| = \frac{23}{9n-21} < \epsilon$$

▢

Definition 1.0.7 ▸ Divergence

A sequence ***diverges*** if it does not converge.

$$\exists(\epsilon > 0)\forall(N \in \mathbb{N})\exists(n \in \mathbb{N})(n > N \wedge |s_n - l| \geq \epsilon)$$

Example 1.0.8 ▸ Diverging Sequence

Prove that $s \coloneqq (1, 0, 1, 0, 0, 1, 0, 0, 0, \ldots)$ does not converge to 0.

Proof. Let $\epsilon \coloneqq 1/2$. Then for all $N \in \mathbb{N}$, there exists $n > N$ such that $s_n = 1$. Then:

$$|s_n - 0| = |1 - 0| > \epsilon$$

Therefore, s does not converge. ▢

1.1 Properties of Limits

A sequence can only converge to one value, not more. That is, if a sequence has a limit, then that limit is unique.

Lemma 1.1.1 ▸ Approximating Zero

Let $x \in \mathbb{R}$. If $x < \epsilon$ for all $\epsilon > 0$, then $x \leq 0$.

Proof. We proceed by contraposition. Suppose $x > 0$. Let $\epsilon \coloneqq x/2 > 0$. Then $x \geq \epsilon = x/2$. ▢

Theorem 1.1.2 ▸ Uniqueness of Limits

Let s_n be a sequence of real numbers. If s_n converges to l_1 and converges to l_2, then $l_1 = l_2$.

Proof. Let $\epsilon > 0$. Since s_n converges to l_1, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - l_1| < \epsilon/2$ for all $n > N_1$. Similarly, since s_n converges to l_2, then there exists $N_2 \in \mathbb{N}$ such that $|s_n - l_2| < \epsilon/2$ for all $n > N_2$.
Let $n \in \mathbb{N}$ where $n > N_1$ and $n > N_2$. Then:

$$|l_1 - l_2| = |l_1 - s_n + s_n - l_2| \leq \underbrace{|l_1 - s_n| + |s_n - l_2|}_{??} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $|l_1 - l_2| < \epsilon$ for all $\epsilon > 0$. Thus, by Lemma 1.1.1, $|l1 - l2| \leq 0$. However, we know that $|l1 - l2| \geq 0$ since it's an absolute value. Thus, we have $|l1 - l2| = 0$, so $l1 = l2$. ▢

Definition 1.1.3 ▸ Sequence Boundedness

A sequence of real numbers (s_n) is:

- ***bounded above*** if there exists $M \in \mathbb{R}$ such that $s_n \leq M$ for all $n \in \mathbb{N}$. We say M is an ***upper bound*** for the sequence (s_n).
- ***bounded below*** if there exists $m \in \mathbb{R}$ such that $m \leq s_n$ for all $n \in \mathbb{N}$. We say m is a ***lower bound*** for the sequence (s_n).
- ***bounded*** if it is both bounded above and below.

Theorem 1.1.4 ▶ Sequence Convergence Implies Boundedness

Let (s_n) be a sequence of real numbers. If (s_n) converges to some $l \in \mathbb{R}$, then (s_n) is bounded.

Proof. Since (s_n) converges to l, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $|s_n - l| < 1$ (applying the definition of convergence with $\epsilon \coloneqq 1$). Let:

$$R \coloneqq \max\{|s_1|, |s_2|, \dots, |s_N|, 1 + |l|\}$$

For $1 \leq n \leq N$, we have $|s_n| \leq R$ by the definition of R. For $n > N$, we can leverage the triangle inequality:

$$|s_n| = |s_n - l + l| \leq |s_n - l| + |l| < 1 + |l| \leq \mathbb{R}$$

Thus, $|s_n| \leq R$ for all $n \in \mathbb{N}$, so (s_n) is bounded. ▢

Theorem 1.1.5 ▶ Properties of Limits

Let (s_n) and (t_n) be convergent sequences of real numbers, and let $s, t \in \mathbb{R}$ such that s_n converges to s and t_n converges to t. Then:

1. For any $c \in \mathbb{R}$, cs_n converges to cs,
2. $s_n + t_n$ converges to $s + t$,
3. $s_n t_n$ converges to st, and
4. if $t_n \neq 0$, then for all n and $t \neq 0$, $\frac{s_n}{t_n}$ converges to $\frac{s}{t}$.

