

ON ROBIN'S INEQUALITY AND THE KANEKO-LAGARIAS INEQUALITY

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ABSTRACT. We prove that Robin's inequality and the Lagarias inequality hold for almost every number, including all numbers not divisible by one of the prime numbers $\{2, 3, 5\}$, primorials, sufficiently big numbers of the form $2^k n$ for odd n and 21-free integers. We also prove that the Kaneko-Lagarias inequality holds for all numbers if and only if it holds for all superabundant numbers.

CONTENTS

| | |
|--|----|
| 1. Preliminaries | 1 |
| 2. Robin's Inequality | 2 |
| 2.1. Sufficiently big numbers not divisible by one of the prime numbers 2, 3, or 5 | 2 |
| 2.2. All numbers not divisible by one of the prime numbers 2, 3, or 5 | 5 |
| 2.3. Primorials and sufficiently big even numbers | 5 |
| 2.4. All 21-free numbers | 6 |
| 2.5. Almost every number | 6 |
| 3. The Lagarias and Kaneko-Lagarias Inequalities | 7 |
| 3.1. Superabundant numbers | 7 |
| 3.2. Connection to Robin's inequality | 9 |
| Acknowledgments | 10 |
| References | 10 |

1. PRELIMINARIES

We denote by $\sigma(n)$ and $\varphi(n)$ the sum of divisors function and Euler's totient function respectively. Robin's inequality ([Rob84]) states that the Riemann hypothesis is equivalent to the assertion that

$$(1.1) \quad \sigma(n) < e^\gamma n \log(\log(n))$$

for all $n > 5040$, where γ denotes the Euler-Mascheroni constant. Similarly, the Lagarias inequality ([Lag02]) states that the Riemann hypothesis is equivalent to the assertion that

$$(1.2) \quad \sigma(n) < H_n + \exp(H_n) \log(H_n)$$

for all $n \geq 1$, where H_n denotes the n -th harmonic number. Lagarias also published an equality that we call the Kaneko-Lagarias inequality (see the acknowledgements in [Lag02]),

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which states that the Riemann hypothesis is equivalent to the assertion that

$$(1.3) \quad \sigma(n) < \exp(H_n) \log(H_n)$$

for all $n > 60$.

2. ROBIN'S INEQUALITY

2.1. Sufficiently big numbers not divisible by one of the prime numbers 2, 3, or

5. Let $p_1 = 2$, $p_2 = 3$, etc. be an enumeration of the prime numbers which we denote by \mathbb{P} . Fix $j \in \mathbb{N}$ and let $q_1 < q_2 < \dots < q_k$ be some prime numbers distinct from p_j . Given $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{N}$, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_k^{\alpha_k}$.

Lemma 2.1. *We have*

$$(2.1) \quad \frac{\sigma(n)}{n} < \prod_{\ell=1}^k \frac{q_\ell}{q_\ell - 1} \leq \prod_{\substack{\ell=1, \dots, j-1 \\ j+1, \dots, k+1}} \frac{p_\ell}{p_\ell - 1} = \frac{n}{\varphi(n)}.$$

Proof. The first inequality follows from the fact that for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$

$$(2.2) \quad \frac{\sigma(p^\alpha)}{p^\alpha} = \frac{p - \frac{1}{p^\alpha}}{p - 1} \nearrow \frac{p}{p - 1} \text{ as } \alpha \rightarrow \infty.$$

The second inequality follows from the fact that $p_i \leq q_i$ for all $1 \leq i \leq k$. □

Note that

$$(2.3) \quad \frac{n}{\varphi(n)} = \left(\prod_{\substack{\ell=1, \dots, j-1 \\ j+1, \dots, k+1}} \frac{p_\ell + 1}{p_\ell} \right) \left(\prod_{\substack{\ell=1, \dots, j-1 \\ j+1, \dots, k+1}} \frac{p_\ell^2}{p_\ell^2 - 1} \right) =: A(k)B(k).$$

