MATH 231: Differential Equations

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Definition ▸ Differential Equation

A **differential equation** is an equation that relates one or more unknown functions and their derivatives.

- **Ordinary Differential Equation (ODE)**: One input
- **Partial Differential Equation (PDE)**: Multiple inputs

Example 1.0.1 ▸ Free Fall (ODE)

Acceleration	$\frac{d^2h}{dt^2} = -g$	$h''(t) = -g$	$g = 9.8\frac{\text{m}}{\text{s}^2}$	
Velocity	$\frac{dh}{dt} = -gt + c_1$	$h'(t) = -gt + c_1$	c_1 is initial velocity	
Position	Solved	$h(t) = -\frac{1}{2}gt^2 + c_1 t + c_2$	c_2 is initial position	

Note: $\dot{h}(t)$ represents the first derivative of $h(t)$, and $\ddot{h}(t)$ represents its second derivative.

Example 1.0.2 ▸ Exponential Growth (ODE)

Diff. Eq.	$\dot{x}(t) = kx(t)$ or $\frac{dt}{dx} = kt$
Solution	$x(t) = x_o e^{kt}$
Computation	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Example 1.0.3 ▸ Heat Equation (PDE)

$h(t, x)$: temperature at time t and location x

$$\frac{\delta h}{\delta t} = \frac{1}{2}\frac{\delta^2 h}{\delta x^2}$$

Note: This is an example of a **partial** differential equation as it has two inputs, t and x.

Definition ▶ Linearity

A differential equation is **linear** if it follows the form:

$$F(x) = a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0(x)y$$

where $a_n(x)$, ..., $a_0(x)$, and $F(x)$ depend only on the independent variable x (i.e. linear equations can only be ODE).

Technique ▶ Solving first order linear ODEs

Take a first order linear ODE in standard form:

$$\frac{dy}{dx} + p(x)y = Q(x)$$

Using a function known as the integrating factor...

$$I(x) = e^{\int p(x)\,dx}$$

We can now find the general solution:

$$y = \frac{1}{I(x)}\left[\int I(x)Q(x)\,dx + C\right]$$

Example 1.0.4 ▶ ${}^{dy}/_{dx} + 2y = 2e^x$

This equation is already in standard form. Let $P(x) = 2$ and $Q(x) = 2e^x$.

$$I(x) = e^{\int P(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$\begin{aligned} y &= \frac{1}{e^{2x}}\left[\int e^{2x} 2e^x\,dx + C\right] \\ &= \frac{1}{e^{2x}}\left[2\int e^{3x}\,dx + C\right] \\ &= \frac{1}{e^{2x}}\left[2\frac{e^{3x}}{3} + C\right] \\ &= \frac{2}{3}e^x + Ce^{-2x} \end{aligned}$$

Technique ▶ Solving Exact Linear Equations

$$M(x,y)dx + N(x,y)dy = 0$$

Check for exactness:

$$\frac{\delta M(x,y)}{\delta y} = \frac{\delta N(x,y)}{\delta x}$$

Then we can express the solution function as such:

$$\frac{\delta f(x,y)}{\delta x} = M(x,y)$$

$$\frac{\delta f(x,y)}{\delta y} = N(x,y)$$

Take the antiderivative of $\frac{\delta f}{\delta x}$ with respect to x

$$f(x,y) = \int M(x,y)\,dx + h(y)$$

$h(y)$ is a generic function that may have been lost in differentiation. To recover $h(y)$, take the derivative of $f(x,y)$ with respect to y

$$\frac{\delta f(x,y)}{\delta y} = N(x,y) + h'(y)$$

Definition ▶ Order

The **order** of a differential equation is defined by the most dominant derivative term.

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First Order Differential Equations

Definition ▸ Separable Equations

A **separable equation** follows the form $\frac{dy}{dx} = f(x)g(y)$. It's easy to work with as it can be rewritten as $\frac{dy}{g(y)} = f(x)dx$

Definition ▸ Homogeneous Differential Equation

A differential equation is **homogeneous** if each term of the equation is the same order.
Example: $\frac{dy}{dx} = \frac{xy}{x^2-y^2}$ Here, xy, x^2, and y^2 are all order 2.

