ANÁLISE NUMÉRICA

Aula nº 9

Matrizes

Exercício 2.20

Verifique se a seguinte matriz é invertível, e calcule a sua inversa.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -2 & 0 & 3 \end{bmatrix}$$

Resolução:

- Transformar a matriz $[A|\, I_3]$ numa matriz em forma de escada.

$$[A|\, I_3] \;=\; \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ L_2 - 2L_1 \\ L_3 + 2L_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 4 & 1 & 2 & 0 & 1 \end{array}\right] L_{23}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & -1 & 1 & 0 & 0 \\ 0 & \boxed{4} & 1 & 2 & 0 & 1 \\ 0 & 0 & \boxed{3} & -2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \frac{1}{4}L_2 \\ \frac{1}{3}L_3 \end{array}$$

A matriz A, de ordem 3×3, na sua forma de escada tem 3 pivôs $\Rightarrow$ A é uma matriz invertível.

- Transformar a matriz $[A|\, I_3]$ numa matriz em escada reduzida

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \end{array}\right] \begin{array}{l} L_1 - 2L_2 \\ L_2 - \frac{1}{4}L_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{12} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \end{array}\right] L_1 + \frac{3}{2}L_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{array}{ccc} -1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{12} & \frac{1}{4} \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{array}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

- A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{12} & \frac{1}{4} \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

Exercício 2.23

Encontre a expressão geral da matriz inversa

$$A^{-1} = \begin{bmatrix} 1 & a & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & d & 1 \end{bmatrix}^{-1}$$

Resolução:

- Colocar a matriz a matriz $[A|\, I_4]$ na forma de escada de linhas

$$[A|\, I_4] \quad = \quad \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & d & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} L_2 - bL_1 \\ \rightarrow \end{array} \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -ab & 1 & 0 & -b & 1 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & d & 1 & 0 & 0 & 0 & 1 \end{array}\right] L_{23}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 1 & 0 \\ 0 & -ab & 1 & 0 & -b & 1 & 0 & 0 \\ 0 & 0 & d & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \rightarrow \\ L_3 + abL_2 \end{array} \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & abc & -b & 1 & ab & 0 \\ 0 & 0 & d & 1 & 0 & 0 & 0 & 1 \end{array}\right] L_4 - dL_3$$

$$\rightarrow \left[\begin{array}{cccc|cccc} \boxed{1} & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & abc & -b & 1 & ab & 0 \\ 0 & 0 & 0 & \boxed{1-abcd} & bd & -d & -abd & 1 \end{array}\right]$$

A matriz A é invertível se e só se $1 - abcd \neq 0$

- Consideremos o caso em que $1 - abcd \neq 0$.

$$\left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & abc & -b & 1 & ab & 0 \\ 0 & 0 & 0 & 1-abcd & bd & -d & -abd & 1 \end{array}\right] \begin{array}{c} L_1 - aL_2 \\ \\ \rightarrow \\ \frac{1}{1-abcd}L_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -ac & 1 & 0 & -a & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & abc & -b & 1 & ab & 0 \\ 0 & 0 & 0 & 1 & \frac{bd}{1-abcd} & -\frac{d}{1-abcd} & -\frac{abd}{1-abcd} & \frac{1}{1-abcd} \end{array}\right] \begin{array}{c} L_1 + acL_4 \\ L_2 - cL_4 \\ L_3 - abcL_4 \\ \rightarrow \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 + \frac{acbd}{1-abcd} & -\frac{acd}{1-abcd} & -a - \frac{a^2cbd}{1-abcd} & \frac{ac}{1-abcd} \\ 0 & 1 & 0 & 0 & -\frac{bcd}{1-abcd} & \frac{cd}{1-abcd} & 1 + \frac{abcd}{1-abcd} & -\frac{c}{1-abcd} \\ 0 & 0 & 1 & 0 & -b - \frac{ab^2cd}{1-abcd} & 1 + \frac{abcd}{1-abcd} & ab + \frac{a^2b^2cd}{1-abcd} & -\frac{abc}{1-abcd} \\ 0 & 0 & 0 & 1 & \frac{bd}{1-abcd} & -\frac{d}{1-abcd} & -\frac{abd}{1-abcd} & \frac{1}{1-abcd} \end{array}\right]$$

A matriz inversa de A é

$$A^{-1} = \begin{bmatrix} \frac{1}{1-abcd} & -\frac{acd}{1-abcd} & -\frac{a}{1-abcd} & \frac{ac}{1-abcd} \\ -\frac{bcd}{1-abcd} & \frac{cd}{1-abcd} & \frac{1}{1-abcd} & -\frac{c}{1-abcd} \\ -\frac{b}{1-abcd} & \frac{1}{1-abcd} & \frac{ab}{1-abcd} & -\frac{abc}{1-abcd} \\ \frac{bd}{1-abcd} & -\frac{d}{1-abcd} & -\frac{abd}{1-abcd} & \frac{1}{1-abcd} \end{bmatrix}$$