We can bound $A(k)$ as follows:

$$(2.4) \quad \log(A(k)) = \sum_{\substack{\ell=1, \dots, j-1 \\ j+1, \dots, k+1}} \log\left(1 + \frac{1}{p_\ell}\right) \leq \sum_{\substack{\ell=1, \dots, j-1 \\ j+1, \dots, k+1}} \frac{1}{p_\ell} = \left(\sum_{\ell=1}^{k+1} \frac{1}{p_\ell} \right) - \frac{1}{p_j}$$

and

$$(2.5) \quad \sum_{\ell=1}^{k+1} \frac{1}{p_\ell} \leq \log(\log(p_{k+1})) + c_1 + \frac{5}{\log(p_{k+1})},$$

where $c_1 \approx .261497$ by Theorem 1.10 in [Ten95]. Thus we obtain

Lemma 2.2. *For all $k \in \mathbb{N}$,*

$$(2.6) \quad A(k) \leq \log(p_{k+1}) \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log(p_{k+1})}\right).$$

Combining [Dus99] and Theorem 3 from [RS62], we obtain the following:

Theorem 2.3. *For $k \geq 6$,*

$$(2.7) \quad k(\log(k) + \log(\log(k)) - 1) < p_k < k(\log(k) + \log(\log(k))).$$

Furthermore, combining Lemma 2.2 and Theorem 2.3, we see that

Lemma 2.4. For $k \geq 6$, $A(k) < C(k)$ where

$$(2.8) \quad C(k) = \log((k+1)(\log(k+1) + \log(\log(k+1)))) \\ \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))}\right).$$

Now, put $m = p_{k+1}\# / p_j$. Our goal is to show the following, since it implies that Robin's inequality for n as above:

Theorem 2.5. For any $j \in \{1, 2, 3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies

$$(2.9) \quad C(k)B(k) < e^\gamma \log(\log(m)).$$

Corollary 2.6. Suppose Theorem 2.5 holds. Then Robin's inequality holds for n as above.

Proof. We calculate

$$(2.10) \quad \frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq A(k)B(k) < C(k)B(k) < e^\gamma \log(\log(m)) \leq e^\gamma \log(\log(n)),$$

where the last inequality follows from the fact that $m \leq n$. \square

Definition 2.7. The *Chebyshev function* is defined as follows:

$$(2.11) \quad \theta(x) = \sum_{p \in \mathbb{P}, p \leq x} \log(p) = \log\left(\prod_{p \in \mathbb{P}, p \leq x} p\right).$$

Theorem 2.8. For $x \geq 529$,

$$(2.12) \quad \prod_{\substack{p \in \mathbb{P} \\ p \leq x}} p = e^{\theta(x)} > e^{x(1 - \frac{1}{2 \log x})} \geq (2.51)^x.$$

Proof. The first inequality is given by (3.14) in [RS62] and the second follows from computations since the function $f(x) = 1 - \frac{1}{2 \log x}$ increases for $x > 1$. \square

Lemma 2.9. For $k \geq 99$,

$$\log(\log(m)) > \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1) \log(2.51) - \log(p_j)) =: D(k).$$

Proof. Noting that $k \geq 99$ implies $p_{k+1} > 529$, we calculate

$$(2.13) \quad \log(\log(m)) = \log\left(\log\left(\frac{p_{k+1}\#}{p_j}\right)\right) > \log\left(\log\left(\frac{(2.51)^{p_{k+1}}}{p_j}\right)\right) \\ = \log(p_{k+1} \log(2.51) - \log(p_j)) \\ > \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1) \log(2.51) - \log(p_j)),$$

where the last inequality uses Theorem 2.3. \square

The following implies Theorem 2.5:

Proposition 2.10. For $j \in \{1, 2, 3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies

$$(2.14) \quad C(k)B(k) < e^\gamma D(k).$$

Proof. Denote

$$(2.15) \quad \tilde{C}(k) = e^{-c_1 + \frac{1}{p_j}} C(k) = \log((k+1)(\log(k+1) + \log(\log(k+1)))) \exp\left(\frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))}\right)$$

and

$$(2.16) \quad \hat{C}(k) = \exp\left(\frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))}\right).$$

