ON ROBIN'S INEQUALITY AND THE KANEKO-LAGARIAS INEQUALITY

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Abstract. We prove that Robin's inequality and the Lagarias inequality hold for almost every number, including all numbers not divisible by one of the prime numbers $\{2, 3, 5\}$, primorials, sufficiently big numbers of the form $2^k n$ for odd n and 21-free integers. We also prove that the Kaneko-Lagarias inequality holds for all numbers if and only if it holds for all superabundant numbers.

Contents

1. Preliminaries

We denote by $\sigma(n)$ and $\varphi(n)$ the sum of divisors function and Euler's totient function respectively. Robin's inequality ([Rob84]) states that the Riemann hypothesis is equivalent to the assertion that

$$\sigma(n) < e^{\gamma} n \log(\log(n)) \tag{1.1}$$

for all $n > 5040$, where γ denotes the Euler-Mascheroni constant. Similarly, the Lagarias inequality ([Lag02]) states that the Riemann hypothesis is equivalent to the assertion that

$$\sigma(n) < H_n + \exp(H_n) \log(H_n) \tag{1.2}$$

for all $n \geq 1$, where H_n denotes the n-th harmonic number. Lagarias also published an equlity that we call the Kaneko-Lagarias inequality (see the acknowledgements in [Lag02]),

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which states that the Riemann hypothesis is equivalent to the assertion that

$$\sigma(n) < \exp(H_n)\log(H_n) \tag{1.3}$$

for all $n > 60$.

2. Robin's Inequality

2.1. **Sufficiently big numbers not divisible by one of the prime numbers 2, 3, or 5.** Let $p_1 = 2$, $p_2 = 3$, etc. be an enumeration of the prime numbers which we denote by $\mathbb{P}$. Fix $j \in \mathbb{N}$ and let $q_1 < q_2 < \cdots < q_k$ be some prime numbers distinct from p_j. Given $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$.

Lemma 2.1. *We have*

$$\frac{\sigma(n)}{n} < \prod_{\ell=1}^{k} \frac{q_\ell}{q_\ell - 1} \leq \prod_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{p_\ell}{p_\ell - 1} = \frac{n}{\varphi(n)}. \tag{2.1}$$

Proof. The first inequality follows from the fact that for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$

$$\frac{\sigma(p^\alpha)}{p^\alpha} = \frac{p - \frac{1}{p^\alpha}}{p-1} \nearrow \frac{p}{p-1} \text{ as } \alpha \to \infty. \tag{2.2}$$

The second inequality follows from the fact that $p_i \leq q_i$ for all $1 \leq i \leq k$. $\square$

Note that

$$\frac{n}{\varphi(n)} = \left(\prod_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{p_\ell + 1}{p_\ell} \right) \left(\prod_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{p_\ell^2}{p_\ell^2 - 1} \right) =: A(k)B(k). \tag{2.3}$$

We can bound $A(k)$ as follows:

$$\log(A(k)) = \sum_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \log\left(1 + \frac{1}{p_\ell}\right) \leq \sum_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{1}{p_\ell} = \left(\sum_{\ell=1}^{k+1} \frac{1}{p_\ell}\right) - \frac{1}{p_j} \tag{2.4}$$

and

$$\sum_{\ell=1}^{k+1} \frac{1}{p_\ell} \leq \log(\log(p_{k+1})) + c_1 + \frac{5}{\log(p_{k+1})}, \tag{2.5}$$

where $c_1 \approx .261497$ by Theorem 1.10 in [Ten95]. Thus we obtain

Lemma 2.2. *For all $k \in \mathbb{N}$,*

$$A(k) \leq \log(p_{k+1}) \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log(p_{k+1})}\right). \tag{2.6}$$

Combining [Dus99] and Theorem 3 from [RS62], we obtain the following:

