Calculus III

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Contents

1 Three-Dimensional Space

In past math classes, we have been used to dealing in $\mathbb{R}^2$ where we work with two degrees of freedom: x and y. Now, we will be working in $\mathbb{R}^3$ with three degrees of freedom: x, y, and z.

1.1 Points

Definition 1.1.1 ▶ Point

A ***point*** in $\mathbb{R}^n$ space is an n-tuple that specifies a location in that space.

$$p = (p_1, \dots, p_n) \in \mathbb{R}^n$$

Definition 1.1.2 ▶ Distance

Given two points $a, b \in \mathbb{R}^n$, the ***distance*** between the two points is defined as:

$$d(a, b) := \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$$

Example 1.1.3 ▶ Distance Between Points

Find the distance between $p_1 = (-1, -1, 4)$ and $p_2 = (-1, 4, -1)$.

$$\begin{aligned} d(p_1, p_2) &= \sqrt{(-1 - (-1))^2 + (4 - (-1))^2 + (-1 - 1)^2} \\ &= \sqrt{0^2 + 5^2 + (-5)^2} \\ &= \sqrt{50} \end{aligned}$$

Definition 1.1.4 ▶ Sphere

Given a point $c = (h, k, l) \in \mathbb{R}^3$, a ***sphere*** is the set of all points $(x, y, z) \in \mathbb{R}^3$ that are a distance r from the point $c = (h, k, l)$.

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Note that all the points of the sphere are equidistant to the center of the sphere. This means the sphere is really a hollow shell.

Example 1.1.5 ▶ Circle

Show that the following quadratic equation represents a circle by rewriting it in standard form. Find the center $c = (h, k)$ and the radius r.

$$x^2 + y^2 + x = 0$$

To solve this, we will have to complete the square:

$$\begin{aligned} x^2 + x + y^2 &= 0 \\ \implies x^2 + x + \frac{1}{4} + y^2 &= \frac{1}{4} \\ \implies \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4} \end{aligned}$$

1.2 Vectors

Definition 1.2.1 ▶ Vector

A ***vector*** is a mathematical object that contains multiple objects of the same type.

$$\vec{v} = \langle v_1, \dots, v_n \rangle \in \mathbb{R}^n$$

As customary in most mathematics textbooks, we will always denote vectors using the little arrow thing. In the context of three-dimensional space, we will only be working with vectors with three components. In addition, we will think of vectors as having a magnitude and direction.

Definition 1.2.2 ▸ Scalar Multiplication

Given a vector $\vec{v}$ and scalar k, we define ***scalar multiplication*** as:

$$k \cdot \vec{v} := \langle kv_1, \dots, kv_n \rangle$$

Note that scalar multiplication is associative, commutative, and distributive.

- $a(b\vec{v}) = b(a(\vec{v})) = (ab)\vec{v}$
- $(k_1 + k_2)\vec{v} = k_1\vec{v} + k_2\vec{v}$
- $k(\vec{v} + \vec{w}) = kv + k\vec{w}$

Definition 1.2.3 ▸ Norm

A vector's ***norm*** is its magnitude or length.

$$\|v\| := \sqrt{v_1^2 + \cdots + v_n^2}$$

Definition 1.2.4 ▸ Unit Vector

A ***unit vector*** is a vector whose magnitude is 1.

We will introduce shorthand notation for the three standard unit vectors:

- $\hat{\imath} := \langle 1, 0, 0 \rangle$
- $\hat{\jmath} := \langle 0, 1, 0 \rangle$
- $\hat{k} := \langle 0, 0, 1 \rangle$

These three vectors form the ***standard basis*** for $\mathbb{R}^3$. That is, we can express any vector in $\mathbb{R}^3$ as a linear combination of $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Technique 1.2.5 ▸ Finding a Unit Vector from a Given Vector

Given a vector $\vec{v} = \langle x, y, z \rangle \in \mathbb{R}^3$, we can find the ***unit vector*** $\vec{u}$ with the same direction by:

$$\vec{u} = \frac{\vec{v}}{\|v\|} = \left\langle \frac{x}{\|v\|}, \frac{y}{\|v\|}, \frac{z}{\|v\|} \right\rangle$$

Definition 1.2.6 ▶ Dot Product

Given two vectors $\vec{a}$ and $\vec{b}$ whose cardinality are both n, we define the ***dot product*** of $\vec{a}$ and $\vec{b}$ as:

$$\vec{a} \cdot \vec{b} \coloneqq a_1 b_1 + \cdots + a_n b_n$$

Like scalar multiplication, dot product is also associative, commutative, and distributive.

