ON ROBIN'S INEQUALITY AND THE KANEKO-LAGARIAS INEQUALITY

IDRIS ASSANI, AIDEN CHESTER, AND ALEX PASCHAL

Abstract. We prove that Robin's inequality and the Lagarias inequality hold for almost every number, including all numbers not divisible by one of the prime numbers $\{2, 3, 5\}$, primorials, sufficiently big numbers of the form $2^k n$ for odd n and 21-free integers. We also prove that the Kaneko-Lagarias inequality holds for all numbers if and only if it holds for all superabundant numbers.

Contents

1. Preliminaries

We denote by $\sigma(n)$ and $\varphi(n)$ the sum of divisors function and Euler's totient function respectively. Robin's inequality ([Rob84]) states that the Riemann hypothesis is equivalent to the assertion that

$$\sigma(n) < e^{\gamma} n \log(\log(n)) \tag{1.1}$$

for all $n > 5040$, where γ denotes the Euler-Mascheroni constant. Similarly, the Lagarias inequality ([Lag02]) states that the Riemann hypothesis is equivalent to the assertion that

$$\sigma(n) < H_n + \exp(H_n)\log(H_n) \tag{1.2}$$

for all $n \geq 1$, where H_n denotes the n-th harmonic number. Lagarias also published an inequality that we call the Kaneko-Lagarias inequality (see the acknowledgements in [Lag02]),

Date: March 23, 2025.
2020 *Mathematics Subject Classification.* 11N56, 11M26.

which states that the Riemann hypothesis is equivalent to the assertion that

$$\sigma(n) < \exp(H_n)\log(H_n) \tag{1.3}$$

for all $n > 60$.

2. Robin's Inequality

2.1. **Sufficiently big numbers not divisible by one of the prime numbers 2, 3, or 5.** Let $p_1 = 2$, $p_2 = 3$, etc. be an enumeration of the prime numbers which we denote by $\mathbb{P}$. Fix $j \in \mathbb{N}$ and let $q_1 < q_2 < \cdots < q_k$ be some prime numbers distinct from p_j. Given $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$.

Lemma 2.1. *We have*

$$\frac{\sigma(n)}{n} < \prod_{\ell=1}^{k} \frac{q_\ell}{q_\ell - 1} \leq \prod_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{p_\ell}{p_\ell - 1} = \frac{n}{\varphi(n)}. \tag{2.1}$$

Proof. The first inequality follows from the fact that for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$

$$\frac{\sigma(p^\alpha)}{p^\alpha} = \frac{p - \frac{1}{p^\alpha}}{p-1} \nearrow \frac{p}{p-1} \text{ as } \alpha \to \infty. \tag{2.2}$$

The second inequality follows from the fact that $p_i \leq q_i$ for all $1 \leq i \leq k$. $\square$

Note that

$$\frac{n}{\varphi(n)} = \left(\prod_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{p_\ell + 1}{p_\ell} \right) \left(\prod_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{p_\ell^2}{p_\ell^2 - 1} \right) =: A(k)B(k). \tag{2.3}$$

We can bound $A(k)$ as follows:

$$\log(A(k)) = \sum_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \log\left(1 + \frac{1}{p_\ell}\right) \leq \sum_{\substack{\ell=1,\ldots,j-1\\ j+1,\ldots,k+1}} \frac{1}{p_\ell} = \left(\sum_{\ell=1}^{k+1} \frac{1}{p_\ell}\right) - \frac{1}{p_j} \tag{2.4}$$

and

$$\sum_{\ell=1}^{k+1} \frac{1}{p_\ell} \leq \log(\log(p_{k+1})) + c_1 + \frac{5}{\log(p_{k+1})}, \tag{2.5}$$

where $c_1 \approx .261497$ by Theorem 1.10 in [Ten95]. Thus we obtain

Lemma 2.2. *For all $k \in \mathbb{N}$,*

$$A(k) \leq \log(p_{k+1}) \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log(p_{k+1})}\right). \tag{2.6}$$

