THIS VASTLY UNDER-APPRECIATED THEORY BY EINSTEIN CAN EXPLAIN WHY THE SKY IS BLUE TO FLUCTUATIONS OF STOCK MARKETS

Einstein-Smoluchowski Theory of Fluctuation

How a little-known theory unveiled the early radical thinking of a once-unknown genius and contributed to the advancement of modern AI models.

Michael Ng · Following
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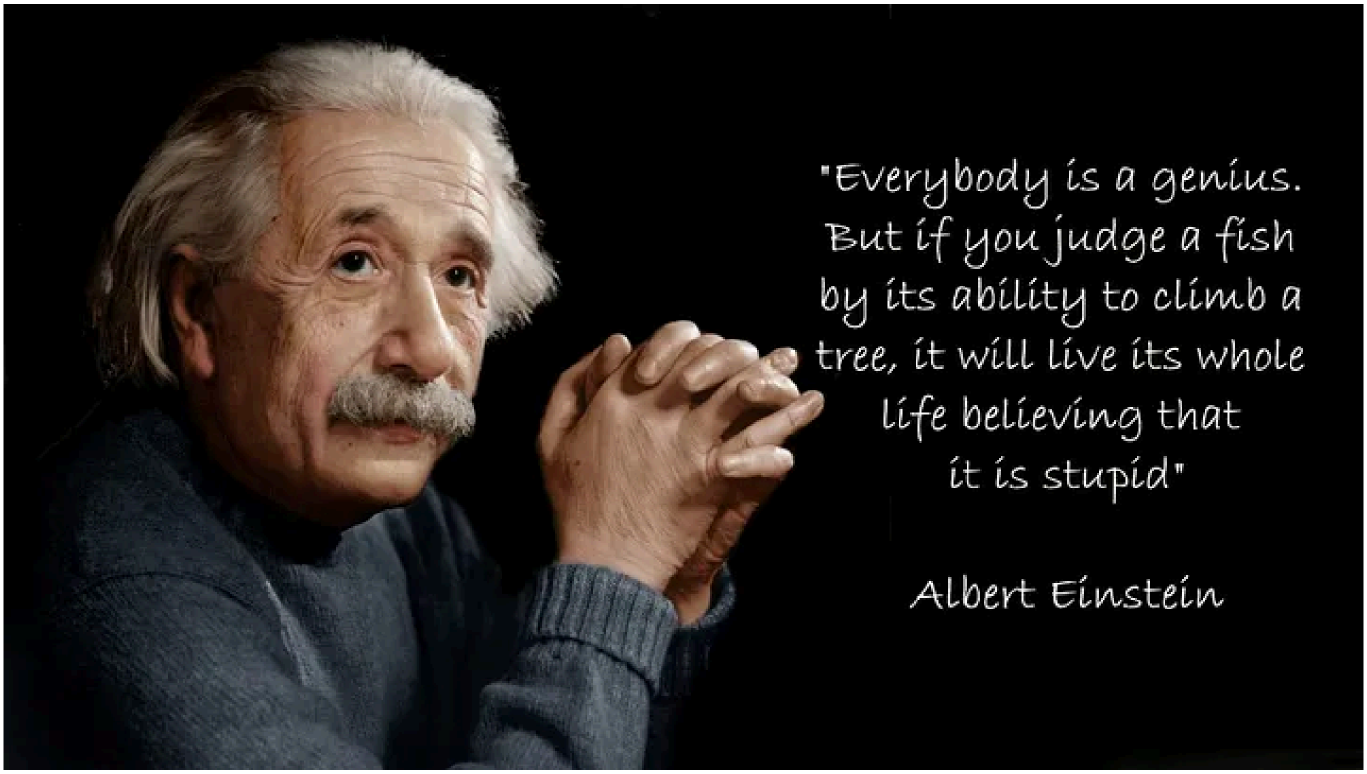
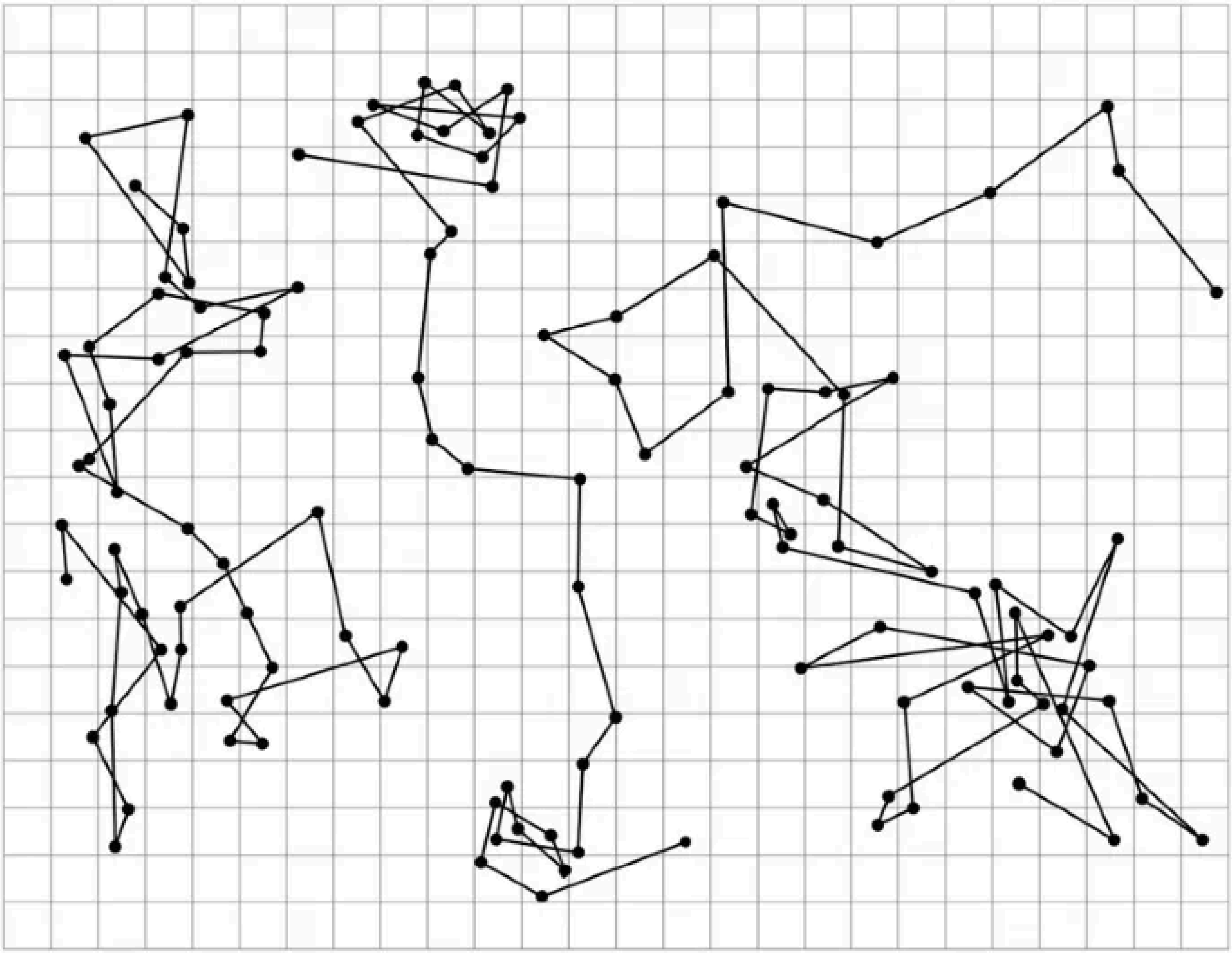


During the quiet hours of the evening on August 16, 1903, Marian von Smoluchowski penned a letter to his mother. "*My dearest Mother,*" he began with earnest excitement, "*I am writing to share the thrilling progress in my research. The mysteries of* ***fluctuations*** *at the microscopic level are beginning to unfold before my eyes.*"

The paper that Smoluchowski, a Professor of Physics at the University of Lemberg (now Lviv, Ukraine), referred to marked the first scientific work that eventually led to the experimental proof of the existence of atoms. Unbeknownst to him at the time, a 25-year-old unknown patent clerk was simultaneously working on the same theory of a mysterious random movement of microscopic particles suspended in a fluid, known as Brownian motion.



