Chapter 1

Continuity

In calculus classes, we are often taught: "f is continuous at c if $\lim_{x \to c} = f(c)$." This is fine for "well-behaving" functions, but consider a function $f : [0, 1] \cup \{2\} \to \mathbb{R}$. It may be tempting to say f is not continuous at 2 because it does not have a limit when x approaches 2. However, for the sake of simplifying future ideas and theorems, we will consider f to be (vacuously) continuous at 2.

Definition 1.0.1 ▸ Isolated Point

Let $A \subseteq \mathbb{R}$. A point $x \in A$ is an ***isolated point*** of A if there exists $r > 0$ such that $B(x, r) \cap A = \{x\}$.

In other words, and isolated point is anything that is not a limit point. For example, in the set $[0, 1] \cup \{2\}$, we would consider 2 to be an isolated point.

Lemma 1.0.2 ▸ Limit/Isolated Point Exclusivity

Let $A \subseteq \mathbb{R}$ and $x \in A$. Then x is **either** a limit point of A or isolated point of A.

Proof. Suppose x is not an isolated point of A. Then, for any $n \in \mathbb{N}$, there exists some value $x_n \in A$ such that $x_n \neq x$, and $x_n \in B(x, 1/n)$. Then (x_n) is entirely contained in $A \setminus \{x\}$, and $|x_n - x| < 1/n$ for any $n \in \mathbb{N}$. That is, x_n converges to x. Therefore, x is a limit point of A. ▢

We upgrade the normal calculus definition of continuity by accounting for any potential isolated points.

Definition 1.0.3 ▶ Continuity at a Point

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then f is ***continuous at*** c if:

1. c is an isolated point of A, or
2. $c \in A'$, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$.

Theorem 1.0.4 ▶ Equivalent Characterizations of Continuity

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. Then the following are equivalent:

(a) f is continuous at c.

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

(c) For all sequences (x_n) contained in A that converge to c, $\lim_{n \to \infty} f(x_n) = f(c)$.

Proof sketch. If c is an isolated point of A, then (a) holds. For $\epsilon > 0$, choose $\delta > 0$ such that $B(c, \delta) \cap A = \{c\}$. If $x \in A$ and $|x - c| < \delta$, then $x = c$, so (b) holds. Similarly, if (x_n) is contained in A and converges to c, then $x_n = c$ for some large enough n. Thus, $\lim_{n \to \infty} f(x_n) = f(c)$.

If instead c is a limit point of A, then we can simply prove the following statements:

- (a) $\implies$ (b) by definition (only need to check $|x - c| = 0$)
- (b) $\implies$ (c) similar to proof of sequential characterization of limits
- (c) $\implies$ (a) similar to the above case

□

Theorem 1.0.5 ▶ Continuity Preservation

Let $A \subseteq \mathbb{R}$, $c \in A$, and $f, g : A \to \mathbb{R}$ that are continuous at c. Then:

(a) For all $\alpha \in \mathbb{R}$, αf is continuous at c.

(b) $f + g$ is continuous at c.

(c) fg is continuous at c.

(d) if $g(c) \neq 0$, then f/g is continuous at c.

Proof of (b). If c is an isolated point of A, then $f + g$ is continuous at c, and we are done. Otherwise, c is a limit point. Then:

$$\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$$

Therefore, $f + g$ is continuous at c. □

For example, the polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ is continuous at every $c \in \mathbb{R}$. To prove this, we would show:

1. $f(x) = x$ is continuous at every $x \in \mathbb{R}$
2. $f(x) = x^k$ is continuous at every $x \in \mathbb{R}$
3. $f(x) = ax^k$ is continuous at every $x \in \mathbb{R}$
4. $f(x) = \sum a_k x^k$ is continuous at every $x \in \mathbb{R}$

If p and q are polynomials and $q(c) \neq 0$, then the rational function p/q is continuous at $c \in \mathbb{R}$. In other words, rational functions are continuous everywhere in their domain.

Definition 1.0.6 ▸ Continuity on a Set

Let $f : A \to \mathbb{R}$, $B \subseteq A$. We say f is ***continuous on*** B if f is continuous at every $x \in B$.

For example, the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = x$ is continuous on $(0,1)$. Interestingly, this function has neither a maximum nor a minimum on this domain. 0 is the infimum of image of f under $(0,1)$, but 0 can never be attained as a function value. The same can be said about 1 as the supremum of the image of f.

