Thermodynamic Formalism in Holomorphic Dynamics

Minicourse by Fabrizio Bianchi and Mary Yan He
Notes by Alexander Paschal

June 2–6, 2025

We will first introduce the main concepts and questions of thermodynamic formalism in the setting of one-dimensional holomorphic dynamics. We will see how the complex setting often allows us to overcome the need for hyperbolicity assumptions usually made to study these problems in more general settings. We will then move to higher dimensions. Holomorphic dynamical systems in higher dimensions exhibit different behaviors from those in dimension 1, as holomorphic maps are not necessarily conformal anymore and several classical theorems in complex analysis no longer hold in several complex variables. We will introduce a volume dimension for measures with positive Lyapunov exponents as a dynamical replacement for the Hausdorff dimension and discuss its applications.

Contents

Lecture 1

Fix a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree d (though the following results also hold for rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$). For example, we could take $f(z) = z^2$. In this case, $\{z \in \mathbb{C} : |z| = 1\}$ is stable under the action of f and is called f's *Julia set*, which is often denoted by $J(f)$ or simply J. Meanwhile, $\{z \in \mathbb{C} : |z| > 1\}$ is unstable and is called f's *Fatou set*. Another example, which gives rise to the famous Mandelbrot set, is $f(z) = z^2 + C$ for some $C > 0$. If $C \gg 0$ (i.e., C is much bigger than 0), then f's limit set is a Cantor set. In general, this f is not uniformly hyperbolic, since the critical point 0 can be found in its Julia set. A symptom of this problem is that there may exist fixed points z with $|f'(z)| = 1$. Such points are classified as *parabolic*, *Siegel*, and *Cremer*, depending on their rationality and f's behavior near them. In dimension 1, both of these problems are "finite" in the sense that the number of critical points and the number of points satisfying $|f'| = 1$ are $\leq d - 1$. This makes dimension 1 nice since this does not hold in general in higher dimensions.

We now view the action of f on $\mathbb{C}$ as a dynamical system. The following theorem is a classical result which we seek to generalize, due in various parts to Lyubich, Freire-Lopez-Mañé, Brolin, Dinh-Sibony, B.-Dinh, etc.:

Theorem 1.1. *There exists a unique measure of maximal entropy μ with $\operatorname{supp}\mu = J$ such that*

(1) for almost every a, $\mu_n := \frac{1}{d^n}\sum_{f^n(b)=a}\delta_b \to \mu$,

(2) $\frac{1}{d^n}\sum_{f^n(z)=z}\delta_x \to \mu$,

(3) $\mathrm{Lyap}(\mu) > 0$, and

(4) for $\delta \approx d$, $|\langle \mu_n - \mu, \varphi\rangle|_{L^\infty} \lesssim \frac{1}{\delta^n}|\varphi|_{L^2}$.

Another classical theorem that we will generalize is due to Mañé and Manning:

Theorem 1.2. *Given μ from Theorem 1.1, we have $h_\mu(f) = \mathrm{Lyap}(\mu)\dim_H(\mu)$, where $\dim_H$ denotes the Hausdorff dimension of a measure.*[1]

Remark 1.3. Theorem 1.2 holds for all ergodic measures ν which are invariant on J. Further, by a theorem of Przytycki, $\mathrm{Lyap}(\nu) \geq 0$ for all such ν.

Let's now build up to trying to sketch the proof of Theorem 1.1(3), falling a little short but communicating the main idea. We define $\log^+(z) := \max(\log(z), 0)$ and

$$G(z) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|.$$

G is known as the *Green function*, and it is Hölder continuous and subharmonic. Let's also define an operator $\mathrm{dd^c}$ by

$$\mathrm{dd^c} u = (\Delta u)\,\mathrm{Leb}_{\mathbb{C}}$$

for $u \in C^2$, where Δ is the Laplacian and $\mathrm{Leb}_{\mathbb{C}}$ is the Lebesgue measure on $\mathbb{C}$.[2] Note that $\mathrm{dd^c}$ can act on continuous functions in a distributional sense. We state without proof that $\mu = \mathrm{dd^c} G$, which we'll use below.

Now for our proof attempt. Note that we can write $f' = d\prod_{j=1}^{d-1}(z - c_j)$, where the c_j's are the critical points of f. We calculate

$$\begin{aligned}
\mathrm{Lyap}(\mu) &= \int \log|f'|\,\mathrm{d}\mu \\
&= \log d + \sum_{j=1}^{d-1} \int \log|z - c_j|\,\mathrm{d}\mu \\
&= \log d + \sum_{j=1}^{d-1} \int \log|z - c_J|\,\mathrm{dd^c} G \\
&= \log d + \sum_{j=1}^{d-1} \delta_{c_j} G \\
&= \log d + \sum_{j=1}^{d-1} G(c_j) \\
&\geq \log d.
\end{aligned} \tag{1.1}$$

The obvious problem is in the third line, where we applied Stokes' theorem without considering the resulting boundary terms. We'll take for granted that if you do this calculation correctly, the resulting quantity is still positive.

Note that $f^*\mu = \mathrm{d}\mu$ and thus $f_*\mu = \mu$ (the definition of this pullback is a little nuanced); i.e., μ is invariant. With this observation, we now seek to generalize Theorem 1.1. Let $\varphi : \mathbb{P}^1 \to \mathbb{R}$ be "well-behaved" (we'll take it to be Hölder for now and define this notion more carefully in Lecture 2). There are two natural generalizations of Theorem 1.1.

(1) Finding a measure μ_φ which maximizes the *pressure* of our system, which is defined to be

$$\sup_\nu \left\{ h_\nu(f) + \int \varphi\,\mathrm{d}\nu \right\},$$

where the sup is taken over all invariant probability measures. Such a measure is known as an *equillibrium state*.

(2) Similarly to Theorem 1.1(1), we'd like to find a measure m_φ which $\frac{1}{d^n}\sum_{f^n(b)=a} e^{\varphi(b)}\delta_b$ converges to for almost every a.

Note that (1) and (2) are separate goals since we do not necessarily have that m_φ is a probability measure. Some prior work in these directions is summarized below:

Theorem 1.4 (Denker, Przyticki, Urbanski, etc.)**.** *Suppose φ is Hölder and satisfies $\Omega(\varphi) := \max_J \varphi - \min_J \varphi < \log d$. Then, there exist unique measures μ_φ, m_φ with $\mu_\varphi = \rho_\varphi m_\varphi$ for some continuous function ρ_φ.*

For $g : \mathbb{P}^1 \to \mathbb{R}$ (or $\mathbb{C}$, but we'll stick to $\mathbb{R}$), we define

$$\mathcal{L}_\phi g(y) := \sum_{f(x)=y} e^{\varphi(x)} g(y), \tag{1.2}$$

[1] We'll define this in Lecture 3.

