MACM 316 Lecture 25 - Chapter 3 Part 2 - Parametric Curves

Alexander Ng

Friday, March 14, 2025

1 Parametric Curves (C3*2-16.3)

The interpolating polynomials and splines that we have previously discussed can only be used to interpolate functions. The techniques can be extended to represent general curves in space that aren't necessarily functions, or even curves which self-intersect.

Suppose we wish to determine a polynomial or a pieceewise polynomial to connect the points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n).$$

in the order given.

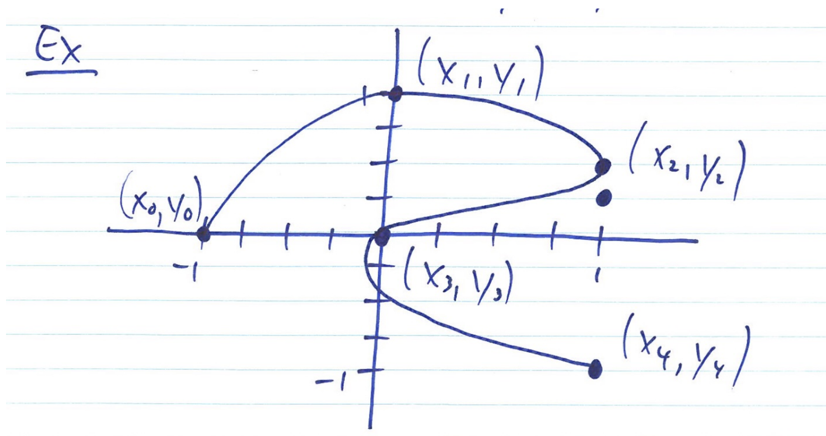
We can define a parameter t with the interval $[t_0, t_n]$ with

$$t_0 < t_1 < \cdots < t_n.$$

and construct approximation functions for x and y separately:

$$x_i = x(t_i), \quad y_i = y(t_i).$$

Ex.



There is flexibility in choosing the points t_i. Suppose we choose them to be evenly spaced over $[0, 1]$. Then we can write

i	0	1	2	3	4
t_i	0	.25	.5	.75	1
x_i	-1	0	1	0	1
y_i	0	1	.5	0	-1

We could apply Lagrange interpolation for x (or y) as a function of t. The Lagrange Interpolating polynomials for x and y are

$$x(t) = \left(\left(\left(64\,t - \frac{352}{3}\right) t + 60\right) t - \frac{1}{3}\right) t - 1$$

$$y(t) = \left(\left(\left(-\frac{64}{3} t + 48\right) t - \frac{116}{3}\right) t + 11\right) t$$

Alternatively, we could use a spline-based interpolation for x and y.

Both of these approaches have the disadvantage that moving a single data point affects the entire curve. We want the geometric property that changing one of the points on the curve only changes one portion of the curve. **i.e.** we want the curve to only be affected locally by changes in the data.

1.1 Piecewise Cubic Hermite Polynomials

Because we want to avoid global changes, we might use a piecewise cubic Hermite polynomial, which is just a spline with specific properties. We use one polynomial for x and one for y, both with respecto to t.

In other words, we specify the endpoints and the derivatives and the endpoints to specify each portion of the curve. Thus, changin ga datapoint will only change the two portions adjacent to the point, which means that smooth curves can be easily and quickly modified.

1.1.1 Uniqueness

Quick answer: No.

1.1.2 The Derivatives and Tangent Lines (C3*2-16.7)

Suppose that the endpoints are at $t = 0$ and $t = 1$, then we only need to satisfy the conditions on the quotients

$$\frac{dy}{dx}(t=0) = \frac{y'(0)}{x'(0)}, \quad \frac{dy}{dx}(t=1) = \frac{y'(1)}{x'(1)}.$$

The actual values of $x'(0)$ and $y'(0)$ can be scaled by a common factor and still satisfy these conditions. The larger the scaling factor, the closer the curve comes to satisfying the tangent line near $(x(0), y(0))$.