Combining [Dus99] and Theorem 3 from [RS62], we obtain the following:

Theorem 2.3. *For $k \geq 6$,*

$$k(\log(k) + \log(\log(k)) - 1) < p_k < k(\log(k) + \log(\log(k))). \tag{2.7}$$

Furthermore, combining Lemma 2.2 and Theorem 2.3, we see that

Lemma 2.4. *For $k \geq 6$, $A(k) < C(k)$ where*

$$\begin{aligned} C(k) = &\log((k+1)(\log(k+1)+\log(\log(k+1)))) \\ &\exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right). \end{aligned} \tag{2.8}$$

Now, put $m = p_{k+1}\#/p_j$. Our goal is to show the following, since it implies that Robin's inequality for n as above:

Theorem 2.5. *For any $j \in \{1,2,3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^{\gamma}\log(\log(m)). \tag{2.9}$$

Corollary 2.6. *Suppose Theorem 2.5 holds. Then Robin's inequality holds for n as above.*

Proof. We calculate

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq A(k)B(k) < C(k)B(k) < e^{\gamma}\log(\log(m)) \leq e^{\gamma}\log(\log(n)), \tag{2.10}$$

where the last inequality follows from the fact that $m \leq n$. □

Definition 2.7. The *Chebyshev function* is defined as follows:

$$\theta(x) = \sum_{p\in\mathbb{P}, p\leq x} \log(p) = \log\left(\prod_{p\in\mathbb{P}, p\leq x} p\right). \tag{2.11}$$

Theorem 2.8. *For $x \geq 529$,*

$$\prod_{\substack{p\in\mathbb{P} \\ p\leq x}} p = e^{\theta(x)} > e^{x\left(1-\frac{1}{2\log x}\right)} \geq (2.51)^x. \tag{2.12}$$

Proof. The first inequality is given by (3.14) in [RS62] and the second follows from computations since the function $f(x) = 1 - \frac{1}{2\log x}$ increases for $x > 1$. □

Lemma 2.9. *For $k \geq 99$,*

$$\log(\log(m)) > \log((k+1)(\log(k+1)+\log(\log(k+1))-1)\log(2.51) - \log(p_j)) =: D(k).$$

Proof. Noting that $k \geq 99$ implies $p_{k+1} > 529$, we calculate

$$\begin{aligned} \log(\log(m)) &= \log\left(\log\left(\frac{p_{k+1}\#}{p_j}\right)\right) > \log\left(\log\left(\frac{(2.51)^{p_{k+1}}}{p_j}\right)\right) \\ &= \log(p_{k+1}\log(2.51) - \log(p_j)) \\ &> \log((k+1)(\log(k+1)+\log(\log(k+1))-1)\log(2.51) - \log(p_j)), \end{aligned} \tag{2.13}$$

where the last inequality uses Theorem 2.3. □

The following implies Theorem 2.5:

Proposition 2.10. *For $j \in \{1,2,3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^{\gamma}D(k). \tag{2.14}$$

Proof. Denote

$$\begin{aligned}\widetilde{C}(k) = e^{-c_1+\frac{1}{p_j}}C(k) = \log((k+1)(\log(k+1)+\log(\log(k+1)))) \\ \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right)\end{aligned} \tag{2.15}$$

and

$$\widehat{C}(k) = \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right). \tag{2.16}$$

Multiplying both sides of (2.14) by $e^{-c_1+\frac{1}{p_j}}p_j^2/(p_j^2-1)$, we obtain

$$\widetilde{C}(k)\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} < \frac{e^{\gamma-c_1+\frac{1}{p_j}}p_j^2}{p_j^2-1}D(k). \tag{2.17}$$

Noting that

$$\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} \nearrow \frac{\pi^2}{6} \text{ as } k\to\infty, \tag{2.18}$$

we see that (2.17) is implied by

$$\widetilde{C}(k) < \frac{6p_j^2 e^{\gamma-c_1+\frac{1}{p_j}}}{\pi^2(p_j^2-1)}D(k) =: E_jD(k). \tag{2.19}$$