[2] The notation $\mathrm{dd^c}$ is motivated by the following standard definitions from complex analysis: $\mathrm{d} = \frac{1}{2}(\partial + \overline{\partial})$ and $\mathrm{d^c} = \frac{1}{2}(\partial - \overline{\partial})$.

the *Perron-Frobenius transfer operator* with weight φ. $\mathcal{L}_\varphi$ satisfies $\mathcal{L}_\varphi \rho_\varphi = \lambda \rho_\varphi$ and $\mathcal{L}_\varphi^* m_\varphi = \lambda m_\varphi$ for some λ, which implies that $f_* \mu_\varphi = \mu_\varphi$ (exercise). Further, we have pressure$(\mu_\varphi) = \log d$, $h_{\mu_\varphi}(f) > 0$, $\operatorname{supp} \mu_\varphi = J$, and $\operatorname{Lyap}(\mu_\varphi) > 0$. Finally, we can apply Mañé's aforementioned formula (Theorem 1.2) to see that $h_{\mu_\varphi}(f) = \operatorname{Lyap}(\mu_\varphi) \dim_H(\mu_\varphi)$.

To finish today's lecture, we state the following theorem:[3]

Theorem 1.5 (B.-Dinh). *For all f as above and every $0 < \gamma \leq 2$, $q > 0$, there exists a norm $|\cdot|_{\log^q} \lesssim |\cdot|_\bullet \lesssim |\cdot|_{C^\gamma}$ depending on f such that, for all φ with $|\varphi|_\bullet < \infty$ and $\Omega(\varphi) < \log d$, we have*

(1) $|\frac{\mathcal{L}_\varphi g}{\lambda}|_\bullet < \beta |g|_\bullet$ for some $\beta < 1$ and all g with $\langle g, m_\varphi \rangle = 0$ and

(2) the operator $t \mapsto \mathcal{L}_{\varphi + t\psi}$ is analytic with respect to the $|\cdot|_\bullet$ norm.

Lecture 2

As promised, we define the $\log^p$ seminorm and space. For any $p > 0$ we let

$$|g|_{\log^p} := \sup_{a,b \in \mathbb{C}} |g(a) - g(b)| (\log^+ d(a,b))^p$$

and say that $g \in \log^p$ when $|g|_{\log^p} < \infty$. Now we can begin.

Our goal is to understand the Perron-Frobenius transfer operator defined in (1.2). More precisely, we'd like to find a Banach space $(E, |\cdot|)$ such that $\mathcal{L}_\varphi : E \to E$

(1) has a spectral gap, and

(2) is analytic in φ; i.e., the map $t \mapsto \mathcal{L}_{\varphi + t\psi}$ is analytic (note the similarity to Theorem 1.5(2)).

As a refresher, we first consider the case where $\varphi = 0$. Recall that every z has d preimages under f counting multiplicities, that, for every n,

$$\frac{1}{d^n} (f^n)^* \delta_z = \frac{1}{d^n} \sum_{f^n(w) = z} \delta_w$$

is a probability measure, and that the limit as $n \to \infty$ satisfies the conditions of Theorem 1.1.

Now let $\varphi : \mathbb{P}^1 \to \mathbb{R}$ and consider the sequence of measures

$$\mu_{\varphi,z,n} := \sum_{f^n(w) = z} e^{S_n \varphi(w)} \delta_w,$$

where $S_n \varphi(w)$ is the n-th *ergodic sum* $\varphi(w) + \varphi(f(w)) + \cdots + \varphi(f^{n-1}(w))$. If we assume that $\Omega(\varphi) = \max_J(\varphi) - \min_J(\varphi) < \log d$, and that, for some $p > 2$ and all $x, y \in \mathbb{P}^1$, we have

$$|\varphi(x) - \varphi(y)| \lesssim \frac{1}{(1 + |\log d(x,y)|)^p}$$

(this is weaker than the Hölder property), then we have the following two theorems:

Theorem 2.1 (Denker-Przytycki-Urbanski, Urbanski-Zdunik, B.-Dinh). *These exists an invariant probability measure μ_φ with $\operatorname{supp} \mu_\varphi = J$, a $\lambda > 0$, and a continuous function $\rho_\varphi : \mathbb{P}^1 \to \mathbb{R}$ such that, for almost every $z \in \mathbb{P}^1$,*

$$\frac{1}{\lambda^n} \mu_{\varphi,z,n} \to \rho_\varphi(z) m_\varphi,$$

where $m_\varphi := \rho_\varphi^{-1} \mu_\varphi$. Further, the points in $f^{-n}(z)$, with weights, are equidistributed with respect to m_φ for large n, and μ_φ is an equillibrium state.

Theorem 2.2. *For all $q > 0, 0 < \gamma \leq 2$, there exists a norm $|\cdot|_\infty + |\cdot|_{\log^q} \leq |\cdot|_\bullet \leq |\cdot|_{C^\gamma}$ depending on f such that, when $|\varphi|_\bullet < \infty$,*

(1) there exists $\beta = \beta(|\varphi|_\bullet) < 1$ such that

$$\left| \frac{\mathcal{L}_\varphi g}{\lambda} - \langle m_\varphi, g \rangle \rho \right|_\bullet \leq \beta |g - \langle m_\varphi, g \rangle \rho|_\bullet, \text{ and}$$

(2) the map $t \mapsto \mathcal{L}_{\varphi + t\psi}$ is analytic in t.

[3] We'll define the $\log^q$ seminorm in Lecture 2.

From these theorems, we obtain numerous statistical properties of our system: when $|\varphi|_\bullet, |u|_\bullet < \infty$, the sequence $u \circ f^n$ is "almost like" i.i.d. random variables on $(\mathbb{P}^1, \mu_\varphi)$ in that it satisfies the local CLT, Berry-Essen theorem, almost sure invariant principle, etc.