Theorem 1.2.7 ▶ Angle Between Vectors

If $\vec{a}$ and $\vec{b}$ are vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cdot \cos(\theta)$$

Proof. TODO: finish proof ▢

Definition 1.2.8 ▶ Parallel, Perpendicular

- Two vectors are ***parallel*** if the angle between the vectors is 0 deg.
- Two vectors are ***perpendicular*** if the angle between the vectors is 90 deg.

Definition 1.2.9 ▶ Vector Orthogonality

We say two vectors are ***orthogonal*** if their dot product is 0.

In other words, $\vec{a}$ and $\vec{b}$ are ***orthogonal*** if $\vec{a} \cdot \vec{b} = 0$.

Given a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$, we have:

$$\frac{\vec{a}}{\|a\|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

where:

- $\alpha = \cos^{-1}\left(\frac{a_1}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the x-axis)
- $\beta = \cos^{-1}\left(\frac{a_2}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the y-axis)
- $\beta = \cos^{-1}\left(\frac{a_3}{\|\vec{a}\|}\right)$ (angle between $\vec{a}$ and the z-axis)

Definition 1.2.10 ▸ Work

If F is a force moving a particle from a point P to a point Q, the ***work*** performed by the force is given by:

$$W = \vec{F} \cdot \overrightarrow{PQ}$$

Example 1.2.11 ▸ Finding Work

Find the work done by a force $\vec{F} = \langle 3, 4, 5 \rangle$ in moving an object from $p = (2, 1, 0)$ to $q = (4, 6, 2)$.

First, we find $\overrightarrow{pq}$ as such:

$$\begin{aligned} \overrightarrow{pq} &= \langle 4 - 2, 6 - 1, 2 - 0 \rangle \\ &= \langle 2, 5, 2 \rangle \end{aligned}$$

Then, we can find work:

$$\begin{aligned} W &= \vec{F} \cdot \overrightarrow{PQ} \\ &= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 \\ &= 36 \end{aligned}$$

1.3 Gradient

Definition 1.3.1 ▸ Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The ***gradient*** of f is a function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\nabla f(x_1, \dots x_n) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

Example 1.3.2 ▸ Gradient

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $f(T, L, \rho) = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

The gradient of $f(T, L, P)$ is denoted

$$\nabla f(T, L, \rho) = \left\langle \frac{\partial f}{\partial T}, \frac{\partial f}{\partial L}, \frac{\partial f}{\partial \rho} \right\rangle$$
$$= \left\langle \frac{1}{4L\sqrt{T\rho}}, -\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}, -\frac{1}{4L}\sqrt{\frac{T}{\rho^3}} \right\rangle$$

We can then calculate gradient as such:

$$\nabla f(2, 1, 1) = \left\langle \frac{1}{4(1)\sqrt{(2)(1)}}, -\frac{1}{2(1)}\sqrt{\frac{2}{1}}, -\frac{1}{(4)(1)}\sqrt{\frac{2}{1}} \right\rangle$$
$$= \left\langle \frac{1}{4\sqrt{2}}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4} \right\rangle$$

Definition 1.3.3 ▸ Directional Derivative

The ***directional derivative*** of $f(x, y, z)$ in the direction of $\vec{a}$ is defined as:

$$\nabla f(x, y, z) \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

Example 1.3.4 ▸ Directional Derivative

If $f(x, y, z) = xy^2z^5$, find the directional derivative of $f(x, y, z)$ at the point $(1, 0, -2)$ in the direction of the unit vector $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$, $\vec{a} = \langle 1, 2, -2 \rangle$.

For this, we calculate $\nabla f(1, 0, -1)$, then calculate the dot product of $\nabla f(1, 0, -1)$ with the unit vector $\vec{u} = \langle 1/3, 2/3, -2/3 \rangle$. Thus, the directional derivative of $f(x, y, z)$ at $(1, 0, -1)$ denoted by $Df(1, 0, -1)$ in the direction of $\vec{u}$ is:

$$Df(1, 0, -1) = \nabla f(1, 2, -2) \cdot \vec{u}$$
$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \vec{u}$$
$$= \langle 0, 0, 0 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$$
$$= 0$$

1.4 Projecting Vectors

Projecting a vector onto another vector

Definition 1.4.1 ▶ Scalar Projection

Given $\vec{a}$ and $\vec{b}$, the ***scalar projection*** of $\vec{b}$ onto $\vec{a}$ is the norm of the vector projection of $\vec{b}$ onto $\vec{a}$.

$$\operatorname{comp}_{\vec{a}} \vec{b} \coloneqq \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|}$$

