Definition 0.0.1 ▸ Cardinality

Cardinality is a measure of the amount of elements in a set, denoted $|A|$. We say two sets have the same cardinality if there exists a bijection between them.

For finite sets, we can think of cardinality as the number of elements in that set. For infinite sets, cardinality can sometimes go against our intuition. For any sets A, B, C:

1. $|A| = |A|$
2. if $|A| = |B|$, then $|B| = |A|$
3. if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Hence, cardinality is an equivalence relation.

Example 0.0.1 ▸ Cardinality of $\mathbb{N}$ and $2\mathbb{N}$

Let $2\mathbb{N} \coloneqq \{2n : n \in \mathbb{N}\}$ (i.e. the set of even natural numbers). Then $|\mathbb{N}| = |2\mathbb{N}|$.

Proof. To show that these two sets have the same cardinality, we need to find some bijection between the sets. Let $f : \mathbb{N} \to 2\mathbb{N}$ be a function defined by $f(n) = 2n$. Note that f is well-defined (i.e. is actually a function) because $f(n) \in 2\mathbb{N}$ for all $n \in \mathbb{N}$. To prove that f is a bijection, we need to prove it is both injective and surjective.

1. Let $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $2n_1 = 2n_2$, so $n_1 = n_2$. Thus, f is injective.
2. Let $m \in 2\mathbb{N}$. Then $m = 2k$ for some $k \in \mathbb{N}$, so $m = 2k = f(k)$ for some $k \in \mathbb{N}$. Thus, f is surjective.

Therefore, f is a bijection, so $|\mathbb{N}| = |2\mathbb{N}|$. □

Example 0.0.2 ▸ Cardinality of Intervals

Let $a, b \in \mathbb{R}$ where $a < b$. Then $|(0,1)| = |(a,b)|$.

Proof. We need to find a bijection from $(0,1)$ to (a,b). We need to "scale" the interval $(0,1)$ to the width of (a,b), then translate it to match (a,b). Define $f : (0,1) \to (a,b)$ by $f(x) = a + (b-a)x$. (We need to check f is well-defined). Let $x \in (0,1)$. Then $0 < x < 1$, so multiplying by $(b-a)$ which is positive gives $0 < (b-a)x < b-a$. Adding a, we get $a < a + (b-a)x < b$. Now we need to show f is a bijection:

1. Let $x_1, x_2 \in (0,1)$ such that $f(x_1) = f(x_2)$. Then $a + (b-a)x_1 = a + (b-a)x_2$. Subtracting a from both sides, we get $(b-a)x_1 = (b-a)x_2$. Since $(b-a) \neq 0$, we

can divide both side by $(b-a)$ to get $x_1 = x_2$.

2. Let $y \in (a, b)$.

> **Scratchwork:** We want to find some $x \in (0,1)$ where $y = f(x) = a + (b - a)x$. Using some algebra to solve for x, we have $x = \frac{y-a}{b-a}$

Let $x = \frac{y-a}{b-a}$. First, we show $x \in (0, 1)$:

$$\begin{aligned} a < y < b \\ \implies \quad 0 < y - a < b - a \\ \implies \quad 0 < \frac{y-a}{b-a} < 1 \end{aligned}$$

Thus, $x \in (0, 1)$. Also:

$$f(x) = a + (b-a)\left(\frac{y-a}{b-a}\right) = a + (y-a) = y$$

Thus, f is surjective.

Therefore, f is a bijective, so $|(0, 1)| = |(a, b)|$. □

Definition 0.0.2 ▶ Power Set

Let A be a set. The ***power set*** of A is the set of all subsets of A.

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

For example, the power set of $\{1, 2, 3\}$ is $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. For any finite set with n elements in it, its power set has 2^n elements in it.

