Introduction to Analysis

UT Knoxville, Spring 2023, MATH 341

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Contents

Preface

These are my notes for the **Introduction to Analysis** course at the University of Tennessee (MATH 341). It is compiled from several sources including lecture notes by Dr. Michael Frazier and Dr. Peter Humphries, as well as online resources such as the Mathematics Stack Exchange.

The first few weeks of the course are spent reviewing the material taught in **Introduction to Abstract Mathematics** (MATH 300): logic, set theory, number systems, and cardinality. They serve as a “primer” for the following material on real analysis.

1 Introduction

Our goal is to understand the theory of real functions in one variable. Specifically, we will deal with functions, limits, sequences, convergence, continuity, differentiation, and integration. The same ideas, concepts, and techniques are used to study more complicated mathematics.

We will primarily focus on the idea of **convergence**. Many computational techniques and algorithms rely on iteration—successive approximations getting closer to an actual solution. In order for those algorithms to work, they need to converge towards an actual solution.

To motivate our quest to learn about convergence, let's look at some classic iterative methods.

Example 1.0.1 ▸ Newton's Method

Given $c > 0$, suppose we want to calculate $\sqrt{c}$. Start with some initial guess $x_1 > 0$.

$$\begin{aligned}
\text{Let} \quad x_2 &:= \frac{1}{2}\left(x_1 + \frac{c}{x_1}\right) \\
\text{Let} \quad x_3 &:= \frac{1}{2}\left(x_2 + \frac{c}{x_2}\right) \\
&\vdots \\
\text{Let} \quad x_{n+1} &:= \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)
\end{aligned}$$

We find that $\lim_{n\to\infty} x_n = \sqrt{c}$.

Does this method work for all $c > 0$ and $x_1 > 0$? Assuming $\lim_{n\to\infty} x_n = x$ converges, then:

$$\begin{aligned}
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) \\
\implies \qquad x &= \frac{1}{2}\left(x + \frac{c}{x}\right) \\
\implies \qquad 2x &= x + \frac{c}{x} \\
\implies \qquad x &= \frac{c}{x} \\
\implies \qquad x^2 &= c \\
\implies \qquad x &= \sqrt{c}
\end{aligned}$$

The above calculation only makes sense if we know the sequence converges. Consider the sequence $x_{n+1} = 6 - x_n$ where $x_1 = 4$. Then:

$$x_1 = 4, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 2, \quad \ldots$$

Since this sequence does not converge, there is no limit when n approaches infinity.

Let's look at a more complicated iterative method.

Example 1.0.2 ▸ Picard's Method

Suppose we had to solve $y' = f(x, y)$ where $y(x_0) = y_0$ (i.e. find a function y that satisfies our two conditions). As it turns out, we can use an iterated method to solve this as well.

- Start with an initial guess $y_1(x)$
- Define $y_{n+1}(x) \coloneqq y_0 + \displaystyle\int_{x_0}^{x} f(t, y_n)\, dt$.

Provided that f and y_0 are "well-behaving", then the sequence of functions $y_n(x)$ converges to the solution $y(x)$.

This idea that an infinite sequence of functions can converge suggests some notion of "distance" between functions. We can use a number of metrics for distance, some possibilities including:

- $\displaystyle\int_a^b |f(x) - g(x)|\, dx$ (total area between the two functions)
- $\sup\{x : x = |f(x) - g(x)|\}$ (max possible "vertical" distance between the two curves)

2

Logic and Proofs

Formal logic is the foundation of mathematics. It enables us to construct logically consistent models by starting with a set of axioms and systematically deducing new statements from them. This process not only helps us to prove known results but also to uncover new mathematical concepts and theorems.

2.1 Basic Logic

Definition 2.1.1 ▶ Statement

A ***statement*** is a claim that is either true or false.

$$p \,:\, \text{some claim}$$

We usually denote statements with a letter like p. For example, we can write "$p : x > 2$", which means p represents the statement "x is greater than 2". Throughout this chapter, we will use p and q to represent arbitrary statements.

Definition 2.1.2 ▶ Conjunction

Logical ***conjunction*** is an operation that takes two statements and produces a new statement that is true only when both input statements are true.

$$p \wedge q \,:\, p \text{ is true } \textbf{and } q \text{ is true}$$

Definition 2.1.3 ▶ Disjunction

Logical ***disjunction*** is an operation that takes two statements and produces a new statement that is true when at least one of the input statements is true.

$$p \vee q \,:\, p \text{ is true } \textbf{or } q \text{ is true}$$

Conjunction and Disjunction follow our intuition of "and" and inclusive "or", respectively. We can visualize the two logical connectives using ***truth tables***.

Example 2.1.4 ▶ Truth Table of Conjunction

p	q	$p \implies q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.1.5 ▶ Truth Table of Disjunction

p	q	$p \implies q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 2.1.6 ▶ Negation

The ***negation*** of a statement is a statement with opposite truth values.

$$\neg p$$

Definition 2.1.7 ▶ Implication

An ***implication*** "p implies q" states "if p is true, then q is true".

$$p \implies q$$

In the implication $p \implies q$, we call p the ***hypothesis*** and q the ***conclusion***. If the hypothesis is false to begin with, then the implication is not really meaningful. Instead of assigning those kinds of implications no truth value, we simply consider them true by convention. These kinds of truths are called ***vacuous truths***.

Example 2.1.8 ▸ Truth Table of Implication

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 2.1.9 ▸ Simple Statements

Let $p : x > 2$ and $q : x^2 > 1$. Consider the following statements:

- "For all real numbers x, $p \implies q$"
  True. If $x > 2$, then $x^2 > 1$.[a]
- "For all real numbers x, $q \implies p$"
  False. Consider $x = 1.1$. Then $x^2 = 1.21 > 1$, but $x = 1.1 < 2$.

[a] This is normally where we would rigorously prove such a statement, but we will omit this for now.

Definition 2.1.10 ▸ Logical Equivalence

p and q are ***logically equivalent*** if $p \implies q$ and $q \implies p$.

$$p \iff q$$

In other words, $p \iff q$ means that p and q share the same truth value. Either p and q are **always both true**, or p and q are **always both false**. Logical equivalence says nothing about the truth of p and q themselves.

We can also say "p if and only if q" or "p iff q" to denote logical equivalence.

Example 2.1.11 ▶ Truth Table of Logical Equivalence

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 2.1.12 ▶ Converse

Given the implication $p \implies q$, its ***converse*** statement is $q \implies p$.

It's important to note that an implication and its converse have no intrinsic equivalence.

Example 2.1.13 ▶ Truth Table of Converse

p	q	$p \implies q$	$q \implies p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Definition 2.1.14 ▶ Contrapositive

Given the implication $p \implies q$, its ***contrapositive*** statement is $\neg q \implies \neg p$.

Unlike the converse, an implication and its contrapositive are logically equivalent. To help our intuition, we can construct a truth table.

Example 2.1.15 ▸ Truth Table of Contrapositive

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg q \implies \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

As we can see, no matter what the truth values of the hypothesis and conclusion are, an implication and its contrapositive always have the same truth values.

> When constructing a truth table, we must include **all** intermediate statements, not just the final statement.

2.2 Proofs and Proof Techniques

While truth tables are a useful tool for evaluating simple statements, they quickly become impractical when dealing with more complex propositions. Moreover, they do not offer insights into the reasoning behind such statements. In contrast, proofs can provide us with a deeper understanding of logical relationships and help us reason about complex statements.

In particular, we often need to prove implications of the form $p \implies q$, where the truth of p guarantees the truth of q. To do so, we can use a variety of proof techniques:

1. ***Direct Proof:*** Assume p is true, then reason that q must be true as well.
2. ***Proof by Contradiction:*** Assume both p and $\neg q$ are true, then logically derive some contradiction.
3. ***Proof by Contrapositive:*** Assume $\neg q$ is true, then reason that $\neg p$ must be true as well.

In mathematical proofs, there are two main types of reasoning: direct and indirect. A direct proof shows a clear path from the premises to the conclusion, providing valuable insights into the underlying mathematics. In contrast, indirect proofs rely on a contradictory hypothesis to establish the truth of the conclusion. While indirect proofs can be useful when a direct proof is not readily available, they may be less insightful since they do not provide much context surrounding the premises.

However, it is worth noting that an indirect proof may be easier to find than a direct proof in certain cases. While a direct proof requires identifying the correct path that leads to the conclusion, an indirect proof only needs to deduce any contradictory statement. Despite this advantage, indirect proofs should be used sparingly and only when a direct proof is not feasible.

Technique 2.2.1 ▸ Proof by Contradiction

To prove $p \implies q$ by contradiction, we carry out the following steps:

1. Assume p is true, and suppose for the sake of contradiction $\neg q$ is true.
2. Logically derive a statement that contradicts something we know to be true.
3. Ultimately conclude that q must be true.

In terms of logic notation, proof by contradiction follows:

$$[(p \wedge (\neg q)) \implies \text{Contradiction}] \implies [p \implies q]$$

Example 2.2.2 ▸ Truth Table of Proof by Contradiction

p	q	$p \implies q$	$\neg q$	$p \wedge (\neg q)$	$\neg [p \wedge (\neg q)]$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	F	T

By the above truth table, we can safely assume the following logical equivalence:

$$(p \implies q) \iff \neg [p \wedge (\neg q)]$$

Technique 2.2.3 ▸ Proof by Contrapositive

To prove $p \implies q$ by contrapositive, we carry out the following steps:

1. Assume $\neg q$ is true.
2. Directly prove that $\neg p$ is true.

In terms of logic notation, proof by contrapositive follows:

$$(\neg q \implies \neg p) \iff (p \implies q)$$

We can actually prove this using proof by contradiction!

Example 2.2.4 ▶ Logical Equivalence of Contrapositive

Given statements p and q, $p \implies q$ and $\neg q \implies \neg p$ are equivalent.

Proof. Assume $p \implies q$. To prove $\neg q \implies \neg p$, we can suppose for contradiction that $\neg q$ and p are both true. But $p \implies q$, so q is true which contradicts $\neg q$. Hence, the assumption that p is true was incorrect. Thus, $\neg q \implies \neg p$.
Assume $\neg q \implies \neg p$. From above, we have $\neg(\neg p) \implies \neg(\neg q)$, so $p \implies q$. $\square$

Example 2.2.5 ▶ Proving Simple Logic Statements

Let p, q, and r be arbitrary statements. Prove that $[p \implies (q \vee r)] \iff [(p \wedge \neg q) \implies r]$.

Proof. Assume $p \implies (q \vee r)$. Suppose $p \wedge \neg q$. Then p is true, so $q \vee r$ is true by assumption. Also, $\neg q$ is true, so r must be true from $q \vee r$.
Assume $(p \wedge \neg q) \implies r$. Suppose p is true. There are two possibilities:

1. If q is true, then $q \vee r$ is true.
2. If $\neg q$ is true, then $p \wedge \neg q$ is true. Thus, r is true by assumption. Hence, $q \vee r$ is true.

$\square$

3 Naive Set Theory

Instead of forming a rigorous, axiomatic basis for sets, we will simply take an informal approach to sets guided by our intuition. Ultimately, our introduction to real analysis does not fiddle with the fine details of set theory, so it's safe to take a naive approach.

3.1 Sets

Definition 3.1.1 ▶ Set

A ***set*** is a collection of distinct objects.

For example, $\mathbb{N} := \{1, 2, 3 \ldots\}$ is the set of all natural numbers, and $\mathbb{Z} := \{\ldots, 1, 2, 3, \ldots\}$ is the set of all integers. It's conventional to use capital letters to denote sets and use lowercase letters to denote elements of sets. Throughout this chapter, we will use A and B to represent arbitrary sets.

Definition 3.1.2 ▶ Membership, $\in$

We write $a \in A$ to mean "a is in A".

Definition 3.1.3 ▶ Subset, $\subseteq$

A is a ***subset*** of B if everything in A is also in B.

$$A \subseteq B \iff \forall(x \in A)(x \in B)$$

Definition 3.1.4 ▶ Set Equality, $=$

A ***equals*** B if A is a subset of B and B is a subset of A.

$$A = B \iff (A \subseteq B \wedge B \subseteq A)$$

Definition 3.1.5 ▶ Proper Subset, $\subsetneq$

A is a ***proper subset*** of B if A is a subset of B but B is not a subset of A.

$$A \subsetneq B \iff (A \subseteq B \land B \nsubseteq A)$$

In other words, A is a proper subset of B if everything in A is also in B, but B has something that A does not.

> Among mathematical texts, the generic subset symbol $\subset$ has no standardized definition. Some use it to represent subset or equal; others use it to represent proper subset. We will simply not use $\subset$ to avoid any ambiguity.

Definition 3.1.6 ▶ Empty Set ($\varnothing$)

The ***empty set*** is the set that contains no elements.

$$\varnothing := \{\}$$

As convention, we assume that $\varnothing$ is a subset of every set, including itself.

Technique 3.1.7 ▶ Proving a Subset Relation

To prove that $A \subseteq B$:

1. Let x be an arbitrary element of A.
2. Show that $x \in B$.

To prove that $A \nsubseteq B$, choose a specific $x \in A$ and show $x \notin B$.

Example 3.1.8 ▶ Proving Simple Subset Relation

Suppose that $A \subseteq B$ and $B \subseteq C$. Prove that $A \subseteq C$.

Proof. Let $x \in A$ be arbitrary. Since $A \subseteq B$, then $x \in B$. Similarly, since $B \subseteq C$, then $x \in C$. Therefore, $A \subseteq C$. □

Definition 3.1.9 ▶ Union

The ***union*** of two sets is the set of all things that are in one or the other set.

$$A \cup B := \{x : x \in A \vee x \in B\}$$

Definition 3.1.10 ▶ Intersection

The ***intersection*** of two sets is the set of all things that are in both sets.

$$A \cap B := \{x : x \in A \wedge x \in B\}$$

More generally, we can apply union and intersection to an arbitrary number of sets, finite or infinite. We use a notation similar to summation using $\sum$. Let Λ be an indexing set, and for each $\lambda \in \Lambda$, let A_λ be a set.

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$$

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\}$$

Example 3.1.11 ▶ Indexed Sets

For $n \in \mathbb{N}$, let $A_n = \left[\frac{1}{n}, 1\right] = \left\{x \in \mathbb{R} : \frac{1}{n} \leq x \leq 1\right\}$. Prove that:

(a) $\bigcup_{n=1}^{\infty} = (0, 1]$

(b) $\bigcap_{n=1}^{\infty} = \{1\}$

Proof of (a). Suppose $x \in \bigcup_{n=1}^{\infty} A_n$. Then there exists $n \in \mathbb{N}$ such that $x \in A_n = \left[\frac{1}{n}, 1\right]$. That is, $0 < \frac{1}{n} \leq x \leq 1$. Therefore, $x \in (0, 1]$.

Suppose $x \in (0, 1]$. Then $x > 0$, so there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. Then $\frac{1}{n_0} \leq x \leq 1$, so $x \in A_{n_0}$. Therefore, $x \in \bigcup_{n=1}^{\infty} A_n$. ▢

Proof of (b). Suppose $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_1 = \{1\}$.

Suppose $x \in \{1\}$. Then $x = 1 \in \left[\frac{1}{n}, 1\right]$ for all $n \in \mathbb{N}$. Therefore, $x \in \bigcap_{n=1}^{\infty} A_n$. ▢

Definition 3.1.12 ▶ Set Minus

The ***set difference*** of two sets is the set of all things that are in first set but not the second set.

$$A \setminus B = \{x \in A : x \notin B\}$$

Definition 3.1.13 ▸ Complement

Let X be a set called the ***universal set***. The ***complement*** of A in X is defined as $X \setminus A$.

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

Theorem 3.1.14 ▸ De Morgan's Laws for Sets

Suppose X is a set, and for any subset S of X, let $S^c = X \setminus S$. Suppose that $A_\lambda \subseteq X$ for every λ belonging to some index set Λ. Prove that:

(a) $\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c$;

(b) $\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$.

Proof of (a). First, let $a \in \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c$. Then, $a \in X \setminus \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$, so $a \in X$ but $a \notin \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$. Thus, $a \notin A_\lambda$ for any $\lambda \in \Lambda$, so $a \in X \setminus A_\lambda$ for all $\lambda \in \Lambda$. In other words, $a \in \bigcap_{\lambda\in\Lambda} A_\lambda^c$.

Next, let $a \in \bigcap_{\lambda\in\Lambda} A_\lambda^c$. Then $a \in A_\lambda^c$ for all $\lambda \in \Lambda$, so $a \in X$ but $a \notin A_\lambda$ for all $\lambda \in \Lambda$. That is, $a \notin \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$. In other words, $a \in \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c$. ▢

Proof of (b). First, let $a \in \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c$. Then, $a \in X \setminus \bigcap_{\lambda\in\Lambda} A_\lambda$, so $a \in X$ but $a \notin \bigcap_{\lambda\in\Lambda} A_\lambda$. That is, $a \notin A_\lambda$ for some $\lambda \in \Lambda$. Thus, $a \in X \setminus A_\lambda$ for some $\lambda \in \Lambda$. Therefore, $a \in \bigcup_{\lambda\in\Lambda} A_\lambda^c$.

Next, let $a \in \bigcup_{\lambda\in\Lambda} A_\lambda^c$. Then $a \in A_\lambda^c$ for some $\lambda \in \Lambda$, so $a \in X$ but $a \notin A_\lambda$ for some $\lambda \in \Lambda$. That is, $a \notin \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. Therefore, $a \in \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c$. ▢

3.2 Functions

We generally think of functions as a "map" or "rule" that assigns numbers to other numbers. For example, $f(x) = 2x$ maps $1 \mapsto 2$, $2 \mapsto 4$, etc. More formally, we define functions in terms of sets.

Definition 3.2.1 ▶ Cartesian Product

Let X and Y be sets. The ***Cartesian product*** of X and Y is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$.

$$X \times Y \coloneqq \{(x, y) : x \in X \land y \in Y\}$$

Definition 3.2.2 ▶ Relation

Let X and Y be sets. A ***relation*** between X and Y is a subset of the Cartesian product $X \times Y$.

Definition 3.2.3 ▶ Function

Let X and Y be sets. A ***function*** from X to Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ where $(x, y) \in f$.

More formally, a ***function*** $f : X \to Y$ is a subset of $X \times Y$ satisfying:

1. $\forall (x \in X)\,[\exists (y \in Y)((x, y) \in f)]$
2. $(x, y_1), (x, y_2) \in f \implies y_1 = y_2$

Given a function $f : X \to Y$, we call X the ***domain*** of f and Y the ***codomain*** of f. Given $x \in X$, we write $f(x)$ to denote the unique element of Y such that $(x, y) \in f$.

$$f(x) = y \iff (x, y) \in f$$

Definition 3.2.4 ▶ Function Image

Let $f : X \to Y$ be a function and $A \subseteq X$. The ***image*** of A under f is the set containing all possible function outputs from all inputs in A.

$$f[A] \coloneqq \{f(a) : a \in A\}$$

Given $f : X \to Y$, we call $f[X]$ the ***range*** of f.

Example 3.2.5 ▶ Function Images

Suppose $f : X \to Y$ is a function, and $A_\lambda \subseteq X$ for each $\lambda \in \Lambda$. Then:

(a) $f\left[\bigcup_{\lambda \in \Lambda} A_\lambda\right] = \bigcup_{\lambda \in \Lambda} f[A_\lambda]$

(b) $f\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] \subseteq \bigcap_{\lambda\in\Lambda} f\left[A_\lambda\right]$

In this example, we will only prove the "forward" direction. That is, we want to show that $f\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] \subseteq \bigcup_{\lambda\in\Lambda} f\left[A_\lambda\right]$.

Proof of (a). Let $y \in f\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right]$. By definition of Function Image, there exists $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$ such that $y = f(x)$. Thus, there exists $\lambda_0 \in \Lambda$ such that $x \in \lambda_0$. That is, $y \in f\left[A_{\lambda_0}\right]$. Therefore, $y \in \bigcup_{\lambda\in\Lambda} f\left[A_\lambda\right]$. ▢

Definition 3.2.6 ▸ Function Inverse Image

Let $f : X \to Y$ be a function and $B \subseteq Y$. The ***inverse image*** of B under f is the set containing all possible function inputs whose output is in B.

$$f^{-1}[B] \coloneqq \{x \in X : f(x) \in B\}$$

Note the following logical equivalence:

$$x \in f^{-1}[B] \iff f(x) \in B$$

Example 3.2.7 ▸ Function Inverse Images

Suppose $f : X \to Y$ is a function, and $B_\lambda \subseteq Y$ for each $\lambda \in \Lambda$. Then:

$$f^{-1}\left[\bigcup_{\lambda\in\Lambda} B_\lambda\right] = \bigcup_{\lambda\in\Lambda} f^{-1}\left[B_\lambda\right]$$

Again, we will only prove the "forward direction".

Proof. Let $x \in f^{-1}\left[\bigcup_{\lambda\in\Lambda} B_\lambda\right]$. Then, $f(x) \in \bigcup_{\lambda\in\Lambda} B_\lambda$. That is, $f(x) \in B_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Thus, $x \in f^{-1}\left[B_{\lambda_0}\right]$, so $x \in \bigcup_{\lambda\in\Lambda} f^{-1}\left[B_\lambda\right]$. ▢

3.3 Injectivity and Surjectivity

Definition 3.3.1 ▸ Injective, One-to-one

A function $f : X \to Y$ is ***injective*** or ***one-to-one*** if no two inputs in X have the same output in Y.

$$\forall(x_1, x_2 \in X)\,[x_1 \neq x_2 \implies f(x_1) \neq f(x_2)]$$

We can also think of injectivity as, "if two inputs have the same output, then the two inputs must be the same". It's really just the contrapositive of our initial definition, which we know must be logically equivalent.

$$\forall(x_1, x_2 \in x)\,[f(x_1) = f(x_2) \implies x_1 = x_2]$$

For example, the function $f(x) = x^2$ is not injective, because $f(-1) = 1$ and $f(1) = 1$. We have two distinct inputs that map to the same output.

Technique 3.3.2 ▸ Proving a Function is Injective

To prove a function $f : X \to Y$ is injective:

1. Let $x_1, x_2 \in X$ where $f(x_1) = f(x_2)$.
2. Reason that $x_1 = x_2$.

Example 3.3.3 ▸ Proving Injectivity

$f(x) = -3x - 7$ is injective.

Proof. Suppose $f(x_1) = f(x_2)$. Then $-3x_1 + 7 = -3x_2 + 7$, so $-3x_1 = -3x_2$. Thus, $x_1 = x_2$, so f is injective. ▢

Example 3.3.4 ▸ Disproving Injectivity

Prove that $f(x) = x^2$ is not injective.

Proof. $f(-1) = 1$ and $f(1) = 1$, but $-1 \neq 1$. Thus, f is not injective. ▢

Definition 3.3.5 ▶ Surjective, Onto

A function $f : X \to Y$ is ***surjective*** or ***onto*** if everything in Y has a corresponding input in X.

$$\forall(y \in Y)\,[\exists(x \in X)(f(x) = y)]$$

Note that $f : X \to f[X]$ is **always** surjective.

Technique 3.3.6 ▶ Proving a Function is Surjective

To prove a function $f : X \to Y$ is surjective:

1. Let $y \in Y$ be arbitrary.
2. "Undo" the function f to obtain $x \in X$ where $f(x) = y$.

Example 3.3.7 ▶ Proving Surjectivity

Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -3x + 7$ is surjective.

