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In a previous paper, we showed that many important quantum information-theoretic phenomena, including the no-cloning and no-broadcasting theorems, are in fact generic in all non-classical probabilistic theories. An exception is teleportation, which most such theories do not support. In this paper, we investigate which probabilistic theories, and more particularly, which composite systems, *do* support a teleportation protocol. We isolate a natural class of composite systems that we term *regular*, and establish necessary and sufficient conditions for a regular tripartite system to support a conclusive, or post-selected, teleportation protocol. We also establish a sufficient condition for deterministic teleportation that yields a large supply of theories, neither classical nor quantum, that support such a protocol.

The standard quantum teleportation protocol [7] allows two parties, Alice and Bob, to transmit an unknown quantum state from Alice's site to Bob's; in compliance with the no-cloning theorem, Alice's copy is destroyed in the process. The protocol assumes that Alice and Bob have access to the two wings, A and B, of a bipartite system $A \otimes B$ in a maximally entangled state, which serves as a kind of quantum channel. The state to be teleported belongs to an auxiliary system A' at Alice's site, which is coupled to her half of the shared system. Alice measures an observable corresponding to the Bell basis on the combined system $A' \otimes A$. Depending upon the result, she instructs Bob (via purely classical signaling) to perform a particular unitary correction on his wing, B, of the shared $A \otimes B$ system. With certainty, Alice now knows that the state of Bob's system is identical to the state (whatever it was) of her ancillary system A'.

The possibility of teleportation is surprising, in view of the no-cloning and no-broadcasting theorems, which prohibit the copying of quantum information. In a previous paper [3], we have shown that both no-cloning and no-broadcasting theorems are in fact quite generic features of essentially any *non-classical* probabilistic theory, and not specifically quantum at all. However, as pointed out in [3, 4], most such theories *do not* allow for teleportation. Classical theories, however, do. The possibility of teleportation can thus be regarded, in some very rough qualitative sense, as a measure of the relative *classicality* (or at any rate, *tameness*) of quantum theory.

In this note, we make some precise statements about *which* probabilistic theories—and more particularly, which tripartite systems—admit teleportation. For simplicity, consider the case in which the three component systems, A', A and B, in the protocol described above are identical. Then an obvious necessary condition for the protocol to succeed is that the cone of unnormalized states in A be isomorphic to the *dual* cone of unnormalized *effects* in A^*—a strong condition that is nevertheless satisfied by both quantum and classical systems. As we shall see, this is sufficient to ground conclusive (or one-outcome post-selected) teleportation. To obtain deterministic teleportation appears to be more difficult; however, where the state space has sufficient symmetry, a sort of deterministic teleportation can always be achieved with respect to a possibly continuously-indexed observable. Specializing to the case in which the state space is symmetric under the action of a finite group, we obtain a wealth of examples of state spaces that are neither classical nor quantum-mechanical, but nevertheless support a genuine deterministic teleportation protocol.

1. Probabilistic Models This section assembles the necessary machinery of generalized probability theory—essentially, the convex sets framework deriving from the work of Mackey [17] and subsequently refined by many authors, notably Davies and Lewis [8], C. M. Edwards [9] and Ludwig [16]. We use more or less the same notation as in [4, 5]; as in the latter, in this paper we consider only probabilistic models having finite-dimensional state spaces.

Abstract State Spaces We model a physical system by an ordered vector space A with a (closed, pointed, generating) positive cone A_+, which we regard as consisting of un-normalized "states". We also posit a distinguished order unit, that is, a linear functional u_A that is *strictly* positive on non-zero positive elements of A; this defines a compact convex set $\Omega_A = u_A^{-1}(1)$ of *normalized* states. We shall call an ordered linear space, equipped with such a functional—more formally: a pair (A, u_A)—an *abstract state space*. If (A, u_A) and (B, u_B) are abstract state spaces, we write $A \leq B$ to indicate that (i) A is a subspace of B; (ii) $A_+ \subseteq B_+$; and (iii) u_A is the restriction of u_B to A_+. Similarly, $A \simeq B$, read "A is isomorphic to B", means that there exists an invertible, positive linear mapping $A \to B$, with a positive inverse, and taking the order unit of A to that of B. Equivalently, such a mapping takes A's normalized state space Ω_A bijectively (and affinely) onto B's normalized state space Ω_B. We refer to an isomorphism $A \to A$ as a *symmetry* of A. A positive linear mapping with positive inverse, but that does not necessarily preserve the order unit, we refer to as an *order isomorphism* between A and B, and we say they are order-isomorphic if such a map exists.

By way of illustration, discrete classical probability theory concerns the case in which A is the space $\mathbb{R}^E$ of all real-valued functions α on a finite set E of measurement outcomes, in the natural point-wise ordering. The order unit is the functional $u_A(\alpha) := \sum_{x \in E} \alpha(x)$, hence the normalized state space Ω_A consists of all probability weights on E. In elementary quantum probability theory, A is the space of Hermitian operators on a complex Hilbert space $\mathbf{H}$, ordered in the usual way; the order unit is the trace, so that Ω_A is the set of density operators.

Physical events (e.g., measurement outcomes) associated with an abstract state space A are represented by *effects*, that is, positive linear functionals $f \in A^*$ with $f(\alpha) \leq 1$ for all $\alpha \in \Omega_A$, or, equivalently, $f \leq u_A$. The understanding is that $f(\alpha)$ represents the *probability* that the event in question will occur when the system's state is α. As indicated above, we wish to restrict our attention here to cases in which the space A is finite-dimensional. Thus we may identify A with A^{**}, so that, for $\alpha \in A$ and $f \in A^*$, we may write $f(\alpha)$ as $\alpha(f)$ whenever it suits us. In the sequel, we shall continue always to denote states by lower case Greek letters, and effects, by lower case Roman letters.

It is helpful to note that the set Ω_A of normalized states actually determines both the ordered space A and the order-unit u_A: one can take A to be the dual of the space of affine real-valued functionals on Ω_A, ordered by the cone of non-negative affine functionals; u_A is simply the constant affine functional on Ω_A with value 1. When describing a particular abstract state space, it is often easiest simply to specify the convex set Ω_A. When we wish to begin with a convex set Ω and reconstruct A in this way, we write $A = A(\Omega)$.

Note that the point-wise ordering of functionals in A^* on Ω_A is exactly the usual dual ordering. There is a natural norm on A^*, namely the supremum norm $\|f\| = \sup_{\alpha \in \Omega_A} |f(\alpha)|$; this gives rise in turn to a norm on A, called the *base norm*, with respect to which $\|\alpha\| = u_A(\alpha)$ for $\alpha \in A_+$. In particular, every normalized state has norm 1, and conversely, a positive element of A having norm 1 is a normalized state (so that the two meanings of "normalized" coincide). In the sequel, we shall write $\widetilde{\alpha}$ for the normalized version of a positive weight $\alpha \in A_+$, i.e.,

$$\widetilde{\alpha} := \frac{\alpha}{\|\alpha\|} = \frac{\alpha}{u_A(\alpha)}$$

It will be convenient to stipulate that $\widetilde{0} = 0$.

Observables Let $(X, \mathcal{B})$ be a measurable space: an X-*valued observable* on a state space A is a weakly countably additive vector measure $F : \mathcal{B} \to A^*$ with $F(X) = u$. This guarantees that if $\alpha \in \Omega$, $B \mapsto F(B)(\alpha)$ is a (finitely additive) probability measure on $\mathcal{B}$. If μ is a given measure on $(X, \mathfrak{B})$, we shall call $f : X \to A^*$ a *density* for F with respect to μ iff, for every $\alpha \in \Omega$ and every set $B \in \mathcal{B}$,

$$\int_B f(x)(\alpha) d\mu(x) = F(B)(\alpha).$$

In the simplest case, where X is a finite set and μ is the counting measure, an X-valued observable amounts to a list $(f_1, ..., f_n)$ of effects with $\sum_i f_i = u$. In the sequel, when we speak of *an observable*, without specifying the value space, this is what we have in mind.

Processes and Dynamics We represent physical *processes* involving an initial system with state space A and a final system with state space B by positive linear mappings $\phi : A \to B$ having the property that $\|\phi(\alpha)\| \leq 1$ for all $\alpha \in \Omega_A$, which is just to say that ϕ is norm contractive, or, equivalently, that $\|\phi\| \leq 1$. In this case, we understand that $\|\phi(\alpha)\| = u(\phi(\alpha))$ represents the probability that the process occurs when the input is α; indeed, we can regard the effect $u \circ \phi \in A^*$ as recording precisely this occurrence. Thus, a family $\{\phi_i | i \in I\}$ of positive linear mappings with $\|\phi_i\| \leq 1$ for all i and $\sum_i \|\phi_i\| = 1$, represents a family of physical processes *one* of which is bound to occur. (Such a family is a (discrete) *instrument* in the sense of [8, 11].)