Brownian Motion — random movement of suspended particles in a fluid: Image from Springer Nature Link

It was only much later that the two recognized each other's work as different approaches to the same underlying phenomenon. In a 1911 paper, Smoluchowski referred to Einstein's contribution as '*a significant advance*'. However, he had not quite understood Einstein's argument. In this paper, Smoluchowski mentioned that the blue sky color is due to two factors: scattering off molecules, according to the incumbent theory known as "*Rayleigh Scattering,*" and scattering resulting from *density fluctuations* based on their independently published theory.

Einstein objected, stating **there is one and only one** cause of scattering:

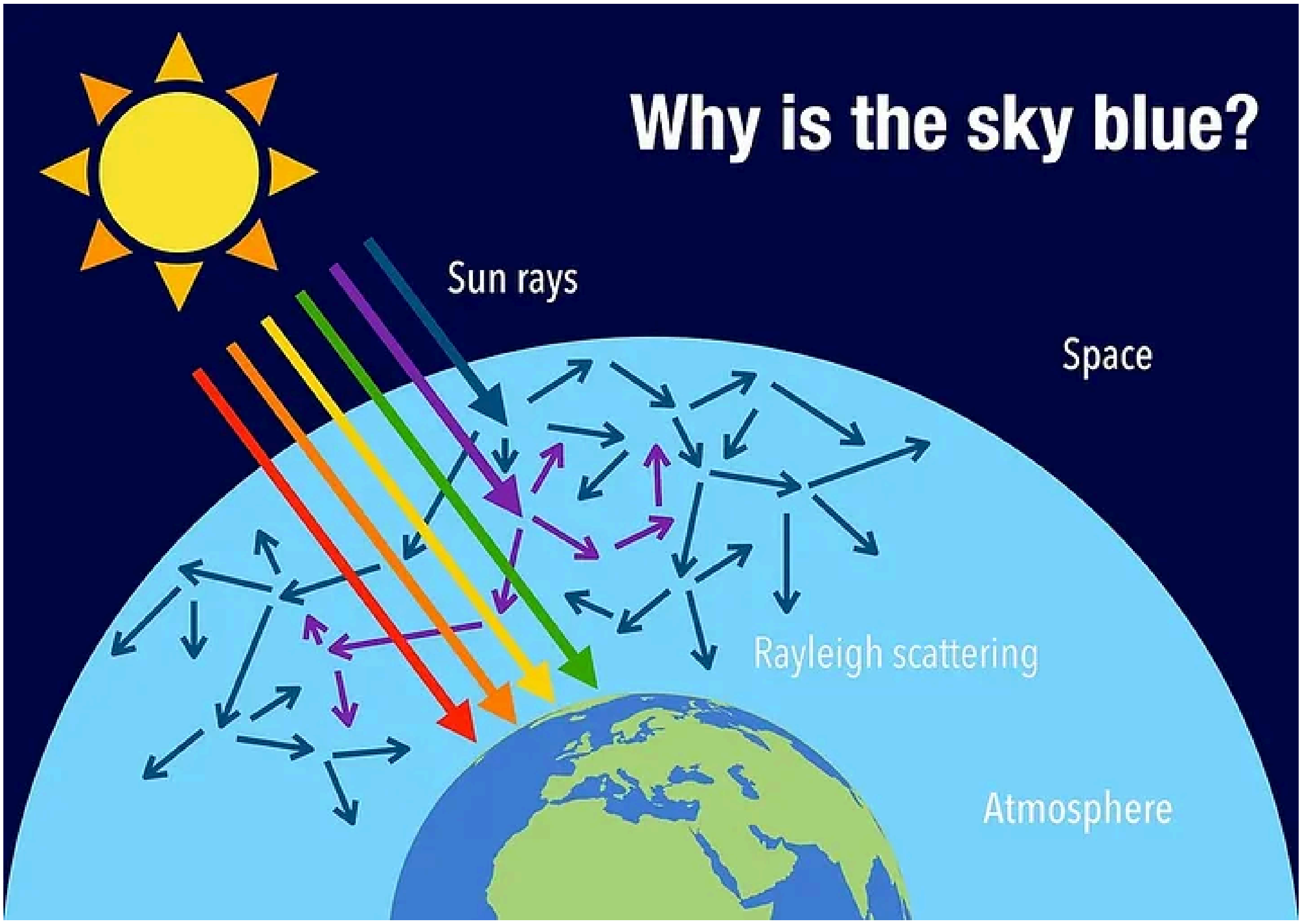
> *"Rayleigh treats a special case of our problem and the agreement between his final formula and my own is no accident."*

Shortly after, Smoluchowski replied,

> ***"You are completely right."***

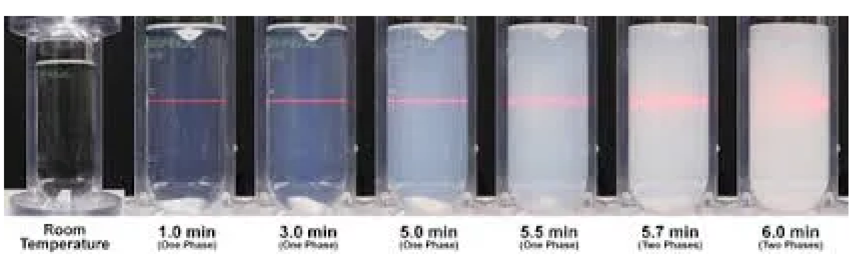
Wait, what?! How could both mechanisms be one and the same?

> *Rayleigh scattering is a phenomenon that explains why the sky looks blue during the day and why the sunsets and sunrises appear red or orange. It relies on the scattering of light by particles that are much smaller than the wavelength of the light. We see blue sky during the day because shorter wavelengths of light such as blue and violet are scattered much more strongly compared to red light.*



Effectiveness of Rayleigh scattering is inversely proportional to the fourth power of the wavelength. Since blue light has shorter wavelength it is scattered about 10 times more effectively — Image Credit: Shutterstock

> *On the other hand, scattering by density fluctuation refers to how light waves scattered when they encounter regions in the medium where the density changes causing variation in the refractive index, which bends the light in different direction. Especially found in critical opalescence, where fluctuations in density near the critical point of a substance (during phase transition) become large and cause the material to scatter light intensely.*



Critical Opalescence demonstration during phase transition: image from [Journal of Chemical Education]

It is from this cryptic mail exchanges that we begin to unravel Einstein's formative genius. But before we can unpack all this, we have to trace back the path Einstein took that led to this groundbreaking yet obscure work.