Multiplying both sides of (2.14) by $e^{-c_1 + \frac{1}{p_j}} p_j^2 / (p_j^2 - 1)$, we obtain

$$(2.17) \quad \tilde{C}(k) \prod_{\ell=1}^{k+1} \frac{p_\ell^2}{p_\ell^2 - 1} < \frac{e^{\gamma - c_1 + \frac{1}{p_j} p_j^2}}{p_j^2 - 1} D(k).$$

Noting that

$$(2.18) \quad \prod_{\ell=1}^{k+1} \frac{p_\ell^2}{p_\ell^2 - 1} \nearrow \frac{\pi^2}{6} \text{ as } k \rightarrow \infty,$$

we see that (2.17) is implied by

$$(2.19) \quad \tilde{C}(k) < \frac{6p_j^2 e^{\gamma - c_1 + \frac{1}{p_j}}}{\pi^2(p_j^2 - 1)} D(k) =: E_j D(k).$$

Raising both sides to the power of e , we see that (2.19) is implied by

$$(2.20) \quad [(k+1)(\log(k+1) + \log(\log(k+1)))]^{\hat{C}(k)} < [(k+1)(\log(k+1) + \log(\log(k+1)) - 1) \log(2.51) - \log(p_j)]^{E_j}.$$

(2.20) is equivalent to

$$(2.21) \quad 1 < [(k+1)(\log(k+1) + \log(\log(k+1)))]^{-\hat{C}(k) + E_j} \left[1 - \frac{(k+1) \log(2.51) + \log(p_j)}{(k+1)(\log(k+1) + \log(\log(k+1)))}\right]^{E_j}.$$

Noting that $E_j > 1$ for $j \in \{1, 2, 3\}$, we see that there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies $-\hat{C}(k) + E_j > \epsilon$ for some $\epsilon \in (0, 1)$. If needed, we can increase K_j so that $k \geq K_j$ implies

$$(2.22) \quad \left[1 - \frac{(k+1) \log(2.51) + \log(p_j)}{(k+1)(\log(k+1) + \log(\log(k+1)))}\right]^{E_j} > \epsilon,$$

and also so that $k \geq K_j$ implies

$$(2.23) \quad 1 < \epsilon [(k+1)(\log(k+1) + \log(\log(k+1)))]^\epsilon,$$

which implies (2.21). □

2.2. All numbers not divisible by one of the prime numbers 2, 3, or 5. Letting $j = 1$ in (2.19), we seek to show that

$$(2.24) \quad \tilde{C}(k) < \frac{8e^{\gamma-c_1+5}}{\pi^2} D(k).$$

Lemma 2.11. *For $k \geq 13042$, $\hat{C}(k) < 1.525$.*

Proof. $\hat{C}(k)$ is decreasing, so the result follows from computation. \square

Denote $f(k) = (k+1)(\log(k+1) + \log(\log(k+1)))$. Applying Lemma 2.11 to (2.24) and performing some algebraic manipulations, our goal reduces to showing that

$$(2.25) \quad \log(f(k)) < \frac{8e^{\gamma-c_1+5}}{\pi^2(1.525)} \log((f(k)-1)\log(2.51) - \log(2)).$$

Raising both sides to the power of e , this becomes

$$(2.26) \quad 1 < f(k)^{2.0166} \left[1 - \frac{(k+1)\log(2.51) - \log(2)}{f(k)} \right]^{1.20166}.$$

The RHS of (2.26) is increasing, and a computation reveals that it holds for $k \geq 13042$. Additionally, using Lemma 2.1, one can check that

$$(2.27) \quad \frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^\gamma \log(\log(m))$$

for $k \geq 3$. Finally, when $k \in \{1, 2\}$, we check that

$$(2.28) \quad \frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq \frac{15}{8} < e^\gamma \log(\log(n))$$

for $n \geq 680$. This confirms the following for $j = 1$:

Theorem 2.12. *For $j \in \{1, 2, 3\}$, Robin's inequality holds for every natural number > 5040 which is not divisible by p_j .*

To confirm Theorem 2.12 when $j \in \{2, 3\}$, one can repeat the above process to see that, for sufficiently big k , (2.19) is satisfied. The cases with smaller k have been verified in [MP20].