Another example, let $f : (0,1) \to \mathbb{R}$ be a function defined by $f(x) = 1/x$. Then f is continuous on $(0,1)$, but again, there is no minimum nor maximum. This time, we only have an infimum for the image of f under $(0,1)$. There is no upper bound for the function values of f.

If instead f were defined on a closed and bounded (i.e. compact) set, then we would have a minimum and maximum for the function values of f. We prove this in the following theorem.

Theorem 1.0.7 ▸ Extreme Value Theorem

Suppose K is a nonempty and compact subset of $\mathbb{R}$, and suppose $f : K \to \mathbb{R}$ is continuous. Then:

(a) f is bounded on K (that is, $f[K]$ is bounded),
(b) there exists $x_0 \in K$ such that $f(x_0) = \sup(f[K])$
(c) there exists $x_1 \in K$ such that $f(x_1) = \inf(f[K])$

Proof of (a). Suppose for contradiction that f is not bounded on K. Then for each $n \in \mathbb{N}$, there must exist $x_n \in K$ such that $|f(x_n)| > n$. Since $K \subseteq \mathbb{R}$ is compact (and thus sequentially compact), there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to some $x \in K$. Since f is continuous, then the sequence $\{f(x_{n_k})\}$ converges to $f(x)$. Since convergent sequences are bounded, then there exists $M \in \mathbb{R}$ such that $|f(x_{n_k})| \leq M$. This contradicts the fact that $|f(x_{n_k})| > n_k \geq k$. Therefore, f must be bounded on K (i.e. $f[K]$ is bounded). ▢

Proof of (b). By (a), we know $f[K]$ is bounded. Since $f[K]$ is also nonempty, then completeness guarantees that $f[K]$ has a supremum in $\mathbb{R}$. By Problem Set 6 # 8, there exists a sequence in $f[K]$ that converges to $\sup(f[K])$. That is, there exists a sequence (x_n) contained in K where the sequence $\{f(x_n)\}$ converges to $\sup(f[K])$. Since K is sequentially compact, there exists a subsequence (x_{n_k}) of (x_n) such that x_{n_k} converges to some $x_0 \in K$. By continuity:

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = \sup f[K]$$

▢

Theorem 1.0.8

Suppose $O \subseteq \mathbb{R}$ is open and $f : O \to \mathbb{R}$. Then f is continuous on O if and only if, for every open set $U \subseteq \mathbb{R}$, $f[U^{-1}]$ is open.

1.1 Uniform Continuity

Definition 1.1.1 ▸ Uniform Continuity

Let $f : A \to \mathbb{R}$ be a function. We say f is ***uniformly continuous*** on A if, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Example 1.1.2 ▸ Simple Uniform Continuity Proof

$f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Proof. ▢

Example 1.1.3 ▸ Simple Uniform Continuity Disproof

$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. $\square$

Theorem 1.1.4

Let K be a compact subset of $\mathbb{R}$, and let $f : K \to \mathbb{R}$ be a continuous function on K. Then f is uniformly continuous on K.

Proof. $\square$

Chapter 2

Differential Calculus

Definition 2.0.1 ▸ Differentiable, Derivative

Let $a, b \in \mathbb{R}$ where $a < b$, let $f : (a, b) \to \mathbb{R}$ be a function, and let $x_0 \in (a, b)$.

- We say f is ***differentiable at*** x_0 if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.
- We say f is ***differentiable on*** I if f is differentiable at every $x \in I$.
- If this limit exists, we define the ***derivative*** of f as $f'(x_0) \coloneqq \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

We can also write the derivative as $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$. In this context, we replace x with $x_0 + h$. This is usually the more familiar form and is referred to as the ***difference quotient***. Without the limit, the difference quotient by itself gives us the slope of the line from $(x_0, f(x_0))$ to $(x_0 + h, f(x_0 + h))$. With the limit, it gives us the slope of the line tangent to f at x_0.

We can think of the derivative $f'(x)$ as:

- definition: the limit of the difference quotient
- graphical: slope of the tangent line
- interpretation: instantaneous rate of change

Example 2.0.2 ▸ Simple Derivative Example

Given $f(x) = x^2$, find $f'(x_0)$.

If $x \neq x_0$, then:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x + x_0)(x - x_0)}{x - x_0} = x + x_0$$

Thus:

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = x_0 + x_0 = 2x_0$$

Theorem 2.0.3 ▶ Differentiability Implies Continuity

If f is differentiable at x_0, then f is continuous at x_0.