Einstein Diffusion Equation

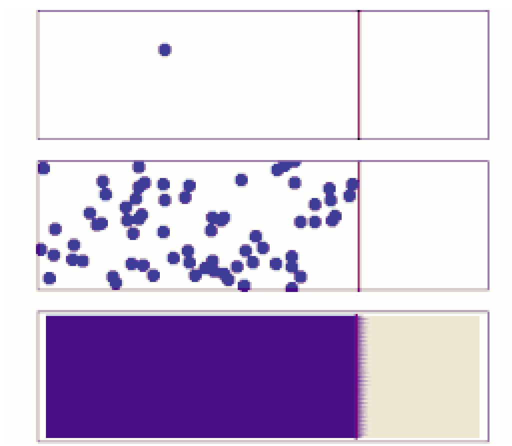
Known as the "*Miracle Year*", 1905 was a remarkable year for the 26-year-old patent clerk. Einstein's rise to fame can be traced back to a few key moments in his life. In six months, at a breakneck pace, Einstein wrote five scientific papers, each worthy of Nobel Prize, and deeply influenced the course of twentieth-century science:

- Discovery of Quanta and the Photoelectric Effect (March 1905)
- Determination of Molecular dimensions — Doctoral Dissertation (April 1905)
- Explanation of Brownian Motion (May 1905)
- Development of the Special Theory of Relativity (June 1905)
- Development of Mass-Energy Equivalence $E = mc^2$ (September 1905)

However, lesser-known were his contributions in statistical physics during his formative years of 1902–04 and again much later in 1906–1910, where Einstein was an undisputed master. In fact, Einstein's most cited paper was neither Relativity nor Quantum Theory, but his April 1905 paper on statistical physics, his doctoral dissertation "*A New Determination of Molecular Dimensions*".

Why did Einstein's PhD dissertation paper (Apr 1905) turn out to be his most cited paper? He introduced a *diffusion equation* and applied it in a very innovative way. In a subsequent paper (May 1905), he applied the diffusion equation to explain and quantify *Brownian motion*, providing the first empirical evidence supporting the existence of atoms and molecules. This step was a crucial foundation for his later achievements.

Diffusion is the process by which particles spread from areas of high concentration to areas of low concentration. This process is driven by random motion of particles and continues until the particles are evenly distributed throughout the system. Examples of diffusion include the dispersal of perfume in a room or the spreading of ink in water.



Diffusion from a microscopic and macroscopic point of view — Source: wikipedia

In the process of diffusion, particles encounter resistive forces, typically resulting from interactions with other particles or the medium through which they move.

Einstein's diffusion equation took into account two opposing forces: a *driving* force of motion and a *resistive* force against the motion experienced by the particles. He combined the random thermal motion of particles with the frictional force they encounter.

This balance between these forces — random thermal motion and the frictional force — is encapsulated in the equation illustrated below:

$$D = \frac{KT}{6\pi r\eta}$$

where

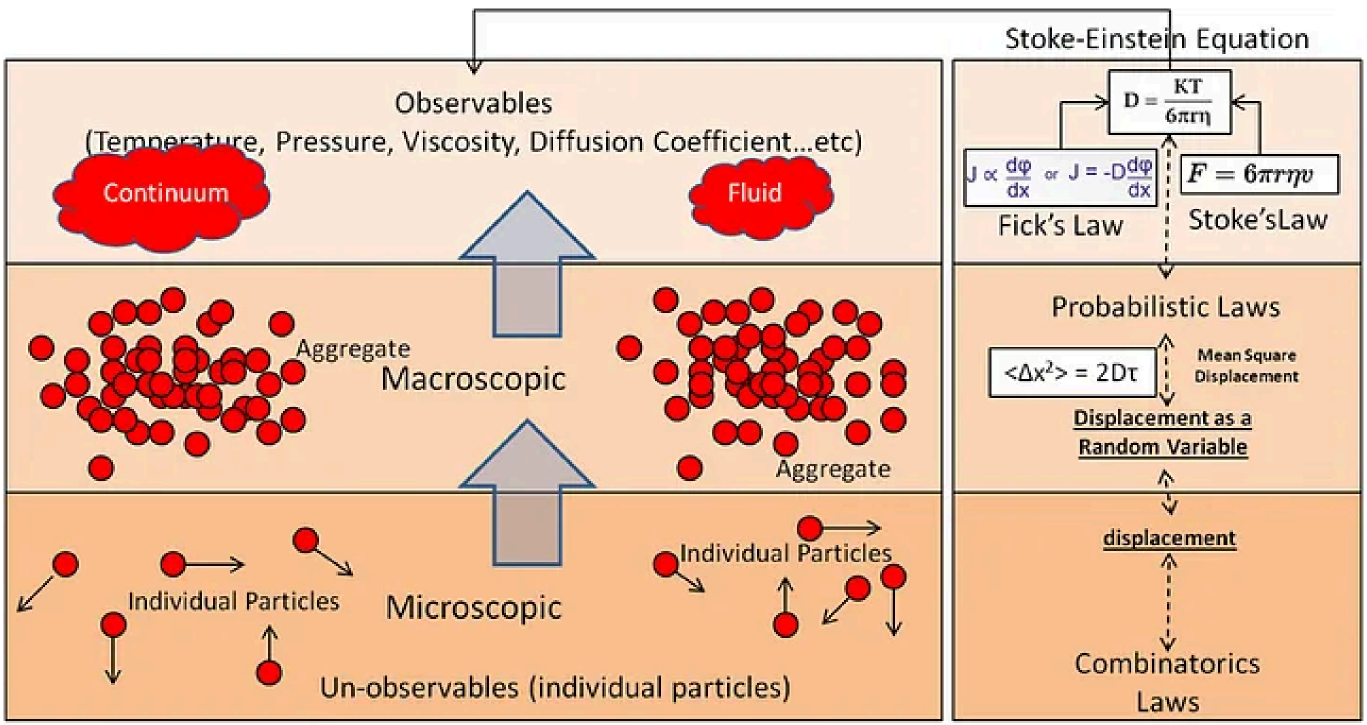
D = Diffusion coefficient
K = Boltzmann's constant
T = Absolute temperature (K)
r = Molecular radius
η = Viscosity of the medium

Stoke-Einstein Equation — image from Quizlet.com

The Stokes-Einstein's equation relates the diffusion coefficient (*D*) of a spherical particle in a fluid to its radius (*r*), the viscosity (*η*) or frictional resistance of the fluid medium, and the temperature (*T*) of the system.

The diffusion equation was not discovered by Einstein; it was originally derived by Adolf Fick in 1855, based on experiments conducted by Thomas Graham in the 1830s. However, Fick's equation was formulated as a set of differential equations from a macroscopic perspective. **Einstein brilliantly reformulated Fick's diffusion equations by connecting the macroscopic observation of diffusion with the microscopic motion of particles, introducing a probabilistic underpinning.** This synthesis is crucial for understanding Einstein's key insight, and you will see this mental toolkit pattern used repeatedly.

Given its importance, let's take a moment to unwind Einstein's steps with the following description. The figure below illustrates that the laws governing the behavior of individual particles at the microscopic level are quite different from the immense collection of these same particles at a larger scale. At the macroscopic level, we can only deal from a statistical and probabilistic perspective. At the human level of observation, Fick's Law and Stokes' Law, therefore, treat matter as if it is a continuum.



Deriving Stoke-Einstein Equation from Fick's and Stoke's laws : image by the author

Hence, Einstein treated the displacement of particles as a random variable and developed equations to describe their average behavior over time. He then connected the diffusion coefficient with the mean squared displacement of particles by combining Fick's Law with Stokes' Law. This relationship allowed for the measurement of the diffusion coefficient by observing the displacement of particles over time.

Einstein also linked the diffusion coefficient to the temperature and viscosity of the fluid, drawing from the kinetic theory of gases. Essentially, he mapped the interactions of microscopic world to the macroscopic world with tangible, measurable properties of matter. Since the microscopic layer is a common layer across all physical phenomena, this allowed the diffusion equation to be applied to a wide range of physical systems. In his paper, Einstein noted:

> *"It is a wonderful feeling to recognize the unifying features of a complex of phenomena which present themselves as quite unconnected to the direct experience of the senses."*

The importance of the diffusion equation cannot be overstated. It has an extremely wide range of applications across various scientific fields, including:

- Explaining Brownian motion
- Statistical mechanics
- Heat transfer
- Chemical reactions
- Biological processes
- Material science
- Environmental science
- Epidemiology
- Quantum computing
- Machine learning

The diffusion equation is a fundamental equation in physics and mathematics that describes how particles, energy, or other quantities spread out over time.