2.3. Primorials and sufficiently big even numbers. Fix $k \in \mathbb{N}$ and let n be odd. We consider numbers of the form $2^k n$. We calculate

$$(2.29) \quad \frac{\sigma(2^k n)}{2^k n} = \frac{\sigma(2^k)}{2^k} \frac{\sigma(n)}{n} < \frac{\sigma(2^k)}{2^k} \frac{n}{\varphi(n)} = \frac{\sigma(2^k)}{2^k} \frac{\varphi(2^k)}{2^k} \frac{2^k n}{\varphi(2^k n)} = \left(1 - \frac{1}{2^{k+1}}\right) \frac{2^k n}{\varphi(2^k n)}.$$

Applying Theorem 15 from [RS62], we know

$$(2.30) \quad \left(1 - \frac{1}{2^{k+1}}\right) \frac{2^k n}{\varphi(2^k n)} < \left(1 - \frac{1}{2^{k+1}}\right) \left(e^\gamma \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right).$$

We ask which n satisfy

$$(2.31) \quad \left(1 - \frac{1}{2^{k+1}}\right) \left(e^\gamma \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right) < e^\gamma (\log(\log(2^k n))).$$

This is equivalent to asking when

$$(2.32) \quad \frac{2.51(2^{k+1} - 1)}{e^\gamma} < (\log(\log(2^k n)))^2$$

holds, which is when

$$(2.33) \quad n > \frac{e^{e^{\sqrt{\frac{2.51(2^{k+1}-1)}{e^\gamma}}}}}{2^k} =: b(k).$$

Thus, we obtain the following:

Theorem 2.13. *Given any $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies (2.33).*

In particular, we have the following:

Corollary 2.14. *If $n \geq 620$ is odd, then Robin's inequality holds for $2n$. Furthermore, Robin's inequality holds for all primorials > 30 .*

Proof. The first statement follows immediately from Theorem 2.13 and the second follows from the computation of primorials < 1240.3 \square

2.4. All 21-free numbers. The results of the previous subsection are based on the inequality in Theorem 15 from [RS62]. This inequality can be improved by using a sharper bound stated in [AN22]:

$$(2.34) \quad \frac{m}{\varphi(m)} < e^\gamma \log(\log(m)) + \frac{.0168}{(\log(\log(m)))^2}$$

for $m \geq 10^{10^{13.11485}} =: C$. Using the same reasoning as in the proof of Theorem 2.13, we derive the following result:

Theorem 2.15. *Given $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies*

$$(2.35) \quad n > \frac{e^{e^{\sqrt[3]{\frac{.0168(2^{k+1}-1)}{e^\gamma}}}}}{2^k} =: \tilde{b}(k).$$

We can thus conclude the following:

Theorem 2.16. *Robin's inequality holds for all 21-free numbers.*

Proof. Let $k \in \mathbb{N}$ and n be an odd natural number. If $n > \tilde{b}(k)$, n satisfies Robin's inequality by Theorem 2.15. If not, note that n satisfies Robin's inequality if $5040 < 2^k n \leq 2^k \tilde{b}(k) \leq C$ by Theorem 13 in [MP20]. Recalling the definition of ℓ -free numbers, we see that if $2^k \tilde{b}(k) < C$, then all $(k+1)$ -free numbers satisfy Robin's inequality. Indeed, setting $k = 20$, we calculate

$$\log(2^{20} \tilde{b}(20)) < 6(10^{11}) < 2.3(10^{13.11485}) < \log(C). \quad \square$$

2.5. Almost every number.

Definition 2.17. The *natural density* of a set E is

$$(2.36) \quad d(E) = \lim_{n \rightarrow \infty} \frac{\#E \cap \{1, 2, \dots, n\}}{n}$$

when the limit exists.

