Initial Value Problems for Ordinary Differential Equations

Many natural scientific & engineering problems can be described in terms of <u>differential equations</u>.

Differential equations give us a way to mathematically express & study quantities with rates of change.

We will be considering methods for treating ordinary differential equations (ODEs). ODEs only consider derivatives with respect to one variable.

Ex Let $y(t)$ denote the number of individuals in a certain population. If this population has a constant growth rate α (the difference between a constant birth rate & death rate) then the differential equation

$$y'(t) = \alpha y(t)$$

with initial condition $y(0) = y_0$ describes the population growth.

Few problems originating from the study of physical phenomena can be solved exactly.

We next consider numerical methods for approximating the solution $y(t)$ to a problem

$$\frac{dy}{dt} = f(t, y) \quad \text{for } a \leq t \leq b$$

subject to the initial condition

$$y(a) = \alpha.$$

The elementary theory of initial value problems.

We want/need some theoretical results, in particular, we would like to show that solutions to equations exist & are unique.

<u>Def</u> A function $f(t,y)$ satisfies a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists such that

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

whenever $(t,y_1),(t,y_2) \in D$.

The constant L is called a Lipschitz constant for f.

Q Does $f(t,y) = ty$ satisfy a Lipschitz condition on $D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}$

If so, what is L?

Ans

Q Does $f(t,y) = -ty + \frac{4t}{y}$ satisfy a Lipschitz condition on $D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}$? If so, what is L?

Def A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever

(t_1, y_1) and (t_2, y_2) belong to D, the point

$$((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$$

also belongs to D for each $\lambda \in [0,1]$.

Thm Suppose $f(t,y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \leq L$$

for all $(t,y) \in D$ then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Pf Let (t,y_1) and (t,y_2) be in D.
Holding t fixed, define $g(y) = f(t,y)$.

Suppose $y_1 < y_2$. Since the line joining (t,y_1) to (t,y_2) lies in D and f is C on D we have $g \in C[y_1, y_2]$.

Furthermore, $g'(y) = \dfrac{\partial f}{\partial y}(t,y)$

Using the Mean Value Theorem on g, a number ξ, with $y_1 < \xi < y_2$ exists so that

$$g(y_2) - g(y_1) = g'(\xi)(y_2 - y_1)$$

Thm Suppose that $D = \{(t,y) : a \le t \le b, -\infty < y < \infty\}$ and that $f(t,y)$ is continuous on D.
If f satisfies a Lipschitz condition on D in the variable y, then the IVP

$$y'(t) = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha$$

has a unique solution $y(t)$ for $a \le t \le b$.

Ex Show that the IVP

$$y' = y\cos t \quad 0 \le t \le 1, \quad y(0) = 1$$

has a unique solution.

Thm

The initial value problem

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \; y(a) = \alpha$$

is said to be well-posed if:

- A unique soln, $y(t)$, to the problem exists, and
- There exists constants $\varepsilon_0 > 0$ & $k > 0$ s.t. for any ε with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is CTS with $|\delta(t)| < \varepsilon$ for all t in $[a,b]$ and when $|\delta_0| < \varepsilon$, the IVP

$$\frac{dz}{dt} = f(t,z) + \delta(t), \quad a \le t \le b,$$

$$z(a) = \alpha + \delta_0$$

has a unique soln $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon$$

for all t in $[a,b]$.

The perturbed problem assumes the possibility of an error

$$|\delta(t)| < \varepsilon$$

being introduced in the statement of the differential equation as well as an error

$$|\delta_0| < \varepsilon$$

being present in the initial condition.

Numerical methods always solve perturbed problems since roundoff errors perturb the original problem.

Thm. Suppose $D = \{(t,y) : a \le t \le b, -\infty < y < \infty\}$

If f is CTS and satisfies a Lipschitz condition in the variable y on the set D, then the IVP

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b$$

$$y(a) = \alpha$$

is well-posed.

Ex Show that the IVP

$$y' = t^2 y + 1, \quad 0 \le t \le 1$$
$$y(0) = 1$$

is well-posed.

Ans

Euler's Method

Our first numerical scheme for IVPs is Euler's Method, a very simple but low order method.

Consider the IVP

$$y' = f(t, y) \quad , \quad a \le t \le b$$
$$y(a) = y_0$$

We will compute an approximation to the solution at the mesh points

$$t_k = a + kh \ , \quad k = 0, 1, \ldots, N$$

where $h = \frac{b-a}{N}$ is called the <u>step size</u>. Here we have assumed h is constant, although variable step sizes are also useful.

To derive, consider a Taylor expansion

$$y(t_{k+1}) = y(t_k + h) = y(t_k) + h y'(t_k) + \frac{h^2}{2} y''(\xi_k)$$

$$= y(t_k) + h f(t_k, y(t_k)) + \frac{h^2}{2} y''(\xi_k)$$

Euler's Method constructs an approximation

$$w_k \approx y(t_k)$$

by dropping the remainder term

$$w_0 = y_0$$

$$w_k = w_{k-1} + h f(t_{k-1}, w_{k-1}) \quad k = 1, 2, \ldots N$$

Geometric interpretation:

At a given point (t_k, w_k) we march for a given finite step size h in a direction given by the slope $f(t_k, w_k)$.