We now work towards a proof of Theorem 2.1. Let's define

$$\mathcal{L}_\varphi^n g(y) := \sum_{f^n(x)=y} e^{S_n\varphi(x)} g(x).$$

By duality we have $\lambda^{-n}\mathcal{L}_\varphi^n g \to \rho_\varphi$. For simplicity, we'll assume $\varphi \in C^2$, $g = 1$ and let $\mathbb{1}_n = \mathcal{L}^n 1$, $\mathbb{1}_n^* = \frac{\mathbb{1}_n}{\lambda^n}$.

The first thing we want is a bound on oscillation given by a bound on $\mathrm{dd^c}$: something like

$$0 \le \mathrm{dd^c} g \le \mathrm{dd^c} h \implies \Omega(g, r) \lesssim \Omega(h, r),$$

where $\Omega(g, r) := \sup\{\Omega_{\mathbb{B}}(g) : \mathbb{B} \text{ is a ball of radius } r\}$. We can, indeed, obtain something (weaker) of this flavor:

Lemma 2.3.

(1) If $0 \le \mathrm{dd^c} g \le \mathrm{dd^c} h$, then $\Omega(g, r) \lesssim \Omega(h, \sqrt{r}) + A\sqrt{r}$ for some constant A.

(2) If $|\mathrm{dd^c} g_n| \le R$ for continuous potentials (g_n), then the famiily (g_n) is equicontinuous.

In the spirit of Lemma 2.3(2), we want to find a uniform (measure) R such that $|\mathrm{dd^c}\mathbb{1}_n^*| \le R$. We calculate,

$$\begin{aligned}
\mathrm{dd^c}\mathbb{1}_n &= \mathrm{dd^c}\left(\sum_{f^n(x)=y} e^{S_n\varphi(x)}\right) \\
&= \sum_{f^n(x)=y} e^{S_n\varphi(x)} \left(\sum_{j=0}^{n-1} \mathrm{dd^c}\varphi(f^j(x)) + \sum_{k,\ell=0}^{n-1} \partial\varphi(f^k(x)) \wedge \bar{\partial}\varphi(f^\ell(x))\right) \\
&= \cdots,
\end{aligned}$$

which gives the bound

$$|\mathrm{dd^c}\mathbb{1}_n^*| \lesssim \sum_{j=0}^{n} \left(e^{\Omega(\varphi)} d\right)^j \Omega(\mathbb{1}_{n-j}^*) |\varphi|_{C^2} f_*^{j-1} \operatorname{Leb}_{\mathbb{C}}. \tag{2.1}$$

We still need to estimate the oscillation of the potential on the RHS of (2.1), for which we use the following lemma:

Lemma 2.4. *Up to addition of a Hölder continuous function, the potential of $\sum_{j=0}^{\infty}(e^{\Omega(\varphi)}/d)^j f_*^{j-1} \operatorname{Leb}_{\mathbb{C}}$ is given by*[4] *$\sum_{j=0}^{n}(e^{\Omega(\varphi)}/d)^j u_j$ where*

(1) u_j is $\frac{\gamma}{d^j}$-Hölder, and

(2) $|u_j|_\infty \lesssim \frac{d^j}{\delta^j}$ for all $\delta < d$.

Lemma 2.4 implies that $\sum_{j=0}^{n}(e^{\Omega(\varphi)}/d)^j u_j \in \log^p$ for all p and thus that $|\mathbb{1}_n^*|_{\log^p} < C_p$ for all n, p. With a little work, one can show that this implies that $\mathbb{1}_n^* \to \rho$, where ρ is the unique equilibrium state of our system, giving Theorem 2.1 modulo several omitted technical details.

Now for Theorem 2.2. We once again first consider the case where $\varphi = 0$. In this case, our desired norm is known.

Definition 2.5 (Dinh-Sibony). Let R^+ (resp. R^-) be the positive (resp. negative) measure given by the Hahn decomposition of $\mathrm{dd^c} g$. We then define $|g|_{\mathrm{DSH}} = \min |R^+|$.

We calculate

$$\left|\frac{f_* g}{d}\right|_{\mathrm{DSH}} \le \frac{1}{d}|f_* R^+ - f_* R^-| = \frac{1}{d}|g|_{\mathrm{DSH}}.$$

If we attempt the same calculations on $\mathrm{dd^c}\mathcal{L}_\varphi g$, we get

$$\mathrm{dd^c}\mathcal{L}_\varphi g(y) \sim \sum_{f(x)=y} e^{\varphi(x)}\, \mathrm{dd^c} g + g(x) e^{\varphi(x)}\, \mathrm{dd^c}\varphi + e^{\varphi(x)}\partial g(x)\bar{\partial}\varphi(x) + e^{\varphi(x)}\partial\varphi(x)\bar{\partial} g(x)$$

The operator $\mathrm{dd^c}$ is complex, but $\mathcal{L}_\varphi$ performs a non-complex perturbation (f_*). This makes the terms in red hard to control, and motivates the search for a weaker norm: one that gives bounds on $\mathrm{dd^c}$ and regularity.

[4] By this we mean $\mathrm{dd^c}\sum_{j=0}^{n}(e^{\Omega(\varphi)}/d)^j u_j = \sum_{j=0}^{\infty}(e^{\Omega(\varphi)}/d)^j f_*^{j-1} \operatorname{Leb}_{\mathbb{C}}$.

As an initial idea, we can consider a norm like $|\cdot|_p :\cong |\cdot|_{\mathrm{DSH}} + |\cdot|_{\log^p}$. In this norm, we have $|\partial g \wedge \overline{\partial} h|_p \leq |g|_p |h|_p$, but we still fall short: note that

$$\frac{dd^c \mathcal{L}_\varphi^n g}{\lambda_n} \lesssim \left(\frac{e^{\Omega(\varphi)}}{d}\right)^n f_*^n \, dd^c g + \sum_{j=1}^n \left(\frac{e^{\Omega(\varphi)}}{d}\right)^j \left|\frac{\mathcal{L}_\varphi^{n-j} g}{\lambda^{n-j}}\right|_\infty f_*^{j-1} \, dd^c \varphi + \cdots,$$

the potential of which is

$$\frac{dd^c \mathcal{L}_\varphi^n g}{\lambda_n} \lesssim \left(\frac{e^{\Omega(\varphi)}}{d}\right)^n f_*^n g + \sum_{j=1}^n \left(\frac{e^{\Omega(\varphi)}}{d}\right)^j \left|\frac{\mathcal{L}_\varphi^{n-j} g}{\lambda^{n-j}}\right|_\infty f_*^{j-1} \varphi + \cdots. \tag{2.2}$$

