Rate of Convergence

Throughout this course, we will study numerical methods that solve a problem by constructing a sequence of (hopefully) better and better approximations which converge to the required solution.

A technique is needed to compare the convergence rates of different methods.

Assume the sequence $\{\alpha_n\}$ converges to α:

$$\lim_{n \to \infty} \alpha_n = \alpha$$

We would like to quantify how quickly α_n tends to α.

Consider an example

$$\alpha_n = \sin\left(\frac{1}{n}\right)$$

We have $\alpha_n \to 0$ as $n \to \infty$
so $\alpha = 0$.

Note that

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$$

is equivalent to

$$\lim_{h\to 0} \sin(h)$$

We will work with the latter
(to avoid working with "∞").

Expand $\sin h$ in a Taylor series in powers of h:

$$\sin h = h - \frac{h^3}{6} + \cdots$$

Thus $\sin h - \alpha = \sin h - 0$
$= h - h^3/6 + \cdots$

$\therefore \sin h \to 0$ like $h \to 0$

We now generalize the underlying concepts.

Consider the following definition:

Suppose $\{\alpha_n\}$ is a sequence that converges to α as $n \to \infty$:

$$\lim_{n\to\infty} \alpha_n = \alpha$$

And assume $\{\beta_n\}$ is a sequence that converges to zero as $n \to \infty$

$$\lim_{n\to\infty} \beta_n = 0.$$

If $|\alpha_n - \alpha| \leq K\,|\beta_n|$

for n large where K is a positive constant then we say

$\{\alpha_n\}$ converges to α

with RATE OF CONVERGENCE

$O(\beta_n)$ (big-Oh of β_n)

If $\alpha_n \to \alpha$ with r.o.c. $O(\beta_n)$ then we sometimes write

$$\alpha_n = \alpha + O(\beta_n)$$

In our previous example, we had the sequence

$$\alpha_n = \sin\left(\frac{1}{n}\right)$$

$$\alpha = 0 \qquad \left(\alpha = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right)\right)$$

Recall that $\alpha_n = \sin\frac{1}{n}$ was like $\frac{1}{n}$ as $n \to \infty$.

So we are comparing with

$$\beta_n = \frac{1}{n}.$$

According to the definition we see that

$\alpha_n = \sin\left(\frac{1}{n}\right)$ convergences to $\alpha = 0$ with rate of convergence

$$O(\beta_n) = \frac{1}{n}.$$

Note: Usually we compare how fast $\alpha_n \to \alpha$

with how fast

$$\beta_n = \frac{1}{n^p} \to 0$$

We are most interested in finding the largest value of p for which

$$\{\alpha_n\} \to \alpha$$

with r.o.c. $O\left(\frac{1}{n^p}\right)$.

Another example: $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right)=1$

We have $\alpha_n = n\sin\left(\frac{1}{n}\right)$

how quickly does α_n tend to α?

$$\alpha = 1$$

Change variables: $h = 1/n$

Now $\lim_{h\to 0} \frac{1}{h}\sin(h) = 1$

So $\frac{1}{h}\sin h - 1 = O(h^p)$. $\boxed{p=?}$

$$\frac{1}{h}\sin(h) - 1 = \frac{1}{h}\left(h - \frac{h^3}{6} + \dots\right) - 1$$

$$= 1 - \frac{h^2}{6} + \dots - 1$$

$$= -h^2/6 + \dots$$

$\therefore \frac{1}{h}\sin(h) - 1 \to 0$ like $-\frac{h^2}{6} \to 0$

and the rate of convergence is

$$O(h^2)$$

or $O\left(\frac{1}{n^2}\right)$.

I expanded around $h_0 = 0$

Why?

We want to know what happens as $h \to 0$ so it makes sense to choose $h_0 = 0$.

What about the higher order term?

$$\frac{1}{h}\sin h - 1 = -\frac{h^2}{6} + c\,h^4 + \cdots$$

Why do we say convergence? $O(h^2)$

As $h \to 0$, the h^2 terms dominate the convergence, since the h^4 term "will be gone to zero" long before the h^2 term.

So the rate of convergence can only be as fast as the h^2 term.