Theorem 2.3. *For $k \geq 6$,*

$$k(\log(k) + \log(\log(k)) - 1) < p_k < k(\log(k) + \log(\log(k))). \tag{2.7}$$

Furthermore, combining Lemma 2.2 and Theorem 2.3, we see that

Lemma 2.4. *For $k \geq 6$, $A(k) < C(k)$ where*

$$\begin{aligned} C(k) &= \log((k+1)(\log(k+1) + \log(\log(k+1)))) \\ &\quad \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))}\right). \end{aligned} \tag{2.8}$$

Now, put $m = p_{k+1}\#/p_j$. Our goal is to show the following, since it implies that Robin's inequality for n as above:

Theorem 2.5. *For any $j \in \{1, 2, 3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^\gamma \log(\log(m)). \tag{2.9}$$

Corollary 2.6. *Suppose Theorem 2.5 holds. Then Robin's inequality holds for n as above.*

Proof. We calculate

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq A(k)B(k) < C(k)B(k) < e^\gamma \log(\log(m)) \leq e^\gamma \log(\log(n)), \tag{2.10}$$

where the last inequality follows from the fact that $m \leq n$. □

Definition 2.7. The *Chebyshev function* is defined as follows:

$$\theta(x) = \sum_{p \in \mathbb{P}, p \leq x} \log(p) = \log\left(\prod_{p \in \mathbb{P}, p \leq x} p\right). \tag{2.11}$$

Theorem 2.8. *For $x \geq 529$,*

$$\prod_{\substack{p \in \mathbb{P} \\ p \leq x}} p = e^{\theta(x)} > e^{x\left(1 - \frac{1}{2\log x}\right)} \geq (2.51)^x. \tag{2.12}$$

Proof. The first inequality is given by (3.14) in [RS62] and the second follows from computations since the function $f(x) = 1 - \frac{1}{2\log x}$ increases for $x > 1$. □

Lemma 2.9. *For $k \geq 99$,*

$$\log(\log(m)) > \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1)\log(2.51) - \log(p_j)) =: D(k).$$

Proof. Noting that $k \geq 99$ implies $p_{k+1} > 529$, we calculate

$$\begin{aligned} \log(\log(m)) &= \log\left(\log\left(\frac{p_{k+1}\#}{p_j}\right)\right) > \log\left(\log\left(\frac{(2.51)^{p_{k+1}}}{p_j}\right)\right) \\ &= \log(p_{k+1}\log(2.51) - \log(p_j)) \\ &> \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1)\log(2.51) - \log(p_j)), \end{aligned} \tag{2.13}$$

where the last inequality uses Theorem 2.3. □

The following implies Theorem 2.5:

Proposition 2.10. *For $j \in \{1, 2, 3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^\gamma D(k). \tag{2.14}$$

Proof. Denote

$$\begin{aligned}\widetilde{C}(k) = e^{-c_1+\frac{1}{p_j}}C(k) = \log((k+1)(\log(k+1)+\log(\log(k+1)))) \\ \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right)\end{aligned} \tag{2.15}$$

and

$$\widehat{C}(k) = \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right). \tag{2.16}$$

Multiplying both sides of (2.14) by $e^{-c_1+\frac{1}{p_j}}p_j^2/(p_j^2-1)$, we obtain

$$\widetilde{C}(k)\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} < \frac{e^{\gamma-c_1+\frac{1}{p_j}}p_j^2}{p_j^2-1}D(k). \tag{2.17}$$

Noting that

$$\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} \nearrow \frac{\pi^2}{6} \text{ as } k\to\infty, \tag{2.18}$$

we see that (2.17) is implied by

$$\widetilde{C}(k) < \frac{6p_j^2e^{\gamma-c_1+\frac{1}{p_j}}}{\pi^2(p_j^2-1)}D(k) =: E_jD(k). \tag{2.19}$$

Raising both sides to the power of e, we see that (2.19) is implied by

$$\begin{aligned}&[(k+1)(\log(k+1)+\log(\log(k+1))]^{\widehat{C}(k)} \\ &< [(k+1)(\log(k+1)+\log(\log(k+1))-1)\log(2.51)-\log(p_j)]^{E_j}.\end{aligned} \tag{2.20}$$