It is hard to control the terms in red, and thus to determine the regularity of the RHS of (2.2). For this, we have the following two results:

Lemma 2.6. *For all $A > 1$, there exists a constant c_p such that $|d^{-j} f_*^j \varphi|_{\log^p} \leq c_p A^j |\varphi|_{\log^p}$.*

Theorem 2.7 (B.-Dinh). *If $|\varphi|_p < \infty$, then $|f_*^j \varphi / d^j|_\infty \to 0$ exponentially (with precise bounds).*

These results imply that

$$\frac{dd^c \mathcal{L}_\varphi^n g}{\lambda_n} \lesssim \left(\frac{e^{\Omega(\varphi)}}{d}\right)^n f_*^n g + \sum_{j=1}^n \left(\frac{e^{\Omega(\varphi)}}{d}\right)^j \left|\frac{\mathcal{L}_\varphi^{n-j} g}{\lambda^{n-j}}\right|_\infty f_*^j \varphi \in \log^q$$

for some explicit $q < p$, so $|\lambda^{-n} \mathcal{L}_\varphi^n g|_q \to 0$ uniformly in g.

Now we seek a spectral gap. Let

$$|R|_{\alpha,p} := \min\{c : R \leq c \sum_j \alpha^j f_*^j S \text{ for some } |S|_p \leq 1\}.$$

We have $|f_* R|_{\alpha,p} \leq \frac{1}{\alpha} |R|_{\alpha,p}$ by definition. This gives

$$\begin{aligned}
\left|\frac{dd^c \mathcal{L}_\varphi^n g}{\lambda_n}\right|_{\alpha,p} &\lesssim \left|\left(\frac{e^{\Omega(\varphi)}}{d}\right)^n f_*^n \, dd^c g + \sum_{j=1}^n \left(\frac{e^{\Omega(\varphi)}}{d}\right)^j \left|\frac{\mathcal{L}_\varphi^{n-j} g}{\lambda^{n-j}}\right|_\infty f_*^{j-1} \, dd^c \varphi\right|_{\alpha,p} \\
&\leq \left(\frac{e^{\Omega(\varphi)}}{\alpha d}\right)^n | \, dd^c g|_{\alpha,p} + \sum_{j=1}^n \left(\frac{e^{\Omega(\varphi)}}{\alpha d}\right)^j \left|\frac{\mathcal{L}_\varphi^{n-j} g}{\lambda^{n-j}}\right|_\infty | \, dd^c \varphi|_{\alpha,p} \\
&\leq c_n | \, dd^c g|_{\alpha,p} \to 0
\end{aligned}$$

We can extend $|\cdot|_{\alpha,p}$ to a norm on continuous functions by letting $|g|_{\alpha,p} := | \, dd^c g|_{\alpha,p}$. We calculate

$$\begin{aligned}
|dd^c(gh)|_{\alpha,p} &= |g \, dd^c h + h \, dd^c g + i\partial g \wedge \overline{\partial} h + i\partial h \overline{\partial} g|_{\alpha,p} \\
&\leq |g|_\infty |dd^c h|_{\alpha,p} + |h|_\infty |dd^c g|_{\alpha,p} + |i\partial g \wedge \overline{\partial} h + i\partial h \wedge \overline{\partial} g|_{\alpha,p},
\end{aligned}$$

but, again, the term in red is hard to control. We thus change the definition of $|\cdot|_{\alpha,p}$ on continuous functions to $|g|_{\alpha,p}^2 := |i\partial g \wedge \overline{\partial} g|_{\alpha,p}$.

The $|\cdot|_{\alpha,p}$ norm satisfies almost everything we want—we even have, in the spirit of Lemma 2.3, that $i\partial g \wedge \overline{\partial} g \leq dd^c h$ implies $\Omega(g,r) \lesssim \Omega(h,r)$, though the proof is quite involved. All we still need is the upper bound in Theorem 2.2. To obtain this γ-Hölder-like norm, we perform an interpolation, defining

$$|g|_{\alpha,p,\gamma} := \min\{c : \text{for all } 0 < \epsilon < 1, \text{ there exist } g_\epsilon^1, g_\epsilon^2 \text{ such that } g = g_\epsilon^1 + g_\epsilon^2, |g_\epsilon^2|_\infty < \infty, \text{ and } |g_\epsilon^1|_{\alpha,p} \leq c(1/\epsilon)^{1/\gamma}\}.$$

Our desired norm is then $|\cdot|_\bullet := |\cdot|_\infty + |\cdot|_{\alpha,p,\gamma}$, completing the sketch of the proof of Theorem 2.2.

For more detail on this part of the minicourse, we refer the reader to [BD24].

Lecture 3

We provide some history on this topic as we proceed to the higher-dimensional realm.

Sullivan proved that a rational map $f : \mathbb{C} \to \mathbb{C}$ with degree ≥ 2 has no wandering domain: letting U be a connected component U, $f^n(U)$ is eventually periodic. He was inspired by an analogous result in geometric group theory known as the Ahlfors finiteness theorem. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$. An example of a Kleinian group is the fundamental group of a closed hyperbolic 3-manifold. The Ahlfors finiteness theorem says that, given a finitely generated Kleinian group Γ with region of discontinuity Ω, Γ/Ω has a finite amount of components, each of which is a compact Riemannian sphere with finitely many points removed.

Since this connection was discovered, there have been many results translating results from geometric group theory to complex dynamics—to fill in "Sullivan's dictionary." This can only be pushed so far though, as there are non-examples such as the following.

Theorem 3.1 (Avila-Lyubich, Buff-Chéritat). *There exists a quadratic polynomial f whose Julia set has positive Lebesgue measure.*

This is not the case for Kleinian groups, as proven by Agol and Calegari-Gabai, who established the Ahlfors measure conjecture (which is equivalent to the tameness conjecture): the limit set of a finitely generated Kleinian group is either the entire Riemannian sphere or has Lebesgue measure 0.

In the last 20 years, this connection has been further explored by generalizing Kleinian groups to Anosov groups (or representations) and rational dynamics to the study of rational maps acting on higher-dimensional complex manifolds. We focus on furthering the connection between these two areas through dimension theory.

Now we move to the higher-dimensional realm. Recall that in the 1-dimensional case, our stated results held for rational maps. In the higher-dimensional case, we work with the following class of functions.

