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Textbooks:
1. https://people.csail.mit.edu/meyer/mcs.pdf
2. Introduction to Probability and Statistics for Engineers and Scientists (Ross).

Sets

Informal definition: An unordered collection of distnict objects (elements). Examples:
$\{a, b, c\}, \{\{a, b\}, \{c, a\}\}, \{1.2, 2.5\}, \{\text{yellow}, \text{red}, \text{green}\}, \{x \mid x \text{ is a capital of a North American country}\}, \{x|x$

An element may not appear twice: $\{a, a, b\} = \{a, b\}$

Unordered: $\{a, b\} = \{b, a\}$

$C = \{x|x \text{ is a color of the rainbow}\}$

Elements of C: $\{\text{red}, \text{orange}, \text{yellow}, \text{green}, \text{blue}, \text{indigo}, \text{violet}\}$

Membership: red $\in C$, brown $\notin C$

Cardinality: Number of elements: $|C| = 7$

Common Sets

- $\emptyset$: The Empty set
- $\mathbb{N}$: Natural numbers $\{0, 1, 2, ...\}$
- $\mathbb{Z}$: Integers $\{-2, -1, 0, 1, 2, ...\}$
- $\mathbb{Q}$: Set of rational numbers that can be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z} \land q \neq 0 -$ $\{-10.1, -1.2.0, 5.5, 15, ...\}$
- $\mathbb{R}$: Real numbers
- $\mathbb{C}$: Complex numbers

Set Comparison: A is a subset of $B \iff$ every element of A is also an element of B

A is a *proper subset* of $B \iff A$ is a subset of B and $A \neq B$

Set Operations

Union: The union of two sets A, B consists of elements appearing in A or B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$

Intersection: The intersection of two sets A, B consists of elements that appear in both A and B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$

Set Difference: The set difference of A, B consists of all elements that are in A, but not in B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \setminus B = B - A = \{4, 5\}$

Complement: Given a domain (or universe) D such that $A \subset D$, the complement of A consistes of all elememnts that are not in A.

$$D = \mathbb{N}, A = \{1, 2, 3\}, A \subset D \land \overline{A} = \{0, 4, 5, 6, ...\}$$

$$A \cup \overline{A} = D, A \cap \overline{A} = \emptyset, A \setminus \overline{A} = A$$

Q: $D = \mathbb{N}, A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Compute $\overline{A \cap B}, (B \setminus A) \cup (A \setminus B)$

- De Morgan's Law: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Power Set: The power set of A is the set of all subsets of A. If $A = \{a, b, c\}$, then

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The number of elements in $P(A)$ is $2^{\{|A|\}}$

- Proof: Every element of A has two choices: either it is in a subset or it is. Therefore, the number of possibilities is equal to the number of possibilities for each element. $2 \times 2 \times 2 \times \ldots$ n times = 2^n
  - TODO: Find a nice proof of this online to really understand it.
  - This is called the product rule, (google this)

Disjoint Sets: Two sets A, B are disjoint if $A \cap B = \emptyset$

Symmetric Difference: The symmetric difference of two sets A, B consists of all elements that are either in A or in B, but not in both.

Q: Show $A \Delta B$ on a Venn diagram. Q: For $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, compute $A \Delta B$

$$A \Delta B = \{x \mid (x \in A \land x \notin B) \lor (x \in B \land x \notin A)\} = \{1, 2, 4, 5\}$$

Relations

The **Cartesiaon Product** of two sets is a set consisting of ordered pairs *(tuples)*, i.e.

$A \times B = \{(a, b) \mid a \in A \land b \in B\}$ if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

i.e. if sets are 1-dimensional objects, the Cartesian Product of two sets can be thought of as 2-dimensional.

Q: Is $A \times B = B \times A$? — No, because tuples are ordered pairs.

Q: If $|A| = m, |B| = n$, then what is $|A \times B|$? — $m * n$, (find a nice proof of this)

In general, $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$ where $(a_1, a_2, \ldots, a_k)$ is referred to as a k-tuple.

$$|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \times |A_2| \times \ldots \times |A_k|$$

(proof by induction)

Laws of Set Theory

Distributive Property: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$z \in A \cap (B \cup C)$

$\iff z \in A \land z \in (B \cup C)$ — by the definition of intersection

$\iff z \in A \land (z \in B \lor z \in C)$ — by the definition of union

Using the distributivity of $\land$ over $\lor$, for binary literals $w, x, y \in \{0, 1\}, x \land (y \lor w) = (x \land y) \lor (x \land w)$. For $x \coloneqq z \in A, y \coloneqq z \in B, w \coloneqq z \in C$,

