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Complementary relationships between entanglement and measurement

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Abstract

Complementary relationships exist among interference properties of particles such as pattern visibility, predictability, and distinguishability. Additionally relationships between average information gain $\bar{G}$ and measurement disturbance F for entangled spin pairs are well established. This article examines whether a similar complementary relationship exists between entanglement and measurement. For qubit systems, both measurements on a single system and measurements on a bipartite system are considered in regard to entanglement. It is proven that $\bar{E} + D \leq 1$ holds, where $\bar{E}$ is the average entanglement after a measurement is made and D is a measure of the measurement disturbance of a single measurement. Assuming measurements on a bipartite system shared by Alice and Bob, it is shown that $\bar{E} + \bar{G} \leq 1$, where $\bar{G}$ is the maximum average information gain that Bob can obtain regarding Alice's result. These results are generalized to arbitrary initial mixed states and non-Hermitian operators. In the case of maximally entangled initial states, it is found that $D \leq E_L$ and $\bar{G} \leq E_L$, where E_L is the loss of entanglement due to measurement by Alice. We conclude that the amount of disturbance and average information gain one can achieve is strictly limited by entanglement.

Keywords: *quantum information, quantum measurement, entanglement, quantum communications, quantum foundations, foundations of physics, quantum cryptography*

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1. Introduction

Combining disparate information from separate measurements allows a deterministic causal description of a system to be possible in classical physics. However, in quantum physics, the results from different types of measurements cannot always be combined, and a quantum physical phenomenon found by observing the same system with different experimental arrangements can be mutually exclusive. The separation between the observer and the observed system in realizing measurements of events may be arranged in many ways, corresponding to different conditions of observations and the type of apparatus that determine the particular aspect of the phenomenon we wish to observe. At the quantum level, the deterministic chain of events in classical physics instead becomes lines of similar possibilities, each weighted by an amplitude for probability of occurrence and concluded by the irreversible click of a detector. In his 1927 lecture at the Volta Congress in Como, Italy [1], Bohr called this logical exclusion of phenomena from different experimental arrangements "complementarity". Bohr's initial attempts to justify the complementarity picture used the Heisenberg uncertainty relations and arguments in terms of disturbances to the system that occur during measurement. However, depending on the particulars of the experiment, complementarity has more generally appeared to be enforced by a variety of other signatures for quantum behavior that have since been identified within quantum mechanics, including entanglement, uncertainty, measurement disturbance, whichway information, visibility, and distinguishability, among others. The roles of these various signatures in quantum interference experiments have turned out to be neither completely logically independent nor purely logical consequences of one another. Recently it was shown that there are numerous ways to dissect complementarity [2, 3]. Bohr's concept of complementarity continues to play a role today in studies of interference involving experimental measurement techniques [4].

A variety of quantitative expressions of wave–particle complementarity relationships have been previously derived, which weigh whichway information against fringe visibility and other quantifiers in interferometric settings. For example, Greenberger et al. [5] derived complementary single-particle duality relationships $V^2 + P^2 \leq 1$ between predictability $P = |\rho_{11} - \rho_{22}|$ and visibility of the interference pattern $V = 2|\rho_{12}|$ for a particle within an interferometer. Englert found that including detectors in the interferometer paths leads to a duality between path distinguishability and visibility $D^2 + V^2 \leq 1$ [6]. It has also been shown that in order to obtain information about a single-qubit state, disturbance of the qubit is necessary. Busch [7] showed that there is a limitation on quantum measurement such that information cannot be obtained without disturbance. Similarly, inequalities have been found between measurement sharpness and disturbance [8], and information gain versus state disturbance was reported by Banaszek [9]. Experimental confirmation of various complementarity relations has been achieved for single quantum

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objects of increasingly larger sizes, approaching the mesoscopic and macroscopic levels [10–15].

Although these results show that various quantities are complementary for a single qubit, the issue of the potential complementary relationships between the entanglement of two qubits and measurement has not been reported. To examine these potential complementary relationships, consider a measurement performed by Alice on one of two qubits, A, which is initially entangled with a second qubit, B, which is held by Bob. Steiner and Rendell [16] showed simulation results indicating that the entanglement of two parties is reduced when one of the particles is subject to a measurement using a particular measurement device model proposed by Gurvitz [17]. This paper theoretically demonstrates that such a loss of entanglement due to measurement is a fundamental property across a general class of positive operator-valued measurements (POVMs) and is not limited to any particular measurement device model.

A property of an entanglement measure or monotone is that the average entanglement that remains after a measurement on a subsystem is made is less than or equal to the initial entanglement [18]. Typically, the amount of average entanglement that remains after the measurement will depend on the disturbance D caused by Alice's measurement, which is closely related to the strength of her measurement. Full loss of average entanglement can occur in the case of a strong measurement with significant disturbance, while no loss of entanglement occurs in the limit of weak measurement with no substantial disturbance. We first examine the degree to which the average entanglement $\bar{E}$ remaining after Alice's measurement is related to the measurement disturbance D. For a general class of POVMs that Alice can apply, it is proven that $\bar{E} + D \leq 1$ between average entanglement $\bar{E}$ and measurement disturbance D. Hence, to the extent that Alice's measurement is strong or has a large disturbance, the entanglement that remains must be sufficiently small and vice versa.

In the prior analysis, Alice's measurement was made on a single qubit that was initially entangled with an ancilla. Additionally, it is desirable to understand the effect that Alice's measurement has on the amount of information that can be gained regarding Alice's outcome from a second measurement on the ancilla B by a second experimenter, Bob. It has been shown previously that by varying the strength of a measurement, complementary tradeoffs between information gain G and measurement disturbance F for entangled spin pairs have been found, where $F^2 + G^2 \leq 1$ [9, 19–22]. For bipartite systems, this is found to extend to a triality relating interference properties and entanglement, $P^2 + V^2 + C^2 \leq 1$, where C is the concurrence measure of entanglement [23]. However, it is desirable to understand the extent to which the remaining average entanglement $\bar{E}$ limits information gain. One might expect that the existence of average entanglement $\bar{E}$ should limit the amount of information that Bob can obtain regarding Alice's measurement result. It is indeed the case; it is proven for arbitrary initial states that the average entanglement $\bar{E}$ and average information gain $\bar{G}$ are complementary in the sense that $\bar{E} + \bar{G} \leq 1$.

In the case of maximally entangled initial states, direct relationships are found such that $D \leq \bar{E}_L$ and $\bar{G} \leq \bar{E}_L$, where $\bar{E}_L \equiv -\Delta\bar{E}$ is the entanglement loss between the initial entanglement of unity and the entanglement after Alice's measurement. Although useful as a bound for maximally entangled states with no initial classical correlation, it is discussed in Section 2.1 why this result does not generalize to arbitrary initial states.

The article is organized as follows: in Section 2.1, further background of the complementary aspects of entanglement and measurement is provided, including why average entanglement should limit disturbance and information gain. This section discusses the form of complementary relationships between entanglement and measurement. The remainder of Section 2 defines the measurement operations used throughout the article, as well as the entanglement measure E, measurement disturbance D, and information gain G. Results are presented in Section 3 assuming two entangled particles. In Section 3.1, Alice makes a measurement on her qubit which has an associated measurement disturbance D. The effect of entanglement is examined relative to the measurement disturbance. In Section 3.2, two-particle measurements by both Alice and Bob are considered, and a tradeoff between entanglement and average information gain is presented. Results are extended to include all mixed initial states in Section 2, Supplementary materials and to non-Hermitian operations in Section 3, Supplementary materials. Conclusions are presented in Section 4.