Proof. Let $y \in Y$ be arbitrary. Let $x := \frac{y-7}{-3}$. Then $x \in \mathbb{R}$, and:

$$\begin{aligned} f(x) &= -3\left(\frac{y-7}{-3}\right) + 7 \\ &= (y-7) + 7 \\ &= y \end{aligned}$$

Therefore, f is surjective. ▢

Definition 3.3.8 ▶ Bijective

A function $f : X \to Y$ is ***bijective*** if it is both injective and surjective.

Definition 3.3.9 ▶ Function Composition

Let $f : X \to Y$ and $g : Y \to Z$ be functions. The ***composition*** of f and g is a function $g \circ f : X \to Z$ defined by:

$$(g \circ f)(x) := g(f(x))$$

Theorem 3.3.10 ▸ Composition Preserves Injectivity and Surjectivity

Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.

(a) If f and g are injective, then $g \circ f$ is injective.

(b) If f and g are surjective, then $g \circ f$ is surjective.

(c) If f and g are bijective, then $g \circ f$ is bijective.

Proof of (a). Let $x_1, x_2 \in X$. Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then, $g(f(x_1)) = g(f(x_2))$. Because g is injective, we have $f(x_1) = f(x_2)$. Because f is injective, we have $x_1 = x_2$. Therefore, $g \circ f$ is injective. □

Proof of (b). Let $z \in Z$. Because g is surjective, there exists an element $y \in Y$ such that $g(y) = z$. Because f is surjective, there exists an element $x \in X$ such that $f(x) = y$. Thus, $(g \circ f)(x) = g(f(x)) = g(y) = z$. Therefore, $g \circ f$ is surjective. □

Proof of (c). We know that from (a) and (b) composition preserves injectivity and surjectivity. Thus, composition must also preserve bijectivity. □

Definition 3.3.11 ▸ Inverse Function

Let $f : X \to Y$ be a bijection. The ***inverse function*** of f is a function $f^{-1} : Y \to X$ defined by:

$$f^{-1} := \{(y, x) \in Y \times X : (x, y) \in f\}$$

The notation for inverse functions conflicts with the notation for inverse images. A key distinction to make it that only bijections can have an inverse function, but we can apply the inverse image to any function. Thus, given a bijection $f : X \to Y$, we know $f^{-1}(f(x)) = x$ for all $x \in X$, and $f(f^{-1}(y)) = y$ for all $y \in Y$.

Example 3.3.12

Let $f : X \to Y$ and $g : Y \to X$ be functions such that $(g \circ f) = x$ for all $x \in X$, and $(f \circ g)(y) = y$ for all $y \in Y$. $f^{-1} = g$.

Proof. todo: finish proof □

4

Number Systems

Our goal is to create an axiomatic basis for the real numbers $\mathbb{R}$. We need to establish axioms for $\mathbb{R}$ and then derive all further properties from the axioms. We would like these axioms to be as minimal and agreeable as possible; however, finding axioms that characterize $\mathbb{R}$ is not easy. Instead, we'll start from the natural numbers $\mathbb{N}$ and expand from there.

4.1 Natural Numbers $\mathbb{N}$ and Induction

How do we define the natural numbers? Listing every natural number is definitely not an option. We could try to define the natural numbers as $\mathbb{N} := \{1, 2, \ldots\}$. However, the "…" is ambiguous. Instead, we can define $\mathbb{N}$ in terms of its properties.

Definition 4.1.1 ▸ Peano Axioms for $\mathbb{N}$

The ***Peano axioms*** are five axioms that can be used to define the natural numbers $\mathbb{N}$.

1. $1 \in \mathbb{N}$
2. Every $n \in \mathbb{N}$ has a successor called $n + 1$.
3. 1 is **not** the successor of any $n \in \mathbb{N}$.
4. If $n, m \in \mathbb{N}$ have the same successor, then $n = m$.
5. If $1 \in S$ and every $n \in S$ has a successor, then $\mathbb{N} \subseteq S$.

Note that there is not one "prescribed" way to do define the natural numbers. This is just the most popular approach.

From the fifth Peano axiom, we can derive a new proof technique for proving statements about consecutive natural numbers.

Theorem 4.1.2 ▸ Principle of Induction (by the Peano Axioms)

Let $P(n)$ be a statement for each $n \in \mathbb{N}$. Suppose that:

1. $P(1)$ is true, and
2. if $P(n)$ is true, then $P(n + 1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $S := \{n \in \mathbb{N} : P(n)\}$. Then $1 \in S$ because $P(1)$ is true. Note that if $n \in S$, then $P(n)$ is true. Hence, $P(n+1)$ is true by assumption, so $n+1 \in S$. By the fifth Peano axiom, we have $\mathbb{N} \subseteq S$. Since S was defined as a subset of $\mathbb{N}$, we have $\mathbb{N} = S$. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$. ▢

A proof by induction has a "domino effect". Imagine a domino for each natural number 1, 2, 3, and so on, arranged in an infinite row. Knocking the 1st domino will knock them all down.

$$\underbrace{P(1)}_{\text{by 1.}} \implies \underbrace{P(2)}_{\text{by 2.}} \implies \underbrace{P(3)}_{\text{by 2.}} \implies \cdots$$

Technique 4.1.3 ▸ Proof by Induction

To prove a statement $P(n)$ for all $n \in \mathbb{N}$, we need to prove two statements:

1. ***Base Case:*** Prove $P(1)$.
2. ***Induction Step:*** Assume $P(n)$ is true from some $n \in \mathbb{N}$, then prove $P(n) \implies P(n+1)$.

It is crucial that we actually use our assumption that $P(n)$ is true in the induction step. Otherwise, our proof is most likely wrong.

Example 4.1.4 ▸ Simple Proof by Induction

Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Proof. Let $P(n)$ be the statement $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Base Case: When $n = 1$, LHS $= 1$ and RHS $= \frac{1(1+1)}{2} = 1$, so $P(1)$ is true.
Induction Step: Assume that $P(n)$ is true for some $n \in \mathbb{N}$. Then:

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

That is, $P(n+1)$ is true. By the Principle of Induction (by the Peano Axioms), $P(n)$ is true for all $n \in \mathbb{N}$. ▢

4.2 Integers $\mathbb{Z}$

From the natural numbers, we can easily construct the integers. First, we assume the existence an operation, addition (+) and multiplication ($\cdot$). On $\mathbb{N}$, we assume addition and multiplication satisfy the following properties for all $a, b, c \in \mathbb{N}$:

- ***Commutativity*** $a + b = b + a$ $a \cdot b = b \cdot a$
- ***Associativity*** $(a + b) + c = a + (b + c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ***Distributivity*** $a \cdot (b + c) = a \cdot b + a \cdot c$
- ***Identity*** $1 \cdot n = n$

Sort out this wonky formatting

We can expand this number system by including:

1. an ***additive identity*** ($n + 0 = n$ for all $n \in \mathbb{N}$)
2. ***additive inverses*** (for all $n \in \mathbb{N}$, add $-n$ so $-n + n = 0$)

From this, we can construct the set of integers.

Definition 4.2.1 ▸ Integers $\mathbb{Z}$

The set of ***integers*** is defined as:

$$\mathbb{Z} := \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$$

Definition 4.2.2 ▸ Even, Odd, Parity

Let $a \in \mathbb{Z}$.

- a is ***even*** if there exists $k \in \mathbb{Z}$ where $a = 2k$.
- a is ***odd*** if there exists $k \in \mathbb{Z}$ where $a = 2k + 1$.
- ***Parity*** describes whether an integer is even or odd.

Theorem 4.2.3 ▸ Parity Exclusivity

Every integer is either even or odd, never both.

TODO: prove this

Example 4.2.4 ▸ Parity of Square

For $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof. We proceed by contraposition. Suppose n is not even. Then n is odd, and thus can be expressed as $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then:

$$\begin{aligned} n^2 &= (2k+1)(2k+1) \\ &= 4k^2 + 4k + 1 \end{aligned}$$

Since the integers are closed under addition and multiplication, then $4k^2 + 4k \in \mathbb{Z}$. Thus, n^2 is odd. ▢

4.3 Rational Numbers $\mathbb{Q}$

We can further expand this number system by the following:

1. Include ***multiplicative inverses*** (for all $n \in \mathbb{Z} \setminus \{0\}$, define ${}^1/_n$ such that $n \cdot {}^1/_n = 1$)
2. Define $m \cdot {}^1/_n := {}^m/_n$ when $n \neq 0$.

From this, we can construct the set of rational numbers.

Definition 4.3.1 ▸ Rational Numbers $\mathbb{Q}$

The set of ***rational numbers*** is defined as:

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in Z \wedge n \neq 0 \right\}$$

To ensure multiplication works as intended, we also define $\frac{m}{n} \cdot \frac{k}{l} := \frac{m \cdot k}{n \cdot l}$.

We say $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ if and only if $m_1 n_1 = m_2 n_2$ where $n_1, n_2 \neq 0$. In other words, $\frac{m_1}{n_1} \sim \frac{m_2}{n_2} \iff m_1 n_2 = m_2 n_1$. Thus, $\mathbb{Q}$ is the set of equivalence classes for this relation.

If $n = kp$ and $m = kq$, where $k, p, q \in \mathbb{Z}$, $k \neq 0$, $q \neq 0$, then:

$$\frac{n}{m} = \frac{kp}{kq} = \frac{k}{p}, \quad \text{because } kpq = kqp$$

If n and m have no common factor (except ± 1), then we say that ${}^n/_m \in \mathbb{Q}$ is in the "lowest terms" or "reduced terms". The set $(Q, +, \cdot)$ forms a field. However, we cannot write $x = {}^n/_m$

where $x^2 = 2$.

Fix this whole section up. Very confusing.

Theorem 4.3.2 ▸ $\sqrt{2}$ is not a Rational Number

$\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose for contradiction $\sqrt{2}$ is a rational number. Then, there exist $n, m \in \mathbb{Z}$ such that $({}^{n}/_{m})^2 = 2$. If $n = kp$ and $m = kq$, then we can "cancel" the common factor k to write ${}^{n}/_{m} = {}^{p}/_{q}$. That is, we can assume that n and m have no (non-trivial) common factors. Now, ${}^{n^2}/_{m^2} = 2$, so by multiplying both sides by m^2, we get $n^2 = 2m^2$. Thus, n^2 is an even number, so n is also even (Example 4.2.4). Then, we can write $n = 2k$ where $k \in \mathbb{Z}$. Then:

$$\begin{aligned} \implies \quad (2k)^2 &= 2m^2 \\ \implies \quad 4k^2 &= 2m^2 \\ \implies \quad 2k^2 &= m^2 \end{aligned}$$

Then m^2 is even, so m is even. Thus, m and n are both even, so they are multiples of 2. This contradicts the fact that we defined ${}^{n}/_{m}$ in the lowest terms. ▢

Does there exist $r \in \mathbb{Q}$ such that $r^2 = 3$?

Definition 4.3.3 ▸ Divides

For $a, b, \in \mathbb{Z}$, we say a ***divides*** b if b is a multiple of a.

$$a \mid b \iff \exists(c \in \mathbb{Z})(b = ac)$$

Theorem 4.3.4 ▸ Division Algorithm

Suppose $a, b \in \mathbb{Z}$. Then $a = kb + r$ where $k \in \mathbb{Z}$ and $r \in \mathbb{Z}$ where $0 \leq r < a$.

Need proof here

Example 4.3.5

If $p \in \mathbb{N}$ and $3 \mid p^2$, then $3 \mid p$.

Proof. By the division algorithm, $p = 3k + j$ where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ where $0 \leq j < 3$. Then, $p^2 = (3k+j)^2 = 9k^2+6kj+j^2$. Suppose that $3 \mid p^2$. Then, $p^2 = 3l = 9k^2+6kj+j^2$.

Thus:

$$j^2 = 3l - 9k^2 - 6kj = 3(l - 3k^2 - 2kj)$$

We have $3 \mid j^2$. Hence, $j \neq 1, j \neq 2$, leaving only $j = 0$. Therefore, $p = 3k + 0$, so $3 \mid p$. ▢

Example 4.3.6 ▸ $\sqrt{3}$ is not a Rational Number

Proof. Suppose for contradiction $\sqrt{3}$ is a rational number. Then, there exist $n, m \in \mathbb{Z}$ such that $({}^{n}/{}_{m})^2$. If n and m share a common factor, then we can "cancel" the common factor to where ${}^{n}/{}_{m} = {}^{kp}/{}_{kq} = {}^{p}/{}_{q}$. Thus, we may assume that n and m have no nontrivial common factor.

$$\begin{aligned} & \left(\frac{n}{m}\right)^2 = 3 \\ \implies \quad & \frac{n^2}{m^2} = 3 \\ \implies \quad & n^2 = 3m^2 \end{aligned}$$

Thus, $3 \mid n^2$, so $3 \mid n$ by the previous lemma. Writing $n = 3k$ for some $k \in \mathbb{Z}$, we have:

$$\begin{aligned} & (3k)^2 = 3m^2 \\ \implies \quad & 9k^2 = 3m^2 \\ \implies \quad & 3k^2 = m^2 \end{aligned}$$

That is, $3 \mid m^2$ so $3 \mid m$. Thus, 3 divides both n and m. This contradicts the fact that we defined ${}^{n}/{}_{m}$ in the lowest terms. ▢

4.4 Fields

Definition 4.4.1 ▸ Field

A ***field*** is a set F with two defined operations, addition and multiplication, satisfying the following for all $a, b, c \in F$:

Axiom	Addition	Multiplication
Associativity	$(a+b)+c = a+(b+c)$	$(ab)c = a(bc)$
Commutativity	$a+b = b+a$	$ab = ba$
Distributivity	$a(b+c) = ab+ac$	$(a+b)c = ac+bc$
Identities	$\exists(0 \in \mathbb{F})(a+0=a)$	$\exists(1 \in \mathbb{F})(1 \neq 0 \wedge 1a = a)$
Inverses	$\exists(-a \in \mathbb{F})(a+(-a)=0)$	$(a \neq 0) \iff \exists(a^{-1} \in \mathbb{F})(aa^{-1}=1)$

All the "standard facts" of arithmetic and algebra in $\mathbb{R}$ follows from these axioms.

$\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are infinite fields, but $\mathbb{Z}_p$ (arithmetic modulo p) is a finite field if p is prime.

More generally, F_q where $q = p^k$ is a finite field.

Theorem 4.4.2 ▸ Facts about Fields

Let F be a field. For all $a, b, c \in F$:

(a) if $a+c = b+c$, then $a = b$
(b) $a \cdot 0 = 0$
(c) $(-a) \cdot b = -(a \cdot b)$
(d) $(-a) \cdot (-b) = a \cdot b$
(e) if $a \cdot c = b \cdot c$ and $c \neq 0$, then $a = b$
(f) if $a \cdot b = 0$, then $a = 0$ or $b = 0$
(g) $-(-a) = a$
(h) $-0 = 0$

Proof of (g).

$$\begin{aligned}
-(-a) &= -(-a) + 0 \\
&= -(-a) + (a + (-a)) \\
&= -(-a) + (-a + a) \\
&= (-(-a) + (-a)) + a \\
&= ((-a) + -(-a)) + a \\
&= 0 + a \\
&= a + 0 \\
&= a
\end{aligned}$$

□

4.5 Ordered Fields

Definition 4.5.1 ▸ Ordered Field

An ***ordered field*** is a field with a relation $<$ such that for all $a, b, c \in F$:

Axiom	Description
Trichotomy	Only one is true: $a < b$, $a = b$, or $b < a$
Transitivity	if $a < b$ and $b < c$ then $a < c$
Additive Property	if $b < c$, then $a + b < a + c$
Multiplicative Property	if $b < c$ and $0 < a$, then $a \cdot b < a \cdot c$

We then define $>$ as the inverse relation of $<$.

Theorem 4.5.2 ▸ Facts about Ordered Fields

- if $a < b$ then $-b < -a$
- if $a < b$ and $c < 0$, then $cb < ca$
- if $a \neq 0$, then $a^2 = a \cdot a > 0$
- $0 < 1$
- if $0 < a < b$ then $0 < 1/b < 1/a$

Although $\mathbb{C}$ is a field, it is not an ordered field. We can certainly define some kind of "order" on $\mathbb{C}$, but there is no way to make it satisfy the four axioms of an ordered field. For example, $i^2 = -1 < 0$, contradicting the fact that any nonzero number's square is greater than 0 in an ordered field.

$\mathbb{R}$ and $\mathbb{Q}$ are ordered fields.

Definition 4.5.3 ▸ Absolute Value

Let F be an ordered field. For $a \in F$, we define the ***absolute value*** of a as:

$$|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

We can think of $|a-b|$ as the distance between a and b. More generally, $|a-b| = d(a,b)$ is the metric we will be using throughout real analysis.

Theorem 4.5.4 ▶ Properties of Absolute Value

- $|a| \geq 0$, $a \leq |a|$, and $-a \leq |a|$
- $|ab| = |a||b|$

Theorem 4.5.5 ▶ Triangle Inequality

Let F be an ordered field. For any $a, b \in F$, $|a+b| \leq |a| + |b|$.

Proof. There are two cases to consider. If $a + b \geq 0$, then:

$$\begin{aligned} |a+b| &= a+b \\ &\leq |a| + b \\ &\leq |a| + |b| \end{aligned}$$

If $a + b < 0$, then:

$$\begin{aligned} |a+b| &= -(a+b) \\ &= -a-b \\ &\leq |a| - b \\ &\leq |a| + |b| \end{aligned}$$

□

4.6 Completeness

Definition 4.6.1 ▶ Bounded Above, Bounded Below, Bounded

Let F be an ordered field, and let $A \subseteq F$.

- A is ***bounded above*** if there exists $b \in F$ such that $a \leq b$ for all $a \in A$. In this context, b is an ***upper bound*** for A.
- A is ***bounded below*** if there exists $c \in F$ such that $c \leq a$ for all $a \in A$. In this context, c is a ***lower bound*** for A.
- A is ***bounded*** if A is bounded above and bounded below.

Example 4.6.2 ▸ Upper and Lower Bounds

Consider the set $(0, 1) := \{x \in \mathbb{R} : 0 < x < 1\}$.

- $(0, 1)$ is bounded above by 1 and any number greater than 1.
- $(0, 1)$ is bounded below by 0 and any negative number.

Consider the set $[3, \infty) := \{x \in \mathbb{R} : 3 \leq x\}$.

- $[3, \infty)$ is not bounded above.
- $[3, \infty)$ is bounded below by 3 and any number less than 3.

Definition 4.6.3 ▸ Maximum, Minimum

Let F be an ordered field, and let $A \subseteq F$.

- If there exists $M \in A$ such that M is an upper bound for A, then M is the ***maximum*** of A, denoted $M = \max A$
- If there exists $m \in A$ such that m is a lower bound for A, then m is the ***minimum*** of A, denoted $m = \min A$.

Note that from the above example, $(0, 1)$ has neither a maximum nor a minimum. However, 3 is the minimum of $[3, \infty)$.

Definition 4.6.4 ▸ Supremum

Let F be an ordered field, and let $A \subseteq F$. $s \in F$ is a ***supremum*** of A if:

1. s is an upper bound for A, and
2. if t is an upper bound for A, then $s \leq t$.

In other words, the supremum is the least upper bound for A. If A has a supremum, then that supremum is unique.

Prove this

Theorem 4.6.5 ▸ Maximum is the Supremum

Let F be an ordered field, and let $A \subseteq F$. If A has a maximum M, then $M = \sup A$.

Proof. Since $M = \max A$, we know M is an upper bound for A. Let t be an upper bound for A. Since $M \in A$, then $t \geq M$. Thus, M is less than or equal to any upper bound t, so $M = \sup A$. $\square$

Example 4.6.6 ▶ Supremum of $(0, 1)$

Prove that $\sup(0, 1) = 1$.

Proof. First, note that 1 is an upper bound for $(0, 1)$. Next, suppose that $t \in \mathbb{Q}$ is an upper bound for $(0, 1)$. Since $0 < 1/2 < 1$, then $0 < 1/2 \leq t$. By transitivity, $t > 0$. Suppose for contradiction $t < 1$. Because $0 < t < 1$, we have $1 < 1 + t < 2$. Dividing across by 2, we have $1/2 < {}^{1+t}/_2 < 1$. That is, ${}^{1+t}/_2 \in (0, 1)$. But $t < 1$, so $2t < 1 + t$. Thus, $t < {}^{1+t}/_2$. This contradicts our assumption that t is an upper bound for $(0, 1)$. Therefore, $t \geq 1$, so $\sup(0, 1) = 1$. ▢

Definition 4.6.7 ▶ Completeness

An ordered field F is ***complete*** if every nonempty subset of F that is bounded above has a supremum in F.

Theorem 4.6.8 ▶ $\mathbb{Q}$ is not complete

Proof sketch. Let $A := \{x \in \mathbb{Q} : x^2 < 2\}$. In other words, $A = \left(-\sqrt{2}, \sqrt{2}\right) \subseteq \mathbb{Q}$. Then A is nonempty and bounded above. Suppose for contradiction that $\mathbb{Q}$ is complete. Then A has a supremum, say $s = \sup(A)$. Consider the following cases:

1. If $s^2 < 2$, let $n \in \mathbb{N}$ such that $(s + 1/n)^2 < 2$. Then $s + 1/n \in A$, contradicting s being an upper bound for A.
2. If $s^2 > 2$, let $n \in \mathbb{N}$ such that $(s - 1/n)^2 > 2$. Then $s - 1/n$ is an upper bound smaller than s, contradicting s being the least upper bound (supremum).
3. If $s^2 = 2$, then $s \notin \mathbb{Q}$ (Theorem 4.3.2).

Thus, $A \subseteq \mathbb{Q}$ does not have a supremum. Therefore, $\mathbb{Q}$ is not complete. ▢

Definition 4.6.9 ▶ Real Numbers $\mathbb{R}$

The ***real numbers*** are a set $\mathbb{R}$ with two operations, $+$ and $\cdot$, and order relation $<$ such that:

1. $(R, +, \cdot)$ is a field,
2. $(\mathbb{R}, +, \cdot, <)$ is an ordered field, and
3. $(\mathbb{R}, +, \cdot, <)$ is complete.

Alternatively, $\mathbb{R}$ can be constructed explicitly using “Dedekind cuts”. Either way, $\mathbb{R}$ is the **only** unique complete ordered field up to isomorphism. That is, if there is some other imposter complete ordered field $\mathbb{R}'$, we can map every element of $\mathbb{R}$ to $\mathbb{R}'$ such that we preserve all the operations and relations between things in $\mathbb{R}$. More formally, there exists an isomorphism $T : \mathbb{R} \to \mathbb{R}'$ where T is bijective, and:

- $T(x + y) = T(x) + T(y)$
- $T(xy) = T(x)T(y)$
- $x < y \iff T(x) < T(y)$

Additionally, $\mathbb{N} \subseteq \mathbb{R}$ where $\mathbb{N}$ satisfies the Peano axioms.

Theorem 4.6.10 ▸ $\sqrt{2}$ is a Real Number

Proof sketch. Let $A \coloneqq \{x \in \mathbb{R} : x^2 < 2\}$.

- Show $A \neq \varnothing$ and A is bounded above
- Completeness says $s \coloneqq \sup A$ exists
- Show $s^2 = 2 \implies s = \sqrt{2} \in \mathbb{R}$.

More generally, if $n, m \in \mathbb{N}$, then $\sqrt[n]{m} \in \mathbb{R}$. □

5

Suprema and Infima

Definition 5.0.1 ▸ Infimum

Let F be an ordered field, and let $A \subseteq F$. s is the ***infimum*** of A if:

1. s is a lower bound for A, and
2. s is greater than every other lower bound for A.

We can prove that the existence of infima is already implied by completeness.

Theorem 5.0.2 ▸ Existence of Infima in $\mathbb{R}$

Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Then A has an infimum in $\mathbb{R}$.