Proof of 1. Let $\epsilon > 0$. Since (s_n) converges to s, then there exists $N \in \mathbb{N}$ such that $|s_n - s| < \frac{\epsilon}{1+|c|}$ for all $n > N$. Then, for all $n > N$, we have:

$$|cs_n - cs| = |c(s_n - s)| = |c||s_n - s| < |c|\frac{\epsilon}{1 + |c|} = \frac{|c|}{1 + |c|}\epsilon < \epsilon$$

▢

Proof of 2. Let $\epsilon > 0$. Since (s_n) converges to s, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2$ for all $n > N$. Similarly, since t_n converges to t, then there exists $N_2 \in \mathbb{N}$ such that $|t_n - t| < \epsilon/2$. Let $N \in \mathbb{N}$ where $N \geq N_1$ and $N \geq N_2$. Then:

$$|(s_n + t_n) - (s + t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

That is, $s_n + t_n$ converges to $s + t$. ▢

Proof of 3. Let $\epsilon > 0$. Since s_n converges to s, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2(|t|+1)$ for all $n > N$. Also, (s_n) converges, so (s_n) is bounded. That is, there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Since t_n converges to t, there exists $N_2 \in \mathbb{N}$ such that $|t_n - t| < \frac{\epsilon}{2(M+1)}$ for all $n > N$. Let $N \in \mathbb{N}$ such that $N \geq N_1$ and $N \geq N_2$. If $n > N$, then:

$$\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&= |s_n(t_n - t) + (s_n - s)t| \\
&\leq |s_n(t_n - t)| + |(s_n - s)t| \\
&= |s_n||t_n - t| + |s_n - s||t| \\
&< M\frac{\epsilon}{2(1+M)} + \frac{\epsilon}{2(1+|t|)}|t| \\
&= \frac{M}{1+M}\frac{\epsilon}{2} + \frac{\epsilon}{2}\frac{|t|}{1+|t|} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

▢

Proof of 4. We will prove that $\frac{1}{t_n}$ converges to $\frac{1}{t}$. Let $\epsilon > 0$. Since t_n converges to t and $t \neq 0$, then there exists $N_1 \in \mathbb{N}$ such that $|t_n - t| < \frac{\epsilon t^2}{2}$. By **??**, there exists $N_2 \in \mathbb{N}$ such that $|t_n| > \frac{|t|}{2}$ for all $n > N_2$. Let $N \in \mathbb{N}$ such that $N > N_1$ and $N > N_2$. Let $n \in \mathbb{N}$ be arbitrary. Then:

$$\begin{aligned}
\left|\frac{1}{t_n} - \frac{1}{t}\right| &= \left|\frac{t - t_n}{t_n t}\right| \\
&= \frac{1}{|t_n|}\frac{1}{|t|}|t - t_n| \\
&< \frac{2}{|t|}\frac{1}{|t|}\frac{\epsilon t^2}{2} \\
&= \epsilon
\end{aligned}$$

By 3, if s_n converges to s, then $\frac{s_n}{t_n} = s_n\left(\frac{1}{t_n}\right)$ converges to $s\left(\frac{1}{t}\right) = \frac{s}{t}$. ▢

lemma 14.6 for bounding in proof of 4.

Explain new notation

Lemma 1.1.6 ▸ Limit of a Constant Sequence

If s_n is a constant sequence $(l, l, l, \ldots)$, then s_n converges to l.

Proof. Let $\epsilon > 0$. For all $n \in \mathbb{N}$, $|s_n - l| = 0 < \epsilon$. $\square$

better

Example 1.1.7 ▸ Using the Properties

Prove $\lim_{n\to\infty} \frac{5n^3-8n^2+15}{7n^3+19n+4} = \frac{5}{7}$.

