MACM 316 Lecture 28

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Friday, March 21, 2025

Lecture Outline

1. Starts with a review of the previous lecture, including:
   (a) Interpolation with two points (Trapezoidal Rule)
   (b) Interpolation with three points (Simpson's Rule)
2. The **error analysis** of numerical integration methods. **(1)**.
   (a) Simpson's Rule, despite using a quadratic approximation, can integrate up to a cubic. (Integrating a quadratic gives a cubic, and the method specifically finds a nice cancellation that allows higher order approximations)
   (b) Definition of the degree of accuracy 1.1
3. It turns out that Simpson's Rule and the Trapezoidal Rule are not the only methods of numerical integration. In fact, there are infinitely many methods of numerical Integration: The **Newton-Cotes Formulas (2)**.
   (a) Even-degree Newton-Cotes formulas are more accurate than odd.
   (b) The formulas are **exact** if the degree of accuracy is greater than or equal to the degree of the function it is approximating.
   (c) The Newton-Cotes formulas are **closed** if the endpoints of the interval are included as nodes. The formulas are **open (2.1)** if the nodes are all contained in the open interval (a, b).

4. Finally, we discuss the **composite numerical integration (3)** method. This method uses Newton-Cotes formulas to approximate the integral of a function over an interval.

1 Error Analysis of Numerical Integration Methods

$f(x)$	Simpson's	Trapezoidal
x	0	0
x^3	nonzero	0

Simpson's Rule can integrate up to a cubic, even though the function used to approximate the integral is a quadratic. This is because in Simpson's Rule, the quadratic approximation is integrated and there is a cancellation that occurs, which allows the quadratic approximation to be used to approximate the integral.

1.1 Degree of Accuracy

Def. The **degree of accuracy** or precision of a quadratic formula is the largest positive integer n such that the formula is exact for x^k when $k = 0, 1, \ldots, n$.

	Degree of Accuracy
Trapezoid Rule	1
Simpson's Rule	3

2 Newton-Cotes Formulas

The Trapezoid and Simpson's Rules are examples of **Newton-Cotes Formulas**. The $(n+1)$-point closed Newton-Cotes formula uses nodes $x_i = x_0 + ih$ for $i = 0, 1, \ldots, n$ where

$$
\begin{aligned}
x_0 &= a, \\
x_n &= b, \\
h &= \frac{b-a}{n}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx &= \int_a^b \sum_{i=0}^{n} h_i(x) f(x_i)\,dx \\
&= \sum_{i=0}^{n} \int_a^b L_i(x) f(x_i)\,dx \\
&= \sum_{i=0}^{n} a_i f(x_i)
\end{aligned}
$$

where $a_i = \displaystyle\int_a^b L_i(x)\,dx$

The formula is closed because the endpoints of the interval are included as nodes. An error analysis of the Newton-Cotes formulas gives an interesting result:

Thm. Suppose that $\displaystyle\sum_{i=0}^{n} a_i f(x_i)$ denotes the $n+1$ point closed Newton-Cotes formula with $x_0 = a, x_n = b$ and $h = \frac{b-a}{n}$. If n is even and $f \in C^{n+2}[a,b]$ then there exists $\xi \in (a,b)$ with

$$
\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\ldots(t-n)\,dt.
$$

If n is odd and $f \in C^{n+1}[a,b]$ then there exists $\xi \in (a,b)$ with

$$
\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{n+1}(\xi)}{(n+1)!} \int_0^n t(t-1)\ldots(t-n)\,dt.
$$

Notice that the degree of precision is $n+1$ and the error is $O(h^{n+3})$ if n is even. If n is odd then the degree of precision is only n and hte error is only $O(h^{n+2})$.

n	**Name**	**Error Term**
1	Trapezoid Rule	$-\frac{h^3}{12}f''(\xi)$
2	Simpson's Rule	$-\frac{h^5}{90}f^{(4)}(\xi)$
3	Simpson's Three-Eighths Rule	$-\frac{3h^5}{80}f^{(4)}(\xi)$
4		$-\frac{h^7}{945}f^{(6)}(\xi)$

