Exercício 2.18

Considere a matriz

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a \\ 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \end{bmatrix}.$$

Transforme a matriz $[A|\, I_4]$ numa matriz em escada de linhas e indique, se possível, a inversa de A.

Resolução:

- Colocar a matriz ampliada $[A|\, I_4]$ na forma de escada de linhas

$$[A|\,I4] \quad = \left[\begin{array}{cccc|cccc} \boxed{1} & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & a & 0 & 1 & 0 & 0 \\ 1 & 1 & a & 1 & 0 & 0 & 1 & 0 \\ 1 & a & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ L_2 - L_1 \\ L_3 - L_1 \\ L_4 - L_1 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} \boxed{1} & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{a-1} & -1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{a-1} & 0 & -1 & 0 & 1 & 0 \\ 0 & \boxed{a-1} & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \quad L_{24}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} \boxed{1} & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{a-1} & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & \boxed{a-1} & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{a-1} & -1 & 1 & 0 & 0 \end{array}\right]$$

Nota: Seja A uma matriz de ordem n e seja A^* a correspondente matriz em escada de linhas. A matriz A é invertível se número de pivôs de A^* for igual a n.

A matriz A é invertível se e só se $a \neq 1$ (número de pivôs da escada de linhas é 4).

- Seja $a \neq 1$. Calcular A^{-1}

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & a-1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & a-1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & a-1 & -1 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} L_1 - \frac{1}{a-1}L_2 \\ \frac{1}{a-1}L_2 \\ \frac{1}{a-1}L_3 \\ \frac{1}{a-1}L_4 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{a}{a-1} & 0 & 0 & -\frac{1}{a-1} \\ 0 & 1 & 0 & 0 & -\frac{1}{a-1} & 0 & 0 & \frac{1}{a-1} \\ 0 & 0 & 1 & 0 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{a-1} & \frac{1}{a-1} & 0 & 0 \end{array}\right] \; L_1 - L_3$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & \frac{a+1}{a-1} & 0 & -\frac{1}{a-1} & -\frac{1}{a-1} \\ 0 & 1 & 0 & 0 & -\frac{1}{a-1} & 0 & 0 & \frac{1}{a-1} \\ 0 & 0 & 1 & 0 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{a-1} & \frac{1}{a-1} & 0 & 0 \end{array}\right] \; L_1 - L_4$$

$$\rightarrow \left[\begin{array}{cccc|cccc} \boxed{1} & 0 & 0 & 0 & \frac{a+2}{a-1} & -\frac{1}{a-1} & -\frac{1}{a-1} & -\frac{1}{a-1} \\ 0 & \boxed{1} & 0 & 0 & -\frac{1}{a-1} & 0 & 0 & \frac{1}{a-1} \\ 0 & 0 & \boxed{1} & 0 & -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & 0 & \boxed{1} & -\frac{1}{a-1} & \frac{1}{a-1} & 0 & 0 \end{array}\right]$$

- A^{-1}

$$A^{-1} = \begin{bmatrix} \frac{a+2}{a-1} & -\frac{1}{a-1} & -\frac{1}{a-1} & -\frac{1}{a-1} \\ -\frac{1}{a-1} & 0 & 0 & \frac{1}{a-1} \\ -\frac{1}{a-1} & 0 & \frac{1}{a-1} & 0 \\ -\frac{1}{a-1} & \frac{1}{a-1} & 0 & 0 \end{bmatrix}, \; a \neq 1$$

Exercício 2.33

Resolva o seguinte sistema de equações pelo método de Gauss. Indique todas as possíveis soluções do sistema:

$$\begin{cases} x + 3y - 2z + 2s = 0 \\ 2x + 6y - 5z - 2r + 4s - 3t = -1 \\ 5z + 10r + 15t = 5 \\ 2x + 6y + 8r + 4s + 18t = 6 \end{cases}$$

Resolução:

- Sistema na forma matricial

$$\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \\ r \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}}_{B}$$

