MACM 316 Lecture 16

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We want to convert from $0 = f(x)$ to $x = g(x)$, which is to convert from a root finding problem to a fixed point problem.

One way to do this, very simply, is to just add x to both sides of the equation.

$$\begin{aligned} x^3 + 4x^2 - 10 &= 0 \\ x^3 + 4x^2 - 10 + x &= x \end{aligned}$$

In general, if your iterative method converges very quickly, you will not have a guarantee of convergence. Therefore, you should use a mix of methods to get a good initial guess and then quickly converge to the fixed point.

1 Convergence

Why do some methods converge and some diverge? Why do they converge with different rates?

Consider a simple example:

ex.

$$g(x) = ax + b.$$

We have

$$
\begin{aligned}
x_1 &= ax_0 + b \\
x_2 &= ax_1 + b = a(ax_0 + b) + b = a^2x_0 + (1 + a)b \\
x_3 &= ax_2 + b = a^3x_0 + (a + a + a^2)b
\end{aligned}
$$

and by induction,

$$
x_n = \begin{cases} a^n x_0 + \left(\frac{1-a^n}{1-a}\right) b & a \neq 1 \\ x_0 + nb & a = 1. \end{cases}.
$$

$$
\therefore \lim_{n\to\infty} x_n = \begin{cases} \frac{1}{1-a} b & |a| < 1 \\ x_0 & a = 1, b = 0. \end{cases}
$$

No proper limit exists for all other values of a, b.

2 Fixed Point Theorem

When does a fixed point iteration converge? How quickly does it converge?

For this, we turn to the **fixed point theorem**.

Thm. Let $g \in C[a, b]$ and suppose $g(x) \in [a, b]$ for all $x \in [a, b]$.

Suppose, in addition, that g' exists on (a, b) and a positive constant $k < 1$ exists with

$$
|g'(x)| \leq k \text{ for all } x \in (a, b).
$$

Then for any number p_0 in (a, b), the sequence defined by

$$
p_n = g(p_{n-1}) \qquad n \geq 1.
$$

converges to the unique fixed point p in $[a, b]$

Proof. Our earlier **Thm.** (existence + uniqueness) tells us that a unique fixed point exists in $[a, b]$. Notice that g maps $[a, b]$ into itself, so the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$ and $p_n \in [a, b]$ for all n.

We may apply the **mean value theorem** to g to show that for any n

$$\begin{aligned}|p_n - p| &= |g(p_{n-1}) - g(p)| \\ &= |g'(c)||p_{n-1} - p| \\ &\leq k|p_{n-1} - p|\end{aligned}$$

Where $c \in (a, b)$. Applying the inequality inductively gives

$$some garbage i haven't written down yet.$$

Since $k < 1$, $\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n |p_0 - p| = 0$

$$\therefore \{p_n\}_{n=0}^{\infty} \text{ converges to } p.$$

This proof also gives us a natural bound for the error.

3 Newton's Method

(or Newton-Raphson Method)

One of the most powerful and well-known methods for solving a root-finding problem

$$f(x) = 0.$$

Pros: Much faster than bisection
Cons: Needs $f'(x)$, not guaranteed to converge
Want: $x = p$ s.t. $f(x) = 0$
Idea: Use slope as well as function values

3.1 Derivation (by Taylor's Thm.)

Want: $x = p$ s.t. $f(x) = 0$

Suppsoe $f \in C^2[a, b]$. Let $\overline{x} \in [a, b]$ be an approximation to p s.t. $f'(\overline{x}) \neq 0$ and $|\overline{x} - p|$ is sufficiently small. Then

$$f(x) = f(\overline{x}) + f'(\overline{x})(x - \overline{x}) + \frac{1}{2}f''(\xi(x))(x - \overline{x})^2 \qquad \xi \text{ lies between } x, \overline{x}.$$

Set $x = p$

$$0 = f(\overline{x}) + f'(\overline{x})(p - \overline{x}) + \frac{1}{2} f''(\xi(p))(p - \overline{x})^2.$$

$p - \overline{x}$ is very small $\implies$ $|p - \overline{x}|$ is even smaller. So we just drop the error term $\frac{1}{2} f''(\xi(p))(p - \overline{x})^2$

$$0 \approx f(\overline{x}) + f'(\overline{x})(p - \overline{x}).$$

Solve for p:

$$\tilde{p} = \overline{x} - \frac{f(\overline{x})}{f'(\overline{x})}.$$

Take $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$
As a stopping criterion, we might use

$$|p_n - p_{n-1}| \leq TOLERANCE \quad \epsilon.$$

Called "absolute error approximation".
We might also use "relative error approximation"

$$\frac{|p_n - p_{n-1}|}{|p_{n-1}|} \leq \epsilon.$$

(1) Newton's method fails:

$$f'(p_n) = 0.$$

$\implies$ method is not effective if f' is equal to zero at p. It will also not perform well if f' is close to 0.

Also we see in the derivation that $|p - \overline{x}|$ needs to be small, which implies we need a good initial guess.

ex.:

Use Newton's Method to compute the square root of a number R. We want to find the roots of $p^2 - R = 0$.

Let

$$f(x) = x^2 - R$$
$$f'(x) = 2x$$

Newton's Method takes the form

$$\begin{aligned} p_n &= p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \\ &= p_{n-1} - \frac{p_{n-1}^2 - R}{2p_{n-1}} \\ &= \frac{1}{2}(p_{n-1} + \frac{R}{p_{n-1}}) \end{aligned}$$

Method is credited to Heron, a Greek Engineer circa 100BC -¿ 100AD.
Try $R - 2$:

$$\begin{aligned} p_0 &= 2 \\ p_1 &= 1.5 \\ p_2 &= 1.416666 \\ p_3 &= 1.41425162 \\ p_4 &= 1.414211356 \qquad \text{12 digits correct} \end{aligned}$$

Newton's method can be shown to converge under reasonable assumptions (smoothness of $f(\cdot)$, $f'(p) \neq 0$ and a good initial guess)