2.1 Open Newton-Cotes Formulas

There are also open Newton-Cotes formulas. here,

$$x_i = x_0 + ih \qquad i = 0, 1, \ldots, n$$
$$x_0 = a + h$$
$$h = \frac{b-a}{n+2}$$

Then the open Newton-Cotes formulas are given by

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i). \tag{1}$$

where $a_i = \displaystyle\int_a^b L_i(x)\,dx$

Note that $x_0 = a + h$ and $x_n = b - h$. The formulas are open because the nodes are all contained in the open interval (a, b). Once again, if n is even, the degree of precision is $(n + 1)$ and the error is $O(h^{n+3})$. If n is odd, the degree of precision is only n and the error is only $O(h^{n+2})$.

Some examples of open Newton-Cotes formulas are:

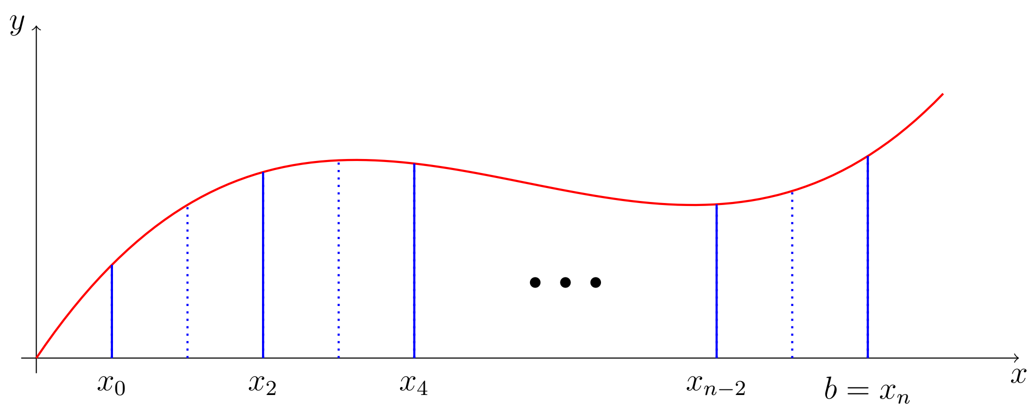
n	
0	$2hf(x_0) + \frac{h^3}{3}f''(\xi), \quad \text{where } \xi \in (a, b)$
1	$\frac{3h}{2}\left[f(x_0) + f(x_1)\right] + \frac{3h^3}{4}f''(\xi), \quad \text{where } \xi \in (a, b)$
2	$\frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2)\right] + \frac{14h^5}{45}f^{(4)}(\xi)$
3	$\frac{5h}{24}\left[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\right] + \frac{95}{144}h^5 f^{(4)}(\xi)$

$n = 0$ is also called the **Midpoint Rule**.

3 Composite Numerical Integration

Typically, we do not apply Newton-Cotes formulas directly onto the interval $[a, b]$. If we did, then high degree formulas would be required to obtain accurate solutions, but as we have already seen, these high degree polynomials would give an oscillatory and innacurate interpolation. To avoid this problem, we prefer a piecewise approach to numerical integration that uses low order Newton-Cotes formulas.

Ex. Simpson's Rule



We divide the interval into an even number of subintervals. Simpson's rule is applied on each consecutive pair of subintervals.

$$\frac{h}{3}\left[f(x_i) + 4f(\frac{x_i + x_{i+2}}{2}) + f(x_{i+2})\right].$$

Take $h = \dfrac{(b-1)}{n}$ and $x_j = a + jh$. Then,

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{j=1}^{\frac{n}{2}} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx \\ &= \sum_{j=1}^{\frac{n}{2}} \left\{\frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] - \frac{h^5}{90} f^{(4)}(\xi_j)\right\} \\ &\quad \text{where } x_{2j-2} < \xi_j < x_{2j} \\ &\quad \text{and } f \in C^4[a, b]\end{aligned}$$

taking into account that $f(x_{2j}), 0 < j < \frac{n}{2}$ appears in 2 terms, this summation can be simplified somewhat

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(x_n)\right] + \text{error}.$$

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