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{5n^3 - 8n^2 + 15}{7n^3 + 19n + 4} &= \lim_{n\to\infty} \frac{5 - {}^{8}/_{n} + {}^{15}/_{n^3}}{7 + {}^{19}/_{n^2} + {}^{4}/_{n^3}} \\
&= \frac{\lim_{n\to\infty} 5 - {}^{8}/_{n} + {}^{15}/_{n^3}}{\lim_{n\to\infty} 7 + {}^{19}/_{n^2} + {}^{4}/_{n^3}} \\
&= \frac{\lim_{n\to\infty} 5 - \lim_{n\to\infty} {}^{8}/_{n} + \lim_{n\to\infty} {}^{15}/_{n^3}}{\lim_{n\to\infty} 7 + \lim_{n\to\infty} {}^{19}/_{n^2} + \lim_{n\to\infty} {}^{4}/_{n^3}}
\end{aligned}$$

Now we can work with each limit independently. Note that $\lim_{n\to\infty} \frac{1}{n^2} = \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right)$, so:

$$\lim_{n\to\infty} \frac{5n^3 - 8n^2 + 15}{7n^3 + 19n + 4} = \frac{5}{7}$$

$\square$

Definition 1.1.8 ▸ Increasing, Decreasing, Monotonic

A sequence of real numbers (s_n) is:

- ***increasing*** if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly increasing*** if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$.
- ***decreasing*** if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly decreasing*** if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$.

If (s_n) satisfies any of these properties, then we say (s_n) is ***monotonic***.

For example, $(s_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ is strictly decreasing and thus monotonic.

Theorem 1.1.9 ▸ Monotone Sequence Theorem

Let (s_n) be a sequence of real numbers.

1. If (s_n) is increasing and bounded above, then (s_n) converges to $\sup\{s_n : n \in \mathbb{N}\}$.
2. If (s_n) is decreasing and bounded below, then (s_n) converges to $\inf\{s_n : n \in \mathbb{N}\}$.

Idea: Assuming s is our limit, we want to find $N \in \mathbb{N}$ such that $|s - s_n| < \epsilon$, or $s - \epsilon < s_n$ for all $n > N$. Then $s - \epsilon < s_n \leq s$ for all $n > N$.

Proof of 1. Let $\epsilon > 0$. Because $\{s_n : n \in \mathbb{N}\}$ is non-empty and bounded above, then it has a supremum. Let $s := \sup\{s_n : n \in \mathbb{N}\}$. Thus, there exists $N \in \mathbb{N}$ such that $s_N > s - \epsilon$ (by the approximation property). Since (s_n) is increasing, we have:

$$\forall(n > N)\,(s - \epsilon < s_N \leq s_n \leq s \quad)$$

Hence, $\epsilon < s_n - s \leq 0$, so $|s_n - s| < \epsilon$. ▢

Proof of 2. Suppose (s_n) is decreasing and bounded below. Then $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$. Moreover, there exists $m \in \mathbb{R}$ such that $s_n \geq m$ for all $n \in \mathbb{N}$. That is, $-s_{n+1} \geq -s_n$ for all $n \in \mathbb{N}$, and $-s_n \leq -m$ for all $n \in \mathbb{N}$. Therefore, $(-s_n)$ is increasing and bounded above. By the first part, we know $(-s_n)$ converges to $\sup\{-s_n : n \in \mathbb{N}\} = -\inf\{s_n : n \in \mathbb{N}\}$. Hence, (s_n) converges to $\inf\{s_n : n \in \mathbb{N}\}$. ▢