Exercício 2.26 a)

Determine a solução do sistema de equações lineares

$$\begin{cases} x + y + 2t = 0 \\ 5x + 6y + 10t = -1 \\ 3x + 2y - z + t = 0 \\ 2x - y + 5t = 3 \end{cases}$$

Resolução:

- Matrizes envolvidas no sistema

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 5 & 6 & 0 & 10 \\ 3 & 2 & -1 & 1 \\ 2 & -1 & 0 & 5 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \text{ e } B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

- Matriz ampliada

$$[A|\,B] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 5 & 6 & 0 & 10 & -1 \\ 3 & 2 & -1 & 1 & 0 \\ 2 & -1 & 0 & 5 & 3 \end{array}\right]$$

- Colocar a matriz ampliada na forma de escada

$$[A|\,B] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 5 & 6 & 0 & 10 & -1 \\ 3 & 2 & -1 & 1 & 0 \\ 2 & -1 & 0 & 5 & 3 \end{array}\right] \begin{array}{l} \\ L_2 - 5L_1 \\ L_3 - 3L_1 \\ L_4 - 2L_1 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -5 & 0 \\ 0 & -3 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \\ L_3 + L_2 \\ L_4 + 3L_2 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

- Passar para sistema

$$\begin{cases} x + y + 2t = 0 \\ y = -1 \\ -z - 5t = -1 \\ t = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y - 2t = 1 \\ y = -1 \\ z = -5t + 1 = 1 \\ t = 0 \end{cases}$$

- Sistema possível determinado. A solução do sistema é $\{(1, -1, 1, 0)\}$.

Exercício 2.46 c)

Encontrar, se existir, uma decomposição LU da matriz C e usar o resultado obtido para encontrar a correspondente inversa.

$$C = \begin{bmatrix} 3 & 6 & 0 \\ 1 & -2 & 0 \\ -4 & 5 & -8 \end{bmatrix}$$

Resolução:

- Colocar a matriz C em escada de linhas

$$C = \begin{bmatrix} \boxed{3} & 6 & 0 \\ 1 & -2 & 0 \\ -4 & 5 & -8 \end{bmatrix} \begin{array}{l} \\ L_2 - \frac{1}{3}L_1 \quad \rightarrow m_{21} = -\frac{1}{3} \\ L_3 + \frac{4}{3}L_1 \quad \rightarrow m_{31} = \frac{4}{3} \end{array} \rightarrow \begin{bmatrix} \boxed{3} & 6 & 0 \\ 0 & \boxed{-4} & 0 \\ 0 & 13 & -8 \end{bmatrix} \begin{array}{l} \\ \\ L_3 + \frac{13}{4}L_2 \quad \rightarrow m_{32} = \frac{13}{4} \end{array}$$

$$\rightarrow \begin{bmatrix} \boxed{3} & 6 & 0 \\ 0 & \boxed{-4} & 0 \\ 0 & 0 & \boxed{-8} \end{bmatrix}$$

- Matriz L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{13}{4} & 1 \end{bmatrix}$$

- Matriz U

$$U = \begin{bmatrix} 3 & 6 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

- Decomposição LU

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{13}{4} & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

- Cálculo de C^{-1}

Descobrir a matriz X de ordem 3×3 que verifica

$$CX = I_3 \Leftrightarrow LUX = I_3 \Leftrightarrow L\left(\underbrace{UX}_{Y}\right) = I_3 \Leftrightarrow LY = I_3$$

Para determinar a matriz X teremos de resolver o sistema de equações

$$\begin{cases} LY = I_3 \\ \\ UX = Y \end{cases}$$

- Cálculo da 1^a coluna de Y ($LY = I_3$)

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{13}{4} & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{11} \\ \frac{1}{3}y_{11} + y_{21} \\ y_{31} - \frac{13}{4}y_{21} - \frac{4}{3}y_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Passando a sistema

$$\begin{cases} y_{11} = 1 \\ \frac{1}{3}y_{11} + y_{21} = 0 \\ y_{31} - \frac{13}{4}y_{21} - \frac{4}{3}y_{11} = 0 \end{cases} \Leftrightarrow \begin{cases} y_{11} = 1 \\ y_{21} = -\frac{1}{3} \\ y_{31} = \frac{13}{4}y_{21} + \frac{4}{3}y_{11} = \frac{1}{4} \end{cases}$$

- Cálculo da 1^a coluna da matrix X $(UX = Y)$

$$\begin{bmatrix} 3 & 6 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3x_{11} + 6x_{21} \\ -4x_{21} \\ -8x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