Definition 1.4.2 ▶ Vector Projection

Given $\vec{a}$ and $\vec{b}$ that are non-zero vectors, the ***vector projection*** of $\vec{b}$ onto the vector $\vec{a}$ is defined by:

$$\operatorname{proj}_{\vec{a}} \vec{b} \coloneqq \operatorname{comp}_{\vec{a}} \vec{b} \frac{\vec{a}}{\|\vec{a}\|}$$

1.5 Cross Product

Definition 1.5.1 ▶ Cross Product

Given two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, the ***cross product*** of $\vec{a}$ and $\vec{b}$ is a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} \coloneqq \vec{c} \quad \text{where} \quad \vec{a} \cdot \vec{c} = 0 \quad \textbf{and} \quad \vec{b} \cdot \vec{c} = 0$$

The cross product is exclusive to vectors in three dimensions.

Technique 1.5.2 ▶ Calculating Cross Product

Let $\vec{a} \coloneqq \langle a_1, a_2, a_3 \rangle$ and $\vec{b} \coloneqq \langle b_1, b_2, b_3 \rangle$ To find $\vec{a} \times \vec{b}$, we:

1. Create a matrix as such:

$$\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

2. Find the determinant of the matrix by cofactor expansion on the first row.

$$\begin{aligned}\vec{a}\times\vec{b} &= \hat{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{\imath}(a_2b_3 - b_2a_3) - \hat{\jmath}(a_1b_3 - b_1a_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \langle a_2b_3 - b_2a_3,\ -a_1b_3 + b_1a_3,\ a_1b_2 - b_1a_2\rangle\end{aligned}$$

- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
- $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$
- If $r\in\mathbb{R}$, then $(r\vec{a})\times\vec{b} = \vec{a}\times(r\vec{b}) = r(\vec{a}\times\vec{b})$
- $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot c$
- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

Theorem 1.5.3

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors in $\mathbb{R}^3$ and θ is the angle between $\vec{a}$ and $\vec{b}$, then:

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta$$

Technique 1.5.4 ▶ Using Cross Product to Calculate Torque

If $\vec{F}$ is a force applied to an object with position vector $\vec{r}$, then the torque $\vec{\mathrm{T}}$ produced by $\vec{F}$ is given by:

$$\vec{\mathrm{T}} \coloneqq \vec{r}\times\vec{F}$$

1.6 Cylinders and Quadratic Surfaces

Definition 1.6.1 ▶ Planar Curve

A ***planar curve*** is any curve that lies on a single plane.

Definition 1.6.2 ▶ Cylinder

Given a planar curve c, the surface in $\mathbb{R}^3$ defined by all parallel lines crossing the curve c is called a ***cylinder***.

Note that our broad definition of cylinder does not require the cylinder to be circular, nor does it require it to be straight. For example, we could have a planar curve defined by $x^2 + y^2 = 1$ and create a circular cylinder with radius 1. We could also have a planar curve defined by $y = x^2$ and create a ***parabolic cylinder***.

Example 1.6.3

Consider the curve $x^2 + y^2 = z^2$. For every z_0 at $x = y = 0$, we have a point, say $p \coloneqq (0, 0, 0)$yh

Definition 1.6.4 ▶ Cone

Definition 1.6.5 ▶ Conic Surface

A ***conic surface*** is a surface that is attained by taking a cross-section of a cone.

There are four types of conic surfaces:

1. The cross-section parallel to the xy-plane is a ***circle***.
2. The cross-section slightly angled from the xy-plane is a ***ellipse***.
3. The cross-section parallel to a generating line is a ***parabola***.
4. The cross-section parallel to the z axis is a ***hyperbola***.

Definition 1.6.6 ▶ Quadratic Surface

A ***quadratic surface*** in $\mathbb{R}^3$ is the set of points whose coordinates satisfy a quadratic polynomial in the variables x, y, z.

For example, the standard equation for a sphere is a quadratic surface.

Definition 1.6.7 ▶ Ellipsoid

An ***ellipsoid*** is a quadratic surface in $\mathbb{R}^3$ defined by the equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

The cross-sections of an ellipsoid with each coordinate plane (xy-plane, xz-plane, yz-plane) is just an ellipse. In other words:

- If we set $z = l$, we get an ellipse in the xy-plane
- If we set $y = k$, we get an ellipse in the xz-plane
- If we set $x = h$, we get an ellipse in the yz-plane

We can also have ***hyperboloids***:

- Type 1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Type 2: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

For type 2 hyperboloids, its cross-section with the xy and xz plane is a hyperbola. Note that there is no cross-section with the yz plane. This is because the equation $-\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ has no solution in the real numbers.

