If Influence Functions are the Answer, Then What is the Question?

Paper review

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Contents

Setting

Consider a prediction task (regression problem) with:

- Input space $\mathcal{X}$;
- Output space $\mathcal{Y}$;
- Training set $\mathcal{D}^n = \{z_i\}_{\{i=1\}}^n$ where $z_i = (x_i, y_i)$ for all $i = 1, \ldots, n$;
- Parameter $\theta \in \Theta := \mathbb{R}^d$;
- $f(\theta; x)$ estimator of $\mathcal{Y}|\mathcal{X}$;
- $l : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ loss function (e.g., $l(y', y) \mapsto \|y' - y\|^2$).

We aim to minimize the training error:

$$L(\theta; \mathcal{D}^n) = \frac{1}{n} \sum_{i=1}^{n} l(f(\theta; x_i), y_i).$$

What happens if we change the importance of a training point $z = (x, y)$ of the dataset?

Call $\hat{\theta} = \arg\min_{\theta \in \Theta} L(\theta; \mathcal{D}^n)$.
How different is it from

$$\hat{\theta}_{\varepsilon,-z} = \arg\min_{\theta \in \Theta} (L(\theta; \mathcal{D}^n) - \varepsilon l(f(\theta; x), y)) \quad ?$$

We can re-train the whole model on $\mathcal{D}^n \setminus \{z\}$ (Leave One Out method), or...

Influence loss difference

Definition

Given $(\overline{x}, \overline{y}) = \overline{z} \in \mathcal{D}^n$, the *influence loss difference* at $\hat{\theta}$ relative to $\overline{z}$ is:

$$\mathcal{Q}(\overline{z}) = \frac{d}{d\varepsilon}\left[L\left(\hat{\theta}; \mathcal{D}^n\right) - \varepsilon l\left(f\left(\hat{\theta}; \overline{x}\right), \overline{y}\right)\right]\Big|_{\varepsilon=\frac{1}{n}}$$

Interpretation: It indicates how much the training error changes when we remove a training data $\overline{z}$.

Influence functions

Definition

Given $(\overline{x}, \overline{y}) = \overline{z} \in \mathcal{D}^n$, the *influence function* at $\hat{\theta}$ relative to $\overline{z}$ is:

$$\mathcal{I}\left(\hat{\theta}; \overline{z}\right) = \lim_{\varepsilon \downarrow 0} \frac{\hat{\theta}_{\varepsilon + \frac{1}{n}, -\overline{z}} - \hat{\theta}_{1/n, -\overline{z}}}{\varepsilon}$$