Definition 3.2. Suppose $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ satisfies $F^{-1}(0) = \{0\}$ and $F = (f_1, f_2, \ldots, f_k)$, where the f_i's are homogeneous polynomials of uniform degree $d \geq 2$. Then F descends to a map $f : \mathbb{CP}^k \to \mathbb{CP}^k$ which we call a *holomorphic endormorphism* with *algebraic degree d*.

Fix a holomorphic endomorphism $f : \mathbb{CP}^k \to \mathbb{CP}^k$. Here are some known results.

Lemma 3.3 (Gromov-Yomdin). $h_{top}(f) = k \log d$.

Theorem 3.4 (Briend-Duval, Lyubich, Freire-Lopez-Mañé). *f a unique measure of maximal entropy.*

Let μ denote f's unique measure of maximal entropy and define the *Julia set* $J(f)$ of f to be $\operatorname{supp} \mu$. Let $\mathcal{M}_J$ be the set of f-invariant ergodic probability measures on $J(f)$, and $\mathcal{M}_J^+$ be the set of those measures in $\mathcal{M}_J$ which have positive Lyapunov exponents. Given $\nu \in \mathcal{M}_J$, there are two constants canonically associated with ν: its *measure-theoretic entropy* $h_\nu(f)$ and its *Lyapunov exponents*, which come from the following theorem.

Theorem 3.5 (Oseledet). *For some $1 \leq \ell \leq k$, there exist $\chi_\ell < \chi_{\ell-1} < \cdots < \chi_1$ such that for ν-a.e. $x \in \mathbb{P}^k$, there exists a splitting of the* complex tangent space *$T_x\mathbb{P}^k$ into subspaces*

$$\{0\} =: (L_{\ell+1})_x \subsetneq (L_\ell)_x \subsetneq \cdots \subsetneq (L_1)_x := T_x\mathbb{P}^k$$

which satisfy $\mathrm{D}f_x(L_j)x = (L_j)_{f(x)}$ and $\lim_{n\to\infty} \frac{1}{n} \log|\mathrm{D}f_x^n v| = \chi_j$ for all $v \in (L_j)_x \setminus (L_{j+1})_x$ and $1 \leq j \leq \ell$.

Let us define the *Hausdorff dimension* of ν to be

$$\dim_H(\nu) = \inf\{\dim_H(X) : X \subseteq \mathbb{P}^k \text{ is Borel and } \nu(X) = 1\}.$$

We have the following theorem of Mañé-Manning.

Theorem 3.6. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ and ν an f-invariant probability measure with positive entropy. Then we have $\dim_H(\nu) = \frac{h_\nu(f)}{\mathrm{Lyap}(\nu)}$, where $\mathrm{Lyap}(\nu)$ denotes the sum of ν's Lyapunov exponents.*

There are some results in smooth dynamics related to this theorem. In dimension 1, this was established by Hofbauer-Raith and Ledrappier. Pesin established that, for diffeomorphism f of a compact manifold M to itself, $h_\nu = \chi_\nu^+$ for all ergodic probability measures on M which are absolutely continuous with respect to Lebesgue measure, where χ_ν^+ denotes the sum of ν's non-negative Lyapunov exponents with multiplicity. Work was also done by Young, Ledrappier-Young, and Barreira-Pesin-Schmeling.

Unfortunately, the Mañé-Manning formula fails in higher dimensions. To see this, consider the map $(z, w) \mapsto (z^2, w^2 + \epsilon i)$ for small $\epsilon > 0$. This map can be extended to a holomorphic endomorphism of $\mathbb{CP}^2$, say, f. By Theorem 3.4, let ν_i be the measure of maximal entropy for the i-th coordinator of this map, and define $\nu := \nu_1 \times \nu_2$. One one hand, we have

$$\dim_H(\nu) = \dim_H(\nu_1) + \dim_H(\nu_2) = \frac{\log 2}{\mathrm{Lyap}(\nu_1)} + \frac{\log 2}{\mathrm{Lyap}(\nu_2)},$$

while, on the other hand, we get

$$\frac{h_\nu(f)}{\mathrm{Lyap}(\nu)} = \frac{2 \log 2}{\mathrm{Lyap}(\nu_1) + \mathrm{Lyap}(\nu_2)}.$$

The Mañé-Manning formula fails because the dynamics of a holomorphic endomorphism $f : \mathbb{CP}^k \to \mathbb{CP}^k$ is not conformal for $k > 1$, and the Hausdorff dimension does not see this. Thus, we generalize the Mañé-Manning formula as follows.

Theorem 3.7 (Bianchi-He)**.** *Let f be a holomorphic endomorphism of algebraic degree ≥ 2 and ν be an f-invariant probability measure with positive Lyapunov exponents. Then,*[5]

$$\mathrm{VD}_f(\nu) = \frac{h_\nu}{2\,\mathrm{Lyap}(\nu)}.$$

There are several consequences of this theorem. First, if all of ν's Lyapunov exponents are equal, then the volume dimension satisfies $2\,\mathrm{VD}_f(\nu) = \dim_H(\nu)$, and we obtain the Mañé-Manning formula. To state the second consequence, let us define the *dynamical dimension* $\mathrm{DD}_J^+(f)$ of f as

$$\mathrm{DD}_J^+(f) := \sup\{\mathrm{VD}_f(\nu) : \nu \in \mathcal{M}_J^+(f)\}$$

and the *pressure* as

$$P_J^+(t) := \sup\{h_\nu(f) - t\,\mathrm{Lyap}(\nu) : \nu \in \mathcal{M}_J^+(f)\}, \quad P_J^+(f) := \inf\{t \in \mathbb{R} : P_J^+(t) = 0\}.$$

We have the following, which generalizes a result of Denker-Urbanski in dimension 1.

Theorem 3.8 (Bianchi-He)**.** *Let f be a holomorphic endomorphism of algebraic degree ≥ 2. Then,*

$$2\,\mathrm{DD}_J^+(f) = P_J^+(f).$$

Given $t \geq 0$, a probability measure ν on $J(f)$ is t-volume-conformal if, for all Borel sets $A \subseteq J(f)$ on which f is invertible, we have $\nu(f(A)) = \int_A |\mathrm{Jac}\, f|^t \, d\nu$. Let $\delta_J(f)$ be the infimum over the $t \geq 0$ such that there exists such a measure on $J(f)$. Then we have

Theorem 3.9 (Bianchi-He)**.** *If f is a hyperbolic holomorphic endomorphism of algebraic degree ≥ 2,*[6] *then we have*

$$\delta_J(f) = P_J^+(f) = 2\,\mathrm{VD}_f(J(f)).$$

Further, there exists a unique ergodic probability measure μ on $J(f)$ such that $\mathrm{VD}_f(\mu) = \mathrm{VD}_F(J(f))$.