Theorem ▸ Solving Homogeneous DEs

We can use a clever substitution to turn a difficult DE into something which can be solved more easily.

$$y = vx, \ dy = vdx + xdv$$

Near the end, we can substitute back.

$$v = \frac{y}{x}$$

3 Mathematical Modeling

Compartmental Analysis

$x(t)$	amount of substance in compartment at time t
$\frac{dx}{dt}$	rate of change of amount of substance in compartment

$$\frac{dx}{dt} = \text{input rate} \ - \ \text{output rate}$$

Population Model

$p(t)$: population at time t

Mathusian's Law: k_1 is birthrate, k_2 is deathrate, $k = (k_1 - k_2)$ is growth rate

$$\frac{dp}{dt} = k_1 p(t) - k_2 p(t) = kp(t)$$

$$\Rightarrow p(t) = p_0 e^{kt}$$

Logistic Law: $A = \frac{k_1}{2}$, $p_1 = \frac{2k_1}{k_3}$ is capacity

$$\frac{dp}{dt} = k1_p(t) - k_3 \frac{p(t)(p(t)-1)}{2}$$

$$\Rightarrow \frac{dp}{dt} = -Ap(p - p_1)$$

$$\Rightarrow \left|1 - \frac{p_1}{p(t)}\right| = ce^{-Ap_1 t}$$

$$\Rightarrow p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$$

Heating and Cooling

Here, we will only consider time and assume that heat is uniform inside of a room.

$T(t)$	temperature inside a building at time t
$\frac{dT}{dt}$	rate of change of temperature in a building
$H(t)$	heat produced by people inside a building
$U(t)$	heat produced by cooling system
$k[M(t) - T(t)]$	heat from outside where k is a constant and $M(t)$ is the temp. outside

$$\frac{dT}{dt} = k\,[M(t) - T(t)] + U(t) + H(t)$$

$$\Rightarrow \frac{dT}{dt} + kT(t) = kM(t) + U(t) + H(t)$$

4 Linear Second-Order Differential Equations

Homogeneous

These equations usually follow a similar form where a, b, c are constants:

$$ay'' + by' + cy = 0$$

We solve the **characteristic equation** to determine our answer:

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three possible cases:

1. $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots, r_1 and r_2
   General Solution: $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
2. $b^2 - 4ac = 0 \Rightarrow$ one real root, r
   General Solution: $y(x) = c_1 e^{rx} + c_2 x e^{rx}$
3. $b^2 - 4ac < 0 \Rightarrow$ two complex roots, $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$
   General Solution: $y(x) = e^{\alpha x}\left[c_1 \cos(\beta x) + c_2 \sin(\beta x)\right]$

Non-Homogeneous (Undetermined Coefficients)

These equations usually follow a similar form where a, b, c are constants:

$$ay'' + by' + cy = G(x)$$

Our general solution is of the form

$$y_g(x) = y_c(x) + y_p(x)$$

- $y_c(x)$ is the general solution of the homogeneous equation when $G(x) = 0$
- $y_p(x)$ is the particular solution which follows a standard form depending on $G(X)$:

Standard guesses for $y_p(x)$:

$G(x)$	Guess for $y_p(x)$
e^{rt}	$Ae^r t$
$\sin(rt)$ or $\cos(rt)$	$A\sin(rt) + B\cos(rt)$
Degree n polynomial	$A_0 + A_1 t + \ldots + A_n t^n$

Since these are guesses, you may need to:

- Multiply by t^s to avoid matching $y_c(x)$
- Add/multiply different types together

Non-Homogeneous (Variation of Parameters)

Our general solution is of the form

$$y_g(x) = y_c(x) + y_p(x)$$

- $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ is the complementary solution
- $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ is the particular solution