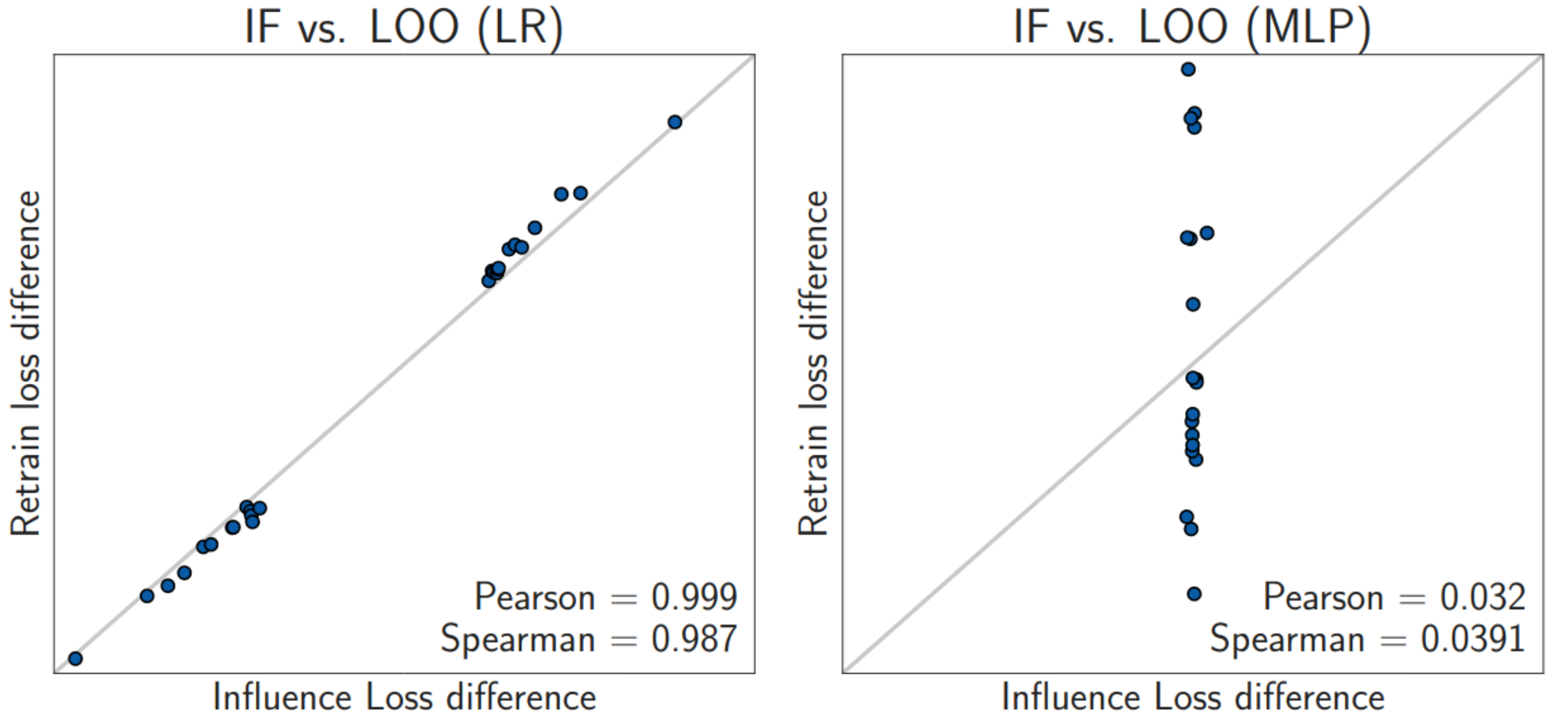
Interpretation: It represents the direction in which the optimal parameter moves when the training error is changed by removing the training data $\overline{z}$.

Assume L is strongly convex. Evaluating influence functions requires heavy computations:

$$\mathcal{I}\left(\hat{\theta}; z\right) = H_{\hat{\theta}}^{-1} \nabla l\left(f\left(\hat{\theta}; x\right), y\right),$$

$$\mathcal{Q}(z) = \nabla l\left(f\left(\hat{\theta}; x\right), y\right)^{\top} H_{\hat{\theta}}^{-1} \nabla l\left(f\left(\hat{\theta}; x\right), y\right),$$

where $H_{\hat{\theta}}$ is the Hessian of L, which can be difficult to compute.



The strong convexity assumption is essential!

Solution?

Solution 1. For iHVP, there are good approximations that only require $O(nd)$ flops instead of $O(n^3)$.

Solution 2. Change point of view: Influence functions are not approximators of LOO retraining, but instead of the proximal Bregman response function (PBRF).

We define the response function in the general setting as:

$$\hat{r}_z(\varepsilon) = \arg\min_{\theta \in \Theta} (L(\theta; \mathcal{D}^n) - \varepsilon l(f(\theta; x), y)).$$

Note that $\hat{r}_z(\varepsilon) = \hat{\theta}_{\varepsilon,-z}$ and $\hat{r}_z(0) = \hat{\theta}$. Since $\hat{r}$ is differentiable at 0, we can define the influence functions as first order approximant of $\hat{r}$. In fact, expanding with Taylor near 0 we get:

$$\hat{r}_{z,\text{lin}}\left(\frac{1}{n}\right) = \hat{r}_z(0) + \left.\frac{d\hat{r}_z}{d\varepsilon}\right|_{\varepsilon=\frac{1}{n}} \left(\frac{1}{n} - 0\right) = \hat{\theta} + \frac{1}{n} H_{\hat{\theta}}^{-1} \nabla l\left(f\left(\hat{\theta}; x\right), y\right).$$

$\boxed{\text{We need } H_\theta \text{ to be positive definite in order to invert it, so } \theta \text{ must be a minimum point.}}$