Proof. If $x \neq x_0$, then $f(x) = f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x - x_0)$. Thus:

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} \left(f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right) \\
&= \left(\lim_{x \to x_0} f(x_0) \right) + \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \left(\lim_{x \to x_0} (x - x_0) \right) \\
&= f(x_0) + f'(x_0) \cdot 0 \\
&= f(x_0)
\end{aligned}$$

Therefore, f is continuous at x_0. □

As we'll see in the next example, the converse statement is not true. That is, continuity does not generally imply differentiability.

Example 2.0.4 ▶ Continuity does not imply differentiability

$f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Proof. We first show f is continuous at $x = 0$. We have $f(0) = 0$, and:

$$\begin{aligned}
\lim_{x \to 0} f(x) &= \lim_{x \to 0} |y| \\
&= 0
\end{aligned}$$

Now to show it is not differentiable, if $x \neq 0$, we have:

$$\frac{f(x) - f(0)}{x - 0} = \frac{absx - 0}{x - 0} = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Then:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} |x| y$$

so its limit as x approaches 0 does not exist. Therefore, f is not differentiable at $x = 0$. ▢

Example 2.0.5 ▸ Piecewise Differentiability Example

Let $f(x) \coloneqq \begin{cases} x^2 \sin {}^1\!/\!_x, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Is f differentiable at $x = 0$?

It turns out that f is differentiable at $x = 0$! However, it may be tempting to give the following **incorrect** proof (assuming we already have the chain rule and product rule):

Incorrect proof. If $x \neq 0$:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

This has no limit as x approaches 0, so $\lim_{x \to 0} f'(x)$ does not exist. ▢

The above approach erroneously hinges on the assumption that the derivative must be continuous (which is not generally true). We must instead use the definition of differentiability.

Correct proof. If $x \neq 0$:

$$\begin{aligned} \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{x^2 \sin {}^1\!/\!_x}{x} \\ &= \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \\ &= 0 \end{aligned}$$

Therefore, f is differentiable at $x = 0$, and $f'(0) = 0$. ▢

This function f is differentiable for every $x \in \mathbb{R}$, but $\lim_{x \to 0} f'(x)$ does not exist! So we have shown f' is not continuous at $x = 0$.

Theorem 2.0.6 ▸ Properties of Differentiation

Suppose $f, g : (a, b) \to \mathbb{R}$ are differentiable at $x_0 \in (a, b)$. Let $c \in \mathbb{R}$. Then cf, $f + g$, and fg are differentiable at x, and if $g'(x) \neq 0$, then f/g is differentiable. Moreover:

(a) $(cf)'(x_0) = cf'(x_0)$

(b) $(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(c) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

(d) $(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$

Proof. To prove (a):

$$\begin{aligned}(cf)'(x_0) &= \lim_{x \to x_0} \frac{cf(x) - cf(x_0)}{x - x_0} \\ &= c \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= cf'(x)\end{aligned}$$

To prove (b):

$$\begin{aligned}(f + g)'(x) &= \lim_{x \to x_0} \frac{(f(x) + g(x)) - (f(x_0) + g(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}\right] \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) + g'(x_0)\end{aligned}$$

To prove (c):

$$\begin{aligned}(fg)'(x) &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot g(x) + f(x) \cdot \frac{g(x) - g(x_0)}{x - x_0}\right] \\ &= \dots\end{aligned}$$

Since f and g were assumed to be differentiable (and thus continuous at x_0), we can apply properties of limits to finally attain:

$$f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$\square$

Theorem 2.0.7 ▶ Chain Rule

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ be arbitrary functions. If f is differentiable at some $x \in (a, b)$ and g is differentiable at $f(x) \in (c, d)$, then $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at x, and:

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

Intuition: When taking $(g \circ f)'$, there are two rates of the change to consider: f' and g', which "compound" one another.