Raising both sides to the power of e, we see that (2.19) is implied by

$$\begin{aligned}&[(k+1)(\log(k+1)+\log(\log(k+1))]^{\widehat{C}(k)} \\ &< [(k+1)(\log(k+1)+\log(\log(k+1))-1)\log(2.51)-\log(p_j)]^{E_j}.\end{aligned} \tag{2.20}$$

(2.20) is equivalent to

$$\begin{aligned}1 < [(k+1)(\log(k+1)+\log(\log(k+1)))]^{-\widehat{C}(k)+E_j} \\ \left[1-\frac{(k+1)\log(2.51)+\log(p_j)}{(k+1)(\log(k+1)+\log(\log(k+1)))}\right]^{E_j}.\end{aligned} \tag{2.21}$$

Noting that $E_j > 1$ for $j\in\{1,2,3\}$, we see that there exists a $K_j\in\mathbb{N}$ such that $k\geq K_j$ implies $-\widehat{C}(k)+E_j>\epsilon$ for some $\epsilon\in(0,1)$. If needed, we can increase K_j so that $k\geq K_j$ implies

$$\left[1-\frac{(k+1)\log(2.51)+\log(p_j)}{(k+1)(\log(k+1)+\log(\log(k+1)))}\right]^{E_j} > \epsilon, \tag{2.22}$$

and also so that $k\geq K_j$ implies

$$1 < \epsilon[(k+1)(\log(k+1)+\log(\log(k+1))]^{\epsilon}, \tag{2.23}$$

which implies (2.21). $\square$

2.2. **All numbers not divisible by one of the prime numbers 2, 3, or 5.** Letting $j = 1$ in (2.19), we seek to show that

$$\widetilde{C}(k) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2} D(k). \tag{2.24}$$

Lemma 2.11. *For $k \geq 13042$, $\widehat{C}(k) < 1.525$.*

Proof. $\widehat{C}(k)$ is decreasing, so the result follows from computation. □

Denote $f(k) = (k+1)(\log(k+1) + \log(\log(k+1))$. Applying Lemma 2.11to (2.24) and performing some algebraic manipulations, our goal reduces to showing that

$$\log(f(k)) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2(1.525)} \log((f(k) - 1)\log(2.51) - \log(2)). \tag{2.25}$$

Raising both sides to the power of e, this becomes

$$1 < f(k)^{2.0166} \left[1 - \frac{(k+1)\log(2.51) - \log(2)}{f(k)} \right]^{1.20166}. \tag{2.26}$$

The RHS of (2.26) is increasing, and a computation reveals that it holds for $k \geq 13042$. Additionally, using Lemma 2.1, one can check that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^{\gamma} \log(\log(m)) \tag{2.27}$$

for $k \geq 3$. Finally, when $k \in \{1, 2\}$, we check that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq \frac{15}{8} < e^{\gamma} \log(\log(n)) \tag{2.28}$$

for $n \geq 680$. This confirms the following for $j = 1$:

Theorem 2.12. *For $j \in \{1, 2, 3\}$, Robin's inequality holds for every natural number > 5040 which is not divisible by p_j.*

To confirm Theorem 2.12 when $j \in \{2, 3\}$, one can repeat the above process to see that, for sufficiently big k, (2.19) is satisfied. The cases with smaller k have been verified in [MP20].

