MACM 316 Lecture 14b - Chapter 2 Part 1

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1 Solutions of Nonlinear Systems

The Basic Problem:

Find a root of $x \in \mathbb{R}$ of an equation of the form $f(x) = 0$ for a given continuous function f.

2 The Bisection Method

In most cases it is not really possible to solve analytically. We will consider iterative methods to approximate the solution. Our first method will be the Bisection Method. We must start with an interval $[a, b]$ with $f(a)$ and $f(b)$ of opposite signs.

By the intermediate value theorem, there exists a number c in (a, b) such that $f(c) = 0$.

Thm. If $f \in C[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) such that $f(c) = k$.

Now set $a_1 = a$ and $b_1 = b$ and let p_1 be the midpoint of $[a_1, b_1]$.

1. Compute the midpoint:
$$p_1 = \frac{1}{2}(a_1 + b_1)$$
2. If $f(p_1) = 0$, then we are done. Set $p = p_1$.
3. If $f(p_1)$ and $f(a_1)$ are of opposite signs, then there must exist a root $p \in (a_1, p_1)$ such that $f(p) = 0$. Set $a_2 = a_1$ and $b_2 = p_1$.

4. Otherwise, if $f(p_1)$ and $f(b_1)$ are of opposite signs, then there must exist a root $p \in (p_1, b_1)$ such that $f(p) = 0$. Set $a_2 = p_1$ and $b_2 = b_1$.

5. Reapply the process to the new interval $[a_2, b_2]$.

6. Once the stopping criteria are satisfied, set the midpoint of the interval as the estimate for the root.

2.1 Possible Stopping Criteria

1. $\frac{b_n - a_n}{2} <$ TOL $\quad \underline{\text{or}} \quad |p_n - p_{n-1}| <$ TOL

   GOOD: Ensures that the returned root value p_n is within tolerance of the exact value p

   GOOD: Easy error analysis

   BAD: Does not ensure that $f(p_n)$ is small.

   BAD: An absolute rather than relative measure of error.

2. $\frac{|p_n - p_{n-1}}{p_n} <$ TOL$, p_n \neq 0$

   Usually preferred over (1) if nothing is known about $f(\cdot)$ or p

3. $|f(p_n)| <$ TOL

   Ensures that $f(p_n)$ is small, but p_n may differ significantly from the true root p.

4. We can also carry out a fixed number of iterations N– This is closely related to (1)

The best stopping criteria will depend on what is known about f and p and on the type of problem. It's often useful to use criteria 2 (relative error test) and criteria 4 (fixed number of steps) together.