2. Background

2.1. Complementary aspects of entanglement and measurement

Schrödinger [24] had noted as early as 1935 that entanglement can result in the interaction of two systems such that "after a time of mutual influence the systems separate again, then they can no longer be described the same way as before, viz., by endowing each of them with a representative state of its own". He argued that this is the principal characteristic trait of quantum theory, stating, "I would not call that one but the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought". It was further found by Schrödinger that a particle cannot be said to be in any particular pure state but must be described by a density matrix [25, 26]. The fact that an overall maximally entangled system cannot be described by a product state of individual subsystems and that there is an observable violation for such systems was later formalized by Bell's inequality and the subsequent experimental verification of entanglement through the violation of Bell's inequality.

To further illustrate the rationale for situations whereby entanglement and measurement can be considered complementary, consider the scenario described in Section 1, where Alice and Bob have Qubits A and B, respectively, which are initially in a maximally entangled state. Each qubit is in a maximally mixed state and lacks a well-defined pure state, as first noted by Schrödinger [25]. In the case that Alice's measurement is very weak, her qubit would be expected to remain, on average, nearly fully entangled and cannot be in a well-defined pure state that correlates with Alice's result. Therefore, one would expect that a second measurement by Bob on Qubit B would not provide any significant information about Alice's measurement due to the persisting entanglement between Qubits A and B. On the other hand, suppose that Alice's measurement is strong and completely destroys the entanglement that initially existed and then after Alice's measurement, both Qubits A and B will be projected into well-defined, correlated pure states. This allows Bob to obtain information regarding Alice's result by measuring Qubit B. In such a case, the amount of average entanglement remaining after Alice's measurement would be expected to limit the amount of information that Bob can obtain regarding Alice's outcome. This is because, to the extent that average entanglement remains after Alice's measurement, this entanglement reflects how much the subsystems remain in a mixed state. Such a mixed state cannot be correlated with Alice's measurement results and thus, strictly limits Bob's ability to obtain information about Alice's outcome. Therefore, a complementary relationship $\bar{E}+\bar{G} \leq 1$ exists between the average entanglement $\bar{E}$ and average information gain $\bar{G}$.

In the relationship above, the average information gain $\bar{G}$ that Bob can obtain regarding Alice's outcome is found to be complementary to the average entanglement $\bar{E}$ after Alice's measurement. One might consider the possibility of deriving complementary relationships between the information that Bob can gain about Alice's measurement and the initial entanglement E of Qubits A and B prior to any measurement by Alice and Bob. However, it does not appear that the initial entanglement E by itself (without further modification) is generally complementary to the information gain. Consider a maximally entangled initial state for which the initial entanglement is $E = 1$. In this case, a measurement conducted by both Alice and Bob in the same basis, where Alice obtains 0 (or 1), will result in Bob obtaining the same result as Alice's, 0 (or 1), resulting in $G = 1$ information gain regarding Alice's outcome. Hence, for such an initial state, we have $E = 1, G = 1$. Now, consider an initial mixed joint Bob–Alice density matrix ρ_{BA} with qubit state $\rho_{BA} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$. Such an initial density matrix has classical correlation in the following sense. A measurement conducted by both Alice and Bob in the computational basis, where Alice obtains 0 (or 1), will be followed by Bob obtaining the same result, 0 (or 1). For such an initial state, the information gain G that Bob can gain regarding Alice's outcome is $G = 1$, and in this case, we have $E = 0, G = 1$. Hence, depending on the initial state, the information gain remains the same, i.e., $G = 1$, yet the entanglement is seen to vary from its potential minimum of $E = 0$ to its potential maximum of $E = 1$. This counterexample of $\rho_{BA} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$ illustrates that the initial entanglement and information gain are not complementary in general.

One can obtain an additional inequality as a lemma to the relationship $\bar{E} + \bar{G} \leq 1$ in the case when one restricts the initial state to a maximally entangled initial state. Starting with $\bar{E} + \bar{G} \leq 1$, this can be rewritten as $\bar{E} + \bar{G} \leq E_i$, where E_i is the initial entanglement of Qubits A and B. Consider that after Alice's measurement, the change in entanglement from the initial to the final state is given by $\Delta\bar{E} = \bar{E} - E_i$. Upon defining the entanglement loss $\bar{E}_L \equiv -\Delta\bar{E}$ between the initial entanglement and final average entanglement and the entanglement after Alice's measurement, we have $\bar{G} \leq \bar{E}_L$. Hence, a direct relationship can be found $\bar{G} \leq \bar{E}_L$ (and by a similar argument $D \leq \bar{E}_L$), where the average information gain must be less than the loss of entanglement. That is, in order to gain information when starting with a maximally entangled state, there must be some corresponding loss of entanglement. Now, from the discussion above regarding the counterexample of initial classical states, such as $\rho_{BA} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$, one can see why such a relationship makes sense. By restricting the initial state to a maximally entangled state, the initial state is completely pure, and such a quantum state appears to be devoid of its mixed classical counterpart, which also conveys some classical information to Bob regarding Alice's outcome. Hence, it is found that the entanglement loss is indeed directly complementary to both measurement disturbance and average information gain for initial maximally entangled states, which are widely utilized in both theory and experiment in quantum information.

2.2. Entanglement quantification

Consider a two-qubit system in an arbitrary pure state given by

$$|\psi_0\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle. \tag{1}$$

This can be rewritten via a Schmidt decomposition as

$$|\psi_0\rangle = \sqrt{\alpha_0}|u_1\rangle \otimes |v_1\rangle + \sqrt{1-\alpha_0}|u_2\rangle \otimes |v_2\rangle, \tag{2}$$

where $|u_i\rangle \in \mathcal{H}_1$ and $|v_i\rangle \in \mathcal{H}_2$, $i = 1, 2$, and $\mathcal{H}_1, \mathcal{H}_2$ are the Hilbert spaces of Qubits 1 and 2, respectively, and we can assume without loss of generality that $\alpha_0 \leq 0.5$. Additionally, the $|u_i\rangle$ are chosen orthogonally in $\mathcal{H}_1$ and similarly for the $|v_i\rangle$ in $\mathcal{H}_2$. Note that $|\psi_0\rangle$ is of the form of the two-qubit ancilla state Eq. (2) when $|u_1\rangle = |0\rangle$, $|v_1\rangle = |0\rangle, |u_2\rangle = |1\rangle, |v_2\rangle = |1\rangle$.

In order to quantify the entanglement throughout the paper, it is known that all entanglement measures of pure states are equivalent in the sense that a one-to-one relationship can be found between any two entanglement measures [27]. The well-known entropy of entanglement, which we denote as E_H, is found as the von Neumann entropy of either reduced density matrix of a bipartite system. The von Neumann entropy is a function of the two eigenvalues of either reduced density matrix $\{\lambda, 1-\lambda\}$, given in bits by

$$E_H(\lambda) = -\lambda \log_2 \lambda - (1-\lambda)\log_2(1-\lambda). \tag{3}$$

Another measure that is equivalent (in the sense given by Donald et al. [27]) to the von Neumann entropy is twice the minimum eigenvalue of the reduced density matrix, which will be denoted as $E(\lambda_{min})$. This is also equivalent to the "geometric measure of entanglement" [28]. Since $\lambda_{\min} + \lambda_{\max} = 1$, if $\lambda \leq 1/2$, then

$$E(\lambda) = 2\lambda, \tag{4}$$

which we have chosen for our entanglement measure, and we will demonstrate that a direct relationship with measurement will be established using this measure. For a state given by the Schmidt decomposition of Eq. (2), one also finds a direct relationship with the smallest Schmidt coefficient α_0, where $E(\lambda) = 2\alpha_0$.