Proof. Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Let B be the set of all lower bounds for A. In other words, $B \coloneqq \{b \in \mathbb{R} : \forall(a \in A)(b < a)\}$. Since A is bounded below, then B is nonempty. Note also that B is bounded above by element of A. By completeness, $s \coloneqq \sup B$ exists. Now, we need to show that $\sup B = \inf A$.

1. Every $a \in A$ is an upper bound for B, and $\sup B$ is the least upper bound for B. Then, $\sup B \leq a$. That is, $\sup B$ is a lower bound for A.
2. Let t be a lower bound for A. Then, by definition of B, it follows that $t \in B$. Then $t \leq \sup B$ as required.

Therefore, $\sup B = \inf A$ in $\mathbb{R}$. ▢

Theorem 5.0.3 ▸ Well-Ordering Principle

Every non-empty subset of $\mathbb{N}$ has a minimum.

Proof. We will use induction. For convenience, let $P(n)$ represent the following statement: "If $A \subseteq \mathbb{N}$ and $A \cap \{1, 2, \ldots, n\} \neq \varnothing$, then A has a minimum."

Base Case: First, we will prove $P(1)$. If $A \subseteq \mathbb{N}$ and $A \cap \{1\} \neq \varnothing$, then $1 \in A$, so A has a minimum.

Induction Step: Assume that $P(n)$ holds for some $n \in N$. Suppose $A \subseteq \mathbb{N}$ and $A \cap \{1, 2, \ldots, n+1\} \neq \varnothing$.

1. If $A \cap \{1, 2, \ldots, n\} \neq \varnothing$, then A has a minimum by $P(n)$.
2. If $A \cap \{1, 2, \ldots, n\} = \varnothing$, then $n + 1 \in A$, so $\min A = n + 1$.

By induction, $P(n)$ holds for all $n \in \mathbb{N}$. If $A \subseteq \mathbb{N}$ and $A \neq \varnothing$, then there exists $m \in A$ such that $m \in \mathbb{N}$. By $P(m)$ (which is true by induction), the set A has a minimum. ▢

Theorem 5.0.4 ▸ Pushing Supremum

Let A be a nonempty subset of $\mathbb{R}$, and let b, c be real numbers.

(a) If $a \leq b$ for all $a \in A$, then $\sup A \leq b$.

(b) If $c \leq a$ for all $a \in A$, then $c \leq \inf A$.

Intuition: Consider the interval $A \coloneqq (0, 1)$. Because $a \leq 1$ for all $a \in (0, 1)$, we have $\sup A \leq 1$. Because $0 \leq a$ for all $a \in (0, 1)$,we have $0 \leq \inf A$.

Proof of (a). Since $a \leq b$ for all $a \in A$, then b is an upper bound for A. By completeness, A has a supremum, and $s \coloneqq \sup A$ is the least upper bound for A. Thus, $s \leq b$. ▢

Proof of (b). ▢

Example 5.0.5

Suppose $A, B \subseteq \mathbb{R}$, $A \neq \varnothing$, $A \subseteq B$, and B is bounded above. Prove that A is bounded above and $\sup A \leq \sup B$.

Proof. Since $A \subseteq B$ and $A \neq \varnothing$, then $B \neq \varnothing$. Also, B is bounded above, so B has a supremum (by completeness). Let $a \in A$ be arbitrary. Then $a \in B$, so $a \leq \sup B$. Thus, A is bounded above, so A has a supremum (by completeness). By Pushing Supremum, $\sup A \leq \sup B$. ▢

Theorem 5.0.6 ▸ Approximation Property of Suprema and Infima

Suppose A is a nonempty subset of $\mathbb{R}$, and $s, r \in \mathbb{R}$. Then:

(a) $s = \sup A$ if and only if (i) s is an upper bound for A, and (ii) for all $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.

(b) $r = \inf A$ if and only if (i) r is a lower bound for A, and (ii) for all $\epsilon > 0$, there exists $a \in A$ such that $a < r + \epsilon$.

Intuition: If we nudge the supremum ever so slightly to the left, then we must have moved past something in A.

Proof of (a). Let $s := \sup A$. Then (i) holds by definition of suprema. To prove (ii), let $\epsilon > 0$. Since $s - \epsilon < s$, then $s - \epsilon$ is not an upper bound for A. Therefore, there exists $a \in A$ such that $s - \epsilon < a$.

Conversely, suppose that (i) and (ii) hold. We need to show $s = \sup A$. From (i), we know that s is an upper bound for A. Now, we need to show that s is the least upper bound. Let t be an upper bound for A. Suppose for contradiction that $t < s$. Let $\epsilon := s - t > 0$. Then $t = s - \epsilon$. By (ii), there exists $a \in A$ such that $a > s - \epsilon = t$. This contradicts t being an upper bound for A. Thus, there is no upper bound less than s. Therefore, $s = \sup A$. ▢

6

Consequences of Completeness

Theorem 6.0.1 ▶ $\mathbb{N}$ is not Bounded Above

Proof. Suppose for contradiction $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is not empty, then $\mathbb{N}$ has a supremum in $\mathbb{R}$. Let $s := \sup \mathbb{N} \in \mathbb{R}$. Then $n \leq s$ for all $n \in \mathbb{N}$. By the Peano axioms, n has a successor $n + 1 \in \mathbb{N}$, so $n + 1 \leq s$ for all $n \in \mathbb{N}$. Therefore, $n \leq s - 1$ for all $n \in \mathbb{N}$. This contradicts s being the least upper bound for $\mathbb{N}$. ▢

Theorem 6.0.2 ▶ Archimedean Principle

Suppose $x, y \in \mathbb{R}$ where $x > 0$. Then, there exists $n \in \mathbb{N}$ such that $nx > y$.

Intuition: This is basically an extension of the fact that $\mathbb{N}$ is not bounded above.

Proof. Since y/x is not an upper bound for $\mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $n > y/x$. Since $x > 0$, then $nx > y$. ▢

Theorem 6.0.3 ▶ Density of $\mathbb{Q}$ in $\mathbb{R}$

Suppose $x, y \in \mathbb{R}$ where $x < y$. Then there exists $r \in Q$ such that $x < r < y$.

Intuition: Given any two different real numbers, there's some rational number between them.

Proof. We will consider three cases:

1. If $x \geq 0$, then $0 \leq x < y$. Since $y - x > 0$, then by the Archimedean Principle, there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$. We want to show there is a natural number between nx and ny. Let $A := \{k \in \mathbb{N} : k > nx\}$. Since $\mathbb{N}$ isn't bounded above, then A is not empty. By the Well-Ordering Principle, A has a minimum. Let $m := \min A$. Then $m > nx$, and $m - 1 \leq nx$. Thus, $m \leq nx + 1$, so:

$$nx < m \leq nx + 1 < ny$$

Dividing across by n yields $x < m/n < y$. Note that $m, n \in \mathbb{N} \subseteq \mathbb{Z}$, so $m/n \in \mathbb{Q}$.

2. If $x < 0$ and $y > 0$, then $x < 0 < y$ where $0 \in \mathbb{Q}$.
3. If $x < 0$ and $y \leq 0$, then $x < y \leq 0$. Multiplying across by -1, we have $-x > -y \geq 0$. By the first case, there must exist $t \in \mathbb{Q}$ where $-y < t < -x$. Multiply across by -1 again to attain $y > -t > x$ where $-t \in \mathbb{Q}$.

This completes the proof. ▢

Theorem 6.0.4 ▸ $\sqrt{2}$ is a Real Number

There exists $s \in \mathbb{R}$ such that $s^2 = 2$.

Proof. Let $A := \{x \in \mathbb{R} : x^2 < 2\}$. Since $0^2 < 2$, then $0 \in A$, so A is not empty. Also, A is bounded above, for example by 2. By completeness, A must have a supremum in $\mathbb{R}$. Let $s := \sup A$. We will use trichotomy to show that $s^2 = 2$.

1. If $s^2 > 2$, then…

   > **Scratchwork:** We need to show that this is not possible, i.e. show there is some $s - 1/n$ that is less than s but is still an upper bound for A. We want $(s - 1/n)^2 > 2$. Then, $s^2 - 2s/n + 1/n^2 > 2$. We can chop off the $1/n^2$, reducing the inequality to $s^2 - 2s/n > 2$. Thus, we need to choose $n > \frac{2s}{s^2-2}$.

   … let $n \in \mathbb{N}$ such that $n > \frac{2s}{s^2-2}$. Then:

$$\begin{aligned} n &> \frac{2s}{s^2-2} \\ \implies \quad s^2 - \frac{2s}{n} &> 2 \\ \implies \quad s^2 - \frac{2s}{n} + \frac{1}{n^2} &> 2 \\ \implies \quad \left(s - \frac{1}{n}\right)^2 &> 2 \end{aligned}$$

   Thus, $s - 1/n$ is an upper bound for A that is less than s. This contradicts s being the supremum for A.
2. If $s^2 < 2$, then…

Scratchwork: Again, we need to show that this is not possible. We know that in this case, $s \in A$, so we need to find another thing in A that is bigger than s. In other words, we want some $(s + 1/n)^2 < 2$. Then, $s^2 + 2s/n + 1/n^2 < 2$. Choose $n > 1/2s$ and $n > \frac{4s}{2-s^2}$.

$$\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2}$$

... let $n \in \mathbb{N}$ such that $n > \max\left\{\frac{1}{2s}, \frac{4s}{2-s^2}\right\}$. Then $\frac{1}{n} < 2s$ and $s^2 + \frac{4s}{n} < 2$. So:

$$\begin{aligned}\left(s + \frac{1}{n}\right)^2 &= s^2 + \frac{2s}{n} + \frac{1}{n^2} \\ &< s^2 + \frac{2s}{n} + \frac{2s}{n} \\ &= s^2 + \frac{4s}{n} < 2\end{aligned}$$

That is, $s + \frac{1}{n} \in A$. This contradicts s being an upper bound for A.

By trichotomy, $s^2 = 2$. ▢

Theorem 6.0.5 ▸ Nested Interval Property

Suppose that for each $n \in \mathbb{N}$, $a_n, b_n \in \mathbb{R}$ with $a_n \leq b_n$, and $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \varnothing$.

Intuition: We can move the two borders of an open interval closer and closer to each other, and it won't be empty.

Proof. Note that $a_n \leq a_{n+1} \leq a_{n+2} \leq \ldots$ and $\ldots \leq b_{n+2} \leq b_{n+1} \leq b_n$. If $k \leq n$, then $a_k \leq a_n \leq b_n$.

- If $k \leq n$, then $a_k \leq a_n \leq b_n$.
- If $k \geq n$, then $a_k \leq b_k \leq b_n$.

That is, $a_k \leq b_n$ for all $k_n \in \mathbb{N}$. Let $A \coloneqq \{a_k : k \in \mathbb{N}\}$. Then A is bounded above, for example by b_1. Also, A is not empty. By completeness, A has a supremum. Let $s \coloneqq \sup A$. Note that since s is an upper bound for A, then $a_n \leq \sup A$ for all $n \in \mathbb{N}$. Also note that $\sup A$ is the least upper bound for A, so $\sup A \leq b_n$ for all $n \in \mathbb{N}$. Thus, $a_n \leq \sup A \leq b_n$ for all $n \in \mathbb{N}$, so $\sup A \in [a_n, b_n]$ for all $n \in \mathbb{N}$. Thus, $\sup A \in \bigcap_{n=1}^{\infty}[a_n, b_n]$, so it is not empty. ▢

The nested interval property is actually false for open intervals!

$$\forall(x \in (0,1))\exists(n \in \mathbb{N})(1/n < x \implies x \notin (0,1/n))$$

7

Cardinality

Definition 7.0.1 ▸ Cardinality

Cardinality is a measure of the amount of elements in a set, denoted $|A|$. We say two sets have the same cardinality if there exists a bijection between them.

For finite sets, we can think of cardinality as the number of elements in that set. For infinite sets, cardinality can sometimes go against our intuition. For any sets A, B, C:

1. $|A| = |A|$
2. if $|A| = |B|$, then $|B| = |A|$
3. if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Hence, equality of cardinalities is an equivalence relation.

Example 7.0.2 ▸ Cardinality of $\mathbb{N}$ and $2\mathbb{N}$

Let $2\mathbb{N} := \{2n : n \in \mathbb{N}\}$ (i.e. the set of even natural numbers). Then $|\mathbb{N}| = |2\mathbb{N}|$.

Proof. To show that these two sets have the same cardinality, we need to find some bijection between the sets. Let $f : \mathbb{N} \to 2\mathbb{N}$ be a function defined by $f(n) = 2n$. Note that f is well-defined (i.e. is actually a function) because $f(n) \in 2\mathbb{N}$ for all $n \in \mathbb{N}$. To prove that f is a bijection, we need to prove it is both injective and surjective.

1. Let $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $2n_1 = 2n_2$, so $n_1 = n_2$. Thus, f is injective.
2. Let $m \in 2\mathbb{N}$. Then $m = 2k$ for some $k \in \mathbb{N}$, so $m = 2k = f(k)$ for some $k \in \mathbb{N}$. Thus, f is surjective.

Therefore, f is a bijection, so $|\mathbb{N}| = |2\mathbb{N}|$. ▢

Example 7.0.3 ▸ Cardinality of Intervals

Let $a, b \in \mathbb{R}$ where $a < b$. Then $|(0, 1)| = |(a, b)|$.

Proof. We need to find a bijection from $(0,1)$ to (a,b). We need to "scale" the interval $(0,1)$ to the width of (a,b), then translate it to match (a,b). Define $f:(0,1)\to(a,b)$ by $f(x)=a+(b-a)x$. (We need to check f is well-defined). Let $x\in(0,1)$. Then $0<x<1$, so multiplying by $(b-a)$ which is positive gives $0<(b-a)x<b-a$. Adding a, we get $a<a+(b-a)x<b$. Now we need to show f is a bijection:

1. Let $x_1, x_2 \in (0,1)$ such that $f(x_1)=f(x_2)$. Then $a+(b-a)x_1 = a+(b-a)x_2$. Subtracting a from both sides, we get $(b-a)x_1=(b-a)x_2$. Since $(b-a)\neq 0$, we can divide both side by $(b-a)$ to get $x_1=x_2$.
2. Let $y\in(a,b)$.

   Scratchwork: We want to find some $x\in(0,1)$ where $y=f(x)=a+(b-a)x$. Using some algebra to solve for x, we have $x=\frac{y-a}{b-a}$

   Let $x=\frac{y-a}{b-a}$. First, we show $x\in(0,1)$:

$$\begin{aligned} & a<y<b \\ \implies\quad & 0<y-a<b-a \\ \implies\quad & 0<\frac{y-a}{b-a}<1 \end{aligned}$$

   Thus, $x\in(0,1)$. Also:

$$f(x)=a+(b-a)\left(\frac{y-a}{b-a}\right)=a+(y-a)=y$$

   Thus, f is surjective.

Therefore, f is a bijective, so $|(0,1)|=|(a,b)|$. □

Definition 7.0.4 ▸ Power Set

Let A be a set. The ***power set*** of A is the set of all subsets of A.

$$\mathcal{P}(A)=\{B : B\subseteq A\}$$

For example, the power set of $\{1,2,3\}$ is $\{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. For any finite set with n elements in it, its power set has 2^n elements in it.

Example 7.0.5 ▸ Cardinality of $\mathbb{N}$ and $\mathcal{P}(N)$

$|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$

Proof. We will show that any function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ cannot be surjective, and thus not bijective. Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any function defined by $f(n) = A_n$. Note $A_n \subseteq \mathbb{N}$, so $A_n \in \mathcal{P}(A)$. Now we will define a set that isn't in $f[\mathbb{N}]$. For each $n \in \mathbb{N}$, if $n \in A_n$, then $n \notin A$, and if $n \notin A_n$, then $n \in A$. More formally, $A \coloneqq \{n \in \mathbb{N} : n \notin A_n\}$. For all $k \in \mathbb{N}$, note that:

- if $k \in A_k$, then $k \notin A$, so $A \neq A_k$, and
- if $k \notin A_k$, then $k \in A_k$, so $A \neq A_k$.

Hence, $A \subseteq \mathbb{N}$, but $f(k) \neq A$ for any $k \in \mathbb{N}$. Thus, f is not surjective. ▢

Definition 7.0.6 ▸ Finite, Countably Infinite, Countable, Uncountable

Let A be a set. We say A is:

- ***finite*** if $A \neq \varnothing$ or $|A| = |\{1, 2, \dots, n\}|$ for some $n \in \mathbb{N}$.
- ***countably infinite*** if $|A| = |N|$.
- ***countable*** if A is finite or countably infinite
- ***uncountable*** if A is not countable

Theorem 7.0.7 ▸ $\mathcal{P}(\mathbb{N})$ is uncountable.

Proof. We know from Example 7.0.5 that $\mathcal{P}(\mathbb{N})$ is not countably infinite. We need to show that $\mathcal{P}(\mathbb{N})$ is not finite. Since $\{1\} \in \mathcal{P}(\mathbb{N})$, then it cannot be empty. Suppose for contradiction $|\{1, 2, \dots, n\}| = |\mathcal{P}(\mathbb{N})|$ for some $n \in \mathbb{N}$, then there exists a bijection $f : \{1, 2, \dots, n\} \to \mathcal{P}(\mathbb{N})$. Define $g : \mathbb{N} \to \{1, 2, \dots, n\}$ by:

$$g(k) = \begin{cases} k, & 1 \leq k \leq n \\ 1, & k > n \end{cases}$$

Then g is surjective, so $f \circ g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is surjective. This contradicts the fact that no such function exists (by Example 7.0.5). ▢

Generally, there is never a bijection from a set to its power set.

Intuition: A set is countable if its elements can be "listed" or "counted". That is, for finite sets:

$$X = \{x_1, x_2, \dots, x_n\} = \{x_k\}_{k=1}^{n}$$

For infinitely countable sets:

$$X = \{x_1, x_2, \dots\} = \{x_k\}_{k=1}^{\infty}$$

If X is finite, then there exists a bijection $f : \{1, 2, \dots, n\} \to X$. Thus, $X = \{f(1), f(2), \dots, f(n)\}$. If X is countably infinite, then there exists a bijection $f : \mathbb{N} \to X$. Thus, $X = \{f(1), f(2), \dots\}$.

Theorem 7.0.8 ▸ Subsets of Countable Sets are Countable

The subset of a countable set is still countable. (i.e. a countable set cannot contain an uncountable subset).

Proof. Let X be a countable set, and let $A \subseteq X$. We will consider two cases. First, if A is finite, then A is countable, and we are done. Otherwise, A is infinite, and hence X is infinite. Then X is countably infinite, so $X = \{x_1, x_2, \dots\} = \{x_k\}_{k=1}^{\infty}$.

Idea: Our set A might look something like $\{x_3, x_4, x_6, \dots\}$. We need to align these indices to 1, 2, 3, and so on. We'll let $k_1 = \min\{3, 4, 6, \dots\}$, let $k_2 = \min\{4, 6, \dots\}$, and so on.

Let $k_1 := \min\{k \in \mathbb{N} : x_k \in A\}$. Let $a_1 := x_{k_1}$. For all $j \in \mathbb{N}$ such that $j > 1$, we define $k_j := \min\{k \in \mathbb{N} : (x_k \in A) \wedge (k > k_{j-1})\}$. Let $a_j := x_{k_j}$. Then $1 \leq k_1 < k_2 < k_3 < \dots$, so k_j approaches infinity. Let $g : \mathbb{N} \to A$ be a function defined by $g(j) = a_j$. We need to show that g is both injective and surjective, and thus a bijection.

- Suppose that $g(j_1) = g(j_2)$ for some $j_1, j_2 \in \mathbb{N}$. Then $a_{j_1} = a_{j_2}$, so $x_{k_{j_1}} = x_{k_{j_2}}$. Then $k_{j_1} = k_{j_2}$, so $j_1 = j_2$. Thus, g is injective.
- Let $a \in A$ Since $A \subseteq X$, then $a \in X$. Thus, $a = x_l$ for some $l \in \mathbb{N}$. Let $m := \min\{j \in \mathbb{N} : k_j \geq l\}$. Since $m \in \{j \in \mathbb{N} : j_k \geq l\}$, then $k_m \geq l$. Also, $m - 1 \notin \{j \in \mathbb{N} : k_j \geq l\}$, so $k_{m-1} < l$. Now, $k_m = \min\{k \in \mathbb{N} : (x_k \in A) \wedge (k > k_{m-1})\}$. But $x_l \in A$, and $l > k_{m-1}$, so $l \in \{k \in \mathbb{N} : (x_k \in A) \wedge (k > k_{m-1})\}$. Thus, $k_m \leq l$, because k_m is the minimum of the set containing l. By trichotomy, $k_m = l$. Therefore:

$$g(m) = a_m = x_{k_m} = x_l = a$$

So g is surjective.

Since g is a bijection, then $|\mathbb{N}| = |A|$, so $|A|$ is countable. ▢

Theorem 7.0.9 ▶ Injectivity and Cardinality

A set A is countable if and only if there exists an injective function $f : A \to \mathbb{N}$.

Proof. First, suppose A is a countable set. We consider two cases:

- If A is countably infinite, then there exists a bijection $f : A \to \mathbb{N}$.
- If A is finite, then there exists a bijection $f : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. Let $g : \{1, 2, \dots, n\} \to \mathbb{N}$ be a function defined by $g(x) = x$ (i.e. an inclusion mapping). Then f and g are both injective, so $g \circ f : A \to \mathbb{N}$ is injective.

Conversely, suppose $f : A \to \mathbb{N}$ is an injection. Then $f[A] \subseteq \mathbb{N}$, so $f[A]$ is countable by Theorem 7.0.8. Define $g : A \to f[A]$ by $g(a) = f(a)$. Then g is injective because f is injective, and g is surjective because $g[A] = f[A]$. Thus, g is a bijection, so $|A| = |f[A]|$. Therefore, A is countable. ▢

Theorem 7.0.10 ▶ $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n, m) = 2^n 3^m$. We now show that f is bijective. To prove f is injective, suppose $f(n_1, m_1) = f(n_2, n_2)$. Then $2^{n_1} 3^{m_1} = 2^{n_2} 3^{m_2}$.

- If $n_1 > n_2$, then $2^{n_1 - n_2} = 3^{m_2 - m_1}$. Since $n_1 > n_2$, we have $n_1 - n_2 > 0$, so $2^{n_1 - n_2} \in \mathbb{N}$. Then also $3^{m_2 - m_1} \in \mathbb{N}$. But $2^{n_1 - n_2}$ is even, and $3^{m_2 - m_1}$ is odd. This contradicts the fact that $2^{n_1 - n_2} = 3^{m_2 - m_1}$.
- If $n_2 > n_1$, then $3^{m_1 - m_2} = 2^{n_2 - n_1}$. By a similar argument, $2^{n_2 - n_1}$ is even and $3^{m_1 - m_2}$ is odd, producing the same contradiction.
- If $n_1 = n_2$, then $2^{n_1} = 2^{n_2}$, so cancelling gives $3^{m_1} = 3^{m_2}$. Thus, $m_1 = m_2$.