Example 0.0.3 ▶ Cardinality of $\mathbb{N}$ and $\mathcal{P}(N)$

$|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$

Proof. We will show that any function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ cannot be surjective, and thus not bijective. Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any function defined by $f(n) = A_n$. Note $A_n \subseteq \mathbb{N}$, so $A_n \in \mathcal{P}(A)$. Now we will define a set that isn't in $f[\mathbb{N}]$. For each $n \in \mathbb{N}$, if $n \in A_n$, then $n \notin A$, and if $n \notin A_n$, then $n \in A$. More formally, $A \coloneqq \{n \in \mathbb{N} : n \notin A_n\}$. For all $k \in \mathbb{N}$, note that:

- if $k \in A_k$, then $k \notin A$, so $A \neq A_k$, and
- if $k \notin A_k$, then $k \in A_k$, so $A \neq A_k$.

Hence, $A \subseteq \mathbb{N}$, but $f(k) \neq A$ for any $k \in \mathbb{N}$. Thus, f is not surjective. ▢

Definition 0.0.3 ▸ Finite, Countably Infinite, Countable, Uncountable

Let A be a set. We say A is:

- ***finite*** if $A \neq \varnothing$ or $|A| = |\{1, 2, \dots, n\}|$ for some $n \in \mathbb{N}$.
- ***countably infinite*** if $|A| = |N|$.
- ***countable*** if A is finite or countably infinite
- ***uncountable*** if A is not countable

Theorem 0.0.1 ▸ $\mathcal{P}(\mathbb{N})$ is uncountable.

Proof. We know from Example 0.0.3 that $\mathcal{P}(\mathbb{N})$ is not countably infinite. We need to show that $\mathcal{P}(\mathbb{N})$ is not finite. Since $\{1\} \in \mathcal{P}(\mathbb{N})$, then it cannot be empty. Suppose for contradiction $|\{1, 2, \dots, n\}| = |\mathcal{P}(\mathbb{N})|$ for some $n \in \mathbb{N}$, then there exists a bijection $f : \{1, 2, \dots, n\} \to \mathcal{P}(\mathbb{N})$. Define $g : \mathbb{N} \to \{1, 2, \dots, n\}$ by:

$$g(k) = \begin{cases} k, & 1 \leq k \leq n \\ 1, & k > n \end{cases}$$

Then g is surjective, so $f \circ g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is surjective. This contradicts the fact that no such function exists (by Example 0.0.3). ▢

Generally, there is never a bijection from a set to its power set.

Intuition: A set is countable if its elements can be "listed" or "counted". That is, for finite sets:

$$X = \{x_1, x_2, \dots, x_n\} = \{x_k\}_{k=1}^{n}$$

For infinitely countable sets:

$$X = \{x_1, x_2, \dots\} = \{x_k\}_{k=1}^{\infty}$$

If X is finite, then there exists a bijection $f : \{1, 2, \dots, n\} \to X$. Thus, $X = \{f(1), f(2), \dots, f(n)\}$. If X is countably infinite, then there exists a bijection $f : \mathbb{N} \to X$.

Thus, $X = \{f(1), f(2), \dots\}$.

Theorem 0.0.2 ▸ Subsets of Countable Sets are Countable

The subset of a countable set is still countable. (i.e. a countable set cannot contain an uncountable subset).

Proof. Let X be a countable set, and let $A \subseteq X$. We will consider two cases. First, if A is finite, then A is countable, and we are done. Otherwise, A is infinite, and hence X is infinite. Then X is countably infinite, so $X = \{x_1, x_2, \dots\} = \{x_k\}_{k=1}^{\infty}$.

> **Idea:** Our set A might look something like $\{x_3, x_4, x_6, \dots\}$. We need to align these indices to 1, 2, 3, and so on. We'll let $k_1 = \min\{3, 4, 6, \dots\}$, let $k_2 = \min\{4, 6, \dots\}$, and so on.

Let $k_1 \coloneqq \min\{k \in \mathbb{N} : x_k \in A\}$. Let $a_1 \coloneqq x_{k_1}$. For all $j \in \mathbb{N}$ such that $j > 1$, we define $k_j \coloneqq \min\{k \in \mathbb{N} : (x_k \in A) \wedge (k > k_{j-1})\}$. Let $a_j \coloneqq x_{k_j}$. Then $1 \leq k_1 < k_2 < k_3 < \dots$, so k_j approaches infinity. Let $g : \mathbb{N} \to A$ be a function defined by $g(j) = a_j$. We need to show that g is both injective and surjective, and thus a bijection.