In many cases, one wants to impose some further constraint on the possible dynamics of a system represented by an abstract state space A. By a *dynamical semigroup* for A, we mean a closed, convex set $\mathfrak{D}_A$ of norm-contractive positive linear mappings $\tau : A \to A$, closed under composition and containing the identity mapping Id_A. We understand $\mathfrak{D}_A$ as representing the set of all physically possible processes on A. (Here, "physically" refers to the use of this framework for abstractly formulating possible physical theories; the framework could also have other applications, so the terminology "operationally possible" might be more accurate. With this caveat, however, we will stick with "physically.") A state space equipped with a distinguished dynamical semigroup, we call a *dynamical model*. Note that any abstract state space can be regarded as a dynamical model if we take $\mathfrak{D}_A$ to be (by default) the set of all norm-contractive positive linear mappings $A \to A$. *In the balance of this paper, we take it as a standing assumption, relaxed only where explicitly noted, that this is the case.*

Self-Duality and Weak Self-Duality In both classical and quantum settings, A carries a natural inner product with respect to which there is a *canonical* order-isomorphism

$A \simeq A^*$. Indeed, in both classical and quantum cases, the positive cone is *self dual*, in that

$$A_+ = A^+ := \{\alpha \in V | \forall \beta \in V_+ \langle \alpha, \beta \rangle \geq 0\}.$$

This property is a very special one, not shared by most abstract state spaces. For an example, let A be three-dimensional, with Ω_A a square. For each side of the square, there is an effect taking the value 1 along that side, with the effects corresponding to opposite sides summing to 1. The dual cone thus also has a square cross-section, so that the cones A_+ and A^*_+ are isomorphic. Nevertheless, A_+ is not self-dual, as A^+ is the image of A_+ under a rotation by $\pi/4$.

In this paper, we shall call a finite-dimensional ordered space *weakly self-dual* iff, as in the example above, there exists an order isomorphism (that is, a bijective, positive linear mapping with positive inverse) $\phi : V \simeq V^*$. This is a far less stringent condition than self-duality. A classical result of Vinberg [23] and Koecher [15] shows that any finite-dimensional self-dual cone that is *homogeneous*, in the sense that any interior point can be mapped to any other by an affine symmetry (automorphism) of the cone, and *irreducible* in the sense that the cone is not a direct sum of simpler cones, is either the cone of positive self-adjoint elements of some full matrix $*$-algebra over the reals, complexes or quaternions, or is the cone generated by a ball-shaped base, or is the set of positive self-adjoint 3×3 matrices over the octonions.

Thus, self-duality, plus irreducibility and homogeneity, brings us within hailing distance of Hilbert space quantum mechanics. One might hope to motivate these conditions in operational terms. In this paper, we make some progress in this direction by identifying *weak* self-duality of a system as a necessary condition for a composite of three copies of the system to support conclusive (probabilistic) teleportation, and a condition not much stronger than homogeneity on the space of normalized states of the system to be teleported, as sufficient for the existence of a tripartite model permitting deterministic teleportation.

2. Composite Systems In order to discuss teleportation protocols, it is important to consider composite systems having, at a minimum, three components: one corresponding to the sender ("Alice"), another to the receiver ("Bob"), and a third, accessible to the sender but entangled with the receiver, to serve as a channel across which the sender's state can be teleported. In this section, we review the account of bipartite state spaces given in [4], and extend it to cover systems having three or more components. In doing so, we identify a non-trivial condition on such composites, which we term *regularity*, that will play an important role in our discussion of teleportation protocols in the sequel.

In order to maintain the flow of discussion, the proofs of several results from this section have been placed in a brief appendix.

Bipartite Systems It will be convenient, in what follows, to identify the algebraic tensor product, $A \otimes B$, of two vector spaces A and B with the space of all bilinear forms on $A^* \times B^*$. In particular, if $\alpha \in A$ and $\beta \in B$, we identify the pure tensor $\alpha \otimes \beta$ with the bilinear form defined by

$$(\alpha \otimes \beta)(a, b) = a(\alpha)b(\beta)$$

for all $a \in A^*$ and $b \in B^*$. If A and B are *ordered* vector spaces, we call a form $\omega \in A \otimes B$ *positive* iff $\omega(a, b) \geq 0$ for all positive functionals $a \in A^*$ and $b \in B^*$. Note that if $\alpha \in A$ and $\beta \in B$ are positive, then $\alpha \otimes \beta$ is a positive form. Note, too, that the set of positive forms is a cone in $A \otimes B$.

Definition 1 *The* maximal tensor product *of ordered vector spaces A and B, denoted $A \otimes_{max} B$, is $A \otimes B$, equipped with the cone of all positive forms. Their* minimal tensor product*, denoted $A \otimes_{min} B$, is $A \otimes B$ equipped with the cone of all positive linear combinations of pure tensors.*

The maximal and minimal tensor products are exactly the injective and projective tensor products discussed by Wittstock in [22]; see also [10, 18]. It is not difficult to show that, in our present finite-dimensional setting, $(A \otimes_{max} B)^* = A^* \otimes_{min} B^*$ and $(A \otimes_{min} B)^* = A^* \otimes_{max} B^*$.

If (A, u_A) and (B, u_B) are abstract state spaces representing two physical systems, then subject to a plausible no-signaling condition and a "local observability" assumption guaranteeing that the correlations between local observables determine the gloabl state (see [4]), the largest sensible model for a bipartite system having physically separated components modeled by A and B is $A \otimes_{max} B$, with order unit given by $u^{AB} = u_A \otimes u_B$. Accordingly, we model a *composite system* with components (A, u_A) and (B, u_B) by the algebraic tensor product of A and B, ordered by *any* cone lying between the maximal and minimal tensor cones, and with order unit $u_{AB} = u_A \otimes u_B$. We shall write AB, generically, for such a state space, denoting the convex set $u_{AB}^{-1}(1)$ of normalized states by Ω_{AB}.

It will be important, below, to remember that all states, in whatever cone we use, can be represented as linear combinations of pure product states, as these span $A \otimes B$. Unless the sets of normalized states for A or B are simplices—that is, unless one system at least is classical—the minimal and maximal tensor products are quite different, with the latter containing many more normalized states than the former. These additional states we term *entangled*; states in $A \otimes_{min} B$, we term *separable*.

Marginal and Conditional States Every state ω in a bipartite system AB has natural *marginal states* $\omega^A \in A$

and $\omega^B \in B$, given respectively by

$$\omega^A(a) = \omega(a \otimes u_B) \text{ and } \omega^B(b) = \omega(u_A \otimes b)$$

for all $a \in A^*$ and $b \in B^*$. We also have un-normalized conditional states, given by

$$\omega_a^B(b) = \omega(a, b) = \omega_b^A(a)$$

and their normalized versions,

$$\widetilde{\omega}_a^B(b) = \frac{\omega(a,b)}{\omega^A(a)} \text{ and } \omega_b^A(a) = \frac{\omega(a,b)}{\omega^B(b)}$$

if the marginal states are non-zero, and set equal to 0 otherwise, so that the expected identities $\omega(a,b) = \omega_b^A(a)\omega^B(b) = \omega^A(a)\omega_a^B(b)$ hold. Using these, it is not difficult to show that, just as in quantum theory, the marginals of an entangled state are necessarily mixed, while those of an unentangled pure state are necessarily pure.

Dynamically Admissible Composites It is reasonable to suppose that, if $\tau_A \in \mathfrak{D}_A$ and $\tau_B \in \mathfrak{D}_B$ are physically admissible processes on A and B, respectively, then, for any state ω on a composite system AB,

$$(\tau_A \otimes \tau_B)(\omega) : a, b \mapsto \omega(\tau_A^* a, \tau_B^* b)$$

is a state of AB. When this is the case, let us say that the composite system AB is *dynamically admissible*. Equivalently, AB is dynamically admissible iff for all $\tau_A \in \mathfrak{D}_A, \tau_B \in \mathfrak{D}_B$, AB_+ is stable under $\tau_A \otimes \tau_B$ acting on $A \otimes B$. Note that both minimal and maximal tensor products are stable under any pure tensor of positive operators, so these are dynamically admissible regardless of the dynamics.

Where $\mathfrak{D}_A$ and $\mathfrak{D}_B$ – as per our standing assumption – comprise *all* norm-contractive positive mappings $A \to A$ and $B \to B$, respectively, AB is dynamically admissible iff its positive cone AB_+ is stable under $\tau_1 \otimes \tau_2$ for *all* positive mappings $\tau_1 : A \to A$ and $\tau_2 : B \to B$. Although the minimal and maximal tensor products $A \otimes_{min} B$ and $A \otimes_{max} B$ both enjoy this property, it is highly non-trivial. Indeed, if $A = \mathcal{B}_h(\mathbf{H})$ and $B = \mathcal{B}_h(\mathbf{K})$, the spaces of self-adjoint operators on Hilbert spaces $\mathbf{H}$ and $\mathbf{K}$, and $AB = \mathcal{B}_h(\mathbf{H} \otimes \mathbf{K})$, the usual quantum-mechanical composite state space, then the cone AB_+ is stable only under products of *completely* positive mappings. However, this difficulty is easily met: one need only define a composite of two dynamical models $(A, \mathfrak{D}_A)$ and $(B, \mathfrak{D}_B)$ to be a model $(AB, \mathfrak{D}_{AB})$ where AB is a dynamically admissible composite of A and B, and $\mathfrak{D}_{AB}$ is a semigroup of norm-contractive positive mappings $AB \to AB$ containing all products $\tau_A \otimes \tau_B$ where $\tau_A \in \mathfrak{D}_B$ and $\tau_B \in \mathfrak{D}_B$. In the balance of this paper, results will be formulated for composites of state spaces, rather than of dynamical models; however, these can easily be modified to accommodate the latter.