Einstein would later apply the principles of diffusion to thermodynamic fluctuations, which became known as the ***Einstein-Smoluchowski Theory of Fluctuation***.

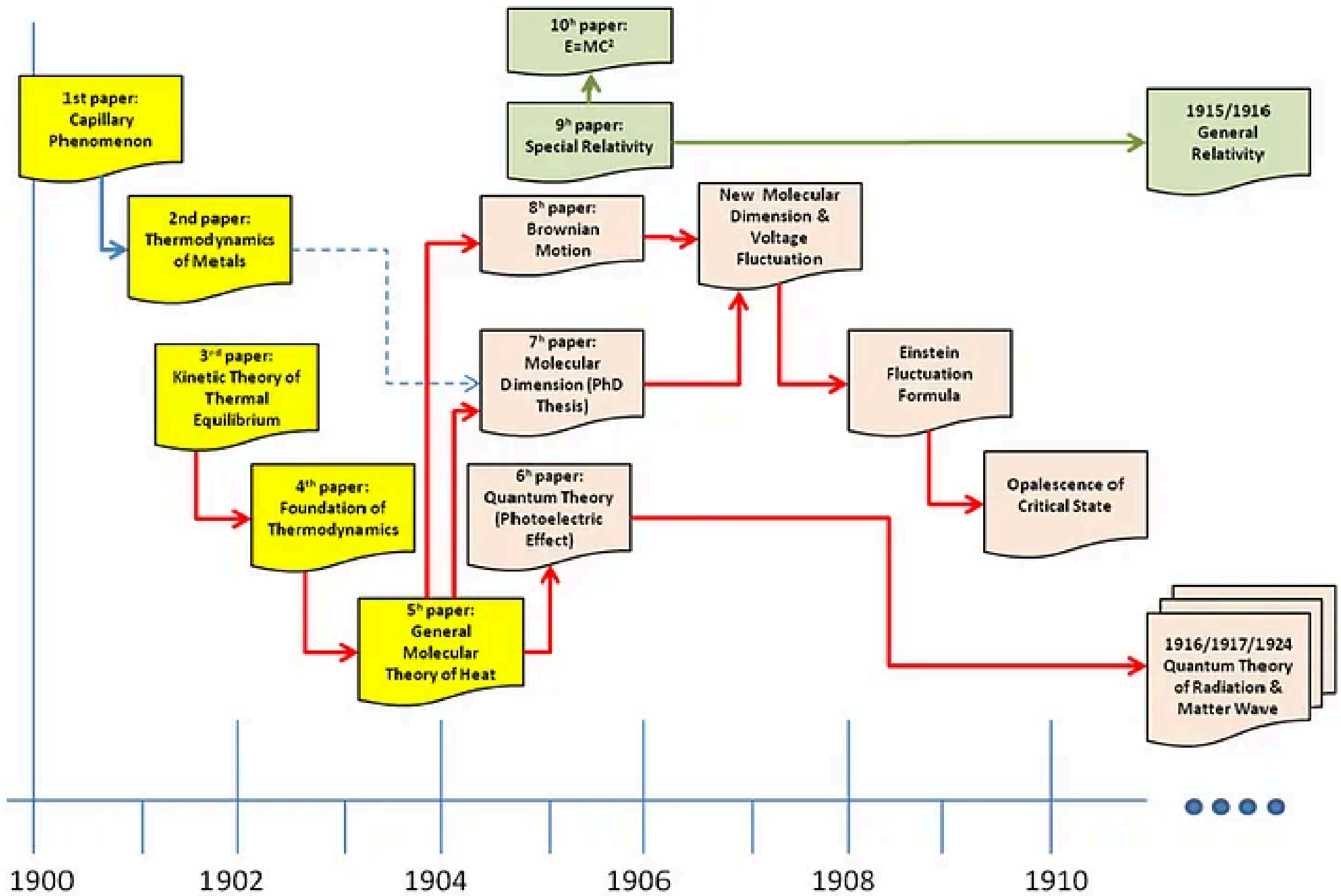
However, in order to understand the *germ* behind how Einstein arrived at that theory, we must trace back to his formative beginnings.

A Historical Detour

After graduating from ETH Zurich in 1900, Einstein faced some initial struggles finding a stable academic position. During this period, he worked as a private tutor and later as a technical assistant at the Swiss Patent Office in Bern. Despite these challenges, he remained preoccupied with the fundamental questions of statistical mechanics. This led to a series of breakthroughs, including theory of Brownian motion, which proved the existence of atoms, as well as to his most important contribution to quantum theory.

> *"I developed statistical mechanics and the molecular-kinetic theory of thermodynamics based on it. My main purpose for doing this was to find facts which would attest to the existence of atoms of definite size."*

The initial goal behind Einstein's first five scientific papers from 1901–1904 was to combine molecular theory with thermodynamics in a manner that extended the work of the masters of the field — Ludwig Boltzmann and Josiah Willard Gibbs. He achieved this by reframing their theoretical frameworks with observable macroscopic properties, such as the viscosity coefficient and the diffusion coefficient. By recasting statistical mechanics in the light of macroscopic properties, he was able to launch his three major discoveries in 1905: the explanation of Brownian motion, the determination of molecular dimensions, and the heuristics of quantum theory — see below figure.



Einstein's scientific papers historical background — Image by the author

Keep in mind that when Einstein was formulating these first five papers between 1901–1904 (shown as yellow boxes in the above figure), the experimental evidence for atoms' existence was still decades in the future. While the laws of classical thermodynamics were well established, the development of statistical mechanics by Boltzmann and Gibbs was still in its infancy and somewhat controversial.

The reason for the skepticism was the lack of evidence for the existence of atoms and the fact that statistical average behavior of the macroscopic phenomena was virtually identical to conventional classical thermodynamics. The only possible way forward was to obtain evidence of deviation (fluctuation) from the mean.

While both Boltzmann, and later Gibbs, considered fluctuations as something inherent to the statistical description itself, they reasoned that for systems with a very high number of degrees of freedom, as is the case for thermodynamic systems, the value of fluctuations is so small that they are practically unobservable and, therefore, irrelevant in equilibrium statistical mechanics. Therefore, it seemed most improbable to experimentally validate the claims of statistical mechanics.

> *It is here that Einstein shows one of his first flashes of boldness and creativity: since fluctuations are characteristic of every statistical description, and since in the usual thermodynamic systems they are unobservable, why not look specifically for a physical system in which they are observable and in which, therefore, the validity of (energy fluctuation) can be tested.*

In the final section of his 1904 paper, Einstein, using some clever mathematics and an intuitive leap, proposes a surprising candidate for verifying statistical mechanics: *black-body radiation*. This choice is unexpected not only because it differs from the typical systems treated within kinetic theory, but also because electromagnetic radiation had previously been explicitly dismissed by Boltzmann and Gibbs as unsuitable for statistical analysis.

Why did Einstein make that risky but enormously fruitful leap of imagination? While it is true that he had noticed a mathematical coincidence between Wien's displacement law and his recently derived energy fluctuation formula (to be described later), this coincidence led him to the path of quantum theory. But I believe there was something more.

Einstein had an obsessive type of curiosity and an uncommon physical intuition that marked his genius. From 1901 to 1904, it seems that he had already been brooding for some time about a mysterious constant (later named the Boltzmann constant, *k* or *k_B*) and why it was showing up in these thermodynamics-related equations.

Throughout his first five scientific papers, Einstein pondered the physical significance of this constant (*k_B*). Historians have often wondered why he took explicit steps, sometimes to an unnecessary degree, to meander around and wrangle various mathematical structures before arriving at a solution. It was to serve his curiosity and arrive at a physical intuition.