In Lecture 4, we'll provide a proof sketch for Theorem 3.7. For the proof of Theorem 3.8 and Theorem 3.9, we refer the reader to [BH24].

Lecture 4

As said last lecture, today we provide a proof sketch for Theorem 3.7. The curious reader is referred to the second chapter of [BH24] for more details.

Let ν be a f-invariant ergodic probability measure with positive Lyapunov exponents $\chi_1 > \chi_2 > \cdots > \chi_\ell > 0$ each with multiplicity $k_1, k_2, \ldots, k_\ell$ respectively. We define

$$\mathcal{O} := \{\hat{x} = (x_n) \in (\mathbb{P}^k)^{\mathbb{Z}} : x_{n+1} = f(x_n) \text{ for all } n\},$$

the *natural extension* of our system. We denote by $\hat{\nu}$ the lift of ν to $\mathcal{O}$. Denoting by $C(f)$ the set of critical points of f, or, more precisely,

$$C(f) := \{x \in \mathbb{P}^k : Df_x \text{ is not invertible}\},$$

we let

$$Z := \{\hat{x} \in \mathcal{O} : x_n \notin C(f) \text{ for all } n\}.$$

Since the Lyapunov exponents of ν are positive, we have $\hat{\nu}(Z) = 1$. Finally, we define $f_{\hat{x}}^{-n}(x_0) = x_{-n}$.

We can now state the Berteloot-Dupart distortion theorem.

Theorem 4.1. *For every $0 < 2\eta < \gamma \ll \chi_\ell$ and $\hat{\nu}$-a.e. $\hat{x} \in Z$, there exist*

(1) an integer $n_{\hat{x}} \geq 1$ and real numbers $h_{\hat{x}} \geq 1$ and $0 < r_{\hat{x}}, \rho_{\hat{x}} \leq 1$,

(2) a sequence $\{\varphi_{\hat{x},n}\}_{n \geq 0}$ of injective holomorphic maps

$$\varphi_{\hat{x},n} : B(x_{-n}, r_{\hat{x}} e^{-n(\gamma+2\eta)}) \to \mathbb{D}^k(\rho_{\hat{x}e^{n\eta}})$$

sending x_{-n} to 0 and satisfying

$$e^{n(\gamma-2\eta)} d(u,v) \leq |\varphi_{\hat{x},n}(u) - \varphi_{\hat{x},n}(v)| \leq e^{n(\gamma+3\eta)} h_{\hat{x}} d(u,v)$$

for every $n \in \mathbb{N}$ and $u, v \in B(x_{-n}, r_{\hat{x}} e^{-n(\gamma+2\eta)})$, and

[5] $\mathrm{VD}_f(\nu)$ is called the *volume dimension* of ν, which we'll define in Lecture 4.

[6] f is *hyperbolic* if there exists $\lambda > 1, C > 0$ such that $|Df_x^n)(v) \geq C\lambda^n |v|$ for all $x \in J(f), v \in T_x\mathbb{P}^k$.

(3) *a sequence* $\{\mathcal{L}_{\hat{x},n}\}_{n\geq 0}$ *of linear maps from* $\mathbb{C}^k$ *to* $\mathbb{C}^k$ *which stabilize each*

$$H_j := \{0\} \times \cdots \times C^{k_j} \times \cdots \times \{0\},$$

satisfy

$$e^{-n\chi_j+n(\gamma-eta)}|v| \leq |\mathcal{L}_{\hat{x},n}(v)| \leq e^{-n\chi_j+n(\gamma+\eta)}|v|$$

for all $n \in \mathbb{N}$ *and* $v \in H_j$*, and such that the diagram*

$$\begin{array}{ccc} B(x_0, r_{\hat{x}}) & \xrightarrow{f_{\hat{x}}^{-n}} & B(x_{-n}, r_{\hat{x}}e^{-n(\gamma+2\eta)}) \\ \downarrow{\scriptstyle \varphi_{\hat{x},0}} & & \downarrow{\scriptstyle \varphi_{\hat{x},n}} \\ \mathbb{D}^k(\rho_{\hat{x}}) & \xrightarrow{\mathcal{L}_{\hat{x},n}} & \mathbb{D}^k(\rho_{\hat{x}}e^{n\eta}) \end{array}$$

commutes for all $n \geq n_{\hat{x}}$

Moreover, the functions $\hat{x} \mapsto h_{\hat{x}}^{-1}, r_{\hat{x}}, \rho_{\hat{x}}$ *are measurable and* η*-slow on* Z.[7]

We now define $Z_\nu \subseteq Z$ to be the full ν-measure set of points $\hat{x} \in Z$ which satisfy the conditions of Theorem 4.1. Given $\hat{x} \in Z_\nu$ and $n \in \mathbb{N}$, we may also define the *dynamical ellipse*

$$\mathcal{E}_{x_n}(r_1, r_2, \ldots, r_k) := \varphi_{\hat{x},n}^{-1} \circ \Phi(\mathbb{B}^k),$$

where $\Phi : \mathbb{C}^k \to \mathbb{C}^k$ is linear with $\Phi((e_j)_x) = r_j(\ell_j)_x$ where $\{e_j\}$ is a basis of $\mathbb{C}^k$ and $(\ell_j)_x$ is a basis of $T_x\mathbb{P}^k$. With these definitions we have the following corollary of Theorem 4.1:

Corollary 4.2. *For every* $0 < \epsilon \ll \chi_{min}$ *sufficiently small,*[8] *there exist* $Z_\nu^*(\epsilon) \subseteq Z_\nu$*,* $n(\epsilon) \in \mathbb{N}$*, and* $r(\epsilon) \in (0,1)$ *such that*

(1) $\hat{\nu}(Z_\nu^*(\epsilon)) > 1 - \epsilon$,

(2) $n_{\hat{x}} \leq n(\epsilon)$ *and* $r_{\hat{x}} \geq r(\epsilon)$ *for all* $\hat{x} \in Z_\nu^*(\epsilon)$*, and*

