1. Expectation of Random Variables

Q: If $R \sim \text{Uniform}(\{v_1, v_2, ..., v_n\})$, compute $\mathbb{E}[R]$.

Range of $R = V = \{v_1, v_2, ..., v_n\}$ and $\Pr[R = v_i] = \frac{1}{n}$. Hence, $\mathbb{E}[R] = \frac{v_1+v_2+...+v_n}{n}$ and the expectation for a uniform random variable is the average of the possible outcomes.

Q: If $R \sim \text{Ber}(p)$, compute $\mathbb{E}[R]$.

Range of R is $\{0, 1\}$ and $\Pr[R = 1] = p$.

$$\mathbb{E}[R] = \sum_{x \in \{0,1\}} x \Pr[R = x] = (0)(1-p) + (1)(p) = p$$

$$\mathbb{E}[R] = p$$

Q: If I_A is the indicator random variable for event A, calculate $\mathbb{E}[I_A]$.

$\text{Range}(I_A) = \{0, 1\}$ and $I_A = 1$ iff A occurs.

$$\mathbb{E}[I_A] = \Pr[I_A = 1](1) + \Pr[I_A = 0](0) = \Pr[A]$$

Hence, for I_A, the expectation is equal to the probability that event A happens.

Q: If $R \sim \text{Geo}(p)$, compute $\mathbb{E}[R]$.

$\text{Range}[R] = \{1, 2, ...\}$ and $\Pr[R = k] = (1-p)^{k-1}p$.

$$\mathbb{E}[R] = \sum_{k=1}^{\infty} k(1-p)^* k - 1)p \Longrightarrow (1-p)\mathbb{E}[R] = \sum_{k=1}^{\infty} k(1-p)^k p$$

$$\Longrightarrow (1-(1-p))\mathbb{E}[R] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p - \sum_{k=1}^{\infty} k(1-p)^k p$$

$$\Longrightarrow \mathbb{E}[R] = \sum_{k=1}^{\infty} (k+1)(1-p)^k - \sum_{k=1}^{\infty} k(1-p)^k + \sum_{k=1}^{\infty} (1-p)^k = 1 + \frac{1-p}{1-(1-p)} = \frac{1}{p}$$

Implies that when tossing a coin multiple times, on average, it will take $\frac{1}{p}$ tosses to get the first heads.

2. Linearity of Expectation.

For two random variables R_1 and R_2, $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$.

In general for n random variables $R_1, R_2, ..., R_n$, and constants $a_1, a_2, ..., a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[R_i]$.

This also means that $\mathbb{E}[aR] = a\mathbb{E}[R]$ for any constant a.

Q: If $R \sim \text{Bin}(n, p)$, compute $\mathbb{E}[R]$.

There are two ways to solve this problem.

1. For a binomial random variable, $\text{Range}[R] = \{0, 1, 2, ..., n\}$ and $\Pr[R = k] = \binom{n}{k} p^k (1-p)^{n-k}$. This leads to a painful computation that ends up showing $\mathbb{E}[R] = \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k}$.
2. The easier way is to use the linearity of expectation: Define R_i to be the indicator random variable that we get a heads in toss i of the coin. Recall that R is the random variable equal to the number of heads in n tosses. Hence,

$$R = R_1 + R_2 + \ldots + R_n$$
$$\Longrightarrow \mathbb{E}[R] = \mathbb{E}[R_1 + R_2 + \ldots + R_n]$$

By linearity of expectation,

$$E[R] = E[R_1] + E[R_2] + \ldots + E[R_n] = \Pr[R_1] + \Pr[R_2] + \ldots + \Pr[R_n] = np$$

Therefore, if the probability of success is p and there are n trials, on average, we expect np of the trials to succeed.