1.2 Subsequences

So far, we've only looked at well-behaving sequences that converge. What about sequences that don't converge? Can we still find some nice properties that describe their behavior? Consider the following divergent sequence:

$$(s_n) := (0, 1, 0, 1, 0, 1, \ldots)$$

What if we had a sequence (t_n) where, for every $n \in \mathbb{N}$, we let $t_n := s_{2n}$. Then we would have:

$$(t_n) = (s_2, s_4, s_6, \ldots) = (1, 1, 1, \ldots)$$

Inside this diverging sequence, we can find a convergent ***subsequence***! Intuitively, we can make a subsequence by "throwing away" terms but keeping the same order. We can formally define a subsequence as follows:

Definition 1.2.1 ▶ Subsequence

Given a sequence (s_n), a ***subsequence*** is any sequence of the form (s_{n_k}) where (n_k) is a strictly increasing sequence of natural numbers.

For example, if we had $s = (s_1, s_2, s_3, s_4, s_5, s_6, s_7, \dots, s_{213}, s_{214}, s_{215}, \dots)$, we can have a subsequence like:

$$(t_n) = (s_3, s_5, s_{213}, \dots)$$

Here, we would have $n_1 = 3$, $n_2 = 5$, $n_3 = 213$, and so on.

Example 1.2.2 ▶ Subsequences

Let $(s_n) := (1, 1/2, 1/3, 1/4, 1/5, 1/6, \dots)$

1. $(t_n) := (1, 1/4, 1/9, 1/16, 1/25)$ is a subsequence of (s_n) where $t_n = 1/n^2$, or $t_n = s_{n^2}$.
2. $(t_n) := (1/5, 1/25, 1/125, \dots)$ is also a subsequence of (s_n) with $t_n = \frac{1}{5^n}$ or $t_n = s_{5^n}$.
3. $(t_n) := (1/7, 1/2, 1/12, 1/6)$ is **not** a subsequence of (s_n) because the indices in s_n are not strictly increasing. We have $n_1 = 7$, but $n_2 = 2$.

In general:

$$(s_n) = (s_1, s_2, s_3, \dots)$$

$$(t_n) = (s_{n_k}) = (s_{n_1}, s_{n_2}, s_{n_3})$$

Lemma 1.2.3 ▶ Indices of Subsequences

If $(s_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(s_n)_{n\in\mathbb{N}}$, then $n_k \geq k$ for all $k \in \mathbb{N}$.

We will use induction.

Base Case: Since $n_1 \in \mathbb{N}$, then $n_1 \geq 1$.

Induction Step: Suppose $n_k \geq k$ for some $k \in \mathbb{N}$. Since $n_{k+1} > n_k$, we have $n_{k+1} \geq n_k + 1 \geq k + 1$.

Hence, $n_k \geq k$ for all $k \in \mathbb{N}$.

Theorem 1.2.4 ▶ Limits of Subsequences

Suppose (s_n) is a sequence of real numbers, and s_n converges to s for some $s \in \mathbb{R}$. If (s_{n_k}) is a subsequence of (s_n), then s_{n_k} converges to s.

Proof. Let $\epsilon > 0$. Since s_n converges to s, then there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for all $n > N$. Suppose $k > N$. By lemma 1.2.3, $n_k \geq k > N$, so $|s_{n_k} - s| < \epsilon$. ▢

1.3 Limit Superior and Inferior

Suppose (s_n) is a bounded sequence. Then there exists $M \in \mathbb{R}$ such that $-M \leq s_n \leq M$ for all $n \in \mathbb{N}$. Let:

$$\begin{aligned}
t_1 &\coloneqq \sup\{s_1, s_2, s_3, \ldots\} = \sup\{s_k : k \geq 1\} \\
t_2 &\coloneqq \sup\{s_2, s_3, s_4, \ldots\} = \sup\{s_k : k \geq 2\} \\
t_3 &\coloneqq \sup\{s_3, s_4, s_5, \ldots\} = \sup\{s_k : k \geq 3\} \\
&\vdots \\
t_n &\coloneqq \sup\{s_n, s_{n+1}, s_{n+2}, \ldots\} = \sup\{s_k : k \geq n\} \\
t_{n+1} &\coloneqq \sup\{s_{n+1}, s_{n+2}, s_{n+3}, \ldots\} = \sup\{s_k : k \geq n+1\}
\end{aligned}$$

Then:

$$-M \leq s_n \leq t_n$$

and:

$$t_{n+1} \leq t_n$$

so (t_n) is bounded below and decreasing. Hence, (t_n) converges by the Monotone Sequence Theorem.