Euler's Method has a straightforward error analysis.

Proof requires a

Lemma: If s & t are positive real numbers & $\{a_i\}_{i=0}^{K}$ is a sequence satisfying

$$a_0 \geq -t/s,$$

$$a_{i+1} \leq (1+s)a_i + t, \quad i = 0, 1, \dots, k$$

then $a_{i+1} \leq e^{(i+1)s}(a_0 + t/s) - t/s$.

pf. see text/notes.

Thm Suppose f is CTS & satisfies a Lipschitz condition with constant L on

$$D = \{(t,y) : a \le t \le b, -\infty < y < \infty\}$$

and a constant M exists with the property that

$$|y''(t)| \le M$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t,y), \quad a \le t \le b$$

$$y(a) = y_0$$

and $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's Method.

Then for each $i = 0, 1, \ldots, N$

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Pf. Clearly true for $i=0$.

Consider $i=1,2,\ldots$

$$
\begin{aligned}
|y(t_{i+1})-w_{i+1}| &= \Big|y(t_i)-w_i+h\big[f(t_i,y_i)-f(t_i,w_i)\big] + \tfrac{h^2}{2}y''(\xi_i)\Big| \\
&\le |y(t_i)-w_i| + h|f(t_i,y_i)-f(t_i,w_i)| + \tfrac{h^2}{2}|y''(\xi_i)| \\
&\le |y(t_i)-w_i| + hL|y(t_i)-w_i| + \tfrac{h^2}{2}M
\end{aligned}
$$

rearrange & apply Lemma to obtain result. ∎

Note that the theorem requires that

$$|y''(t)| \le M$$

The second derivative $y''(t)$ may not be known, but if $\partial f/\partial t$ and $\partial f/\partial y$ exist:

$$
\begin{aligned}
y''(t) &= \frac{d}{dt}y'(t) = \frac{d}{dt}f(t,y(t)) \\
&= \frac{\partial f}{\partial t}(t,y(t)) + \frac{\partial f}{\partial y}(t,y(t))\cdot f(t,y(t)).
\end{aligned}
$$

Ex What value of h is needed to ensure that

$$|y(t_i) - w_i| \le 0.1$$

for the IVP

$$y' = \frac{2}{t} y + t^2 e^t \qquad 1 \le t \le 2.$$
$$y(1) = 0$$

You are given $y''(t) = (2 + 4t + t^2)e^t - 2e$

ANS

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We need a way to compare the efficiency of different approximation methods

One approach is to compare how much the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation.

Def The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i)$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - (y(t_i) + h\phi(t_i, y(t_i)))}{h}$$
$$= \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i))$$
$$i = 0, 1, \ldots, N-1.$$

Ex <u>Euler's method</u>

The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hf(t_i, w_i)$$

has local truncation error

$$\tau_{i+1}(h) =$$

Local truncation errors are called local because they measure the accuracy of the method at a specific step, assuming the method was exact at the previous steps.

We obviously want the local truncation error to be small.

Often, methods for solving ordinary differential equations are derived so that the errors are of the form

$$O(h^p)$$

for the largest p possible, while keeping the number of operations reasonable.

How to obtain improved accuracy?

ie a larger p in the $O(h^p)$ local truncation error.

Suppose we want to approximate the solution to the IVP

$$y' = f(t, y) \qquad a \le t \le b$$
$$y(0) = \alpha$$

where $y(t) = C^{(n+1)}[a, b]$.

One approach is to expand the solution in terms of its n^{th} Taylor polynomial about t_i:

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i)$$

$$+ \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + R$$

where $R = \dfrac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$

If we drop the remainder term, we obtain the

Taylor Method of Order n

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$$
$$i = 0, 1, \ldots, N-1$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

Note: Euler's Method is just Taylor's Method of order one.

Q Use Taylors method of order two to approximate the solution for the IVP

$$y' = t e^{3t} - 2y, \quad 0 \le t \le 1$$
$$y(0) = 0 \qquad \text{with } h = 0.5$$

__Ans__

We have

$$w_1 = w_0 + h(t_0 e^{3t_0} - 2w_0) + \frac{h^2}{2}(t_0 e^{3t_0} + e^{3t_0} + 4w_0)$$
$$= 0.125$$

$$w_2 = w_1 + h(t_1 e^{3t_1} - 2w_1) + \frac{h^2}{2}(t_1 e^{3t_1} + e^{3t_1} + 4w_1)$$
$$\approx 2.02323897$$

Suppose that we want to determine an approximation at some intermediate point

eg, for some $t \in (t_{i-1}, t_i)$

The local truncation error for Taylor's method of order n is easily derived

$$y_{i+1} = y_i + h f(t_i, y_i) + h^2 f'(t_i, y_i) + \dots + \frac{h^n}{n!} f^{(n-1)}(t_i, y_i)$$
$$+ \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

where $y_i \equiv y(t_i)$

Thus the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) - \frac{h}{2} f'(t_i, y_i)$$
$$- \dots - \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, y_i)$$
$$= \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

Thus if $y \in C^{n+1}[a,b]$

$\Rightarrow y^{(n+1)}(t) = f^{(n)}(t, y(t))$ is bounded

& $\tau_i = O(h^n)$ for each $i = 1, 2, \dots, N$.