The standard form equation for a parabola is:

$$z - l = \pm c\left[(x-l)^2 + (y-k)^2\right]$$

TODO: more quadratic surfaces here

2

Vector Functions

Definition 2.0.1 ▸ Vector Function

A ***vector function*** $\vec{r}(t)$ is a function that maps each $t \in \mathbb{R}$ to a corresponding vector in $\mathbb{R}^n$. In other words:

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

For example, we can have $\vec{r}(t) = \langle 2 - 3t, 5 - 7t, t \rangle$. Then $\vec{r}(t)$ is a vector function that parameterizes a line that passes through $(2, 5, 0)$ and with vector direction $\vec{v} = \langle -3, -7, 1 \rangle$.

Example 2.0.2 ▸ Circle Vector Function

Let $\vec{r}_1(t) = \langle \cos t, \sin t \rangle$ where $0 \leq t < 2\pi$. This function's graph is a circle of radius 1. We can think of this as $\cos^2 t + \sin^2 t = 1$.
Similarly, consider $\vec{r}_2(t) = \langle 5 \cos t, 8 \sin t \rangle$. Thus, $x/5 = \cos t$ and $y/8 = \sin t$. Using the equation $cos^2 t + sin^2 t = 1$, we now have $(x/5)^2 + (y/8)^2 = 1$. It's an ellipse.

If a vector function is given by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then the domain of $\vec{r}(t)$ is the intersection of the domains of f, g, h. This is denoted as $\mathrm{Dom}(\vec{r}(t))$.

2.1 Limits and Continuity

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector function, we say that:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Recall L'Hopital's rule:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Only applies if the left-hand side is an indeterminate form (i.e. a fraction whose denominator is 0).

Definition 2.1.1 ▸ Continuity

We say a function $f : X \to Y$ is ***continuous*** at some value a if:

1. $a \in X$,
2. $\lim_{x\to a} f(x)$ exists, and
3. $f(a) = \lim_{x\to a} f(x)$

Similarly, a vector function is continuous if it satisfies the above conditions.

Recall that for a function $f : \mathbb{R} \to \mathbb{R}$, the derivative at some value x can be generalized by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Similarly, the derivative of a vector function at some value x can be generalized as:

$$\vec{r}(t) = \lim_{h\to 0} \frac{\vec{r}(x+h) - \vec{r}(x)}{h}$$

Thus, if $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, we have:

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Theorem 2.1.2 ▸ Properties of the differentials of vector functions

Let $\vec{u}(t) = \langle f_1(t), g_1(t), h_1(t)\rangle$ and $\vec{v}(t) = \langle f_2(t), g_2(t), h_2(t)\rangle$ where $f_1, g_1, h_1, f_2, g_2, h_2$ are differentiable. Let c be a scalar. Then:

- $\frac{d}{dt}\vec{u}(t) = \langle {f_1}'(t), {g_1}'(t), {h_1}'(t)\rangle$
- $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$
- $\frac{d}{dt}c\vec{v}(t) = c\frac{d}{dt}\vec{v}(t)$
- $\frac{d}{dt}f(t)\vec{v}(t) = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$
- $\frac{d}{dt}\vec{u}(t) \cdot \vec{v}(t) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}\vec{u}(t) \times \vec{v}(t) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ (note that order here is sensitive)
- $\frac{d}{dt}\vec{u}(f(t)) = \vec{u}'(f(t))f'(t)$

If we consider the curve C formed by all the terminal points of the graph of a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we call $\vec{r}(t)$ a ***parameterization*** of the curve C, and we call $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ the ***displacement vector***.

Example 2.1.3

Find the parametric equations of the tangent line to the helix with parameterization $\vec{r}(t) := \langle 2\cos t, \sin t, t\rangle$ at the point $(0, 1, \pi/2)$

$$\vec{r}'\left(\frac{\pi}{2}\right) = \left\langle -2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1\right\rangle$$

Then the parameterized line is:

$$\left\langle 0, 1, \frac{\pi}{2}\right\rangle + t\langle -2, 0, 1\rangle$$

2.2 Integrability

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $t \in \mathbb{R}$, and $x(t)$, $y(t)$, and $z(t)$ are integrable on the interval $[a, b]$, then:

$$\begin{aligned}\int_a^b \vec{r}(t)\,dt &= \int_a^b \langle x(t), y(t), z(t)\rangle\,dt \\ &= \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle\end{aligned}$$

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ where $t \in [a, b]$, and if $\vec{r}$ is a parameterization of a curve C, then $\vec{v}(t) = \vec{r}'(t)$ is the ***velocity*** vector, and $\vec{a}(t) = \vec{r}''(t)$ is the ***acceleration vector***. Also, $\|\vec{r}'(t)\|$ is the ***speed***, and $\|\vec{r}''(t)\|$ is the ***acceleration***.

