MACM 316 Lecture 21

Alexander Ng

Friday, February 28, 2025

1 Continued from Lecture 20

Our next task is to develop estimates for the error. As it turns out, the form of the error (but not necessarily the magnitude) resembles that of the n^{th} Taylor Polynomial.

1.1 Error Estimates

Thm. (3.3 of Text)

Suppose $x_0, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each $x \in [a, b]$, a number $\xi(x) \in (a, b)$ exists with the property

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n).$$

n-th Langrange Interpolating Polynomial

Recall: If f has $(n + 1)$ continuous derivatives on $[a, b]$ and $P(x)$ is the interpolating polynomial of degree $\leq n$ for f at the points $x_0, \ldots, x_n$, then,

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^{n} (x - x_k).$$

Ex. suppose you need to construct six-decimal-place tables for the common, or base-10, logarithm function from $x = 1$ to $x = 10$ in a way that linear interpolation is accurate within 10^{-6} of the true value. Determine a bound for the step size for this table.

i	x_i	$f(x_i)$
0	3.2	22.0
1	2.7	17.8
2	1.0	14.2
3	4.8	38.3
4	5.6	51.7

Based on the following data:

find approximations to $f(3)$ using the 2^{nd} and 3^{rd} Lagrange interpolating polynomials.

Soln. we will use x_0, x_1 and x_3 to build the 2^{nd} Lagrange interpreting polynomial.

$$\begin{aligned}
P_2(3) &= \frac{(3-2.7)(3-4.8)}{(3.2-2.7)(3.2-4.8)}(22.0) \\
&\quad + \frac{(3-3.2)(3-4.8)}{(2.7-3.2)(2.7-4.8)}(17.8) \\
&\quad + \frac{(3-3.2)(3-2.7)}{(4.8-3.2)(4.8-2.7)}(38.3) \\
&\approx 20.27
\end{aligned}$$

We will use x_0, x_1, x_2, x_3 to build the 3^{rd} Lagrange interpolating polynomial.

$$\begin{aligned}
P_3(3) &= \frac{(3.0-2.7)(3.0-1.0)(3.0-4.8)}{(3.2-2.7)(3.2-1.0)(3.2-4.8)}(22.0) \\
&\quad + \frac{(3.0-3.2)(3.0-1.0)(3.0-4.8)}{(2.7-3.2)(2.7-1.0)(2.7-4.8)}(17.8) \\
&\quad + \frac{(3.0-3.2)(3.0-2.7)(3.0-4.8)}{(1.0-3.2)(1.0-2.7)(1.0-4.8)}(14.2) \\
&\quad + \frac{(3.0-3.2)(3.0-2.7)(3.0-1.0)}{(4.8-3.2)(4.8-2.7)(4.8-1.0)}(38.3) \\
&\approx 20.21
\end{aligned}$$

Notice that:

1. We do not know the derivative values of f, therefore we cannot apply the error formula. However, we can make an estimate for the error by examining polynomials of different degrees by using different nodes.

2. The P_2-calculation was not used to reduce the work in calculating P_3. We want to find a way to use previous degrees of P, especially since the previous point implies that we will examine the results for Lagrange polynomials of varying degrees.

We want to examine polynomials based on different nodes. In the last example, we considered the polynomial based on the nodes x_0, x_1 and x_3

We will call this polyomial $\mathbf{P_{013}(x)}$

We also considered the polynomial based on the nodes x_0, x_1, x_2, x_3

We will call this polyomial $\mathbf{P_{0123}(x)}$

Similarly, we make the following definition:

Let f be a function defined at $x_0, x_1, x_2, \ldots, x_n$ and suppose that $m_1, m_2, \ldots, m_k$ are k distinct integers with $0 \leq m_i \leq n$ and for each i.

The Lagrange Interpolating Polynomial that agrees with f at $x_m, x_{m2}, \ldots, x_{mk}$ is denoted

$$P_{m_1, m_2, \ldots, m_k}.$$

Using this notation,

$$P_0(x) = f(x_0)$$

$$P_1(x) = \frac{(x - x_1)f(x_0) + (x - x_0)f(x_1)}{x_0 - x_1}$$

and

$$\begin{aligned} P_{0,1}(x) &= \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1) \\ &= \frac{(x - x_1)P_0(x) - (x - x_0)P_1(x)}{x_0 - x_1} \end{aligned}$$

So $P_{0,1}(x)$ can be recursively defined in terms of $P_0(x)$ and $P_1(x)$.

More generally:

Thm. Let f be defined at $x_0, x_1, \ldots, x_k$ and x_j and x_i be two distinct numbers in this set. Then

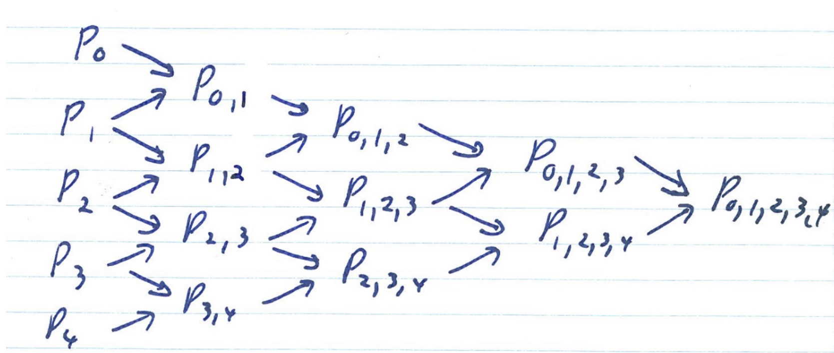
$$P_{0,1,\ldots,k}(x) = \frac{(x - x_j)P_{0,1,\ldots,j,j+1,\ldots,n}(x) - (x - x_i)P_{0,1,\ldots,i-1,i+1,\ldots,n}(x)}{x_i - x_j}.$$

Proof. I left out the proof.

The corresponding procedure is called Neville's Method. Here, values for each interpolating polynomial are generated using previous calculations.

Ex. $P_{0,1}(x) = \dfrac{(x - x_1)P_0 - (x - x_0)P_1}{(x_0 - x_1)}$ is derived from $P_0 + P_1$. Correspondingly, $P_{1,2}(x)$ is derived from P_1 and P_2.

Written as a table:



Ex. suppose $x_j = j$ for $j = 0, 1, 2, 3$ and it is known that

$$\begin{aligned}
P_{0,1}(x) &= 2x + 1 \\
P_{0,2}(x) &= x + 1 \\
P_{1,2,3}(2.5) &= 3
\end{aligned}$$

Find $P_{0123}(2.5)$.

We have P_{123} and we need another quadratic to find our P_{0123}

$$P_{0,1,2} = \frac{(x - x_1)P_{02}(x) - (x - x_2)P_{01}(x)}{x_2 - x_1}.$$

We evaluate $P_{02}(2.5) = 2.25$ and $P_{01}(2.5) = 6$

We have $x_1 = 1, x_2 = 2, x = 2.5$

$$P_{012}(2.5) = 2.25.$$

$$P_{0123}(2.5) = \frac{(2.25 - x_0)P_{123}(2.25) - (2.25 - x_3)P_{012}(2.25)}{x_3 - x_0} = 2.875$$