Before long, with his fifth scientific paper in 1904, Einstein had arrived at his conclusion. The energy variance $<(\Delta E)^2>$ is a measure of the *stability* and *variability* of the thermodynamic system at the human scale of observability; and the Boltzmann constant serves as a bridge between the microscopic world of atoms and the macroscopic world of thermodynamics. The smaller the value of the Boltzmann constant (*k_B* = 1.380649×10^−23 joules per kelvin), the lower the variability of the system and therefore the higher the stability. But this variability or stability is only an illusion. It depends on the level or scale at which you are making the observation. It turns out that in a black body experimental cavity, the relative scaling is much closer to the microscopic level, making it possible to observe the variation.

It was only after this realization that Einstein was able to discover his fluctuation theory. In essence, he had discovered a world described at different scale or layers of observability. The fluctuation theory is a mathematical and analytical tool that allows him to map from one layer (frame of reference) to another — ***foreshadowing his future discovery of the theory of relativity!***

Fluctuation Theory

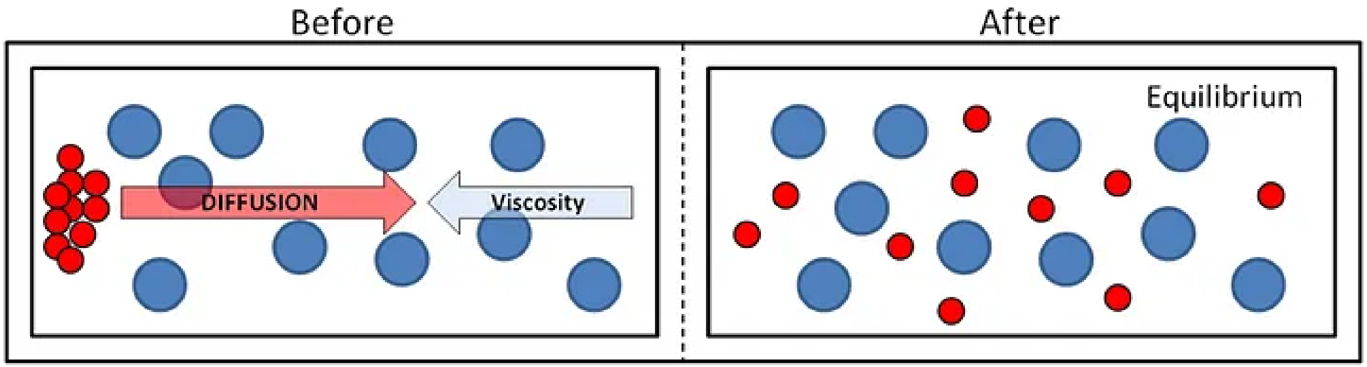
Einstein's fluctuation theory wasn't derived directly from his diffusion formulation. Rather, it arose from a deeper understanding of the connection between diffusion and the underlying randomness of the molecular motion.

The first inkling came when Einstein was in the process of formulating his 5th paper in 1904 — *General Molecular Theory of Heat.* In this paper, he recognized that when he had developed his version of the diffusion equation, which describes how the concentration of particles spread out over time due to random motion, he was able to relate the diffusion coefficient (D) to the mean square displacement $<\Delta x^2>$of the particles over a given time τ.

$$D = \frac{\langle \Delta x^2 \rangle}{2\tau}$$

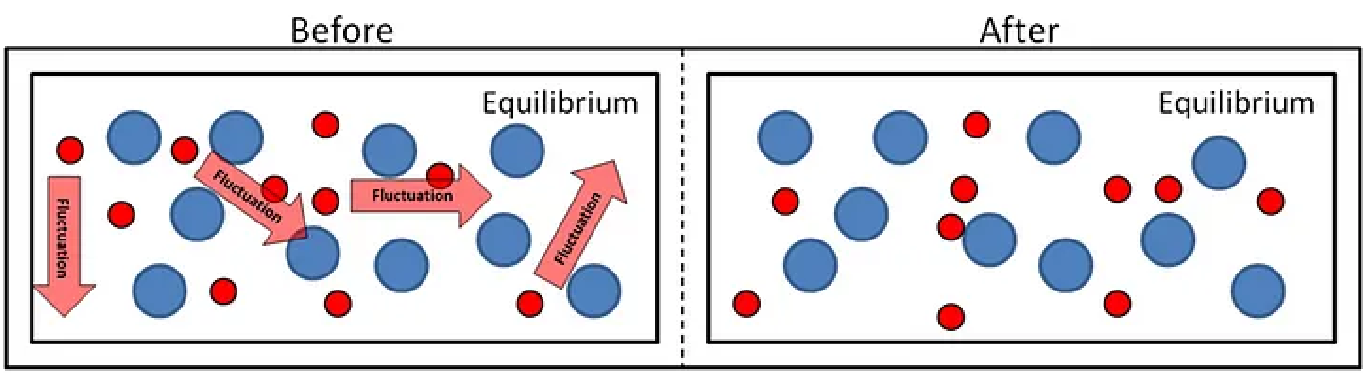
Diffusion coefficient as a function of mean square displacement and time: image by the author

Einstein then realized that diffusion is a process driving a system towards equilibrium. At equilibrium, the concentration of particles is uniform throughout the fluid. Diffusion results from the random motion of particles induced by thermal energy.



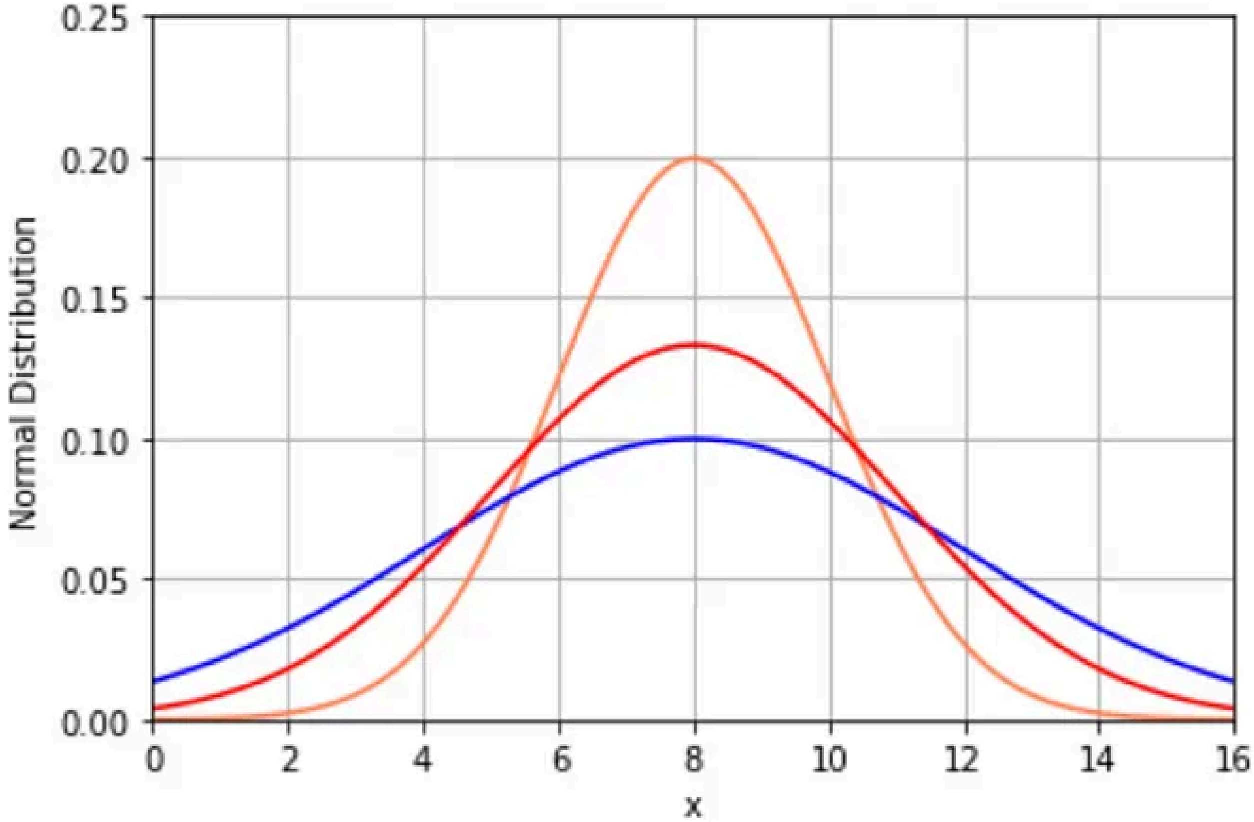
Diffusion is driving force toward Equilibrium: Image by the author

However, this thermal energy doesn't cease when the system reaches equilibrium. At least, that's what the equation indicated to him. So, what exactly is this thermal motion doing at the equilibrium state of the system? It remains the source of the random motion of these particles, manifesting as fluctuations in the displacement, even in a state of equilibrium.