Definition 2.2.1 ▸ Unit Tangent Vector

Given $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, the ***unit tangent vector*** is a unit vector that is tangent to the curve defined by $\vec{r}(t)$.

$$T(t) := \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Definition 2.2.2 ▶ Normal Vector

Given $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, the ***normal vector*** is a unit vector that

finish definition

$$N(t) := \frac{\vec{r}''(t)}{\|\vec{r}''(t)\|}$$

Definition 2.2.3 ▶ Osculating Plane, Binormal Vector

At each point t, $T(t)$ and $N(t)$ determine a plane called the ***osculating plane***. An equation of the osculating plane to C at $P_0 = \vec{r}(t)$ has a normal direction $B(t) := T(t) \times N(t)$ called the ***binormal vector*** to the curve C at p_0. The plane determined by $N(t)$ and $B(t)$ is called the ***normal plane*** to the curve C at terminal point of $\vec{r}(t) = p_0$, and it has normal direction given by $T(t)$.

2.3 Arc Length and Curvature

If a curve in the xy-plane is parameterized by $x = f(t)$ and $y = g(t)$ for $t \in [a, b]$, and f and g are differentiable on $[a, b]$, then the total arc length of the curve is given by:

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Similarly, if C is a curve in $\mathbb{R}^3$ parameterized by a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $t \in [a, b]$. Then the total arc length of the curve is given by:

Arc Length of a Curve in $\mathbb{R}^3$

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

Definition 2.3.1 ▶ Curvature, Osculating Circle

The ***curvature*** of a curve at some point is how much it curves at that point. More specifically, it's the inverse of the radius of the ***osculating circle***, tangent to the point that most closely "matches" the curve. This circle is called the ***osculation circle***, and it lies in the osculating plane.

Formally, if C is a curve parameterized by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $t \in [a, b]$, and f, g, h are differentiable twice on $[a, b]$, then the ***curvature*** of C is given by:

$$k(t) := \|\frac{d}{ds}\vec{T}(t)\| = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

A lower curvature at some point actually means that it curves more.

Theorem 2.3.2

The curvature given by a vector function $\vec{r}(t) : \mathbb{R} \to \mathbb{R}^3$ is given by:

$$k(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

If C is a planar curve given by $y = f(x)$, then $k(t) = \frac{\|f''(t)\|}{(1+[f'(t)]^2)^{3/2}}$

Definition 2.3.3 ▶ Torsion

The ***torsion*** of a curve C at a point $\bar{r}(t)$ (the endpoint of $\vec{r}(t)$) is a measure of how much C departs from the osculating plane.

$$\tau := \frac{d}{ds}\vec{B}(t) \cdot \vec{N}(t)$$

Theorem 2.3.4

If C is a "smooth" curve parameterized by $\vec{r}(t)$, then the torsion is given by:

$$\tau := \frac{\left[\vec{r}'(t) \times \vec{r}''(t)\right] \cdot \vec{r}'''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|}$$

Tangential component of $\vec{a}$ is given by:

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|}$$

2.4 Projectile Motion

Suppose we launch some projectile from an initial position $(0,0)$, initial angle θ, and initial velocity $\vec{v}_0$. Disregarding air resistance, the only force acting on this projectile is gravity. Thus, the force vector is given by $\vec{F} = \langle 0, -g, 0 \rangle$ where g is a gravitational constant. Then, the velocity at any time t is given by:

$$\vec{t} = Q1RFSDA$$

3 Partial Derivatives

3.1 Functions of Several Variables

We will only be concerned with functions in the real numbers.

Definition 3.1.1 ▸ Function of n Variables

A ***function of n variables*** is a function $f : X \to Y$ where $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}$. In other words, it maps ordered pairs of real numbers $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^2$ to some real number $y \in \mathbb{R}$. In this context, we call $x_1, \ldots, x_n$ ***independent variables*** and call y a ***dependent variable***.

We will mostly be concerned with functions of two variables, mapping ordered pairs of real numbers to a unique real number. These functions exist in a three-dimensional space, and are thus relatively easy to visualize.