2.3. Measurement operations

Let us consider a protocol in which Bob and Alice each have a qubit in the initial pure state $|\psi_0\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, represented by the Schmidt decomposition of Eq. (2), where $\alpha_0 \leq 0.5$, and $|u_i\rangle$ are chosen orthogonally in $\mathcal{H}_1$ and similarly the $|v_i\rangle$ in $\mathcal{H}_2$. We define projection onto a pure state as a strong measurement (a sharp measurement was defined by Busch [7] as a projection-valued measurement which can include higher-rank projections). Alice makes a measurement with two possible outcomes $\{0, 1\}$ on the qubit defined on $\mathcal{H}_2$. A two-outcome POVM that Alice can apply is found as

$$\begin{aligned} M_0^{(A)} &\equiv \frac{1}{\sqrt{1+2\Lambda(1+\Lambda)}} \begin{bmatrix} \Lambda + \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \Lambda + \sin^2\theta \end{bmatrix}, \\ M_1^{(A)} &\equiv \frac{1}{\sqrt{1+2\Lambda(1+\Lambda)}} \begin{bmatrix} \Lambda + \sin^2\theta & -e^{i\phi}\sin\theta\cos\theta \\ -e^{-i\phi}\sin\theta\cos\theta & \Lambda + \cos^2\theta \end{bmatrix}, \end{aligned} \tag{5}$$

where $0 \leq \phi \leq 2\pi$, $0 \leq \Lambda$, and $\theta, \phi, \Lambda \in R$, where R denotes the real numbers and $M_0^{(A)\dagger} M_0^{(A)} + M_1^{(A)\dagger} M_1^{(A)} = I$. Note that any set of matrices $M_i^{(A)} \in S^2$, $i = 0, 1$, where S^2 denotes the set of 2×2 positive semi-definite matrices that also satisfy $M_0^{(A)\dagger} M_0^{(A)} + M_1^{(A)\dagger} M_1^{(A)} = I$, is described by the class defined by Eq. (5). We restrict $\Lambda + \cos^2\theta \geq \Lambda + \sin^2\theta$ so that $M_0^{(A)}$ projects with a higher probability into state $|0\rangle$ than $|1\rangle$ and vice versa for $M_1^{(A)}$. This is achieved when $0 \leq \theta \leq \frac{\pi}{4}$. When outcome $i \in \{0, 1\}$ occurs corresponding to $M_i^{(A)}$, the partial density matrices of Alice and Bob are denoted as $\rho_{A,i}$ and $\rho_{B,i}$, respectively. Here, $p_{A,i}$ is the probability of Alice obtaining the result $M_i^{(A)}$, $i = 0, 1$, $p_{A,i} = Tr\left(M_i^{(A)\dagger} M_i^{(A)} |\psi_0\rangle\langle\psi_0|\right)$.

2.4. Measurement disturbance and information gain

The effects of measurement will be quantified using several quantities, assuming an initially entangled state shared by Alice and Bob [19–22]. In the case that Alice makes a single measurement, the disturbance D will be used to quantify the effect of the measurement, as discussed below. In cases where both Alice and Bob make measurements on their respective qubits of an entangled state, a bipartite measure of information gain G will be defined utilizing related work in [19–22].

Quantification of measurement disturbance

In the case of two entangled particles shared by Alice and Bob, when measurement is made by Alice on her qubit, the effect of the measurement will be quantified using the measurement disturbance or quality factor F as utilized in [19–22]. In order to examine the tradeoff between entanglement and measurement, we want to determine the extent to which the initial entanglement is affected by the measurement. For our purposes, we define the measurement disturbance as $D = 1 - F$, which was utilized by Zhu et al. [19] and is shown in Section 4, Supplementary material as $D = \frac{1}{1+2\Lambda(1+\Lambda)}$. Note that the disturbance is neither a function of the degree of superposition of the qubit nor the parameters θ and ϕ in Eq. (5), but rather it only enters through the strength or weakness of the measurement, which is a function of Λ.

Information gain in two-particle measurements

A second quantity that was utilized in quantifying the effect of measurement in [19–22] is the information gain. This will be presented and extended for use in two-particle measurements, which will be applied in Section 3.2. For single-qubit weak measurement in both [19, 21], the information gain is defined by

$$G = 1 - |\langle 1|\ \psi_{P,\uparrow}\rangle|^2 - |\langle 0|\ \psi_{P,\downarrow}\rangle|^2 . \tag{6}$$

As noted in [19, 21], in addition to showing that G represents the precision of the measurement, Eq. (6) represents a probability of error. For simplicity, Hu et al. [21] made an assumption that there is symmetric ambiguity, meaning that $|\langle 1|\ \psi_{P,\uparrow}\rangle|^2 = |\langle 0|\ \psi_{P,\downarrow}\rangle|^2$. Note that the probability of error, whereby a spin $|\uparrow\rangle$ ($|\downarrow\rangle$) is measured by a result $|1\rangle$ ($|0\rangle$), is given by

$$P_e = |\langle 1|\ \psi_{P,\uparrow}\rangle|^2 \beta + |\langle 0|\ \psi_{P,\downarrow}\rangle|^2 (1 - \beta) , \tag{7}$$

where β is the probability of spin $|\uparrow\rangle$ occurring from Eq. 4.1, Supplementary material. In the case that $|\langle 1|\psi_{P,\uparrow}\rangle|^2 = |\langle 0|\psi_{P,\downarrow}\rangle|^2 \equiv \alpha$, this simplifies to $P_e = \alpha$ or $G = 1 - 2\alpha$ or $G = 1 - 2P_e$. Note that since $0 \leq P_e \leq 1/2$ for optimal measurements, one can see that $0 \leq G \leq 1$. Let us define the following extension of the gain for single-particle measurement when $|\langle 1|\psi_{P,\uparrow}\rangle|^2 \neq |\langle 0|\psi_{P,\downarrow}\rangle|^2$ as $G = 1 - 2P_e$ or

$$G = 1 - 2\,|\langle 1|\ \psi_{P,\uparrow}\rangle|^2 \beta - 2\,|\langle 0|\ \psi_{P,\downarrow}\rangle|^2 (1 - \beta) , \tag{8}$$

which reduces to Eq. (6) when $|\langle 1|\ \psi_{P,\uparrow}\rangle|^2 = |\langle 0|\ \psi_{P,\downarrow}\rangle|^2$. This definition can be further generalized as follows: suppose Alice and Bob share an arbitrary initial bipartite two-qubit state, and Alice first makes a measurement in the basis $\left\{M_0^{(A)}, M_1^{(A)}\right\}$. Bob's task is to attempt to determine, as accurately as possible, in the sense of minimizing the probability of error, what Alice's measurement result was. Bob is generally assumed to know the operators $\left\{M_0^{(A)}, M_1^{(A)}\right\}$ but not Alice's specific result. In this case, we will utilize the measure of information gain defined as $G = 1 - 2P_e$. Now, this measure of gain can be applied directly to the previously specified measurement operations. Let us define $G_i \equiv 1 - 2P_{e|A=i}$, where $P_{e|A=i}$ is the probability of error given that Alice obtains result $i \in \{0, 1\}$, and let us define $\pi_{A,i}$ as the probability of Alice obtaining result $i \in \{0, 1\}$. The average information gain that Bob obtains about Alice's qubit is then found as $\bar{G} = \pi_{A,0}G_0 + \pi_{A,1}G_1$.

3. Results

We will now consider that both Alice and Bob share an entangled state. In the first case discussed in Section 3.1, the effect of the entanglement when Alice makes a measurement with some disturbance D defined in Section 2.4 will be considered. In Section 3.2, both Alice and Bob will make measurements with the goal of maximizing the information gain, as defined in Section 2.4. In the case of two-qubit measurement, without loss of generality, the initial state can be taken to be of the form $\sqrt{\alpha}|0\rangle \otimes |0\rangle + \sqrt{1-\alpha}|1\rangle \otimes |1\rangle$, as shown in Section 1, Supplementary material. Alice applies the measurement consisting of the operators $M_i^{(A)}$ of Eq. (5), where the entanglement will be reduced from its original value of $2\alpha_i$ to one of two possible final values, depending on the outcome of Alice's measurement $\{0, 1\}$. The average entanglement that is obtained is given by

$$\bar{E} = \pi_{A,0}E(\rho_{B,0}) + \pi_{A,1}E(\rho_{B,1}) , \tag{9}$$

where $\rho_{B,i}$ is the partial density matrix of Bob when outcome i occurs (note that since $E(\rho_{B,i}) = E(\rho_{A,i})$ for bipartite systems, one can equivalently use Alice's partial density matrix to compute the entanglement). In the case of one-particle measurement, the measurement refers only to Alice's measurement, and the goal is to determine the tradeoff between the measurement disturbance D and the average entanglement $\bar{E}$.

