MACM 316 Lecture 17

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1 Newton's Method

Needs $f'(p_k)$ to exist.

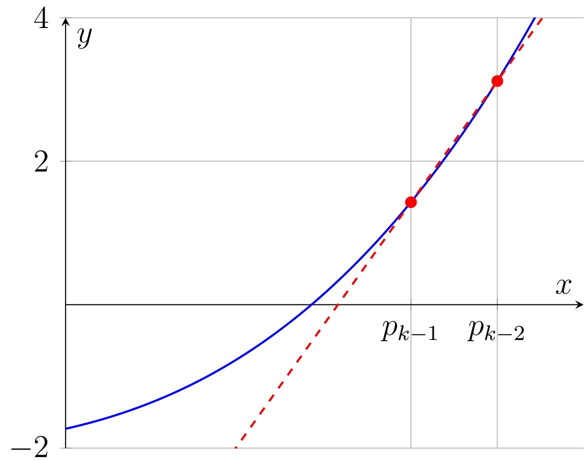
We might have that:

- f' is not available
- f' is expensive

We might choose to approximate:

$$f'(p_k) \approx \frac{f(p_{k-1}) - f(p_{k-2})}{p_{k-1} - p_{k-2}}.$$

This gives the secant method:



both newton's method and the secant method have the limitation that they may diverge when the initial guess is not sufficiently close to the root. In bisection, we used the idea of bracketing the root at each step to ensure convergence.

2 False Position

If instead of considering a midpoint approximation for the root ($p_k \approx p_0 + \frac{p_1 - p_0}{2}$), we consider a secant approximation for the root (based on the endpoints of the interval). This is called the **Method of False Position**.

$$p_k \approx p_1 - f(p_1)\frac{p_1 - p_0}{f(p_0) - f(p_1)}.$$

3 oops, skipped a bunch of pages

4 Error Analysis

We want to be able to give a more precise description of how a method converges to a the solution. For example, consider finding a root for the polynomial

$$x^3 + 4x^2 - 10 = 0.$$

with two different methods (A and B). Suppose that the errors produced by these methods are as given below.

	Method A	**Method B**
$\lvert p - p_0\rvert$	0.134769987	0.134769987
$\lvert p - p_1\rvert$	0.078276245	0.008103332
$\lvert p - p_2\rvert$	0.037310791	0.000003811
$\lvert p - p_3\rvert$	0.019771639	0.000000000
$\lvert p - p_4\rvert$	0.009940240	to all significant digits

Table 1: Comparison of Methods A and B

Notice that the error for method A decreases by a constant factor (about 2) at each iteration. For method B, the error drops off much more quickly. The error at step n is roughly proportional to the error at step $n-1$, squared.

Both of these behaviours can be quantified.

Def. Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$ then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

Notice that:

- a sequence with high order of convergence converges more rapidly than a sequence with a lower order.
- the constant λ affects the speed of convergence, but is not as important as the order (α).
- We want a large α. $\alpha \geq 1$ is sufficient.
  If $\alpha = 0.5$, for example, then the error at k is proportional to the square root of the error at $k-1$. This is not a good behaviour.

4.1 Common Cases

For different values of α, we observe the following convergence behaviors:

- $\alpha = 1$, $\lambda < 1$: Linearly convergent.

$$\mathcal{E}_{n+1} \approx \lambda \mathcal{E}_n$$

- $\alpha = 2$: Quadratically convergent.

$$\mathcal{E}_{n+1} \approx \lambda \mathcal{E}_n^2$$

- α does not have to be an integer. For example, the secant method has:

$$\alpha = \frac{1+\sqrt{5}}{2} < 2.$$

In order to truly understand the behaviour of a method, we need to find both the order and the asymptotic error constant.

5 Thm (2.7 of Text)

Let $g \in C[a,b]$ s.t. $g(x) \in [a,b]$ for all $x \in [a,b]$.

Suppose, in addition, that g' is continuous on (a,b) and a constant $0 \leq k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a,b)$.

If $g'(p) \neq 0$, then for any number p_0 in $[a,b]$ the sequence

$$p_n = g(p_{n-1}) \text{ for } n \geq 1.$$

converges only linearly to the unique fixed point p in $[a,b]$.

Proof. We know from the fixed point theorem that the sequence converges to p. since g' exists on $[a,b]$ we can apply the mean value theorem to g:

$$\underbrace{g(p_n) - g(p)}_{p_{n+1}-p} = g'(\xi_n)(p_n - p).$$

where ξ_n is between p_n and p. Thus,

$$\frac{p_{n+1} - p}{p_n - p} = g'(\xi_n).$$

and fixed point iteration gives linear convergence with asymptotic error constant $|g'(p)|$ whenever $g'(p) \neq 0$.