MACM 316 Lecture 35 - End of Chapter 5

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1 Higher order Runge-Kutta Methods (24.8)

Third order Runge-Kutta methods are not commonly used. However, fourth order Runge-Kutta methods are widely used and derived in a similar fashion. Greater complexity results from having to compare terms through h^4 and this gives a set of 11 equations in 13 unknowns. The set of equations can be solved with 2 unknowns being chosen arbitrarily.

The most commonly used set of values leads to the following algorithm:

$$\begin{aligned}
w_{n+1} &= w_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\
k_1 &= hf(t_n, w_n) \\
k_2 &= hf\left[t_n + \frac{1}{2}h, w_n + \frac{1}{2}k_1\right] \\
k_3 &= hf\left[t_n + \frac{1}{2}h, w_n + \frac{1}{2}k_2\right] \\
k_4 &= hf[t_n + h, w_n + k_3]
\end{aligned}$$

The main computational effort in applying Runge-Kutta methods is the evaluation of f. In the second order methods, the local truncation error is $\mathcal{O}(h^2)$ and the cost is two functional evaluations per step. The Runge-Kutta method of order four requires four evaluations per step and the local truncation error is $\mathcal{O}(h^4)$.

We may wonder about higher order formulas...

1.1 Error Analysis of Higher Order Runge-Kutta Methods (24.9)

Butcher has shown that the following relationship holds:

evaluations	2	3	4	$5 \leq n \leq 7$	$8 \leq n \leq 9$	$10 \leq n$
Best LTE	$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^{n-1})$	$O(h^{n-2})$	$O(h^{n-3})$

This indicates why methods of order ≤ 5 are often used rather than higher order methods with a larger step size.