Bipartite states and effects as operators Elements of the tensor product $A \otimes B$ and of its dual $(A \otimes B)^*$ can be regarded as operators $A^* \to B$ and $A \to B^*$, respectively. Indeed, every $f \in (A \otimes B)^*$ induces a linear mapping $\widehat{f} : A \to B^*$, uniquely defined by the condition that

$$\widehat{f}(\alpha)(\beta) = f(\alpha \otimes \beta).$$

The mapping $f \mapsto \widehat{f}$ is a linear isomorphism. Note also that, if f is positive, then so is $\widehat{f}$ (though not conversely, unless we use the maximal tensor product). Similarly, any $\omega \in A \otimes B$ induces a linear mapping $\widehat{\omega} : A^* \to B$, uniquely defined by the condition that

$$\widehat{\omega}(f)(g) = (f \otimes g)(\omega)$$

for all $f, g \in V^*$. Again, the mapping $\omega \mapsto \widehat{\omega}$ is a linear isomorphism. Also, since elements of the maximal tensor product $A \otimes_{max} B$ are precisely those corresponding to positive bilinear forms, $\widehat{\omega}$ will be a positive operator, regardless of which tensor product we use. In the special case in which ω is a pure tensor, say $\omega = \beta \otimes \gamma$, we have

$$\widehat{(\beta \otimes \gamma)}(f) = f(\beta)\gamma.$$

In the sequel, we shall write $\widehat{AB}$ for the set of operators $\widehat{\omega}$ corresponding to $\omega \in AB$, ordered by the cone of operators $\widehat{\omega}$ with $\omega \in AB_+$. For example, $A \widehat{\otimes_{max}} B$ is simply the space $\mathcal{L}(A, B)$, ordered by the cone of positive operators.

Note that the operator $\widehat{\omega}$ corresponding to a *normalized* state in AB has the property that $\widehat{\omega}(u)(u) = 1$, i.e., $\widehat{\omega}(u)$ is a state. Conversely, given a positive linear mapping $\phi : A^* \to B$ with the property that $\phi(u_A)$ is a state, the bilinear form $\omega(a, b) := \phi(a)(b)$ defines an element of the maximal tensor product, with $\phi = \widehat{\omega}$. It is useful to note ([10], Equation 16) that any positive operator $\phi : A^* \to B$ has operator norm (induced by the above-defined order-unit and base norms on A^* and B) given by

$$\|\phi\| = \|\phi(u)\|_B$$

where $\| \cdot \|_B$ denotes the base-norm on B; hence, bipartite states correspond exactly to positive operators of norm 1.

Similarly, if f is a bipartite effect in $A^* \otimes_{max} A^*$, then the mapping $\widehat{f} : A \to A^*$ takes any state α to the effect $\widehat{f}(\alpha)(\beta) = f(\alpha \otimes \beta)$. Evidently, this is no greater than unity on Ω, so we have $\widehat{f}(\alpha) \leq u$ for all $\alpha \in \Omega$; conversely, any such positive mapping defines a bipartite effect.

Multi-partite Systems Up to a point, the foregoing considerations readily extend to composite systems involving more than two components. Suppose

$(A_1, u_1), ..., (A_n, u_n)$ are abstract state spaces. As above, call an n-linear form on $A_1^* \times \cdots \times A_n^*$ *positive* iff it takes non-negative values on all n-tuples $f = (f_1, ..., f_n)$ of positive functionals $f_i \in A_i^*$. Given states $\alpha_i \in A_{i+}$ for $i = 1, ..., n$, the product state $\alpha_1 \otimes \cdots \otimes \alpha_n$, defined by $(\otimes_i \alpha_i)(f) = \Pi_i \alpha_i(f_i)$, is obviously positive in this sense.

Definition 2 *A* composite *of state spaces (A_i, u_i), $i = 1, ..., n$, is any space A of n-linear forms on $A_1^* \cdots A_n^*$, ordered by any cone of positive forms containing all product states, and with with order-unit given by $u = u_1 \otimes \cdots \otimes u_n$.*

This is equivalent to saying that A contains all product states, and A^* contains all product effects. Examples of composites of, say, three spaces A, B and C would include $A \otimes_{max} B \otimes_{max} C$, $A \otimes_{min} B \otimes_{min} C$, and mixed composites such as $A \otimes_{min} (B \otimes_{max} C)$. Extending the terminology of the previous section, we shall call a composite A of state spaces (A_i, u_i) *dynamically admissible* iff A_+ is stable under mappings of the form $\bigotimes_i \tau_i$ where $\tau_i : A_i \to A_i$ are arbitrary positive mappings. A product of dynamical models $(A_i, \mathfrak{D}_i)$ is a dynamical model $(A, \mathfrak{D})$ where A is a dynamically admissible model of $A_1, ..., A_n$ and $\mathfrak{D}$ is a dynamical semigroup that includes all products of mappings $\tau_i \in \mathfrak{D}_i$.

Regular composites Suppose now that A is a composite of $A_1, ..., A_n$, and that $J \subseteq \{1, ..., n\}$. Given a list of positive linear functionals $f = (f_i) \in \Pi_{i \in I \setminus J} A_i^*$ and a state $\omega \in A_+$, we may define a $|J|$-linear form ω_f^J on $\Pi_{j \in J} A_j^*$ by setting

$$\omega_f^J(g) = \omega(f \otimes g),$$

where $(f \otimes g)_i$ is g_i if $i \in J$ and f_i otherwise. We refer to ω_f^J as a *partially evaluated* state. The set of such partially-evaluated states ω_f^J generates a cone in $\bigotimes_{j \in J} A_j$; together with the order unit $\otimes_{j \in J} u_j$, this defines an abstract state space A^J, which we call the *J-partial sub-system*, and which we take to represent the subsystem corresponding to the set of elementary systems A_j with $j \in J$.

In the simplest cases, we should expect that that a composite of "elementary" systems $A_1, ..., A_n$ can equally be regarded as a composite of complex sub-systems A^J obtained through an arbitrary coarse-graining of the index set $I = \{1, ..., n\}$. This suggests the following

Definition 3 *A composite A of state spaces $A_1, ..., A_n$ is* regular *iff, for all partitions $\{J_1, ..., J_k\}$ of $\{1, ..., n\}$, A is a composite, in the sense of Definition 1, of the partial systems $A^{J_1}, ..., A^{J_k}$.*

Equivalently, A is a regular composite of $A_1, ..., A_n$ iff for all partitions $J_1, .., J_k$ of $\{1, .., n\}$, and for all sequences of states $\mu_k \in A^{J_k}$, the product state $\bigotimes_k \mu_k$ belongs to A, and for all sequences of effects $f_k \in (A^{J_k})^*$, the product effect $\bigotimes_k f_k$ belongs to A^*.

We regard regularity as an eminently reasonable restriction on a model of a composite physical system, at least in cases in which the components retain their separate identities (so that the systems are "separated"). As we shall see in the sequel, regularity is sufficient to ground a weak analogue of a teleportation protocol, which we call *remote evaluation*. In the balance of this section, we collect some examples of regular composites, and adduce some technical results concerning the notion of regularity.

As a matter of notational convenience, we'll write ABC for a composite of three systems A, B and C, denoting by AB, BC, and AC the three bipartite subsystems. In this case, the condition that ABC be regular amounts to requiring that

$$AB \otimes_{min} C \leq ABC \leq AB \otimes_{max} C$$

and similarly A and BC and for AC and B. Equivalently, we require that

$$AB \otimes_{min} C \leq ABC \text{ and } (AB)^* \otimes_{min} C^* \leq (ABC)^*.$$

As an example, let us show that the mixed tensor product

$$A \otimes_{min} (B \otimes_{max} C)$$

is a regular composite of A, B and C. The only interesting coarse-grainings here are $\{\{A, B\}, \{C\}\}$ and $\{\{A, C\}, \{B\}\}$. To analyze the first of these, suppose that $\omega = \sum_i t_i \alpha_i \otimes \mu_i$ where $\alpha_i \in A_+$ and $\mu_i \in (B \otimes_{max} C)_+$. Then for all $c \in C^*$,

$$\omega_c^{AB} = \sum_i t_i \alpha_i \otimes \widehat{\mu}_i(c),$$

a positive linear combination of positive elements of A and B; hence, $\omega_c^{AB} \in (A \otimes_{min} B)_+$, so $AB = A \otimes_{min} B$. It follows that, if $\gamma \in C_+$, we have

$$\begin{aligned} \omega_c^{AB} \otimes \gamma &\in (A \otimes_{min} B \otimes_{min} C)_+ \\ &\leq ((A \otimes_{min} B) \otimes_{max} C)_+ = (AB \otimes_{max} C)_+ \ . \end{aligned}$$

A similar argument applies to the bipartition $\{\{A, C\}, B\}$.