Passando a sistema

$$\begin{cases} 3x_{11} + 6x_{21} = 1 \\ -4x_{21} = -\frac{1}{3} \\ -8x_{31} = \frac{1}{4} \end{cases} \Leftrightarrow \begin{cases} x_{11} = \frac{1}{6} \\ x_{21} = \frac{1}{12} \\ x_{31} = -\frac{1}{32} \end{cases}$$

- Cálculo da 2^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{13}{4} & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{12} \\ \frac{1}{3}y_{12} + y_{22} \\ y_{32} - \frac{13}{4}y_{22} - \frac{4}{3}y_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Passando a sistema

$$\begin{cases} y_{12} = 0 \\ \frac{1}{3}y_{12} + y_{22} = 1 \\ y_{32} - \frac{13}{4}y_{22} - \frac{4}{3}y_{12} = 0 \end{cases} \Leftrightarrow \begin{cases} y_{12} = 0 \\ y_{22} = 1 \\ y_{32} = \frac{13}{4}y_{22} + \frac{4}{3}y_{12} = \frac{13}{4} \end{cases}$$

- Cálculo da 2^a coluna da matrix X $(UX = Y)$

$$\begin{bmatrix} 3 & 6 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{13}{4} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3x_{12} + 6x_{22} \\ -4x_{22} \\ -8x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{13}{4} \end{bmatrix}$$

Passando a sistema

$$\begin{cases} 3x_{12} + 6x_{22} = 0 \\ -4x_{22} = 1 \\ -8x_{32} = \frac{13}{4} \end{cases} \Leftrightarrow \begin{cases} x_{12} = \frac{1}{2} \\ x_{22} = -\frac{1}{4} \\ x_{32} = -\frac{13}{32} \end{cases}$$

- Cálculo da 3^a coluna de Y $(LY = I_3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{4}{3} & -\frac{13}{4} & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_{13} \\ \frac{1}{3}y_{13} + y_{23} \\ y_{33} - \frac{13}{4}y_{23} - \frac{4}{3}y_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Passando a sistema

$$\begin{cases} y_{13} = 0 \\ \frac{1}{3}y_{13} + y_{23} = 0 \\ y_{33} - \frac{13}{4}y_{23} - \frac{4}{3}y_{13} = 1 \end{cases} \Leftrightarrow \begin{cases} y_{13} = 0 \\ y_{23} = 0 \\ y_{33} = 1 \end{cases}$$

- Cálculo da 3^a coluna da matrix X $(UX = Y)$

$$\begin{bmatrix} 3 & 6 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3x_{13} + 6x_{23} \\ -4x_{23} \\ -8x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Passando a sistema

$$\begin{cases} 3x_{13} + 6x_{23} = 0 \\ -4x_{23} = 0 \\ -8x_{33} = 1 \end{cases} \Leftrightarrow \begin{cases} x_{13} = 0 \\ x_{23} = 0 \\ x_{33} = -\frac{1}{8} \end{cases}$$

- C^{-1}

$$C^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{12} & -\frac{1}{4} & 0 \\ -\frac{1}{32} & -\frac{13}{32} & -\frac{1}{8} \end{bmatrix}$$

Exercício 2.50 b)

Encontre uma decomposição de Cholesky para a matriz dos coeficientes do sistema de equações lineares seguinte, e use a decomposição para resolver o sistema

$$\begin{cases} 2x_1 - x_2 - x_3 = -4 \\ -x_1 + 2x_2 + x_3 = 7 \\ -x_1 + x_2 + 6x_3 = -1 \end{cases}$$

Resolução:

- As matrizes envolvidas no sistema são

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 6 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ e } B = \begin{bmatrix} -4 \\ 7 \\ -1 \end{bmatrix}$$

- A é simétrica

$$A^T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 6 \end{bmatrix} = A$$

- A é definida positiva

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 6 \end{bmatrix} \begin{matrix} \rightarrow \\ L_2 + \frac{1}{2}L_1 \\ L_3 + \frac{1}{2}L_1 \end{matrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{11}{2} \end{bmatrix} \begin{matrix} \\ \\ L_3 - \frac{1}{3}L_2 \end{matrix} \rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{16}{3} \end{bmatrix} = A^*$$

A é simétrica e todos os elementos da diagonal principal de A^* são positivos $\Rightarrow$ A é definida positiva.