(2.20) is equivalent to

$$\begin{aligned}1 < [(k+1)(\log(k+1)+\log(\log(k+1)))]^{-\widehat{C}(k)+E_j} \\ \left[1-\frac{(k+1)\log(2.51)+\log(p_j)}{(k+1)(\log(k+1)+\log(\log(k+1)))}\right]^{E_j}.\end{aligned} \tag{2.21}$$

Noting that $E_j > 1$ for $j \in \{1,2,3\}$, we see that there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies $-\widehat{C}(k)+E_j > \epsilon$ for some $\epsilon \in (0,1)$. If needed, we can increase K_j so that $k \geq K_j$ implies

$$\left[1-\frac{(k+1)\log(2.51)+\log(p_j)}{(k+1)(\log(k+1)+\log(\log(k+1)))}\right]^{E_j} > \epsilon, \tag{2.22}$$

and also so that $k \geq K_j$ implies

$$1 < \epsilon[(k+1)(\log(k+1)+\log(\log(k+1))]^{\epsilon}, \tag{2.23}$$

which implies (2.21). $\square$

2.2. **All numbers not divisible by one of the prime numbers 2, 3, or 5.** Letting $j = 1$ in (2.19), we seek to show that

$$\widetilde{C}(k) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2} D(k). \tag{2.24}$$

Lemma 2.11. *For $k \geq 13042$, $\widehat{C}(k) < 1.525$.*

Proof. $\widehat{C}(k)$ is decreasing, so the result follows from computation. □

Denote $f(k) = (k+1)(\log(k+1) + \log(\log(k+1))$. Applying Lemma 2.11to (2.24) and performing some algebraic manipulations, our goal reduces to showing that

$$\log(f(k)) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2(1.525)} \log((f(k) - 1)\log(2.51) - \log(2)). \tag{2.25}$$

Raising both sides to the power of e, this becomes

$$1 < f(k)^{2.0166} \left[1 - \frac{(k+1)\log(2.51) - \log(2)}{f(k)}\right]^{1.20166}. \tag{2.26}$$

The RHS of (2.26) is increasing, and a computation reveals that it holds for $k \geq 13042$. Additionally, using Lemma 2.1, one can check that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^\gamma \log(\log(m)) \tag{2.27}$$

for $k \geq 3$. Finally, when $k \in \{1, 2\}$, we check that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq \frac{15}{8} < e^\gamma \log(\log(n)) \tag{2.28}$$

for $n \geq 680$. This confirms the following for $j = 1$:

Theorem 2.12. *For $j \in \{1, 2, 3\}$, Robin's inequality holds for every natural number > 5040 which is not divisible by p_j.*

To confirm Theorem 2.12 when $j \in \{2, 3\}$, one can repeat the above process to see that, for sufficiently big k, (2.19) is satisfied. The cases with smaller k have been verified in [MP20].

2.3. **Primorials and sufficiently big even numbers.** Fix $k \in \mathbb{N}$ and let n be odd. We consider numbers of the form $2^k n$. We calculate

$$\frac{\sigma(2^k n)}{2^k n} = \frac{\sigma(2^k)}{2^k}\frac{\sigma(n)}{n} < \frac{\sigma(2^k)}{2^k}\frac{n}{\varphi(n)} = \frac{\sigma(2^k)}{2^k}\frac{\varphi(2^k)}{2^k}\frac{2^k n}{\varphi(2^k n)} = \left(1 - \frac{1}{2^{k+1}}\right)\frac{2^k n}{\varphi(2^k n)}. \tag{2.29}$$

Applying Theorem 15 from [RS62], we know

$$\left(1 - \frac{1}{2^{k+1}}\right)\frac{2^k n}{\varphi(2^k n)} < \left(1 - \frac{1}{2^{k+1}}\right)\left(e^\gamma \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right). \tag{2.30}$$

We ask which n satisfy

$$\left(1 - \frac{1}{2^{k+1}}\right)\left(e^\gamma \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right) < e^\gamma(\log(\log(2^k n))). \tag{2.31}$$

This is equivalent to asking when

$$\frac{2.51(2^{k+1} - 1)}{e^\gamma} < (\log(\log(2^k n)))^2 \tag{2.32}$$

holds, which is when

$$n > \frac{e^{e^{\sqrt{\frac{2.51(2^{k+1}-1)}{e^{\gamma}}}}}}{2^k} =: b(k). \tag{2.33}$$

Thus, we obtain the following:

Theorem 2.13. *Given any $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies* (2.33).