Definition 3.1.2 ▸ Graph

Let f be a function of two variables with domain D. The ***graph*** of f is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $(x, y) \in D$ and $z = f(x, y)$

The graph of a function gives us a really nice visualization of three-dimensional functions. Take for instance a function in two variables defined as $f(x, y) = 6 - 3x - 2y$.

finish example

Definition 3.1.3 ▸ Level Curve

Let f be a function of two variables. A ***level curve*** of f is a curve with equation $f(x, y) = k$ for a fixed k in the range of f.

Formally, let $f : A \to B$ be a function where $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}$. Let $z \in f[A]$ (the range of f). A ***level curve*** is the set of all (x, y, z) such that $f(x, y) = z$.

We call a collection of level curves a ***contour map***. Contour maps are most descriptive when the level curves are equally spaced; that is, we choose equally spaced z values.

3.2 Limits and Continuity

Definition 3.2.1 ▶ Limits for functions of two variables

Let f be a function of two variables with domain D. We say the ***limit*** as f approaches (a,b) is L if, for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $(x,y) \in D$, if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$.

$$\forall(\epsilon > 0)\exists(\delta > 0)\forall((x,y) \in D)\left(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \epsilon\right)$$

make sure this definition is correct

In other words, for any arbitrarily small distance ϵ, there is some disc of points around (x,y) such that every point in that disc maps to something within ϵ distance of the limit L.

Example 3.2.2 ▶ Proving the Limit

Let $f(x,y) = \frac{3x^2}{x^2+y^2}$. Prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Intuition: We may be tempted to try a bunch of curves approaching $(0,0)$ and show that those curves follow the limit. However, this would have to work for every possible curve, so this method is not exhaustive. Instead, we need to leverage the formal definition of the limit.

Note that $x^2 \leq x^2 + y^2$. If $(x,y) \neq (0,0)$, then we can divide both sides by $x^2 + y^2$ to attain $\frac{x^2}{x^2+y^2} \leq 1$. Multiplying both sides by 3, we get $\frac{3x^2}{x^2+y^2} \leq 3$. Also, since $y \neq 0$, then $|y| > 0$. Thus, we can multiply both sides by $|y|$ to get:

$$|f(x,y)| = \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \leq 3|y|$$

Also note that the distance between $(0,0)$ and (x,y) is given by $\sqrt{x^2+y^2}$. Thus, we also have:

$$3|y| \leq 3\sqrt{x^2+y^2} < 3\delta < \epsilon$$

Thus, we want to choose some $\delta \leq \frac{\epsilon}{3}$.

Proof. Let $\epsilon > 0$. Let $\delta \leq \frac{\epsilon}{3}$. Then, for all $\sqrt{x^2+y^2} < \delta = \frac{\epsilon}{3}$, we have:

$$\begin{aligned}
& \sqrt{x^2+y^2} < \frac{\epsilon}{3} \\
\implies \quad & |y| < \frac{\epsilon}{3} \\
\implies \quad & 3|y| < \epsilon \\
\implies \quad & \frac{3x^2|y|}{x^2+y^2} < \epsilon \\
\implies \quad & \left|\frac{3x^2y}{x^2+y^2}\right| < \epsilon \\
\implies \quad & \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon \\
\implies \quad & |f(x,y) - 0| < \epsilon
\end{aligned}$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. ▢

The calculation of limits can be greatly simplified by using the properties of limits:

Theorem 3.2.3 ▸ Properties of Limits

Suppose we had had:

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = l_f \quad \textbf{and} \quad \lim_{(x,y,z)\to(a,b,c)} g(x,y,z) = l_g$$

Then:

1. $\lim_{(x,y,z)\to(a,b,c)} [f(x,y,z) + g(x,y,z)] = l_f + l_g$
2. For a constant $c \in \mathbb{R}$, $\lim_{(x,y,z)\to(a,b,c)} [cf(x,y,z)] = cl_f$
3. $\lim_{(x,y,z)\to(a,b,c)} [f(x,y,z) \cdot g(x,y,z)] = l_f \cdot l_g$
4. If $l_g \neq 0$, then $\lim_{(x,y,z)\to(a,b,c)} \left[\frac{f(x,y,z)}{g(x,y,z)}\right] = \frac{l_f}{l_g}$.

Definition 3.2.4 ▸ Continuity

Let f be a function of two variables with domain D. We say that f is ***continuous*** at a point $(a,b) \in D$ if:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say that f is ***continuous*** on D if f is continuous at every point $(a,b) \in D$.

Intuitively, we can think of a function being continuous if it has no gaps or sudden breaks.

> All polynomials are continuous on $\mathbb{R}^2$, and all rational functions are continuous on their domains.