In the case of two-particle measurement, Alice similarly applies the measurement consisting of the operators $M_i^{(A)}$ of Eq. (5), for which the average entanglement is obtained in Eq. (9). At this point, Bob makes a measurement, attempting to maximize his information gain regarding Alice's result. That is, he desires to make a measurement that will be maximally correlated with Alice's result so that his probability of error is minimized. Bob is assumed to know Alice's measurement basis. In order to optimize the probability of error, Bob must measure in a basis that is optimal in terms of minimizing the probability of error. A solution to Bob's basis problem is given by Bergou [29]. Bergou's construction is utilized to optimize the information gain for two-particle measurement.

Note that Alice and Bob can apply local unitary operations to their respective qubits without changing the entanglement.

3.1. Two entangled particles: single measurement by Alice

In this section, we will assume that Bob and Alice each have a particle of an entangled state, and Alice desires to measure her qubit. As we will see, Alice's measurement will generally reduce the entanglement of the shared state of Alice and Bob to a final value that depends on the strength of the measurement. Alice will apply the general measurement operators $\left\{M_0^{(A)}\ , M_1^{(A)}\right\}$ of the form shown in Eq. (5). One might also desire, for completeness, to consider the effect of Alice's macroscopic measurement device on the results. To this end, the von Neumann measurement scheme, which includes consideration of measurement device pointer states, is considered in Section 4, Supplementary material. It is shown in Section 4, Supplementary material that particle interaction with a measurement device, followed by projection within the measurement device, is completely equivalent to Alice using a direct measurement via her operators $\left\{M_0^{(A)}\ , M_1^{(A)}\right\}$. Given an initial state and measurement operators of the form shown in Eq. (5), it is also found that $F = \frac{\Lambda}{\sqrt{1+2\Lambda(1+\Lambda)}}$ which is only a function of the strength or weakness Λ of Alice's measurement operators. Due to the equivalence found in Section 4, Supplementary material, it will henceforth be considered that Alice makes a measurement on her qubit using the operators $\left\{M_0^{(A)}\ , M_1^{(A)}\right\}$, without further reference to the von Neumann measurement technique.

The entanglement after Alice's initial measurement is given by Eq. (9). The probability of Alice obtaining result $i \in \{0,1\}\,, \pi_{A,i}$ and $E\left(\psi_{A,i}\right)$ are now evaluated:

$$\pi_{A,i} = Tr\left[|\psi_0\rangle\langle\psi_0|\left(I\otimes M_i^{(A)}\right)^\dagger\left(I\otimes M_i^{(A)}\right)\right]. \tag{10}$$

Substituting for $|\psi_0\rangle$ and $M_i^{(A)}$, and simplifying,

$$\begin{aligned}\pi_{A,0} &= \frac{1+2\Lambda\left(1+\Lambda\right)+\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]}{2+4\Lambda\left(1+\Lambda\right)},\\ \pi_{A,1} &= \frac{1}{2}-\frac{\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]}{2+4\Lambda\left(1+\Lambda\right)}.\end{aligned} \tag{11}$$

The entanglement depends on the particular outcome that Alice finds. The density matrix that Bob obtains in case the outcome $i \in \{0,1\}$ occurs is found by

$$\rho_{B,i} = \frac{Tr_A\left[\left(I\otimes M_i^{(A)}\right)|\psi_0\rangle\langle\psi_0|\left(I\otimes M_i^{(A)}\right)^\dagger\right]}{\pi_{A,i}}, \tag{12}$$

where Tr_A denotes the partial trace operation on Alice's Hilbert space. Substituting and reducing, it is found that

$$\begin{aligned}\rho_{B,0} &= \begin{bmatrix} \frac{2a\left(\Lambda^2+(1+2\Lambda)\cos[\theta]^2\right)}{1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} & \frac{\sqrt{-((-1+a)a)}e^{-i\phi}(1+2\Lambda)\sin[2\theta]}{1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} \\ \frac{\sqrt{-((-1+a)a)}e^{i\phi}(1+2\Lambda)\sin[2\theta]}{1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} & \frac{(-1+a)(-1-2\Lambda(1+\Lambda)+(1+2\Lambda)\cos[2\theta])}{1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} \end{bmatrix},\\ \rho_{B,1} &= \begin{bmatrix} \frac{a\left(-1-2\Lambda(1+\Lambda)+(1+2\Lambda)\cos[2\theta]\right)}{-1-2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} & \frac{\sqrt{-((-1+a)a)}e^{-i\phi}(1+2\Lambda)\sin[2\theta]}{-1-2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} \\ \frac{\sqrt{-((-1+a)a)}e^{i\phi}(1+2\Lambda)\sin[2\theta]}{-1-2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} & \frac{(-1+a)(1+2\Lambda(1+\Lambda)+(1+2\Lambda)\cos[2\theta])}{-1-2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta]} \end{bmatrix}.\end{aligned} \tag{13}$$

The entanglement $E\left(\rho_{B,i}\right)$ is found as twice the minimum eigenvalue of the corresponding matrices in Eq. (13). These are found to simplify to

$$\begin{aligned}E\left(\rho_{B,0}\right) &= 1-\frac{2\sqrt{16\left(-1+a\right)a\Lambda^2\left(1+\Lambda\right)^2+\left(1+2\Lambda\left(1+\Lambda\right)+\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]\right)^2}}{2+4\Lambda\left(1+\Lambda\right)+2\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]},\\ E\left(\rho_{B,1}\right) &= 1-\frac{2\sqrt{16\left(-1+a\right)a\Lambda^2\left(1+\Lambda\right)^2+\left(1+2\Lambda\left(1+\Lambda\right)-\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]\right)^2}}{2+4\Lambda\left(1+\Lambda\right)-2\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]}.\end{aligned} \tag{14}$$

The average entanglement can now be computed by substituting Eqs. (11) and (14) into Eq. (9). Upon simplifying, one obtains for $\bar{E}$

$$-\frac{1}{2+4\Lambda\left(1+\Lambda\right)}\left(\begin{array}{c}-2-4\Lambda-4\Lambda^2+\sqrt{16\left(-1+a\right)a\Lambda^2\left(1+\Lambda\right)^2+\left(1+2\Lambda\left(1+\Lambda\right)-\left(-1+2a\right)\left(1+2\Lambda\right)\cos\left[2\theta\right]\right)^2}\\ +\sqrt{16\left(-1+a\right)a\Lambda^2\left(1+\Lambda\right)^2+\left(1+2\Lambda\left(1+\Lambda\right)+\left(-1+2a\right)\left(1+2\Lambda\right)cos\left[2\theta\right]\right)^2}\end{array}\right). \tag{15}$$

Now, consider the function $H \equiv \bar{E}+D-1$. Note that both $\bar{E}$ and D are independent of ϕ. Hence, H is independent of $\phi.H$ can be shown to simplify to

$$-\frac{1}{2+4\Lambda(1+\Lambda)}\left(\begin{array}{c}-2+\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)-(-1+2a)(1+2\Lambda)\cos[2\theta])^2}\\+\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta])^2}\end{array}\right). \tag{16}$$

Since $\bar{E}+D\leq 1$ iff $H\leq 0$, and since $-\frac{1}{2+4\Lambda(1+\Lambda)}<0$, we can multiply both sides by $-\frac{1}{2+4\Lambda(1+\Lambda)}$, and it is needed to be shown that $\hat{H}(a,\Lambda,\theta)\geq 0$, where

$$\begin{aligned}\hat{H}(a,\Lambda,\theta)=-2&+\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)-(-1+2a)(1+2\Lambda)\cos[2\theta])^2}\\&+\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta])^2}.\end{aligned} \tag{17}$$

