MACM 316 Lecture 31

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IDK why but he starts off with this example

1 Quadrature Formula Example

Find the constants c_0, c_i and x such that the quadrature formula

$$\int_{-1}^{0} f(x)\,dx = c_0 f(-1) + c_i f(x_1).$$

has the highest degree of precision possible.
ans.

Function f	Equation
$f(x) = 1$	$\int_{-1}^{0} 1\,dx = c_0 + c_i$
$f(x) = x$	$\int_{-1}^{0} x\,dx = \frac{x^2}{2}\Big\vert_{-1}^{0} = c_0(-1) + c_i x_1 = -\frac{1}{2}$
$f(x) = x^2$	$\int_{-1}^{0} x^2\,dx = c_0 + c_i x_1^2 = \frac{1}{3}$

Next, we do some substutitions:

1. substitute (1) into (2) to eliminate c_0

2. substitute (1) into (3) to eliminate c_0

   We now have 2 equations for $c_i + x_1$.
   Solve: $c_0 = \frac{1}{4}, c_i = \frac{3}{4}, x_i = -\frac{1}{3}$
   Could it be exact for cubics?
   LHS $= \int_{-1}^{0} x^3\,dx = -\frac{1}{4}$
   RHS $= -\underbrace{c_0}_{\frac{1}{4}} + \underbrace{c_i}_{\frac{3}{4}}\underbrace{x_i^3}_{-\frac{1}{3}} \neq$ LHS
   So, no, it cannot be exact for cubics.

2 Legendre Polynomials

This approach can be used to obtain the nodes and coefficients for larger n, but Legendre polynomials can be used to obtain them more easily.

The Legendre Polynomials are defined according to the following two properties:

1. $P_n(x)$ is a polynomial of degree n.
   $\implies P_0(x) = 1$

2. $\int_{-1}^{1} P(x)P_n(x)\,dx = 0$ whenever $P(x)$ is a polynomial of degree less than n.

The first few Legendre polynomials are

$P_n(x)$	
$P_0(x)$	1
$P_1(x)$	x
$P_2(x)$	$x^2 - \frac{1}{3}$
$P_3(x)$	$x^3 - \frac{3}{5}x$
$P_4(x)$	$x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

Some properties:

- The roots of these polynomials are distinct
- The roots of these polynomials lie in $(-1, 1)$

- The P_n's are symmetrical about the origin
  $\implies$ the roots are symmetrical about the origin

- The roots of the n^{th} degree Legendre polynomial have the property that they are the nodes needed to produce an integral approximation formula that gives the exact result for any polynomial of degree less than $2n$.

Thm. Suppose $x_1, x_2, \ldots, x_n$ are the roots of the n^{th} degree Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$ the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\, dx.$$

If $P(x)$ is any polynomial of degree less than $2n$ then

$$\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i).$$