Theorem 0.0.1 ▸ $\mathbb{N}$ is not Bounded Above

Proof. Suppose for contradiction $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is not empty, then $\mathbb{N}$ has a supremum in $\mathbb{R}$. Let $s := \sup \mathbb{N} \in \mathbb{R}$. Then $n \leq s$ for all $n \in \mathbb{N}$. By the Peano axioms, n has a successor $n + 1 \in \mathbb{N}$, so $n + 1 \leq s$ for all $n \in \mathbb{N}$. Therefore, $n \leq s - 1$ for all $n \in \mathbb{N}$. This contradicts s being the least upper bound for $\mathbb{N}$. ▢

Theorem 0.0.2 ▸ Archimedean Principle

Suppose $x, y \in \mathbb{R}$ where $x > 0$. Then, there exists $n \in \mathbb{N}$ such that $nx > y$.

Intuition: This is basically an extension of the fact that $\mathbb{N}$ is not bounded above.

Proof. Since y/x is not an upper bound for $\mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $n > y/x$. Since $x > 0$, then $nx > y$. ▢

Theorem 0.0.3 ▸ Density of $\mathbb{Q}$ in $\mathbb{R}$

Suppose $x, y \in \mathbb{R}$ where $x < y$. Then there exists $r \in Q$ such that $x < r < y$.

Intuition: Given any two different real numbers, there's some rational number between them.

Proof. We will consider three cases:

1. If $x \geq 0$, then $0 \leq x < y$. Since $y - x > 0$, then by the Archimedean Principle, there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$. We want to show there is a natural number between nx and ny. Let $A := \{k \in \mathbb{N} : k > nx\}$. Since $\mathbb{N}$ isn't bounded above, then A is not empty. By the ??, A has a minimum. Let $m := \min A$. Then $m > nx$, and $m - 1 \leq nx$. Thus, $m \leq nx + 1$, so:

$$nx < m \leq nx + 1 < ny$$

   Dividing across by n yields $x < m/n < y$. Note that $m, n \in \mathbb{N} \subseteq \mathbb{Z}$, so $m/n \in \mathbb{Q}$.

2. If $x < 0$ and $y > 0$, then $x < 0 < y$ where $0 \in \mathbb{Q}$.

3. If $x < 0$ and $y \leq 0$, then $x < y \leq 0$. Multiplying across by -1, we have $-x > -y \geq 0$. By the first case, there must exist $t \in \mathbb{Q}$ where $-y < t < -x$. Multiply across by -1 again to attain $y > -t > x$ where $-t \in \mathbb{Q}$.

This completes the proof. ▢

Theorem 0.0.4 ▸ $\sqrt{2}$ is a Real Number

There exists $s \in \mathbb{R}$ such that $s^2 = 2$.

Proof. Let $A := \{x \in \mathbb{R} : x^2 < 2\}$. Since $0^2 < 2$, then $0 \in A$, so A is not empty. Also, A is bounded above, for example by 2. By completeness, A must have a supremum in $\mathbb{R}$. Let $s := \sup A$. We will use trichotomy to show that $s^2 = 2$.

1. If $s^2 > 2$, then...

   Scratchwork: We need to show that this is not possible, i.e. show there is some $s - 1/n$ that is less than s but is still an upper bound for A. We want $(s - 1/n)^2 > 2$. Then, $s^2 - 2s/n + 1/n^2 > 2$. We can chop off the $1/n^2$, reducing the inequality to $s^2 - 2s/n > 2$. Thus, we need to choose $n > \frac{2s}{s^2-2}$.

   ... let $n \in \mathbb{N}$ such that $n > \frac{2s}{s^2-2}$. Then:

$$\begin{aligned} & n > \frac{2s}{s^2 - 2} \\ \implies \quad & s^2 - \frac{2s}{n} > 2 \\ \implies \quad & s^2 - \frac{2s}{n} + \frac{1}{n^2} > 2 \\ \implies \quad & \left(s - \frac{1}{n}\right)^2 > 2 \end{aligned}$$

   Thus, $s - 1/n$ is an upper bound for A that is less than s. This contradicts s being the supremum for A.

2. If $s^2 < 2$, then...

   Scratchwork: Again, we need to show that this is not possible. We know that in this case, $s \in A$, so we need to find another thing in A that is bigger than s. In other words, we want some $(s + 1/n)^2 < 2$. Then, $s^2 + 2s/n + 1/n^2 < 2$. Choose $n > 1/2s$ and $n > \frac{4s}{2-s^2}$.

$$\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2}$$

... let $n \in \mathbb{N}$ such that $n > \max\left\{\frac{1}{2s}, \frac{4s}{2-s^2}\right\}$. Then $\frac{1}{n} < 2s$ and $s^2 + \frac{4s}{n} < 2$. So:

$$\begin{aligned}\left(s + \frac{1}{n}\right)^2 &= s^2 + \frac{2s}{n} + \frac{1}{n^2} \\ &< s^2 + \frac{2s}{n} + \frac{2s}{n} \\ &= s^2 + \frac{4s}{n} < 2\end{aligned}$$

That is, $s + \frac{1}{n} \in A$. This contradicts s being an upper bound for A.

By trichotomy, $s^2 = 2$. ▢

Theorem 0.0.5 ▸ Nested Interval Property

Suppose that for each $n \in \mathbb{N}$, $a_n, b_n \in \mathbb{R}$ with $a_n \leq b_n$, and $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \varnothing$.

Intuition: We can move the two borders of an open interval closer and closer to each other, and it won't be empty.

Proof. Note that $a_n \leq a_{n+1} \leq a_{n+2} \leq \ldots$ and $\ldots \leq b_{n+2} \leq b_{n+1} \leq b_n$. If $k \leq n$, then $a_k \leq a_n \leq b_n$.

- If $k \leq n$, then $a_k \leq a_n \leq b_n$.
- If $k \geq n$, then $a_k \leq b_k \leq b_n$.

That is, $a_k \leq b_n$ for all $k_n \in \mathbb{N}$. Let $A \coloneqq \{a_k : k \in \mathbb{N}\}$. Then A is bounded above, for example by b_1. Also, A is not empty. By completeness, A has a supremum. Let $s \coloneqq \sup A$. Note that since s is an upper bound for A, then $a_n \leq \sup A$ for all $n \in \mathbb{N}$. Also note that $\sup A$ is the least upper bound for A, so $\sup A \leq b_n$ for all $n \in \mathbb{N}$. Thus, $a_n \leq \sup A \leq b_n$ for all $n \in \mathbb{N}$, so $\sup A \in [a_n, b_n]$ for all $n \in \mathbb{N}$. Thus, $\sup A \in \bigcap_{n=1}^{\infty}[a_n, b_n]$, so it is not empty. ▢

The nested interval property is actually false for open intervals!

$$\forall(x \in (0,1))\exists(n \in \mathbb{N})(1/n < x \implies x \notin (0, 1/n))$$