Goal: Find $u_1(x)$ and $u_2(x)$ such that $y_p(x)$ is a valid solution

Tip: Arbitrarily impose $u_1{}'(x)y_1(x) + u_2{}'(x)y_2(x) = 0$

Now apply the following substitution:

$$k(t) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1{}'(t) & y_2{}'(t) \end{pmatrix}$$

$$u_1 = -\frac{1}{a}\int \frac{y_2(t)f(t)}{k(t)}dt$$

$$u_2 = \frac{1}{a}\int \frac{y_1(t)f(t)}{k(t)}dt$$

Alternatively, impose the same constraint but make this substitution instead:

- $y'(x) = u_1(x)y_1{}'(x) + u_2(x)y_2{}'(x)$
- $y''(x) = u_1{}'(x)y_1{}'(x) + u_1(x)y_1{}''(x) + u_2{}'(x)y_2{}'(x) + u_2(x)y_2{}''(x)$

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Laplace Transformation

Definition ▶ Laplace Transform

Let $f(t)$ be a function defined on $[0, \infty)$. The **Laplace transform** of f is defined as:

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

We use F to denote the Laplace transform of f:

$$F(S) = \mathscr{L}\{f\}(s)$$

Notice the Laplace transform is an improper integral, so we need to consider what conditions will cause it to diverge.

Example 5.0.1 ▶ $\mathscr{L}\{e^{at}(s)\}$

We simply plug e^{at} as $f(t)$ in our formula:

$$\begin{aligned}
\mathscr{L}\{e^{at}\}(s) &= \int_0^\infty e^{-st} e^{at}\, dt \\
&= \int_0^\infty e^{(a-s)t}\, dt \\
&= \left. \frac{e^{(a-s)t}}{a-s} \right|_{t=0}^{\infty} \\
&= \begin{cases} 0 - \frac{1}{a-s} = \frac{1}{s-a}, & s > a \\ \text{Diverges}, & s \le a \end{cases}
\end{aligned}$$

Thus, $\mathscr{L}\{e^{at}\}(s) = \frac{1}{s-a}$ if $s > a$.

Common Laplace Transforms

$f(t)$	$\mathscr{L}\{f\}(s)$	Conditions
1	$\dfrac{1}{s}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$	$s > a$

Properties of Laplace Transforms

Linearity	$\mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$
Constant c	$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$
Translation	$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s-a)$
1st Derivative	$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$
2nd Derivative	$\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$
nth Derivative	$\mathscr{L}\left\{f^{(n)}\right\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$t^n f(t)$	$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\left(\mathscr{L}\{f\}(s)\right)$

Example 5.0.2 ▸ $f(t) = sin(bt)$ for some $b \neq 0$

$$\begin{aligned}
\mathscr{L}\{f\}(s) &= \int_0^\infty e^{-st}\sin(bt)\,dt \\
&= \lim_{M\to\infty}\int_0^M e^{-st}\sin(bt)\,dt \\
&= \lim_{M\to\infty}\left[-\frac{e^{-st}}{b}\cos(bt)\right]_{b=0}^{M} - \frac{s}{b}\lim_{M\to\infty}\int_0^M \cos(bt)e^{-st}\,dt \\
&= \frac{1}{b} - \frac{s}{b}\lim_{M\to\infty}\int_0^M e^{-st}\cos(bt)\,dt \\
&= \frac{1}{b} - \frac{s}{b}\lim_{M\to\infty}\left[\left[\frac{e^{-st}}{b}\sin(bt)\right]_{b=0}^{M} + \frac{s}{b}\int_0^M \sin(bt)e^{-st}\,dt\right] \\
&= \frac{1}{b} - \frac{s}{b}\left[0 + \frac{s}{b}\int_0^\infty e^{-st}\sin(bt)\,dt\right] \\
&= \frac{1}{b} - \frac{s^2}{b^2}(L)(f)(s) \\
\Rightarrow \left[1+\frac{s^2}{b^2}\right]\mathscr{L}\{f\}(s) &= \frac{1}{b} \\
\Rightarrow \frac{b^2+s^2}{b^2}\mathscr{L}\{f\}(s) &= \frac{1}{b} \\
\Rightarrow \mathscr{L}\{f\}(s) &= \frac{b}{b^2+s^2} \text{ if } s > 0
\end{aligned}$$

