If Influence Functions are the Answer, Then What is the Question?

Paper review

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Contents

Setting

Consider a prediction task (regression problem) with:

- Input space $\mathcal{X}$;
- Output space $\mathcal{Y}$;
- Training set $\mathcal{D}^n = \{z_i\}_{\{i=1\}}^n$ where $z_i = (x_i, y_i)$ for all $i = 1, ..., n$, $X = (x_1, ..., x_n)$, $Y = (y_1, ..., y_n)$;
- Parameter $\theta \in \Theta := \mathbb{R}^d$;
- $f(\theta; x)$ estimator of $\mathcal{Y}|X$;
- $l : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ loss function (e.g., $l(y', y) \mapsto \|y' - y\|^2$).

We aim to minimize the training error:

$$L(\theta; \mathcal{D}^n) = \frac{1}{n} \sum_{i=1}^{n} l(f(\theta; x_i), y_i).$$

Goal

What happens if we change the importance of a training point $z = (x, y)$ of the dataset?

Call $\hat{\theta} = \arg\min_{\theta \in \Theta} L(\theta; \mathcal{D}^n)$.
How different is it from

$$\hat{\theta}_{\varepsilon, -z} = \arg\min_{\theta \in \Theta} (L(\theta; \mathcal{D}^n) - \varepsilon l(f(\theta; x), y)) \quad ?$$

We can re-train the whole model on $\mathcal{D}^n \setminus \{z\}$ (Leave One Out method), or...

Influence loss difference

Definition

Given $(\overline{x}, \overline{y}) = \overline{z} \in \mathcal{D}^n$, the *influence loss difference* relative to $\overline{z}$ is:

$$\mathcal{Q}(\overline{z}) = \frac{d}{d\varepsilon}[L(\theta; \mathcal{D}^n) - \varepsilon l(f(\theta; \overline{x}), \overline{y})]\Big|_{\varepsilon=\frac{1}{n}}$$

Interpretation: It indicates how much the training error changes when we remove a training data $\overline{z}$.

Definition

Given $(\overline{x}, \overline{y}) = \overline{z} \in \mathcal{D}^n$, the *influence function* relative to $\overline{z}$ is:

$$\mathcal{I}(\overline{z}) = \lim_{\varepsilon \downarrow \frac{1}{n}} \frac{\hat{\theta}_{\varepsilon, -\overline{z}} - \hat{\theta}_{1/n, -\overline{z}}}{\varepsilon}$$

Interpretation: It represents the direction in which the optimal parameter moves when the training error is changed by removing the training data $\overline{z}$.