- Colocar a matriz ampliada na forma de escada

$$[A|\,B] \;=\; \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ L_2-2L_1 \\ \\ L_4-2L_1 \end{array}$$

$$\to \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ L_3-5L_2 \\ L_4-4L_2 \end{array}$$

$$\to \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 20 & 0 & 30 & 10 \\ 0 & 0 & 0 & 16 & 0 & 30 & 10 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{10}L_3 \\ \\ \end{array}$$

$$\to \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & \boxed{2} & 0 & 3 & 1 \\ 0 & 0 & 0 & 16 & 0 & 30 & 10 \end{array}\right] \begin{array}{l} \\ \\ \\ L_4-8L_3 \end{array}$$

$$\to \left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & \boxed{2} & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{2} & 2 \end{array}\right]$$

- Passar para sistema

$$\begin{cases} x+3y-2z+2s=0 \\ z-2r-3t=-1 \\ 2r+3t=1 \\ 2t=2 \end{cases} \Leftrightarrow \begin{cases} x=-3y+2z-2s=-3y-2s \\ z=2r+3t-1=0 \\ r=\frac{-3t+1}{2}=-1 \\ t=1 \end{cases}$$

O sistema é possível indeterminado e a sua solução é da forma

$$\{(-3y-2s, y, 0, -1, s, 1)\,,\ y, s \in \mathbb{R}\}$$

Exercício 2.39

Calcule a matriz inversa de

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 8 & 5 \\ 3 & 7 & 5 \end{bmatrix}.$$

Determine, com a ajuda desta inversa, uma matriz X solução de :

$$AX = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- Cálculo de $A^{-1} \rightarrow$ Matriz ampliada $[A|\, I_3]$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 3 & 8 & 5 & 0 & 1 & 0 \\ 3 & 7 & 5 & 0 & 0 & 1 \end{array}\right]$$

- Colocar a matriz ampliada $[A|\, I_3]$ na forma de escada de linhas

$$[A|\, I_3] = \left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ \boxed{3} & 8 & 5 & 0 & 1 & 0 \\ 3 & 7 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} L_1 - \frac{2}{3}L_2 \\ \\ L_3 - L_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 0 & -\frac{4}{3} & -\frac{1}{3} & 1 & -\frac{2}{3} & 0 \\ \boxed{3} & 8 & 5 & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & 0 & -1 & 1 \end{array}\right] L_1 - \frac{4}{3}L_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & \boxed{-\frac{1}{3}} & 1 & \frac{2}{3} & -\frac{4}{3} \\ \boxed{3} & 8 & 5 & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & 0 & -1 & 1 \end{array}\right] L_{13}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 0 & \boxed{-1} & 0 & 0 & -1 & 1 \\ \boxed{3} & 8 & 5 & 0 & 1 & 0 \\ 0 & 0 & \boxed{-\frac{1}{3}} & 1 & \frac{2}{3} & -\frac{4}{3} \end{array}\right] L_{21}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{3} & 8 & 5 & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & 0 & -1 & 1 \\ 0 & 0 & \boxed{-\frac{1}{3}} & 1 & \frac{2}{3} & -\frac{4}{3} \end{array}\right] \begin{array}{c} L_1 + 8L_2 \\ -L_2 \\ -3L_3 \end{array}$$

- Colocar a matriz ampliada $[A|\, I_3]$ na forma de escada de linhas reduzida

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{3} & 0 & 5 & 0 & -7 & 8 \\ 0 & \boxed{1} & 0 & 0 & 1 & -1 \\ 0 & 0 & \boxed{1} & -3 & -2 & 4 \end{array}\right] L_1 - 5L_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{3} & 0 & 0 & 15 & 3 & -12 \\ 0 & \boxed{1} & 0 & 0 & 1 & -1 \\ 0 & 0 & \boxed{1} & -3 & -2 & 4 \end{array}\right] \frac{1}{3}L_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 5 & 1 & -4 \\ 0 & \boxed{1} & 0 & 0 & 1 & -1 \\ 0 & 0 & \boxed{1} & -3 & -2 & 4 \end{array}\right]$$

- Matriz inversa de A

$$A^{-1} = \begin{bmatrix} 5 & 1 & -4 \\ 0 & 1 & -1 \\ -3 & -2 & 4 \end{bmatrix}$$

