Chapter 1

Open and Closed Sets

We will describe some concepts that generalize open/closed intervals. This chapter also serves as a very light introduction to topology—specifically, the topology of the real number line.

1.1 Open Sets

Definition 1.1.1 ▸ Open Set

Intuitively, set is ***open*** if it does not contain any of its "boundary points", such as minimum or maximum.

More formally, $A \subseteq \mathbb{R}$. We say A is ***open*** if, for all $x \in A$, there exists $r > 0$ such that $(x - r, x + r) \subseteq A$.

$$\forall(x \in A)\exists(r > 0)\,((x - r, x + r) \subseteq A)$$

Example 1.1.2 ▸ $[0, 1)$ is not open

The interval $[0, 1)$ is not open.

Proof. $0 \in [0, 1)$, but $(0 - r, 0 + r) \not\subseteq [0, 1)$ for any $r > 0$. ▢

Definition 1.1.3 ▸ Open Ball

We call the interval $(x - r, x + r)$ the ***open ball*** of radius r centered at x, notated as $B(x, r)$ or $B_r(x)$.

$$B(x,r) = B_r(x) = (x-r, x+r)$$

This new notation lets us write ideas more succinctly. For example, $\mathbb{R}$ is open. Given any $x \in \mathbb{R}$, then any $r > 0$ will give us $B(x,r) \in \mathbb{R}$. Also, $\varnothing$ is vacuously open.

Lemma 1.1.4 ▸ Open Intervals are Open Sets

Let $a, b \in \mathbb{R}$ where $a < b$. Then (a,b) is an open set.

Proof. Let $c \coloneqq \frac{a+b}{2}$, and let $R \coloneqq \frac{b-a}{2}$. Then $(a,b) = B(c,R)$. Let $x \in B(c,R)$. Then $|x-c| < R$. Let $r \coloneqq R - |x-c| > 0$. We now prove $B(x,r) \subseteq B(c,R)$. Let $y \in B(x,r)$. Then $|x-y| < r$, so:

$$|y-c| = |y-x+x-c| \le |y-x| + |x-c| < r + |x-c| = R - |x-c| + |x-c| = R$$

Hence, $y \in B(c,R) = (a,b)$. Therefore, (a,b) is an open set. ▢

As we prove below, an arbitrary union of open sets is itself an open set.

Theorem 1.1.5 ▸ Union of Open Sets is Open

Suppose Λ is a set, and for each $\lambda \in \Lambda$, O_λ is an open subset of $\mathbb{R}$. Then $\bigcup_{\lambda \in \Lambda} O_\lambda$ is an open set.

Proof. Let $x \in \bigcup_{\lambda \in \Lambda} O_\lambda$. Then there exists some $\lambda_0 \in \Lambda$ such that $x \in O_{\lambda_0}$. Since O_{λ_0} is open, there exists $r > 0$ such that:

$$(x-r, x+r) \subseteq O_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

▢

The intersection of open sets is more troublesome. Countable intersections of open sets may not be open. For example, let $A_n \coloneqq \left(-\frac{1}{n}, \frac{1}{n}\right)$ for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ is not open!

Theorem 1.1.6 ▸ Finite Intersection of Open Sets is Open

Let $n \in \mathbb{N}$, and let $O_1, O_2, \dots, O_n$ be open subsets of $\mathbb{R}$. Then $\bigcap_{k=1}^{n} O_k$ is open.

Proof. Let $x \in \bigcap_{k=1}^{n} O_k$. Then $x \in O_k$ for $k = 1, 2, \dots, n$. Then, for each $k \in \{1, 2, \dots, n\}$, there must be some radius $r_k > 0$ such that $B(x, r_k) \subseteq O_k$. Since there are only finitely many open sets, we can take the minimum radius. Let $r \coloneqq \min\{r_1, r_2, \dots, r_n\}$. Then, $r \leq r_k$ for each $k \in \{1, 2, \dots, n\}$. Hence:

$$B(x, r) \subseteq B(x, r_k) \subseteq O_k \quad \text{for all } k \in \{1, 2, \dots, n\}$$

Therefore, $B(x, r) \subseteq \bigcap_{k=1}^{n} O_k$, so it is open. ▢

Note how the above theorem only works by taking the minimum radius of all the open sets. We can only take this minimum radius because there are only a finite number of open sets.

1.2 Closed Sets

Definition 1.2.1 ▸ Closed Set

Intuitively, a set is ***closed*** if it contains all of its "boundary points".

More formally, a set $E \subseteq \mathbb{R}$ is ***closed*** if every convergent sequence (s_n) where $s_n \in E$ for all $n \in \mathbb{N}$ satisfies $\lim_{n\to\infty} s_n \in E$.

Example 1.2.2 ▸ $(0, 1]$ is not closed

The interval $[0, 1)$ is not closed.

Proof. Consider the sequence $(s_n) \coloneqq 1/n$. Then (s_n) converges to 0, but $0 \notin (0, 1]$. ▢

Note that this interval $(0, 1]$ is neither open nor closed! It is wrong to think of open/closed as strictly one or the other (i.e. openness and closedness are not mutually exclusive). Moreover, a set can be both open and closed, going against the intuition of open and closed sets.

Example of set that is open and closed (clopen)

Lemma 1.2.3 ▶ Closed Intervals are Closed Sets

Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b]$ is a closed set.