Now, $\hat{H}$ in Eq. (17) is a function of a,Λ,θ. It will now be shown that $\underset{a}{argmin}\ \hat{H}(a,\Lambda,\theta)$, where $a\in\left[0,\frac{1}{2}\right]$ occurs at $a=1/2$. Taking the derivative of $\hat{H}(a,\Lambda,\theta)$ with respect to a, it is found

$$\begin{aligned}\frac{\partial\hat{H}(a,\Lambda,\theta)}{\partial a}&=-\frac{1}{2+4\Lambda(1+\Lambda)}\\&\left(\begin{array}{c}\frac{2\left(4(-1+2a)\Lambda^2(1+\Lambda)^2-(1+4\Lambda+6\Lambda^2+4\Lambda^3)\cos[2\theta]+(-1+2a)(1+2\Lambda)^2\cos[2\theta]^2\right)}{\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)-(-1+2a)(1+2\Lambda)\cos[2\theta])^2}}\\+\frac{2\left(4(-1+2a)\Lambda^2(1+\Lambda)^2+(1+4\Lambda+6\Lambda^2+4\Lambda^3)\cos[2\theta]+(-1+2a)(1+2\Lambda)^2\cos[2\theta]^2\right)}{\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta])^2}}\end{array}\right).\end{aligned} \tag{18}$$

We want to determine where the local minimal values of $\hat{H}(a,\Lambda,\theta)$ occur. Setting the above to zero, we can multiply through by $-(2+4\Lambda(1+\Lambda))$, and this will occur when

$$\begin{aligned}&\frac{2\left(4(-1+2a)\Lambda^2(1+\Lambda)^2-(1+4\Lambda+6\Lambda^2+4\Lambda^3)\cos[2\theta]+(-1+2a)(1+2\Lambda)^2\cos[2\theta]^2\right)}{\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)-(-1+2a)(1+2\Lambda)\cos[2\theta])^2}}\\&+\frac{2\left(4(-1+2a)\Lambda^2(1+\Lambda)^2+(1+4\Lambda+6\Lambda^2+4\Lambda^3)\cos[2\theta]+(-1+2a)(1+2\Lambda)^2\cos[2\theta]^2\right)}{\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)+(-1+2a)(1+2\Lambda)\cos[2\theta])^2}}=0.\end{aligned} \tag{19}$$

One can verify that the above reduces identically to zero when $a=1/2$, for all $0\leq\Lambda\leq\infty$, $0\leq\theta\leq\frac{\pi}{4}$ and that furthermore $a=1/2$ corresponds to a minimum of $\hat{H}(a,\Lambda,\theta)$ where $\frac{\partial^2\hat{H}(a,\Lambda,\theta)}{\partial a^2}>0$ can be shown. Furthermore, it is seen from Eq. (17) that this reduces to

$$\hat{H}(1/2,\Lambda,\theta)=4\Lambda. \tag{20}$$

Hence, $\hat{H}(a,\Lambda,\theta)\geq\hat{H}(1/2,\Lambda,\theta)=4\Lambda\geq 0$, and the result $\bar{E}+D\leq 1$ is proven for all pure initial states.

Note that when the initial state is a maximally entangled initial state, the inequality $\bar{E}+D\leq 1$ can be rewritten as $\bar{E}+D\leq E_i$, where E_i is the initial entanglement of Qubits A and B. After Alice's measurement, the change in entanglement from the initial to the final is given by $\Delta\bar{E}=\bar{E}-E_i$. Upon defining the average entanglement loss $\bar{E}_L\equiv-\Delta\bar{E}$ between the initial entanglement and final average entanglement, we have the direct relationship between distortion and entanglement loss $D\leq\bar{E}_L$. That is, for initial maximally entangled states, as Alice causes distortion D in order to obtain information about a particle during the process of measurement, such measurement inevitably leads to a corresponding loss of entanglement between Alice's particle and Bob's particle. An example application of this loss in the area of quantum eavesdropping will be examined in Section 4.

3.2. Two entangled particles: measurements by Alice and Bob

In the prior section, two entangled particles of Alice and Bob were considered, with Alice making a weak measurement on her qubit. However, no measurement was made by Bob in the prior section. In this subsection, measurements by both Alice and Bob are considered. Alice first makes a measurement in the basis $\left\{M_0^{(A)}, M_1^{(A)}\right\}$. Bob's task is to attempt to determine as accurately as possible, in the sense of minimizing the probability of error, what Alice's measurement result was. It is assumed that Bob knows the basis $\left\{M_0^{(A)}, M_1^{(A)}\right\}$ that Alice uses in her measurement but not Alice's result.

We desire to determine the tradeoff between the average entanglement and the average information gain, and will show that $\bar{E}+\bar{G}\leq 1$. In order to determine $\bar{G}$, the minimum average probability of error needs to be determined. It was found that by substituting $b=\sin^2\theta$ and $1-b=\cos^2\theta$ into Eq. (5), the resulting expressions for the two-particle measurement problem are simplified. Previously, $0\leq\theta\leq\frac{\pi}{4}$, which corresponds to $0\leq b\leq 1/2$. Hence, in this section, the following equivalent form for the measurement operators is utilized:

$$M_0^{(A)} \equiv \frac{1}{\sqrt{1+2\Lambda(1+\Lambda)}}\begin{bmatrix} \Lambda+1-b & e^{i\phi}\sqrt{(1-b)\,b} \\ e^{-i\phi}\sqrt{(1-b)\,b} & \Lambda+b \end{bmatrix}, \quad M_1^{(A)} \equiv \frac{1}{\sqrt{1+2\Lambda(1+\Lambda)}}\begin{bmatrix} \Lambda+b & -e^{i\phi}\sqrt{(1-b)\,b} \\ -e^{-i\phi}\sqrt{(1-b)\,b} & \Lambda+1-b \end{bmatrix}. \tag{21}$$

As in the prior section, the initial state is given by $|\psi_0\rangle = \sqrt{\alpha_0}|0\rangle \otimes |0\rangle + \sqrt{1-\alpha_0}|1\rangle \otimes |1\rangle$. Several equations for the case of two-particle measurement are the same as those for one-particle measurement. The equivalent forms for Eqs. (11)–(14) are found by substituting $1-2b$ for $\cos[2\theta]$ and $2\sqrt{b\,(1-b)}$ for $\sin[2\theta]$.

Now, after Alice makes her measurement, there are two possible outcomes. Bob desires to determine Alice's measurement as accurately as possible, in the sense of minimizing the probability of error of his measurement outcome compared with Alice's result. After Alice's measurement, Bob will be left with one of two possible density matrices $\rho_{B,0}$ or $\rho_{B,1}$, as given in Eq. (13). The objective of Bob is to measure his density matrix in a manner that his outcome maximally correlates with Alice's result, thereby minimizing the average probability of error. Knowing the initial state and Alice's measurement operators but not Alice's result, Bob needs to find the optimal measurement operators $\left\{M_0^{(B)}, M_1^{(B)}\right\}$. Bob's measurement operators that optimally discriminate between $\rho_{B,0}$ and $\rho_{B,1}$ were found theoretically by Bergou [29]. The optimal solution is found by first constructing the matrix T

$$T \equiv \pi_{A,1}\rho_{B,1} - \pi_{A,0}\rho_{B,0}, \tag{22}$$

and decomposing $T = \sum_{k=1}^{D} \lambda_{T,k}|\phi_k\rangle\langle\phi_k|$, where the states ϕ_k denote orthonormal eigenstates corresponding to the eigenvalues $\lambda_{T,k}$ of T. Bergou assumes that the eigenvalues are numbered in the following manner:

$$\begin{aligned} &\lambda_{T,k} < 0 \text{ for } 1 \le k \le k_0, \\ &\lambda_{T,k} > 0 \text{ for } k_0 \le k \le D, \\ &\lambda_{T,k} = 0 \text{ for } D \le k \le D_S. \end{aligned} \tag{23}$$

The optimal POVMs are found by Bergou to be given by

$$\begin{aligned} \Pi_1 &= \sum_{k=1}^{k_0-1} |\phi_k\rangle\langle\phi_k|, \\ \Pi_2 &= \sum_{k=k_0}^{D_S} |\phi_k\rangle\langle\phi_k|. \end{aligned} \tag{24}$$

where $\Pi_1 + \Pi_2 = I$, and the expression for Π_2 has been supplemented by orthogonal eigenstates corresponding to the eigenvalues $\lambda_{T,k} = 0$.