- Resolução da equação matricial

$$
\begin{aligned}
AX &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&\Leftrightarrow \underbrace{A^{-1}A}_{I_3} X = A^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&\Leftrightarrow I_3 X = \begin{bmatrix} 5 & 1 & -4 \\ 0 & 1 & -1 \\ -3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&\Leftrightarrow X = \begin{bmatrix} -4 & 1 & 5 \\ -1 & 1 & 0 \\ 4 & -2 & -3 \end{bmatrix}
\end{aligned}
$$

Exercício 2.43

1. Encontrar, pelo método de Gauss e substituição inversa, a inversa da matriz $A \in M_{3\times 3}(\mathbb{R})$

$$
A = \begin{bmatrix} 1 & 3 & -1 \\ -1 & -2 & -1 \\ -3 & -8 & 2 \end{bmatrix}.
$$

Resolução:

- Colocar em escada a matriz ampliada $[A|\, I_3]$

$$
\begin{aligned}
[A|\, I_3] &= \left[\begin{array}{ccc|ccc} \boxed{1} & 3 & -1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 1 & 0 \\ -3 & -8 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ L_2 + L_1 \\ L_3 + 3L_1 \end{array} \\
&\rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 3 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \\ L_3 - L_2 \end{array} \\
&\rightarrow \left[\begin{array}{c|c} \underbrace{\begin{array}{ccc} \boxed{1} & 3 & -1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & \boxed{1} \end{array}}_{A^*} & \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{array}}_{B^*} \end{array}\right]
\end{aligned}
$$

- Substituição inversa

Seja

$$
X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, A^* = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{ e } B^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}
$$

Resolução do sistema $A^*X = B^*$

$$\begin{bmatrix} x_{11}+3x_{21}-x_{31} & x_{12}+3x_{22}-x_{32} & x_{13}+3x_{23}-x_{33} \\ x_{21}-2x_{31} & x_{22}-2x_{32} & x_{23}-2x_{33} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\Leftrightarrow$$

$$\left\{\begin{array}{l} x_{11}+3x_{21}-x_{31}=1 \\ x_{12}+3x_{22}-x_{32}=0 \\ x_{13}+3x_{23}-x_{33}=0 \\ \\ x_{21}-2x_{31}=1 \\ x_{22}-2x_{32}=1 \\ x_{23}-2x_{33}=0 \\ \\ x_{31}=2 \\ x_{32}=-1 \\ x_{33}=1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x_{11}=1-3x_{21}+x_{31}=-12 \\ x_{12}=-3x_{22}+x_{32}=2 \\ x_{13}=-3x_{23}+x_{33}=-5 \\ \\ x_{21}=1+2x_{31}=5 \\ x_{22}=1+2x_{32}=-1 \\ x_{23}=2x_{33}=2 \\ \\ x_{31}=2 \\ x_{32}=-1 \\ x_{33}=1 \end{array}\right.$$

- A matriz inversa de A é

$$A^{-1} = \begin{bmatrix} -12 & 2 & -5 \\ 5 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$$

Exercício 2.43

2. Encontrar, pelo método de Gauss e substituição inversa, a inversa da matriz $C \in M_{3\times 3}(\mathbb{R})$

$$C = \begin{bmatrix} -1 & 2 & -1 \\ -4 & 1 & -6 \\ -6 & 1 & -9 \end{bmatrix}.$$

Resolução:

- Colocar em escada a matriz ampliada $[C|\, I_3]$

$$[C|\, I_3] = \left[\begin{array}{ccc|ccc} \boxed{-1} & 2 & -1 & 1 & 0 & 0 \\ -4 & 1 & -6 & 0 & 1 & 0 \\ -6 & 1 & -9 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ L_2 - 4L_1 \\ L_3 - 6L_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{-1} & 2 & -1 & 1 & 0 & 0 \\ 0 & \boxed{-7} & -2 & -4 & 1 & 0 \\ 0 & -11 & -3 & -6 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ L_3 - \frac{11}{7}L_2 \end{array}$$

$$\rightarrow \left[\begin{array}{c|c} \underbrace{\begin{array}{ccc} \boxed{-1} & 2 & -1 \\ 0 & \boxed{-7} & -2 \\ 0 & 0 & \boxed{\frac{1}{7}} \end{array}}_{C^*} & \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ -4 & 1 & 0 \\ \frac{2}{7} & -\frac{11}{7} & 1 \end{array}}_{B^*} \end{array}\right]$$

