Counting Recap

Product Rule: For sets $A_1, A_2, ..., A_m$, $|A_1 \times A_2 \times ... \times A_m| = \prod_{i=1}^{m} |A_i|$

Sum Rule: If $A_1, A_2, ..., A_m$ are **disjoint** sets, then, $|A_1 \cup A_2 \cup ... \cup A_m| = \sum_{i=1}^{m} |A_i|$

Counting Sequences using the Generalized Product Rule

Question 1: Suppose we have p prizes to be handed amongst the set A of n students. What are the number of ways in which we can distribute the prizes?

For each of the p prizes, we can give it to any of the n students in A. We have n choices for each prize, therefore the number of ways in which we can distribute the prizes is $n \times n \times ... \times n = n^p$.

In other words, what is the number of ways in which we can distribute a set of distint prizes can be distributed amongst another set of distinct students, where a student may be given zero, one or more than one prizes?

Question 2: Suppose we have p prizes to be handed amongst the set A of n students. What are the number of ways in which we can distribute the prizes such that each prize is given to a different student? Assume that $n \geq p$.

For the first prize, we can give it to any of the n students in A. For the second prize, we can give it to any of the $n - 1$ students in A, and so on. We can generalize this to say:

$$n \times (n-1) \times ... \times (n-p+1) = C(n,p) = \frac{n!}{(n-p)!}$$

Generalized Product Rule: If S is the set of length k sequences, such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times ... \times n_k$. If $n_1 = n_2 = ... = n_k = n$, then we recover the product rule.

Counting - Example

Q: A dollar bill is "defective" if some digit appears more than once in the 8-digit serial number. What is the fraction of non-defective bills?

In order to compute the fraction of non-defective bills, we need to compute the quantity

$$\frac{|\text{serial numbers with all different digits}|}{|\text{possible serial numbers}|}$$

For computing |possible serial numbers|, each digit can be one of 10 numbers, hence, using the product, rule |possible serial numbers| = 10^8.

To compute |serial numbers with all different digits|, the first digit can be one of 10 numbers. Once the first digit is chosen, the second one can be chosen in 9 ways, and so on. By the generalized product rule, |serial numbers with all different digits| $= 10 \times 9 \times ... \times 3 = \frac{10!}{2!} = 1,814,400$.

The fraction of non-defective bills is $1,814,\frac{400}{10^8} = 1.8144\%$.

Permutations

A permutation of a set S is a sequence of length $|S|$ that contains every element of S exactly once. Permutations of $\{a, b, c\}$ are $(a, b, c), (a, c, b)$...

Q: Given a set of size n, what is the total number of permutations?

Considering a sequence of length n, the first entry can be chosen in n ways. Since each element can only be chosen once, the second entry can be chosen in $n - 1$ ways, and so on. By the generalized product rule, the number of permutations is $n \times n - 1 \times ... \times 1 = n!$.

Factorial: $n! := n \times (n - 1) \times ... \times 1$. By convention, $0! = 1$.

How big is $n!$? The stirling approximation for $n!$ is $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. This approximation is exponential in n, meaning the order of $n!$ grows exponentially with n.

Q: Which is bigger? $n!$ or $n(n - 1)(n + 2)(n - 3)!$?

$n! = n \times (n - 1)! = n \times (n - 1) \times (n - 2) \times (n - 3)!$

Comparing the two, you can cancel out n, $(n - 1)$ and $(n - 3)!$ from each side, and you're left with $(n - 2)$ vs. $(n + 2)$. Therefore, the right hand side is larger.

Q: In how many ways can we arrange n people in a line?

$n!$, because this question is equivalent to asking "how many permutations of a set of size n are there?"

Division Rule

A k-to-1 function maps exactly k elements of the domain to every element of the codomain.

If $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$. i.e. the number of elements in A is exactly k times the number of elements in B, because there exists exactly k elements that map to each element of B.

Example: E is the set of ears in this room, and P is the set of people. Then f mapping the ears to people is a 2-to-1 function. Hence, $|E| = 2|P|$.

Q: if $f : A \to B$ is a k-to-1 function, and $g : B \to C$ is a m-to-1 function, then what is $\frac{|A|}{|C|}$?

$|A| = k|B|$, and $|B| = m|C|$. Therefore, $\frac{|A|}{|C|} = km$.

Q: if $f : A \to B$ is a k-to-1 function, and $g : C \to B$ is a m-to-1, then what is $\frac{|A|}{|C|}$?

$|A| = k|B|$, and $|C| = m|B|$. $\frac{k|B|}{m|B|} = \frac{k}{m}$.

Arrangements around a Round Table

Q: In how many ways can we arrange n people around a round table? Two seatings are considered to be the same *arrangement* if each person sits with the same person on their left in both seatings.

The number of seatings is $n!$ (permutations of n people vs n seat numbers).

You may create n identical arrangements for each seating by rotating the seat numbers once clockwise. Therefore, the number of seatings is $n \times$ the number of arrangements. You can then say that $f : S \to A$ is an n-to-1 function where S is the set of seat numbers, and A is the set of arrangements. Therefore, $|S| = n|A| \to |A| = \frac{n!}{n} = (n - 1)!$.

You can also think of it as something like this: To create a unique seating (an arrangement), you need to fix one person to kill rotational symmetry. One choice is taken away from us, and therefore we have (n-1) choices left, which is $n - 1!$.