2.3. **Primorials and sufficiently big even numbers.** Fix $k \in \mathbb{N}$ and let n be odd. We consider numbers of the form $2^k n$. We calculate

$$\frac{\sigma(2^k n)}{2^k n} = \frac{\sigma(2^k)}{2^k} \frac{\sigma(n)}{n} < \frac{\sigma(2^k)}{2^k} \frac{n}{\varphi(n)} = \frac{\sigma(2^k)}{2^k} \frac{\varphi(2^k)}{2^k} \frac{2^k n}{\varphi(2^k n)} = \left(1 - \frac{1}{2^{k+1}}\right) \frac{2^k n}{\varphi(2^k n)}. \tag{2.29}$$

Applying Theorem 15 from [RS62], we know

$$\left(1 - \frac{1}{2^{k+1}}\right) \frac{2^k n}{\varphi(2^k n)} < \left(1 - \frac{1}{2^{k+1}}\right) \left(e^{\gamma} \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right). \tag{2.30}$$

We ask which n satisfy

$$\left(1 - \frac{1}{2^{k+1}}\right) \left(e^{\gamma} \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right) < e^{\gamma}(\log(\log(2^k n))). \tag{2.31}$$

This is equivalent to asking when

$$\frac{2.51(2^{k+1} - 1)}{e^{\gamma}} < (\log(\log(2^k n)))^2 \tag{2.32}$$

holds, which is when

$$n > \frac{e^{e^{\sqrt{\frac{2.51(2^{k+1}-1)}{e^\gamma}}}}}{2^k} =: b(k). \tag{2.33}$$

Thus, we obtain the following:

Theorem 2.13. *Given any $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies* (2.33)*.*

In particular, we have the following:

Corollary 2.14. *If $n \geq 620$ is odd, then Robin's inequality holds for $2n$. Furthermore, Robin's inequality holds for all primorials > 30.*

Proof. The first statement follows immediately from Theorem 2.13 and the second follows from the computation of primorials < 1240.3 □

2.4. **All 21-free numbers.** The results of the previous subsection are based on the inequality in Theorem 15 from [RS62]. This inequality can be improved by using a sharper bound stated in [AN22]:

$$\frac{m}{\varphi(m)} < e^\gamma \log(\log(m)) + \frac{.0168}{(\log(\log(m)))^2} \tag{2.34}$$

for $m \geq 10^{10^{13.11485}} =: C$. Using the same reasoning as in the proof of Theorem 2.13, we derive the following result:

Theorem 2.15. *Given $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies*

$$n > \frac{e^{e^{\sqrt[3]{\frac{.0168(2^{k+1}-1)}{e^\gamma}}}}}{2^k} =: \tilde{b}(k). \tag{2.35}$$

We can thus conclude the following:

Theorem 2.16. *Robin's inequality holds for all 21-free numbers.*

Proof. Let $k \in \mathbb{N}$ and n be an odd natural number. If $n > \tilde{b}(k)$, n satisfies Robin's inequality by Theorem 2.15. If not, note that n satisfies Robin's inequality if $5040 < 2^k n \leq 2^k \tilde{b}(k) \leq C$ by Theorem 13 in [MP20]. Recalling the definition of ℓ-free numbers, we see that if $2^k \tilde{b}(k) < C$, then all $(k+1)$-free numbers satify Robin's inequality. Indeed, setting $k = 20$, we calculate

$$\log(2^{20}\tilde{b}(20)) < 6(10^{11}) < 2.3(10^{13.11485}) < \log(C).$$ □

2.5. **Almost every number.**

Definition 2.17. The *natural density* of a set E is

$$d(E) = \lim_{n\to\infty} \frac{\#E \cap \{1, 2, \ldots, n\}}{n} \tag{2.36}$$

when the limit exists.

Theorem 2.18. *Denote by $\mathcal{R}$ the set of numbers satisfying Robin's inequality. Then the natural density of $\mathcal{R}$ is* 1*.*

Proof. We will prove that the natural density of $\mathcal{R}^c$ is 0. Fix $\epsilon > 0$. Let $E_k = \{2^k n : n \in \mathbb{N}_{\text{odd}}, n \leq b(k)\}$ and note that $\mathcal{R}^c \subseteq \bigcup_{k\geq 1} E_k$ by Theorem 2.12 and Theorem 2.13.[1] Pick M

[1] Here $\mathbb{N}_{\text{odd}} := \{1, 3, \ldots\}$.