At Equilibrium, thermal energy continue to drive random motion of the particles in the form of fluctuation — Image by the author

Einstein used statistical mechanics to link these fluctuations to the diffusion coefficient. He argued that the probability of a fluctuation is related to the work required to create it. This work, in turn, is associated with the free energy of the system. Consequently, the greater the available free energy, the larger the fluctuation (variance).



In statistical mechanics, the probability of a system being in a particular state is related to the energy of that state. This relationship is described by the Boltzmann distribution:
P(E) ∝ exp(-E/k_B*T)

- where P(E) is the probability of the system having energy E
- k_B is Boltzmann's constant
- T is the temperature

A fluctuation in a system can be considered as a deviation from the equilibrium state, corresponding to a change in the system's energy. Work, being a form of energy transfer, is often require to create such a fluctuation. This work moves the system away from its equilibrium state.

Through this reasoning, Einstein arrived at his famous formula for ***energy fluctuations***:

$$\langle E^2\rangle - \langle E\rangle^2 \equiv \langle(\Delta E)^2\rangle = (k\_B * T^2) / (\partial\langle E\rangle/\partial T)$$

Where:

- $\langle(\Delta E)^2\rangle$ is the mean square fluctuation in the energy of the particles.
- $\langle E\rangle$ is the average energy of the particles.

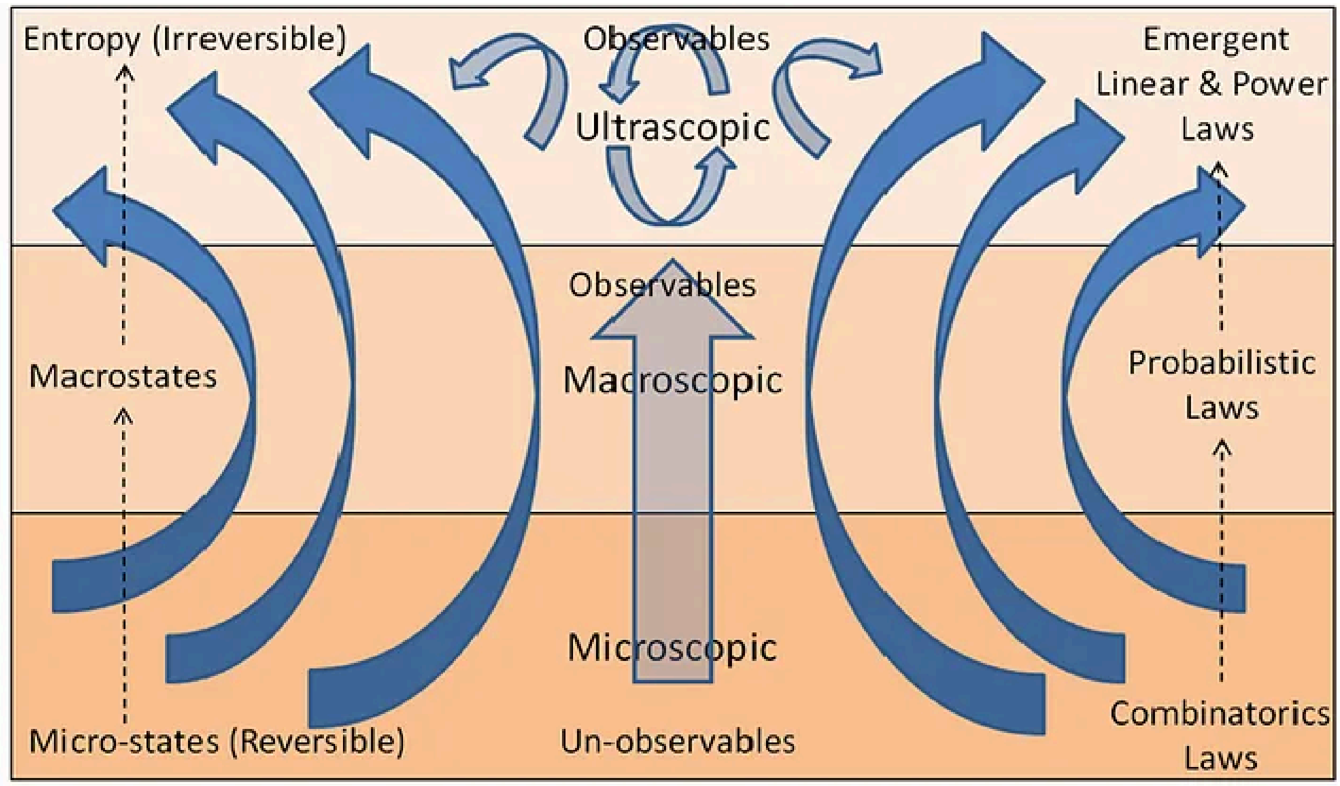
Einstein's physical insight led him to determine that term $\langle(\Delta E)^2\rangle$ is a measure for the thermal stability of the system. The larger the fluctuations, the lower the system's degree of stability. Thus, the Boltzmann constant *k_B* also determines the thermal stability of the system.

Furthermore, Einstein noticed something peculiar about the fluctuation equation he derived. He observed that….

> *… it does not contain any quantities which remind one of the assumptions on which the theory is based.*

In other words, the relationship he had just found, which started out from his diffusion theory, does not contain any quantity reminiscent of the assumptions on which the theory is based — such as diffusion, viscosity, and molecular radius. Somehow, along the way from these microscopic events to the macroscopic observations to the ultrascopic phenomenon, these parameters *washed away* in the aggregate form of the emergent patterns we observe in nature (e.g. snowflakes, stock markets).

This observation intrigued Einstein, as it suggested a deeper, more fundamental principle underlying the phenomenon. The equation's generality implied that the relationship between fluctuations and equilibrium could be universally applied across various systems, regardless of the specific details of their microscopic interactions."



Microscopic to Macroscopic states — Image by the author

There is a certain aesthetic eloquence to this mathematical generality. It means this conceptual framework of fluctuation can be applied to any systems undergoing transitions between non-equilibrium and equilibrium. Notably, Einstein developed not just a theory for a specific phenomenon but a theoretical framework, akin to the *Fourier Transform*, *Z-Transform*, *Brouwer Fixed-Point Theorem*, or *Noether's Theorem*.

These analytical tools, including the Fluctuation-Dissipation Theorem, provide profound insights into the behavior of physical systems and are indispensable for solving complex problems in various scientific disciplines.

From 1905 to 1924, Einstein's prolific output of scientific papers revealed multiple fields where the fluctuation theory could be applied. His groundbreaking achievement provided deep insights into the nature of macroscopic and ultrascopic states of matter and laid the foundation for numerous areas of physics, chemistry, material science, economics, and even modern machine learning models.