Proximal response functions

In order for the influence functions to be computable in the MLP case, we need to address the hessian inverse. This can be done by approximating $H_{\hat{\theta}}$ with the Gauss-Newton Hessian (GNH) and adding a damping term to ensure GNH is invertible:

$$\mathcal{J}^{\dagger}\left(\hat{\theta}; z\right) = \left(J_{y\hat{\theta}}^{\top} H_{\hat{\theta}} J_{y\hat{\theta}} + \lambda \boldsymbol{I}\right)^{-1} \nabla l\left(f\left(\hat{\theta}; x\right), y\right),$$

where $J_{y\hat{\theta}}$ is the Jacobian of $F(\theta) = (f(\theta; x_1), ..., f(\theta; x_n))$ in $\hat{\theta}$.
We can get the previous formula by linearizing near 0:

$$\begin{aligned} \hat{r}_{z,\text{damp}}(\varepsilon) &= \arg\min_{\theta \in \Theta} L(\theta; \mathcal{D}^n) - \varepsilon l(f(\theta; x), y)) + \frac{\lambda}{2} \|\theta - \hat{\theta}\|^2, \\ \hat{r}_{z,\text{damp,lin}}(1/n) &\approx \hat{\theta} + \frac{1}{n} \mathcal{J}^{\dagger}\left(\hat{\theta}; z\right). \end{aligned}$$

What is PBRF?

In practice, θ is not a minimum point for L.
However, we can consider another training error for which the early arrested parameter θ^s is optimal:

$$\mathcal{L}(\theta; \theta^s, \mathcal{D}^n) = \frac{1}{n} \sum_{i=1}^{n} D_{l^{(i)}}(f(\theta; x_i), f(\theta^s; x_i)),$$

where $D_{l^{(i)}}(y, y') = l(y, y_i) - l(y', y_i) - \nabla_1 l(y', y_i)^\top (y - y')$ is called Bregman difference.
We can then define the PBRF as:

$$r^b_{z,\mathrm{damp}}(\varepsilon) = \arg\min_{\theta \in \Theta} \mathcal{L}(\theta; \theta^s, \mathcal{D}^n) - \varepsilon l(f(\theta, x), y) + \frac{\lambda}{2} \|\theta - \theta^s\|^2.$$

The optimal solution for the linearised PBRF is the same as the influence function estimation:

$$r^b_{z,\mathrm{damp},\mathrm{lin}}(1/n) = \theta^s + \frac{1}{n}\mathcal{J}^\dagger(\theta^s; z).$$

Therefore, influence functions do NOT depict the retraining with LOO algorithm using L as training error.
Instead they estimate what happens after training from θ^s using as empirical risk:

$$\mathcal{L}(\theta; \theta^s, \mathcal{D}^n) - \frac{1}{n} l(f(\theta, x), y) + \frac{\lambda}{2}\|\theta - \theta^s\|^2.$$

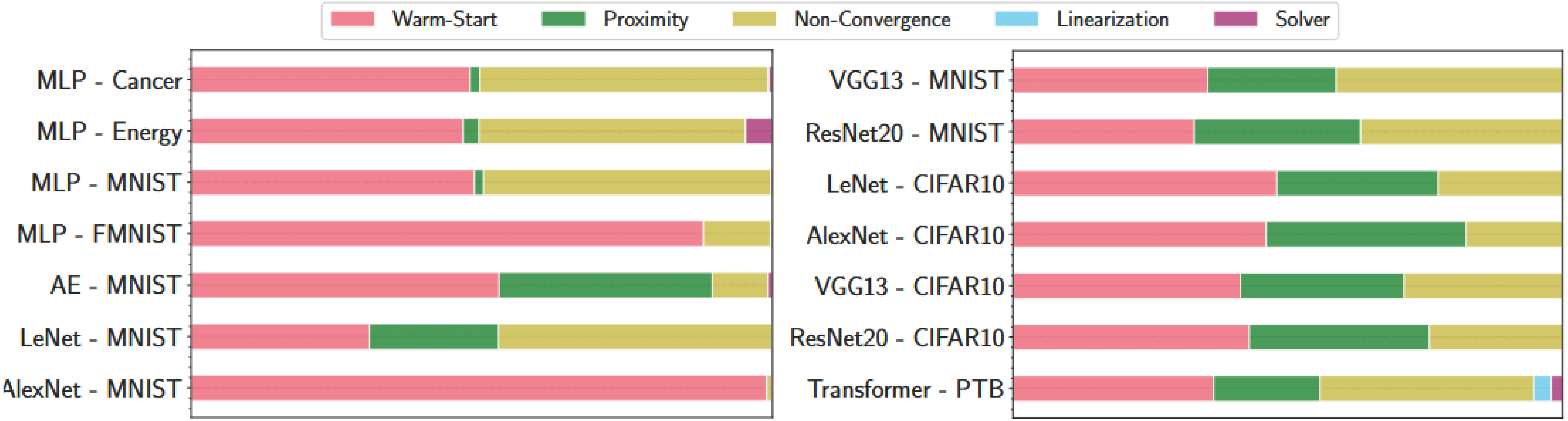
Error decomposition

We can decompose the approximation error of the influence functions in 5 categories:

- **Warm-start gap**: LOO starts from a random parameter (cold start), while IF are related to θ^s; we can then converge to another "optimal" point;
- **Proximity gap**: the factor $\|\theta - \theta^s\|$ induces the warm start not to move far away from θ^s;
- **Non-convergence gap**: in practice we almost never start from a fully trained network;
- **Linearization error**: produced by approximating the Taylor expansion at first order;
- **Solver error**: algorithms used to compute iHVP are approximated.

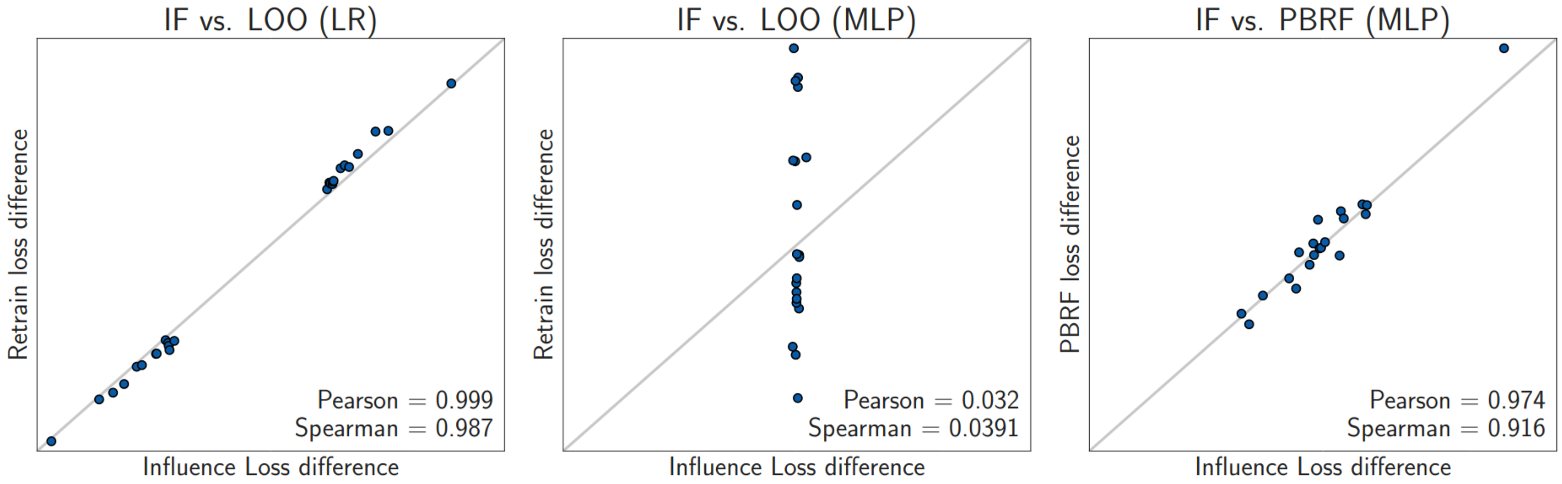
The PBRF method annihilates the first three components.

Error ratio


Warm-Start
Proximity
Non-Convergence
Linearization
Solver
MLP - Cancer
MLP - Energy
MLP - MNIST
MLP - FMNIST
AE - MNIST
LeNet - MNIST
AlexNet - MNIST
VGG13 - MNIST
ResNet20 - MNIST
LeNet - CIFAR10
AlexNet - CIFAR10
VGG13 - CIFAR10
ResNet20 - CIFAR10
Transformer - PTB

Summary

Influence functions seem not to work well for NNs, as the loss function is non-convex. In reality, they are approximating the result of PBRF instead of LOO retraining.



[“If Influence Functions are the Answer, Then What is the Question?”, J. Bae, N. Ng, A. Lo, M. Ghassemi, R. Grosse, 2022]
[“On the Accuracy of Influence Functions for Measuring Group Effects”, Koh et al., 2019]
[“Understanding Black-box Predictions via Influence Functions”, Koh and Liang, 2020]