0.1. Recap

$\text{Range}(R)$ is the set of values that R actually takes with positive probability. It is related to V, but some values in V may never occur.

A **distribution** can be specified by its probability desntiy function (PDF, denoted by f).

- **Bernoulli distribution**: $f_{p(0)} = 1 - p$, $f_{p(1)} = p$. Example: when tossing a coin such that $\Pr[\text{heads}] = p$, random variable R is equal to 1 if we get a heads, and 0 otherwise. In this case, R follows the Bernoulli distribution, i.e. $R \sim \text{Ber}(p)$.
- **Uniform distribution**: if $R : S \to V$, then for all $v \in V$, $f(v) = \frac{1}{|V|}$. Example: when tossing an n -sided die, random variable R is the number that comes up on the die. $V = \{1, 2, ..., n\}$. In this case, R follows the Uniform distribution, i.e. $R \sim \text{Uniform}(1, n)$.
- **Binomial distribution**: $f_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$. Example: when tossing n independent coins such that $\Pr[\text{heads}] = p$, random variable R is the number of heads in n coin tosses. In this case, R follows the Binomial distribution, i.e. $R \sim \text{Bin}(n, p)$.
- **Geometric distribution**: $f_p(k) = (1-p)^{k-1} p$. Example: when repeatedly tossing a coin such that $\Pr[\text{heads}] = p$, random variable R is the number of tosses needed to get the first heads. In this case, R follows the geometric distribution i.e. $R \sim \text{Geo}(p)$.

1. Expectation of Random Variables

Recall that a random variable R is a total function from $S \to V$.

Definition: Expectation of R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally,

$$\mathbb{E}[R] = \sum_{\omega \in S} \Pr[\omega] R[\omega]$$

$\mathbb{E}[R]$ is also known as the "expected value" or the "mean" of the random variable R.

- Example: In a class, suppose we uniformly at random choose a student and define a r.v. R equal to the percentage of marks they scored in an exam. The sample space S is the student ID number of the student we picked, and $R : S \to [0, 100]$. $\mathbb{E}[R]$ is the class average.
- An r.v. does not necessarily achieve its expected value. Intuitively, consider doing the "experiment" (throw a dice and record the number) multiple times. This average of the numbers we record will tend to $\mathbb{E}[R]$ as the number of experiments becomes large. So the expectation can be thought of as a "long-term" average.

Alternate definition: $\mathbb{E}[R] = \sum_{x \in \text{ Range}(R)} x \Pr[R = x]$.

- This definition does not depend on the sample space.