Chapter 1

Turing Machines

The ***Turing machine*** is a much more powerful and accurate model for describing general purpose computers. It can solve any problem that a real computer can do. However, there are still some problems that a Turing machine cannot solve.

Turing machines model memory with a sequence of 0s and 1s called a ***tape***. The machine has a "tape head" which can move across the tape, reading or writing one bit at a time.

Definition 1.0.1 ▸ Turing Machine (TM)

A ***Turing machine (TM)*** is an abstract model that can generalize any computational problem or task.

More formally, a ***Turing machine*** is defined as a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ where:

- Q is a finite set of states,
- Σ is the input alphabet not containing the ***blank symbol*** $\textvisiblespace$,
- Γ is the ***tape alphabet*** where $\textvisiblespace \in \Gamma$ and $\Sigma \subseteq \Gamma$,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function,
- $q_0 \in Q$ is the start state,
- $q_a \in Q$ is the accept state, and
- $q_r \in Q$ is the reject state where $q_a \neq q_r$.

A Turing machine M computes as follows:

1. M receives an input string $w := w_1 \dots w_n \in \Sigma^*$ on the leftmost n squares of the tape, and the rest of the tape is blank.
2. The tape head starts on the leftmost square of the tape. Since Σ does not contain the

blank symbol, the first blank the machine sees marks the end of the input.

3. The machine computes according to the transition function δ. If it tries to move the tape head left off the left-end of the tape, it just stays in the same place for that move.
4. Computation continues until it reachers either the accept or reject state, at which point it halts.

As a Turing machine computes, three things may change: the current state, current tape contents, and current head location. These are collectively referred to as the ***configuration*** of the Turing machine.

Definition 1.0.2 ▸ Turing Machine Configuration

Given a Turing machine, its ***configuration*** refers to its current state, current tape contents, and current head location. For a state q and two strings u and v, we write $u\ q\ v$ to denote the configuration where the current state is q, the current tape contents are uv, and the current head location is the first symbol of v.

Definition 1.0.3 ▸ Yield

Given a Turing machine and two configurations C_1 and C_2, we say C_1 ***yields*** C_2 if the machine can legally move from C_1 to C_2 in a single step.

- The ***start configuration*** using input w and the initial state is q_0 is $\epsilon q_0 w$.
- A ***accepting configuration*** is any configuration with state q_a.
- A ***rejecting configuration*** is any configuration with state q_r.
- Accepting and rejecting configurations are called ***halting configurations*** and cannot yield further configurations.

Definition 1.0.4 ▸ Decider

We say a Turing machine is a ***decider*** if it halts on all possible inputs.

Definition 1.0.5 ▸ Turing-decidable, Turing-recognizable

We say a language is ***Turing-decidable*** if there exists some Turing machine that decides it. We say a language is ***Turing-recognizable*** if there exists some Turing machine that

recognizes it.