Recap

- **Sample Space S:** A countable set of possible outcomes
- **Outcome $w \in S$:** A possible "thing" that can happen
- **Event E:** A subset of the sample space. E.g., $E = \{6\}$ or $E = \{3, 6\}$

Event Operations

Union of Events

- Since the event E is a set, all set theory operations are available.
- Suppose $E, F \in S$. The union $E \cup F$ consists of outcomes that are either in E or in F (standard union operation). Formally:

$$G = E \cup F = \{w : w \in E \vee w \in F\}$$

I.e., G ocdurs if either E or F ocurrs.

- We can define the union between more than two events in the same way we defined union between more than two sets. $G = E_1 \cup E_2 \cup ... \cup E_n$ "happens" when *at least one* of the events E_i ocurrs.

Intersection of Events

- We define the intersection in a similar way:

$$G = E \cap F = \{w : w \in E \wedge w \in F\}$$

- "G occurs if both E and F occur."

Mutually Exclusive and Complement Events

- **Mutually Exclusive:** If E, F are two events such that $E \cap F = \emptyset$, then E and F are mutually exclusive.
- **Complement:** If E is an event, then E^c is defined such that $E \cup E^c = S$ and $E^c \cap E = \emptyset$.
  - Example: We threw one dice and want to get a 6. $E = \{6\}$ and $E^c = \{1, 2, 3, 4, 5\}$
- Two complement events are mutually exclusive, but two mutually exclusive events are not necessarily complements of each other.
- Subset: If $E \subset F$, then $E \Longrightarrow F$.

Axioms of Probability

- **Probability Function:** A *total function* $P : S \to [0, 1]$

$$\forall w \in S, 0 \leq P[w] \leq 1; \sum_{w \in S} P[w] = 1$$

- **Probability Space:** The outcome space S together with the probability function
- **Union:** For mutually exclusive events, $E_1, E_2, ..., E_n$, (i.e. sets E_i are disjoint), then $P[E_1 \cup E_2 \cup ... \cup E_n] = P[E_1] + P[E_2] + ... + P[E_n]$

There's a proof for this but I'm lazy so I'll just say it's the sum rule.

Back to Throwing Dice

Q: Suppose we throw a standard dice. What is the probability that the number that comes up is 6

$$S = \{1, 2, 3, 4, 5, 6\}$$
$$P[1] = P[2] = P[3] = P[4] = P[5] = P[6]$$
$$P[S] = 1 \Longrightarrow \sum_{w \in S} P[w] = 1$$
$$\Longrightarrow P[6] = \frac{1}{6}$$

Q: Suppose we throw a standard dice, what is the probability that we get either a 3 or a 6?

$$E = \{3\}, F = \{6\}, G = E \cup F = \{3, 6\}$$
$$E \cap F = \emptyset$$
$$\Longrightarrow E \text{ and } F \text{ are mutually exclusive}$$
$$\Longrightarrow P[G] = P[E] + P[F] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Probability Rules

- **Complement Rule:** $P[E] = 1 - P[E^c]$
- **Inclusion-Exclusion Rule:** For any two events E, F, $P[E \cup F] = P[E] + P[F] - P[E \cap F]$
- **Union Bound:** For any two events E, F, $P[E \cup F] \leq P[E] + P[F]$
  - For any events $E_1, E_2, ..., E_n$,

$$P[E_1 \cup E_2 \cup ... \cup E_n] \leq \sum_{i=1}^{n} P[E_i]$$

- **Monotonicity Rule:** For events A and B, if $A \subset B$, then $P[A] < P[B]$