Example 5.0.3 ▸ $f(t) = e^{at}$ for some a

$$\begin{aligned}
\mathscr{L}\{e^{at}\}(s) &= \int_0^\infty e^{-st}e^{at}\,dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \lim_{M\to\infty}\left[\frac{1}{a-s}e^{(a-s)t}\right]_{t=0}^{M} \\
&= \frac{1}{a-s}\lim_{M\to\infty}\left[e^{(a-s)M}-1\right] \\
&= \frac{1}{s-a} \text{ if } s > a
\end{aligned}$$

Example 5.0.4 ▸ Multi-Case Function

Let $f(t) = \begin{cases} 2 & \text{if } 0 < t < 5, \\ 0 & \text{if } 5 \le t < 10, \\ e^{at} & \text{if } t \ge 10 \end{cases}$

$$\begin{aligned}
\mathscr{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^5 e^{-st} 2\, dt + \int_5^{10} e^{-st} 0\, dt + \int_0^\infty e^{-st} e^{at}\, dt \\
&= 2\int_0^5 e^{-st}\, dt + 0 + \int_{10}^\infty e^{(4-s)t}\, dt \\
&= -\frac{2}{s} \lim_{M\to\infty} \left[e^{-st}\right]_{t=0}^5 + \lim_{M\to\infty} \left[\frac{e^{(4-s)t}}{4-s}\right]_{t=10}^M \\
&= -\frac{2}{s}(e^{-5s} - 1) + \frac{1}{4-s} \lim_{M\to\infty} \left[e^{(4-s)M} - e^{(4-s)10}\right] \\
&= -\frac{2e^{-5s}}{s} + \frac{2}{s} + \frac{e^{-10(s-4)}}{s-4} \text{ if } s > 4
\end{aligned}$$

Theorem ▸ Linearity

For two functions f_1 and f_2, and any constants c_1 and c_2, we have:

$$\mathscr{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathscr{L}\{f_1\} + c_2 \mathscr{L}\{f_2\}$$

Example 5.0.5 ▸ Linearity

Let $f(t) = 11 + 10e^{4t} + 20\sin(2t)$

$$\begin{aligned}
\mathscr{L}\{f\} &= \mathscr{L}\{11\} + \mathscr{L}\{10e^{4t}\} + \mathscr{L}\{20\sin(2t)\} \\
&= 11\mathscr{L}\{1\} + 10\mathscr{L}\{e^{4t}\} + 20\mathscr{L}\{\sin(2t)\} \\
&= \frac{11}{s} + \frac{10}{s-4} + 20\frac{2}{s^2+2} \text{ if } s > 4 \\
&= \frac{11}{s} + \frac{10}{s-4} + \frac{40}{s^2+4} \text{ if } s > 4
\end{aligned}$$

Theorem ▸ Existence of Laplace Transform

If $f(t)$ is continuous and exponential order with constant c, then

$$\mathscr{L}\{f(t)\}(s) = F(s)$$

is defined for all $f > c$.

Theorem ▸ Existence of Laplace Transformation

Theorem: Assume that there is $\alpha > 0$ such that

$$|f(t)| \leq Me^{at} \text{ when } t \geq T$$

Then $\mathscr{L}\{f\}$ exists.

Intuition:

$$\begin{aligned}\mathscr{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt\end{aligned}$$

$$\left|\int_T^\infty e^{-st} f(t)\, dt\right| \leq \int_T^\infty e^{-st} |f(t)|\, dt \leq M \int_T^\infty e^{(\alpha-s)t}\, dt < \infty$$

Technique ▸ Solving an Initial Value Problem

1. Take Laplace transform of both sides of the equation.
2. Reduce the equation to $\mathscr{L}\{y\} = \ldots$
3. $y = \mathscr{L}^{-1}\{\ldots\}$

Example 5.0.6 ▶ Initial Value Problem

$y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12$

1. Take Laplace transform of both sides.

$$\mathscr{L}\{y'' - 2y' + 5y\} = \mathscr{L}\{-8e^{-t}\}$$

2. We can use linearity to rewrite this as:

$$\mathscr{L}\{y''\}(s) - 2\mathscr{L}\{y'\}(s) + 5\mathscr{L}\{y\}(s) = -\frac{8}{s+1}$$

Let $Y(s) \coloneqq \mathscr{L}\{y\}(s)$.

