MACM 316 Lecture 15

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Bisection Method Example

He starts with an example on the bisection method. I will provide one later on.

Lowkey I missed notes from page 5.7 to 5.10 but I'll add them later. (page 5 of chapter 2 part 1 notes)

1 Fixed Point Iteration

We wish to find the roots of an equation $f(p) = 0$. We will be focusing on methods that <u>iterate</u> to find the root:

$$p_{n+1} = g(p_n)$$

We start by considering the fixed point problem.

Def. A fixed point p is the value of p such that $g(p) = p$.

We can form a fixed pt. problem from a root finding problem.

$$f(p) = 0$$

Find a fixed point problem.

e.g. set $g(x) = f(x) + x$

$g(p) = f(p) + p$

$p = g(p)$

there are many choices of g

Try $g(x) = f^3(x) + x$

Notice that fixed point problems and root finding problems are equivalent. (???)

1.1 Example

We are given $g(p) = p$. Formulate a root finding problem.

$$f(x) = g(x) - x$$

Now we have $f(p) = 0$. We now have a root finding problem.

$$f(p) = 0 \implies g(p) = p$$

i.e. f has a root p implies g has a fixed point p.

$$g(p) = p \implies f(p) = 0.$$

i.e. g has a fixed point p implies f has a root p.
There are many possible choices for g: example $g(x) - x = (f(x))^3$
Our ultimate goal is to find functions with fixed points.

2 Theorems

Existence and Uniqueness

$\star$If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$ then $g(x)$ has a fixed point in $[a, b]$.

$\star\star$ Suppose, in addition, that $g'(x)$ exists on (a, b) and that a positive constant $k < 1$ exists with

$$|g'(x)| \leq k < 1 \text{ for all } x \in (a, b).$$

then the fixed point in $[a, b]$ is unique

Proof. ($\star$): Existence

If $g(a) = a$ or $g(b) = b$ then g has a fixed point at an endpoint.

Suppose not, then it must be true that $g(a) > a$ and $g(b) < b$.

Define $h(x) = g(x) - x$. Then h is continuous on $[a, b]$ (adding two continuous functions yields a continuous function) and

$$h(a) = g(a) - a > 0 \text{ and } h(b) = g(b) - b < 0.$$

The **IVT** implies that there exists a $p \in (a, b)$ for which $h(p) = 0$

This $g(p) - p = 0 \implies p$ is a fixed point of g

Proof. ($\star\star$): Uniqueness

Suppose, in addition,

$$\forall x \in (a,b), |g'(x)| \leq k < 1.$$

and that p and q are both fixed points in $[a,b]$ with $p \neq q$.
By the **MVT**, a number c eixsts between p and q such that

$$\frac{g(p)-g(q)}{p-q} = g'(c).$$

then

$$\begin{aligned}
|p-q| &= |g(p)-g(q)| \\
&= |g'(c)||p-q| \\
&\leq k|p-q| \\
&< |p-q|
\end{aligned}$$

Which is a contradiction
This contradiction must come from the assumption that $p \neq q$
$\therefore p = q$ and the fixed point is unique.
We want to approximate the fixed point of a function g.
IDEA

- choose an initial approximation p_0
- generate a squence $\{p_n\}_{n=0}^{\infty}$ such that $p_n = g(p_{n-1}); n \geq 1$

If the sequence converges to p and g is continuous;

$$\begin{aligned}
p &\equiv \lim_{n\to\infty} p_n \\
&= \lim_{n\to\infty} g(p_{n-1}) \\
&= g(\lim_{n\to\infty} p_{n-1}) \\
&= g(p)
\end{aligned}$$

This gives the Fixed Point Algorithm:
See next notes package