Consider the function $f(x, y) = \frac{3x^2y}{x^2+y^2}$. This function is defined every except at $(0, 0)$. Thus, we have a gap discontinuity at $(0, 0)$. We could define a function like:

$$g(x) =$$

piecewise function where g(x) = 0

3.3 Partial Derivatives

Let's first consider a function f in one variable. In Calculus I, we define the derivative of f at some point x as:

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We are usually taught to think of this as the "slope of the line tangent to f at the point x"

Now let f be a function in two variables. If $f(x, y) = z$, we define the ***partial derivative with respect to*** x as follows.

$$\frac{\partial}{\partial x} f(x, y) := \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

In the above, we treat y as a constant and only care about how f changes as x changes. If instead we wanted the partial derivative with respect to y, we would have:

$$\frac{\partial}{\partial y} f(x, y) := \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Definition 3.3.1 ▶ Partial Derivative

Let $f : A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = z$ for some $z \in B$. The ***partial derivative*** with respect to x_k is defined as:

$$\frac{\partial}{\partial x_k} f(x_1, \dots, x_n) \coloneqq \lim_{h \to 0} \frac{f(x_1, \dots, x_k + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

Maybe add a neat image here for the geometric interpretation of partial derivatives

We can also write $\frac{\partial}{\partial x} f(x, y)$ as f_x or $D_x f$. If we wanted to take the partial derivative with respect to x twice, we write that as:

$$\frac{\partial^2}{\partial x^2} f(x, y)$$

We can also take a partial derivative with respect x first, then partial derivative respect to y. This is written as:

$$\frac{\partial^2}{\partial y \partial x} f(x, y)$$

Theorem 3.3.2 ▶ Clairaut's Theorem

Suppose that f is a function in two variables that is defined on some disk D containing the point (a, b). If f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

3.4 Tangent Planes

If $z = f(x, y)$ defines a surface S in $\mathbb{R}^3$ by

finish this part

Let $\vec{n}$ denote the normal direction of the tangent plane. Given $f(x, y)$ and some point (x_0, y_0), then:

$$\vec{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$$

Thus, the equation of the tangent plane at (x_0, y_0) is:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

Another way to describe the normal direction is to first define $F(x, y, z) = f(x, y) - z$. Then, we have:

$$\vec{n} = \nabla F(x, y, z) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y), -1 \right\rangle$$

Definition 3.4.1 ▸ Differentiable

A

Formally, let f be a function of two variables defined as $f(x, y) = z$. f is ***differentiable*** at (a, b) if:

Finish definition

Intuitively, a function is differentiable at (x_0, y_0) if its tangent plane is always a "pretty good" approximation for the function near (x_0, y_0).

Theorem 3.4.2 ▸ Differentiability

If partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

More generally, if partial derivatives f_x and f_y exist and are continuous for every point (x, y), then f is differentiable.

3.5 Chain Rule

From Calculus I, if f and g are functions that are differentiable on their domains, and the output of g is always in the domain of f, then $f \circ g$ is also differentiable on its domain. By the Chain Rule, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

The above only applies to functions of the form $f : \mathbb{R} \to \mathbb{R}$. We can generalize the chain rule for any function $f : \mathbb{R}^n \to \mathbb{R}$ as follows:

Theorem 3.5.1 ▸ Generalized Chain Rule

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. So, $f(x_1, x_2, \dots, x_n) \in \mathbb{R}$. Suppose that each x_j is a differentiable function in m variables, so $x_j(t_1, t_2, \dots, t_m) \in \mathbb{R}$. The partial derivative of f with respect to t_k is as follows:

$$\frac{\partial f}{\partial t_k} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t_k} \right)$$

We can also write:

jacobian matrix, chain rule in terms of gradients

3.6 Directional Derivatives

Definition 3.6.1 ▶ Directional Derivative

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. The ***directional derivative*** of f at a point $(x_1, \dots, x_n)$ in the direction of a unit vector $\vec{u} := \langle u_1, \dots, u_n \rangle$ is defined as:

$$D_{\vec{u}} f(x_1, \dots, x_n) = \lim_{h \to 0} \frac{f(x_0 + hu_1, \dots, x_n + hu_n) - f(x_0, \dots, x_n)}{h}$$

Theorem 3.6.2

If f is differentiable at (x_0, y_0), then $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$.

If $w = F(x, y, z)$ is a differentiable function, then a level curve for F at k can be described as:

$$S_k := \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = k\}$$

If a curve C is contained in S_k, we write this as $C \subseteq S_K$, and has a vector function parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. Then $w(t) = F(\overline{r}(t)) = F(x(t), y(t), z(t))$ where $\overline{r}(t)$ denotes the end point of $\vec{r}(t)$.