Theorem 2.18. *Denote by \mathcal{R} the set of numbers satisfying Robin's inequality. Then the natural density of \mathcal{R} is 1.*

Proof. We will prove that the natural density of \mathcal{R}^c is 0. Fix $\epsilon > 0$. Let $E_k = \{2^k n : n \in \mathbb{N}_{\text{odd}}, n \leq b(k)\}$ and note that $\mathcal{R}^c \subseteq \bigcup_{k \geq 1} E_k$ by Theorem 2.12 and Theorem 2.13.¹ Pick M

¹Here $\mathbb{N}_{\text{odd}} := \{1, 3, \dots\}$.

so that $\sum_{k=M+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. For $n \in \mathbb{N}$ we calculate

$$(2.37) \quad \frac{\#\mathcal{R}^c \cap \{1, 2, \dots, n\}}{n} \leq \frac{\#\bigcup_{k \geq 1} E_k \cap \{1, 2, \dots, n\}}{n} = \frac{\sum_{k \geq 1} \#E_k \cap \{1, 2, \dots, n\}}{n} \\ = \frac{\sum_{k=1}^M \#E_k \cap \{1, 2, \dots, n\} + \sum_{k=M+1}^{\infty} \#E_k \cap \{1, 2, \dots, n\}}{n},$$

where the first equality follows from the fact that the E_k 's are disjoint. Noting that $\sum_{k=1}^M \#E_k \cap \{1, 2, \dots, n\} < \infty$ for all $n \in \mathbb{N}$, we see that we can pick N so that $n \geq N$ implies that the RHS of (2.37) is $< \epsilon$, completing our proof. \square

3. THE LAGARIAS AND KANEKO-LAGARIAS INEQUALITIES

3.1. Superabundant numbers. Let $\Gamma(x)$ denote the gamma function. We define two functions:

$$(3.1) \quad H(x) = \int_0^1 \frac{t^x - 1}{t - 1} dt, \\ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

ψ is known as the *digamma function*. One can verify that H is smooth for $x \geq 1$ and that $H(n) = H_n$ for all $n \in \mathbb{N}$. It's also easy to see that ψ , known as the digamma function, satisfies

$$(3.2) \quad H(x) = \psi(x+1) + \gamma.$$

Lemma 3.1. *For all $x \geq 1$,*

$$(3.3) \quad H(x) < \log(x) + \gamma + \frac{1}{2x}.$$

Proof. By (2.2) from [Alz97],

$$(3.4) \quad \psi(x) < \log(x) - \frac{1}{2x}$$

for all $x \geq 1$. Then we use (3.2) and $\psi(x+1) = \psi(x) + \frac{1}{x}$ to finish. \square

Lemma 3.2. *For all $x \geq 4$,*

$$(3.5) \quad H(x) < \frac{2 \log(x)}{1 + \frac{6}{\pi^2 x}}.$$

Proof. By Lemma 3.1, it suffices to show that

$$(3.6) \quad \log(x) + \gamma + \frac{1}{2x} < \frac{2 \log(x)}{1 + \frac{6}{\pi^2 x}}$$

for $x \geq 4$. By arithmetic manipulations, (3.6) becomes

$$(3.7) \quad \frac{1}{\pi^2 x - 6} \left(\gamma \pi^2 x + \frac{\pi^2}{2} + 6\gamma + \frac{3}{x} \right) < \log(x).$$

Computation reveals that (3.7) holds for $x = 4$, and the LHS of (3.7) is decreasing while the RHS is increasing so we obtain the result. \square

Lemma 3.3. *The following hold:*

- (a) *For all $n > 1$, $H_{n+1} \leq \frac{n}{\log(n)}$.*
- (b) *For all $x \geq 4$, $\log(H(x)) \leq \frac{x}{2 \log(x)}$.*