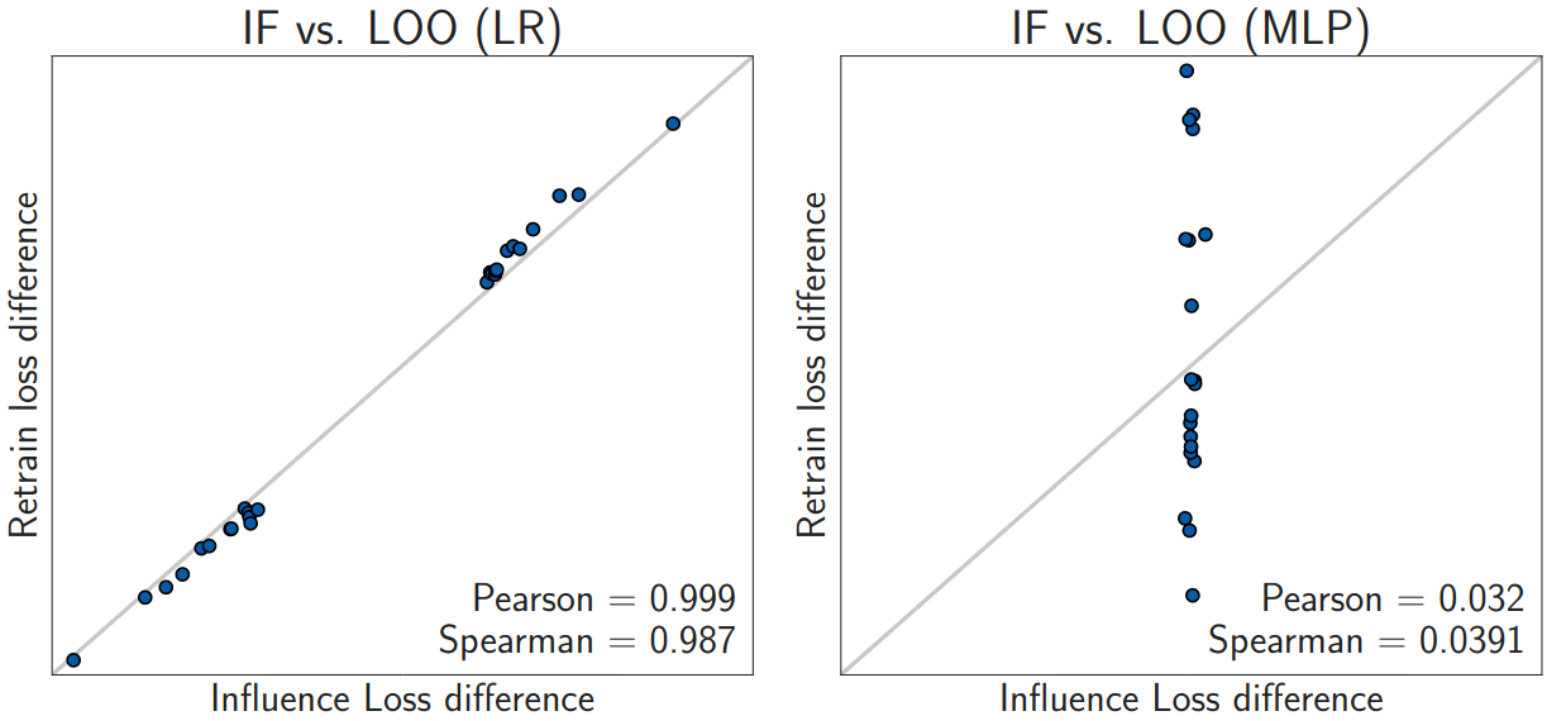
Assume L is strongly convex. Evaluating influence functions requires heavy computations:

$$\mathcal{I}(z) = \frac{1}{n} H_{\hat{\theta}}^{-1} \nabla l\Big(f\Big(\hat{\theta}; x\Big), y\Big),$$

$$\mathcal{Q}(z) = \frac{1}{n} \nabla l\Big(f\Big(\hat{\theta}; x\Big), y\Big)^{\top} H_{\hat{\theta}}^{-1} \nabla l\Big(f\Big(\hat{\theta}; x\Big), y\Big),$$

where $H_{\hat{\theta}}$ is the Hessian of L, which can be difficult to compute.

Problems



The strong convexity assumption is essential!

Solution 1. For iHVP, there are good approximations that only require $O(nd)$ flops instead of $O(n^3)$.

Solution 2. Change point of view: Influence functions are not approximators of LOO retraining, but instead of the proximal Bregman response function (PBRF).

We define the response function in the general setting as:

$$\hat{r}_z(\varepsilon) = \arg\min_{\theta\in\Theta}(L(\theta;\mathcal{D}^n) - \varepsilon l(f(\theta;x),y)).$$

Note that $\hat{r}_z(\varepsilon) = \hat{\theta}_{\varepsilon,-z}$ and $\hat{r}_z(0) = \hat{\theta}$. Since (by IFT) $\hat{r}$ is differentiable at 0, we can define the influence functions as first order approximant of $\hat{r}$. In fact, expanding with Taylor near 0 we get:

$$\hat{r}_{z,\text{lin}}\left(\frac{1}{n}\right) = \hat{r}_z(0) + \left.\frac{d\hat{r}_z}{d\varepsilon}\right|_{\varepsilon=0}(\varepsilon - 0) = \hat{\theta} + \frac{1}{n}H_{\hat{\theta}}^{-1}\nabla l\Big(f\Big(\hat{\theta};x\Big),y\Big).$$

We need H_θ to be positive definite in order to invert it, so θ must be a minimum point.