In the next section (see Corollary 1), we'll show that $A \otimes_{max} (B \otimes_{min} C)$ is also regular. An example of a non-regular composite is

$$(A \otimes_{min} A) \otimes_{max} (A \otimes_{min} A)$$

where A is weakly self-dual. This follows from considerations involving entanglement swapping, as discussed in section 6; we postpone further discussion of this example until then.

The following lemma collects a number of facts about composites and regular composites that will be used freely—and often tacitly— in the sequel. (For a proof, see the appendix.)

Lemma 1 *Let A be a composite of systems $A_1, ..., A_n$. Then*

(a) If $K \subseteq J \subseteq \{1, ..., n\}$, then $(A^J)^K = A^K$.

(b) If A is regular, then $(A^J)_+ = \{\omega_u^J | \omega \in A_+\}$.

(c) If A is regular, so is A^J for every $J \subseteq \{1, ..., n\}$.

Probabilistic Theories Roughly, by a *probabilistic theory*, we mean a class $\mathcal{C}$ of probabilistic models—that is, abstract state-spaces—closed under some construction or constructions whereby systems can be composed. Examples would include the class of all classical systems (i.e., systems with simplicial state spaces), the class of all quantum systems with the usual quantum-mechanical state space, the class obtained by forming the maximal tensor products of quantum systems, the convexified version of Spekkens' "toy theory" [21], etc. In principle, this idea might be given a precise category-theoretic formulation (something we expect to pursue in a subsequent paper); here, we content ourselves with a more informal treatment.

Consider a class $\mathcal{C}$ of state spaces equipped with a specific coupling $A, B \mapsto A \circledast B$, where $A \circledast B$ is a composite of A and B. We shall call $\circledast$ *associative* if for all $A, B, C \in \mathcal{C}$, $A \circledast (B \circledast C) \simeq (A \circledast B) \circledast C)$ under the obvious association mapping (defined on product states by $\alpha \otimes (\beta \otimes \gamma) \mapsto (\alpha \otimes \beta) \otimes \gamma$. The straightforward but tedious proof of the following can be found in the appendix:

Proposition 1 *If $\circledast$ is associative, then for all $A_1, ..., A_n \in \mathcal{C}$, $A_1 \circledast \cdots \circledast A_n$ is a regular composite of $A_1, ..., A_n$.*

It follows that composites constructed using only the maximal, or only the minimal, tensor product are regular, as are composite quantum systems. For later purposes, if $\mathcal{C}$ is a class of abstract state spaces closed under an associative coupling $\circledast$ preserving isomorphism, we shall call the pair $(\mathcal{C}, \circledast)$ a *monoidal theory*. (By *preserving isomorphism*, we mean that if $A \simeq B$ and $C \simeq D$, then $(A \circledast C) \simeq (B \circledast D)$.) It is by no means obvious that every sensible theory must be monoidal, however – for instance, we may wish to consider theories in which one can form tripartite systems of the form $A \otimes_{min} (B \otimes_{max} C)$, in which there is maximal entanglement between B and C, but no entanglement at all between A and either B or C. There is certainly precedent for such mixed tensorial constructions, e.g., in Hardy's causaloid framework for quantum gravity [12]. On the other hand, considerations involving entanglement swapping, as spelled out in section 5, place some nontrivial restrictions on non-monoidal theories.

Remark: In the interest of clarity, it will sometimes be helpful in the sequel to adorn an element of a factor in a tensor product with a superscript indicating to which factor it belongs, writing, for instance, $\alpha \otimes \beta$ or $\alpha^A \otimes \beta^B$ for product states in $A \otimes B$, or f^{AB} for an arbitrary bipartite effect in $(A \otimes B)^*$. On occasion, both ornamented and unornamented forms—e.g., ω and ω^{AB}—may occur in the same calculation; when they do, they refer to the same object.

3. Conclusive Teleportation Suppose ABC is a composite of state spaces A, B and C. If f is an effect on AB and ω is a state in BC, then we have positive linear mappings $\widehat{f} : A \to B^*$ and $\widehat{\omega} : B^* \to C$. Their composite, $\widehat{\omega} \circ \widehat{f}$, is a positive operator $A \to C$. If ABC is a *regular* composite, we also have, for any state $\alpha \in A$ and any effect $c \in C^*$, that $\alpha \otimes \omega$ is a state in ABC and $f \otimes c$ is an effect in $(ABC)^*$. We now make a technically trivial but crucial observation:

Lemma 2 *With notation as above, the un-normalized conditional state of $\alpha \otimes \omega$ given an effect $f \in AB$ is*

$$(\alpha^A \otimes \omega^{BC})_f^C = \widehat{\omega}(\widehat{f}(\alpha)).$$

Proof: As pure tensors generate BC, it is sufficient to check this in the case that $\omega = \beta \otimes \gamma$. Then, for any $b \in B^*$, $\widehat{\omega}(b) = \beta(b)\gamma$ (using, here, our convention of identifying a state space with its double dual). Note also that $f(\alpha \otimes \beta) = \beta(\widehat{f}(\alpha))$. Hence, for any $c \in C^*$, $(f \otimes c)(\alpha \otimes \omega) = f(\alpha \otimes \beta)\gamma(c) = \beta(\widehat{f}(\alpha))\gamma(c) = \widehat{\omega}(\widehat{f}(\alpha))(c)$. $\square$

Corollary 1 *For any state spaces A, B and C,*

(i) There is a canonical embedding

$$A \otimes_{min} (B \otimes_{max} C) \leq (A \otimes_{min} B) \otimes_{max} C.$$

(ii) The composite $(A \otimes_{min} B) \otimes_{max} C$ is regular.

Proof: By Lemma 2, any product state $\alpha \otimes \omega$ with $\alpha \in A_+$ and $\omega \in (B \otimes_{max} C)_+$ yields a positive bilinear form on $(A \otimes_{min} B)^* \times C^*$, namely, $(\alpha \otimes \beta)(f, c) = c(\widehat{\omega}(\widehat{f}(\alpha)))$. Hence, we have a natural positive linear mapping $A \otimes_{min} (B \otimes_{max} C) \to ((A \otimes_{min} B)^* \otimes_{min} C^*)^*$; the last is isomorphic to $(A \otimes_{min} B) \otimes_{max} C$. This establishes (i).

To show that $(A \otimes_{min} B) \otimes_{max} C$ is regular, we first observe that $BC = B \otimes_{max} C$. Indeed, let $\mu \in B \otimes_{max} C$, and let $\widehat{\mu}$ be the associated positive operator $B^* \to C$. Let α be some fixed state in A. Given $f \in (A \otimes_{min} B)^* \simeq \mathcal{L}_+(A, B^*)$ and $c \in C^*$, set

$$\omega(f, c) = \widehat{\mu}(\widehat{f}(\alpha))(c) :$$

this is bilinear in f and in c, and positive where both f and c are positive, and so, defines an element $\omega \in (A \otimes_{min} B) \otimes_{max} C$. We now observe that the reduced state $\omega^{BC}_{u_A}$, evaluated on a pair of effects $(b,c) \in B^* \times C^*$, yields

$$\begin{aligned} \omega^{BC}_{u_A}(b,c) &= \omega(u_A, b, c) \\ &= \widehat{\mu}(\widehat{(u_A \otimes b)}(\alpha))(c) \\ &= \widehat{\mu}(b)(c) = \mu(b,c). \end{aligned}$$

Thus, $\omega^{BC}_{u_A} = \mu$. This shows that $B \otimes_{max} C \leq BC$; the reverse inclusion is trivial, so $BC \simeq B \otimes_{max} C$, as claimed. We now have, by part (i), that

$$\begin{aligned} A \otimes_{min} (BC) &= A \otimes_{min} (B \otimes_{max} C) \\ &\leq (A \otimes_{min} B) \otimes_{max} C = ABC. \end{aligned}$$

Obviously, we have $ABC \leq A \otimes_{max} (B \otimes_{max} C) = A \otimes_{max} BC$. The corresponding result for the coarse-graining $\{\{AC\},\{B\}\}$ follows similarly (or by symmetry), and that for $\{\{A\},\{B,C\}\}$ is trivial, so so ABC is regular. □

We can interpret Lemma 2 in information-processing terms as follows. Suppose two parties, Alice and Bob, have access to systems A and B, respectively. Suppose, moreover, that Alice's system consists of two subsystems, A_1 and A_2, with A_1 in an unknown state α. If the total Alice-Bob system is represented by a regular composite $AB = A_1A_2B$, then if f is an effect on A and ω is a known state on A_2B, we may prepare A_1A_2B in the joint state $\alpha \otimes \omega$: if Alice performs a measurement on $A = A_1A_2$ having f as a possible outcome, then, conditional upon securing this outcome, the conditional state of B is, up to normalization, $\widehat{\omega}(\widehat{f}(\alpha))$. Thus, we may say that Alice has evaluated a known mapping, namely $\widehat{\omega} \circ \widehat{f}$, on an unknown input α, simply by securing f as a measurement outcome. In the sequel, we refer to this protocol as *remote evaluation.*

This is obviously reminiscent of a teleportation protocol. Indeed, conclusive teleportation can be regarded as the special case of remote evaluation in which the mapping $\widehat{\omega} \circ \widehat{f}$ is invertible. Suppose that $\eta : A_1 \simeq B$ is a fixed isomorphism between Alice's system A_1 and Bob's system B (allowing us to say what we mean by saying a state of B is the same as a state of A_1). Suppose, further, that the unknown state α is recoverable from the *normalized* conditional state $(\widetilde{\alpha \otimes \omega})^B_f$ by means of a physically admissible process τ, depending on f but not on α: upon securing a measurement outcome corresponding to f, Alice can then instruct Bob to make the correction τ; once this is done, she is certain that the conditional state of Bob's system B — whatever it is — is identical (up to η) to the original, but unknown, state α.