Proof. (a) We can manually verify the inequality for $n \leq 6$. Noting that

$$(3.8) \quad H_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} \leq 1 + \int_1^{n+1} \frac{dt}{t} = 1 + \log(n+1),$$

it suffices to show that

$$(3.9) \quad \log(x)(\log(x+1)+1) \leq x.$$

Put $g(t) = e^t - t^2 - t - 1$. We see that $g(2) > 0$ and that $g'(t) = e^t - 2t - 1 > 0$ for $t \geq 2$, so $g(t) > 0$ for $t \geq 2$. For $x \geq e^2 - 1$ we have

$$(3.10) \quad 0 < g(\log(x+1)) = x+1 - (\log(x+1))^2 - \log(x+1) - 1 < x - \log(x)(\log(x+1)+1).$$

(b) For $x \geq 4$, note that the function mapping $x \mapsto \frac{x}{\log(x)}$ is increasing. If $n \leq x < n+1$, then

$$(3.11) \quad H_n \leq H(x) < H_{n+1} \leq \frac{n}{\log(n)} \leq \frac{x}{\log(x)}.$$

For $y > 2$ we see that $\log(y) < \frac{y}{2}$, so let $y = H(x)$ and apply (3.11) finish. \square

Lemma 3.4. For $x \geq 4$,

$$(3.12) \quad H(x) \log(H(x)) < \frac{x^2}{x + \frac{6}{\pi^2}}.$$

Proof. Apply Lemma 3.1 and Lemma 3.3. \square

Lemma 3.5. For $x \geq 4$,

$$(3.13) \quad H'(x) > \frac{H(x) \log(H(x))}{x^2}.$$

Proof. We will use (51) from [FD14] which states that

$$(3.14) \quad \frac{1}{\psi'(x)} \leq x + \frac{6}{\pi^2} - 1$$

for $x \geq 1$. We calculate

$$(3.15) \quad H'(x) = \psi'(x+1) \geq \frac{1}{x+6\pi^2} > \frac{H(x) \log(H(x))}{x^2},$$

where the equality follows from taking the derivative of (3.2) and the second inequality follows from Lemma 3.4. \square

Proposition 3.6. The function

$$(3.16) \quad g(x) = \frac{\exp(H(x)) \log(H(x))}{x}$$

is increasing for $x \geq 4$.

Proof. We start with (3.5) from [Lag02]:

$$(3.17) \quad \begin{aligned} H_n &= \log(n) + \gamma + \int_n^\infty \frac{x - \lfloor x \rfloor}{x^2} dx \\ \implies \exp(H_n) &= e^\gamma n \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2} dx\right) \\ \implies \frac{\exp(H_n) \log(H_n)}{n} &= e^\gamma \log(H_n) \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2} dx\right). \end{aligned}$$

Given $k \in \mathbb{N}$, put

$$(3.18) \quad g_k(x) = e^\gamma \log(H(x)) \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} dt\right)$$

so that $\lim_{k \rightarrow \infty} g_k(x) = g(x)$. We compute

$$(3.19) \quad g'_k(x) = e^\gamma \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} dt\right) \left(\frac{H'(x)}{H(x)} + \log(H(x)) \left(-\frac{x - \lfloor x \rfloor}{x^2}\right)\right),$$

so $g'_k(x) > 0$ if and only if

$$(3.20) \quad \frac{H'(x)}{H(x)} + \log(H(x)) \left(-\frac{x - \lfloor x \rfloor}{x^2}\right) \geq \frac{H'(x)}{H(x)} - \frac{\log(H(x))}{x^2} > 0,$$

which is the content of Lemma 3.5. Thus, $g(x)$ is the limit of monotonically increasing functions and is therefore monotonically increasing. \square

Corollary 3.7. *The sequence*

$$(3.21) \quad \left\{ \frac{\exp(H_n) \log(H_n)}{n} \right\}_{n=1}^\infty$$

is monotonically increasing.

Proof. Proposition 3.6 gives the result for $n \geq 4$ and we can manually check the smaller cases. \square

Definition 3.8. A number n is *superabundant* if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.