Proximal response functions

In order for the influence functions to be computable in the MLP case, we need to address the hessian inverse. This can be done by approximating $H_{\hat{\theta}}$ with the Gauss-Newton Hessian (GNH) and adding a damping term to ensure GNH is invertible:

$$\mathcal{I}^{\dagger}(z) = \frac{1}{n}\Big(J_{y\hat{\theta}}^{\top} H_{\hat{\theta}} J_{y\hat{\theta}} + \lambda \boldsymbol{I}\Big)^{-1} \nabla l\Big(f\Big(\hat{\theta}; x\Big), y\Big),$$

where $J_{y\hat{\theta}}$ is the Jacobian of $F(\theta) = (f(\theta; x_1), ..., f(\theta; x_n))$ in $\hat{\theta}$.
We can get the previous formula by linearizing near 0:

$$\hat{r}_{z,\text{damp}}(\varepsilon) \qquad = \arg\min_{\theta \in \Theta} L(\theta; \mathcal{D}^n) - \varepsilon l(f(\theta; x), y)) + \frac{\lambda}{2} \|\theta - \hat{\theta}\|^2,$$

$$\hat{r}_{z,\text{damp,lin}}(1/n) \approx \hat{\theta} + \mathcal{I}^{\dagger}.$$

What is PBRF?

In practice, θ is not a minimum point for L.
However, we can consider another training error for which the early arrested parameter θ^s is optimal:

$$\mathcal{L}(\theta; \theta^s, \mathcal{D}^n) = \frac{1}{n} \sum_{i=1}^{n} D_{l^{(i)}}(f(\theta; x_i), f(\theta^s; x_i)),$$

where $D_{l^{(i)}}(y, y') = l(y, y_i) - l(y', y_i) - \nabla_1 l(y', y_i)^\top (y - y')$ is called Bregman difference.
We can then define the PBRF as:

$$r^b_{z,\mathrm{damp}}(\varepsilon) = \arg\min_{\theta \in \Theta} \mathcal{L}(\theta; \theta^s, \mathcal{D}^n) - \varepsilon l(f(\theta, x), y) + \frac{\lambda}{2} \|\theta - \theta^s\|^2.$$

The optimal solution for the linearised PBRF is the same as the influence function estimation:

$$r^b_{z,\text{damp,lin}}(1/n) = \theta^s + \mathcal{I}^\dagger(z).$$

Therefore, influence functions do NOT depict the retraining with LOO algorithm using L as training error.
Instead they estimate what happens after training from θ^s using as empirical risk:

$$\mathcal{L}(\theta; \theta^s, \mathcal{D}^n) - \frac{1}{n} l(f(\theta, x), y) + \frac{\lambda}{2} \|\theta - \theta^s\|^2.$$

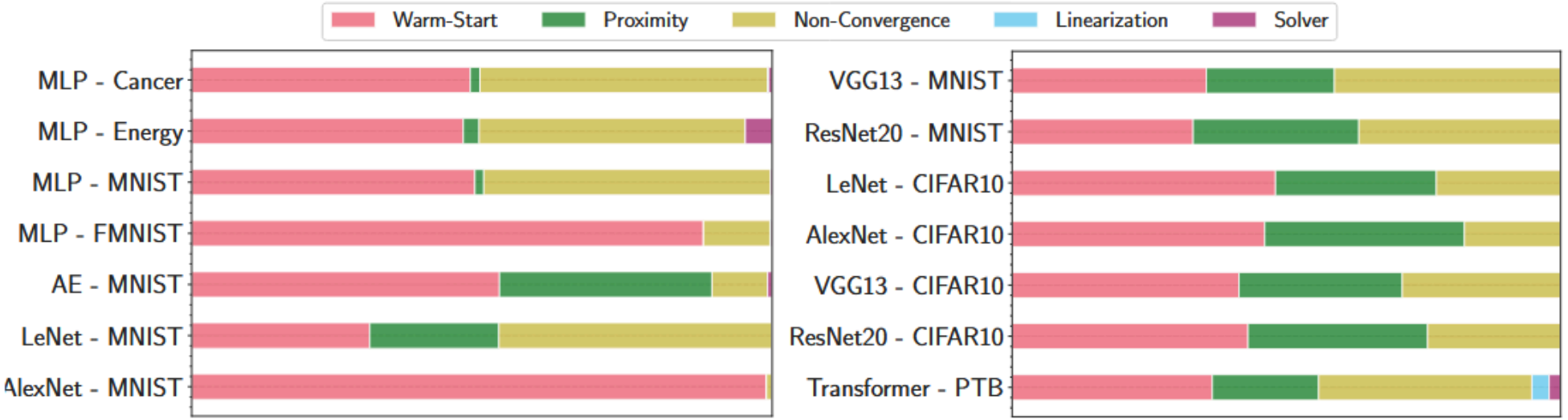
Error decomposition

We can decompose the approximation error of the influence functions in 5 categories:

- **Warm-start gap**: LOO starts from a random parameter (cold start), while IF are related to θ^s; we can then converge to another "optimal" point;
- **Proximity gap**: the factor $\|\theta - \theta^s\|$ induces the warm start not to move far away from θ^s;
- **Non-convergence gap**: in practice we almost never start from a fully trained network;
- **Linearization error**: produced by approximating the Taylor expansion at first order;
- **Solver error**: algorithms used to compute iHVP are approximated.

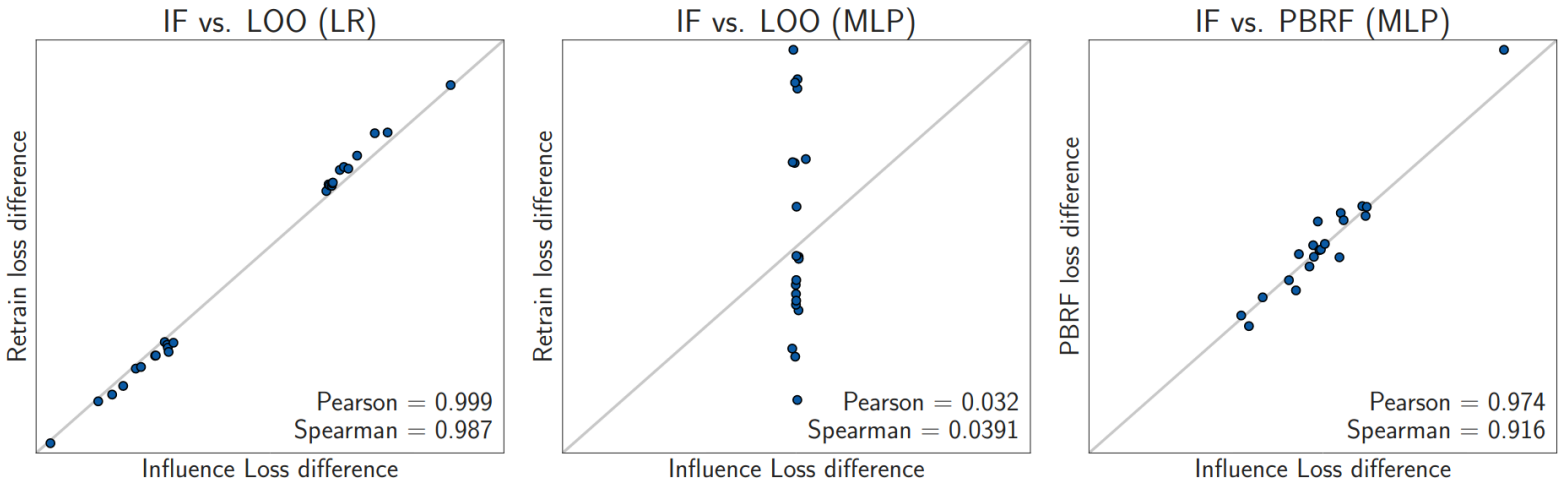
The PBRF method annihilates the first three components.

Error ratio


Warm-Start
Proximity
Non-Convergence
Linearization
Solver
MLP - Cancer
MLP - Energy
MLP - MNIST
MLP - FMNIST
AE - MNIST
LeNet - MNIST
AlexNet - MNIST
VGG13 - MNIST
ResNet20 - MNIST
LeNet - CIFAR10
AlexNet - CIFAR10
VGG13 - CIFAR10
ResNet20 - CIFAR10
Transformer - PTB

Summary

Influence functions seem not to work well for NNs, as the loss function is non-linear.
In reality, they are approximating the result of PBRF instead of LOO retraining.



["If Influence Functions are the Answer, Then What is the Question?", J. Bae, N. Ng, A. Lo, M. Ghassemi, R. Grosse, 2022]
["On the Accuracy of Influence Functions for Measuring Group Effects", Koh et al., 2019]
["Understanding Black-box Predictions via Influence Functions", Koh and Liang, 2020]