For example, he discovered that *concentration* fluctuations follow a similar form:

$$\langle(\Delta N)^2\rangle = \bar{N} * (k\_B * T) / (\partial\mu/\partial N)$$

Where:

- $\langle(\Delta N)^2\rangle$ is the mean square fluctuation in the number of particles
- $\bar{N}$ is the average number of particles
- μ is the chemical potential

This formula demonstrates that the magnitude of fluctuations is proportional to the temperature and the compressibility of the system. It provides a fundamental link between microscopic fluctuations and macroscopic thermodynamic properties.

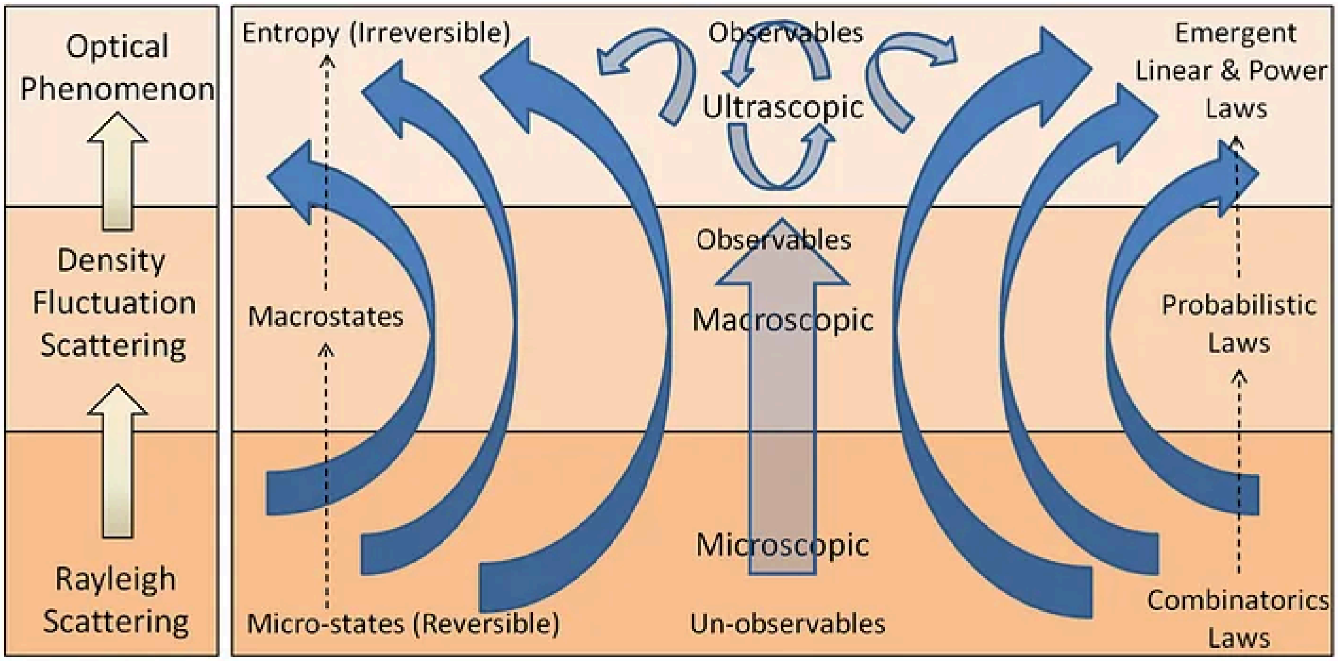
Einstein would later discover other fluctuation formulations in different areas, including:

up to the development of quantum theory.

- **Electrical charge fluctuation:** Leading up to his patented invention of an electrostatic induction machine designed to amplify small voltages.
- **Density fluctuation:** Explaining critical opalescence, a phenomenon observed near the critical point of a substance.

Do you recall at the beginning of the article where I mentioned about Einstein's cryptic remark to Smoluchowsk about there being **one and only one cause of scattering:** Rayleigh scattering and density fluctuation?

Now, we are ready to unravel this puzzling statement. What Einstein meant was that the different scattering mechanisms are the same underlying phenomenon, viewed from different levels. In other words, from microscopic frame of reference of the to the macroscopic or ultrascopic frame of reference. Rayleigh scattering occurs at the microscopic level of individual molecules, while density fluctuation scattering happens at a more aggregate level of macroscopic or ultrascopic phenomenon (see figure below).



Rayleigh scattering viewed at microscopic level vs. Density fluctuation scattering at the macroscopic level: Image by the author

Concluding Remarks

Many decades later, researchers began to understand why this fluctuation theory, now known in its more general form as the *Fluctuation-Dissipation Theorem* (FDT)*,* is so pervasive in nature.

Examples include:

- ***Financial Market Fluctuation***: The random walk theory is a well-known application of fluctuation, explaining the random movement in stock prices, currency exchange rates, and more. At the individual level, fluctuations results from market dynamics of investor behavior, market indicators, and economic news.
- ***Population Dynamics***: In ecology, fluctuation theory is used to understand population fluctuations and cycles observed in many species. These fluctuations can be influenced by factors like predation, food availability, and environmental conditions.

- ***Solid State Physics***: The Fluctuation-Dissipation theorem is used to relate the fluctuations in a system to tis response to external perturbations. This theorem helps in understanding various properties of materials, such as electrical conductivity and thermal properties.
- ***Diffusion Model of AI***: In the context of AI, fluctuation theory can be applied to *diffusion models* used in machine learning. These models simulate the process of diffusion to generate new data points, such as images or text. Fluctuation theory helps in understanding the *variability* and *stability* of the generated data.

All the above examples involve systems transitioning between equilibrium states. The Fluctuation-Dissipation Theorem (FDT) establishes a link between two phenomena in a system at thermal equilibrium: **fluctuations** and **responses to external perturbations**.

- **Fluctuations**: These are random variations in the properties of a system due to thermal energy, even when the system is in equilibrium. For example, small, random changes in the position of particles within a fluid.
- **External Perturbations**: These are external forces or influences applied to the system that disturb its equilibrium state. For example, applying an external electric field to a fluid.

> *The FDT essentially states that the same underlying mechanisms that cause these spontaneous fluctuations (random changes) in the system are also responsible for how the system responds when external forces are applied. In other words, by studying the natural, random fluctuations within a system, we can predict how the system will react when an external force is introduced.*

In simpler terms, it's like saying that the way a system "jitters" on its own is related to how it "shakes" when you poke it.