In fact, we can distinguish two situations: one in which the correction operation τ is certain to succeed, and another in which it may fail, but in which this failure will be apparent to Bob. In the latter case, the teleportation protocol has an additional step: Alice must wait for Bob to report the success of the correction. We shall refer to these as *strong* and *weak* conclusive teleportation, respectively. Notice that the standard (one-outcome postselected) quantum teleportation protocol is an instance of a strong teleportation protocol.

We make this language precise as follows. To avoid needess repetition, here and in the balance of this paper A_1A_2B denotes a regular composite of state spaces A_1, A_2 and B with A_1 isomorphic to B by a fixed isomorphism $\eta : A_1 \simeq B$; and f is an effect on A_1A_2 and ω is a state in A_2B.

Definition 4 *We say that the pair (f,ω) is a* conclusive teleportation protocol *on A_1A_2B iff there exists a norm-contractive linear mapping $\tau : B \to B$, called a* correction*, such that, for every normalized state $\alpha \in \Omega_{A_1}$,*

$$\tau((\widetilde{\alpha \otimes \omega})^B_f) = t_\alpha \eta(\alpha)$$

for some constant $t_\alpha > 0$. If τ can be so chosen that $t_\alpha = 1$ for all α, we say that the protocol (f,ω) is strong.

By Lemma 2, the conditional state $(\widetilde{\alpha \otimes \omega})^B_f$ can be expressed as $\widehat{\omega}(\widehat{f}(\alpha))/u(\widehat{\omega}(\widehat{f}(\alpha)))$. Let

$$\mu := \widehat{\omega} \circ \widehat{f} : A_1 \to B,$$

noting that this is a norm-contractive positive mapping. Then (f,ω) is a teleportation protocol iff there exists a norm-contractive positive mapping $\tau : B \to B$ with

$$\tau(\mu(\alpha)) = t_\alpha \|\mu(\alpha)\| \eta(\alpha)$$

for all $\alpha \in \Omega_A$. Notice that $t_\alpha = u(\tau(\widetilde{\mu(\alpha)}))$, i.e., t_α is the probability that the correction τ *succeeds* in the conditional state $\widetilde{\mu(\alpha)})$. Accordingly, a strong protocol is one for which there exists a correction that is certain to succeed.

Theorem 1, below, gives a complete characterization of conclusive teleportation protocols, strong or otherwise, in terms of the mapping $\mu = \widehat{\omega} \circ \widehat{f}$. We require an easy preliminary

Lemma 3 *Let A and B be any abstract state spaces. Let $\phi, \psi : A \to B$ be any two linear mappings with ψ injective. If, for every $\alpha \in \Omega_A$, there is a constant $k(\alpha)$ such that $\phi(\alpha) = k(\alpha)\psi(\alpha)$, then in fact $k(\alpha) \equiv k$, a constant not depending on α.*

Proof: Let α and β be distinct, and hence, linearly independent, elements of Ω_A, and consider $\gamma = (\alpha + \beta)/2$. Then we have

$$\phi(\gamma) = k(\gamma)\psi(\gamma) = (k(\gamma)/2)(\phi(\alpha) + \phi(\beta))$$

and also

$$\phi(\gamma) = (\phi(\alpha) + \phi(\beta))/2 = (k(\alpha)\psi(\alpha) + k(\beta)\psi(\beta))/2.$$

Thus,

$$(k(\alpha) - k(\gamma))\psi(\alpha) + (k(\beta) - k(\gamma))\psi(\beta) = 0.$$

Since ψ is injective, $\psi(\alpha)$ and $\psi(\beta)$ are linearly independent in B; hence, $k(\alpha) - k(\gamma) = k(\beta) - k(\gamma) = 0$, whence $k(\alpha) = k(\beta)$. □

Recall that an *order-isomorphism* between abstract state spaces is a positive linear bijection with a positive inverse, while an *isomorphism* also preserves normalization.

Theorem 1 *Let $\mu := \widehat{\omega} \circ \widehat{f} : A_1 \to B$. Then*

(a) (f, ω) is a conclusive teleportation protocol iff μ is an order isomorphism; in this case $\tau = s(\eta \circ \mu^{-1})$ where $s \leq 1/\|\mu^{-1}\| \leq 1$.

(b) (f, ω) is a strong teleportation protocol iff μ is proportional to an isomorphism; in this case, the correction τ is a symmetry of B.

Proof:

(a) Suppose first that (f, ω) is a teleportation protocol. Then there exists a positive, norm-contractive mapping $\tau : B \to B$ such that, for all $\alpha \in \Omega_{A_1}$, there

$$\tau(\mu(\alpha)) = t_\alpha \|\mu(\alpha)\| \eta(\alpha)$$

for some constant $t_\alpha > 0$. As η is injective, Lemma 4 implies that

$$t_\alpha \|\mu(\alpha)\| \equiv s,$$

a constant independent of α. Note that, as τ is norm-contractive, $s < 1$. Since Ω_{A_1} spans A_1, we have $\tau \circ \mu = s\eta$. It follows that $\tau : B \to B$ is a surjective linear mapping. As we are working in finite dimensions, this implies that τ is invertible; we have

$$\tau^{-1} = \mu \circ \frac{1}{s}\eta^{-1},$$

which is positive. Thus, τ is an order-isomorphism. It follows $\mu = \tau^{-1} \circ s\eta$ is also an order-isomorphism.

For the converse, suppose that μ is an order-isomorphism. Then $\eta \circ \mu^{-1}$ is also an order-isomorphism. Let

$$\tau := s(\eta \circ \mu^{-1})$$

where $s < 1/\|\mu^{-1}\|$. As $\|\eta\| = 1$, we have $\|\tau\| \leq s\|\eta\|\|\mu^{-1}\| < 1$, so τ is norm-contractive. Now $\tau \circ \mu = s\eta$. For all α, let $t_\alpha = s/\|\mu(\alpha)\|$ (noting that $\|\mu(\alpha)\| > 0$, since μ is injective), so that

$$\tau(\mu(\alpha)) = s\eta(\alpha) = t_\alpha \|\mu(\alpha)\| \eta(\alpha).$$

(b) Suppose first that $\mu = k\phi$ for some isomorphism $\phi : A_1 \to B$. Then $\phi : A_1 \to B$ and some positive constant k. Let $\tau = \eta \circ \phi^{-1}$: then $\tau(\mu(\alpha)) = k\eta(\alpha)$ for all α. Since $k = \|\mu(\alpha)\|$ for all α, we have a strong teleportation protocol.

For the converse, suppose (f, ω) is a strong teleportation protocol. Thus, there exists a norm-contractive positive mapping $\tau : B \to B$ such that, for all $\alpha \in \Omega_{A_1}$,

$$\tau(\mu(\alpha)) = \|\mu(\alpha)\|\eta(\alpha).$$

We claim that τ is a symmetry. To see this, let $\Gamma = \mu(\Omega_A) := \{\mu(\alpha) | \alpha \in \Omega_{A_1}\}$, and set $\widetilde{\Gamma} = \{\widetilde{\gamma} | \gamma \in \Gamma\}$; note that this set is a convex subset of Ω_B.[24] Now, $\tau(\tilde{\mu}(\alpha)) = \eta(\alpha) \in \Omega_B$, so τ effects an affine bijection of Γ onto Ω_B. It follows that the affine span of Γ equals that of Ω, whence, that τ preserves the affine span of the latter – which is exactly the hyperplane $u_B^{-1}(1)$. As τ is positive, it also preserves the positive cone B_+, whence, τ preserves $B_+ \cap u^{-1}(1) = \Omega_B$. Thus, τ is a symmetry, as claimed. It remains to show that μ is proportional to an isomorphism. But as we have

$$\tau(\mu(\alpha)) = \|\mu(\alpha)\|\eta(\alpha),$$

we also have

$$\mu(\alpha) = \|\mu(\alpha)\|\tau^{-1}(\eta(\alpha))$$

for all $\alpha \in \Omega_{A_1}$. Invoking Lemma 4, we see that $\|\mu(\alpha)\| \equiv k$, a constant independent of α – whence, $\mu = k\tau^{-1} \circ \eta$. □

Remarks: (1) For Bob to be able to apply the correction mapping τ, the latter must belong to the dynamical semigroup $\mathfrak{D}_B$. Given our simplifying assumption is that $\mathfrak{D}_B$ comprises all norm-contractive positive mappings on B, this is automatic, but in a treatment using more general dynamical models, it would need to be assumed as part of the definition of a teleportation protocol.