- Suppose that $g(j_1) = g(j_2)$ for some $j_1, j_2 \in \mathbb{N}$. Then $a_{j_1} = a_{j_2}$, so $x_{k_{j_1}} = x_{k_{j_2}}$. Then $k_{j_1} = k_{j_2}$, so $j_1 = j_2$. Thus, g is injective.
- Let $a \in A$ Since $A \subseteq X$, then $a \in X$. Thus, $a = x_l$ for some $l \in \mathbb{N}$. Let $m \coloneqq \min\{j \in \mathbb{N} : k_j \geq l\}$. Since $m \in \{j \in \mathbb{N} : j_k \geq l\}$, then $k_m \geq l$. Also, $m - 1 \notin \{j \in \mathbb{N} : k_j \geq l\}$, so $k_{m-1} < l$. Now, $k_m = \min\{k \in \mathbb{N} : (x_k \in A) \wedge (k > k_{m-1})\}$. But $x_l \in A$, and $l > k_{m-1}$, so $l \in \{k \in \mathbb{N} : (x_k \in A) \wedge (k > k_{m-1})\}$. Thus, $k_m \leq l$, because k_m is the minimum of the set containing l. By trichotomy, $k_m = l$. Therefore:
$$g(m) = a_m = x_{k_m} = x_l = a$$
So g is surjective.

Since g is a bijection, then $|\mathbb{N}| = |A|$, so $|A|$ is countable. ▢

Theorem 0.0.3 ▸ Injectivity and Cardinality

Suppose $A = \varnothing$. Then A is countable if and only if there exists an injective function $f : A \to \mathbb{N}$.

Proof. First, suppose A is a countable set. We consider two cases:

- If A is countably infinite, then there exists a bijection $f : A \to \mathbb{N}$.
- If A is finite, then there exists a bijection $f : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. Let $g : \{1, 2, \dots, n\} \to \mathbb{N}$ be a function defined by $g(x) = x$ (i.e. an inclusion mapping). Then f and g are both injective, so $g \circ f : A \to \mathbb{N}$ is injective.

Conversely, suppose $f : A \to \mathbb{N}$ is an injection. Then $f[A] \subseteq \mathbb{N}$, so $f[A]$ is countable by Theorem 0.0.2. Define $g : A \to f[A]$ by $g(a) = f(a)$. Then g is injective because f is injective, and g is surjective because $g[A] = f[A]$. Thus, g is a bijection, so $|A| = |f[A]|$. Therefore, A is countable. $\square$

Theorem 0.0.4 ▸ $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n, m) = 2^n 3^m$. We now show that f is bijective. To prove surjectivity, suppose $f(n_1, m_1) = f(n_2, n_2)$. Then $2^{n_1} 3^{m_1} = 2^{n_2} 3^{m_2}$.

- If $n_1 > n_2$, then $2^{n_1 - n_2} = 3^{m_2 - m_1}$. Since $n_1 > n_2$, we have $n_1 - n_2 > 0$, so $2^{n_1 - n_2} \in \mathbb{N}$. Then also $3^{m_2 - m_1} \in \mathbb{N}$. But $2^{n_1 - n_2}$ is even, and $3^{m_2 - m_1}$ is odd. This contradicts the fact that $2^{n_1 - n_2} = 3^{m_2 - m_1}$.
- If $n_2 > n_1$, then $3^{m_1 - m_2} = 2^{n_2 - n_1}$. By a similar argument, $2^{n_2 - n_1}$ is even and $3^{m_1 - m_2}$ is odd, producing the same contradiction.
- If $n_1 = n_2$, then $2^{n_1} = 2^{n_2}$, so cancelling gives $3^{m_1} = 3^{m_2}$. Thus, $m_1 = m_2$.

Hence, $(n_1, m_1) = (n_2, m_2)$, so f is injective. $\square$