Hence, $(n_1, m_1) = (n_2, m_2)$, so f is injective. By Theorem 7.0.9, $\mathbb{N} \times \mathbb{N}$ is countable. Also, $\mathbb{N} \times \mathbb{N}$ is infinite, so $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. ▢

Theorem 7.0.11 ▶ Collection of Countable Sets

Suppose that for all $k \in \mathbb{N}$, A_k is a countable set. Then $\cup_k A_k \in \mathbb{N}$ is countable. (i.e. a countable union of countable sets is countable)

Proof. Let $k \in \mathbb{N}$. A_k is countable, so A_k can be listed as such:

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, a_{13}, a_{14}, \ldots\} = \{a_j\}_{j\in\mathbb{N}} \\ A_2 &= \{a_{21}, a_{22}, a_{23}, a_{24}, \ldots\} \\ &\vdots \\ A_k &= \{a_{k1}, a_{k2}, a_{k3}, a_{k4}, \ldots\} \end{aligned}$$

> We want to define some function $f : \bigcup_{k\in\mathbb{N}} A_k \to \mathbb{N}\times\mathbb{N}$ where $f(a_{kj}) = (k, j) \in \mathbb{N}\times\mathbb{N}$. However, we need to consider the possibility that the sets A_k are not disjoint. If $a_{12} = a_{34}$, then $a(_{12}) = (1, 2)$ and $f(a_{34}) = (3, 4)$.

Given $a \in \bigcup_{k\in\mathbb{N}} A_k$, let $k(a) := \min\{k \in \mathbb{N} : a \in A_k\}$. If $a \in A_{k(a)}$, then there is a unique $j(a) \in \mathbb{N}$ such that $a = a_{k(a)j(a)}$. Now define $f : \bigcup_{k\in\mathbb{N}} A_k \to \mathbb{N}\times\mathbb{N}$ by $f(a) = (k(a), j(a))$. We must show that f is injective. Let $x, y \in \bigcup_{k\in\mathbb{N}} A_k$ such that $f(x) = f(y)$. That is, $(k(x), j(x)) = (k(y), j(y))$. Then $x = a_{k(x)j(x)} = a_{k(y)j(y)} = y$. By Theorem 7.0.10, there exists some injection $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. Hence, $g \circ f : \bigcup_{k\in\mathbb{N}} A_k \to \mathbb{N}$ is injective. By Theorem 7.0.9, $\bigcup_{k\in\mathbb{N}} A_k$ is countable. $\square$

This theorem shows that, in order to prove a countable union of countable sets is countable, we just need to show that each set in the union is countable. We'll use this in our proof that the set of rational numbers is a countable set.

Theorem 7.0.12 ▶ $\mathbb{Q}$ is Countable

Proof. Let $\mathbb{Q}^+ := \{r \in \mathbb{Q} : r > 0\}$, and let $\mathbb{Q}^- := \{r \in \mathbb{Q} : r < 0\}$. First, we'll prove that $\mathbb{Q}^+$ is countable. Let $f : \mathbb{Q}^+ \to \mathbb{N}\times\mathbb{N}$ be a function defined as $f(r) = (p, q)$ such that $r = p/q$ where $p, q \in \mathbb{N}$ and p shares no common factors with q. To show f is injective, let $r_1, r_2 \in \mathbb{Q}$ where $f(r_1) = f(r_2)$. Then $r_1 = p_1/q_1, r_2 = p_2/q_2$ where $p_1, q_1 \in \mathbb{N}$ with no common factors, and $p_2, q_2 \in \mathbb{N}$ with no common factors. Thus, $(p_1, q_1) = (p_2, q_2)$, so $p_1 = p_2$ and $q_1 = q_2$. Thus, $r_1 = p_1/q_1 = p_2/q_2 = r_2$, so f is injective. Since there exists an injection $g \in \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, then $g \circ f : \mathbb{Q}^+ \to \mathbb{N}$ is injective. Thus, $\mathbb{Q}^+$ is countable.
Next, we'll prove that $\mathbb{Q}^-$ is countable. Let $h : \mathbb{Q}^- \to \mathbb{Q}^+$ by $h(r) = -r$. We show h is injective. If $h(r_1) = h(r_2)$ where $r_1, r_2 \in \mathbb{Q}^-$, then $-r_1 = -r_2$, so $r_1 = r_2$. Thus, h is injective. From above, there exist an injection $\phi : \mathbb{Q}^+ \to \mathbb{N}$. Hence, $h \circ \phi : \mathbb{Q}^{-1} \to \mathbb{N}$ is injective.

Finally, $\{0\}$ is countable because it is finite. Since $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ is a countable union of countable sets, then $\mathbb{Q}$ is countable. $\square$

This means we can "list" the rational numbers (disregarding order) as $\mathbb{Q} = \{r_1, r_2, \dots\} = \{r_n\}_{n\in\mathbb{N}}$.

Theorem 7.0.13 ▸ $\mathbb{R}$ is Uncountable

Proof. Suppose for contradiction $\mathbb{R}$ is countable. Then $\mathbb{R}$ can be "listed" as $\mathbb{R} = \{x_1, x_2, \dots\} = \{x_n\}_{n\in\mathbb{N}}$. We will define a sequence of non-empty closed intervals $\{I_k\}_{k\in\mathbb{N}}$ such that $I_{k+1} \subseteq I_k$ and $x_k \notin I_k$ for all $k \in \mathbb{N}$. Let $I_0 := [0, 1]$. Divide I_0 into three equal closed intervals. Then, at least one of these three intervals does not contain x_1. Choose such an interval and call it I_1. Divide I_1 into there equal closed intervals. Then, at least one of these three intervals does not contain x_2. Choose such an interval and call it I_2. Given I_k for some $k \in \mathbb{N}$, divide I_k into three equal closed intervals, then choose the interval that does not contain x_{k+1} and call it I_{k+1}. By induction, we have $\{I_k\}_{k=1}^{\infty}$ where each I_k is a (nonempty) closed interval, and $I_{k+1} \subseteq I_k$ for each $k \in \mathbb{N}$. By the Nested Interval Property, $\bigcap_{k\in\mathbb{N}} I_k$ is not empty, so there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{k\in\mathbb{N}} I_k$. Since $x \in \mathbb{R}$, we have $x = x_n$ for some $n \in \mathbb{N}$ (by our supposition that $\mathbb{R}$ is countable). However, we constructed I_n such that $x_n \notin I_n$, so $x_n \notin \bigcap_{k\in\mathbb{N}} I_k$. This contradiction renders our initial supposition false. Therefore, $\mathbb{R}$ is uncountable. $\square$

Theorem 7.0.14 ▸ Irrational Numbers are Uncountable

Proof. Suppose for contradiction $\mathbb{R} \setminus \mathbb{Q}$ is countable. Then $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$, so $\mathbb{R}$ is a countable union of countable sets, making $\mathbb{R}$ countable. This contradicts the fact that $\mathbb{R}$ is uncountable, so $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. $\square$

7.1 Additional Remarks

Definition 7.1.1 ▸ Algebraic Number, Transcandental Number

If α is a root of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ where each $a_i \in \mathbb{Z}$, then α is called an ***algebraic number***. If α is not algebraic, then we call it a ***transcendental number***.

The set of all algebraic numbers is countable, so "most" real numbers are transcendental.

- Even though $\mathbb{Q}$ is dense in $\mathbb{R}$, there are "more" irrational numbers than rational numbers

- The set $\{\sqrt[n]{m} : n, m \in \mathbb{N}\}$ is countable, so "most" real numbers are not radicals.
- The set of algebraic numbers is countable, so "most" real numbers are transcendental.
- $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, but the set $\{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is countable.
- If there exists an injection $f : A \to B$, then we say $|A| \leq |B|$. If there exists an injection from A to B, but there does not exist an injection from B to A, then we say $|A| < |B|$.

Theorem 7.1.2 ▸ Schroeder-Bernstein

If there exists an injection $f : A \to B$ and an injection $g : B \to A$, then there exists a bijection $h : A \to B$.

Theorem 7.1.3 ▸ Continuum Hypothesis

There is no cardinality between $|\mathbb{N}|$ and $|\mathbb{R}|$.

In Zermelo-Fraenkel with Choice (ZFC) set theory, the continuum hypothesis cannot be proven to be true nor false. In 1938, Godel proved that the continuum hypothesis is consistent with ZFC. In 1963, Colen proved that the negation of the continuum hypothesis is also consisten with ZFC.

Just because you can write a description of a set does not mean that the set exists nor makes sense.

- For example, let $A := \left\{\bigcup\{B : B \text{ is a set}\}\right\}$. Then $\mathcal{P}(A) \subseteq A$, so $|\mathcal{P}(A)| \leq A < |\mathcal{P}(A)|$.
- Another example: let $B := \{\text{all sets}\}$. Let $C := \{A : A \notin A\}$. Is $C \in C$?

8

Sequences and Convergence

Definition 8.0.1 ▸ Sequence

A ***sequence*** is an ordered list of real numbers.

$$s = (s_1, s_2, s_3, s_4, \ldots)$$

Formally, a ***sequence*** is a function $s : \mathbb{N} \to \mathbb{R}$. We write s_n to denote $s(n)$.

We can define a sequence using an expression, like $s_n := n^2$. Then $s = (1, 4, 9, 16, \ldots)$. Also, we can informally define a sequence in terms of its elements, like $s = (3, 1, 4, 1, 5, 9, \ldots)$. We could just have a random sequence like $s := (12.3, e^2, 1 - \pi, 10000. \ldots)$.

Let's consider how we can formalize the definitions of limits and convergence. Consider the sequence $s_n := 1/n$, then $(s_n) = (1, 1/2, 1/3, 1/4, \ldots)$. We have an intuitive idea that, as n gets bigger, then $1/n$ gets closer to 0. We can say that this sequence "converges" to 0.

Now consider the sequence $s := (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots)$. Does this sequence converge? This really depends on our definition of convergence. We might define this as, "s_n gets close to l as n gets large". It certainly matches our intuition, but what exactly does "close to l" mean? Maybe we could say, "$|s_n - l|$ gets small as n gets large". More precisely, this might be "for all $\epsilon > 0$, $|s_n - l| < \epsilon$ when n is large". That "n is large" is still imprecise. Fixing that part, we get the formal definition for convergence:

Definition 8.0.2 ▸ Convergence

Let $s := (s_n)_{n \in \mathbb{N}}$ be a sequence of real numbers, and let $l \in \mathbb{R}$. We say s_n ***converges*** to l if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - l| < \epsilon$ for all $n > N$.

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(n > N)(|s_n - l| < \epsilon)$$

Like in the approximation property, we use ϵ to denote some arbitrarily tiny value that's really really close to 0, but not actually 0. We can also write $\lim_{n\to\infty} s_n = l$ or $s_n \to l$ to mean s_n converges to l.

Technique 8.0.3 ▸ Proving Convergence

To prove that a sequence s converges to l, we carry out the following steps:

1. As some scratch work, solve the inequality $|s_n - l| < \epsilon$ for n.
2. In the formal proof, let $\epsilon > 0$, and let N be greater than the solved thing. Let $n > N$, then work towards $|s_n - l| < \epsilon$.

Make this explanation better

Example 8.0.4 ▸ $1/n$ converges to 0

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Intuition: Since we're proving something for all $\epsilon > 0$, let's start by choosing some arbitrary $\epsilon > 0$. Next, we need to choose some $N \in \mathbb{N}$ where $|s_n - l| < \epsilon$ for all $n > N$. Thus:

$$\begin{aligned} |s_n - l| &< \epsilon \\ \left|\frac{1}{n} - 0\right| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ n &> \frac{1}{\epsilon} \end{aligned}$$

So we choose $N > \frac{1}{\epsilon}$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ where $N > 1/\epsilon$. If $n > N > 1/\epsilon$, then $1/n < \epsilon$. Thus:

$$|s_n - l| = |1/n - 0| = 1/n < \epsilon$$

Therefore, s converges to 0. ▢

Example 8.0.5

Prove that $\lim_{n\to\infty} \frac{2n+3}{3n+7} = \frac{2}{3}$.

Intuition: This time, we want to choose some $N \in \mathbb{N}$ such that $|s_n - l| < \epsilon$. Thus:

$$\begin{aligned}
\left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| &< \epsilon \\
\left|\frac{6n+9-6n-14}{9n+21}\right| &< \epsilon \\
\frac{5}{9n+21} &< \epsilon \\
\frac{5}{\epsilon} &< 9n+21 \\
\frac{1}{9}\left(\frac{5}{\epsilon} - 21\right) &< n
\end{aligned}$$

Thus, we want to choose $N > {}^1\!/_9\,({}^5\!/_\epsilon - 21)$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > {}^1\!/_9\,({}^5\!/_\epsilon - 21)$. If $n > N > {}^1\!/_9\,({}^5\!/_\epsilon - 21)$, then:

$$\begin{aligned}
9n &> {}^5\!/_\epsilon - 21 \\
9n &> \frac{5}{\epsilon} - 21 \\
9n + 21 &> \frac{5}{\epsilon} \\
\frac{5}{9n+21} &< \epsilon
\end{aligned}$$

Thus:

$$\begin{aligned}
|s_n - l| &= \left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| \\
&= \left|\frac{6n+9-6n-14}{9n+21}\right| \\
&= \frac{5}{9n+21} \\
&< \epsilon
\end{aligned}$$

□

The above proof chooses a sort of "optimal" or "best possible" N. We could have thrown away the 21 in the denominator, and the inequality we're aiming for will still be the same.

Alternate proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{5}{9\epsilon}$. If $n > N > \frac{5}{9\epsilon}$, then $\frac{5}{9n} < \epsilon$, so $\frac{5}{9n+21} < \frac{5}{9n} < \epsilon$. Then:

$$|s_n - l| = \left|\frac{2n+3}{3n+7} - \frac{2}{3}\right| = \frac{5}{9n+21} < \epsilon$$

□

Example 8.0.6

Prove that $\lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}$.

Intuition: Here, we have to be careful about throwing away terms.

$$\begin{aligned} |s_n - l| &< \epsilon \\ \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| &< \epsilon \\ \left|\frac{6n+9-6n+14}{9n-21}\right| &< \epsilon \\ \frac{23}{|9n-21|} &< \epsilon \end{aligned}$$

We want $9n - 21 > 0$, so we must have $n \geq 3$. We can apply this restriction on n to get rid of the absolute value:

$$\begin{aligned} \frac{23}{9n-21} &< \epsilon \\ \frac{23}{\epsilon} &< 9n - 21 \\ \frac{1}{9}\left(\frac{23}{\epsilon} + 21\right) &< n \end{aligned}$$

Thus, we want to choose some $N > \frac{1}{9}\left(\frac{23}{\epsilon} + 21\right)$ and $N \geq 3$.

Proof. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \frac{1}{9}\left(\frac{23}{\epsilon} + 21\right)$. Then $N > \frac{21}{9}$, and since N is a natural number, then $N \geq 3$. Let $n \in \mathbb{N}$ where $n > N$. Then:

$$\begin{aligned} 9n &> \frac{23}{\epsilon} + 21 \\ 9n - 21 &> \frac{23}{\epsilon} \\ \epsilon &> \frac{23}{9n-21} \end{aligned}$$

Thus:

$$|s_n - l| = \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \left|\frac{23}{9n-21}\right| = \frac{23}{9n-21} < \epsilon$$

□

Definition 8.0.7 ▶ Divergence

A sequence ***diverges*** if it does not converge.

$$\exists(\epsilon > 0)\forall(N \in \mathbb{N})\exists(n > N)(|s_n - l| \geq \epsilon)$$

Example 8.0.8 ▶ Diverging Sequence

Prove that $s = (1, 0, 1, 0, 0, 1, 0, 0, 0, \ldots)$ does not converge to 0.

Proof. Let $\epsilon = 1/2$. Then for all $N \in \mathbb{N}$, there exists $n > N$ such that $s_n = 1$. Then:

$$|s_n - 0| = |1 - 0| > \epsilon$$

Therefore, s does not converge. ▢

8.1 Properties of Limits

A sequence can only converge to one value, not more. That is, if a sequence has a limit, then that limit is unique.

Lemma 8.1.1 ▶ Approximating Zero

Let $x \in \mathbb{R}$. If $x < \epsilon$ for all $\epsilon > 0$, then $x \leq 0$.

Proof. We proceed by contraposition. Suppose $x > 0$. Let $\epsilon := x/2 > 0$. Then $x \geq \epsilon = x/2$. ▢

Theorem 8.1.2 ▶ Uniqueness of Limits

Let s_n be a sequence of real numbers. If s_n converges to l_1 and converges to l_2, then $l_1 = l_2$.

Proof. Let $\epsilon > 0$. Since s_n converges to l_1, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - l_1| < \epsilon/2$ for all $n > N_1$. Similarly, since s_n converges to l_2, then there exists $N_2 \in \mathbb{N}$ such that $|s_n - l_2| < \epsilon/2$ for all $n > N_2$.

Let $n \in \mathbb{N}$ where $n > N_1$ and $n > N_2$. Then:

$$|l_1 - l_2| = |l_1 - s_n + s_n - l_2| \leq \underbrace{|l_1 - s_n| + |s_n - l_2|}_{\text{Triangle Inequality}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $|l_1 - l_2| < \epsilon$ for all $\epsilon > 0$. Thus, by Lemma 8.1.1, $|l1 - l2| \leq 0$. However, we know that $|l1 - l2| \geq 0$ since it's an absolute value. Thus, we have $|l1 - l2| = 0$, so $l1 = l2$. ▢

Definitions of bounds for sequences, show that convergent implies boundedness

Split the theorem below into four separate theorems?

Theorem 8.1.3 ▸ Properties of Limits

Let (s_n) and (t_n) be convergent sequences of real numbers, and let $s, t \in \mathbb{R}$ such that s_n converges to s and t_n converges to t. Then:

1. For any $c \in \mathbb{R}$, cs_n converges to cs,
2. $s_n + t_n$ converges to $s + t$,
3. $s_n t_n$ converges to st, and
4. if $t_n \neq 0$, then for all n and $t \neq 0$, $\frac{s_n}{t_n}$ converges to $\frac{s}{t}$.

Proof of 1. Let $\epsilon > 0$. Since (s_n) converges to s, then there exists $N \in \mathbb{N}$ such that $|s_n - s| < \frac{\epsilon}{1+|c|}$ for all $n > N$. Then, for all $n > N$, we have:

$$|cs_n - cs| = |c(s_n - s)| = |c||s_n - s| < |c|\frac{\epsilon}{1 + |c|} = \frac{|c|}{1 + |c|}\epsilon < \epsilon$$

▢

Proof of 2. Let $\epsilon > 0$. Since (s_n) converges to s, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2$ for all $n > N$. Similarly, since t_n converges to t, then there exists $N_2 \in \mathbb{N}$ such that $|t_n - t| < \epsilon/2$. Let $N \in \mathbb{N}$ where $N \geq N_1$ and $N \geq N_2$. Then:

$$|(s_n + t_n) - (s + t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

That is, $s_n + t_n$ converges to $s + t$. ▢

Proof of 3. Let $\epsilon > 0$. Since s_n converges to s, then there exists $N_1 \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2(|t| + 1)$ for all $n > N$. Also, (s_n) converges, so (s_n) is bounded. That is, there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Since t_n converges to t, there exists $N_2 \in \mathbb{N}$ such that

$|t_n - t| < \frac{\epsilon}{2(M+1)}$ for all $n > N$. Let $N \in \mathbb{N}$ such that $N \geq N_1$ and $N \geq N_2$. If $n > N$, then:

$$\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&= |s_n(t_n - t) + (s_n - s)t| \\
&\leq |s_n(t_n - t)| + |(s_n - s)t| \\
&= |s_n||t_n - t| + |s_n - s||t| \\
&< M\frac{\epsilon}{2(1+M)} + \frac{\epsilon}{2(1+|t|)}|t| \\
&= \frac{M}{1+M}\frac{\epsilon}{2} + \frac{\epsilon}{2}\frac{abst}{1+|t|} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

$\square$

Proof of 4. We will prove that $\frac{1}{t_n}$ converges to $\frac{1}{t}$. Let $\epsilon > 0$. Since t_n converges to t and $t \neq 0$, then there exists $N_1 \in \mathbb{N}$ such that $|t_n - t| < \frac{\epsilon t^2}{2}$. By **??**, there exists $N_2 \in \mathbb{N}$ such that $|t_n| > \frac{|t|}{2}$ for all $n > N_2$. Let $N \in \mathbb{N}$ such that $N > N_1$ and $N > N_2$. Let $n \in \mathbb{N}$ be arbitrary. Then:

$$\begin{aligned}
\left|\frac{1}{t_n} - \frac{1}{t}\right| &= \left|\frac{t - t_n}{t_n t}\right| \\
&= \frac{1}{|t_n|}\frac{1}{|t|}|t - t_n| \\
&< \frac{2}{|t|}\frac{1}{|t|}\frac{\epsilon t^2}{2} \\
&= \epsilon
\end{aligned}$$

By 3, if s_n converges to s, then $\frac{s_n}{t_n} = s_n\left(\frac{1}{t_n}\right)$ converges to $s\left(\frac{1}{t}\right) = \frac{s}{t}$. $\square$

Lemma 8.1.4 ▸ Limit of a Constant Sequence

If s_n is a constant sequence $(l, l, l, \ldots)$, then s_n converges to l.

Proof. Let $\epsilon > 0$. For all $n \in \mathbb{N}$, $|s_n - l| = 0 < \epsilon$. $\square$

lemma 14.6 for bounding in proof of 4.

Explain new notation

better

Example 8.1.5 ▸ Using the Properties

Prove $\lim_{n\to\infty} \frac{5n^3-8n^2+15}{7n^3+19n+4} = \frac{5}{7}$.

Proof.

$$\begin{aligned}\lim_{n\to\infty} \frac{5n^3-8n^2+15}{7n^3+19n+4} &= \lim_{n\to\infty} \frac{5 - {}^{8}/_{n} + {}^{15}/_{n^3}}{7 + {}^{19}/_{n^2} + {}^{4}/_{n^3}} \\ &= \frac{\lim_{n\to\infty} 5 - {}^{8}/_{n} + {}^{15}/_{n^3}}{\lim_{n\to\infty} 7 + {}^{19}/_{n^2} + {}^{4}/_{n^3}} \\ &= \frac{\lim_{n\to\infty} 5 - \lim_{n\to\infty} {}^{8}/_{n} + \lim_{n\to\infty} {}^{15}/_{n^3}}{\lim_{n\to\infty} 7 + \lim_{n\to\infty} {}^{19}/_{n^2} + \lim_{n\to\infty} {}^{4}/_{n^3}}\end{aligned}$$

Now we can work with each limit independently. Note that $\lim_{n\to\infty} \frac{1}{n^2} = \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right)$, so:

$$\lim_{n\to\infty} \frac{5n^3-8n^2+15}{7n^3+19n+4} = \frac{5}{7}$$

$\square$

Definition 8.1.6 ▸ Increasing, Decreasing, Monotonic

A sequence (s_n) is:

- ***increasing*** if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly increasing*** if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$.
- ***decreasing*** if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly decreasing*** if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$.

If (s_n) satisfies any of these properties, then we say (s_n) is ***monotonic***.

For example, $(s_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ is strictly decreasing and thus monotonic.

Theorem 8.1.7 ▸ Monotone Sequence Theorem

Let (s_n) be a sequence of real numbers.

1. If (s_n) is increasing and bounded above, then (s_n) converges to $\sup\{s_n : n \in \mathbb{N}\}$.
2. If (s_n) is decreasing and bounded below, then (s_n) converges to $\inf\{s_n : n \in \mathbb{N}\}$.

Idea: Assuming s is our limit, we want to find $N \in \mathbb{N}$ such that $|s - s_n| < \epsilon$, or $s - \epsilon < s_n$ for all $n > N$. Then $s - \epsilon < s_n \leq s$ for all $n > N$.