(2) If (f, ω) is a teleportation protocol on A_1A_2B, then we can regard it also as a teleportation protocol on $A_1 \otimes_{min} (A_2 \otimes_{max} B)$, as the latter is regular, f is an effect on $A_1 \otimes_{min} A_2$, and ω is a state in $A_2 \otimes_{max} B$. Thus, all teleportation protocols involving regular composites of A_1, A_2 and B live, so to speak, in $A_2 \otimes_{min} (A_2 \otimes_{max} B)$.

One can regard non-strong conclusive teleportation protocols as *inherently inefficient*. The question arises, whether an inefficient protocol can always be replaced with one that is perfectly efficient. We show that this is always possible when the composite is dynamically admissible. (Recall under our standing assumption, a composite is dynamically admissible iff its positive cone is closed under products of positive mappings on the factors.)

Corollary 2 *Suppose A_1A_2B is dynamically admissible. If (f, ω) is a conclusive teleportation protocol with correction τ, then let $\omega' \in A_2B$ be the state defined, for all*

$a \in A_2^*$ and $b \in B^*$, by

$$\omega'(a,b) = \omega(\widetilde{a, \tau(b)}).$$

Then (f, ω') is a strong conclusive teleportation protocol, requiring no correction.

Proof: Since (f, ω) is a conclusive teleportation protocol, there exists a positive mapping $\tau : B \to B$ such that

$$\tau \circ \widehat{\omega} \circ \widehat{f} = s\eta$$

for some constant s. Since A_1A_2B is dynamically admissible, $\omega' \in A_2B$. It is easily verified that $\widehat{\omega}' = (\tau \circ \omega)/\|\tau \circ \omega\|$; hence,

$$\widehat{\omega}' \circ \widehat{f} = \frac{s}{\|\tau \circ \omega\|}\eta.$$

Thus, $\widehat{\mu}' := \widehat{\omega}' \circ \widehat{f}$ is proportional to a symmetry, so $(f, \widehat{\omega})$ is a strong conclusive teleportation protocol, by Theorem 1. Moreover, as the symmetry in question is η itself, no correction is required. □

It follows from Theorem 1 that if a bipartite state ω on A_2B and a bipartite state f on A_1A_2 supply a conclusive teleportation protocol, then the positive linear mappings $\widehat{f}$ and $\widehat{\omega}$ are respectively injective and surjective. We can be somewhat more precise about the geometry of the situation. Let us say that a *compression* on an ordered space V is a positive mapping $P : V \to V$ such that $P^2 = P$. Equivalently, P's range, $P(V)$, is an ordered subspace of V, and $P(\alpha) = \alpha$ for all $\alpha \in P(V)$. As an example, let K be a cube, and let F be a face thereof; the obvious affine surjection $K \to F$ extends to a compression $V(K) \to V(F)$.

Suppose now that (f, ω) is a conclusive teleportation protocol on A_1A_2B with an order-isomorphic correction $\tau : B \to B$, so that

$$\tau \circ \widehat{\omega} \circ \widehat{f} = s\eta$$

for some constant $s > 0$. Then $\widehat{f} : A_1 \to A_2^*$ is an order-embedding, and and $\widehat{\omega} : A_2^* \to B$ is a positive surjection. Let

$$P := \widehat{f} \circ \eta^{-1} \circ \tau \circ \widehat{\omega} : A_2^* \to A_2^* :$$

an easy computation shows that P is a compression in the above-defined sense, with range equal to the image of $\widehat{f}$.

Conversely, suppose we are given an effect f such that $\widehat{f} : A_1 \to A_2^*$ taking A_1 order-isomorphically onto the range of a compression $P : A_2^* \to A_2^*$. Let $\widehat{f}^+ : \mathrm{Ran}(P) \to A_1$ be the inverse of $\widehat{f}$'s co-restriction to $\mathrm{Ran}(P)$, and let $\alpha_o = \widehat{f}^+(u_{A_2})$, i.e, the unique element of A_{1+} such that $\widehat{f}(\alpha_o) = P(u_{A_2})$. Define

$$\widehat{\omega'} := \frac{1}{\|\alpha_o\|}\eta \circ \widehat{f}^+ \circ P.$$

Then $\widehat{\omega'}(u_{A_2}) = \eta(\alpha)/\|\alpha_o\| \in \Omega_B$ (since η is an isomorphism, hence norm-preserving), whence, $\widehat{\omega'}$ corresponds to a normalized state ω' in $A_2 \otimes_{max} B$. The pair (f, ω') gives us a *strong*—and correction-free—teleportation protocol on $A_1 \otimes_{min} (A_2 \otimes_{max} B)$. If A_1A_2B is dynamically admissible, then $\omega' \in A_2B$, and indeed, is precisely the state ω' defined in Corollary 2.

Summarizing:

Theorem 2 *Let A_1, A_2 and $B \simeq A_1$ be abstract state spaces with $B \simeq A_1$. A regular composite A_1A_2B supports a conclusive teleportation protocol iff there exists an effect f on A_1A_2, a state ω in A_2B, and a compression $P : A_2^* \to A_2^*$ such that $\widehat{f}$, co-restricted to Ran(P), is an order-isomorphism $A_1 \simeq$ Ran(P) and $\widehat{\omega}$, restricted to Ran(P), is an order-isomorphism Ran$(P) \simeq B$.*

Corollary 3 *$A_1 \otimes_{min} (A_2 \otimes_{max} A_1)$ supports conclusive teleportation with $\eta(\alpha) = \alpha$ for all α iff $A_1 \leq A_2^*$ is the range of a compression $P : A_2^* \to A_2^*$.*

Proof: Suppose first that we have a compression $P : A_2^* \to A_1^*$: regarding P as a positive surjection $\pi : A_2^* \to A_1$, and letting $\iota : A_1 \to A_2^*$ be the positive inclusion mapping, we have $\pi \in (A_2 \widehat{\otimes_{max}} A_1)$ and $\iota \in (A_1 \widehat{\otimes_{min}} A_2)^*$. As $(\pi \circ \iota)(\alpha) = \alpha$ for all $\alpha \in A_1$, Theorem 1 tells us that $A_1A_2A_1 = A_1 \otimes_{min} (A_2 \otimes_{max} A_1)$ supports conclusive teleportation.

Conversely, if $A_1 \otimes_{min} (A_2 \otimes_{max} A_1)$ supports conclusive teleportation, then by Corollary 2, there exist positive operators $\widehat{\omega} : A_2^* \to A_1$ and $\widehat{f} : A_1 \to A_2^*$ with $\widehat{\omega} \circ \widehat{f} : A_1 \to A_1$ an isomorphism, in which case $P := \widehat{f} \circ \widehat{\omega}$ is a compression. □

We also have

Corollary 4 *Let A_1A_2B be a regular composite of three pairwise isomorphic, weakly self-dual state spaces. If A_2B contains a state ω with $\widehat{\omega} : A_2^* \simeq B$, then A_1A_2B supports conclusive teleportation. In particular, $A_1 \otimes_{min} (A_2 \otimes_{max} B)$ supports conclusive teleportation.*

Remark: As observed above, the standing assumption that for a system A, its dynamical semigroup $\mathfrak{D}_A$ is the set of all positive maps on A, strongly restricts the nature of dynamically admissible tensor products, and is, for example, incompatible with the usual quantum tensor product. However, our definitions and results concerning teleportation are easily adapted to the setting of regular composites of arbitrary dynamical models: as noted above, the definition of a teleportation protocol in that setting requires that the correction mapping τ^B on B belong to the dynamical semigroup $\mathfrak{D}_B$; with this modification, one has one has obvious analogues of Theorems 1 and 2, and of Corollary 2.

4. Deterministic Teleportation As in the previous section, A_1A_2B is a regular composite of three state spaces A_1, A_2 and B, with B isomorphic to A_1. In order for A_1A_2B to support a *deterministic* teleportation protocol, we require a bipartite state $\omega \in A_2B$ and an *observable* $\{f_1, ..., f_n\}$ on A_1A_2 such that for every state α in A_1 and for each i, the state α is recoverable from the conditional state of $\alpha \otimes \omega$ given outcome (effect) f_i.