- Substituição inversa

Seja

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, C^* = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -7 & -2 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \text{ e } B^* = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ \frac{2}{7} & -\frac{11}{7} & 1 \end{bmatrix}$$

Cálculo de $C^*X = B^*$

$$\begin{bmatrix} 2x_{21} - x_{11} - x_{31} & 2x_{22} - x_{12} - x_{32} & 2x_{23} - x_{13} - x_{33} \\ -7x_{21} - 2x_{31} & -7x_{22} - 2x_{32} & -7x_{23} - 2x_{33} \\ \frac{1}{7}x_{31} & \frac{1}{7}x_{32} & \frac{1}{7}x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ \frac{2}{7} & -\frac{11}{7} & 1 \end{bmatrix}$$

$$\Leftrightarrow$$

$$\begin{cases} 2x_{21} - x_{11} - x_{31} = 1 \\ 2x_{22} - x_{12} - x_{32} = 0 \\ 2x_{23} - x_{13} - x_{33} = 0 \\ \\ -7x_{21} - 2x_{31} = -4 \\ -7x_{22} - 2x_{32} = 1 \\ -7x_{23} - 2x_{33} = 0 \\ \\ \frac{1}{7}x_{31} = \frac{2}{7} \\ \frac{1}{7}x_{32} = -\frac{11}{7} \\ \frac{1}{7}x_{33} = 1 \end{cases} \Leftrightarrow \begin{cases} x_{11} = -1 + 2x_{21} - x_{31} = -3 \\ x_{12} = 2x_{22} - x_{32} = 17 \\ x_{13} = 2x_{23} - x_{33} = -11 \\ \\ x_{21} = \frac{4}{7} - \frac{2}{7}x_{31} = 0 \\ x_{22} = -\frac{1}{7} - \frac{2}{7}x_{32} = 3 \\ x_{23} = -\frac{2}{7}x_{33} = -2 \\ \\ x_{31} = 2 \\ x_{32} = -11 \\ x_{33} = 7 \end{cases}$$

- A matriz inversa de C é

$$C^{-1} = \begin{bmatrix} -3 & 17 & -11 \\ 0 & 3 & -2 \\ 2 & -11 & 7 \end{bmatrix}$$

Exercício 2.43

3. Encontrar, pelo método de Gauss e substituição inversa, a inversa da matriz $D \in M_{3\times 3}(\mathbb{R})$

$$D = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 8 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

Resolução:

- Colocar em escada a matriz ampliada $[D|\, I_3]$

$$[D|\, I_3] = \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ -1 & 8 & 6 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] L_2 + L_1$$

$$\to \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & \boxed{9} & 8 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] L_3 - \frac{1}{9}L_2$$

$$\to \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & \boxed{9} & 8 & 1 & 1 & 0 \\ 0 & 0 & \boxed{\frac{1}{9}} & -\frac{1}{9} & -\frac{1}{9} & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{9}L_2 \\ 9L_3 \end{array}$$

- $[D|\, I_3]$ na forma escada de linhas reduzida

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{8}{9} & \frac{1}{9} & \frac{1}{9} & 0 \\ 0 & 0 & \boxed{1} & -1 & -1 & 9 \end{array}\right] \begin{array}{l} L_1 - L_2 \\ L_2 - \frac{8}{9}L_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & \frac{10}{9} & \frac{8}{9} & -\frac{1}{9} & 0 \\ 0 & \boxed{1} & 0 & 1 & 1 & -8 \\ 0 & 0 & \boxed{1} & -1 & -1 & 9 \end{array}\right] \; L_1 - \frac{10}{9}L_3$$

$$\rightarrow \left[\begin{array}{c|c} \underbrace{\begin{array}{ccc} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{array}}_{I_3} & \underbrace{\begin{array}{ccc} 2 & 1 & -10 \\ 1 & 1 & -8 \\ -1 & -1 & 9 \end{array}}_{D^{-1}} \end{array}\right]$$

- A matriz inverrsa de D é

$$D^{-1} = \begin{bmatrix} 2 & 1 & -10 \\ 1 & 1 & -8 \\ -1 & -1 & 9 \end{bmatrix}$$

Decomposição LU

Suponha-se que ao longo do processo de eliminação de Gauss que transformou o sistema

$$AX = B$$

no sistema equivalente

$$A^*X = B^*$$

não houve troca de linhas.