so that $\sum_{k=M+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. For $n \in \mathbb{N}$ we calculate

$$\begin{aligned} \frac{\#\mathcal{R}^c \cap \{1, 2, \ldots, n\}}{n} &\leq \frac{\# \bigcup_{k \geq 1} E_k \cap \{1, 2, \ldots, n\}}{n} = \frac{\sum_{k \geq 1} E_k \cap \{1, 2, \ldots, n\}}{n} \\ &= \frac{\sum_{k=1}^{M} \#E_k \cap \{1, 2, \ldots, n\} + \sum_{k=M+1}^{\infty} \#E_k \cap \{1, 2, \ldots, n\}}{n}, \end{aligned} \tag{2.37}$$

where the first equality follows from the fact that the E_k's are disjoint. Noting that $\sum_{k=1}^{M} \#E_k \cap \{1, 2, \ldots, n\} < \infty$ for all $n \in \mathbb{N}$, we see that we can pick N so that $n \geq N$ implies that the RHS of (2.37) is $< \epsilon$, completing our proof. □

3. The Lagarias and Kaneko-Lagarias Inequalities

3.1. **Superabundant numbers.** Let $\Gamma(x)$ denote the gamma function. We define two functions:

$$\begin{aligned} H(x) &= \int_0^1 \frac{t^x - 1}{t - 1} \, \mathrm{d}t, \\ \psi(x) &= \frac{\Gamma'(x)}{\Gamma(x)}. \end{aligned} \tag{3.1}$$

ψ is known as the *digamma function*. One can verify that H is smooth for $x \geq 1$ and that $H(n) = H_n$ for all $n \in \mathbb{N}$. It's also easy to see that ψ, known as the digamma function, satisfies

$$H(x) = \psi(x + 1) + \gamma. \tag{3.2}$$

Lemma 3.1. *For all $x \geq 1$,*

$$H(x) < \log(x) + \gamma + \frac{1}{2x}. \tag{3.3}$$

Proof. By (2.2) from [Alz97],

$$\psi(x) < \log(x) - \frac{1}{2x} \tag{3.4}$$

for all $x \geq 1$. Then we use (3.2) and $\psi(x + 1) = \psi(x) + \frac{1}{x}$ to finish. □

Lemma 3.2. *For all $x \geq 4$,*

$$H(x) < \frac{2 \log(x)}{1 + \frac{6}{\pi^2 x}}. \tag{3.5}$$

Proof. By Lemma 3.1, it suffices to show that

$$\log(x) + \gamma + \frac{1}{2x} < \frac{2 \log(x)}{1 + \frac{6}{\pi^2 x}} \tag{3.6}$$

for $x \geq 4$. By arithmetic manipulations, (3.6) becomes

$$\frac{1}{\pi^2 x - 6} \left(\gamma \pi^2 x + \frac{\pi^2}{2} + 6\gamma + \frac{3}{x} \right) < \log(x). \tag{3.7}$$

Computation reveals that (3.7) holds for $x = 4$, and the LHS of (3.7) is decreasing while the RHS is increasing so we obtain the result. □

Lemma 3.3. *The following hold:*

(a) *For all $n > 1$, $H_{n+1} \leq \frac{n}{\log(n)}$.*
(b) *For all $x \geq 4$, $\log(H(x)) \leq \frac{x}{2 \log(x)}$.*

Proof. (a) We can manually verify the inequality for $n \leq 6$. Noting that

$$H_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} \leq 1 + \int_1^{n+1} \frac{\mathrm{d}t}{t} = 1 + \log(n+1), \tag{3.8}$$

it suffices to show that

$$\log(x)(\log(x+1)+1) \leq x. \tag{3.9}$$

Put $g(t) = e^t - t^2 - t - 1$. We see that $g(2) > 0$ and that $g'(t) = e^t - 2t - 1 > 0$ for $t \geq 2$, so $g(t) > 0$ for $t \geq 2$. For $x \geq e^2 - 1$ we have

$$0 < g(\log(x+1)) = x+1-(\log(x+1))^2 - \log(x+1) - 1 < x - \log(x)(\log(x+1)+1). \tag{3.10}$$