Definition 5 *Let A_1A_2B be a regular composite of A_1, A_2 and B with $B \simeq A_1$ via a fixed isomorphism $\eta : A_1 \to B$. If ω is a state in A_2B and $E = (f_1, ..., f_n)$ is an observable on $A = A_1A_2$, we shall say that the pair (E, ω) realizes a* deterministic teleportation protocol *iff, for each effect $f_i \in E$, the pair (f_i, ω) realizes a strong conclusive teleportation protocol.*

The idea is that, upon measuring E and obtaining outcome f_i, Alice instructs Bob to apply a suitable correction τ_i; the conditional state of B is then $\eta(\alpha)$. Note that, by Theorem 1, the correction τ_i must be a symmetry of B.

At present, it is not clear to us exactly what conditions on the pair A_1, A_2 will be necessary in order to secure a deterministic teleportation protocol. However, Theorem 2 below provides a wealth of examples of systems which, while weakly self-dual, are neither classical nor quantum, but can nevertheless by combined so as to support a deterministic teleportation protocol. In particular, self-duality is not necessary for deterministic teleportation.

In what follows, let A be an abstract state space carrying an action of a finite group G that preserves the state space Ω. Note that there is a canonical dual action of G on A^* given by

$$(ga)(\alpha) = a(g^{-1}\alpha)$$

for all $g \in G$, $a \in A^*$, and $\alpha \in A$. Note, too, that the order-unit $u = u_A$ is invariant under this action, i.e,. $gu = u$ for all $g \in G$. A state ω is called *G-equivariant* if for all $g \in G$ and all effects $a \in A^*$ we have

$$g\widehat{\omega}(a) = \widehat{\omega}(ga) \ . \tag{1}$$

Theorem 3 *Let A be weakly self-dual, and suppose G is a finite group acting on A, in such a way that (i) G acts transitively on the extreme points of Ω, and (ii) there exists a G-equivariant isomorphism $A^* \simeq A$. Then $A \otimes_{min} (A \otimes_{max} A)$ supports a deterministic teleportation protocol.*

For an example, consider the state space obtained by taking Ω to be a unit square in $\mathbb{R}^3$, displaced one unit from the origin; A_+ is the cone generated by this square base. As observed earlier, with respect to the usual inner product, A^* can be represented as $\mathbb{R}^3$ with cone obtained by rotating A_+ by $\pi/4$. This gives us an order-isomorphism $A^* \to A$ that is equivariant with respect to the the natural action of $\mathbb{Z}_4$ on Ω; as this last is transitive on the vertices of the latter, Theorem 2 tells us that $A \otimes_{min} (A \otimes_{max} A)$ will support a deterministic teleportation protocol. Similar considerations show that the same conclusion holds whenever Ω_A is any regular polygon.

For the proof of Theorem 3, we need an easy lemma.

Lemma 4 *Let A and G be as in Theorem 3. Then there exists a unique invariant normalized state $\omega_o \in \Omega_A$.*

Proof: Notice, first, that there is certainly at least one fixed state, namely $(1/|G|)\omega_o = \sum_{g \in G} g\alpha_o$, where α_o is any one extreme state. To see that there can be no more than one such state, let Γ denote the set of G-fixed points of Ω. Observe that Γ is an affine section of Ω; hence, if Γ contains more than a single point, it contains an affine line, which must intersect the topological boundary of Ω. Let α be a fixed state belonging to this boundary: equivalently, α is fixed, and belongs to a proper face of Ω. Let F be the smallest face containing α: for each $g \in G$, gF is again a face containing α, so $F \subseteq gF$. In other words, F is invariant. But since F is a proper face and G acts transitively on Ω's extreme points, this is impossible. $\square$

Proof of Theorem 3: Let A, G and ω_o be as above. By assumption, there is an equivariant order-isomorphism $\phi : A^* \to A$; normalizing if necessary, we can assume that $\phi = \widehat{\omega}$ for some bipartite state on AB. We claim that $\widehat{\omega}(u) = \omega_o$. Indeed, for all $g \in G$, we have

$$g\widehat{\omega}(u) = \widehat{\omega}(gu) = \widehat{\omega}(u).$$

Thus, $\widehat{\omega}(u)$ is G-invariant; but there is only one invariant state, namely ω_o.

Now, for all $g \in G$, let $f_g \in (A \otimes_{max} A)^*$ correspond to the operator

$$\widehat{f_g} = \frac{1}{|G|}\widehat{\omega}^{-1} \circ g.$$

We claim that $E = \{f_g\}$ is an observable, and (E, ω_o) realizes a strong deterministic teleportation protocol. To see this, note that for every $\alpha \in A$, $\frac{1}{|G|}\sum_{g \in G} g\alpha$ is a G-invariant state, and hence, by Lemma 3, equals ω_o. Thus,

$$\begin{aligned} \sum_{g \in G} f_g(\alpha) &= \sum_{g \in G} \frac{1}{|G|}\widehat{\omega}^{-1}(g\alpha) \\ &= \widehat{\omega}^{-1}\left(\frac{1}{|G|}\sum_{g \in G} g\alpha\right) \\ &= \widehat{\omega}^{-1}(\omega_o) = u \end{aligned}$$

(appealing, in the last step, to the fact that $\widehat{\omega}(u) = \omega_o$). So $\sum_{g\in G} f_g = u$, i.e., $g \mapsto f_g$ is an observable. Moreover,

$$\widehat{\omega}(\widehat{f_g}(\alpha)) = \widehat{\omega}(\widehat{\omega}^{-1}(g\alpha)) = g\alpha.$$

Thus, $\widehat{\omega} \circ \widehat{f_g}$ acts as the group element $g \in G$ – and hence, in particular, has a norm-preserving inverse. □

Remarks: If the group G is compact, we can replace the discrete observable $\{f_g | g \in G\}$ in Theorem 2 by the continuous G-valued density $g \mapsto f_g := \int_G \omega^{-1} \circ g \; d\mu(g)$, where μ is the normalized Haar measure on G. While it is far from clear that we should want to regard this as a "continuously indexed observable" in any literal sense, it may be that discrete, coarse-grained versions of the effect-valued measure $B \mapsto \int_{g\in B} f_g d\mu(g)$ (B ranging over Borel subsets of G) can each underwrite some form of approximate teleportation protocol, of which a deterministic protocol is in some sense the limiting case. We defer exploration of this possibility to a future paper.

Also note that homogeneity of A implies that the group of base-preserving automorphisms of A, which is finite or compact, acts transitively on the extreme points of Ω_A, so homogeneous weakly self-dual state spaces are good candidates for supporting the deterministic teleportation protocol described in Theorem 2, or its continuous analogue.

5. Entanglement Swapping Consider a scenario in which Alice and Bob each possess one wing of two non-local, bipartite systems, say $S_1 = A_1B_1$ and $S_2 = A_2B_2$. We may model this situation by supposing that the total system, S, is a composite of the four components A_1, A_2, B_1 and B_2. We then have, in addition to the two non-local marginal systems S_1 and S_2, two local systems, $A = A_1A_2$ and $B = B_1B_2$ corresponding to Alice and Bob, respectively.

Suppose now that f is an effect on $A = A_1A_2$ and μ and ω are states in $S_1 = A_1B_1$ and $S_2 = A_2B_2$, respectively. We have corresponding positive operators $\widehat{f} : A_1 \to A_2^*$, $\widehat{\omega} : A_2^* \to B_2$, and $\widehat{\mu}^* : B_1^* \to A_1$ (the dual of $\widehat{\mu} : A_1^* \to B_1$). Composing, we obtain a positive operator $\widehat{\omega} \circ \widehat{f} \circ \widehat{\mu}^* : B_1^* \to B_2$, corresponding to a sub-normalized state in $B_1 \otimes_{max} B_2$. The question arises, does this belong to the marginal state space $B = B_1B_2$? Equivalently, can we implement the mapping in question by (un-normalized) conditionalization on the outcome of a measurement on A?

If S is a *regular* composite of A_1, A_2, B_1 and B_2, the answer is yes: $\mu \otimes \omega$ is then a legitimate state on $S = AB$, whence, for all $f \in A^*$, the partially evaluated state $(\mu \otimes \omega)_B(f) = (\mu \otimes \omega)(f \otimes \; - \;)$ lies in B. Now notice the following analogue of Lemma 1 (proved in the same way, i.e,. by checking it on pure tensors):

Lemma 5 *With notation as above,*

$$(f^A \otimes g^B)(\mu^{S_1} \otimes \omega^{S_2}) = g^B(\widehat{\omega} \circ \widehat{f} \circ \widehat{\mu}^*).$$

It follows that

$$\widehat{\omega} \circ \widehat{f} \circ \widehat{\mu}^* = (\mu \otimes \omega)_B(f) \in B,$$

as claimed. This is analogous to the remote evaluation protocol of Section 3: conditional upon Alice securing a measurement outcome corresponding to f^A, the conditional state of Bob's system $B = B_1 \otimes B_2$ corresponds to the operator $\widehat{\omega} \circ \widehat{f^A} \circ \widehat{\mu}^*$. We might call this *state-pivoting*, as one can easily verify that the marginal state of B_1 is undisturbed.