Definition 1.3.1 ▸ Limit Superior, Limit Inferior

Let (s_n) be a bounded sequence of real numbers. The ***limit superior*** is defined as:

$$\limsup s_n \coloneqq \lim_{n\to\infty} \sup\{s_k : k \geq n\}$$

Similarly, the ***limit inferior*** is defined as:

$$\liminf s_n \coloneqq \lim_{n\to\infty} \inf\{s_k : k \geq n\}$$

Example 1.3.2

Define $s_n := \begin{cases} 3 + \frac{1}{n}, & n \text{ is even} \\ 1 - \frac{1}{n} & n \text{ is odd} \end{cases}$

$(s_n) = (0, 3 + 1/2, 2/3, 3 + 1/4, 4/5, 3 + 1/6)$

Let's try to calculate the limit superior of s_n. Define (t_n) as follows:

$$
\begin{aligned}
t_1 &:= \sup\{s_1, s_2, s_3, \ldots\} = 3 + \frac{1}{2} \\
t_2 &:= \sup\{s_2, s_3, s_4, \ldots\} = 3 + \frac{1}{2} \\
t_3 &:= \sup\{s_3, s_4, s_5, \ldots\} = 3 + 1/4 \\
t_4 &:= \sup\{s_4, s_5, s_6, \ldots\} = 3 + 1/4 \\
t_5 &:= \sup\{s_5, s_6, s_7, \ldots\} = 3 + 1/6 \\
&\vdots
\end{aligned}
$$

We can see that $\limsup s_n = \lim_{n\to\infty} \sup\{s_k : k \geq n\} = 3$. We might refer to 3 as the "largest limit point".

Now let's try to calculate the limit inferior of s_n. Define (r_n) as follows:

$$
\begin{aligned}
r_1 &:= \inf\{s_1, s_2, s_3, \ldots\} = 0 \\
r_2 &:= \inf\{s_2, s_3, s_4, \ldots\} = \frac{2}{3} \\
r_3 &:= \inf\{s_3, s_4, s_5, \ldots\} = \frac{2}{3} \\
r_4 &:= \inf\{s_4, s_5, s_6, \ldots\} = \frac{4}{5} \\
&\vdots
\end{aligned}
$$

We can see that $\liminf s_n = \lim_{n\to\infty} \inf\{s_k : k \geq n\} = 1$. We might refer to 1 as the "smallest limit point".

Theorem 1.3.3

Suppose (s_n) is a bounded sequence of real numbers, and suppose that (s_{n_k}) is a convergent subsequence of (s_n). Then $\liminf s_n \leq \lim_{k\to\inf} s_{n_k} \leq \limsup s_n$.

Proof. Let $r_n := \inf\{s_k : k \geq n\}$ and $t_n := \sup\{s_k : k \geq n\}$. Then $r_n \leq s_n \leq t_n$ for all $n \in \mathbb{N}$. In particular, $r_{n_k} \leq s_{n_k} \leq t_{n_k}$ for all $k \in \mathbb{N}$. By (todo: theroem), $\lim_{k\to\infty} r_{n_k} = \lim_{n\to\infty} r_n$. Note that $\lim_{n\to\infty} r_n = \liminf s_n$, and $\lim_{k\to\infty} t_{n_k} = \lim_{n\to\infty} t_n = \limsup s_n$. By the (todo: problem set squeeze theorem), we have:

$$\liminf s_n = \lim_{k\to\infty} r_{n_k} \leq lim_{k\ to\infty} s_{n_k} \leq \lim_{k\to\infty} t_{n_k} = \limsup s_n$$

▢

Theorem 1.3.4 ▸ Bolzano-Weierstrass Theorem

Suppose (s_n) is a bounded sequence of real numbers. The (s_n) has a subsequence that coverges to $\limsup s_n$, and (s_n) has a subsequence that converges to $\liminf s_n$.