Proof of 1. Let $\epsilon > 0$. Because $\{s_n : n \in \mathbb{N}\}$ is non-empty and bounded above, then it has a supremum. Let $s := \sup\{s_n : n \in \mathbb{N}\}$. Thus, there exists $N \in \mathbb{N}$ such that $s_N > s - \epsilon$ (by the approximation property). Since (s_n) is increasing, we have:

$$\forall (n > N)\,(s - \epsilon < s_N \leq s_n \leq s \quad)$$

Hence, $\epsilon < s_n - s \leq 0$, so $|s_n - s| < \epsilon$. ▢

Proof of 2. Suppose (s_n) is decreasing and bounded below. Then $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$. Moreover, there exists $m \in \mathbb{R}$ such that $s_n \geq m$ for all $n \in \mathbb{N}$. That is, $-s_{n+1} \geq -s_n$ for all $n \in \mathbb{N}$, and $-s_n \leq -m$ for all $n \in \mathbb{N}$. Therefore, $(-s_n)$ is increasing and bounded above. By the first part, we know $(-s_n)$ converges to $\sup\{-s_n : n \in \mathbb{N}\} = -\inf\{s_n : n \in \mathbb{N}\}$. Hence, (s_n) converges to $\inf\{s_n : n \in \mathbb{N}\}$. ▢

8.2 Subsequences

So far, we've only looked at well-behaving sequences that converge. What about sequences that don't converge? Can will still find some nice properties that describe their behavior?

$$(s_n) = (0, 1, 0, 1, 0, 1, \dots)$$

Consider a sequence (t_n) where $t_n = s_{2n}$. That is:

$$(t_n) = (s_2, s_4, s_6, \dots) = (1, 1, 1, \dots)$$

Inside this diverging sequence, we can find a convergent ***subsequence***! Intuitively, we can make a subsequence by "throwing away" terms but keeping the same order. We can formally define a subsequence as follows:

Definition 8.2.1 ▶ Subsequence

Given a sequence (s_n), a ***subsequence*** is any sequence of the form $(t_k)_{k \in \mathbb{N}}$ where $t_k = s_{n_k}$ for all $k \in \mathbb{N}$, $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$, and $n_k < n_{k+1}$ for all $k \in \mathbb{N}$.

For example, if we had $s = (s_1, s_2, s_3, s_4, s_5, s_6, s_7, \dots, s_{213}, s_{214}, s_{215}, \dots)$, we can have a subsequence like:

$$(t_n) = (s_3, s_5, s_{213}, \ldots)$$

Here, we would have $n_1 = 3$, $n_2 = 5$, $n_3 = 213$, and so on.

Example 8.2.2 ▸ Subsequences

Let $(s_n) := (1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots)$

1. $(t_n) := (1, 1/4, 1/9, 1/16, 1/25)$ is a subsequence of (s_n) where $t_n = 1/n^2$, or $t_n = s_{n^2}$.
2. $(t_n) := (1/5, 1/25, 1/125, \ldots)$ is also a subsequence of (s_n) with $t_n = \frac{1}{5^n}$ or $t_n = s_{5^n}$.
3. $(t_n) := (1/7, 1/2, 1/12, 1/6)$ is **not** a subsequence of (s_n) because the indices in s_n are not strictly increasing. We have $n_1 = 7$, but $n_2 = 2$.

In general:

$$(s_n) = (s_1, s_2, s_3, \ldots)$$

$$(t_n) = (s_{n_k}) = (s_{n_1}, s_{n_2}, s_{n_3})$$

Lemma 8.2.3 ▸ Indices of Subsequences

If $(s_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(s_n)_{n\in\mathbb{N}}$, then $n_k \geq k$ for all $k \in \mathbb{N}$.

We will use induction.

Base Case: Since $n_1 \in \mathbb{N}$, then $n_1 \geq 1$.

Induction Step: Suppose $n_k \geq k$ for some $k \in \mathbb{N}$. Since $n_{k+1} > n_k$, we have $n_{k+1} \geq n_k + 1 \geq k + 1$.

Hence, $n_k \geq k$ for all $k \in \mathbb{N}$.

Theorem 8.2.4 ▸ Limits of Subsequences

Suppose (s_n) is a sequence of real numbers, and s_n converges to s for some $s \in \mathbb{R}$. If (s_{n_k}) is a subsequence of (s_n), then s_{n_k} converges to s.

Proof. Let $\epsilon > 0$. Since s_n converges to s, then there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for all $n > N$. Suppose $k > N$. By lemma 8.2.3, $n_k \geq k > N$, so $|s_{n_k} - s| < \epsilon$. ▢

8.3 Limit Superior and Inferior

Suppose (s_n) is a bounded sequence. Then there exists $M \in \mathbb{R}$ such that $-M \leq s_n \leq M$ for all $n \in \mathbb{N}$. Let:

$$\begin{aligned}
t_1 &:= \sup\{s_1, s_2, s_3, \ldots\} = \sup\{s_k : k \geq 1\} \\
t_2 &:= \sup\{s_2, s_3, s_4, \ldots\} = \sup\{s_k : k \geq 2\} \\
t_3 &:= \sup\{s_3, s_4, s_5, \ldots\} = \sup\{s_k : k \geq 3\} \\
&\vdots \\
t_n &:= \sup\{s_n, s_{n+1}, s_{n+2}, \ldots\} = \sup\{s_k : k \geq n\} \\
t_{n+1} &:= \sup\{s_{n+1}, s_{n+2}, s_{n+3}, \ldots\} = \sup\{s_k : k \geq n+1\}
\end{aligned}$$

Then:

$$-M \leq s_n \leq t_n$$

and:

$$t_{n+1} \leq t_n$$

so (t_n) is bounded below and decreasing. Hence, (t_n) converges by the Monotone Sequence Theorem.

Definition 8.3.1 ▸ Limit Superior, Limit Inferior

Let (s_n) be a bounded sequence of real numbers. The ***limit superior*** is defined as:

$$\limsup s_n := \lim_{n\to\infty} \sup\{s_k : k \geq n\}$$

Similarly, the ***limit inferior*** is defined as:

$$\liminf s_n := \lim_{n\to\infty} \inf\{s_k : k \geq n\}$$

Example 8.3.2

Define $s_n := \begin{cases} 3 + \frac{1}{n}, & n \text{ is even} \\ 1 - \frac{1}{n} & n \text{ is odd} \end{cases}$

$(s_n) = (0, 3 + 1/2, 2/3, 3 + 1/4, 4/5, 3 + 1/6)$
Let's try to calculate the limit superior of s_n. Define (t_n) as follows:

$$\begin{aligned}
t_1 &:= \sup\{s_1, s_2, s_3, \ldots\} = 3 + \frac{1}{2} \\
t_2 &:= \sup\{s_2, s_3, s_4, \ldots\} = 3 + \frac{1}{2} \\
t_3 &:= \sup\{s_3, s_4, s_5, \ldots\} = 3 + 1/4 \\
t_4 &:= \sup\{s_4, s_5, s_6, \ldots\} = 3 + 1/4 \\
t_5 &:= \sup\{s_5, s_6, s_7, \ldots\} = 3 + 1/6 \\
&\vdots
\end{aligned}$$

We can see that $\limsup s_n = \lim_{n\to\infty} \sup\{s_k : k \geq n\} = 3$. We might refer to 3 as the "largest limit point".
Now let's try to calculate the limit inferior of s_n. Define (r_n) as follows:

$$\begin{aligned}
r_1 &:= \inf\{s_1, s_2, s_3, \ldots\} = 0 \\
r_2 &:= \inf\{s_2, s_3, s_4, \ldots\} = \frac{2}{3} \\
r_3 &:= \inf\{s_3, s_4, s_5, \ldots\} = \frac{2}{3} \\
r_4 &:= \inf\{s_4, s_5, s_6, \ldots\} = \frac{4}{5} \\
&\vdots
\end{aligned}$$

We can see that $\liminf s_n = \lim_{n\to\infty} \inf\{s_k : k \geq n\} = 1$. We might refer to 1 as the "smallest limit point".

Theorem 8.3.3

Suppose (s_n) is a bounded sequence of real numbers, and suppose that (s_{n_k}) is a convergent subsequence of (s_n). Then $\liminf s_n \leq \lim_{k\to\inf} s_{n_k} \leq \limsup s_n$.

Proof. Let $r_n := \inf\{s_k : k \geq n\}$ and $t_n := \sup\{s_k : k \geq n\}$. Then $r_n \leq s_n \leq t_n$ for all $n \in \mathbb{N}$. In particular, $r_{n_k} \leq s_{n_k} \leq t_{n_k}$ for all $k \in \mathbb{N}$. By (todo: theroem), $\lim_{k\to\infty} r_{n_k} = \lim_{n\to\infty} r_n$. Note that $\lim_{n\to\infty} r_n = \liminf s_n$, and $\lim_{k\to\infty} t_{n_k} = \lim_{n\to\infty} t_n = \limsup s_n$. By the (todo: problem set squeeze theorem), we have:

$$\liminf s_n = \lim_{k\to\infty} r_{n_k} \le lim_{k\ to\infty} s_{n_k} \le \lim_{k\to\infty} t_{n_k} = \limsup s_n$$

▢

Theorem 8.3.4 ▸ Bolzano-Weierstrass Theorem

Suppose (s_n) is a bounded sequence of real numbers. The (s_n) has a subsequence that coverges to $\limsup s_n$, and (s_n) has a subsequence that converges to $\liminf s_n$.

Intuition:

- Let $t_k \coloneqq \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$, so $\limsup s_n = \lim_{k\to\infty} t_k$.
- For each $k \in \mathbb{N}$ we can find some $n_k \ge k$ such that $t_k - 1/k < s_{n_k}$.
- Thus, $-1/k < s_{n_k} - t_k \le 0$, so $|s_{n_k} - t_k| < 1/k$
- By (todo: problem set), $s_{n_k} - t_k \to 0$, so $s_{n_k} = s_{n_k} - t_k + t_k \to \limsup s_n$.
- But: we need $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. So we need to choose n_k inductively!

Proof for limsup. We will choose a subsequence of (s_n) that converges to $\limsup s_n$. For each $k \in \mathbb{N}$, let $t_k \coloneqq \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ For convenience, let $P(n)$ be the statement "there exists $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $|s_{n_k} - t_{1+n_{k-1}}| < \frac{1}{k}$." We define $n_0 \coloneqq 0$.
Base Case: Let $t_1 \coloneqq \sup\{s_1, s_2, \ldots\}$. By the approximation property (todo ref), there exists $n_1 \in \mathbb{N}$ such that $t_1 - 1 < s_{n_1} \le t_1$. Subtracting across by t_1, we have $-1 < s_{n_1} - t_1 \le 0$. Thus, $|s_{n_1} - t_1| < 1$.
Induction Step: Now we aim to prove $P(k-1) \implies P(k)$. There exists $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$, and:

$$\begin{aligned}
& t_{1+n_{k-1}} - \frac{1}{k} < s_{n_k} \le t_{1+n_{k-1}} \\
\implies & -\frac{1}{k} < s_{n_k} - t_{1+n_{k-1}} \le 0 \\
\implies & |s_{n_k} - t_{1+n_{k-1}}| < \frac{1}{k}
\end{aligned}$$

That is, $\lim_{k\to\infty}(s_{n_k} - t_{1+n_{k-1}}) = 0$. Since $n_k > n_{k-1}$ for all $k \in \mathbb{N}$, (s_{n_k}) is a subsequence of (s_n). But $(t_{1+n_{k-1}})$ is a subsequence of (t_k), so:

$$\lim_{k\to\infty} t_{1+n_{k-1}} = \lim_{k\to\infty} t_k = \limsup s_n$$

Thus:

$$s_{n_k} = s_{n_k} - t_{1+n_{k-1}} + t_{1+n_{k-1}}$$

so:

$$\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} \left(s_{n_k} - t_{1+n_{k-1}}\right) + \lim_{k\to\infty} t_{1+n_{k-1}} = 0 + \limsup s_n$$

Therefore, (s_{n_k}) is a subsequence of (s_n) that converges to $\limsup s_n$. ▢

Theorem 8.3.5 ▸ Convergence iff $\limsup = \liminf$

Let (s_n) be a bounded sequence of real numbers. Then (s_n) converges if and only if $\liminf s_n = \limsup s_n$

Proof. First, suppose s_n converges to some $s \in \mathbb{R}$. By the Bolzano-Weierstrass Theorem, there exists a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \limsup s_n$. But s_n converges to s, so s_{n_k} also converges to s. That is, $s = \lim_{k\to\infty} s_{n_k} = \limsup s_n$. By the same reasoning, we have $s = \liminf s_n$. Hence, $\liminf s_n = \lim s_n = \limsup s_n$.
Conversely, suppose $\liminf s_n = \limsup s_n$. Let $r_n := \inf\{s_k : k \geq n\}$ and $t_n := \sup\{s_k : k \geq n\}$. Then $r_n \leq s_n \leq_n$ for all $n \in \mathbb{N}$. Then $\lim r_n = \liminf s_n = \limsup s_n = \lim t_n$. Therefore, by the Squeeze Theorem (todo: ref), s_n converges to $\liminf s_n$. ▢

8.4 Cauchy Sequences

To show that a sequence (s_n) converges using the definition of limit, we need to know what limit is beforehand. Consider the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^3}$$

This sequence of partial sums converges, but its limit is unknown. Certainly we can get a decimal approximation for this value, but there is no known "closed" form of this value.

Definition 8.4.1 ▸ Cauchy Sequence

We say a sequence is ***Cauchy*** if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ for all $n > N$ and $m > M$.

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(n > N, m > N)\,(|s_n - s_m| < \epsilon)$$

In other words, a sequence is Cauchy if **all** terms in the tail can be made arbitrarily close to each other. Or, for any arbitrarily small distance, there exists some "tail" of the sequence that exists entirely within that distance. This definition circumvents any mention of a specific "limit". But we can prove that any Cauchy sequence of real numbers is convergent, and vice versa.

Lemma 8.4.2 ▸ Convergent Sequences are Cauchy

If a sequence of real numbers converges, then that sequence is Cauchy.

Proof. Let (s_n) be a convergent sequence of real numbers. Let $\epsilon > 0$. Since (s_n) converges to some $s \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2$ for all $n > N$. If $n > N$ and $m > N$, then:

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Lemma 8.4.3

Suppose (s_n) is a Cauchy sequence, and that (s_{n_k}) is a convergent subsequence of (s_n) where s_{n_k} converges to some $s \in \mathbb{R}$. Then (s_n) converges, and $\lim s_n = s$.

Proof. Let $\epsilon > 0$. Since (s_n) is Cauchy, then there exists some $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon/2$ for all $n > N$ and $m > N$. Since (s_{n_k}) converges to s, there exists $N_1 \in \mathbb{N}$ such that $|s_{n_k} - s| < \frac{\epsilon}{2}$ for all $k > N_1$. Let $k \in \mathbb{N}$ where $k > N$ and $k > N_1$. Since $n_k \geq k$, then $n_k > N$ and $n_k > N_1$. For all $n > k$, we have:

$$|s_n - s| = |s_n - s_{n_k} + s_{n_k} - s| \leq \underbrace{|s_n - s_{n_k}|}_{n_1, n_k > N} + \underbrace{|s_{n_k} - s|}_{n_k > N_1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Lemma 8.4.4 ▸ Cauchy Sequences are Bounded

If a sequence of real numbers is Cauchy, then that sequence is bounded.

Proof. Let (s_n) be a Cauchy sequence of real numbers. Then there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < 1$ for all $n > N$ and $m > N$. Let $m := N + 1$. Then, for all $n > N$, we have:

$$|s_n| = |s_n - s_m + s_m| \leq |s_n - s_m| + |s_m| < 1 + |s_m| = 1 + |s_{N+1}|$$

Thus, for all $n \in \mathbb{N}$, we have:

$$|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_N|, 1 + |s_{N+1}|\}$$

Therefore, (s_n) is bounded. ▢

Theorem 8.4.5 ▸ Cauchy Criterion

A sequence of real numbers converges if and only if it is Cauchy.

Proof. By Lemma 8.4.2, we know that convergence implies Cauchy.
If (s_n) is Cauchy, then by Lemma 8.4.4, (s_n) is bounded. By the Bolzano-Weierstrass Theorem, (s_n) has a convergent subsequence. By Lemma 8.4.3, (s_n) converges. ▢

Our definition of completeness in $\mathbb{R}$ predicates on a notion of order between the elements. Specifically, we said $\mathbb{R}$ is complete because every subset of $\mathbb{R}$ that is bounded above has a supremum. What does it mean to say $\mathbb{R}^2$ is complete?

Definition 8.4.6 ▸ Completeness (in terms of Cauchy Sequences)

A (metric) space is ***complete*** if every Cauchy sequence converges to a point in the space.

The intuition is the same: there are no points "missing" from the space.

9 Open and Closed Sets

We will describe some concepts that generalize open/closed intervals. This chapter also serves as a very light introduction to topology—specifically, the topology of the real number line.

9.1 Open Sets

Definition 9.1.1 ▶ Open Set

Intuitively, set is ***open*** if it does not contain any of its "boundary points", such as minimum or maximum.

More formally, we say $A \subseteq \mathbb{R}$ is ***open*** if, for all $x \in A$, there exists $r > 0$ such that $(x - r, x + r) \subseteq A$.

$$\forall (x \in A) \exists (r > 0) \, ((x - r, x + r) \subseteq A)$$

Example 9.1.2 ▶ $[0, 1)$ is not open

The interval $[0, 1)$ is not open.

Proof. $0 \in [0, 1)$, but $(0 - r, 0 + r) \nsubseteq [0, 1)$ for any $r > 0$. □

Definition 9.1.3 ▶ Open Ball

We call the interval $(x - r, x + r)$ the ***open ball*** of radius r centered at x, notated as $B(x, r)$ or $B_r(x)$.

$$B(x, r) = B_r(x) = (x - r, x + r)$$

This new notation lets us write ideas more succinctly. For example, $\mathbb{R}$ is open. Given any $x \in \mathbb{R}$, then any $r > 0$ will give us $B(x, r) \in \mathbb{R}$. Also, $\varnothing$ is vacuously open.

Lemma 9.1.4 ▶ Open Intervals are Open Sets

Let $a, b \in \mathbb{R}$ where $a < b$. Then (a, b) is an open set.

Proof. Let $c := \frac{a+b}{2}$, and let $R := \frac{b-a}{2}$. Then $(a, b) = B(c, R)$. Let $x \in B(c, R)$. Then $|x - c| < R$. Let $r := R - |x - c| > 0$. We now prove $B(x, r) \subseteq B(c, R)$. Let $y \in B(x, r)$. Then $|x - y| < r$, so:

$$|y - c| = |y - x + x - c| \leq |y - x| + |x - c| < r + |x - c| = R - |x - c| + |x - c| = R$$

Hence, $y \in B(c, R) = (a, b)$. Therefore, (a, b) is an open set. ▢

As we prove below, an arbitrary union of open sets is itself an open set.

Theorem 9.1.5 ▶ Union of Open Sets is Open

Suppose Λ is a set, and for each $\lambda \in \Lambda$, O_λ is an open subset of $\mathbb{R}$. Then $\bigcup_{\lambda \in \Lambda} O_\lambda$ is an open set.

Proof. Let $x \in \bigcup_{\lambda \in \Lambda} O_\lambda$. Then there exists some $\lambda_0 \in \Lambda$ such that $x \in O_{\lambda_0}$. Since O_{λ_0} is open, there exists $r > 0$ such that:

$$(x - r, x + r) \subseteq O_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

▢

The intersection of open sets is more troublesome. Countable intersections of open sets may not be open. For example, let $A_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$ for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ is not open!

Theorem 9.1.6 ▶ Finite Intersection of Open Sets is Open

Let $n \in \mathbb{N}$, and let $O_1, O_2, \ldots, O_n$ be open subsets of $\mathbb{R}$. Then $\bigcap_{k=1}^{n} O_k$ is open.

Proof. Let $x \in \bigcap_{k=1}^{n} O_k$. Then $x \in O_k$ for $k = 1, 2, \ldots, n$. Then, for each $k \in \{1, 2, \ldots, n\}$, there must be some radius $r_k > 0$ such that $B(x, r_k) \subseteq O_k$. Since there are only finitely many open sets, we can take the minimum radius. Let $r := \min\{r_1, r_2, \ldots, r_n\}$. Then, $r \leq r_k$ for each $k \in \{1, 2, \ldots, n\}$. Hence:

$$B(x, r) \subseteq B(x, r_k) \subseteq O_k \quad \text{for all } k \in \{1, 2, \ldots, n\}$$

Therefore, $B(x, r) \subseteq \bigcap_{k=1}^{n} O_k$, so it is open. ▢

Note how the above theorem only works by taking the minimum radius of all the open sets. We can only take this minimum radius because there are only a finite number of open sets.

9.2 Closed Sets

Definition 9.2.1 ▸ Closed Set

Intuitively, a set is ***closed*** if it contains all of its "boundary points".

More formally, a set $E \subseteq \mathbb{R}$ is ***closed*** if every convergent sequence (s_n) where $s_n \in E$ for all $n \in \mathbb{N}$ satisfies $\lim_{n\to\infty} s_n \in E$.

Example 9.2.2 ▸ $(0, 1]$ is not closed

The interval $[0, 1)$ is not closed.

Proof. Consider the sequence $(s_n) := 1/n$. Then (s_n) converges to 0, but $0 \notin (0, 1]$. ▢

Note that this interval $(0, 1]$ is neither open nor closed! It is wrong to think of open/closed as strictly one or the other (i.e. openness and closedness are not mutually exclusive). Moreover, a set can be both open and closed (or ***clopen***), going against the intuition of open and closed sets. There are only two clopen sets in the real numbers: $\mathbb{R}$ itself, and $\varnothing$.

Lemma 9.2.3 ▸ Closed Intervals are Closed Sets

Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b]$ is a closed set.

Proof. Let (s_n) be an arbitrary convergent sequence of real numbers where $a \leq s_n \leq b$ for all $n \in \mathbb{N}$. Since (s_n) is convergent, then $\lim_{n\to\infty} s_n$ exists. By the properties of limits, we have:

$$\lim_{n\to\infty} a \leq \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} b$$

Hence, $\lim_{n\to\infty} s_n \in [a, b]$. Therefore, $[a, b]$ is a closed set. ▢

Theorem 9.2.4 ▸ Intersection of Closed Sets is Closed

Let Λ be a set, and let $E_\lambda \subseteq \mathbb{R}$ be closed for all $\lambda \in \Lambda$. Then $\bigcap_{\lambda \in \Lambda} E_\lambda$ is a closed set.

Proof. Let (s_n) be an arbitrary convergent sequence of real numbers entirely contained within $\bigcap_{\lambda \in \Lambda} E_\lambda$. Since (s_n) is convergent, then $\lim_{n\to\infty} s_n$ exists. Let l denote that limit. Let $\lambda \in \Lambda$ be arbitrary. Then $s_n \in E_\lambda$ for all $n \in \mathbb{N}$. Since E_λ is closed, then any convergent sequence contained in E_λ has its limit in E_λ. Thus, $\lim_{n\to\infty} s_n \in E_\lambda$. Since $\lambda \in \Lambda$ is arbitrary, then $\lim_{n\to\infty} s_n \in E_\lambda$ for all $\lambda \in \Lambda$. Therefore, $s \in \bigcap_{\lambda \in \Lambda} E_\lambda$, so this set is closed. $\square$

Similar to the intersection of open sets, the union of closed sets is guaranteed to be closed if it is a finite union. For example, the union $\left(\bigcup_{n\in\mathbb{N}} [1/n, 1]\right) = (0, 1]$ is not closed!

Theorem 9.2.5 ▸ Finite Union of Closed Sets is Closed

Let $n \in \mathbb{N}$, and let $E_1, E_2, \dots, E_n$ be closed subsets of $\mathbb{R}$. Then $\bigcup_{k=1}^{n} E_k$ is a closed set.

A direct proof of this theorem can be found in the textbook.