Where the operation $\widehat{\omega} \circ \widehat{f}$ can be reversed, this protocol can be used to transfer the state μ from subsystem S_1 to subsystem B, as in conventional entanglement-swapping Indeed, suppose that (i) $A_1 = B_2$, (ii) there exists a conclusive teleportation protocol for the tripartite system $A_1A_2B_2$— i.e., that we can find a state $\widehat{\omega}$ in S_2 and an effect f in A^* such that $\widehat{\omega} \circ \widehat{f}$ is proportional to the identity operator on A_1. Then, for any $\mu \in S_2$, Lemma 4 tells us that

$$(\mu \otimes \omega)_f = \mu :$$

That is, conditional on the occurrence of f in some measurement by Alice on system A, the state of Bob's system B is μ. In this situation, we may say that μ has been teleported from S_1 *through* ω to B.

The same considerations also allow us to convert an effect f on A into a sub-normalized state on B. Indeed, if S_1 and S_2 contain states η_1 and η_2, respectively, corresponding to order-isomorphisms $\widehat{\eta}_i : B_i^* \simeq A_i$ for $i = 1, 2$, then the mapping $\widehat{f} \mapsto \widehat{\eta}_1 \circ \widehat{f} \circ \widehat{\eta}_2^*$ gives us an order-preserving linear injection from A^* to B. Pursuing this a bit further, let $(\mathcal{C}, \circledast)$ be a monoidal theory, as defined in Section 2. Let us say that a state-space $A \in \mathcal{C}$ is $\mathcal{C}$*-self dual* iff there exists a state $\eta \in A \circledast A$ with $\widehat{\eta} : A^* \to A$ an isomorphism, and $\widehat{\eta}^{-1} : A \to A^*$ corresponding to an effect in $(A \circledast A)^*$. It follows from the above, with $A_1 = A_2$ and $B_1 = B_2$, that if A and B are $\mathcal{C}$-self dual, then so is $A \circledast B$.

Four-part disharmonies The entanglement-swapping protocol described above can be applied negatively, to show that certain four-part composites aren't regular.

Example: Consider any four non-classical state spaces A_1, A_2, B_1 and B_2 with $B_2 \simeq A_1$. If A_1, A_2 and B_2 support a conclusive teleportation protocol (in particular, if all three are isomorphic and weakly self-dual) then the composite

$$S := (A_1 \otimes_{min} A_2) \otimes_{max} (B_1 \otimes_{min} B_2)$$

cannot be regular. Indeed, arguing as in the proof of Corollary 1, we see that the reduced system $B := B_1B_2$

is precisely $B_1 \otimes_{min} B_2 \simeq B_1 \simeq A_1 \otimes_{min} B_1$, while $S_1 := A_1B_1$ is $A_1 \otimes_{max} B_1$. Since A_1 and B_1 are non-classical, we can find an entangled state $\omega \in A_1 \otimes_{max} B_1$. If the composite were regular, we could apply the entanglement-swapping protocol of Lemma 5 to pivot ω to an entangled state on $B = B_1 \otimes_{min} B_2$—which is absurd, as the latter contains no entangled states.

A similar disharmony obtains between the maximal and the usual tensor products of quantum systems [6]. Consider a situation in which two quantum-mechanical systems, represented by state spaces A and B, are coupled by means of the maximal tensor product to form $A \otimes_{max} B$. Suppose also that A and B are themselves composite systems, say $A = A_1 \otimes A_2$ and $B = B_1 \otimes B_2$, where $\otimes$ is the usual quantum-mechanical tensor product. Then an application of Lemma 5 shows that if ω is a maximally entangled state on $A_2 \otimes B_2$ and $\rho \in A_1 \otimes_{max} B_1$ is what we might call an *ultra-entangled* state of $A_1 \otimes B_1$—that is, a state of the maximal tensor product not belonging to A_1B_1—then conditional on a suitable maximally entangled outcome for a measurement on A, one finds that ρ has apparently been teleported through ω, and now resides in $B_1 \otimes B_2$—which is absurd, as the latter is an ordinary composite quantum system hosting no ultra-entangled states.

6. Conclusions and Prospectus We have established necessary and sufficient conditions for a composite of three probabilistic models to admit a conclusive teleportation protocol. We have also provided a class of examples illustrating that deterministic teleportation can be supported by weakly self-dual probabilistic models that are far from being either classical or quantum-mechanical. Along the way, we have developed tools for manipulating regular composites that are likely to be useful in any systematic study of categories of probabilistic models, and particularly categories equipped with more than a single tensor product.

It remains an open problem to find non-trivial necessary and sufficient conditions for a deterministic teleportation protocol to exist. Theorem 3 is a step in this direction; however, one would like a sharp criterion for the existence of a G-equivariant isomorphism $A^* \simeq A$, where G is a finite or, more generally, compact group acting transitively on the extreme points of Ω_A.

Looking further ahead, one would like to consider in detail the categorical structure of probabilistic theories subject to precise axioms governing remote evaluation, teleportation, etc., making contact with the rapidly developing theory of information processing in compact-closed categories [1, 2, 19, 20].

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[24] To spell this out, suppose $\gamma_1, \gamma_2 \in \Gamma$ with normalized versions $\tilde{\gamma}_1 = t_1\gamma_1, \tilde{\gamma}_2 = t_2\gamma_2 \in \tilde{\Gamma}$. Now consider a convex combination

$$\alpha = p\widetilde{\gamma}_1 + q\widetilde{\gamma}_2$$

where $p, q \geq 0$ with $p + q = 1$. Then

$$\alpha = pt_1\gamma_1 + qt_2\gamma_2.$$

Let

$$\gamma = \frac{pt_1}{pt_1 + qt_2}\gamma_1 + \frac{pt_2}{pt_1 + qt_2}\gamma_2 \in \Gamma$$

and note that

$$\alpha = (pt_1 + qt_2)\gamma = \widetilde{\gamma} \in \widetilde{\Gamma}.$$

Appendix: proofs from section 3

Proof of Lemma 1 (a) Let $\mu \in (A^J)_+$ be a positive linear combination $\mu = \sum_p t_p(\omega_p)^J_{a^p}$ of reduced states, where for all p, $\omega_p \in A$ and $a^p = (a^p_i) \in \Pi_{i \in I \setminus J} A^*_i$. Then for any $b = (b_j) \in \Pi_{j \in J \setminus K}$, we have $\mu^K_b = \sum_p t_p(\omega_p)^J_{a^p}(b) = \sum_p (\omega_p)^K_{a^p \otimes b} \in A^K$. It follows that $((A^J)^K)_+ \subseteq (A^K)_+$. For the converse, let $\omega \in A_+$: for any $a = (a_i) \in \Pi_{i \in I} A_i$, we have $a = b \otimes c$ where $b = (b_j) \in \Pi_{j \in J \setminus K} A_j$ and $c = (c_k) \in \Pi_{k \in K} A_k$. Thus, $\omega^K_a = (\omega^J_b)^K_c \in ((A^J)^K)_+$.

For (b), suppose $\omega \in A$ and $a = (a_i)$ in $\Pi_{i \not\in I} A^*_i$. Pick any $c = (b_j) \in \Pi_{j \in J} A^*_j$: we can set

$$\alpha = \omega^J_a \in A_J \text{ and } \beta = \omega^{I-J}_b \in A_{I-J}.$$

If A is regular, we then have $\alpha \otimes \beta \in A$, whence,

$$\alpha = (\alpha \otimes \gamma)^J_{u_{I-J}}.$$

Part (c) follows from (a) and (b). $\square$

Proof of Proposition 1 Let $A = \bigodot_{i \in I} A_i$. We first show that, for any set $J \subseteq I$, $A^J = \bigodot_{j \in J} A_j)$. By assumption, we have

$$A \simeq (\bigodot_{j \in J} A_j) \odot (\bigodot_{k \in I \setminus J} A_k) \geq (\bigodot_{j \in J} A_j) \otimes_{min} (\bigodot_{k \in I \setminus J} A_k).$$

It follows that, for every $\mu \in \bigodot_{j \in J} A_j$, and for any $\nu \in \bigodot_{k \in I \setminus J} A_k$, $\mu \otimes \nu \in A$; hence,

$$\mu = (\mu \otimes \nu)^J_{\otimes_{k \in I \setminus J} u_k} \in A_J.$$

Thus, $\bigodot_{j \in J} A_j \leq A^J$.

For the reverse inclusion, note that we also have

$$(\bigodot_{j \in J} A_j) \odot (\bigodot_{k \in I \setminus J} A_k) \leq (\bigodot_{j \in J} A_j) \otimes_{max} (\bigodot_{k \in I \setminus J} A_k);$$

hence, for any $\omega \in A$ and any $f \in (\bigodot_{k \in K} A^k)^*$—in particular, for any $f = (f_k)_{k \in I \setminus J}$—we have $\omega^J_f \in \bigodot_{j \in J} A_j$. The rest of the proof now proceeds easily. If $J_1, ..., J_m$ is a partition of I, then we have $A = \bigodot_{p=1}^m (\bigodot_{j \in J_p} A_p) = \bigodot_{p=1}^m A^{J_p}$. Since $\odot$ is a coupling, this last is a composite of A^{J_p}, $p = 1, ..., m$. $\square$