MACM 316 Lecture 10

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1 Speical Types of Matrices

Where can Gauassian Elimination be performe without row exchanges?

1.1 Strictly Diagonally Dominant Matrices

Def. An $n \times n$ matrix A is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1; j \neq i}^{n} |a_{ij}| \text{ for all } i = 1, \ldots, n$$

Example

$$\begin{bmatrix} 3h & h & -h \\ 4 & 10 & 4 \\ 1 & 1 & -3 \end{bmatrix}$$

This matrix is strictly diagonally dominant when $h \neq 0$.

1.1.1 Thm. 1

A strictly diagonally dominant matrix A is nonsingular.

1.1.2 Thm. 2

Let A be a strictly diagonally dominant matrix. Then Gaussian Elimination can be performed on any linear system of the form $Ax = b$ to obtain its unique solution without row or column interchanges, and the computations are stable to the growth of roundoff error.

1.2 Symmetric, Positive Definite Matrices

Def. A matrix A is <u>positive definite</u> if

$$\forall x \neq 0 (x^T Ax > 0)$$

Example

Find all values of α for which the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}$$

is positive definite.

Ans.

$$\begin{aligned} x^T Ax &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x_1^2 + x_2^2 + \alpha x_3^2 - 2x_1x_3 + 2x_2x_3 \\ &= (x_1 - x_3)^2 + (x_2 + x_3)^2 + (\alpha - 2)x_3^2 \end{aligned}$$

A is positive definite $\iff \alpha > 2$.

Some necessary conditions for an $n \times n$ matrix to be positive definite u include:

(a) A is nonsingular

(b) $a_{ii} > 0$ for each $i = i...n$

(c) ???

(d) $(a_{ij})^2 < a_{ii}a_{jj}$ for each $i \neq j$

HOWEVER: These are not sufficient conditions for positive definiteness. they are necessary but not sufficient.

We would like necessary and sufficient conditions for a matrix to be positive definite.

1.3 Leading Principle Submatrices

A leading principle submatrix of a matrix A is a matrix of the form

$$A_k = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{bmatrix}$$

for some $1 \leq k \leq n$.

Based on this definition, we have the following theorem:

1.3.1 Thm. 3

A symmetrix matrix A is positive definite if and only if each of its leading principle submatrices has a positive determinant.

As it turns out, we don't need to carry out row exchanges when Gaussian Elimination is used on a symmetric, positive definite matrix.

1.3.2 Thm. 4

A symmetric matrix A is positive definite if and only if Gaussian Elimination withour row exchanges can be performed on the linear system $Ax = b$ with all the pivot elements positive. Moreover, in this case, the computations are stable with respect to the growth of roundoff error.

Corollary: The matrix A is symmetric positive definite if and only if A can be factored in the form LDL^T where L is lower triangular with 1's on its diagonal and D is a diagonal matrix with positive diagonal entries.

A modification of the LU factorization algorithm can be made to factor a symmetric positive definite matrix into the form

$$A = LDL^T$$

This LDL^T factorization only requires $\frac{n^3}{6}+n^2-\frac{7n}{6}$ multiplications/divisions, and $\frac{n^3}{6}-\frac{n}{6}$ additions/subtractions. This is only half the number of operations as LU factorization.

A version of this algorithm can also be constructed for matrices that are symmetric but not positive definite.

Corollary 2: The matrix A is positive definite if and only if A can be factored int he form LL^T where L is lower triangular with nonzero diagonal entries. Once again, a modification of the LU factorization algorithm can be made. This method, called Choleski's Algorithm, factors a symmetric positive definite matrix into the form

$$A = LL^T$$

Choleski's Algorithm only requires $\frac{n^3}{6}+\frac{n^2}{2}-\frac{2n}{3}$ multiplications/divisions, and $\frac{n^3}{6}-\frac{n}{6}$ additions/subtractions, which is even less than the LDL^T factorization. However, for small n, Choleski's Algorithm may be slower because it requires n square roots to be computed.

1.4 Band Matrices

Another important class of matrices that arise in a wide variety of applications are band matrices. A band matrix is a matrix of the form