MACM 316 Lecture 18

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Friday, February 14, 2025

1 Thm (2.7 of Text)

Let $g \in C[a, b]$ s.t. $g(x) \in [a, b]$ for all $x \in [a, b]$.

Suppose, in addition, that g' is continuous on (a, b) and a constant $0 \leq k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$.

If $g'(p) \neq 0$, then for any number p_0 in $[a, b]$ the sequence

$$p_n = g(p_{n-1}) \text{ for } n \geq 1.$$

converges only linearly to the unique fixed point p in $[a, b]$.

Proof. We know from the fixed point theorem that the sequence converges to p. since g' exists on $[a, b]$ we can apply the mean value theorem to g:

$$\underbrace{g(p_n) - g(p)}_{p_{n+1}-p} = g'(\xi_n)(p_n - p).$$

where ξ_n is between p_n and p. Thus,

$$\frac{p_{n+1} - p}{p_n - p} = g'(\xi_n).$$

and fixed point iteration gives linear convergence with asymptotic error constant $|g'(p)|$ whenever $g'(p) \neq 0$.

Proof. ...

Thus $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|$ and fixed-point iteration gives linear convergence with asymptotic error constant $|g'(p)|$ whenever $g'(p) \neq 0$.

Method A was the fixed-point iteration method defined by the iteration function

$$g(x) = \frac{1}{2}(10 - x^3)^{1/2}.$$

Notice that:

$$\begin{aligned} g'(p = 1.365230013) &= -\frac{3}{4}x^2(10 - x^3)^{-1/2} \\ &= -0.51 \neq 0 \end{aligned}$$

so the theorem applies if we consider the interval $[1, 1.5]$ and we see that linear convergence is obtained. On the other hand, Method B was the fixed point iteration method defined by the iteration function

$$g(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}.$$

This method gave quadratic convergence, but the theorem cannot be applied because

$$g'(p) = 0.$$

We saw last day that higher order convergence for fixed point method can occur only when $g'(p) = 0$. It is possible under certain reasonable conditions to obtain quadratic convergence...

2 Theorem (2.8 of Text)

Let p be a solution of the equation $x = g(x)$.

Suppose $g'(p) = 0$ and g'' is continuous and strictly bounded by M on an open interval I containing p. Then, there exists a $\delta > 0$ such that for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$ when $n \geq 1$ converges at least quadratically to p.

Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Proof. ... (see lecture notes)

Thus the sequence $\{p_n\}_{n=0}^{\infty}$ converges quadratically if $g''(p) \neq 0$ and higher order convergent if $g''(p) = 0$. Also, we know $|g''| < M$ so

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

So the idea behind finding iteration methods with a high order of convergence is to look for schemes whose derivatives are zero at the fixed point.

3 Newton's Method

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ g'(x) &= 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} \\ &= \frac{f(x)f''(x)}{[f'(x)]^2} \end{aligned}$$

$\therefore g'(p) = 0$ provided $f'(p) \neq 0$.
$\therefore$ Newton's Method satisfies the derivative condition for **Thm.** 2.8.

Let's take another look at Newton's Method. Consider using Newton's Method to find the roots of

$$p^3 - p^3 - p + 1 = 0.$$

Newton's Method here is

$$p_{n+1} = p_n = \frac{p_n^3 - p_n^2 - p_n + 1}{3p_n^2 - 2p_n - 1}.$$

Starting from $p_0 = 1.1$ we find

Iteration	**Value**
p_0	1.1
p_1	1.05116...
p_2	1.02589...
p_3	1.01303...
p_4	1.00653...
p_5	1.00327...
$\vdots$	$\vdots$

Table 1: Numerical Iterations

Which is very slow (Linear) convergence to the root (which is $p = 1$).

Why is this?

In Newton's Method, we need to find $f'(p) \neq 0$ to obtaine quadratic convergence. Notice that

$$f'(p) = 3p^2 - 2p - 1|_{p=1} = 0.$$

So the theorem doesn't hold. Moreover, factoring f:

$$f(x) = (x - 1)^2(x + 1).$$

we see that $x = 1$ is a zero with multiplicity of 2.

Def. A solution p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$ we can write $f(x) = (x - p)^m q(x)$ where $\lim_{x \to p} q(x) \neq 0$.

Simple zeros are those that have multiplicity 1.

Thus Newton's Method can only be applied to simple zeros of a function. Identification fo the multiplicity of a zero is often made easier by the two following theorems.

Thm. 2.10

$f \in C'[a,b]$ has a simple zero at p in (a,b) if and only if $f(p) = 0$ but $f'(p) \neq 0$.

Thm. 2.11

The function $f \in C^m[a,b]$ has a zero of multiplicity m at p if and only if

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p).$$

but $f^{(m)}(p) \neq 0$.

We want to obtain quadratic convergence with Newton's Method for multiple roots.

One approach is to define a new function

$$\mu(x) = f\frac{(x)}{f'(x)}.$$

We assume p is a zero of multiplicity m and $f(x) = (x-p)^m q(x)$ where $q(p) \neq 0$. Then,

$$\begin{aligned}
\mu(x) &= \frac{(x-p)^m}{m(x-p)^{m-1}q(x) + q'(x)(x-p)^m} \\
&= \frac{(x-p)q(x)}{mq(x) + q'(x)(x-p)} \\
&= (x-p)\frac{q(x)}{mq(x) + q'(x-p)}
\end{aligned}$$

$$q(p) \neq 0 \quad \text{therefore, } \mu(p) \text{ has a simple root at } x = p$$

so $\mu(p) = 0$, but $\dfrac{q(p)}{mq(p) + q'(p)(p-p)} = \dfrac{1}{m} \neq 0$. and p is a zero of multiplicity 1 of $\mu(x)$.

Good:

- Quadratic convergence for all roots

Bad:

- Need f''

- μ is more expensive to work with
- μ might give more roundoff error