A similar situation holds for

$$(x(1), y(1)).$$

1.1.3 Guide Points (Cubic Bezier Curves)

To simplify the process of specifying the slopes and to obtain a unique curve, commercial software commonly specifies a second point called a “guidepoint” which lies on the desired tangent line.

Suppose the endpoints are

$$(x(0), y(0)), \text{ and } (x(1), y(1)).$$

and let the guidepoints be

$$(x_0 + \alpha_0, y_0 + \beta_0), \text{ and } (x_1 + \alpha_1, y_1 + \beta_1).$$

we will insist that x satisfies

$$x(0) = x_0 \qquad x'(0) = \alpha_0$$
$$x(1) = x_1 \qquad x'(1) = \alpha_1$$

and that y satisfies

$$y(0) = y_0 \qquad y'(0) = \beta_0$$
$$y(1) = y_1 \qquad y'(1) = \beta_1$$

Now there is a unique solution for x and y:

$$\begin{aligned} x(t) = {} & [2(x_0 - x_1) + (\alpha_0 + \alpha_1)]\, t^3 \\ & + [3(x_1 - x_0) - (\alpha_1 + 2\alpha_0)]\, t^2 \\ & + \alpha_0 t + x_0 \end{aligned}$$

$$\begin{aligned} y(t) = {} & [2(y_0 - y_1) + (\beta_0 + \beta_1)]\, t^3 \\ & + [3(y_1 - y_0) - (\beta_1 + 2\beta_0)]\, t^2 \\ & + \beta_0 t + y_0 \end{aligned}$$

Popular graphics programs will typically use a slightly modified form for x and y. They use Bézier polynomials which scale the derivatives x' and y' by a factor of **3** at the endpoints.

$$\begin{aligned} x(t) = {} & [2(x_0 - x_1) + (\alpha_0 + \alpha_1)]\, t^3 \\ & + [3(x_1 - x_0) - (\alpha_1 + 2\alpha_0)]\, t^2 \\ & + \mathbf{3}\alpha_0 t + x_0 \end{aligned}$$

$$\begin{aligned} y(t) = {} & [2(y_0 - y_1) + (\beta_0 + \beta_1)]\, t^3 \\ & + [3(y_1 - y_0) - (\beta_1 + 2\beta_0)]\, t^2 \\ & + \mathbf{3}\beta_0 t + y_0 \end{aligned}$$

1.1.4 Fun Notes

Generally, given that you don't have two points on the same point, i.e. you don't have $(x(0), y(0)) = (x(1), y(1))$, you can guarantee that the output curve will be continuous with a continuous derivative. (i.e. $P \in C^1[a, b]$).

2 Numerical Differentiation (C4*1-17.1)

We also need to approximate the derivatives of functions. One approach is to differentiate Lagrange polynomial approximations.

Suppose $x_0, x \in (a, b)$ and $f \in C^2[a, b]$.

Now

$$\begin{aligned} f(x) &= P_{0,1}(x) + \frac{1}{2!}(x - x_0)(x - x_1)f''(\xi(x)) \\ &= \frac{f(x_0)(x - x_1)}{x_0 - x_1} + \frac{f(x_1)(x - x_0)}{x_1 - x_0} + \frac{(x - x_0)(x - x_1)}{2!} f''(\xi(x)) \\ &\text{where } \xi(x) \in [a, b] \end{aligned}$$

Now differentiate:

$$\begin{aligned} f'(x) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} + D_x \left[\frac{(x - x_0)(x - x_1)}{2!} f''(\xi(x)) \right] \\ &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{2(x - x_0)(x - x_1)}{2} f''(\xi(x)) \\ &\quad + \frac{(x - x_0)(x - x_1)}{2} D_x \left(f''(\xi(x)) \right) \end{aligned}$$

2.1 More remarks

Chapter 4 also considers numerical integration. We know how to integrate polynomials, so we can just integrate the lagrange polynomial and call it a day.