Now, T can be written in closed form via Eqs. (11)–(13) as follows:

$$T = \begin{pmatrix} \dfrac{a(-1+2b)(1+2\Lambda)}{1+2\Lambda(1+\Lambda)} & -\dfrac{2\sqrt{(-1+a)\,a\,(-1+b)\,b}\,e^{-i\phi}(1+2\Lambda)}{1+2\Lambda(1+\Lambda)} \\ -\dfrac{2\sqrt{(-1+a)\,a\,(-1+b)\,b}\,e^{i\phi}(1+2\Lambda)}{1+2\Lambda(1+\Lambda)} & \dfrac{(-1+a)(-1+2b)(1+2\Lambda)}{1+2\Lambda(1+\Lambda)} \end{pmatrix}. \tag{25}$$

The eigenstates are given by

$$|\phi_1\rangle = \begin{pmatrix} \dfrac{4\sqrt{\dfrac{(-1+a)a(-1+b)b}{1+\dfrac{16(-1+a)a(-1+b)b}{\left(-1+2b+\sqrt{1-4(1-2a)^2b+4(1-2a)^2b^2}\right)^2}}}\,e^{-i\phi}}{-1+2b+\sqrt{1-4(1-2a)^2b+4(1-2a)^2b^2}} \\ \dfrac{1}{\sqrt{1+\dfrac{16(-1+a)a(-1+b)b}{\left(-1+2b+\sqrt{1-4(1-2a)^2b+4(1-2a)^2b^2}\right)^2}}} \end{pmatrix},$$

$$|\phi_2\rangle = \begin{pmatrix} -\dfrac{4\sqrt{\dfrac{(-1+a)a(-1+b)b}{1+\dfrac{16(-1+a)a(-1+b)b}{\left(1-2b+\sqrt{1-4(1-2a)^2b+4(1-2a)^2b^2}\right)^2}}}\,e^{-i\phi}}{1-2b+\sqrt{1-4(1-2a)^2b+4(1-2a)^2b^2}} \\ \dfrac{1}{\sqrt{1+\dfrac{16(-1+a)a(-1+b)b}{\left(1-2b+\sqrt{1-4(1-2a)^2b+4(1-2a)^2b^2}\right)^2}}} \end{pmatrix}, \tag{26}$$

with corresponding eigenvalues

$$\begin{aligned}\lambda_{T,1} &= \frac{\left(1-2a-2b+4ab-\sqrt{-4\left(-a+a^2\right)+\left(-1+2a+2b-4ab\right)^2}\right)\left(1+2\Lambda\right)}{2\left(1+2\Lambda+2\Lambda^2\right)},\\ \lambda_{T,2} &= \frac{\left(1-2a-2b+4ab+\sqrt{-4\left(-a+a^2\right)+\left(-1+2a+2b-4ab\right)^2}\right)\left(1+2\Lambda\right)}{2\left(1+2\Lambda+2\Lambda^2\right)}.\end{aligned} \tag{27}$$

One can see by inspection that $\lambda_{T,1} \leq 0$ and $\lambda_{T,2} \geq 0$ with $0 \leq a \leq \frac{1}{2}, 0 \leq b \leq \frac{1}{2}$. Hence, the conditions established in Bergou are met, and the optimal POVM elements Π_0, Π_1 can be computed as

$$\Pi_0 = \begin{pmatrix} \frac{1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}}{2\sqrt{1+4(1-2a)^2(-1+b)b}} & \frac{4\sqrt{\frac{(-1+a)a(-1+b)b}{\left(1+\frac{16(-1+a)a(-1+b)b}{\left(-1+2b+\sqrt{1+4(1-2a)^2(-1+b)b}\right)^2}\right)^2}}e^{-i\phi}}{-1+2b+\sqrt{1+4(1-2a)^2(-1+b)b}} \\ \frac{4\sqrt{\frac{(-1+a)a(-1+b)b}{\left(1+\frac{16(-1+a)a(-1+b)b}{\left(-1+2b+\sqrt{1+4(1-2a)^2(-1+b)b}\right)^2}\right)^2}}e^{i\phi}}{-1+2b+\sqrt{1+4(1-2a)^2(-1+b)b}} & \frac{1}{1+\frac{16(-1+a)a(-1+b)b}{\left(1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}\right)^2}} \end{pmatrix}, \tag{28}$$

$$\Pi_1 = \begin{pmatrix} \frac{-1+2b+\sqrt{1+4(1-2a)^2(-1+b)b}}{2\sqrt{1+4(1-2a)^2(-1+b)b}} & -\frac{4\sqrt{\frac{(-1+a)a(-1+b)b}{\left(1+\frac{16(-1+a)a(-1+b)b}{\left(1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}\right)^2}\right)^2}}e^{-i\phi}}{1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}} \\ -\frac{4\sqrt{\frac{(-1+a)a(-1+b)b}{\left(1+\frac{16(-1+a)a(-1+b)b}{\left(1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}\right)^2}\right)^2}}e^{i\phi}}{1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}} & \frac{1}{1+\frac{16(-1+a)a(-1+b)b}{\left(1-2b+\sqrt{1+4(1-2a)^2(-1+b)b}\right)^2}} \end{pmatrix}.$$

From these equations, $\left\{M_0^{(B)}, M_1^{(B)}\right\}$ can be computed by decomposing $\Pi_i = M_i^{(B)\dagger} M_i^{(B)}$ (there may be more than a single implementation of $\left\{M_0^{(B)}, M_1^{(B)}\right\}$ that gives the optimal POVM $\{\Pi_0\ ,\ \Pi_1\}$). Now, recall that the gain is $G_i \equiv 1 - 2P_{e|A=i}$, where $P_{e|A=i}$ is the probability of error given that Alice obtains result $i \in \{0,1\}$, and the average gain is found as $\bar{G} = \pi_{A,0}G_0 + \pi_{A,1}G_1$. We can then compute $P_{e|A=0} = Tr\left[\Pi_1 \rho_{B,1}\right]$ and $P_{e|A=1} = Tr\left[\Pi_0 \rho_{B,2}\right]$. Upon substituting the closed-form expressions above, we obtain

$$\begin{aligned}P_{e|A=0} &= \frac{1}{2}\left(1+\frac{-8a^2\left(-1+b\right)b\left(1+2\Lambda\right)-\left(-1+2b\right)\left(b+2b\Lambda+\Lambda^2\right)+a\left(-1-6b+4b\left(-3+\Lambda\right)\Lambda-2\Lambda\left(1+\Lambda\right)+8b^2\left(1+2\Lambda\right)\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(b+2b\Lambda+\Lambda^2-a\left(-1+2b\right)\left(1+2\Lambda\right)\right)}\right),\\ P_{e|A=1} &= \frac{1}{2}\left(1-\frac{8a^2\left(-1+b\right)b\left(1+2\Lambda\right)+\left(-1+2b\right)\left(b+2b\Lambda-\left(1+\Lambda\right)^2\right)-a\left(1+2\Lambda+2\Lambda^2+8b^2\left(1+2\Lambda\right)-2b\left(5+10\Lambda+2\Lambda^2\right)\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(\left(1+\Lambda\right)^2-b\left(1+2\Lambda\right)+a\left(-1+2b\right)\left(1+2\Lambda\right)\right)}\right).\end{aligned} \tag{29}$$