In particular, we have the following:

Corollary 2.14. *If $n \geq 620$ is odd, then Robin's inequality holds for $2n$. Furthermore, Robin's inequality holds for all primorials > 30.*

Proof. The first statement follows immediately from Theorem 2.13 and the second follows from the computation of primorials < 1240.3 □

2.4. **All 21-free numbers.** The results of the previous subsection are based on the inequality in Theorem 15 from [RS62]. This inequality can be improved by using a sharper bound stated in [AN22]:

$$\frac{m}{\varphi(m)} < e^{\gamma} \log(\log(m)) + \frac{.0168}{(\log(\log(m)))^2} \tag{2.34}$$

for $m \geq 10^{10^{13.11485}} =: C$. Using the same reasoning as in the proof of Theorem 2.13, we derive the following result:

Theorem 2.15. *Given $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies*

$$n > \frac{e^{e^{\sqrt[3]{\frac{.0168(2^{k+1}-1)}{e^{\gamma}}}}}}{2^k} =: \tilde{b}(k). \tag{2.35}$$

We can thus conclude the following:

Theorem 2.16. *Robin's inequality holds for all 21-free numbers.*

Proof. Let $k \in \mathbb{N}$ and n be an odd natural number. If $n > \tilde{b}(k)$, n satisfies Robin's inequality by Theorem 2.15. If not, note that n satisfies Robin's inequality if $5040 < 2^k n \leq 2^k \tilde{b}(k) \leq C$ by Theorem 13 in [MP20]. Recalling the definition of ℓ-free numbers, we see that if $2^k \tilde{b}(k) < C$, then all $(k+1)$-free numbers satify Robin's inequality. Indeed, setting $k = 20$, we calculate

$$\log(2^{20}\tilde{b}(20)) < 6(10^{11}) < 2.3(10^{13.11485}) < \log(C).$$ □

2.5. **Almost every number.**

Definition 2.17. The *natural density* of a set E is

$$d(E) = \lim_{n\to\infty} \frac{\#E \cap \{1, 2, \dots, n\}}{n} \tag{2.36}$$

when the limit exists.

Theorem 2.18. *Denote by $\mathcal{R}$ the set of numbers satisfying Robin's inequality. Then the natural density of $\mathcal{R}$ is* 1.

Proof. We will prove that the natural density of $\mathcal{R}^c$ is 0. Fix $\epsilon > 0$. Let $E_k = \{2^k n : n \in \mathbb{N}_{\text{odd}}, n \leq b(k)\}$ and note that $\mathcal{R}^c \subseteq \bigcup_{k\geq 1} E_k$ by Theorem 2.12 and Theorem 2.13.[1] Pick M

[1] Here $\mathbb{N}_{\text{odd}} := \{1, 3, \dots\}$.

so that $\sum_{k=M+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. For $n \in \mathbb{N}$ we calculate

$$\begin{aligned}\frac{\#\mathcal{R}^c \cap \{1,2,\ldots,n\}}{n} &\leq \frac{\#\bigcup_{k\geq 1} E_k \cap \{1,2,\ldots,n\}}{n} = \frac{\sum_{k\geq 1} E_k \cap \{1,2,\ldots,n\}}{n} \\ &= \frac{\sum_{k=1}^{M} \#E_k \cap \{1,2,\ldots,n\} + \sum_{k=M+1}^{\infty} \#E_k \cap \{1,2,\ldots,n\}}{n},\end{aligned} \tag{2.37}$$

where the first equality follows from the fact that the E_k's are disjoint. Noting that $\sum_{k=1}^{M} \#E_k \cap \{1,2,\ldots,n\} < \infty$ for all $n \in \mathbb{N}$, we see that we can pick N so that $n \geq N$ implies that the RHS of (2.37) is $< \epsilon$, completing our proof. □