(3) *for all* $t, t_1, \ldots, t_k \in (0,1]$*,* $n \geq n(\epsilon)$*,* $\hat{x} \in Z_\nu^*(\epsilon)$*, and* $y, w \in B(x_0, r(\epsilon))$*, we have*
   (a) $e^{-n(\mathrm{Lyap}(\nu)+k\epsilon)} \leq |\mathrm{Jac}\, f_{\hat{x}}^{-n}(y)| \leq e^{-n(\mathrm{Lyap}(\nu)-k\epsilon)}$,
   (b) $e^{-kn\epsilon} \leq |\mathrm{Jac}\, f_{\hat{x}}^{-n}(y)||\mathrm{Jac}\, f_{\hat{x}}^{-n}(w)|^{-1} \leq e^{kn\epsilon}$,
   (c) $\mathcal{E}_{x_{-n}}(t_j r(\epsilon)e^{-n(\chi_j+\epsilon)}) \subseteq f_{\hat{x}}^{-n}(\mathcal{E}_{x_0}(t_j r(\epsilon))) \subseteq \mathcal{E}_{x_{-n}}(t_j r(\epsilon)e^{-n(\chi_j-\epsilon)})$,
   (d) $(tr(\epsilon))^{2k}e^{-2n(\mathrm{Lyap}(\nu)+k\epsilon)} \leq \mathrm{Vol}(\mathcal{E}_{x_{-n}}(tr(\epsilon)e^{-n\chi_j})) \leq (tr(\epsilon))^{2k}e^{-2n(\mathrm{Lyap}(\nu)-k\epsilon)}$*, and*
   (e) $(tr(\epsilon))^{2k}e^{-2n(\mathrm{Lyap}(\nu)+k\epsilon)} \leq \mathrm{Vol}(f_{\hat{x}}^{-n}(B(x_0, tr(\epsilon)))) \leq (tr(\epsilon))^{2k}e^{-2n(\mathrm{Lyap}(\nu)-k\epsilon)}$.

We can now define the aforementioned U sets. Let

$$M := \log \sup_{x\in\mathbb{P}^k} \sup_{v\in\mathbb{C}^k} \frac{|Df_x(v)|}{|v|},$$

which satisfies $f(B(x_0, r)) \subseteq B(f(x_0, e^M r))$; M is something like a Lipschitz constant. Given $N \geq 0$, $x \in \mathbb{P}^k$, and $\kappa, \epsilon > 0$, the set $U = U(N, x, \kappa, \epsilon)$ is such that

(1) $f^N(U) = B(f^N(x), \kappa e^{-NM\epsilon})$, and

(2) $f^N|_U$ is injective.

There is an obvious issue with this definition in that we do not know if there exist sets satisfying this condition. To resolve this, define the projection $\pi : \mathcal{O} \to \mathbb{P}^k$ by $\pi(\hat{x}) = x_0$. Then we have the following lemma, whose proof follows from Corollary 4.2.

Lemma 4.3. *There exists an exhaustion* $\{Z_\nu^*(\epsilon)\}_{\epsilon>0}$ *of* Z_ν *with* $Z_\nu^*(\epsilon) \to Z_\nu$ *as* $\epsilon \searrow 0$ *such that for each fixed* ϵ *with* $0 < \epsilon < \chi_{min}$*, there exists* $n(\epsilon) \in \mathbb{N}$ *and* $r(\epsilon \in (0,1)$ *such that for all* $x \in \pi(Z_\nu^*(\epsilon))$*,* $N \geq n(\epsilon)$*, and* $0 < \kappa < r(\epsilon)$*, the set* $U(N, x, \kappa, \epsilon)$ *is well-defined and we have*

(1) $\mathcal{E}_x(\kappa e^{-N(\chi_j+(2M+1)\epsilon)}) \subseteq U(N, x, \kappa, \epsilon) \subseteq \mathcal{E}_x(\kappa e^{-N(\chi_j-\epsilon)})$,

(2) $\kappa^{2k}e^{-2N(\mathrm{Lyap}(\nu)+k(2M+1)\epsilon)} \leq \mathrm{Vol}(U(N, x, \kappa, \epsilon)) \leq \kappa^{2k}e^{-2N(\mathrm{Lyap}(\nu)-k\epsilon)}$*, and*

(3) *for all* $y, w \in U(N, x, \kappa, \epsilon)$*, we have* $e^{-N(2M+1)k\epsilon} \leq |\mathrm{Jac}\, f^N(y)||\mathrm{Jac}\, f^N(w)|^{-1} \leq e^{N(2M+1)k\epsilon}$.

We have now built up to defining the aforementioned volume dimension. Letting $Y \subseteq \pi(Z_\nu)$ and $\epsilon > 0$, we put $Y^\epsilon := Y \cap \pi(Z_\nu^*(\epsilon))$ and define

$$\Lambda_\alpha^\epsilon(Y^\epsilon) := \limsup_{\kappa\to 0} \lim_{N^*\to\infty} \inf_{\{U_j\}} \sum_j \mathrm{Vol}(U_j)^\alpha,$$

[7] Denoting the shift map on $\mathcal{O}$ by T, we say that a function $f : \mathcal{O} \to (0,1]$ is η*-slow* if, for all $\hat{x} \in \mathcal{O}$, we have $e^{-\eta}f(\hat{x}) \leq f(T(\hat{x})) \leq e^{\eta}f(\hat{x})$.

[8] Here χ_{min} denotes the smallest Lyapunov exponent of ν.

where the inf is taken over all covers $\{U_j\}$ of Y^ϵ with $U_j = U(N_j, x, \kappa, \epsilon)$ for some $N_j \geq N^* \geq n(\epsilon)$. The *volume dimension of Y at level ϵ* is then defined to be

$$\mathrm{VD}^{\epsilon}_{f,\nu}(Y^\epsilon) := \sup\{\alpha : \Lambda^{\epsilon}_{\alpha}(Y^\epsilon) = \infty\} = \inf\{\alpha : \Lambda^{\epsilon}_{\alpha}(Y^\epsilon) = 0\},$$

where the second inequality can be proven. Next, the *volume dimension of Y* is defined as

$$\mathrm{VD}_f(\nu) := \mathrm{VD}_{f,\nu}(Y) = \limsup_{\epsilon\to 0} \mathrm{VD}^{\epsilon}_{f,\nu}(Y^\epsilon),$$

where this limsup can be proven to be a lim. Finally, the *volume dimension* is defined to be

$$\mathrm{VD}_f(\nu) := \inf\{\mathrm{VD}_{f,\nu}(Y) : Y \text{ is a Borel subset of } \pi(Z_\nu) \text{ with } \nu(Y) = 1\}.$$