The direct proof here is rather wordy and awkward. We will first establish a concrete relationship between open and closed sets, then leverage that to prove this theorem "indirectly".

Theorem 9.2.6 ▸ Complement of an Open Set is Closed

Let $O \subseteq \mathbb{R}$ be open. Then $\mathbb{R} \setminus O$ is closed.

Proof. Let (x_n) be an arbitrary convergent sequence entirely contained within $R \setminus O$. Let $l_x \coloneqq \lim_{n\to\infty} x_n$. Suppose for contradiction that $l_x \notin \mathbb{R} \setminus O$. Then $l_x \in O$. Since O is open, there exists some radius $r > 0$ such that $B(l_x, r) \in O$. Since (x_n) converges to l_x, then there exists $N \in \mathbb{N}$ such that $|x_n - l_x| < r$ for all $n > N$. That is, $x_n \in B(l_x, r) \subseteq O$ for all $n > N$. This contradicts $x_n \in \mathbb{R} \setminus O$. Thus, $l_x \in \mathbb{R} \setminus O$, so $\mathbb{R} \setminus O$ is closed. $\square$

Theorem 9.2.7 ▸ Complement of a Closed Set is Open

Let $C \subseteq \mathbb{R}$ be closed. Then $\mathbb{R} \setminus C$ is open.

Proof. Let $x \in \mathbb{R} \setminus C$. We must prove the following statement:

$$\exists (n \in \mathbb{N}) \left(B\left(x, 1/n\right) \subseteq \mathbb{R} \setminus C\right)$$

Suppose for contradiction the negation of the previous statement holds. That is:

$$\forall(n \in \mathbb{N})\,(B\,(x, {}^{1}\!/\!{}_{n}) \not\subseteq \mathbb{R} \setminus C)$$

In other words, for all $n \in \mathbb{N}$, there exists $x_n \in B(x, {}^{1}\!/\!{}_{n})$ such that $x_n \in C$. Hence, the sequence (x_n) satisfies $x_n \in C$ for all $n \in \mathbb{N}$ and $|x_n - x| < {}^{1}\!/\!{}_{n}$. Thus, (x_n) converges to x. However, C is closed, and (x_n) is a sequence in C, so $x \in C$. This contradicts $x \in \mathbb{R} \setminus C$. Therefore, our original statement holds, so $\mathbb{R} \setminus C$ is open. ▢

Combining the two above theorems, we can infer a pretty useful relationship between open and closed sets.

$$A \text{ is open} \iff \mathbb{R} \setminus A \text{ is closed}$$

$$B \text{ is closed} \iff \mathbb{R} \setminus B \text{ is open}$$

We can apply this relationship to directly prove that the finite union of closed sets is closed (Theorem 9.2.5).

Proof of Theorem 9.2.5. By De Morgan's Laws, we have:

$$\mathbb{R} \setminus \left(\bigcup_{k=1}^{n} E_k \right) = \bigcap_{k=1}^{n} (\mathbb{R} \setminus E_k)$$

Since each $\mathbb{R} \setminus E_k$ is open, the finite intersection $\bigcap_{k=1}^{n}(\mathbb{R} \setminus E_k)$ is also open. Hence, $\mathbb{R} \setminus \left(\bigcup_{k=1}^{n} E_k\right)$ is open, so $\bigcup_{k=1}^{n} E_k$ is closed. ▢

9.3 Interior and Closure

Definition 9.3.1 ▸ Interior of a Set

Intuitively, the ***interior*** of a set is the set containing its elements but not its boundary points.

Formally, for $A \subseteq \mathbb{R}$, the ***interior*** of A is the set A° defined as:

$$A^\circ := \{x \in A : \exists(r > 0)(B(x, r) \subseteq A)\}$$

Definition 9.3.2 ▸ Closure of a Set

Intuitively, the ***closure*** of a set is the set containing its elements and boundary points.

Formally, for $A \subseteq \mathbb{R}$, the ***closure*** of A is the set $\overline{A}$ defined as:

$$\overline{A} \coloneqq \{x \in \mathbb{R} : \exists\,(\text{sequence }(x_n))\forall(n \in \mathbb{N})(x_n \in A) \text{ and } x_n \to x\}$$

For example, let's consider the closure the open interval $A \coloneqq (0, 1)$.

- We can take the constant sequence of $1/2$ contained in $(0, 1)$ which converges to $1/2$, so $1/2 \in \overline{A}$.
- We can take the sequence $1/n+1$ contained in $(0, 1)$ which converges to 0, so $0 \in \overline{A}$.
- We can take the sequence $1 - 1/n+1$ contained in $(0, 1)$ which converges to 1, so $1 \in \overline{A}$.

Theorem 9.3.3 ▸ Properties of Closures of Sets

Let $A \subseteq \mathbb{R}$. Then:

(i) $A \subseteq \overline{A}$,
(ii) $\overline{A}$ is closed,
(iii) $A = \overline{A}$ if and only if A is closed,
(iv) $\overline{A} = \overline{\overline{A}}$,
(v) if $F \subseteq \mathbb{R}$ is closed and $A \subseteq F$, then $\overline{A} \subseteq F$, and
(vi) $\overline{A} = \bigcap\{F \subseteq \mathbb{R} : F \text{ is closed, and } A \subseteq F\}$

Proof. The proofs for i through vi are as follows:

(i) Let $x \in A$, and let $(x_n) \coloneqq (x, x, x, \dots)$. Then $x_n \in A$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$. Therefore, $x \in \overline{A}$.

(ii) Let (x_n) be a sequence contained within $\overline{A}$ that converges to some $x \in \mathbb{R}$. For each $x_n \in \overline{A}$, there exists $y_n \in A$ such that $|y_n - x_n| < 1/n$. Then:

$$|y_n - x| = |y_n - x_n + x_n - x| \leq |y_n - x_n| + |x_n - x| < 1/n + |x_n - x|$$

Thus, $y_n - x \to 0$, so $y_n \to x$. Therefore, $x \in \overline{A}$.

(iii) First, suppose $A = \overline{A}$. Then by (ii), $\overline{A}$ is closed, so A is closed. Conversely, suppose A is closed. Then by (i), $A \subseteq \overline{A}$. Now we show $\overline{A} \subseteq A$. Let $x \in \overline{A}$. By definition, there exists a sequence (x_n) contained in A that converges to x. Since A is closed, then $x \in A$, so $\overline{A} \subseteq A$. Thus, $\overline{A} = A$.

(iv) By (ii), $\overline{A}$ is closed. By (iii), $\overline{A} = \overline{\overline{A}}$.

(v) Let $x \in \overline{A}$. By definition, there exists a sequence (x_n) contained in A that converges to x. Since $A \subseteq F$, then (x_n) is also contained in F. Since F is closed, then $x \in F$.

(vi) By (v), if F is closed and $A \subseteq F$, then $\overline{A} \subseteq F$. Therefore, $\overline{A} \subseteq \bigcap\{F \subseteq \mathbb{R} : F \text{ is closed, and } A \subseteq F\}$. By (ii), $\overline{A}$ is closed, and by (i), $A \subseteq \overline{A}$. Thus, we have:

$$\bigcap\{F \subseteq \mathbb{R} : F \text{ is closed, and } A \subseteq F\} \subseteq \overline{A}$$

▢

These properties can make it easier to prove statements about closures.

Example 9.3.4 ▸ Using Properties of Closure

If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof. By (i), $A \subseteq B \subseteq \overline{B}$, and by (ii), $\overline{B}$ is closed. Thus, by (v), $\overline{A} \subseteq \overline{B}$. ▢

10 Compact Sets

In this chapter, we describe the idea of ***compactness*** for sets. This idea will prove tremendously useful in future chapters as it will enable us to bring a “finite quality” to an otherwise infinite set or idea. It turns out that, for metric spaces like the real numbers, compactness is equivalent to closed and bounded.

However, we will approach this idea from two (seemingly different) notions: sequential compactness which deals closed sets, and open cover compactness (or simply “compact”) that deals with open sets. Having these different approaches provides us with several avenues for proving things about compactness. We’ll dedicate a section to either approach, and ultimately show that these two ideas are equivalent in the real numbers (this is not generally true in all topological spaces).

10.1 Sequential Compactness

Definition 10.1.1 ▸ Sequential Compactness

A set $K \subseteq \mathbb{R}$ is ***sequentially compact*** if every sequence contained within K also has a subsequence that converges to an element of K.

Example 10.1.2 ▸ Closed Intervals are Sequentially Compact

Let $a, b \in \mathbb{R}$ where $a \leq b$. Then $[a, b]$ is sequentially compact.

Intuition: We want to strictly use the definition of sequential compactness. So we will consider an arbitrary subsequence of $[a, b]$, and then find a subsequence that converges to something in $[a, b]$.

Proof. Let (x_n) be an arbitrary sequence entirely contained within $[a, b]$. Then (x_n) is bounded, so the Bolzano-Weierstrass Theorem guarantees the existence of a subsequence (x_{n_k}) of (x_n) that x_{n_k} converges to some $x \in \mathbb{R}$. Since $[a, b]$ is a closed set, then every sequence contained in $[a, b]$ converges to a number in $[a, b]$. Then we know (x_{n_k}) converges to some number in $[a, b]$. $\square$

Example 10.1.3 ▶ Open Sets are not Sequentially Compact

$(0, 1]$ is not sequentially compact.

Intuition: Our strategy is to find a sequence contained within $(0, 1]$ that converges to something outside of $(0, 1]$.

Proof. Consider the sequence $(x_n) := 1/n$. Then the sequence is entirely contained within $(0, 1]$, but it converges to 0 which is not in $(0, 1]$. Note that any arbitrary subsequence of (x_n) also converges to 0.

Reference to therorem 8.2.4 limits of subsequences

□

Theorem 10.1.4 ▶ Sequentially compact just means closed and bounded

Let $E \subseteq \mathbb{R}$. Then E is sequentially compact if and only if E is both closed and bounded.

Proof. First, suppose E is sequentially compact. Let (x_n) be an arbitrary sequence contained within E that converges to some $x \in \mathbb{R}$. Since E is sequentially compact, then there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to some $y \in E$. By (todo: Proposition 15.2 whatever this ref is), we have:

$$\lim_{k\to\infty} x_{n_k} = x$$

Since limits are unique (as proven in todo), $x = y$. Thus, (x_n) converges to $y \in E$, so E is closed.

To prove E is bounded, we proceed by contradiction. Suppose E is not bounded. Then, for each $N \in \mathbb{N}$, there exists $x_n \in E$ such that $|x_n| > n$. Since E is sequentially compact, then there exists a subsequence (x_{n_k}) of (x_n) such that x_{n_k} converges to a point in E. Note that $|x_{n_k}| > n_k \geq k$, so the sequence (x_{n_k}) is unbounded and thus is divergent. This contradicts the fact that (x_{n_k}) does converge. Thus, our supposition that E is not bounded was false, so E is in fact bounded.

Conversely, suppose that E is both closed and bounded. Let (x_n) be an arbitrary sequence entirely contained within E. Since E is bounded, then there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. By the Bolzano-Weierstrass Theorem, there exists a subsequence (x_{n_k}) of (x_n) where (x_{n_k}) converges to some $x \in \mathbb{R}$. Since E is closed and (x_{n_k}) is contained within E, then its limit x must be in E. Therefore, E is sequentially compact. □

10.2 Open Cover Compactness

Definition 10.2.1 ▸ Open Cover

Let $A \subseteq \mathbb{R}$. An ***open cover*** of A is a collection of open sets such that A is a subset of the union of that collection. We say that the collection ***covers*** A.

In other words, every number in A is in at least one of the open sets in the collection of open sets.

Example 10.2.2 ▸ Open Cover Example #1

$[0,1]$ is covered by $\{B(x, {}^{1}\!/_{10}) : x \in [0,1]\}$.

Proof. First note that $B(x, {}^{1}\!/_{10})$ is open for all $x \in [0,1]$. Let $a \in [0,1]$ be arbitrary. Then:

$$a \in B(a, {}^{1}\!/_{10}) \in \{B(x, {}^{1}\!/_{10}) : x \in [0,1]\}$$

So it covers $[0,1]$. ▢

Example 10.2.3 ▸ Open Cover Example #2

$(0,1)$ is covered by $\{({}^{x}\!/_{2}, 1) : x \in (0,1)\}$.

Proof. Note that $({}^{x}\!/_{2}, 1)$ is open for all $x \in (0,1)$. Let $a \in (0,1)$. Then:

$$a \in ({}^{a}\!/_{2}, 1) \in \{({}^{x}\!/_{2}, 1) : x \in (0,1)\}$$

So it covers $(0,1)$. ▢

Definition 10.2.4 ▸ Subcover

Given an open cover of a set, a ***subcover*** is a subset of the open cover that covers the set.

More formally, let $A \subseteq \mathbb{R}$, and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of A. Then $\{O_\lambda : \lambda \in \Lambda'\}$ is a ***subcover*** of $\{O_\lambda : \lambda \in \Lambda\}$ if $\Lambda' \subseteq \Lambda$ and $A \subseteq \bigcup_{\lambda \in \Lambda'} O_\lambda$.

In other words, a subcover is created by throwing away sets from the original cover, and the subcover still covers the original set. Also note that a cover is also one of its own subcovers. We say that a subcover is ***finite*** if there are only finitely many sets in the collection. Finiteness in this context does not refer to the cardinality of the open sets in the collection, but rather the collection itself.

Example 10.2.5 ▶ Open Cover Example #1 Revisited

Then open cover $\{(x - 1/10, x + 1/10) : x \in [0,1]\}$ of $[0,1]$ has a finite subcover.

For example, we can take $\{B(0, 1/10), B(1/10, 1/10), \dots, B(1, 1/10)\}$ which only contains 11 open balls and is therefore a finite subcover of $[0,1]$.

Example 10.2.6 ▶ Open Cover Example #2 Revisited

The open cover $\{(x/2, 1) : x \in (0,1)\}$ of $(0,1)$ does not have a finite subcover.

Proof. Suppose for contradiction that there exists a finite subcover. Then, for some $n \in \mathbb{N}$, there exists $x_1, x_2, \dots, x_n \in (0,1)$. such that $(0,1) \subseteq \bigcup_{i=1}^{n} (x_i/2, 1)$. Since there are finitely many "x's", let $y \coloneqq \min\{x_1, \dots, x_n\}$. Then, for all $i \in \{1, 2, \dots, n\}$, we have:

$$0 < \frac{y}{4} \leq \frac{x_i}{4} < \frac{x_i}{2}$$

so $y/4 \notin (x_i/2, 1)$. Thus, we found some $y \in (0,1)$ that is not in the supposed finite subcover. This contradicts the fact that the subcover must cover the entirety of $(0,1)$. Therefore, there does not exist any finite subcover. ▢

Definition 10.2.7 ▶ Open Cover Compactness

A set $K \subseteq \mathbb{R}$ is ***open cover compact*** if every open cover of K has a finite subcover.

For example, the interval $(0,1)$ is not compact by Example 10.2.6. We found an open cover of $(0,1)$ that does not have a finite subcover.

Example 10.2.8 ▶ Any finite set of points in $\mathbb{R}$ is open cover compact

Proof. Let $A \subseteq \mathbb{R}$ be a finite set. Then $A = \{x_1, x_2, \dots, x_n\}$ for some $n \in \mathbb{N}$. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of A. For each $i \in \{1, 2, \dots, n\}$, we have $x_i \in A$. Also note that $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$, so x_i is contained in some O_{λ_i} for some $\lambda_i \in \Lambda$. Then $\{O_{\lambda_i} : i \in \{1, 2, \dots, n\}\}$ is a finite subcover of A, so A is compact. ▢

Example 10.2.9 ▶ Open Cover Compactness Example #1

Let $A := \{1/n : n \in \{2, 3, 4, \ldots\}\}$. Then A is not open cover compact.

Proof. Consider the open cover:

$$\left\{\left(\frac{1}{n+1}, \frac{1}{n-1}\right) : n \in \{2, 3, 4, \ldots\}\right\}$$

For each $n_o \in \{2, 3, 4, \ldots\}$, note that $1/n_0 \in (1/n_0 + 1, 1/n_0 - 1)$, but if $n \neq n_0$ then $1/n_0 \notin (1/n + 1, 1/n - 1)$. That is, we cannot remove any of the open intervals from the open cover, so there is no finite subcover. Therefore, A is not open cover compact. $\square$

Surprisingly, we can add a single point to the set in the above example and make it a compact set.

Example 10.2.10 ▶ Open Cover Compactness Example #2

Let $A := \{0\} \cup \{1/n : n \in \{2, 3, 4, \ldots\}\}$. Then A is open cover compact.

Proof. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of A. Since $0 \in A$, then there must be some $\lambda_0 \in \Lambda$ such that $0 \in O_{\lambda_0}$. Since this O_{λ_0} is an open set, then there exists some $r > 0$ such that $B(0, r) \subseteq O_{\lambda_0}$. Let $N \in \mathbb{N}$ such that $N > 1/r$. Then for any $n > N$, we have $n > 1/r > 0$. In particular, $0 < 1/n < r$. That is, for any $n > N$, $1/n \in B(0, r) \subseteq O_{\lambda_0}$. For $n \in \{2, 3, \ldots, N\}$, there exists some $\lambda_n \in \Lambda$ such that $1/n \in O_{\lambda_n}$. We can now create a finite subcover as follows:

$$\{O_{\lambda_0}, O_{\lambda_2}, O_{\lambda_3}, \ldots, O_{\lambda_N}\}$$

which is a finite subcover for A. Since our choice of the initial open cover was arbitrary, then every open cover of A has such a finite subcover. Therefore, A is open cover compact. $\square$

Theorem 10.2.11 ▶ Compactness implies boundedness

If $K \subseteq \mathbb{R}$ is open cover compact, then K is bounded.

Proof. Consider the following open cover of K:

$$O := \{(-n, n) : n \in \mathbb{N}\}$$

Since K is compact, there exists a finite subset of O that still covers K (i.e. a finite subcover), which can be of the form:

$$\{(-n_i, n_i) : i \in \{1, 2, \dots, m\}\} \quad \text{such that} \quad m, n_1, n_2, \dots, n_m \in \mathbb{N}$$

Note $K \subseteq \bigcup_{i=1}^{m} (-n_i, n_i)$. Let $N := \max\{n_1, n_2, \dots, n_m\}$. Then:

$$K \subseteq \bigcup_{i=1}^{m} (-n_i, n_i) \subseteq (-N, N)$$

Therefore, K is bounded above by N and bounded below by $-N$, so K is bounded. $\square$

Theorem 10.2.12 ▸ Compactness implies closedness

If $K \subseteq \mathbb{R}$ is open cover compact, then K is closed.

Proof. We will show that $\mathbb{R} \setminus K$ is open. Let $x \in \mathbb{R} \setminus K$. For all $r > 0$, define O_r as:

$$O_r := \mathbb{R} \setminus [x - r, x + r]$$

Then for all $r > 0$, O_r is open. Also, $\bigcup_{r>0} O_r = \mathbb{R} \setminus \{x\}$, so $K \subseteq \mathbb{R} \setminus \{x\} = \bigcup_{r>0} O_r$. That is, $\{O_r : r > 0\}$ is an open cover for K. Since K is compact, there exists a finite subcover. Thus, for some $n \in \mathbb{N}$, there exists $r_1, r_2, \dots, r_n$ such that $K \subseteq \bigcup_{i=1}^{n} O_{r_i}$. Let $r := \min\{r_1, r_2, \dots, r_n\}$. Then $\bigcup_{i=1}^{n} O_{r_i} \subseteq O_r$. Since $K \subseteq O_r$, we have:

$$\mathbb{R} \setminus K \supseteq O_r^c = [x - r, x + r] \supseteq (x - r, x + r)$$

so $\mathbb{R} \setminus K$ is open. Therefore, K is closed. $\square$

Theorem 10.2.13 ▸ Closed subsets of compact sets are compact

Let $K \subseteq \mathbb{R}$. If K is open cover compact and $E \subseteq K$ is closed, then E is open cover compact.

Proof. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for E. Since E is closed, $\mathbb{R} \setminus E$ is open. Thus, the collection $\{O_\lambda : \lambda \in \Lambda\} \cup \{\mathbb{R} \setminus E\}$ is an open cover for K. Since K is compact, there exists a finite subcover for K from $\{O_\lambda : \lambda \in \Lambda\} \cup \{\mathbb{R} \setminus E\}$. This subcover can either be:

1. $\{O_{\lambda_i} : i \in \{1, 2, \dots, n\}\}$ for some $n \in \mathbb{N}$, in which case we have $E \subseteq K \subseteq \bigcup_{i=1}^{n} O_{\lambda_i}$, so $E \subseteq \bigcup_{i=1}^{n} O_{\lambda_i}$, or
2. $\{O_{\lambda_i} : i \in \{1, 2, \dots, n\}\} \cup \{\mathbb{R} \setminus E\}$, in which case we have $E \subseteq K \subseteq \left(\bigcup_{i=1}^{n} O_{\lambda_i}\right) \cup \mathbb{R} \setminus E$.

If $x \in E$, then $x \notin \mathbb{R} \setminus E$, so $x \in \bigcup_{i=1}^{n} O_{\lambda_i}$. Thus, $E \subseteq \bigcup_{i=1}^{n} O_{\lambda_i}$.
In either case, the collection $\{O_{\lambda_i} : i \in \{1, 2, \dots, n\}\}$ is a finite subcover of E. Therefore, E is compact. $\square$

Theorem 10.2.14 ▶ Every closed interval is compact

Let $a, b \in \mathbb{R}$ where $a < b$. Then $[a, b]$ is open cover compact.

Proof. Let $I := [a, b]$, and let $O := \{O_\lambda : \lambda \in \Lambda\}$ be an open cover of I. Suppose for contradiction there does not exist any finite subcover of I from O. Let $I^l := \left[a, \frac{a+b}{2}\right]$, and let $I^r := \left[\frac{a+b}{2}, b\right]$. Then I^l and/or I^r do not have a finite subcover from O. Choose such an interval and call it I_1. Repeat indefinitely for all $n \in \mathbb{N}$.

> We claim there exists a sequence (I_j) of closed intervals such that:
> 1. $I_j \subseteq I_{j-1} \subseteq I$, where $I_0 = I = [a, b]$,
> 2. $\ell(I_j) = \frac{\ell(I)}{2^j}$, where $\ell([c, d]) = d - c$ (i.e. the “length” of the interval), and
> 3. each I_j does not have a finite subcover from O.

For each $j \in \mathbb{N}$, let $P(j)$ denote the statement: “all three properties hold for I_j.” Then the interval I_1 defined above satisfies the three properties.

Maybe elaborate here

Now suppose that $P(j)$ is true for some $j \in \mathbb{N}$. Then I_j does not have a finite subcover from O. Note that I_{j+1} is one of I_j^l or I_j^r which does not have a finite subcover from O.

1. $I_{j+1} \subseteq I_j \subseteq I$, so the first property holds.
2. $\ell(I_{j+1}) = \frac{1}{2}\ell(I_j) = \frac{\ell(I)}{2^{j+1}}$, so the second property holds.
3. I_j does not have a finite subcover from O, so the third property holds.