3. The Lagarias and Kaneko-Lagarias Inequalities

3.1. **Superabundant numbers.** Let $\Gamma(x)$ denote the gamma function. We define two functions:

$$\begin{aligned} H(x) &= \int_0^1 \frac{t^x - 1}{t-1}\,\mathrm{d}t, \\ \psi(x) &= \frac{\Gamma'(x)}{\Gamma(x)}. \end{aligned} \tag{3.1}$$

ψ is known as the *digamma function*. One can verify that H is smooth for $x \geq 1$ and that $H(n) = H_n$ for all $n \in \mathbb{N}$. It's also easy to see that ψ, known as the digamma function, satisfies

$$H(x) = \psi(x+1) + \gamma. \tag{3.2}$$

Lemma 3.1. *For all $x \geq 1$,*

$$H(x) < \log(x) + \gamma + \frac{1}{2x}. \tag{3.3}$$

Proof. By (2.2) from [Alz97],

$$\psi(x) < \log(x) - \frac{1}{2x} \tag{3.4}$$

for all $x \geq 1$. Then we use (3.2) and $\psi(x+1) = \psi(x) + \frac{1}{x}$ to finish. □

Lemma 3.2. *For all $x \geq 4$,*

$$H(x) < \frac{2\log(x)}{1 + \frac{6}{\pi^2 x}}. \tag{3.5}$$

Proof. By Lemma 3.1, it suffices to show that

$$\log(x) + \gamma + \frac{1}{2x} < \frac{2\log(x)}{1 + \frac{6}{\pi^2 x}} \tag{3.6}$$

for $x \geq 4$. By arithmetic manipulations, (3.6) becomes

$$\frac{1}{\pi^2 x - 6}\left(\gamma\pi^2 x + \frac{\pi^2}{2} + 6\gamma + \frac{3}{x}\right) < \log(x). \tag{3.7}$$

Computation reveals that (3.7) holds for $x = 4$, and the LHS of (3.7) is decreasing while the RHS is increasing so we obtain the result. □

Lemma 3.3. *The following hold:*

(a) *For all $n > 1$, $H_{n+1} \leq \frac{n}{\log(n)}$.*
(b) *For all $x \geq 4$, $\log(H(x)) \leq \frac{x}{2\log(x)}$.*

Proof. (a) We can manually verify the inequality for $n \leq 6$. Noting that

$$H_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} \leq 1 + \int_1^{n+1} \frac{\mathrm{d}t}{t} = 1 + \log(n+1), \tag{3.8}$$

it suffices to show that

$$\log(x)(\log(x+1)+1) \leq x. \tag{3.9}$$

Put $g(t) = e^t - t^2 - t - 1$. We see that $g(2) > 0$ and that $g'(t) = e^t - 2t - 1 > 0$ for $t \geq 2$, so $g(t) > 0$ for $t \geq 2$. For $x \geq e^2 - 1$ we have

$$0 < g(\log(x+1)) = x+1-(\log(x+1))^2 - \log(x+1) - 1 < x - \log(x)(\log(x+1)+1). \tag{3.10}$$

(b) For $x \geq 4$, note that the function mapping $x \mapsto \frac{x}{\log(x)}$ is increasing. If $n \leq x < n+1$, then

$$H_n \leq H(x) < H_{n+1} \leq \frac{n}{\log(n)} \leq \frac{x}{\log(x)}. \tag{3.11}$$

For $y > 2$ we see that $\log(y) < \frac{y}{2}$, so let $y = H(x)$ and apply (3.11) finish. □