Now, given $x \in \pi(Z^*_\nu(\epsilon))$, $0 < \kappa < r(\epsilon)$, and $N \geq n(\epsilon)$, let us set

$$\delta_x(\epsilon, \kappa, N) := \frac{\log(\nu(U(N, x, \kappa, \epsilon)))}{\log(\mathrm{Vol}(U(N, x, \kappa, \epsilon)))}.$$

We want to show that for ν-a.e. x, we have

$$\liminf_{\epsilon\to 0}\liminf_{\kappa\to 0}\liminf_{N\to\infty} \delta_x(\epsilon,\kappa,N) = \limsup_{\epsilon\to 0}\limsup_{\kappa\to 0}\limsup_{N\to\infty} \delta_x(\epsilon,\kappa,N) = \frac{h_\nu(f)}{2\,\mathrm{Lyap}(\nu)}.$$

It suffices to show the following.

Theorem 4.4. *Let $0 < \epsilon \ll \chi_{min}$ be sufficiently small. Then, for $\hat{\nu}$-a.e. $\hat{x} \in Z^*_\nu(\epsilon)$ and all $0 < \kappa < r(\epsilon)$, there exist integers $m_1(\epsilon, \hat{x}) \geq n(\epsilon)$ and $m_2(\epsilon, \kappa) \geq 0$ such that, for all $N \geq m_1(\epsilon, \hat{x}) + m_2(\epsilon, \kappa)$, we have*

$$\frac{h_\nu(f)}{2\,\mathrm{Lyap}(\nu)} - C\epsilon \leq \delta_{x_0}(\epsilon, \kappa, N) \leq \frac{h_\nu(f)}{2\,\mathrm{Lyap}(\nu)} + C\epsilon,$$

where $C > 0$ is a constant independent of $\epsilon, \hat{x}, \kappa$.

Let us take the following as a given and try to derive Theorem 4.4 from it.

Proposition 4.5. *For all $0 < \epsilon \leq \chi_{min}$, there exist two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ with $h_\nu(f, \mathcal{P}_1) \geq h_\nu(f) - \epsilon$ and four constants b_E, b_F (which are independent of ϵ) and $c_E, c_F > 0$ (which may depend on ϵ) such that, for $\hat{\nu}$-a.e. $\hat{x} \in Z^*_\nu(\epsilon)$, there exists an integer $m(\epsilon, \hat{x}) \geq n(\epsilon)$ such that, for all $n \geq m(\epsilon, \hat{x})$, we have $E(n) := \mathcal{E}_{x_0}(c_E e^{-n(\chi_j + b_E\epsilon)}) \subseteq \mathcal{P}_1^N(x_0)$ and $P_2^n(x_0) \subseteq F(n) := \mathcal{E}_{x_0}(c_F e^{-n(\chi_j - b_F\epsilon)})$.*[9]

For all $N \geq n(\epsilon)$ and $0 < \kappa < r(\epsilon)$, define

$$n_E(N,\kappa) := \min_{J(f)} \left\lfloor \frac{(\chi_j - \epsilon)N + \log(c_E) - \log(k)}{\chi_j + b_E\epsilon} \right\rfloor, \quad n_F(N,\kappa) := \max_{J(f)} \left\lceil \frac{[\chi_j + (2m+2)\epsilon]N + \log(c_F) - \log(k)}{\chi_j - b_F\epsilon} \right\rceil.$$

By Lemma 4.3 and Proposition 4.5, we have that, for every $0 < \kappa < r(\epsilon)$,

$$\mathcal{P}_2^{n_F(N,k)}(x_0) \subseteq F(n_F(N,\epsilon)) \subseteq U(N, x_0, \kappa, \epsilon) \subseteq E(n_E(N,\kappa)) \subseteq \mathcal{P}_1^{n_E(N,\kappa)}(x_0) \tag{4.1}$$

for all $N \geq m(\epsilon, \hat{x})$, and

$$\kappa^{2k} e^{-2N(\mathrm{Lyap}(\nu) + k(2M+2)\epsilon)} \leq \mathrm{Vol}(U(N, x_0, \kappa, \epsilon)) \leq \kappa^{2k} e^{-2N(\mathrm{Lyap}(\nu) - k\epsilon)} \tag{4.2}$$

for all $N \geq n(\epsilon)$. By (4.1) and the Shannon-McMillan-Breiman theorem, there exist $m'(\epsilon, \hat{x}) \geq m(\epsilon, \hat{x})$ and $m''(\epsilon, \kappa) \gg 1$ such that

$$(h_\nu(f) - 2\epsilon)\left(\lim_{N\to\infty} \frac{n_E(N,\kappa)}{N} - \epsilon\right) \leq -\frac{\log(\nu(U(N, x_0, \kappa, \epsilon)))}{N} \leq (h_\nu(f) + \epsilon)\left(\lim_{N\to\infty} \frac{n_F(N,\kappa)}{N} + \epsilon\right) \tag{4.3}$$

for all $N \geq m'(\epsilon, \hat{x}) + m''(\epsilon, \kappa)$. Further, by (4.2), there exists $m'''(\epsilon, \kappa) \gg 1$ such that

$$2(\mathrm{Lyap}(\nu) - k\epsilon) - \epsilon \leq -\frac{\log(\mathrm{Vol}(U(N, x_0, \kappa, \epsilon)))}{N} \leq 2(\mathrm{Lyap}(\nu) + (2M+2)\epsilon) + \epsilon. \tag{4.4}$$

Theorem 4.4 then follows from (4.3), (4.4), and the definitions of $n_E(N,\kappa)$ and $n_F(N,\kappa)$.

References

[BH24] Fabrizio Bianchi and Yan Mary He, *A Mané-Manning formula for expanding measures for endomorphisms of $\mathbb{P}^k$*, Transactions of the American Mathematical Society **377** (2024), no. 11, 8179–8219.

[BD24] Fabrizio Bianchi and Tien-Cuong Dinh, *Equilibrium States of Endomorphisms of $\mathbb{P}^k$: Spectral Stability and Limit Theorems*, Geometric and Functional Analysis **34** (2024), 1006–1051.

[9] Here $\mathcal{P}^n$ is the partition generated by $\mathcal{P}, f^{-1}(\mathcal{P}), f^{-2}(\mathcal{P}), \ldots, f^{-n}(\mathcal{P})$.