(b) For $x \geq 4$, note that the function mapping $x \mapsto \frac{x}{\log(x)}$ is increasing. If $n \leq x < n+1$, then

$$H_n \leq H(x) < H_{n+1} \leq \frac{n}{\log(n)} \leq \frac{x}{\log(x)}. \tag{3.11}$$

For $y > 2$ we see that $\log(y) < \frac{y}{2}$, so let $y = H(x)$ and apply (3.11) finish. □

Lemma 3.4. *For $x \geq 4$,*

$$H(x)\log(H(x)) < \frac{x^2}{x + \frac{6}{\pi^2}}. \tag{3.12}$$

Proof. Apply Lemma 3.1 and Lemma 3.3. □

Lemma 3.5. *For $x \geq 4$,*

$$H'(x) > \frac{H(x)\log(H(x))}{x^2}. \tag{3.13}$$

Proof. We will use (51) from [FD14] which states that

$$\frac{1}{\psi'(x)} \leq x + \frac{6}{\pi^2} - 1 \tag{3.14}$$

for $x \geq 1$. We calculate

$$H'(x) = \psi'(x+1) \geq \frac{1}{x + 6\pi^2} > \frac{H(x)\log(H(x))}{x^2}, \tag{3.15}$$

where the equality follows from taking the derivative of (3.2) and the second inequality follows from Lemma 3.4. □

Proposition 3.6. *The function*

$$g(x) = \frac{\exp(H(x))\log(H(x))}{x} \tag{3.16}$$

is increasing for $x \geq 4$.

Proof. We start with (3.5) from [Lag02]:

$$\begin{aligned} H_n &= \log(n) + \gamma + \int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x \\ \implies \exp(H_n) &= e^\gamma n \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x\right) \\ \implies \frac{\exp(H_n)\log(H_n)}{n} &= e^\gamma \log(H_n) \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x\right). \end{aligned} \tag{3.17}$$

Given $k \in \mathbb{N}$, put

$$g_k(x) = e^{\gamma} \log(H(x)) \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} \,\mathrm{d}t\right) \tag{3.18}$$

so that $\lim_{k\to\infty} g_k(x) = g(x)$. We compute

$$g_k'(x) = e^{\gamma} \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2} \,\mathrm{d}t\right)\left(\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x - \lfloor x \rfloor}{x^2}\right)\right), \tag{3.19}$$

so $g_k'(x) > 0$ if and only if

$$\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x - \lfloor x \rfloor}{x^2}\right) \geq \frac{H'(x)}{H(x)} - \frac{\log(H(x))}{x^2} > 0, \tag{3.20}$$

which is the content of Lemma 3.5. Thus, $g(x)$ is the limit of monotonically increasing functions and is therefore monotonically increasing. □

Corollary 3.7. *The sequence*

$$\left\{\frac{\exp(H_n)\log(H_n)}{n}\right\}_{n=1}^{\infty} \tag{3.21}$$

is monotonically increasing.

Proof. Proposition 3.6 gives the result for $n \geq 4$ and we can manually check the smaller cases. □

Definition 3.8. A number n is *superabundant* if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.

Theorem 3.9. *If there are counterexamples to the Kaneko-Lagarias inequality, the smallest such counterexample is a superabundant number.*

Proof. Suppose, for sake of contradiction, that m is the smallest counterexample to the Kaneko-Lagarias inequality and that m is not superabundant. Let n be the greatest superabundant number $< m$. We calculate,

$$\frac{\sigma(n)}{n} > \frac{\sigma(m)}{n} \geq \frac{\exp(H_m)\log(H_m)}{m} > \frac{\exp(H_n)\log(H_n)}{n}, \tag{3.22}$$

so $n < m$ violates the Kaneko-Lagarias inequality: a contradiction. □

3.2. **Connection to Robin's inequality.**

Theorem 3.10. *If Robin's inequality holds for some $n \in \mathbb{N}$, then the Kaneko-Lagarias inequality holds for n.*