Lemma 3.4. *For $x \geq 4$,*

$$H(x)\log(H(x)) < \frac{x^2}{x + \frac{6}{\pi^2}}. \tag{3.12}$$

Proof. Apply Lemma 3.1 and Lemma 3.3. □

Lemma 3.5. *For $x \geq 4$,*

$$H'(x) > \frac{H(x)\log(H(x))}{x^2}. \tag{3.13}$$

Proof. We will use (51) from [FD14] which states that

$$\frac{1}{\psi'(x)} \leq x + \frac{6}{\pi^2} - 1 \tag{3.14}$$

for $x \geq 1$. We calculate

$$H'(x) = \psi'(x+1) \geq \frac{1}{x + 6\pi^2} > \frac{H(x)\log(H(x))}{x^2}, \tag{3.15}$$

where the equality follows from taking the derivative of (3.2) and the second inequality follows from Lemma 3.4. □

Proposition 3.6. *The function*

$$g(x) = \frac{\exp(H(x))\log(H(x))}{x} \tag{3.16}$$

is increasing for $x \geq 4$.

Proof. We start with (3.5) from [Lag02]:

$$\begin{aligned} H_n &= \log(n) + \gamma + \int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x \\ \implies \exp(H_n) &= e^\gamma n \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x\right) \\ \implies \frac{\exp(H_n)\log(H_n)}{n} &= e^\gamma \log(H_n) \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x\right). \end{aligned} \tag{3.17}$$

Given $k \in \mathbb{N}$, put

$$g_k(x) = e^\gamma \log(H(x)) \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} \, \mathrm{d}t\right) \tag{3.18}$$

so that $\lim_{k\to\infty} g_k(x) = g(x)$. We compute

$$g_k'(x) = e^\gamma \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} \, \mathrm{d}t\right)\left(\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x - \lfloor x \rfloor}{x^2}\right)\right), \tag{3.19}$$

so $g_k'(x) > 0$ if and only if

$$\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x - \lfloor x \rfloor}{x^2}\right) \geq \frac{H'(x)}{H(x)} - \frac{\log(H(x))}{x^2} > 0, \tag{3.20}$$

which is the content of Lemma 3.5. Thus, $g(x)$ is the limit of monotonically increasing functions and is therefore monotonically increasing. □

Corollary 3.7. *The sequence*

$$\left\{\frac{\exp(H_n)\log(H_n)}{n}\right\}_{n=1}^\infty \tag{3.21}$$

is monotonically increasing.

Proof. Proposition 3.6 gives the result for $n \geq 4$ and we can manually check the smaller cases. □

Definition 3.8. A number n is *superabundant* if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.

Theorem 3.9. *If there are counterexamples to the Kaneko-Lagarias inequality, the smallest such counterexample is a superabundant number.*

Proof. Suppose, for sake of contradiction, that m is the smallest counterexample to the Kaneko-Lagarias inequality and that m is not superabundant. Let n be the greatest superabundant number $< m$. We calculate,

$$\frac{\sigma(n)}{n} > \frac{\sigma(m)}{n} \geq \frac{\exp(H_m)\log(H_m)}{m} > \frac{\exp(H_n)\log(H_n)}{n}, \tag{3.22}$$

so $n < m$ violates the Kaneko-Lagarias inequality: a contradiction. □

3.2. Connection to Robin's inequality.

Theorem 3.10. *If Robin's inequality holds for some $n \in \mathbb{N}$, then the Kaneko-Lagarias inequality holds for n.*

Proof. We use the approximation

$$H_n \geq \log(n) + \gamma + \frac{1}{2n+1} \tag{3.23}$$

to calculate

$$\frac{\exp(H_n)\log(H_n)}{n} \geq \frac{e^{\gamma + \frac{1}{2n+1}} n \log\left(\log(n) + \gamma + \frac{1}{2n+1}\right)}{n} > e^\gamma \log(\log(n)), \tag{3.24}$$

which implies the result. □

Note that we obtain the same result for the Lagarias inequality.

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