Theorem 3.9. *If there are counterexamples to the Kaneko-Lagarias inequality, the smallest such counterexample is a superabundant number.*

Proof. Suppose, for sake of contradiction, that m is the smallest counterexample to the Kaneko-Lagarias inequality and that m is not superabundant. Let n be the greatest superabundant number $< m$. We calculate,

$$(3.22) \quad \frac{\sigma(n)}{n} > \frac{\sigma(m)}{n} \geq \frac{\exp(H_m) \log(H_m)}{m} > \frac{\exp(H_n) \log(H_n)}{n},$$

so $n < m$ violates the Kaneko-Lagarias inequality: a contradiction. \square

3.2. Connection to Robin's inequality.

Theorem 3.10. *If Robin's inequality holds for some $n \in \mathbb{N}$, then the Kaneko-Lagarias inequality holds for n .*

Proof. We use the approximation

$$(3.23) \quad H_n \geq \log(n) + \gamma + \frac{1}{2n+1}$$

to calculate

$$(3.24) \quad \frac{\exp(H_n) \log(H_n)}{n} \geq \frac{e^{\gamma + \frac{1}{2n+1}} n \log\left(\log(n) + \gamma + \frac{1}{2n+1}\right)}{n} > e^\gamma \log(\log(n)),$$

which implies the result. \square

Note that we obtain the same result for the Lagarias inequality.

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REFERENCES

- [AF09] Amir Akbary and Zachary Friggstad. “Superabundant Numbers and the Riemann Hypothesis”. In: *The American Mathematical Monthly* 116.3 (2009), pp. 273–275.
- [AE44] L. Alaoglu and P. Erdos. “On Highly Composite and Similar Numbers”. In: *Transactions of the American Mathematical Society* 56.3 (1944), pp. 448–469.
- [Alz97] Horst Alzer. “On Some Inequalities for the Gamma and Psi Functions”. In: *Mathematics of Computation* 66.217 (1997), pp. 373–389.
- [AN22] Christian Axler and Jean-Louis Nicolas. “Large values of $\frac{n}{\varphi(n)}$ ”. In: *Acta Arithmetica* 209 (2022), pp. 357–383.
- [BT15] Kevin Boughan and Tim Trudgian. “Robin’s inequality for 11-free integers”. In: *Integers* 15 (2015).
- [Bri06] Keith Briggs. “Abundant Numbers and the Riemann Hypothesis”. In: *Experimental Mathematics* 15.2 (2006), pp. 251–256.
- [Cho+07] YoungJu Choie et al. “On Robin’s criterion for the Riemann hypothesis”. In: *Journal de Théorie des Nombres de Bordeaux* 19.2 (2007), pp. 357–372.
- [Dus99] Pierre Dusart. “The k -th Prime is Greater than $k(\log(k)() + \log(\log(k)) - 1)$ for $k \geq 2$ ”. In: *Mathematics of Computation* 68.225 (1999), pp. 411–415.
- [FD14] M. R. Farhangdoost and M. Kargar Dolatabadi. “New Inequalities for Gamma and Digamma Functions”. In: *Journal of Applied Mathematics* 2014 (2014), pp. 1–7.
- [Lag02] Jeffrey C. Lagarias. “An Elementary Problem Equivalent to the Riemann Hypothesis”. In: *The American Mathematical Monthly* 109.6 (2002), pp. 534–543.
- [MP20] Thomas Morrill and David Platt. “Robin’s inequality for 20-free integers”. In: *Integers* 21 (2020).
- [Rob84] Guy Robin. “Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann”. In: *Journal de Mathématiques Pures et Appliquées* 63 (1984), pp. 187–213.
- [RS62] J. Barkley Rosser and Lowell Schoenfeld. “Approximate formulas for some functions of prime numbers”. In: *Illinois Journal of Mathematics* 6.1 (1962), pp. 64–94.
- [SP12] P. Sole and M. Planat. “The Robin inequality for 7-free integers”. In: *Integers* 12 (2012), pp. 301–309.
- [Ten95] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*. French. Vol. 46. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995, pp. xvi+448.

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