Então, a matriz A admite a factorização $A = LU$ em que L é a matriz **triangular inferior de diagonal unitária**

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & -m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & -m_{n2} & -m_{n3} & \cdots & 1 \end{bmatrix},$$

e U é a **matriz triangular superior** A^* obtida no final daquele processo de eliminação de Gauss.

Nota: Chamamos m_{ij} ao multiplicador m usado na operação elementar $L_i + mL_j$, em que $i > j$.

Exercício 2.46

1. Encontrar, se existir, uma decomposição LU da matriz A e usar o resultado obtido para encontrar a correspondente inversa.

$$A = \begin{bmatrix} 3 & 9 & -6 \\ -1 & -1 & 10 \\ -2 & -4 & 13 \end{bmatrix}$$

Resolução:

- Colocar a matriz A em escada de linhas

$$A = \begin{bmatrix} \boxed{3} & 9 & -6 \\ -1 & -1 & 10 \\ -2 & -4 & 13 \end{bmatrix} \begin{array}{ll} L_2 + \frac{1}{3}L_1 & \rightarrow m_{21} = \frac{1}{3} \\ L_3 + \frac{2}{3}L_1 & \rightarrow m_{31} = \frac{2}{3} \end{array}$$

$$\rightarrow \begin{bmatrix} \boxed{3} & 9 & -6 \\ 0 & \boxed{2} & 8 \\ 0 & 2 & \boxed{9} \end{bmatrix} \; L_3 - L_2 \quad \rightarrow m_{32} = -1$$

- A matriz A é equivalente à matriz

$$A \rightarrow \begin{bmatrix} \boxed{3} & 9 & -6 \\ 0 & \boxed{2} & 8 \\ 0 & 0 & \boxed{1} \end{bmatrix} = U$$

- A matriz L é uma matriz triangular inferior com diagonal unitária

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix}$$

- U é a matriz triangular superior obtida

$$U = \begin{bmatrix} 3 & 9 & -6 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

- Factorização LU

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 9 & -6 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

- Cálculo de A^{-1}

  Descobrir a matriz X de ordem 3×3 que verifica

$$AX = I_3 \Leftrightarrow LUX = I_3 \Leftrightarrow L\left(\underbrace{UX}_{Y}\right) = I_3 \Leftrightarrow LY = I_3.$$

  Temos de resolver o sistema

$$\begin{cases} LY = I_3 \\ UX = Y \end{cases}.$$

- Seja Y matriz de ordem 3×3. Cálculo da 1^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{11} \\ y_{21} - \frac{1}{3}y_{11} \\ y_{21} - \frac{2}{3}y_{11} + y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} y_{11} = 1 \\ y_{21} - \frac{1}{3}y_{11} = 0 \\ y_{21} - \frac{2}{3}y_{11} + y_{31} = 0 \end{cases} \Leftrightarrow \begin{cases} y_{11} = 1 \\ y_{21} = \frac{1}{3} \\ y_{31} = \frac{1}{3} \end{cases}$$

- Cálculo da 1^a coluna de $X, (UX = Y)$

$$\begin{bmatrix} 3 & 9 & -6 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3x_{11} + 9x_{21} - 6x_{31} \\ 2x_{21} + 8x_{31} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 3x_{11} + 9x_{21} - 6x_{31} = 1 \\ 2x_{21} + 8x_{31} = \frac{1}{3} \\ x_{31} = \frac{1}{3} \end{cases} \Leftrightarrow \begin{cases} x_{11} = \frac{9}{2} \\ x_{21} = -\frac{7}{6} \\ x_{31} = \frac{1}{3} \end{cases}$$

- Cálculo da 2^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{12} \\ y_{22} - \frac{1}{3}y_{12} \\ y_{22} - \frac{2}{3}y_{12} + y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} y_{12} = 0 \\ y_{22} - \frac{1}{3}y_{12} = 1 \\ y_{22} - \frac{2}{3}y_{12} + y_{32} = 0 \end{cases} \Leftrightarrow \begin{cases} y_{12} = 0 \\ y_{22} = 1 \\ y_{32} = -1 \end{cases}$$

