# MACM 316 Lecture 33

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## 1 Euler's Method

This section starts with Euler's Method Error Analysis. The analysis is straightforward, and interesting because it can be extended to the higher order methods that will be discussed in later sections. To derive the proof of convergence, we need the following

*Lemma*: If $s$ and $t$ are positive real numbers, $\{a_i\}_{i=0}^{k}$ is a sequence satisfying

$$\begin{aligned} a_0 &\geq -\frac{t}{s} \\ a_{i+1} &\leq (1+s)a_i + t \quad i = 0, 1, \ldots, k \end{aligned}$$

then $a_{i+1} \leq e^{(i+1)s}\left(a_0 + \dfrac{t}{s}\right) - \dfrac{t}{s}$

The proof of the lemme is not important, but it is included in section 23.5 of the Chapter 5 lecture notes.

### 1.1 Theorem 1 (23.6)

Suppose $f$ is continuous and satisifes a Lipschitz condition with constant $L$ on

$$D = \{(t, y) : a \leq t \leq b, -\infty < y < \infty\}.$$

and that a constant $M$ exists with the property that

$$|y''(t)| \leq M.$$

Let $y(t)$ denote the unique solution to the initial value problem

$$y' = f(t, y); \quad y(a) = y_0, a \leq t \leq b.$$

and $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's Method.

Then, for each $i = 0, \ldots, N$,

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

*Proof.* (23.7) (I didn't add it yet)

Note that the theorem requires that

$$|y''(t)| \leq M.$$

The second derivative $y''(t)$ may not be known, but if $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ exist,

$$\begin{aligned} y''(t) = \frac{d}{dt}y'(t) &= \frac{df}{dt}(t, y(t)) \\ &= \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t)) \end{aligned}$$

### 1.1.1 Example (23.8)

What value of $h$ is needed to ensure that $|y(t_i) - w_i| \leq 0.1$ for the initial value problem

$$\begin{cases} y' = \frac{2}{t}y + t^2 e^t & 1 \leq t \leq 2 \\ y = 0 & t = 1. \end{cases}.$$

You are given $y''(t) = (2 + 4t + t^2)e^t - 2e$

*Soln.* (23.9) $y''(t)$ is increasing and positive on $[1, 2]$, so

$$\begin{aligned} |y''(t)| &\leq |y''(2)| \\ &= 14e^2 - 2e \\ &= 98.0102 \end{aligned}$$

$$\begin{aligned} \text{since} \quad &\left|\frac{\partial}{\partial y}\left(\frac{2}{t}y + t^2 e^2\right)\right| \\ &\leq \left|\frac{2}{t}\right| \\ &\leq 2 \end{aligned}$$

a Lipschitz Constant for $f(t, y) = \frac{2}{t}y + t^2e^t$ is $L = 2$.

$$\therefore |y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i-1)} - 1\right] \le 0.1.$$

we need to choose $h$ so that $\frac{98.0102h}{4}\left[e^{2(2-1)} - 1\right] \le 0.1.$

$$\implies h \le \frac{0.4}{98.0102(e^2 - 1)} = 0.00064.$$

# 2 The Difference Method

We ned a way to compare the efficiency of different approximation methods. The difference method compares how much the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation.

**Def.** The Difference Method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i) \end{aligned}$$

has a local trunctation error

$$\begin{aligned} \tau_{i+1}(h) &= \frac{y(t_{i+1}) - (y(t_i) + h\phi(t_i, y(t_i)))}{h} \\ &= \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)) \qquad\qquad i = 0, 1, \ldots, N-1 \end{aligned}$$

## 2.1 Example: Euler's Method (23.11)

The difference method for Euler's Method has $\phi = f$.

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i) = w_i + hf(t_i, w_i) \end{aligned}$$

has local truncation error

$$\begin{aligned} \tau_{i+1}(h) &= \frac{y(t_{i+1}) - y(t_i)}{h} - f(t_i, y(t_i)) \\ &= \frac{y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi)}{den} \qquad\qquad \text{we do a taylor expansion} \end{aligned}$$

Local truncation errors are called local because they measure hte accuracy of the method at a specific step, assuming the method was exact at the previous steps. We obviously want the local truncation error to be small. Often, methods for solving ODE's are derived so that the local truncation errors are of the form

$$O(h^p).$$

for the largest possible $p$, while keeping the number of operations reasonable.

### 2.2 How to obtain improved accuracy?

**i.e.** a larger $p$ in the $O(h^p)$ local truncation error.

Suppose we want to approximate the solution to the ivp

$$\begin{cases} y' = f(t, y) & a \leq t \leq b \\ y = \alpha & t = 0. \end{cases}.$$

where $y(t) \in C^{(n+1)}[a, b]$

One approach is to expand the solution in terms of its $n^{th}$ Taylor Polynomial about $t_i$.

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + R \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + R \end{aligned}$$

where $\quad R = \dfrac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$

# 3 The Taylor Method of Order $n$

If we drop the remainder term, we obtain the **Taylor Method of Order** $n$.

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hT^{(n)}(t_i, w_i) & i = 0, 1, \ldots, N-1. \end{cases}$$

where $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^n}{n!} f^{(n)}(t_i, w_i)$ is the $n^{th}$ Taylor Polynomial of $f$ about $t_i$.

Note: *Euler's Method is equivalent to Taylor's Method of Order* 1.