Thus, $P(j + 1)$ is true. By the Principle of Induction, $P(n)$ is true for all $n \in \mathbb{N}$. By the Nested Interval Property (todo: ref), we have:

$$\bigcap_{j \in \mathbb{N}} I_j \neq \varnothing$$

so there exists some $x \in \bigcap_{j \in \mathbb{N}} I_j \subseteq [a, b]$. Since O is an open cover for $[a, b]$, there exists an index $\lambda_x \in \Lambda$ such that $x \in O_{\lambda_x}$. Since O_{λ_x} is open, there exists some radius $r > 0$ such that $B(x, r) \subseteq O_{\lambda_x}$. Choose $n \in \mathbb{N}$ such that $\ell(I_n) < r$. Since $x \in I_n$, we have $I_n \subseteq B(x, r) \subseteq O_{\lambda_x}$, so I_n is covered by just one open set in the open cover O. This contradicts the claim that there does not exist any finite subcover of I_n from O, also contradicting the assumption that $[a, b]$ has no finite subcover. Therefore, $[a, b]$ is open cover compact. $\square$

Theorem 10.2.15 ▸ Heine-Borel Theorem

A set $E \subseteq \mathbb{R}$ is open cover compact if and only if E is both closed and bounded.

Proof. The forward direction follows from Theorems 10.2.12 and 10.2.11. For the converse direction, suppose $E \subseteq \mathbb{R}$ is closed and bounded. Since E is bounded, there exists $M \in \mathbb{R}$ such that for all $x \in E$, $-M \leq x \leq M$. Thus, $E \subseteq [-M, M]$. By Theorem 10.2.14, E is compact. $\square$

For any $E \subseteq \mathbb{R}$, the following statements are equivalent:

1. E is sequentially compact.
2. E is closed and bounded.
3. E is open cover compact.

11

Limits of Functions

In this chapter, we give precise meaning to the familiar notation $\lim_{x\to c} f(x)$, as well as intuition behind the formalization. For example, it is obvious to see that $\lim_{x\to 4}(5x+1) = 21$. We can simply plug in the value 4 for x and attain the answer. Not so obvious, we also have $\lim_{x\to 0} \frac{\sin x}{x} = 1$. In this example, we can't just plug in the value 0 as we can't divide by 0. However, we still have a limit value at the point 0.

We need some notion of f being defined "near" c despite the possibility that it may not actually be defined at c.

11.1 Introduction

We start by introducing the definition of a limit for sort of "well-behaving" functions. It's simple to parse but does not generalize to all real functions.

Definition 11.1.1 ▸ Limit of a Function (for "well-behaving" functions)

Let $c, r, L \in \mathbb{R}$ where $r > 0$, and let $f : B(c, r) \to \mathbb{R}$ be a function. We write $\lim_{x\to c} f(x) = L$ to mean: for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

$$\lim_{x\to c} f(x) = L \iff \forall(\epsilon > 0)\,[\exists(\delta > 0)(0 < |x - c| < \delta \implies |f(x) - L| < \epsilon)]$$

That is, we need to choose $\delta > 0$ so that $f(x)$ is within ϵ of L when x is within δ of c.

Example 11.1.2 ▸ Simple Function Limit Proof

Prove that $\lim_{x\to 4}(5x+1) = 21$.

Intuition: As always for convergence proofs, we let ϵ be some arbitrary real number greater than 0. Now we do some scratch work to find an appropriate δ value.

$$\begin{aligned} |(5x+1)-21| &= |5x-20| \\ &= 5|x-4| \end{aligned}$$

We need $0 < |x-4| < \delta$, so we choose $\delta \leq \epsilon/5$.

Proof. Let $\epsilon > 0$. Choose $\delta := \epsilon/5$. If $0 < |x-4| < \delta$, then:

$$\begin{aligned} |(5x+1)-21| &= |5x-20| \\ &= 5|x-4| \\ &< 5\delta \\ &= \epsilon \end{aligned}$$

which completes the proof. $\square$

Example 11.1.3 ▶ Limit of Piecewise Function

Let $f(x) := \begin{cases} x^2, & x \neq 4 \\ 0, & x = 4 \end{cases}$. Prove that $\lim_{x \to 4} f(x) = 16$.

Intuition: We want $|f(x) - L| < \epsilon$. When $x \neq 4$, we have:

$$\begin{aligned} |f(x)-L| &= |x^2-16| \\ &= |(x+4)(x-4)| \\ &= |x+4||x-4| \end{aligned}$$

We need some estimate for $|x+4|$. If we suppose $|x-4| < 1$, then:

$$-1 < x-4 < 1 \implies 7 < x+4 < 9$$

So $|x+4| < 9$. Thus:

$$|f(x)-L| = |x+4||x-4| < 9|x-4|$$

So we choose $\delta \leq \epsilon/9$.

Proof. Let $\epsilon > 0$. Let $\delta = \min\{1, \epsilon/9\}$. If $0 < |x-4| < \delta$, then $|x-4| < 1$, and $|x-4| < \epsilon/9$. Thus:

$$\begin{aligned} |x-4| < 1 &\implies 3 < x < 5 \\ &\implies 7 < x+4 < 9 \end{aligned}$$

That is, $|x+4| < 9$, so:

$$\begin{aligned} |f(x) - L| &= |x^2 - 16| \\ &= |(x+4)(x-4)| \\ &= |x+4||x-4| \\ &< 9\delta \\ &= \epsilon \end{aligned}$$

which completes the proof. □

Example 11.1.4 ▸ Simple Function Limit Disproof

Let $f(x) \coloneqq \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Intuition: Approaching from the left-hand side, we have $f(x) = -1$, but approaching from the right-hand side, we have $f(x) = 1$.

Proof. Let $\epsilon \coloneqq 1/2$. Suppose for contradiction that $\lim_{x\to 0} f(x) = L$. Then there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $|f(x) - L| < 1/2$. For all $x_1 \in \mathbb{R}$ where $0 < x_1 < \delta$, we have $f(x_1) = 1$, so $|f(x_1) - L| = |1 - L| < 1/2$, so $1/2 < L < 3/2$. For all $x_2 \in \mathbb{R}$ such that $-\delta < x_2 < 0$, we have $f(x_2) = -1$. Thus, $|f(x_2) - L| = |-1 - L|$, so $-3/2 < L < -1/2$. No such L exists, contradicting our supposition that such an L did exist. Therefore, $\lim_{x\to 0} f(x)$ does not exist. □

Example 11.1.5 ▸ Dirichlet Function

Let $f(x) \coloneqq \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. This function has no limit for any $x \in \mathbb{R}$.

Example 11.1.6 ▶ Topologists' Sine Curve

Consider the function $f : B(0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin {}^1\!/\!_x$.

Graph here

$\sin \theta = 0$ when $\theta = \pi, 2\pi, 3\pi, \ldots$. And $\sin {}^1\!/\!_x = 0$ when $x = {}^1\!/\!_\pi, {}^2\!/\!_\pi, {}^3\!/\!_\pi, \ldots$. The limit as x approaches 0 does not exist!.

11.2 Limit Points

Suppose $A = (0, 1) \cup \{2\}$. If a function is defined only on A, then there is no notion of a limit as x approaches 2. The function isn't defined for values "near" 2.

Definition 11.2.1 ▶ Limit Point

Let $A \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is a ***limit point*** of A if there exists a sequence (x_n) contained in $A \setminus \{x\}$ that converges to x. We write A' to denote the set of all limit points of A.

In set A defined above, the number 2 is **not** a limit point. That is, $2 \in A$, but $2 \notin A'$. In fact, $A' = [0, 1]$ because 0 and 1 (and any number in between) are limit points of A. In general, $A \nsubseteq A'$, and $A' \nsubseteq A$.

With this definition, we can give a more generalized definition of a limit.

Definition 11.2.2 ▶ Limit of a Function (in terms of limit points)

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in A'$. We write $\lim_{x \to c} = L$ to mean: for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

$$\lim_{x \to c} = L \iff \forall(\epsilon > 0)\,[\exists(\delta > 0)(0 < |x - c| < \delta \implies |f(x) - L| < \epsilon)]$$

Theorem 11.2.3 ▶ Characterization of Function Limits using Sequences

Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A'$, and $L \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = L$ is equivalent to saying: for all sequences (x_n) contained in $A \setminus \{c\}$ that converge to c, $\lim_{n \to \infty} f(x_n) = L$.

Intuition: This theorem relates the definition of sequential limits with the definition for functional limits.

Proof. First, suppose $\lim_{x \to c} = L$. Let (x_n) be an arbitrary sequence contained in $A \setminus \{c\}$

that converges to c. We prove that the sequence $\{f(x_n)\}$ converges to L. Let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x)-L| < \epsilon$. Since x_n converges to c, there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - c| < \delta$. Note that $x_n \neq c$, so $0 < |x_n - c| < \delta$. Thus, $|f(x_n) - L| < \epsilon$ for all $n \in \mathbb{N}$.

Improve clarity of above proof.

We prove the converse implication by contraposition. Suppose that $\lim_{x\to c} f(x) \neq L$. Then there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists $x \in A$ where $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$. Thus, for each $n \in \mathbb{N}$, there exists $x_n \in A$ where $0 < |x_n - c| < 1/n$ but $|f(x_n) - L| \geq \epsilon$. So (x_n) is contained in $A \setminus \{c\}$ and converges to c. Thus, $\lim_{n\to\infty} f(x_n) \neq L$. ▢

Theorem 11.2.4 ▸ Uniqueness of Function Limits

Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A'$, and $L_1, L_2 \in \mathbb{R}$. If $\lim_{x\to c} L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

Proof. Since $c \in A'$, there exists a sequence (x_n) contained in $A\setminus\{c\}$ that converge to c. By Theorem 11.2.3, we have $\lim_{n\to\infty} f(x_n) = L_1$ and $\lim_{n\to\infty} f(x_n) = L_2$. Since sequential limits must be unique, then $L_1 = L_2$. (todo: ref to sequence limit uniqueness) ▢

Theorem 11.2.5 ▸ Algebraic Properties of Limits

Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, $c \in A'$, $L, M \in \mathbb{R}$, and $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$. Then:

(i) for all $\alpha \in \mathbb{R}$, $\lim_{x\to c} \alpha f(x) = \alpha L$.
(ii) $\lim_{x\to c}(f(x) + g(x)) = L + M$
(iii) $\lim_{x\to c}(f(x)g(x)) = LM$
(iv) if $M \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

Proof of (iii). Let (x_n) be a sequence contained in $A \setminus \{c\}$ that converges to c. Then $\lim_{n\to\infty} f(x_n) = L$. and $\lim_{n\to\infty} g(x_n) = M$. By limit properties for sequences:

$$\lim_{n\to\infty} f(x_n)g(x_n) = \left(\lim_{n\to\infty} f(x_n)\right)\left(\lim_{n\to\infty} g(x_n)\right) = LM$$

▢

By Theorem 11.2.3, $\lim_{x\to c} f(x)g(x) = LM$.

12 Continuity

In calculus classes, we are often taught: "f is continuous at c if $\lim_{x\to c} = f(c)$." This is fine for "well-behaving" functions, but consider a function $f : [0,1] \cup \{2\} \to \mathbb{R}$. It may be tempting to say f is not continuous at 2 because it does not have a limit when x approaches 2. However, for the sake of simplifying future ideas and theorems, we will consider f to be (vacuously) continuous at 2.

Definition 12.0.1 ▸ Isolated Point

Let $A \subseteq \mathbb{R}$. A point $x \in A$ is an ***isolated point*** of A if there exists $r > 0$ such that $B(x,r) \cap A = \{x\}$.

In other words, and isolated point is anything that is not a limit point. For example, in the set $[0,1] \cup \{2\}$, we would consider 2 to be an isolated point.

Lemma 12.0.2 ▸ Limit/Isolated Point Exclusivity

Let $A \subseteq \mathbb{R}$ and $x \in A$. Then x is **either** a limit point of A or isolated point of A.

Proof. Suppose x is not an isolated point of A. Then, for any $n \in \mathbb{N}$, there exists some value $x_n \in A$ such that $x_n \neq x$, and $x_n \in B(x, 1/n)$. Then (x_n) is entirely contained in $A \setminus \{x\}$, and $|x_n - x| < 1/n$ for any $n \in \mathbb{N}$. That is, x_n converges to x. Therefore, x is a limit point of A. ▢

We upgrade the normal calculus definition of continuity by accounting for any potential isolated points.

Definition 12.0.3 ▸ Continuity at a Point

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is ***continuous at*** c if:

1. c is an isolated point of A, or
2. $c \in A'$, $\lim_{x\to c} f(x)$ exists, and $\lim_{x\to c} f(x) = f(c)$.

Theorem 12.0.4 ▸ Equivalent Characterizations of Continuity

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then the following are equivalent:

(a) f is continuous at c.

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

(c) For all sequences (x_n) contained in A that converge to c, $\lim_{n\to\infty} f(x_n) = f(c)$.

Proof sketch. If c is an isolated point of A, then (a) holds. For $\epsilon > 0$, choose $\delta > 0$ such that $B(c,\delta) \cap A = \{c\}$. If $x \in A$ and $|x - c| < \delta$, then $x = c$, so (b) holds. Similarly, if (x_n) is contained in A and converges to c, then $x_n = c$ for some large enough n. Thus, $\lim_{n\to\infty} f(x_n) = f(c)$.

If instead c is a limit point of A, then we can simply prove the following statements:

- (a) $\implies$ (b) by definition (only need to check $|x - c| = 0$)
- (b) $\implies$ (c) similar to proof of sequential characterization of limits
- (c) $\implies$ (a) similar to the above case

$\square$

Theorem 12.0.5 ▸ Continuity Preservation

Let $A \subseteq \mathbb{R}$, $c \in A$, and $f, g : A \to \mathbb{R}$ that are continuous at c. Then:

(a) For all $\alpha \in \mathbb{R}$, αf is continuous at c.

(b) $f + g$ is continuous at c.

(c) fg is continuous at c.

(d) if $g(c) \neq 0$, then f/g is continuous at c.

Proof of (b). If c is an isolated point of A, then $f + g$ is continuous at c, and we are done. Otherwise, c is a limit point. Then:

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c)$$

Therefore, $f + g$ is continuous at c. $\square$

For example, the polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ is continuous at every $c \in \mathbb{R}$. To prove this, we would show:

1. $f(x) = x$ is continuous at every $x \in \mathbb{R}$
2. $f(x) = x^k$ is continuous at every $x \in \mathbb{R}$
3. $f(x) = ax^k$ is continuous at every $x \in \mathbb{R}$

4. $f(x) = \sum a_k x^k$ is continuous at every $x \in \mathbb{R}$

If p and q are polynomials and $q(c) \neq 0$, then the rational function p/q is continuous at $c \in \mathbb{R}$. In other words, rational functions are continuous everywhere in their domain.

Definition 12.0.6 ▶ Continuity on a Set

Let $f : A \to \mathbb{R}$, $B \subseteq A$. We say f is ***continuous on*** B if f is continuous at every $x \in B$.

For example, the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = x$ is continuous on $(0,1)$. Interestingly, this function has neither a maximum nor a minimum on this domain. 0 is the infimum of image of f under $(0,1)$, but 0 can never be attained as a function value. The same can be said about 1 as the supremum of the image of f.

Another example, let $f : (0,1) \to \mathbb{R}$ be a function defined by $f(x) = 1/x$. Then f is continuous on $(0,1)$, but again, there is no minimum nor maximum. This time, we only have an infimum for the image of f under $(0,1)$. There is no upper bound for the function values of f.

If instead f were defined on a closed and bounded (i.e. compact) set, then we would have a minimum and maximum for the function values of f. We prove this in the following theorem.

Theorem 12.0.7 ▶ Extreme Value Theorem

Suppose K is a nonempty and compact subset of $\mathbb{R}$, and suppose $f : K \to \mathbb{R}$ is continuous. Then:

(a) f is bounded on K (that is, $f[K]$ is bounded),

(b) there exists $x_0 \in K$ such that $f(x_0) = \sup(f[K])$

(c) there exists $x_1 \in K$ such that $f(x_1) = \inf(f[K])$

Proof of (a). Suppose for contradiction that f is not bounded on K. Then for each $n \in \mathbb{N}$, there must exist $x_n \in K$ such that $|f(x_n)| > n$. Since $K \subseteq \mathbb{R}$ is compact (and thus sequentially compact), there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to some $x \in K$. Since f is continuous, then the sequence $\{f(x_{n_k})\}$ converges to $f(x)$. Since convergent sequences are bounded, then there exists $M \in \mathbb{R}$ such that $|f(x_{n_k})| \leq M$. This contradicts the fact that $|f(x_{n_k})| > n_k \geq k$. Therefore, f must be bounded on K (i.e. $f[K]$ is bounded). ▢

Proof of (b). By (a), we know $f[K]$ is bounded. Since $f[K]$ is also nonempty, then completeness guarantees that $f[K]$ has a supremum in $\mathbb{R}$. By Problem Set 6 # 8, there exists a sequence in $f[K]$ that converges to $\sup(f[K])$. That is, there exists a sequence (x_n)

contained in K where the sequence $\{f(x_n)\}$ converges to $\sup(f[K])$. Since K is sequentially compact, there exists a subsequence (x_{n_k}) of (x_n) such that x_{n_k} converges to some $x_0 \in K$. By continuity:

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = \sup f[K]$$

▢

Theorem 12.0.8

Suppose $O \subseteq \mathbb{R}$ is open and $f : O \to \mathbb{R}$. Then f is continuous on O if and only if, for every open set $U \subseteq \mathbb{R}$, $f[U^{-1}]$ is open.

12.1 Uniform Continuity

Recall from Theorem 12.0.4 where we described equivalent characterizations of continuity, we can say $f : A \to \mathbb{R}$ is continuous at $c \in A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in A$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. That is:

$$\forall(\epsilon > 0)\exists(\delta > 0)\forall(x \in A)(|x - c| < \delta \implies |f(x) - f(c)| < \epsilon)$$

The δ value in the above description of continuity depends on not only on f and ϵ, but also the value of c. Our current idea of continuity is a very local property; the choice of δ can vary greatly. Uniform continuity extends this idea by "unfixing" that c value. That is, we try to say that the function has the same degree of continuity at every point, so one choice of δ works for all points on the function.

Would be nice to have the graphic from inclass notes

Definition 12.1.1 ▸ Uniform Continuity

Let $f : A \to \mathbb{R}$ be a function. We say f is ***uniformly continuous*** on A if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for every $x, y \in A$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

$$\forall(\epsilon > 0)\exists(\delta > 0)\forall(x, y \in A)\,(|x - y| < \delta \implies |f(x) - f(y)| < \epsilon)$$

Once again, the choice of δ depends on f and ϵ, but not any specific point in the domain.

Example 12.1.2 ▸ Simple Uniform Continuity Proof

$f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Proof. Let $\epsilon > 0$. Choose $\delta := \epsilon$. Let $x, y \in \mathbb{R}$ (the domain of f). If $|x - y| < \delta$, then:

$$|f(x) - f(y)| = |x - y| < \delta = \epsilon$$

Therefore, f is uniformly continuous. ▢

Example 12.1.3 ▸ Simple Uniform Continuity Disproof

$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Intuition: We may think that x^2 would be a "well-behaving" function, but since its graph gets steeper, we would have to adjust our δ depending on which point on the graph we chose. The further out we go, we can find huge jumps in the function value for tiny steps in the x values.

Proof. Let $\epsilon := 1$, and let $\delta > 0$. Choose $x := 2/\delta > 0$ and $y := x + \delta/2 > 0$. Then $|x - y| = \delta/2 < \delta$, and:

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| > |x - y|(x) = \frac{2}{\delta} \cdot \frac{\delta}{2} = 1 = \epsilon$$

Therefore, f is not uniformly continuous. ▢

Theorem 12.1.4

Let K be a compact subset of $\mathbb{R}$, and let $f : K \to \mathbb{R}$ be a continuous function on K. Then f is uniformly continuous on K.

Proof. Let $\epsilon > 0$. Since f is continuous on K, then for any $z \in K$, there exists $\delta_z > 0$ such that for any $x \in K$, if $|x - z| < \delta_z$, then $|f(x) - f(z)| < \epsilon/2$. Let $I_z := B(z, \delta_z/2) = (z - \delta_z/2, z + \delta_z/2)$. Since it is open and $K \subseteq \bigcup_{z \in K} I_z$, then $\{I_z : z \in K\}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{I_{z_1}, I_{z_2}, \ldots, I_{z_n}\}$ for some $n \in \mathbb{N}$. So we have n different radii we can choose from. Let $\delta := \min\left\{\frac{\delta_{z_1}}{2}, \frac{\delta_{z_2}}{2}, \ldots, \frac{\delta_{z_n}}{2}\right\}$. Let $x, y \in K$ such that $|x - y| < \delta$.

> It is important that we chose δ before choosing x and y.

Since $x \in K \subseteq \bigcup_{i=1}^{n} I_{z_i}$, there exists $j \in \{1, 2, \dots, n\}$ such that $x \in I_{z_j}$. Also, $|x - y| < \delta < \frac{\delta_{z_j}}{2}$. Thus:

$$|y - z_j| = |y - x + x - z_j| \leq |y - x| + |x - z_j| < \frac{\delta_{z_j}}{2} + \frac{\delta_{z_j}}{2} = \delta_{z_j}$$

That is, $x, y \in B(z_j, \delta_{z_j})$.

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(z_j) + f(z_j) - f(y)| \\ &\leq |f(x) - f(z_j)| + |f(z_j) - f(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

which completes the proof. ▢

13 Differential Calculus

Definition 13.0.1 ▶ Differentiable, Derivative

Let $a, b \in \mathbb{R}$ where $a < b$, let $f : (a, b) \to \mathbb{R}$ be a function, and let $x_0 \in (a, b)$.

- We say f is ***differentiable at*** x_0 if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists.
- We say f is ***differentiable on*** I if f is differentiable at every $x \in I$.
- If this limit exists, we define the ***derivative*** of f as $f'(x_0) \coloneqq \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

We can also write the derivative as $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$. In this context, we replace x with $x_0 + h$. This is usually the more familiar form and is referred to as the ***difference quotient***. Without the limit, the difference quotient by itself gives us the slope of the line from $(x_0, f(x_0))$ to $(x_0 + h, f(x_0 + h))$. With the limit, it gives us the slope of the line tangent to f at x_0.

We can think of the derivative $f'(x)$ as:

- definition: the limit of the difference quotient
- graphical: slope of the tangent line
- interpretation: instantaneous rate of change

Example 13.0.2 ▶ Simple Derivative Example

Given $f(x) = x^2$, find $f'(x_0)$.

If $x \neq x_0$, then:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x + x_0)(x - x_0)}{x - x_0} = x + x_0$$

Thus:

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = x_0 + x_0 = 2x_0$$

Theorem 13.0.3 ▸ Differentiability Implies Continuity

If f is differentiable at x_0, then f is continuous at x_0.

Proof. If $x \neq x_0$, then $f(x) = f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x - x_0)$. Thus:

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\left(f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right) \\
&= \left(\lim_{x\to x_0} f(x_0)\right) + \left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right)\left(\lim_{x\to x_0}(x-x_0)\right) \\
&= f(x_0) + f'(x_0)\cdot 0 \\
&= f(x_0)
\end{aligned}$$

Therefore, f is continuous at x_0. ▢

As we'll see in the next example, the converse statement is not true. That is, continuity does not generally imply differentiability.

Example 13.0.4 ▸ Continuity does not imply differentiability

$f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Proof. We first show f is continuous at $x = 0$. We have $f(0) = 0$, and:

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0}|y| \\
&= 0
\end{aligned}$$

Now to show it is not differentiable, if $x \neq 0$, we have:

$$\frac{f(x)-f(0)}{x-0} = \frac{absx - 0}{x-0} = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Then:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}|x|y$$

so its limit as x approaches 0 does not exist. Therefore, f is not differentiable at $x = 0$. ▢

Example 13.0.5 ▸ Piecewise Differentiability Example

Let $f(x) \coloneqq \begin{cases} x^2 \sin {}^1\!/\!_x, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Is f differentiable at $x = 0$?