Intuition:

- Let $t_k := \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$, so $\limsup s_n = \lim_{k\to\infty} t_k$.
- For each $k \in \mathbb{N}$ we can find some $n_k \geq k$ such that $t_k - 1/k < s_{n_k}$.
- Thus, $-1/k < s_{n_k} - t_k \leq 0$, so $|s_{n_k} - t_k| < 1/k$
- By (todo: problem set), $s_{n_k} - t_k \to 0$, so $s_{n_k} = s_{n_k} - t_k + t_k \to \limsup s_n$.
- But: we need $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. So we need to choose n_k inductively!

Proof for limsup. We will choose a subsequence of (s_n) that converges to $\limsup s_n$. For each $k \in \mathbb{N}$, let $t_k := \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ For convenience, let $P(n)$ be the statement "there exists $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $|s_{n_k} - t_{1+n_{k-1}}| < \frac{1}{k}$." We define $n_0 := 0$.
Base Case: Let $t_1 := \sup\{s_1, s_2, \ldots\}$. By the approximation property (todo ref), there exists $n_1 \in \mathbb{N}$ such that $t_1 - 1 < s_{n_1} \leq t_1$. Subtracting across by t_1, we have $-1 < s_{n_1} - t_1 \leq 0$. Thus, $|s_{n_1} - t_1| < 1$.
Induction Step: Now we aim to prove $P(k-1) \implies P(k)$. There exists $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$, and:

$$\begin{aligned}
& t_{1+n_{k-1}} - \frac{1}{k} < s_{n_k} \leq t_{1+n_{k-1}} \\
\implies \quad & -\frac{1}{k} < s_{n_k} - t_{1+n_{k-1}} \leq 0 \\
\implies \quad & |s_{n_k} - t_{1+n_{k-1}}| < \frac{1}{k}
\end{aligned}$$

That is, $\lim_{k\to\infty}\left(s_{n_k} - t_{1+n_{k-1}}\right) = 0$. Since $n_k > n_{k-1}$ for all $k \in \mathbb{N}$, (s_{n_k}) is a subsequence of (s_n). But $(t_{1+n_{k-1}})$ is a subsequence of (t_k), so:

$$\lim_{k\to\infty} t_{1+n_{k-1}} = \lim_{k\to\infty} t_k = \limsup s_n$$

Thus:

$$s_{n_k} = s_{n_k} - t_{1+n_{k-1}} + t_{1+n_{k-1}}$$

so:

$$\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty}\left(s_{n_k} - t_{1+n_{k-1}}\right) + \lim_{k\to\infty} t_{1+n_{k-1}} = 0 + \limsup s_n$$

Therefore, (s_{n_k}) is a subsequence of (s_n) that converges to $\limsup s_n$. ▢

Theorem 1.3.5 ▸ Convergence iff $\limsup = \liminf$

Let (s_n) be a bounded sequence of real numbers. Then (s_n) converges if and only if $\liminf s_n = \limsup s_n$

Proof. First, suppose s_n converges to some $s \in \mathbb{R}$. By the Bolzano-Weierstrass Theorem, there exists a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \limsup s_n$. But s_n converges to s, so s_{n_k} also converges to s. That is, $s = \lim_{k\to\infty} s_{n_k} = \limsup s_n$. By the same reasoning, we have $s = \liminf s_n$. Hence, $\liminf s_n = \lim s_n = \limsup s_n$.
Conversely, suppose $\liminf s_n = \limsup s_n$. Let $r_n := \inf\{s_k : k \geq n\}$ and $t_n := \sup\{s_k : k \geq n\}$. Then $r_n \leq s_n \leq_n$ for all $n \in \mathbb{N}$. Then $\lim r_n = \liminf s_n = \limsup s_n = \lim t_n$. Therefore, by the Squeeze Theorem (todo: ref), s_n converges to $\liminf s_n$. ▢

1.4 Cauchy Sequences

To show that a sequence (s_n) converges using the definition of limit, we need to know what limit is beforehand. Consider the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^3}$$

This sequence of partial sums converges, but its limit is unknown. Certainly we can get a decimal approximation for this value, but there is no known "closed" form of this value.