- Cálculo da 2^a coluna de $X, (UX = Y)$

$$\begin{bmatrix} 3 & 9 & -6 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3x_{12} + 9x_{22} - 6x_{32} \\ 2x_{22} + 8x_{32} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 3x_{12} + 9x_{22} - 6x_{32} = 0 \\ 2x_{22} + 8x_{32} = 1 \\ x_{32} = -1 \end{cases} \Leftrightarrow \begin{cases} x_{12} = -\frac{31}{2} \\ x_{22} = \frac{9}{2} \\ x_{32} = -1 \end{cases}$$

- Cálculo da 3^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{13} \\ y_{23} - \frac{1}{3}y_{13} \\ y_{23} - \frac{2}{3}y_{13} + y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} y_{13} = 0 \\ y_{23} - \frac{1}{3}y_{13} = 0 \\ y_{23} - \frac{2}{3}y_{13} + y_{33} = 1 \end{cases} \Leftrightarrow \begin{cases} y_{13} = 0 \\ y_{23} = 0 \\ y_{33} = 1 \end{cases}$$

- Cálculo da 3^a coluna de $X, (UX = Y)$

$$\begin{bmatrix} 3 & 9 & -6 \\ 0 & 2 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3x_{13} + 9x_{23} - 6x_{33} \\ 2x_{23} + 8x_{33} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 3x_{13} + 9x_{23} - 6x_{33} = 0 \\ 2x_{23} + 8x_{33} = 0 \\ x_{33} = 1 \end{cases} \Leftrightarrow \begin{cases} x_{13} = 14 \\ x_{23} = -4 \\ x_{33} = 1 \end{cases}$$

- A matriz inversa de A é

$$A^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{31}{2} & 14 \\ -\frac{7}{6} & \frac{9}{2} & -4 \\ \frac{1}{3} & -1 & 1 \end{bmatrix}$$

Exercício 2.46

2. Encontrar, se existir, uma decomposição LU da matriz B e usar o resultado obtido para encontrar a correspondente inversa.

$$B = \begin{bmatrix} 8 & 24 & 16 \\ 1 & 12 & 11 \\ 4 & 13 & 19 \end{bmatrix}$$

Resolução:

- Colocar a matriz B em escada de linhas

$$\begin{aligned} B &= \begin{bmatrix} 8 & 24 & 16 \\ 1 & 12 & 11 \\ 4 & 13 & 19 \end{bmatrix} \begin{matrix} \\ L_2 - \frac{1}{8}L_1 \quad \rightarrow m_{21} = -\frac{1}{8} \\ L_3 - \frac{1}{2}L_1 \quad \rightarrow m_{31} = -\frac{1}{2} \end{matrix} \\ &\rightarrow \begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 1 & 11 \end{bmatrix} \begin{matrix} \\ \\ L_3 - \frac{1}{9}L_2 \quad \rightarrow m_{32} = -\frac{1}{9} \end{matrix} \end{aligned}$$

- A matriz B é equivalente à matriz

$$B \rightarrow \begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 0 & 10 \end{bmatrix} = U$$

- A matriz L é uma matriz triangular inferior com diagonal unitária

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{8} & 1 & 0 \\ \frac{1}{2} & \frac{1}{9} & 1 \end{bmatrix}$$

- U é a matriz triangular superior obtida

$$U = \begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

- A factorização LU da matriz B é:

$$B = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{8} & 1 & 0 \\ \frac{1}{2} & \frac{1}{9} & 1 \end{bmatrix} \begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

- Cálculo de B^{-1}

Descobrir a matriz X de ordem 3×3 que verifica

$$BX = I_3 \Leftrightarrow LUX = I_3 \Leftrightarrow L\left(\underbrace{UX}_{Y}\right) = I_3 \Leftrightarrow LY = I_3$$