It turns out that f is differentiable at $x = 0$! However, it may be tempting to give the following **incorrect** proof (assuming we already have the chain rule and product rule):

Incorrect proof. If $x \neq 0$:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

This has no limit as x approaches 0, so $\lim_{x\to 0} f'(x)$ does not exist. $\square$

The above approach erroneously hinges on the assumption that the derivative must be continuous (which is not generally true). We must instead use the definition of differentiability.

Correct proof. If $x \neq 0$:

$$\begin{aligned} \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{x^2 \sin {}^1\!/\!_x}{x} \\ &= \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \\ &= 0 \end{aligned}$$

Therefore, f is differentiable at $x = 0$, and $f'(0) = 0$. $\square$

This function f is differentiable for every $x \in \mathbb{R}$, but $\lim_{x\to 0} f'(x)$ does not exist! So we have shown f' is not continuous at $x = 0$.

Theorem 13.0.6 ▸ Properties of Differentiation

Suppose $f, g : (a, b) \to \mathbb{R}$ are differentiable at $x_0 \in (a, b)$. Let $c \in \mathbb{R}$. Then cf, $f + g$, and fg are differentiable at x, and if $g'(x) \neq 0$, then ${}^f\!/\!_g$ is differentiable. Moreover:

(a) $(cf)'(x_0) = cf'(x_0)$

(b) $(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(c) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

(d) $({}^f\!/\!_g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$

Proof. To prove (a):

$$\begin{aligned}
(cf)'(x_0) &= \lim_{x\to x_0} \frac{cf(x) - cf(x_0)}{x - x_0} \\
&= c \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= cf'(x)
\end{aligned}$$

To prove (b):

$$\begin{aligned}
(f+g)'(x) &= \lim_{x\to x_0} \frac{(f(x) + g(x)) - (f(x_0) + g(x_0))}{x - x_0} \\
&= \lim_{x\to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right] \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(x_0) + g'(x_0)
\end{aligned}$$

To prove (c):

$$\begin{aligned}
(fg)'(x) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot g(x) + f(x) \cdot \frac{g(x) - g(x_0)}{x - x_0} \right] \\
&= \dots
\end{aligned}$$

Since f and g were assumed to be differentiable (and thus continuous at x_0), we can apply properties of limits to finally attain:

$$f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

□

Theorem 13.0.7 ▶ Chain Rule

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ be arbitrary functions. If f is differentiable at some $x \in (a, b)$ and g is differentiable at $f(x) \in (c, d)$, then $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at x, and:

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

Intuition: When taking $(g \circ f)'$, there are two rates of the change to consider: f' and g', which "compound" one another.

Proof sketch.

$$\begin{aligned}(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \right)\end{aligned}$$

The idea is the first fraction approaches $g'(f(x_0))$, and the second fraction approaches $f'(x_0)$. However, if $f(x) - f(x_0) = 0$, then the first fraction is invalid. To circumvent this, we can redefine differentiability as a multiplicative property. Precisely, we can say a function f is ***differentiable*** at x to mean:

$$f(x + h) - f(x) = f'(x) \cdot h + \epsilon(h) \cdot h$$

where $\epsilon(h)$ approaches 0 as h approaches 0. Intuitively, this definition verifies that we can well approximate the function at that point using a linear function. The $\epsilon(h) \cdot h$ term denotes the error in the linear approximation, which should become negligible ▢

Definition 13.0.8 ▶ Local/Global Maxima/Minima (Extreme Values)

Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, and $f : I \to \mathbb{R}$ be a function. We say f has a:

- ***local maximum*** at x_0 if there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) \cap I$, $f(x) \leq f(x_0)$.
- ***local minimum*** at x_0 if there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) \cap I$, $f(x) \geq f(x_0)$.
- ***global maximum*** at x_0 if for all $x \in I$, $f(x) \leq f(x_0)$.
- ***global minimum*** at x_0 if for all $x \in I$, $f(x) \geq f(x_0)$.

Theorem 13.0.9 ▸ Fermat's Theorem

Let $f : I \to \mathbb{R}$ be a function. If f has a local minimum or local maximum at $x_0 \in I$, then either:

(a) x_0 is an endpoint of I, or

(b) f is not differentiable at x_0, or

(c) f is differentiable at x_0, and $f'(x_0) = 0$.

Proof. Suppose f has a local maximum at x_0. Then there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) \cap I$, $f(x) \leq f(x_0)$. We prove that, if neither (a) nor (b) are true, then (c) must be true. Suppose x_0 is not an endpoint of I, and suppose that f is differentiable at x_0. Let $x \in B(x_0, \delta) \cap I$ be arbitrary.

- If $x > x_0$, then $x - x_0 > 0$ and $f(x) - f(x_0) \leq 0$. Hence, $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$, so $f'(x_0) = \lim_{x \to 0} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$.
- If $x < x_0$, then $x - x_0 < 0$ and $f(x) - f(x_) \leq 0$. Hence, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$, so $f'(x_0) = \lim_{x \to 0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$.

By trichotomy, $f'(x_0) = 0$. ▢

Theorem 13.0.10 ▸ Rolle's Theorem

Let $a, b \in \mathbb{R}$ where $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = 0$ and $f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since $[a, b]$ is compact and f is continuous, the Extreme Value Theorem states that f attains both its maximum and minimum on $[a, b]$.

- If both the maximum and minimum of f occur at the endpoints a and b, then maximum and minimum of $f[(a, b)]$ is 0. Thus, $f(x) = 0$ for all $x \in [a, b]$. Thus, $f'(x) = 0$ for all $x \in (a, b)$, so we can take c to be any value in (a, b).
- Otherwise, either the maximum or the minimum occurs at some point $c \in (a, b)$. By Fermat's Theorem, we have $f'(c) = 0$.

Since the above cases are exhaustive, the proof is complete. ▢

Theorem 13.0.11 ▸ Mean Value Theorem

Let $a, b \in \mathbb{R}$ where $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $l : [a,b] \to \mathbb{R}$ be the function of the line through $(a, f(a))$ and $(b, f(b))$. That is, for any $x \in [a,b]$:

$$l(x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Note that $l'(x) = \frac{f(b)-f(a)}{b-a}$. Let $g : [a,b] \to \mathbb{R}$ be defined for every $x \in [a,b]$ by:

$$g(x) := f(x) - l(x)$$

Then g is continuous on $[a,b]$, and g is differentiable on (a,b). Also note $g(a) = 0$ and $g(b) = 0$. By Rolle's Theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$. We then have:

$$0 = g'(c) = f'(c) - l'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Adding across by $\frac{f(b)-f(a)}{b-a}$, we have $f'(c) = \frac{f(b)-f(a)}{b-a}$. ▢

The Mean Value Theorem has tons of application in both calculus and real analysis.

Example 13.0.12 ▸ Positive derivative means increasing

If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).

Intuition: This seems like a fairly obvious result, but to prove it rigorously, we can apply the Mean Value Theorem.

Proof. If $a < x < y < b$, then there exists $c \in (a,b)$ where $\frac{f(y)-f(x)}{y-x} = f'(c)$. Thus, $f(y) - f(x) > 0$ for any choice of $x, y \in (a,b)$ where $y > x$. Therefore, f is strictly increasing. ▢

14 Riemann Integration

Recall that the main motivating problem in Calculus II was to find the area under the graph of a function $y = f(x)$ on some interval $[a, b]$. The idea was to take a lot of rectangles reaching from the x-axis to points of the graph, then combine the areas of those rectangles to create a rough estimate. As we chop the graph up into more rectangles, we get closer and closer to the actual area under the graph.

14.1 Riemann Sums

Definition 14.1.1 ▸ Partition

A ***partition*** P of $[a, b]$ is an ordered set of points $P := \{x_0, x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

Graphic

For $j \in \{1, 2, \dots, n\}$, we define $I_j := [x_{j-1}, x_j]$ and $\Delta x_j := x_j - x_{j-1}$. Then we have:

$$\sum_{j=1}^{n} x_j = b - a$$

Definition 14.1.2 ▸ Riemann Sum

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and P is a partition of $[a, b]$ with n pieces (i.e. $P = \{x_0, x_1, \dots, x_n\}$). For $j \in \{1, 2, \dots, n\}$, define:

$$\begin{aligned} m_j(f, P) &:= \inf\big(f[I_j]\big) = \inf\{f(x) : x \in I_j\} \\ M_j(f, P) &:= \sup\big(f[I_j]\big) = \sup\{f(x) : x \in I_j\} \end{aligned}$$

The ***lower Riemann sum*** is defined as:

$$L(f,P) \coloneqq \sum_{j=1}^{n} m_j(f,P) \cdot \Delta x_j$$

The ***upper Riemann sum*** is defined as:

$$U(f,P) \coloneqq \sum_{j=1}^{n} M_j(f,P) \cdot \Delta x_j$$

Definition 14.1.3 ▸ Refinement

Let $P \coloneqq \{x_0, x_1, \dots, x_n\}$ and $Q \coloneqq \{y_0, y_1, \dots, y_m\}$ be partitions of $[a,b]$. We say Q is a ***refinement*** of P if $P \subseteq Q$, in which case $n \leq m$.

That is, for all $j \in \{1, 2, \dots, n\}$, there exists $k_j \in \mathbb{N}$ such that $x_j = y_{k_j}$. Note that $x_0 = y_0$, so $k_0 = 0$.

Lemma 14.1.4

Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and P and Q are partitions of $[a,b]$ where Q is a refinement of P. Then $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

Intuition: When we refine our partition P, the lower sum increases, and the upper sum decreases. The infimum can only increase, and the supremum can only decrease.

Proof. Let $P \coloneqq \{x_0, x_1, \dots, x_n\}$ and $Q \coloneqq \{y_0, y_1, \dots, y_m\}$. Then:

$$\begin{aligned}
U(f,Q) &= \sum_{l=1}^{m} M_l(f,Q) \cdot \Delta y_l \\
&= \sum_{j=1}^{n} \left(\sum_{l=k_{j-1}+1}^{k_j} M_l(f,Q) \cdot \Delta y_l \right) \\
&\leq \sum_{j=1}^{n} \left(\sum_{l=k_{j-1}+1}^{k_j} M_j(f,P) \cdot \Delta y_l \right) \\
&= \sum_{j=1}^{n} \left(M_j(f,P) \sum_{l=k_{j-1}+1}^{k_j} \Delta y_l \right) \\
&= \sum_{j=1}^{n} M_j(f,P) \cdot \Delta x_j \\
&= U(f,P)
\end{aligned}$$

Note that:

$$\begin{aligned}
M_l(f,Q) &= \sup\{f(x) : y_{l-1} \leq x \leq y_l\} \\
&\leq \sup\{f(x) : x_{j-1} \leq x \leq x_j\} \\
&= M_j(f,P)
\end{aligned}$$

□

Lemma 14.1.5 ▸ Lower sums are always smaller than upper sums

Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let P and Q be partitions of $[a,b]$. Then $L(f,P) \leq U(f,Q)$.

Intuition: Any lower Riemann sum is smaller than (or equal to) any upper Riemann sum.

Proof. Note that the set $P \cup Q$ is a refinement of both P and Q (because $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$). By Lemma 14.1.4:

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q)$$

which completes the proof. ▢

14.2 Riemann Integration

Definition 14.2.1 ▸ Riemann Integral

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. We define ***lower Riemann integral*** of f on $[a,b]$ as:

$$L(f) \coloneqq \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

We define ***upper Riemann integral*** of f on $[a,b]$ as:

$$U(f) \coloneqq \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

Note that for any partitions P and Q of $[a,b]$, $L(f,P) \le U(f,Q)$. Taking the supremum on the left:

$$L(f) \le U(f,Q)$$

Taking the infimum on the right:

$$L(f) \le U(f)$$

fix wording of "taking the supremum"

Definition 14.2.2 ▸ Riemann Integrable

We say f is ***Riemann integrable*** if on $[a,b]$ if $L(f) = U(f)$, in which case we define:

$$\int_a^b f(x)\,dx \coloneqq L(f) = U(f)$$

For any set A, we write $\mathcal{R}(A)$ to denote the set of all Riemann integrable functions with domain A.

Example 14.2.3 ▸ Simple Riemann Integration

Show that $x^2 \in \mathcal{R}([0,1])$ (i.e. x^2 is Riemann integrable on $[0,1]$), and find $\int_0^1 x^2\,dx$.

Proof. For any $n \in \mathbb{N}$, consider the "regular partition" P_n of $[0,1]$ where:

$$P_n := \{0/n, 1/n, 2/n, \dots, n/n\}$$

In other words, the regular partition evenly splits the interval $[0,1]$. Then $x_0 = 0$, and for any $j \in \{1, 2, \dots, n\}$, we have $x_j = j/n$ and $\Delta x_j = 1/n$. Also:

$$m_j(f, P_n) := \inf(f[I_j]) = \left(\frac{j-1}{n}\right)^2$$

$$M_j(f, P_n) := \sup(f[I_j]) = \left(\frac{j}{n}\right)^2$$

Thus:

$$\begin{aligned} L(f, P_n) &= \sum_{j=1}^{n} m_j(f, P_n) \cdot \Delta x_j \\ &= \sum_{j=1}^{n} \left(\frac{j-1}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{j=1}^{n} (j-1)^2 \\ &= \frac{1}{n^3} \sum_{j=0}^{n-1} j^2 \\ &= \frac{1}{n^3} \sum_{j=1}^{n-1} j^2 \end{aligned}$$

$$\begin{aligned} U(f, P_n) &= \sum_{j=1}^{n} M_j(f, P_n) \cdot \Delta X_j \\ &= \sum_{j=1}^{n} \left(\frac{j}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{j=1}^{n} j^2 \end{aligned}$$

Recall from Problem Set 6 (todo: ref) that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. Therefore:

$$L(f, P_n) = \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

$$U(f, P_n) = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Now:

$$L(f, P_n) \le \sup\{L(f, P) : P \text{ is a partition of } [0,1]\} = L(f)$$

$$U(f, P_n) \ge \inf\{U(f, P) : P \text{ is a partition of } [0,1]\} = U(f)$$

so:

$$\lim_{n\to\infty} L(f, P_n) \le L(f)$$

$$\lim_{n\to\infty} U(f, P_n) \ge U(f)$$

and hence:

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

Thus, $U(f) \le 1/3 \le L(f) \le U(f)$, so $U(f) = L(f) = 1/3$. Therefore, $x^2 \in \mathcal{R}([0,1])$, and $\int_0^1 x^2\, dx = 1/3$. ▢

Lemma 14.2.4 ▸ Criterion for Riemann Integrability

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if, for all $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

$$f \in \mathcal{R}([a,b]) \iff \forall(\epsilon > 0)\exists(P)\,[U(f,P) - L(f,P) < \epsilon]$$

Proof. Suppose that the right-hand side of the equivalence holds. Then for each $n \in \mathbb{N}$, there exists a partition P_n of $[a,b]$ such that $U(f,P_n) - L(f,P_n) < 1/n$. Then, for any $n \in \mathbb{N}$:

$$U(f) \le U(f,P_n) < L(f,P_n) + 1/n \le L(f) + 1/n$$

That is, $U(f) < L(f) + 1/n$ for any $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we get $U(f) \le L(f)$. Since $L(f) \le U(f)$, then by trichotomy, $L(f) = U(f)$. Thus, f is Riemann integrable on $[a,b]$.

Suppose that the left-hand side of the equivalence holds (that is, f is Riemann integrable). Then $L(f) = U(f)$. Let $\epsilon > 0$.

> Our goal here is to find a partition P such that $U(f, P) - L(f, P) < \epsilon$.

Since $L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$, there exists a partition Q_1 of $[a, b]$ such that $L(f, Q_1) > L(f) - \epsilon/2$. This is guaranteed by the approximation property (todo: ref). Similarly, since $U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$, there exists a partition Q_2 of $[a, b]$ such that $U(f, Q_2) < U(f) + \epsilon/2$. Let $Q := Q_1 \cup Q_2$, which is the common refinement of Q_1 and Q_2. Then, from Lemma 22.4 (todo: ref):

$$\underbrace{U(f, Q) \leq U(f, Q_2)}_{Q \text{ is a refinement of } Q_2} < \underbrace{U(f) + \frac{\epsilon}{2} = L(f) + \frac{\epsilon}{2}}_{U(f)=L(f)} < \underbrace{\left(L(f, Q_1) + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} \leq L(f, Q) + \epsilon}_{Q \text{ is a refinement of } Q_1}$$

Subtracting across by $L(f, Q)$, we have $U(f, Q) - L(f, Q) < \epsilon$. ▢

Theorem 14.2.5 ▸ Continuous Functions are Riemann Integrable

Let $f : [a, b] \to \mathbb{R}$ be a function. If f is continuous, then f is Riemann integrable.

Proof. Let $\epsilon > 0$. We will find a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Since the interval $[a, b]$ is a compact set and f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$ and bounded on $[a, b]$ (todo: ref proposition 20.9 and proposition 19.13). Hence, there exists $\delta > 0$ such that, for all $x, y \in [a, b]$ where $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Let $P := \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ such that, for each $j \in \{1, 2, \dots, n\}$, $\Delta x_j < \delta$.

> To be clear, we can say "chose a P such that …", but there might not exist such a P. To be especially clear that this choice of P exists, consider the regular partition of $[a, b]$ where $n \in \mathbb{N}$ satisfies $\frac{b-a}{n} < \delta$.

Then:

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{j=1}^{n} M_j(f, P)\Delta x_j - \sum_{j=1}^{n} m_j(f, P)\Delta x_j \\ &= \sum_{j=1}^{n} \left(M_j(f, P) - m_j(f, P)\Delta x_j\right) \end{aligned}$$

Since $[x_{j-1}, x_j]$ is compact and f is continuous on $[x_{j-1}, x_j]$, then by the Extreme Value Theorem, there exists $a_j, b_j \in [x_{j-1}, x_j]$ such that $f(a_j) = m_j(f,P)$, and $f(b_j) = M_j(f,P)$. Then $|b_j - a_j| \leq |x_j - x_{j-1}| < \delta$, so $|f(b_j) - f(a_j)| < \frac{\epsilon}{b-a}$. Thus:

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{j=1}^{n} \left(M_j(f,P) - m_j(f,P)\Delta x_j\right) \\
&= \sum_{j=1}^{n} \left(f(b_j) - f(a_j)\right)\Delta x_j \\
&< \sum_{j=1}^{n} \left(\frac{\epsilon}{b-a}\right)\Delta x_j \\
&= \frac{\epsilon}{b-a}\sum_{j=1}^{n} \Delta x_j \\
&= \frac{\epsilon}{b-a}(b-a) \\
&= \epsilon
\end{aligned}$$

□

Example 14.2.6 ▸ Dirichlet Function is not Riemann Integrable

Define $f : [0,1] \to \mathbb{R}$ by:

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q} \cup [0,1] \\ 0, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cup [0,1] \end{cases}$$

Show that f is not Riemann integrable.

Proof. Let $P := \{x_0, x_1, \dots, x_n\}$ be a partition of $[0,1]$. Note that on any interval $[x_{j-1}, x_j]$, there exists a rational number y and irrational number z such that $f(y) = 1$ and $f(z) = 0$. Hence, $m_j(f,P) = 0$ and $M_j(f,P) = 1$, so:

$$L(f,P) = \sum_{j=1}^{n} m_j(f,P)\Delta x_j = \sum_{j=1}^{n} 0 = 0$$

$$U(f,P) = \sum_{j=1}^{n} M_j(f,P)\Delta x_j = \sum_{j=1}^{n} \Delta x_j = 1 - 0 = 1$$

Thus, $L(f) = 0$ and $U(f) = 1$, so f is not Riemann integrable. ▢

14.3 Fundamental Theorem of Calculus

Theorem 14.3.1 ▸ Fundamental Theorem of Calculus (Part I)

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function, and let $F : [a, b] \to \mathbb{R}$ be a function that's continuous on $[a, b]$, differentiable on (a, b), and for any $x \in (a, b)$, $F'(x) = f(x)$. Then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof. Let $P \coloneqq \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then:

$$\begin{aligned} F(b) - F(a) &= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \cdots + (F(x_1) - F(x_0)) \\ &= \sum_{j=1}^{n} \big(F(x_j) - F(x_{j-1})\big) \end{aligned}$$

Since we assumed that F is continuous on $[x_{j-1}, x_j]$ and differentiable on (x_{j-1}, x_j), we can apply the Mean Value Theorem to find some $c_j \in (x_{j-1}, x_j)$ such that:

$$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1}) = f(c_j)(x_j - x_{j-1}) = f(c_j)\Delta x_j$$

We can apply this in our initial calculation as follows:

$$F(b) - F(a) = \sum_{j=1}^{n} \big(F(x_j) - F(x_{j-1})\big) = \sum_{j=1}^{n} f(c_j)\Delta x_j$$

Now:

$$L(f, P) = \sum_{j=1}^{n} m_j(f, P)\Delta x_j \le \sum_{j=1}^{n} f(c_j)\Delta x_j \le \sum_{j=1}^{n} M_j(f, P)\Delta x_j = U(f, P)$$

That is, $L(f, P) \le F(b) - F(a) \le U(f, P)$ for **any** partition P of $[a, b]$. Hence:

$$\sup\{L(f, P) : P \text{ partitions } [a, b]\} \le F(b) - F(a) \le \inf\{U(f, P) : P \text{ partitions } [a, b]\}$$

Thus:

$$L(f) \le F(b) - F(a) \le U(f)$$

Since we assumed f to be Riemann integrable, then $L(f) = U(f) = \int_a^b f(x)\,dx$. Therefore:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

□

In loose terms, the above theorem states that integration cancels out differentiation. What happens if we differentiate an integral?

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x)\,dx = \frac{\mathrm{d}}{\mathrm{d}x}(\text{some number}) = 0$$

Theorem 14.3.2 ▸ Fundamental Theorem of Calculus (Part II)

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. For all $x \in [a, b]$, define:

$$F(x) := \int_a^x f(t)\,dt$$

If $x_0 \in (a, b)$ and f is continuous at x_0, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Intuition: We may be more familiar with a slight rewriting of the theorem which states:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,dt = f(x)$$

Proof sketch.

$$\begin{aligned} F(x) - F(x_0) &= \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt \\ &= \int_{x_0}^x f(t)\,dt \\ &\approx f(x_0)(x - x_0) + \mathcal{O}\left((x - x_0)^2\right) \end{aligned}$$

Dividing across by $x - x_0$, we have:

$$\frac{F(x) - F(x_0)}{x - x_0} \approx f(x_0) + \mathcal{O}(x - x_0)$$

If we take the limit as x approaches x_0, we have:

$$\begin{aligned} F'(x_0) &= \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \left(f(x_0) + \mathcal{O}(x - x_0)\right) \\ &= f(x_0) \end{aligned}$$

$\square$

Example 14.3.3 ▸ Picard Method Revisited

Solve $y'(x) = f(x, y(x))$ where $y_0 := y(x_0)$.

Introduce a "dummy variable," say t:

$$y'(t) = f(t, y(t))$$

Integrate from x_0 to x:

$$\int_{x_0}^{x} y'(t)\, dt = \int_{x_0}^{x} f(t, y(t))\, dt$$

Apply the Fundamental Theorem of Calculus (Part I):

$$\begin{aligned} y(x) - y(x_0) &= \int_{x_0}^{x} f(t, y(t))\, dt \\ \implies \qquad y(x) &= y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \end{aligned}$$

Define an "integral operator," say T. For a function z, let $Tz(x) := z(x_0) + \int_{x_0}^{x} f(t, z(t))\, dt$. Note that $y(x)$ satisfies $Ty(x) = y(x)$, so we call y a "fixed point" of the operator T. Find a fixed point of T (i.e. find y such that $Ty = y$).

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