The information gains are now computed as

$$\begin{aligned}G_0 &= -\frac{-8a^2\left(-1+b\right)b\left(1+2\Lambda\right)-\left(-1+2b\right)\left(b+2b\Lambda+\Lambda^2\right)+a\left(-1-6b+4b\left(-3+\Lambda\right)\Lambda-2\Lambda\left(1+\Lambda\right)+8b^2\left(1+2\Lambda\right)\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(b+2b\Lambda+\Lambda^2-a\left(-1+2b\right)\left(1+2\Lambda\right)\right)},\\ G_1 &= \frac{8a^2\left(-1+b\right)b\left(1+2\Lambda\right)+\left(-1+2b\right)\left(b+2b\Lambda-\left(1+\Lambda\right)^2\right)-a\left(1+2\Lambda+2\Lambda^2+8b^2\left(1+2\Lambda\right)-2b\left(5+10\Lambda+2\Lambda^2\right)\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(\left(1+\Lambda\right)^2-b\left(1+2\Lambda\right)+a\left(-1+2b\right)\left(1+2\Lambda\right)\right)}.\end{aligned} \tag{30}$$

Note that $\bar{E}+\bar{G}=\pi_{A,0}E\left(\rho_{B,0}\right)+\pi_{A,1}E\left(\rho_{B,1}\right)+\pi_{A,0}G_0+\pi_{A,1}G_1=\pi_{A,0}\left(E\left(\rho_{B,0}\right)+G_0\right)+\pi_{A,1}\left(E\left(\rho_{B,1}\right)+G_1\right)$. Using $E\left(\rho_{B,i}\right)$ from Eq. (14) and G_i from above with $i=0,1$, the following is obtained:

$$\begin{aligned}
E(\rho_{B,0})+G_0 = &-\frac{-8a^2\left(-1+b\right)b\left(1+2\Lambda\right)-\left(-1+2b\right)\left(b+2b\Lambda+\Lambda^2\right)+a\left(-1-6b+4b\left(-3+\Lambda\right)\Lambda-2\Lambda\left(1+\Lambda\right)+8b^2\left(1+2\Lambda\right)\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(b+2b\Lambda+\Lambda^2-a\left(-1+2b\right)\left(1+2\Lambda\right)\right)}\\
&+2\left(\frac{1}{2}+\frac{\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)+(-1+2a)(1-2b)(1+2\Lambda))^2}}{4a(-1+2b)(1+2\Lambda)-4(b+2b\Lambda+\Lambda^2)}\right),\\
E(\rho_{B,1})+G_1 = &\frac{8a^2\left(-1+b\right)b\left(1+2\Lambda\right)+\left(-1+2b\right)\left(b+2b\Lambda-\left(1+\Lambda\right)^2\right)-a\left(1+2\Lambda(1+\Lambda)+8b^2(1+2\Lambda)-2b(5+2\Lambda(5+\Lambda))\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(\left(1+\Lambda\right)^2-b\left(1+2\Lambda\right)+a\left(-1+2b\right)\left(1+2\Lambda\right)\right)}\\
&+2\left(\frac{1}{2}-\frac{\sqrt{16(-1+a)a\Lambda^2(1+\Lambda)^2+(1+2\Lambda(1+\Lambda)+(-1+2a)(1-2b)(1+2\Lambda))^2}}{2+4\Lambda(1+\Lambda)+2(-1+2a)(-1+2b)(1+2\Lambda)}\right).
\end{aligned} \tag{31}$$

Upon substituting for $\pi_{A,i}$ from Eq. (11)$(i=0,1)$ into $\pi_{A,0}(E(\rho_{B,0})+G_0)+\pi_{A,1}(E(\rho_{B,1})+G_1)-1=\bar{E}+\bar{G}-1$ and reducing, $\bar{E}+\bar{G}-1$ is obtained as:

$$\frac{f_1\left(a,b,\Lambda\right)-f_2\left(a,b,\Lambda\right)-f_3\left(a,b,\Lambda\right)}{\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}\left(1+2\Lambda\left(1+\Lambda\right)\right)}, \tag{32}$$

where

$$f_1\left(a,b,\Lambda\right)=\left(1+2\Lambda+4\left(1-2a\right)^2\left(-1+b\right)b\left(1+2\Lambda\right)\right),$$

$$\begin{aligned}
f_2(a,b,\Lambda) \quad = \quad &[(1+4(1-2a)^2(-1+b)b)((a+b-2ab)^2+4(a+b-2ab)^2\Lambda+2(a(-1+4a)+b+2(1-4a)ab\\
&+2(1-2a)^2b^2)\Lambda^2+4(-1+2a)(a-b)\Lambda^3+(1-2a)^2\Lambda^4)]^{1/2},
\end{aligned}$$

$$\begin{aligned}
f_3(a,b,\Lambda) \quad = \quad &[(1+4(1-2a)^2(-1+b)b)((b+2b\Lambda-(1+\Lambda)^2)^2+a^2(-4b(1+2\Lambda)^2+4(b+2b\Lambda)^2+(1+2\Lambda(1+\Lambda))^2)\\
&-2a(2(b+2b\Lambda)^2+(1+\Lambda)^2(1+2\Lambda(1+\Lambda))-b(1+2\Lambda)(3+2\Lambda(3+\Lambda))]^{\frac{1}{2}}.
\end{aligned}$$

Now, we desire to prove that $\bar{E}+\bar{G}\leq 1$ or $\bar{E}+\bar{G}-1\leq 0$. Note that the prefactor $\left(\sqrt{1+4\left(1-2a\right)^2\left(-1+b\right)b}(1+2\Lambda\left(1+\Lambda\right)\right)^{-1}$ is always greater than or equal to zero when $0\leq a\leq\frac{1}{2}, 0\leq b\leq\frac{1}{2}$. Hence, we can multiply both sides of $\bar{E}+\bar{G}-1\leq 0$ by the prefactor, obtaining the inequality

$$f_1\left(a,b,\Lambda\right)-f_2\left(a,b,\Lambda\right)-f_3\left(a,b,\Lambda\right)\leq 0. \tag{33}$$

We desire to determine if $f_1-f_2-f_3\leq 0$. In order to determine this, consider the following lemma which is easily proven. Let $h\left(x\right)$ and $g\left(x\right)$ be real functions defined on a set $x\in S\subset R$, where R denotes the set of real numbers. Suppose $h\left(x\right)\geq 0, x\in S$. If $h^2\left(x\right)\geq g^2\left(x\right)$, then $h\left(x\right)\geq g\left(x\right)$. Now, the above inequality can be rewritten as $f_2+f_3\geq f_1$. It can be seen that $f_2\geq 0, f_3\geq 0, f_2+f_3\geq 0$; hence, if $\left(f_2+f_3\right)^2\geq {f_1}^2$, then it follows that $f_2+f_3\geq f_1$.