5.1 Laplace Transform of Discontinuous Functions

Definition ▶ Unit Step Function

$$u(t) \coloneqq \begin{cases} 1, & t < 0 \\ 0, & 0 < t \end{cases}$$

Definition ▶ Rectangular Window Function

$$\Pi_{a,b}(t) \coloneqq u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & b < t \end{cases}$$

Theorem ▶ Translation

If $a > 0$, then:

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathscr{L}\{f\}(s)$$

Conversely:

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$$

5.2 Dirac Delta Functions

In physics, we can model impulses as an "instantaneous force". How do we represent this in mathematics? We use the **dirac delta function** to represent an instantaneous thing.

Definition ▸ Dirac Delta Function

$$\delta(x) = \lim_{b \to 0} \frac{a}{|b|\pi} e^{-x/b^2}$$

Informally, we can think of this function as:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

The Dirac delta function is a distribution that satisfies $\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$.

Theorem ▸ Identity

For any $f : \mathbb{R} \to \mathbb{R}$, we have:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$$

Proof. Let $s \coloneqq t - a$ and $ds \coloneqq dt$. Using integration by substitution, we have:

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt &= \int_{-\infty}^{\infty} f(s+a)\delta(s)\,ds \\ &= f(0+a) \end{aligned}$$

□

Example 5.2.1 ▶ Laplace of Dirac Delta Function

What is $\mathscr{L}\{\delta\}$?

$$\begin{aligned}\mathscr{L}\{\delta\} &= \int_0^\infty e^{-st}\delta(t)\,dt \\ &= \int_{-\infty}^\infty e^{-st}\delta(s)\,ds \\ &= e^{-0\cdot s} \\ &= 1\end{aligned}$$

Theorem ▶

For any $a \in \mathbb{R}$, $\mathscr{L}\{\delta(t-a)\} = e^{-as}$.

Example 5.2.2 ▶ Spring Force System

$$\begin{cases} y'' + 9y = 3\delta(t-\pi) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Let $Y \coloneqq \mathscr{L}\{y\}$. Then:

$$\begin{aligned}\mathscr{L}\{y''\} + 9\mathscr{L}\{y\} &= 3\mathscr{L}\{\delta(t-\pi)\} \\ s^2Y - sy(0) - y'(0) + 9Y &= 3e^{-\pi s} \\ (s^2+9)Y &= 3e^{-\pi s} + s\end{aligned}$$

Thus, $Y(s) = \frac{3e^{-\pi s}}{s^2+9} + \frac{s}{s^2+9}$. Therefore, $y(t) = u(t-\pi)\sin(3(t-\pi)) + cos(3t)$.

6

Qualitative Study

Consider the ODE $y'' = f(y)$ where f is a given function.

- $y'' + (1 - y^2)y = 0 \implies f(y) = -(1 - y^2)y$
- $y'' = y^2 \implies f(y) = y^2$

Definition ▸ Energy

$$E(t) = \frac{1}{2}y'(t)^2 - F(y)$$

where F is an antiderivative of f. In other words, $F'(y) = f(y)$.

Theorem ▸ Energy

If $y(t)$ is a solution of $y'' = f(y)$, then $E'(t) = 0$. In particular, $E(t) = C$ where C is some constant.

Proof. Consider the definition of the energy equation:

$$E(t) = \frac{1}{2}y'(t)^2 - F(y)$$

Differentiating with respect to t, we get:

$$\begin{aligned} E'(t) &= \frac{1}{2} \cdot 2y'(t)y''(t) - f(y(t))y'(t) \\ &= y'(t)\left[\underbrace{y''(t) - f(y(t))}_{=0}\right] \end{aligned}$$