Definition 3.6.3 ▶ Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The ***gradient*** of f is a vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\nabla f(x_1, \dots, x_n) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

- The directional derivative of f at $\vec{x}$ in the direction of a unit vector $\vec{u}$ is given by $D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$.
- $\nabla f(\vec{x})$ points in the direction of maximum rate of increase of f at $\vec{x}$, and that maximum rate of change is $|\nabla f(\vec{x})|$.
- $\nabla f(\vec{x})$ is perpendicular to the level curve or level surface of f through x.

Technique 3.6.4 ▸ Finding Maximum Rate of Change at a Point

Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, the maximum rate of change at a point $(x_1, \dots, x_n)$ is given by:

$$\|\nabla f(x_1, \dots, x_n)\|$$

3.7 Maximum and Minimum Values

Definition 3.7.1 ▸ Local Maximum, Local Minimum

Let $f : A \to B$ be a function where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}$, and let $p \coloneqq (a_1, \dots, a_n) \in A$. We say there is a:

- ***local maximum*** at p if there exists some $r > 0$ such that, for all points $q \in A$ within r distance of p, $f(p) \geq f(q)$. Then $f(p)$ is the ***local maximum value***.
- ***local minimum*** at p if there exists some $r > 0$ such that, for all points $q \in A$ within r distance of p, $f(p) \leq f(q)$. Then $f(p)$ is the ***local minimum value***.

Definition 3.7.2 ▸ Critical Point

Let $f : A \to B$ be a function where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}$, and let $p \coloneqq (a_1, \dots, a_n) \in A$. We say p is a ***critical point*** of f if either:

- **all** of the partial derivatives at p is 0, or
- at least one partial derivative does not exist.

It can be proven that local maximums and minimums only happen at critical points; however, critical points are not guaranteed to be local minimums nor maximums.

Definition 3.7.3 ▸ Saddle Point

Let $f : A \to B$ be a function where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}$, and let $p \coloneqq (a_1, \dots, a_n) \in A$. We say p is a ***saddle point*** if it is a critical point but is neither a local maximum nor local minimum.

To classify whether a critical point is a local minimum, local maximum, or saddle point, we can use the following theorem:

Theorem 3.7.4 ▶ Second Derivative Test

Let f be a function of two variables. Suppose p is a critical point of f. Let:

$$D \coloneqq \frac{\partial^2 f(a,b)}{\partial x^2} \cdot \frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial x \, \partial y}\right)^2$$

Definition 3.7.5 ▶ Absolute Maximum, Absolute Minimum

Let $f : A \to B$ be a function where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}$, and let $p \coloneqq (a_1, \dots, a_n) \in A$. Then $f(p)$ is the:

- ***absolute maximum*** of f if $f(p) \geq f(q)$ for all $q \in \mathbb{R}^n$.
- ***absolute minimum*** of f if $f(p) \leq f(q)$ for all $q \in \mathbb{R}^n$.

3.8 Lagrange Multipliers

The Lagrange multipliers method is used to maximize or minimize a general function (say $f(x, y, z)$) subject to a constraint of the form $g(x, y, z) = k$.

Recall that the directional derivative in the direction of a unit vector $\vec{u}$ is defined as:

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

We can rewrite this as:

$$D_{\vec{u}} f(x, y) = \|\nabla f(x, y)\| \|\vec{u}\| \cos\theta$$

where θ is the angle between the gradient and $\vec{u}$.

Technique 3.8.1 ▶ Finding Extreme Values using Lagrange Multipliers

Say we have a function $f(x, y)$ and some constraint $g(x, y) = z$, and wanted to find its extreme values. To apply the method, we:

1. Set up the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$.
2. Solve for λ, x, and y.
3. Plug in the values you have found for x and y into the function f to find the extreme values.

Example 3.8.2

Find the extreme values of f

Get examples from paper notes into here

4

Multiple Integrals

Recall the intuition behind integrals: we are chopping up a function's graph into little bits and summing the area of each of those bits. The same intuition can be applied to multiple integrals. Instead of our bits being slivers of the two-dimensional graph, we will instead have our bits be very skinny rectangular prisms of the three-dimensional graph.

4.1 Double Integrals

Definition 4.1.1 ▸ Double Integral

The ***double integral*** of f over a rectangle R is:

$$\iint_R f(x,y)dA =$$

formal definition

Theorem 4.1.2

If $f(x,y)$ is a continuous function defined on a rectangle $R := [a,b] \times [c,d]$, then the limit always exists, and

$$\iint_R f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

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