Calculus III

UT Knoxville, Spring 2023, MATH 341

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January 30, 2023

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Introduction

Much of our focus will be on Stoke's Theorem.

Three-Dimensional Space

In past math classes, we have been used to dealing in $\mathbb{R}^2$ where we work with two degrees of freedom: x and y. Now, we will be working in $\mathbb{R}^3$ with three degrees of freedom: x, y, and z.

2.1 Points

Definition 2.1.1 ▶ Point

A **point** in $\mathbb{R}^n$ space is an n-tuple that specifies a location in that space.

$$p = (p_1, \dots, p_n) \in \mathbb{R}^n$$

Definition 2.1.2 ▶ Distance

Given two points $a, b \in \mathbb{R}^n$, the **distance** between the two points is defined as:

$$d(a, b) := \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$$

Example 2.1.1 ▶ Distance Between Points

Find the distance between $p_1 = (-1, -1, 4)$ and $p_2 = (-1, 4, -1)$.

$$\begin{aligned} d(p_1, p_2) &= \sqrt{(-1 - (-1))^2 + (4 - (-1))^2 + (-1 - 1)^2} \\ &= \sqrt{0^2 + 5^2 + (-5)^2} \\ &= \sqrt{50} \end{aligned}$$

Definition 2.1.3 ▶ Sphere

Given a point $c = (h, k, l) \in \mathbb{R}^3$, a **sphere** is the set of all points $(x, y, z) \in \mathbb{R}^3$ that are a distance r from the point $c = (h, k, l)$.

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Note that all the points of the sphere are equidistant to the center of the sphere. This means the sphere is really a hollow shell.

Example 2.1.2 ▶ Circle

Show that the following quadratic equation represents a circle by rewriting it in standard form. Find the center $c = (h, k)$ and the radius r.

$$x^2 + y^2 + x = 0$$

To solve this, we will have to complete the square:

$$x^2 + x + y^2 = 0$$
$$\implies x^2 + x + \frac{1}{4} + y^2 = \frac{1}{4}$$
$$\implies \left(x + \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

Definition 2.1.4 ▶ Cylinder

Given a planar curve c, the surface in $\mathbb{R}^3$ defined by all parallel lines crossing the curve c is called a **cylinder**.

Similarly, the set of all points (x, y, z) such that x and y satisfy $(x - h)^2 + (y - k)^2 = r^2$ forms a circular cylinder. Note that our broad definition of cylinder does not require the cylinder to be circular.

Regarding the quadrants:

$$Q1 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$$
$$Q4 = \{(x, y) \in \mathbb{R}^2 : x > 0, y < 0\}$$

2.2   Vectors

Definition 2.2.1 ▶ Vector

A **vector** is a mathematical object that contains multiple objects of the same type.

$$\vec{v} = \langle v_1, \ldots, v_n \rangle \in \mathbb{R}^n$$

As customary in most mathematics textbooks, we will always denote vectors using the little arrow thing. In the context of three-dimensional space, we will only be working with vec-

tors with three components. In addition, we will think of vectors as having a magnitude and direction.

> **Definition 2.2.2 ▸ Scalar Multiplication**
>
> Given a vector $\vec{v}$ and scalar k, we define **scalar multiplication** as:
>
> $$k \cdot \vec{v} := \langle kv_1, \dots, kv_n \rangle$$

Note that scalar multiplication is associative, commutative, and distributive.

- $a(b\vec{v}) = b(a(\vec{v})) = (ab)\vec{v}$

- $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$

- $k(\vec{v} + \vec{w}) = kv + k\vec{w}$

> **Definition 2.2.3 ▸ Norm**
>
> A vector's **norm** is its magnitude or length.
>
> $$\|v\| := \sqrt{v_1^2 + \cdots + v_n^2}$$

> **Definition 2.2.4 ▸ Unit Vector**
>
> A **unit vector** is a vector whose magnitude is 1.

We will introduce shorthand notation for the three standard unit vectors:

- $\hat{i} := \langle 1, 0, 0 \rangle$

- $\hat{j} := \langle 0, 1, 0 \rangle$

- $\hat{k} := \langle 0, 0, 1 \rangle$

These three vectors form the **standard basis** for $\mathbb{R}^3$. That is, we can express any vector in $\mathbb{R}^3$ as a linear combination of $\hat{i}, \hat{j}, \hat{k}$.

> **Technique 2.2.1 ▸ Finding a Unit Vector from a Given Vector**
>
> Given a vector $\vec{v} = \langle x, y, z \rangle \in \mathbb{R}^3$, we can find the **unit vector** $\vec{u}$ with the same direction by:
>
> $$\vec{u} = \frac{\vec{v}}{\|v\|} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|}, \frac{z}{\|v\|} \right\rangle$$

> **Definition 2.2.5 ▶ Dot Product**
>
> Given two vectors $\vec{a}$ and $\vec{b}$ whose cardinality are both n, we define the **dot product** of $\vec{a}$ and $\vec{b}$ as:
> $$\vec{a} \cdot \vec{b} := a_1 b_1 + \cdots + a_n b_n$$

Like scalar multiplication, dot product is also associative, commutative, and distributive.

> **Theorem 2.2.1**
>
> If $\vec{a}$ and $\vec{b}$ are vectors and c is the angle between $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cdot \cos(c)$.

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