MACM 316 Lecture 3 - Reducing Roundoff Error and Taylor Series Approximation

Alexander Ng

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1 Reducing Roundoff Error

One way to reduce roundoff error is to minimize the number of floating-point operations.

1.1 Polynomial Evaluation Using Nested Multiplication

Consider evaluating the polynomial:

$$f(z) = 1.01z^4 - 4.62z^3 - 3.11z^2 + 12.2z - 1.99.$$

We can rewrite this expression using nested multiplication:

$$\begin{aligned} f(z) &= (1.01z^3 - 4.62z^2 - 3.11z + 12.2)z - 1.99 \\ &= ((1.01z^2 - 4.62z - 3.11)z + 12.2)z - 1.99 \\ &= \dots \end{aligned}$$

By factoring out z as much as possible, we reduce the total number of floating-point operations, minimizing error accumulation.

2 Cancellation Errors

2.1 Quadratic Formula and Cancellation Errors

Consider solving the quadratic equation:

$$ax^2 + bx + c = 0.$$

Using the quadratic formula:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Suppose $b = 600$, $a = c = 1$. The issue arises because $-b$ is close in magnitude to $+\sqrt{b^2 - 4ac}$, causing significant cancellation error in x_1.

2.2 Reformulating to Reduce Cancellation

We rationalize the numerator:

$$x_1 = \frac{(-b + \sqrt{b^2 - 4ac})}{2a} \times \frac{(-b - \sqrt{b^2 - 4ac})}{(-b - \sqrt{b^2 - 4ac})}.$$

This simplifies to:

$$x_1 = \frac{b^2 - (b^2 - 4ac)}{2a(-b - \sqrt{b^2 - 4ac})}.$$

Now, the cancellation error is eliminated. If $b = -600$, the same issue occurs with x_2, and we apply the same rationalization technique.

3 Review of Taylor Series

Taylor's theorem is fundamental for numerical approximations.

3.1 Definition of Taylor Series

Given a function $f(x)$ that is sufficiently smooth on $[a, b]$, we can approximate $f(x)$ with a Taylor polynomial $P_n(x)$:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Here, $x_0, x \in [a, b]$.

3.2 Conditions for Taylor Series Expansion

For $f(x)$ to have a valid Taylor series expansion:

- $f \in C^n[a, b]$ (i.e., f, f', f'', ..., f^n must be continuous).
- $f^{(n+1)}$ must exist on $[a, b]$.

3.3 Error in Taylor Approximation

The error in Taylor series approximation is given by:

$$f(x) = P_n(x) + R_n(x),$$

where the remainder term $R_n(x)$ satisfies:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

for some $c \in (x_0, x)$. The approximation is most accurate when x is close to x_0.

4 Example: Third-Order Taylor Polynomial for $\sin(x)$

Find $P_3(x)$ for $f(x) = \sin(x)$ centered at $x_0 = 0$.

$$\begin{aligned} P_3(x) &= f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 \\ &= x - \frac{x^3}{6}. \end{aligned}$$

4.1 Error Analysis

The remainder term for $n = 3$ is:

$$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4.$$

Since $f^{(4)}(x) = \sin(x)$, we have:

$$R_3(x) = \frac{\sin(c)}{24}x^4.$$

For $x = 0.1$:

$$|R_3(0.1)| \leq \frac{|\sin(0.1)|}{24}(0.1)^4 < 4.2 \times 10^{-7}.$$

This shows the high accuracy of the Taylor series approximation.