Runge-Kutta Methods

Taylor methods are seldom used in practice because they require the computation and evaluation of the derivatives of $f(t,y)$. These evaluations can be complicated and expensive.

Runge-Kutta methods have the high local truncation error of the Taylor methods but do not need to compute and evaluate derivatives of $f(t,y)$.

We now derive a class of Runge Kutta methods.

We take the increment to w to be a weighted average of F-values:

Now derive a scheme by choosing the 4 parameters a, b, α, β to minimize the local truncation error.

Re-writing

$$w_{n+1} = w_n + ah\, f(t_n, w_n) + bh\, f(t_n + \alpha h, w_n + \beta h f(t_n, w_n))$$

The local truncation error is

$$\tau_{n+1}(h) = \frac{y_{n+1} - y_n}{h} - af(t_n, y_n) - bf(t_n + \alpha h, y_n + \beta h f(t_n, y_n))$$

Apply a Taylor series:

$$y_{n+1} = y_n + h f(t_n, y_n) + \frac{h^2}{2} f'(t_n, y_n) + O(h^3)$$

$$f(t_n + \alpha h, y_n + \beta h f(t_n, y_n))$$
$$= f(t_n, y_n) + f_t(t_n, y_n)\alpha h$$
$$+ f_y(t_n, y_n) f(t_n, y_n)\beta h + O(h^2)$$

Some examples of second order Runge-Kutta methods

$$\left.\begin{array}{l} a=0 \\ b=1 \\ \alpha=1/2 \\ \beta=1/2 \end{array}\right\} \Rightarrow w_{n+1}=w_n+hf\left(t_n+\frac{h}{2}, w_n+\frac{h}{2}f(t_n,w_n)\right)$$

Midpoint method.

$$\left.\begin{array}{l} a=\frac{1}{2} \\ b=1/2 \\ \alpha=1 \\ \beta=1 \end{array}\right\} \Rightarrow w_{n+1}=w_n+\frac{h}{2}\left[f(t_n,w_n)+f(t_{n+1}, w_n+hf(t_n,w_n))\right]$$

Modified Euler

$$\left.\begin{array}{l} a=\frac{1}{4} \\ b=3/4 \\ \alpha=2/3 \\ \beta=2/3 \end{array}\right\} \Rightarrow w_{n+1}=w_n+\frac{h}{4}\left[f(t_n,w_n)+3f\left(t_n+\frac{2}{3}h, w_n+\frac{2}{3}hf_n\right)\right]$$

Heun's method

errors

t_i	$y(t_i)$	midpoint Rule	modified Euler	Heun's method
0.0	0.5000000	0	0	0
0.2	0.8292986	0.0012986	0.0032986	0.0000542
0.4	1.2140877	0.0027277	0.0071677	0.0001127
0.6	1.6489406	0.0042814	0.0116982	0.0001747
0.8	2.1272295	0.0059453	0.0169938	0.0002390
1.0	2.6408591	0.0076923	0.0231715	0.0003035
1.2	3.1799415	0.0094781	0.0303627	0.0003653
1.4	3.7324000	0.0112346	0.0387138	0.0004197
1.6	4.2834838	0.0128620	0.0483866	0.0004608
1.8	4.8151763	0.0142177	0.0595577	0.0004797
2.0	5.3054720	0.0151025	0.0724173	0.0004648

Third order Runge-Kutta methods are less commonly used.

Fourth order Runge-Kutta methods are widely used & are derived in a similar fashion.

Greater complexity in the derivations:

- have to compare terms through h^4
- gives a set of 11 equations and 13 unknowns
- can be solved, w/ 2 free parameters

Most common choice:

$$w_{n+1} = w_n + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$K_1 = h\, f(t_n, w_n)$$
$$K_2 = h\, f(t_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}K_1)$$
$$K_3 = h\, f(t_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}K_2)$$
$$K_4 = h\, f(t_n + h, w_n + K_3)$$

t_i	$y(t_i)$	Heun's method	RK4
0.0	0.5000000	0	0
0.2	0.8292986	0.0000542	0.0000053
0.4	1.2140877	0.0001127	0.0000114
0.6	1.6489406	0.0001747	0.0000186
0.8	2.1272295	0.0002390	0.0000269
1.0	2.6408591	0.0003035	0.0000364
1.2	3.1799415	0.0003653	0.0000474
1.4	3.7324000	0.0004197	0.0000599
1.6	4.2834838	0.0004608	0.0000743
1.8	4.8151763	0.0004797	0.0000906
2.0	5.3054720	0.0004648	0.0001089

Note :

Evaluation per step	2	3	4	$5 \le n \le 7$	$8 \le n \le 9$	$n \ge 10$
Best LTE						