- Seja Y matriz de ordem 3×3. Cálculo da 1^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{8} & 1 & 0 \\ \frac{1}{2} & \frac{1}{9} & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{11} \\ \frac{1}{8}y_{11} + y_{21} \\ \frac{1}{2}y_{11} + \frac{1}{9}y_{21} + y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} y_{11} = 1 \\ \frac{1}{8}y_{11} + y_{21} = 0 \\ \frac{1}{2}y_{11} + \frac{1}{9}y_{21} + y_{31} = 0 \end{cases} \Leftrightarrow \begin{cases} y_{11} = 1 \\ y_{21} = -\frac{1}{8} \\ y_{31} = -\frac{35}{72} \end{cases}$$

- Cálculo da 1^a coluna de $X, (UX = Y)$

$$\begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{8} \\ -\frac{35}{72} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 8x_{11} + 24x_{21} + 16x_{31} \\ 9x_{21} + 9x_{31} \\ 10x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{8} \\ -\frac{35}{72} \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 8x_{11} + 24x_{21} + 16x_{31} = 1 \\ 9x_{21} + 9x_{31} = -\frac{1}{8} \\ 10x_{31} = -\frac{35}{72} \end{cases} \Leftrightarrow \begin{cases} x_{11} = \frac{17}{144} \\ x_{21} = \frac{5}{144} \\ x_{31} = -\frac{7}{144} \end{cases}$$

- Cálculo da 2^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{8} & 1 & 0 \\ \frac{1}{2} & \frac{1}{9} & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{12} \\ \frac{1}{8}y_{12} + y_{22} \\ \frac{1}{2}y_{12} + \frac{1}{9}y_{22} + y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} y_{12} = 0 \\ \frac{1}{8}y_{12} + y_{22} = 1 \\ \frac{1}{2}y_{12} + \frac{1}{9}y_{22} + y_{32} = 0 \end{cases} \Leftrightarrow \begin{cases} y_{12} = 0 \\ y_{22} = 1 \\ y_{32} = -\frac{1}{9} \end{cases}$$

- Cálculo da 2^a coluna de $X, (UX = Y)$

$$\begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{9} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 8x_{12} + 24x_{22} + 16x_{32} \\ 9x_{22} + 9x_{32} \\ 10x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{9} \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 8x_{12} + 24x_{22} + 16x_{32} = 0 \\ 9x_{22} + 9x_{32} = 1 \\ 10x_{32} = -\frac{1}{9} \end{cases} \Leftrightarrow \begin{cases} x_{12} = -\frac{31}{90} \\ x_{22} = \frac{11}{90} \\ x_{32} = -\frac{1}{90} \end{cases}$$

- Cálculo da 3^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{8} & 1 & 0 \\ \frac{1}{2} & \frac{1}{9} & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{13} \\ \frac{1}{8}y_{13} + y_{23} \\ \frac{1}{2}y_{13} + \frac{1}{9}y_{23} + y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \Leftrightarrow \begin{cases} y_{13} = 0 \\ \frac{1}{8}y_{13} + y_{23} = 0 \\ \frac{1}{2}y_{13} + \frac{1}{9}y_{23} + y_{33} = 1 \end{cases} \Leftrightarrow \begin{cases} y_{13} = 0 \\ y_{23} = 0 \\ y_{33} = 1 \end{cases}$$

- Cálculo da 3^a coluna de $X, (UX = Y)$

$$\begin{bmatrix} 8 & 24 & 16 \\ 0 & 9 & 9 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 8x_{13} + 24x_{23} + 16x_{33} \\ 9x_{23} + 9x_{33} \\ 10x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 8x_{13} + 24x_{23} + 16x_{33} = 0 \\ 9x_{23} + 9x_{33} = 0 \\ 10x_{33} = 1 \end{cases} \Leftrightarrow \begin{cases} x_{13} = \frac{1}{10} \\ x_{23} = -\frac{1}{10} \\ x_{33} = \frac{1}{10} \end{cases}$$

- A matriz inversa de B é

$$B^{-1} = \begin{bmatrix} \frac{17}{144} & -\frac{31}{90} & \frac{1}{10} \\ \frac{5}{144} & \frac{11}{90} & -\frac{1}{10} \\ -\frac{7}{144} & -\frac{1}{90} & \frac{1}{10} \end{bmatrix}$$