- Determinar a matriz L de ordem 3×3 triangular inferior

$$A = LL^T = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Leftrightarrow$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 6 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Determinar as entradas da matriz L

Nota: vamos exigir que todas as entradas l_{ii} sejam positivas.

$$\begin{cases} l_{11}^2 = 2 \\ l_{11}l_{21} = -1 \\ l_{11}l_{31} = -1 \\ l_{21}^2 + l_{22}^2 = 2 \\ l_{21}l_{31} + l_{22}l_{32} = 1 \\ l_{31}^2 + l_{32}^2 + l_{33}^2 = 6 \end{cases} \Rightarrow \begin{cases} l_{11} = \sqrt{2} \\ l_{21} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ l_{31} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ l_{22}^2 = 2 - l_{21}^2 = 2 - \frac{1}{2} = \frac{3}{2} \\ \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} l_{32} = 1 \\ l_{33}^2 = 6 - l_{31}^2 - l_{32}^2 \end{cases} \Leftrightarrow \begin{cases} l_{11} = \sqrt{2} \\ l_{21} = -\frac{\sqrt{2}}{2} \\ l_{31} = -\frac{\sqrt{2}}{2} \\ l_{22}^2 = \frac{3}{2} \\ l_{32} = \frac{\sqrt{6}}{6} \\ l_{33}^2 = 6 - l_{31}^2 - l_{32}^2 \end{cases} \Rightarrow \begin{cases} l_{11} = \sqrt{2} \\ l_{21} = -\frac{\sqrt{2}}{2} \\ l_{31} = -\frac{\sqrt{2}}{2} \\ l_{22} = \frac{\sqrt{6}}{2} \\ l_{32} = \frac{\sqrt{6}}{6} \\ l_{33} = \frac{4\sqrt{3}}{3} \end{cases}$$

- A matriz L é

$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{4\sqrt{3}}{3} \end{bmatrix}$$

- Factorização de Cholesky

$$A = \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{4\sqrt{3}}{3} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \sqrt{2} & -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 0 & 0 & \frac{4}{3}\sqrt{3} \end{bmatrix}}_{L^T}$$

- Solução do sistema $AX = B$

Descobrir a matriz X de ordem 3×1 que verifica

$$AX = B \Leftrightarrow LL^TX = B \Leftrightarrow L\left(\underbrace{L^TX}_{Y}\right) = B \Leftrightarrow LY = B$$

Para determinar a matriz X teremos de resolver o sistema

$$\begin{cases} LY = B \\ L^TX = Y \end{cases}$$

- Descobrir a matriz Y de ordem 3×1 que verifica $LY = B$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{4\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \sqrt{2}y_1 \\ \frac{1}{2}\sqrt{6}y_2 - \frac{1}{2}\sqrt{2}y_1 \\ \frac{4}{3}\sqrt{3}y_3 - \frac{1}{2}\sqrt{2}y_1 + \frac{1}{6}\sqrt{6}y_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -1 \end{bmatrix}$$

Resolver o sistema

$$\begin{cases} \sqrt{2}y_1 = -4 \\ \frac{1}{2}\sqrt{6}y_2 - \frac{1}{2}\sqrt{2}y_1 = 7 \\ \frac{4}{3}\sqrt{3}y_3 - \frac{1}{2}\sqrt{2}y_1 + \frac{1}{6}\sqrt{6}y_2 = -1 \end{cases} \Leftrightarrow \begin{cases} y_1 = -2\sqrt{2} \\ y_2 = \frac{5\sqrt{6}}{3} \\ y_3 = -\frac{7\sqrt{3}}{6} \end{cases}$$

- Descobrir a matriz X de ordem 3×1 que verifica $L^T X = Y$

$$\begin{bmatrix} \sqrt{2} & -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 0 & 0 & \frac{4}{3}\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ \frac{5\sqrt{6}}{3} \\ -\frac{7\sqrt{3}}{6} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \sqrt{2}x_1 - \frac{1}{2}\sqrt{2}x_2 - \frac{1}{2}\sqrt{2}x_3 \\ \frac{1}{2}\sqrt{6}x_2 + \frac{1}{6}\sqrt{6}x_3 \\ \frac{4}{3}\sqrt{3}x_3 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ \frac{5\sqrt{6}}{3} \\ -\frac{7\sqrt{3}}{6} \end{bmatrix}$$

Resolver o sistema

$$\begin{cases} \sqrt{2}x_1 - \frac{1}{2}\sqrt{2}x_2 - \frac{1}{2}\sqrt{2}x_3 = -2\sqrt{2} \\ \\ \frac{1}{2}\sqrt{6}x_2 + \frac{1}{6}\sqrt{6}x_3 = \frac{5\sqrt{6}}{3} \\ \\ \frac{4}{3}\sqrt{3}x_3 = -\frac{7\sqrt{3}}{6} \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{5}{8} \\ \\ x_2 = \frac{29}{8} \\ \\ x_3 = -\frac{7}{8} \end{cases}$$

- A solução do sistema é $\left\{\left(-\frac{5}{8}, \frac{29}{8}, -\frac{7}{8}\right)\right\}$.