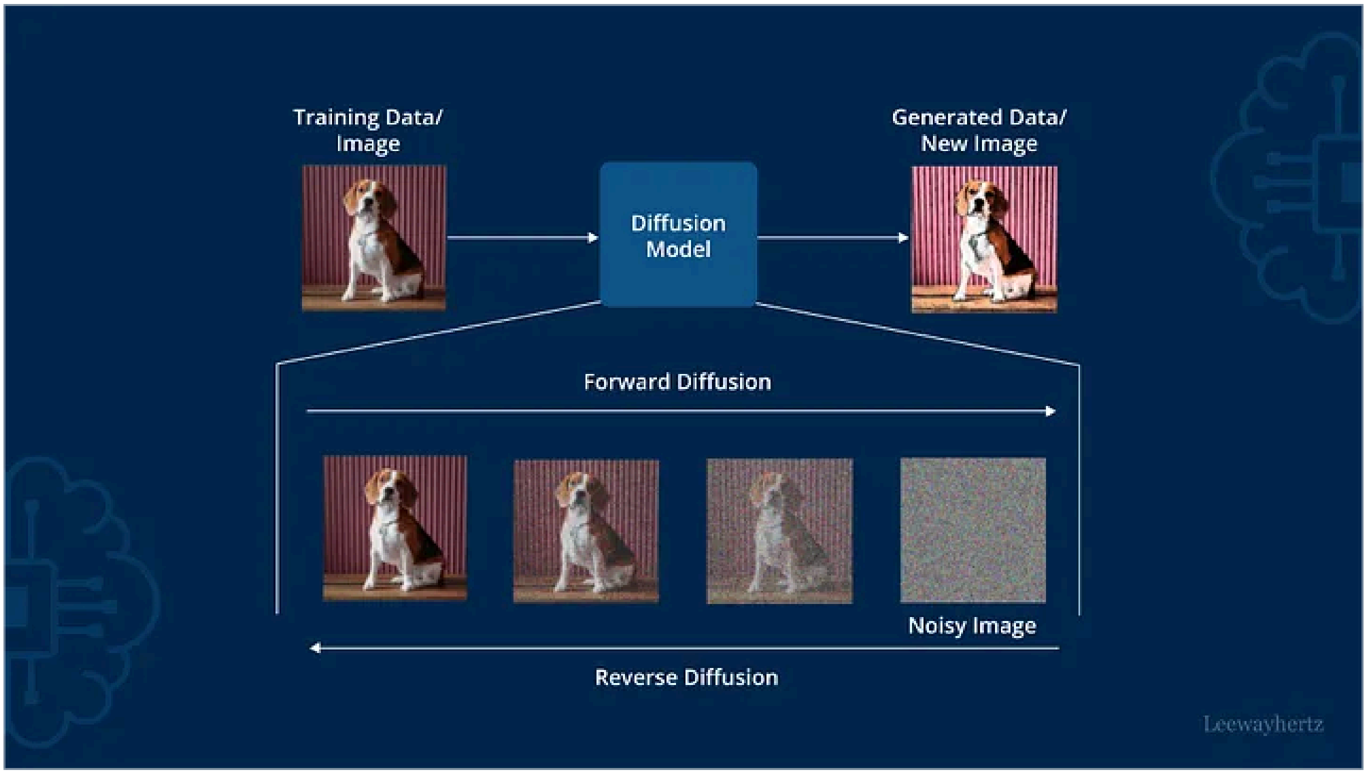
Appendix on the AI Diffusion Model:

Now, let us explore how Einstein's fluctuation theory can be applied in the field of AI.

Diffusion models in machine learning are a type of generative model that generate data by gradually transforming noise into coherent data through a process called denoising. The process involves two main steps:

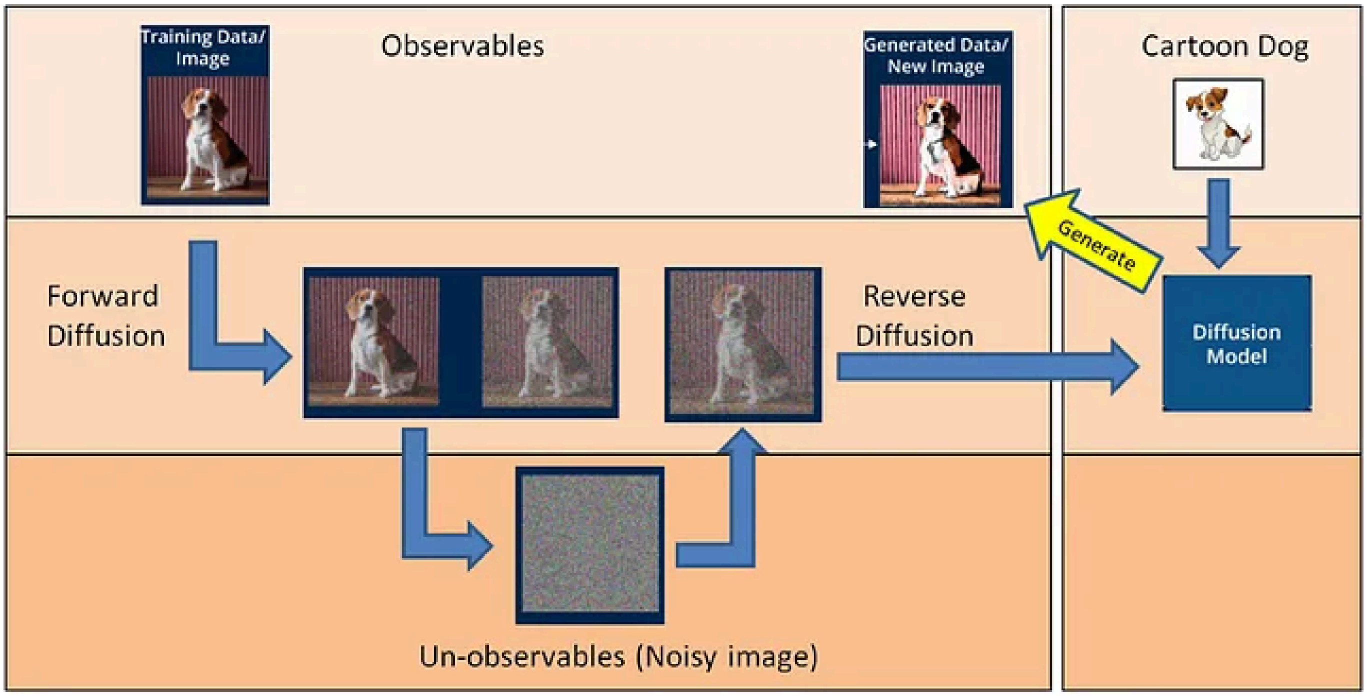
1. **Forward Process**: This involves progressively adding noise to the data until it becomes nearly random. The key word here is '*nearly,*' not 'totally' random. The objective is to model the distribution of noisy data while keeping the essential core data intact.
2. **Reverse Process**: This step involves learning to reconstruct the original data from the noisy data by reversing the noise addition process. This is achieved through a denoising algorithm that gradually removes noise to generate coherent data.

By applying fluctuation theory, we can better understand the variability and stability of the generated data, thus enhancing the robustness and reliability of diffusion models in AI.



In a seminal 2015 paper named "The Deep Unsupervised Learning using Nonequilibrium Thermodynamics," Sohl-Dickstein et al. first introduced diffusion models in deep learning: Image credit LeewayHertz

These models have gained popularity for their ability to generate new high-quality images, text, and other types of data from existing data. Let us now see how this is possible through the lens developed from Einstein's theory of diffusion and fluctuation — illustrated in the diagram below.



AI diffusion model of transformation by stochastic processes from one layer o another : Image by the author

The forward diffusion process adds noise to the image, moving it from the observable top layer to the unobservable bottom layer. However, the image with added noise retains its core essentials. This preservation allows the process to be reversed, resulting in a core essential image combined with the cartoon dog characteristics. Consequently, the new image is generated based on the core original image, now enhanced with cartoon characteristics.

References:

[1] Abraham Pais, *Subtle is the Lord: The Science and the Life of Albert Einstein*, Oxford University Press, 1982.

[2] Luis Navarro Veguillas, *The Lesser-Known Albert Einstein*, Springer Nature, Oct 2023.

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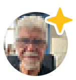

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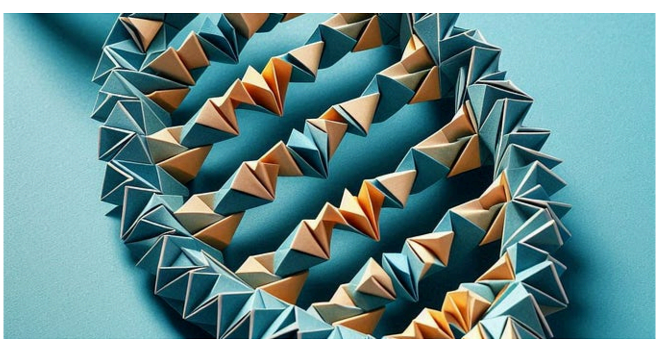

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