Proof. We use the approximation

$$H_n \geq \log(n) + \gamma + \frac{1}{2n+1} \tag{3.23}$$

to calculate

$$\frac{\exp(H_n)\log(H_n)}{n} \geq \frac{e^{\gamma + \frac{1}{2n+1}} n \log\left(\log(n) + \gamma + \frac{1}{2n+1}\right)}{n} > e^{\gamma}\log(\log(n)), \tag{3.24}$$

which implies the result. □

Note that we obtain the same result for the Lagarias inequality.

Acknowledgments

The first author thanks Jeff Lagarias for his comments and the references he provided. The third author thanks Keith Briggs for providing his code used to compute superabundant numbers, Perry Thompson and Owen McAllister for their help implementing it in Rust, and Jean-Louis Nicolas for sharing the paper [AN22].

References

[AF09] Amir Akbary and Zachary Friggstad. "Superabundant Numbers and the Riemann Hypothesis". In: *The American Mathematical Monthly* 116.3 (2009), pp. 273–275.

[AE44] L. Alaoglu and P. Erdos. "On Highly Composite and Similar Numbers". In: *Transactions of the American Mathematical Society* 56.3 (1944), pp. 448–469.

[Alz97] Horst Alzer. "On Some Inequalities for the Gamma and Psi Functions". In: *Mathematics of Computation* 66.217 (1997), pp. 373–389.

[AN22] Christian Axler and Jean-Louis Nicolas. "Large values of $\frac{n}{\varphi(n)}$". In: *Acta Arithmetica* 209 (2022), pp. 357–383.

[BT15] Kevin Boughan and Tim Trudgian. "Robin's inequality for 11-free integers". In: *Integers* 15 (2015).

[Bri06] Keith Briggs. "Abundant Numbers and the Riemann Hypothesis". In: *Experimental Mathematics* 15.2 (2006), pp. 251–256.

[Cho+07] YoungJu Choie et al. "On Robin's criterion for the Riemann hypothesis". In: *Journal de Théorie des Nombres de Bordeaux* 19.2 (2007), pp. 357–372.

[Dus99] Pierre Dusart. "The k-th Prime is Greater than $k(\log(k)() + \log(\log(k)) - 1)$ for $k \geq 2$". In: *Mathematics of Computation* 68.225 (1999), pp. 411–415.

[FD14] M. R. Farhangdoost and M. Kargar Dolatabadi. "New Inequalities for Gamma and Digamma Functions". In: *Journal of Applied Mathematics* 2014 (2014), pp. 1–7.

[Lag02] Jeffrey C. Lagarias. "An Elementary Problem Equivalent to the Riemann Hypothesis". In: *The American Mathematical Monthly* 109.6 (2002), pp. 534–543.

[MP20] Thomas Morrill and David Platt. "Robin's inequality for 20-free integers". In: *Integers* 21 (2020).

[Rob84] Guy Robin. "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann". In: *Journal de Mathématiques Pures et Appliquées* 63 (1984), pp. 187–213.

[RS62] J. Barkley Rosser and Lowell Schoenfeld. "Approximate formulas for some functions of prime numbers". In: *Illinois Journal of Mathematics* 6.1 (1962), pp. 64–94.

[SP12] P. Sole and M. Planat. "The Robin inequality for 7-free integers". In: *Integers* 12 (2012), pp. 301–309.

[Ten95] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory.* French. Vol. 46. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995, pp. xvi+448.

I. Assani, Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, *E-mail address:* `assani@email.unc.edu`

A. Chester, University of North Carolina at Chapel Hill, *E-mail address:* `achester@unc.edu`

A. Paschal, University of North Carolina at Chapel Hill, *E-mail address:* `ampasch@unc.edu`