MACM 316 Lecture 23 - Chapter 3 Part 2

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Lecture Review

In this lecture, he begins with a review of somet things that we covered in lecture 22 and 21, which he was absent for and had a TA teach for him.

0.1 Divided Differences

Assume that $f(x)$ is known at several points along the x-axis. We do not assume that the x's are evenly spaced or even that the values are arranged in any particular order.

We write the (any arbitrary) n^{th} degree polynomial in the following way:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\ldots(x - x_{n-1}).$$

Def. $P_n(x)$ is an interpolating polynomial for $f(x)$ at $x_0, x_1, \ldots, x_n$ if we choose a_i such that $P_n(x) = f(x))$ at the $n + 1$ known points.

We want to find our a_i to make $P_n(x)$ an interpolating polynomial, and we can determine our a_I by using what are called the "divided differences."

0.1.1 The Zeroth Divided Difference

First, we determine $a_0 : P_n(x_0) = f(x_0) = a_0$, and define the zero$^{\text{th}}$ (degree) divided difference as

$$f[x_i] = f(x_i),$$

which is just the value of f at x_i.

0.1.2 The First Divided Difference

Now we need to determine a_1:

$$P_n(x_1) = f(x_1) = a_0 + (x_1 - x_0)a_1.$$

Rearranging, we get

$$a_i = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0}.$$

which allows us to determine a_1 by using the zero$^{\text{th}}$ divided difference. We define the first divided difference of f with respect to x_i and x_{i+1} as

$$f[x_i, x_{i+1}] = f[x_{i+1}, x_i] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

0.1.3 The Second Divided Difference

Similarly, the second divided difference is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

0.1.4 The Kth Divided Difference

In a similar fashion to the evaluation of a_0 and a_1, we can show

$$a_k = f[x_0, x_1, \ldots, x_k].$$

This gives Newton's Interpolatory divided difference formula

$$\begin{aligned} P_n(x) = {} & f[x_0] + f[x_0, x_1](x - x_0) \\ & + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ & + \ldots \\ & + f[x_0, x_1, \ldots, x_n](x - x_0) \ldots (x - x_{n-1}) \end{aligned}$$

0.1.5 Disadvantages of Divided Differences

- $P_n(x)$ could be expensive to evaluate
- $P_n(x)$ does go through the datapoints, but between each datapoint, we could have large oscillations, which we expect as the generic behaviour of large degree polynomials.

  This is especially true near endpoints, where our datapoints are sparse and sections where our function is not smooth.

1 Better Interpolating Polynomials - Ch2. Pt3. (14.1)

The Lagrange Polynomial can oscillate wildly, except where contained between datapoints that are in close proximity. We want our interpolating polynomial to have the same "shape" as the function at the datapoints. In other words, we want the tangent lines (first derivative) and the function itself to agree at (x_i, f_1)

1.1 Proof from (14.2).

This section is directly copied from the Chapter 3 Part 2 pdf.

Thm. If $f \in C^1[a,b]$ and $x_0, \ldots, x_n \in [a,b]$ are distinct, the interpolating polynomial of least degree agreeing with f at $x_0, \ldots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by

$$H(x) = \sum_{j=0}^{n} f(x_j) H_j(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_j(x)$$

where

$$H_j(x) = \left[1 - 2(x - x_j) L_j'(x_j)\right] L_j^2(x)$$

and

$$\hat{H}_j(x) = (x - x_j) L_j^2(x)$$

where $L_j(x)$ denotes the Lagrange coefficient polynomial of degree n.

Proof. See text for a derivation showing that $H(x)$ agrees with f and $H'(x)$ agrees with f' at $x_0, x_1, \ldots, x_n$.

To show the uniqueness of this polynomial:

Suppose that $P(x)$ is another polynomial with

$$P(x_j) = f(x_j), \quad P'(x_j) = f'(x_j) \quad \text{for } j = 0, \ldots, n$$

and that the degree of $P(x)$ is at most $2n+1$.

Let $D(x) = H(x) - P(x)$. Then $D(x)$ is a polynomial of degree at most $2n+1$.

The zeros at each $x_0, x_1, \ldots, x_n$ are equal to

$$D(x) = (x - x_0)^2 \cdots (x - x_n)^2 Q(x)$$

Either $D(x)$ is of degree $2n+2$ or more, which would be a contradiction, <u>or</u>

$$Q(x) \equiv 0 \Rightarrow D(x) \equiv 0 \Rightarrow P(x) = H(x)$$

which implies that the polynomial is unique. □

1.2 The Newton Interpolatory divided difference formula

Unfortunately, a direct application of the theorem requires us to evaluate the Lagrange Polynomials and their derivatives, which is tedious even for small values of n. Instead, we use the Newton Interpolatory divided difference formula:

Select a new sequence of nodes $z_0, z_1, \ldots, z_{2n+1}$

$$z_{2i} = z_{2i+1} = x_i.$$

We have a small problem.

$$f[z_{2i}] = f[z_{2i+1}] = f(x_i).$$

$$f[z_{2i}, z_{2i+1}] = \frac{f[z_{2i+1}] - f[z_{2i}]}{z_{2i+1} - z_{2i}}.$$

This gives us $\frac{0}{0}$, which is not defined. Instead, we think of taking the limit as z_{2i} approaches z_{2i+1}:

$$f[z_{2i}, z_{2i+1}] = \lim_{z_{2i} \to z_{2i+1}} \frac{f[z_{2i+1}] - f[z_{2i}]}{z_{2i+1} - z_{2i}}.$$

Thus, the derivative values are used in place of the undefined first divided differences, otherwise Newton's divided differences are produced as usual.

z	**f(z)**	**first divided differences**
$z_0 = x_0$	$f[z_0] = f[x_0]$	
		$f[z_0, z_1] = f'(x_0)$
$z_1 = x_0$	$f[z_1] = f[x_0]$	
		$f[z_1, z_2]$
$z_2 = x_1$	$f[z_2] = f[x_1]$	
		$f[z_2, z_3] = f'(x_1)$
$z_3 = x_1$	$f[z_3] = f[x_1]$	

1.3 The Hermite Polynomial

The Hermite polynomial is then defined as

$$H(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k] \prod_{j=0}^{k-1} (x - z_j).$$

2 Splines

Previously, we saw how to compute an approximation to a function over some finite interval using a single polynomial. We increased the degree of the polynomial if we wanted more accuracy.