Taylor Series (Review)

Taylor's theorem is one of the most important tools for this course.

Assuming a function is sufficiently smooth on an interval $[a,b]$ then we can construct a polynomial approximation $P_n(x)$ to $f(x)$ as follows:

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

where $x_1, x_0 \in [a,b]$.

How smooth does $f(x)$ have to be?

- $f \in C^n[a,b] \Rightarrow f, f', f'', \ldots, f^{(n)}$ all continuous.
- $f^{(n+1)}$ must exist on $[a,b]$.

Q. How good is this approximation?
What error is made?

We have $f(x) = P_n(x) + \text{Error}$
$= P_n(x) + R_n(x)$

Taylor's thm says
$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

where ξ is between x_0 and x

Note: Approximation will be best where x is close to x_0

x_0 is known as the expansion point.

Example

Find the third Taylor polynomial $P_3(x)$ for $f(x) = \sin(x)$ with $x_0 = 0$.

Soln. $P_3(x) = f(0) + f'(0)x$

$$+ \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{3!} x^3$$

We need f', f'', f''':

$f'(x) = \cos x \Rightarrow f'(0) = \cos(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos(0) = -1$

also $f(0) = 0$.

$\Longrightarrow P_3(x) = x - x^3/6$.

What about the error?

$$R_3(x) = \frac{f''''(\xi)}{4!} x^4$$

$f''''(x) = \sin x$

$$\Longrightarrow R_3(x) = \frac{\sin \xi}{24} x^4$$

Use the bound to estimate how big the error could be:

$$R_3(\pi/2) = \frac{\sin(\xi)}{24}\left(\frac{\pi}{2}\right)^4 \quad \text{where } \xi \in (0, \pi/2)$$

We know $|\sin \xi| \leq 1$ for all $\xi \in (0, \pi/2)$

$$\Rightarrow \left|R_3\left(\frac{\pi}{2}\right)\right| \leq \frac{1}{24}\left(\frac{\pi}{2}\right)^4$$

$$\approx 0.25$$

<u>Actual Error</u>

$$\left| f\left(\frac{\pi}{2}\right) - P_3\left(\frac{\pi}{2}\right)\right| = \left|\sin\frac{\pi}{2} - \left(\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^3\Big/6\right)\right|$$

$$\approx \left|1 - (1.57 - 0.645)\right|$$

$$= 0.075$$

The remainder term gives a bound on the actual error, but it is much larger.

Notice

actual error $\leq$ error bound.

Often the bound gives more useful information near the point of expansion.

Another Useful form of Taylor's Thm.

We have

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots$$

Let $x = x_0 + h$. Then

$$f(x_0+h) = f(x_0) + f'(x_0)(x_0+h-x_0) + \frac{f''(x_0)}{2}(x_0+h-x_0)^2 + \cdots$$

$$\Rightarrow f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + \cdots$$

Example : Use P_1 (linear approximation) to find an approximation to $\sqrt{16.1}$

Soln : Let $f(x) = \sqrt{x}$.
We wish to compute $f(16.1)$.

We know $f(x_0+h) \approx f(x_0) + h\, f'(x_0)$

$x_0 = 16, \quad h = 0.1$

$$f(16.1) = f(16+0.1)$$
$$\approx f(16) + (0.1)\, f'(16)$$
$$= \sqrt{16} + (0.1)\left(\tfrac{1}{2}\right)\frac{1}{\sqrt{16}}$$
$$= 4 + \frac{(0.05)}{4} = 4.0125$$

$f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$

Thus $\sqrt{16.1} \approx 4.0125$

Exact value $\sqrt{16.1}$ is
4.0124805...