# CMPT 210 — Practice Problems for Lectures 17–20

Conditional Expectation, Independence, Variance, Covariance, Correlation

Unless otherwise stated, all random variables are discrete.

## Problem 1: Conditional Expectation Warm-Up (L17)

Let $X$ be the result of rolling a fair six-sided die, so $X \in \{1, 2, 3, 4, 5, 6\}$ and $\Pr[X = k] = \frac{1}{6}$.

(a) Compute $E[X]$ directly from the definition.

(b) Let $A$ be the event "$X$ is even". Compute the conditional pmf $\Pr[X = k \mid A]$ for $k = 1, \ldots, 6$.

(c) Compute $E[X \mid A]$.

(d) Let $B$ be the event "$X \geq 4$". Compute $E[X \mid B]$.

(e) Verify the Law of Total Expectation with the partition $\{A, A^c\}$ by checking

$$E[X] = E[X \mid A]\Pr(A) + E[X \mid A^c]\Pr(A^c).$$

## Problem 2: Law of Total Expectation (L17)

A factory produces light bulbs. Each day, the factory is either in a *good* state (event $G$) or a *bad* state (event $B$). On a randomly chosen day:

$$\Pr(G) = 0.8, \qquad \Pr(B) = 0.2.$$

Let $D$ be the number of defective bulbs in a batch of 100 bulbs produced that day. Assume:

$$D \mid G \sim \text{Bin}(100, 0.01), \qquad D \mid B \sim \text{Bin}(100, 0.10).$$

(a) Express $E[D \mid G]$ and $E[D \mid B]$ in terms of the parameters of the binomial distribution.

(b) Use the Law of Total Expectation to compute $E[D]$.

(c) Suppose the manager only observes the total number of defects $D$, not whether the day was good or bad. Explain in words what $E[D]$ represents and why the Law of Total Expectation is useful here.

**Optional (challenge).** Let $C$ be the number of *non-defective* bulbs in the batch. Express $E[C]$ in terms of $E[D]$ and verify your answer using the binomial parameters.

## Problem 3: Independence vs. Dependence (L18)

We toss three independent fair coins. Let the sample space be

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\},$$

each outcome with probability 1/8.

Define random variables:

$$C = \text{number of heads in the three tosses,}$$

$$M = \begin{cases} 1, & \text{if all three coins match (HHH or TTT),} \\ 0, & \text{otherwise.} \end{cases}$$

(a) List the range of $C$ and of $M$.

(b) Compute $\Pr[C = 3]$ and $\Pr[M = 1]$.

(c) Compute $\Pr(C = 3, M = 1)$.

(d) Are $C$ and $M$ independent? Justify carefully using the definition of independence for random variables.

Now define $H_1$ to be the indicator of the event "first toss is heads":

$$H_1 = \begin{cases} 1, & \text{if the first toss is H,} \\ 0, & \text{otherwise.} \end{cases}$$

(e) Compute $\Pr(H_1 = 1)$, $\Pr(M = 1)$, and $\Pr(H_1 = 1, M = 1)$.

(f) Are $H_1$ and $M$ independent? Justify using the definition of independence for r.v.'s.

## Problem 4: Variance Basics (L18–L19)

Let $X$ be a random variable with $E[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

(a) Using the *definition* of variance

$$\text{Var}(X) = E\Big[(X - E[X])^2\Big],$$

expand the square and show that

$$\text{Var}(X) = E[X^2] - (E[X])^2.$$

(This is the "alternate definition" of variance.)

(b) Let $Y = aX + b$, where $a, b$ are constants. Using the alternate definition $\text{Var}(Y) = E[Y^2] - (E[Y])^2$, prove that

$$\text{Var}(aX + b) = a^2 \,\text{Var}(X).$$

(c) Suppose $Z$ is defined as

$$Z = \begin{cases} -1, & \text{with probability } \frac{1}{2}, \\ +1, & \text{with probability } \frac{1}{2}. \end{cases}$$

Compute $E[Z]$ and $\text{Var}(Z)$ directly from the definitions.

(d) Let $W = 10Z + 5$. Compute $E[W]$ and $\text{Var}(W)$ in two ways:

(i) Directly from the definition of expectation and variance.

(ii) Using the linearity of expectation and the scaling rule you proved in part (b).

Check that the answers match.

## Problem 5: Variance of Sums (L19–L20)

Let $X_1, X_2, \ldots, X_n$ be pairwise independent random variables with finite variance. Define

$$S = \sum_{i=1}^{n} X_i.$$

(a) Show that

$$\mathrm{Var}(S) = \sum_{i=1}^{n} \mathrm{Var}(X_i) \quad \text{if the } X_i \text{ are pairwise independent.}$$

(Hint: first prove the formula for $n = 2$: $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2)$, then generalize.)

(b) Now suppose $X_i$ are independent Bernoulli random variables with parameters $p_i$, i.e. $\Pr(X_i = 1) = p_i$, $\Pr(X_i = 0) = 1 - p_i$. Let $C = \sum_{i=1}^{n} X_i$ be the total number of "successes".

(i) Express $E[C]$ in terms of the $p_i$.

(ii) Express $\mathrm{Var}(C)$ in terms of the $p_i$.

(c) In the special case $p_i = p$ for all $i$, identify the distribution of $C$ and rewrite your expressions for $E[C]$ and $\mathrm{Var}(C)$ using the standard formulas for that distribution.

## Problem 6: Covariance and Correlation (L19–L20)

Let $X$ and $Y$ be random variables. Recall

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y],$$

and

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

(a) Show that $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.

(b) Suppose $X$ and $Y$ are independent. Prove that $\mathrm{Cov}(X, Y) = 0$ using the definition above.

(c) Let $A$ and $B$ be events, and define indicator random variables

$$I_A = \begin{cases} 1, & \text{if event } A \text{ occurs,} \\ 0, & \text{otherwise,} \end{cases} \qquad I_B = \begin{cases} 1, & \text{if event } B \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Show that

$$\mathrm{Cov}(I_A, I_B) = \Pr(A \cap B) - \Pr(A)\Pr(B).$$

(d) Using part (c), argue what it means (in terms of probabilities of $A$ and $B$) if $\mathrm{Cov}(I_A, I_B) > 0$, and what it means if $\mathrm{Cov}(I_A, I_B) < 0$. (Give a short, intuitive explanation.)

**Optional (correlation extremes).**

(e) Let $Y = X$. Show that $\mathrm{Corr}(X, X) = 1$.

(f) Let $Y = -X$. Assuming $\mathrm{Var}(X) > 0$, show that $\mathrm{Corr}(X, -X) = -1$.