Proof sketch.

$$\begin{aligned}
(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\
&= \lim_{x \to x_0} \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \right)
\end{aligned}$$

The idea is the first fraction approaches $g'(f(x_0))$, and the second fraction approaches $f'(x_0)$. However, if $f(x) - f(x_0) = 0$, then the first fraction is invalid. To circumvent this, we can redefine differentiability as a multiplicative property. Precisely, we can say a function f is ***differentiable*** at x to mean:

$$f(x + h) - f(x) = f'(x) \cdot h + \epsilon(h) \cdot h$$

where $\epsilon(h)$ approaches 0 as h approaches 0. Intuitively, this definition verifies that we can well approximate the function at that point using a linear function. The $\epsilon(h) \cdot h$ term denotes the error in the linear approximation, which should become negligible $\square$

Definition 2.0.8 ▸ Local/Global Maxima/Minima (Extreme Values)

Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, and $f : I \to \mathbb{R}$ be a function. We say f has a:

- ***local maximum*** at x_0 if there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) \cap I$, $f(x) \leq f(x_0)$.
- ***local minimum*** at x_0 if there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) \cap I$, $f(x) \geq f(x_0)$.
- ***global maximum*** at x_0 if for all $x \in I$, $f(x) \leq f(x_0)$.
- ***global minimum*** at x_0 if for all $x \in I$, $f(x) \geq f(x_0)$.

Theorem 2.0.9 ▸ Fermat's Theorem

Let $f : I \to \mathbb{R}$ be a function. If f has a local minimum or local maximum at $x_0 \in I$, then either:

(a) x_0 is an endpoint of I, or

(b) f is not differentiable at x_0, or

(c) f is differentiable at x_0, and $f'(x_0) = 0$.

Proof. Suppose f has a local maximum at x_0. Then there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) \cap I$, $f(x) \leq f(x_0)$. We prove that, if neither (a) nor (b) are true, then (c) must be true. Suppose x_0 is not an endpoint of I, and suppose that f is differentiable at x_0. Let $x \in B(x_0, \delta) \cap I$ be arbitrary.

- If $x > x_0$, then $x - x_0 > 0$ and $f(x) - f(x_0) \leq 0$. Hence, $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$, so $f'(x_0) = \lim_{x \to 0} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$.
- If $x < x_0$, then $x - x_0 < 0$ and $f(x) - f(x_) \leq 0$. Hence, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$, so $f'(x_0) = \lim_{x \to 0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$.

By trichotomy, $f'(x_0) = 0$. ▢

Theorem 2.0.10 ▸ Rolle's Theorem

Let $a, b \in \mathbb{R}$ where $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = 0$ and $f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since $[a, b]$ is compact and f is continuous, the Extreme Value Theorem states that f attains both its maximum and minimum on $[a, b]$.

- If both the maximum and minimum of f occur at the endpoints a and b, then maximum and minimum of $f[(a,b)]$ is 0. Thus, $f(x) = 0$ for all $x \in [a,b]$. Thus, $f'(x) = 0$ for all $x \in (a,b)$, so we can take c to be any value in (a,b).
- Otherwise, either the maximum or the minimum occurs at some point $c \in (a,b)$. By Fermat's Theorem, we have $f'(c) = 0$.

Since the above cases are exhaustive, the proof is complete. ▢

Theorem 2.0.11 ▸ Mean Value Theorem

Let $a, b \in \mathbb{R}$ where $a < b$, and let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $l : [a,b] \to \mathbb{R}$ be the function of the line through $(a, f(a))$ and $(b, f(b))$. That is, for any $x \in [a,b]$:

$$l(x) \coloneqq f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Note that $l'(x) = \frac{f(b)-f(a)}{b-a}$. Let $g : [a,b] \to \mathbb{R}$ be defined for every $x \in [a,b]$ by:

$$g(x) \coloneqq f(x) - l(x)$$

Then g is continuous on $[a,b]$, and g is differentiable on (a,b). Also note $g(a) = 0$ and $g(b) = 0$. By Rolle's Theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$. We then have:

$$0 = g'(c) = f'(c) - l'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

Adding across by $\frac{f(b)-f(a)}{b-a}$, we have $f'(c) = \frac{f(b)-f(a)}{b-a}$. ▢

The Mean Value Theorem has tons of application in both calculus and real analysis.

Example 2.0.12 ▸ Positive derivative means increasing

If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).

Intuition: This seems like a fairly obvious result, but to prove it rigorously, we can apply the Mean Value Theorem.

Proof. If $a < x < y < b$, then there exists $c \in (a,b)$ where $\frac{f(y)-f(x)}{y-x} = f'(c)$. Thus, $f(y) - f(x) > 0$ for any choice of $x, y \in (a,b)$ where $y > x$. Therefore, f is strictly increasing. ▢