Definition 1.4.1 ▸ Cauchy Sequence

We say a sequence is ***Cauchy*** if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ for all $n > N$ and $m > M$.

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(n > N, m > N)\,(|s_n - s_m| < \epsilon)$$

In other words, a sequence is Cauchy if **all** terms in the tail can be made arbitrarily close to each other. Or, for any arbitrarily small distance, there exists some "tail" of the sequence that exists entirely within that distance. This definition circumvents any mention of a specific "limit". But we can prove that any Cauchy sequence of real numbers is convergent, and vice versa.

Lemma 1.4.2 ▸ Convergent Sequences are Cauchy

If a sequence of real numbers converges, then that sequence is Cauchy.

Proof. Let (s_n) be a convergent sequence of real numbers. Let $\epsilon > 0$. Since (s_n) converges to some $s \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2$ for all $n > N$. If $n > N$ and $m > N$, then:

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Lemma 1.4.3

Suppose (s_n) is a Cauchy sequence, and that (s_{n_k}) is a convergent subsequence of (s_n) where s_{n_k} converges to some $s \in \mathbb{R}$. Then (s_n) converges, and $\lim s_n = s$.

Proof. Let $\epsilon > 0$. Since (s_n) is Cauchy, then there exists some $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon/2$ for all $n > N$ and $m > N$. Since (s_{n_k}) converges to s, there exists $N_1 \in \mathbb{N}$ such that $|s_{n_k} - s| < \frac{\epsilon}{2}$ for all $k > N_1$. Let $k \in \mathbb{N}$ where $k > N$ and $k > N_1$. Since $n_k \geq k$, then $n_k > N$ and $n_k > N_1$. For all $n > k$, we have:

$$|s_n - s| = |s_n - s_{n_k} + s_{n_k} - s| \leq \underbrace{|s_n - s_{n_k}|}_{n_1, n_k > N} + \underbrace{|s_{n_k} - s|}_{n_k > N_1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Lemma 1.4.4 ▸ Cauchy Sequences are Bounded

If a sequence of real numbers is Cauchy, then that sequence is bounded.

Proof. Let (s_n) be a Cauchy sequence of real numbers. Then there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < 1$ for all $n > N$ and $m > N$. Let $m \coloneqq N + 1$. Then, for all $n > N$, we have:

$$|s_n| = |s_n - s_m + s_m| \leq |s_n - s_m| + |s_m| < 1 + |s_m| = 1 + |s_{N+1}|$$

Thus, for all $n \in \mathbb{N}$, we have:

$$|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_N|, 1 + |s_{N+1}|\}$$

Therefore, (s_n) is bounded. ▢

Theorem 1.4.5 ▸ Cauchy Criterion

A sequence of real numbers converges if and only if it is Cauchy.

Proof. By Lemma 1.4.2, we know that convergence implies Cauchy.
If (s_n) is Cauchy, then by Lemma 1.4.4, (s_n) is bounded. By the Bolzano-Weierstrass Theorem, (s_n) has a convergent subsequence. By Lemma 1.4.3, (s_n) converges. ▢

Our definition of completeness in $\mathbb{R}$ predicates on a notion of order between the elements. Specifically, we said $\mathbb{R}$ is complete because every subset of $\mathbb{R}$ that is bounded above has a supremum. What does it mean to say $\mathbb{R}^2$ is complete?

Definition 1.4.6 ▸ Completeness (in terms of Cauchy Sequences)

A (metric) space is ***complete*** if every Cauchy sequence converges to a point in the space.

The intuition is the same: there are no points "missing" from the space.