Proof. Let (s_n) be an arbitrary convergent sequence of real numbers where $a \leq s_n \leq b$ for all $n \in \mathbb{N}$. Since (s_n) is convergent, then $\lim_{n\to\infty} s_n$ exists. By the properties of limits, we have:

$$\lim_{n\to\infty} a \leq \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} b$$

Hence, $\lim_{n\to\infty} s_n \in [a, b]$. Therefore, $[a, b]$ is a closed set. $\square$

Theorem 1.2.4 ▶ Intersection of Closed Sets is Closed

Let Λ be a set, and let $E_\lambda \subseteq \mathbb{R}$ be closed for all $\lambda \in \Lambda$. Then $\bigcap_{\lambda\in\Lambda} E_\lambda$ is a closed set.

Proof. Let (s_n) be an arbitrary convergent sequence of real numbers entirely contained within $\bigcap_{\lambda\in\Lambda} E_\lambda$. Since (s_n) is convergent, then $\lim_{n\to\infty} s_n$ exists. Let l denote that limit. Let $\lambda \in \Lambda$ be arbitrary. Then $s_n \in E_\lambda$ for all $n \in \mathbb{N}$. Since E_λ is closed, then any convergent sequence contained in E_λ has its limit in E_λ. Thus, $\lim_{n\to\infty} s_n \in E_\lambda$. Since $\lambda \in \Lambda$ is arbitrary, then $\lim_{n\to\infty} s_n \in E_\lambda$ for all $\lambda \in \Lambda$. Therefore, $s \in \bigcap_{\lambda\in\Lambda} E_\lambda$, so this set is closed. $\square$

Similar to the intersection of open sets, the union of closed sets is guaranteed to be closed if it is a finite union. For example, the union $\left(\bigcup_{n\in\mathbb{N}} [1/n, 1]\right) = (0, 1]$ is not closed!

Theorem 1.2.5 ▶ Finite Union of Closed Sets is Closed

Let $n \in \mathbb{N}$, and let $E_1, E_2, \ldots, E_n$ be closed subsets of $\mathbb{R}$. Then $\bigcup_{k=1}^{n} E_k$ is a closed set.

A direct proof of this theorem can be found in the textbook.

The direct proof here is rather wordy and awkward. We will first establish a concrete relationship between open and closed sets, then leverage that to prove this theorem "indirectly".

Theorem 1.2.6 ▶ Complement of an Open Set is Closed

Let $O \subseteq \mathbb{R}$ be open. Then $\mathbb{R} \setminus O$ is closed.

Proof. Let (x_n) be an arbitrary convergent sequence entirely contained within $R \setminus O$. Let $l_x := \lim_{n\to\infty} x_n$. Suppose for contradiction that $l_x \notin \mathbb{R} \setminus O$. Then $l_x \in O$. Since O is open, there exists some radius $r > 0$ such that $B(l_x, r) \in O$. Since (x_n) converges to l_x, then there exists $N \in \mathbb{N}$ such that $|x_n - l_x| < r$ for all $n > N$. That is, $x_n \in B(l_x, r) \subseteq O$ for all $n > N$. This contradicts $x_n \in \mathbb{R} \setminus O$. Thus, $l_x \in \mathbb{R} \setminus O$, so $\mathbb{R} \setminus O$ is closed. ▢

Theorem 1.2.7 ▸ Complement of a Closed Set is Open

Let $E \subseteq \mathbb{R}$ be closed. Then $\mathbb{R} \setminus E$ is open.

Proof. Let $x \in \mathbb{R} \setminus E$. We must prove the following statement:

$$\exists(n \in \mathbb{N})\,(B\,(x, 1/n) \subseteq \mathbb{R} \setminus E)$$

Suppose for contradiction the negation of the previous statement holds. That is:

$$\forall(n \in \mathbb{N})\,(B\,(x, 1/n) \nsubseteq \mathbb{R} \setminus E)$$

Then, for all $n \in \mathbb{N}$, there exists $x_n \in B(x, 1/n)$ such that $x_n \in E$. Hence, the sequence (x_n) satisfies $x_n \in E$ for all $n \in \mathbb{N}$ and $|x_n - x| < 1/n$.

Finish Proof

▢

Combining the two above theorems, we can infer a pretty useful relationship between open and closed sets.

redo proof that union of closed sets is closed

1.3 Closure

Definition 1.3.1 ▸ Closure of a Set

For $A \subseteq \mathbb{R}$, the ***closure*** of A is the set:

For example, the closure of the interval $(0, 1)$ is

Theorem 1.3.2 ▶ Properties of Closures of Sets

Let $A \subseteq \mathbb{R}$. Then:

(i) $A \subseteq \overline{A}$,

(ii) $\overline{A}$ is closed,

(iii) $A = \overline{A}$ if and only if A is closed,

(iv) $\overline{\overline{A}} = \overline{A}$,

(v) if $F \subseteq \mathbb{R}$ is closed and $A \subseteq F$, then $\overline{A} \subseteq F$, and

(vi) $\overline{A} = \bigcap \{F \subseteq \mathbb{R} : F \text{ is closed, and } A \subseteq F\}$

These properties can make it easier to prove statements about closures.

Example 1.3.3 ▶ Using Properties of Closure

If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof. ▢

The corresponding idea for open sets is the ***interior*** of a set.