Now, $\left(f_2+f_3\right)^2-{f_1}^2\geq 0$ can be written as $-\left(f_2+f_3\right)^2+{f_1}^2\leq 0$ or expanding this expression:

$$-2(1+4(1-2a)^2(-1+b)b)\left[\begin{array}{l}
\Lambda^2\left(1+\Lambda\right)^2+b\left(1+2\Lambda\right)^2-\left(b+2b\Lambda\right)^2+a\left(-4b\left(1+2\Lambda\right)^2+4\left(b+2b\Lambda\right)^2-\left(1+2\Lambda\left(1+\Lambda\right)\right)^2\right)\\
+a^2\left(4b\left(1+2\Lambda\right)^2-4\left(b+2b\Lambda\right)^2+\left(1+2\Lambda\left(1+\Lambda\right)\right)^2\right)\\
+(((a+b-2ab)^2+4(a+b-2ab)^2\Lambda+2(a(-1+4a)+b+2(1-4a)ab+2(1-2a)^2b^2)\Lambda^2\\
+4(-1+2a)(a-b)\Lambda^3+(1-2a)^2\Lambda^4)((-1+a+b-2ab)^2+4(-1+a+b-2ab)^2\Lambda\\
+2(3+a(-7+4a)-5b+2(7-4a)ab+2(1-2a)^2b^2)\Lambda^2\\
+4(-1+2a)(-1+a+b)\Lambda^3+(1-2a)^2\Lambda^4))^{1/2}
\end{array}\right]\leq 0.$$

The prefactor $-2\left(1+4\left(1-2a\right)^2\left(-1+b\right)b\right)$ is always less than or equal to zero when $0 \le a \le \frac{1}{2}, 0 \le b \le \frac{1}{2}$. Dividing both sides by the prefactor, the following inequality is obtained:

$$-2(1+4(1-2a)^2(-1+b)b)\left[\begin{array}{l}\Lambda^2\left(1+\Lambda\right)^2+b\left(1+2\Lambda\right)^2-\left(b+2b\Lambda\right)^2+a\left(-4b\left(1+2\Lambda\right)^2+4\left(b+2b\Lambda\right)^2-\left(1+2\Lambda\left(1+\Lambda\right)\right)^2\right)\\ +a^2\left(4b\left(1+2\Lambda\right)^2-4\left(b+2b\Lambda\right)^2+\left(1+2\Lambda\left(1+\Lambda\right)\right)^2\right)\\ +(((a+b-2ab)^2+4(a+b-2ab)^2\Lambda+2(a(-1+4a)+b+2(1-4a)ab+2(1-2a)^2b^2)\Lambda^2\\ +4(-1+2a)(a-b)\Lambda^3+(1-2a)^2\Lambda^4)((-1+a+b-2ab)^2+4(-1+a+b-2ab)^2\Lambda\\ +2(3+a(-7+4a)-5b+2(7-4a)ab+2(1-2a)^2b^2)\Lambda^2\\ +4(-1+2a)(-1+a+b)\Lambda^3+(1-2a)^2\Lambda^4))^{1/2}\end{array}\right]\le 0. \tag{34}$$

Denote the terms h_1 and h_2 as

$$\begin{array}{l} h_1=\Lambda^2(1+\Lambda)^2+b(1+2\Lambda)^2-(b+2b\Lambda)^2+a(-4b(1+2\Lambda)^2+4(b+2b\Lambda)^2-(1+2\Lambda(1+\Lambda))^2)\\ +a^2(4b(1+2\Lambda)^2-4(b+2b\Lambda)^2+(1+2\Lambda(1+\Lambda))^2),\\ h_2=(((a+b-2ab)^2+4(a+b-2ab)^2\Lambda+2(a(-1+4a)+b+2(1-4a)ab+2(1-2a)^2b^2)\Lambda^2\\ \quad +4(-1+2a)(a-b)\Lambda^3+v(1-2a)^2\Lambda^4)((-1+a+b-2ab)^2+4(-1+a+b-2ab)^2\Lambda\\ \quad +2(3+a(-7+4a)-5b+2(7-4a)ab+2(1-2a)^2b^2)\Lambda^2+4(-1+2a)(-1+a+b)\Lambda^3+(1-2a)^2\Lambda^4))^{1/2}. \end{array} \tag{35}$$

It is then desired to show that $h_1+h_2 \ge 0$. Note that h_2 can be seen to be greater than or equal to zero when $0 \le a \le \frac{1}{2}, 0 \le b \le \frac{1}{2}$. Now, if ${h_1}^2 \ge (-h_2)^2$, then $h_1 \ge -h_2$. It can be verified that ${h_1}^2+(-h_2)^2$ factorizes as follows:

$${h_1}^2+(-h_2)^2=4\left(1-2a\right)^2\left(-1+a\right)a\left(-1+b\right)b\left(1+2\Lambda\left(2+\Lambda\left(3+2\Lambda\right)\right)\right)^2. \tag{36}$$

When $0\ a \le \frac{1}{2}, 0 \le b \le \frac{1}{2}$, note that the RHS of the equation above is always greater than or equal to zero since $(1-2a)^2 \ge 0, (-1+a)(-1+b) \ge 0$ and $(1+2\Lambda(2+\Lambda(3+2\Lambda)))^2 \ge 0$. Hence, ${h_1}^2+(-h_2)^2 \ge 0$, and the proof that $\bar{E}+\bar{G} \le 1$ is complete.

4. Conclusions

Assuming an initial entangled two-particle state shared by Alice and Bob, both measurement on a single system by Alice and measurement on both systems by both Alice and Bob have been considered in regard to the effect on the entanglement. For single-particle measurement by Alice, it has been proven that the sum of the entanglement and the measurement disturbance, quantified by D, is less than or equal to unity, i.e., $\bar{E}+D \le 1$. For the case of two-particle measurement, it has been shown that both the average entanglement and the average information gained through measurement, denoted as $\bar{G}$, cannot both be arbitrarily high, i.e., $\bar{E}+\bar{G} \le 1$.

The results were initially proven for arbitrary initial pure states and extended to arbitrary mixed states in Section 2, Supplementary material. Additionally, non-Hermitian measurement implementations of a given POVM were considered in Section 3, Supplementary material, and the complementary results were proven to continue to hold.

In the case of maximally entangled initial states, it is found that entanglement loss is directly complementary to both measurement disturbance and average information gain. This provides a simple and easily applicable formula for such initial states which are widely utilized in both theory and experiment in quantum information, often utilizing maximally entangled states.

One might consider what applications such bounds have. For example, let us consider the potential application of these bounds in secure communication. Suppose that Alice desires to transmit classical information, specifically a classically encrypted bit stream, to Bob through a medium. Suppose that Alice uses quantum degrees of freedom, such as $|H\rangle$ and $|V\rangle$ photon polarization, to represent classical bits of 0 and 1, respectively. Suppose that the medium is not secure and is subject to be tapped by an eavesdropper, Eve. Eve is allowed to make measurements on the qubits and potentially retransmit the qubits.

In the case when Alice transmits classical bits to Bob, Eve can intercept the bits, learn them, and then retransmit them without detection. It does not appear that, at least classically, Alice and Bob can detect the presence or absence of an eavesdropper, such as Eve, since Eve retransmits the bits precisely as she receives them. Consider the following protocol which utilizes a simplified version of a quantum protocol proposed by Humble [30] to illustrate how Eve can be detected by Alice and Bob using quantum entanglement. Suppose that Alice generates two maximally entangled polarization qubits at some rate and interleaves one of these within her encrypted classical data stream in a manner initially known only to Alice and Bob. Alice and Bob then perform a Bell experiment on a given number of pairs to characterize the violation. Now, if Eve is eavesdropping, she will have measured some of these interleaved particles sent to Bob, and we can apply our bound $D \le \bar{E}_L$, where $\bar{E}_L \equiv -\Delta\bar{E}$ is the entanglement loss or $\bar{E}_f \le 1-D$. If Eve makes strong measurements corresponding to $\Lambda \to 0$ or $D \to 1$, we see that $\bar{E}_f \to 0$ results immediately, and the initial entanglement will necessarily be destroyed by Eve in the process of eavesdropping. Hence, Alice and Bob can reliably detect the presence of Eve if she uses strong measurements. Note that Eve can utilize weak measurements to minimize her disturbance. Given some background channel noise level, it may be advantageous for Eve to lower her disturbance in a manner that makes it more difficult for Alice and Bob to detect her. This approach, however, will lower her ability to make reliable measurements of the channel. Other papers have considered this and other schemes, including the use of continuous entanglement [31]. An experimental demonstration of confidential communication, albeit with a modified version of these techniques, was recently reported by Gong et al. [32].

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Mathematica was used for deriving the equations, and Maple was employed to verify these algebraic results. MATLAB was utilized to double-check the closed-form equations numerically.

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