(We can only use the properties of binary literals because we can basically equate z being in or not in a set S to assigning a variable x to be 0 or 1)

$\iff (z \in A \land z \in B) \lor (z \in A \land z \in C)$

$\iff z \in (A \cap B) \lor z \in (A \cap C)$

$\iff z \in (A \cap B) \cup (A \cap C)$

Sequences

Examples: $(a, b, a), (1, 3, 4), (4, 3, 1)$

An element may appear twice, and the order of the elements does matter.
- $(a, a, b) \neq (a, b)$
- $(a, b) \neq (b, a)$

Q: What is the size of $(1, 2, 2, 3)$? What is the size of $\{1, 2, 2, 3\}$?
- TODO: answer the question

Sets and Sequences: The Cartesnian product of sets $S \times T \times U$ is a set consisting of all sequences where the first component is drawn from S, the second is drawn from T, and the third is drawn from U.

$$S \times T \times U = \{(s, t, u) \mid s \in S, t \in T, u \in U\}$$

Q: For set $S = \{0, 1\}, S^3 = \{S \times S \times S\}$. Enumerate S^3. What is |S^3|?
- Enumeration: $S^3\} = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$
- $2 \times 2 \times 2 = 8$

Formula:

$$|S \times T \times U| = |S| \times |T| \times |U|$$

$$|S^n| = |S|^n$$

Counting Sets — Using the Sum Rule

Q: Let R be the set of rainy days, S be the set of sunny days, and H be hte set of really hot days in 2023. A bad day can be either rainy, snowy or really hot. What is the number of good days?

Let B be the set of bad days. $B = R \cup S \cup H$, and we want to estimate $|\overline{B}|$. $|D| = 365$. $|\overline{B}| = |D| - |B| = 365 - |R \cup S \cup H|$.

Since the sets R, S, H are disjoint, $|R \cup S \cup H| = |R| + |S| + |H|$.. hence, the number of good days $= 365 - |R| - |S| - |H|$

$$B \cup G = D, |B \cup G| = |D| = 365$$

B and G are disjoint, so we can apply the sum rule:

$$|B| + |G| = |D|$$
$$\Longrightarrow |G| = 365 - |B| = 365 - (|R| + |S| + |H|)$$

Sum Rule: if $A_1, A_2, ..., A_m$ are disjoint sets, $|A_1 \cup A_2 \cup ..., \cup A_m| = \sum_{i=1}^{m} |A_i|$

Counting Sequences — Using the Product Rule

Q: Suppose the university offers Math courses (M), CS courses (C), and Statistics courses (S). We need to pick one course from each subject, Math, CS and Statistics. What is the number of ways we can select the 3 courses?

We can use the cartesian product, so the number of ways we can select three courses is $|M \times C \times S| = |M| \times |C| \times |S|$

Formally:
The number of ways to select the 3 courses is
$=$ the number of sequences of the form $(m, c, s) \mid m \in M, c \in C, s \in S$

$$M \times C \times S = \{(m, c, s) \mid m \in M, c \in C, s \in S\}$$

$$|M \times C \times S| = |M| \times |C| \times |S|$$

Counting — Example 1

Q: What is the number of length n passwords that can be generated if each character in the password is only allowed to be a lowercase letter?

The number of passwords $=$ the number of sequences of length n of the form $(a, c, d, ...) =$ $\{a, b, ..., z\} \times \{a, b, ..., z\} \times ...$ n times. Therefore, the number of passwords of length n is $|\{a, b, ..., z\} \times \{a, b, ..., z\} \times ...| = |\{a, b, ..., z\}|^n = 26 \times 26 \times ... \times 26 = 26^n$

Counting — Example 2

Q: What is the number of passwords that can be generated if
1. The first character is only allowed to be a lowercase letter,
2. Each subseqnet character in the password is allowed to be lower-case letter or digit
3. The password may be between 6 and 8 characters long

Answer: $26 * 36^7 + 26 * 36^6 + 26 * 36^5 = 2,095,636,727,808$

Let $L = \{a, b, ..., z\}$ and $D = \{0, 1, ..., 9\}$. Using the equivalences between sequences and products of sets, the set of passwords of length n is given by $P_n = L \times (L \cup D)^{n-1}$.

Since the total set of passwords $P = P_6 \cup P_7 \cup P_8$, and a password can be either of length 6, 7 or 8, sets P_6, P_7, P_8 are disjoint. Using the sum rule, we can write $|P| = |P_6| + |P_7| + |P_8| = |L| \times [(|L| + |D|)^5 \times (1 + (|L| + |D|) + (|L| + |D|)^2)] = 26 \times 36^5 \times [1 + 36 + 36^2] =$ $2,095,636,727,808$