MACM 316 Lecture 11 - Chapter 6.2

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1 More Special Matrices

1.1 Band Matrices

Another important class of matrices that arise in a wide variety of applications are called band matrices. These matrices concentrate all their nonzero entries about the diagonal.

1.1.1 Definition

An $n \times n$ matrix is called a band matrix if there exist integers p and q such that $1 < p, q < n$ having the property that $a_{ij} = 0$ whenever $i + p \leq j$ or $j + q \leq i$.

The bandwidth of a band matrix is defined as $w = p + q - 1$. We subtract 1 from our count because we do not want to double count the diagonal.

We will focus on the important case of tridiagonal matrices, which are band matrices with $p = q = 2$, i.e., the matrix is tridiagonal if the nonzero entries are on the main diagonal and the diagonals above and below the main diagonal.

Suppose A can be factored into the triangular matrices L and U. Suppose that the matrices can be found in the form:

$$L = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & & \\ & & \ddots & \\ 0 & \dots & l_{n,n-1} & l_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ & & \ddots & \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

The zero entries of A are automatically generated by LU. Multiplying $A = LU$, we also find the following conditions:

1.
$$a_{11} = l_{11}, \quad a_{i,i-1} = l_{i,i-1}, \quad i = 2, 3, \dots, n$$

2.
$$a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii}, \quad i = 2, 3, \dots, n$$

3.
$$a_{i,i+1} = l_{ii}u_{i,i+1}, \quad i = 1, 2, \dots, n-1$$

This system is straightforward to solve: (1) gives us l_{11} and the off-diagonal entries of L. (2) and (3) are used alternately to obtain the remaining entries of L and U. This solution technique is often referred to as **Crout Factorization**.

If we count up the number of operations, we find:

- $(5n-4)$ multiplications/divisions - $(3n-3)$ additions/subtractions

Crout Factorization can be applied to a matrix that is positive definite or one that is strictly diagonally dominant. See the text for another general case where it can be applied.