□

Example 6.0.1 ▸ Application

$$\frac{1}{2}(y'(t))^2 - f(y) = K \implies y'(t) = 2\left[F(y) + K\right]$$

so

$$y'(t) = \pm\sqrt{2\left[F(y) + K\right]}$$

Separation of variables:

$$\pm\frac{dy}{\sqrt{2\left[F(y) + K\right]}} = dt$$

Then:

$$t = \pm\int\frac{dy}{\sqrt{2\left[F(y) + K\right]}} + C$$

Example 6.0.2 ▸

$$y'' = 6y^2$$

Here, $f(y) = 6y^2$, so

$$\begin{aligned} t &= \pm\int\frac{dy}{\sqrt{2\left[2y^3 + K\right]}} \\ &= \pm \in \frac{dy}{2\sqrt{y^3 + K_1}} \end{aligned}$$

Let's consider for the sake of simplicity $K_1 = 0$. Then:

$$\begin{aligned} t &= \frac{1}{2}\int\frac{dy}{y^{3/2}} + C \\ &= \frac{1}{2}\int y^{-3/2}\,dy + C \\ &= \frac{1}{2}\cdot\frac{1}{-1/2}y^{-1/2} + C \\ &= C - y^{-1/2} \end{aligned}$$

Then $y^{-1/2} = C - t \implies y^{-1} = (c-t)^2$, so $y = \frac{1}{(C-t)^2}$

6.1 Free Mechanical Vibration

The mass-spring system can be modelled with the following equation:

$$my'' + by' + ky = F_{ex}$$

where:

- m: inertia (mass)
- b: damping constant
- k: stiffness
- F_{ex}: external forces

In free vibration, we would have $F_{ex} = 0$. Hence, we will only consider:

$$my'' + by' + ky = 0$$

Goal: Dynamic solutions $y(t)$. In particular, what does $y(t)$ look like as t increases?

6.1.1 Undamped Case

$b = 0$

$$my'' + ky = 0$$

Characteristic Equation: $mr^2 + k = 0$, so $r_{1,2} = \pm i\sqrt{k/m}$. Denote $\omega = \sqrt{k/m}$ called **angular frequency**. Then, general solution is:

$$y(t) = c_1 \underbrace{\cos(\omega t)}_{y_1(t)} + c_2 \underbrace{\sin(\omega t)}_{y_2(t)}$$

But how does $y(t)$ behave?

$$\begin{aligned} y(t) &= \sqrt{c_1^2 + c_2^2}\left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos(\omega t) + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin(\omega t)\right] \\ &= \sqrt{c_1^2 + c_2^2}\,[\cos(\omega t)\sin(\phi) + \sin(\omega t)\cos(\phi)] \\ &= A\sin(\omega t + \phi) \end{aligned}$$

where:

- $A = \sqrt{c_1^2 + c_2^2}$ is an amplitude
- $\omega = \sqrt{k/m}$ is angular frequency
- $\phi = \tan^{-1}(c_1/c_2)$
- $\omega/2\pi$ is natural frequency
- $2\pi/\omega$ is period

6.1.2 Overdamped Case

We consider a spring system "overdamped" if $b^2 > 4mk$. Then:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Note that $r_1 < 0$ and $r_2 < 0$.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

In general, as t increases, $y(t)$ will exponentially approach 0. Sometimes $y(t)$ may have one local maximum or minimum.

6.1.3 Underdamped Case

We consider a spring system "underdamped" if $b^2 < 4km$. Then:

$$r_{1,2} = \underbrace{-\frac{b}{2m}}_{\alpha} \pm i \underbrace{\frac{\sqrt{4km - b^2}}{2m}}_{\beta}$$

$$\begin{aligned} y(t) &= e^{\alpha t}\left[c1 \cos(\beta t) + c_2 \sin(\beta t)\right] \\ &= Ae^{\alpha t} \sin(\beta t + \phi) \end{aligned}$$

As t increases, $y(t)$ will oscillate but still approach 0.

6.1.4 Critically Damped

We consider a spring system "critically damped" if $b^2 = 4km$. Then:

$$r_1 = r_2 = -\frac{b}{2m}$$

$$\begin{aligned} y(t) &= c_1 e^{-\frac{b}{2m}t} + c_2 t e^{-\frac{b}{2m}t} \\ &= e^{-\frac{bt}{2m}}(c_1 + c_2 t) \end{aligned}$$