# MACM 316 Lecture 27

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## 1 Overview

The most important parts of this lecture are the following:

1. **Richardson's Extrapolation**: A method for obtaining higher order accuracy from lower order formulas. link: 2
   (a) 2.0.1 illustrates the procedure of building the richardson extrapolation table.
   (b) 2.1 illustrates the use of Richardson's Extrapolation to obtain a higher order approximation to the integral of $\sin x$.
2. **Numerical Integration**: A method for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain. link: 3
   (a) 3.1 illustrates the procedure of using a 2 point integration formula.
   (b) 3.1.1 illustrates the procedure of using the weighted mean value theorem for integrals.
   (c) 3.2 illustrates the procedure of using a 3 point integration formula.

## 2 Richardson's Extrapolation (C4*1-17.10)

When the error depends on some parameter such as the step size $h$ and the dependency is predictable, we can often derive higher order accuracy from low order formulas. To illustrate the procedure, assume we have an approximation $N(h)$ to some quantity $M$. Assume this approximation has an order $h$ truncation error and that we know the expression for the first few terms of the trunction error,

$$M = N(h) + k_1 h + k_2 h^2 + k_3 h^3 + \dots \tag{1}$$

where the $k_i$'s are constants, $h$ is a positive parameter and $N(h)$ is an $O(h)$ approximation to $M$. We can repeat the calculation with a parameter $\frac{h}{2}$:

$$M = N\left(\frac{h}{2}\right) + \frac{k_1}{2}h + \frac{k_2}{4}h^2 + \frac{k_3}{8}h^3 + \dots. \tag{2}$$

We want to obtain a higher order method by using some combination of these results. Subtracting (1) from twice (2) gives:

$$M = \left[2N\left(\frac{h}{2}\right) - N(h)\right] + k_2\left(\frac{h^2}{2} - h^2\right) + k_3\left(\frac{h^3}{4} - h^3\right) + \dots \tag{3}$$

which is an $O(h^2)$ approximation formula for $M$. For ease of notiation,

$$\text{Let } N_2(h) = 2N\left(\frac{h}{2}\right) - N(h).$$

Generally, if $M$ can be written as

$$M = N(h) + \sum_{j=1}^{m-1} K_j h^j + O(h^m),.$$

then for each $j = 2, 3, \dots, m$, we have an $O(h^j)$ approximation of the form

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{2^{j-1} - 1}..$$

In practice, because of how $N_j$ is defined, higher order approximations can be systematically derived from lower order approximations.

### 2.0.1 Building the Extrapolation Table (18.1)

$$
\begin{tikzcd}
O(h) & O(h^2) & O(h^3) & O(h^4) \\
N_1(h) \arrow[dr] & & & \\
N_1\left(\frac{h}{2}\right) \arrow[r] \arrow[dr] & N_2\left(\frac{h}{2}\right) \arrow[dr] & & \\
N_1\left(\frac{h}{4}\right) \arrow[r] \arrow[dr] & N_2\left(\frac{h}{4}\right) \arrow[r] \arrow[dr] & N_3\left(\frac{h}{2}\right) \arrow[dr] & \\
N_1\left(\frac{h}{8}\right) \arrow[r] & N_2\left(\frac{h}{8}\right) \arrow[r] & N_3\left(\frac{h}{4}\right) \arrow[r] & N_4(h)
\end{tikzcd}
$$

The cost of building the extrapolation table is $O(n)$ where $n$ is the inverse degree of the error term $(O(h^n))$. Each subsequent iteration of $N$ is defined solely by its previous iteration, therefore, the cost of building the extrapolation is the number of initial calculations of $N_1$ required to build out the table. Some pitfalls of this approach is that the calculation of $N_{j+1}$ requires a subtraction of two numbers that get closer and closer as the degree of the error term increases. This is a problem for higher order calculations, as you may run out of precision and introduce substantial machine error.

Extrapolation can be used whenever the truncation error for a formula has the form

$$\sum_{j=1}^{m-1} K_j h^{\alpha_j} + O(h^{\alpha_m}).$$

for constants $K_J$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_m$.

### 2.0.2 When $N_j(h)$ is an $O(h^{2j})$ approximation

Suppose $N_j(h)$ is an $O(h^{2j})$ approximation of $M$. Then, from the definition:

$$
\begin{aligned}
M &= N_j(h) + O(h^{2j}) \qquad \text{we add another term of } M: && (4)\\
&= N_j(h) + k_j(h^{2j}) + O(h^{2j+2}) && (5)\\
&= N_j(\frac{h}{2}) + k_j h^{2j} + O(h^{2j+2}) && (6)
\end{aligned}
$$

$2^{2j}$ (5) $-$ (6) gives:

$$M = N_j(\frac{h}{2}) + \frac{N_j(\frac{h}{2}) - N_j(h)}{2^{2j-1}} + O(h^{2j+2})$$

$$\boxed{\therefore N_{j+1}(h) \equiv N_j\left(\frac{h}{2}\right) + \frac{N_j\left(\frac{h}{2}\right) - N_j(h)}{4^{j-1}}}.$$

is an $O(h^{2j+2})$ approximation of $M$. Then, the table becomes

$$O(h^2) \quad O(h^4) \quad O(h^6) \quad O(h^8)$$

## 2.1 Richardson's Extrapolation Example (18.1)

**Ex.** The following data gives approximations to the integral

$$M = \int_0^{\infty} \sin x \, dx \tag{7}$$

$$N_1\left(\frac{h}{2}\right) = 1.570796, \quad N_1\left(\frac{h}{2}\right) = 1.896119$$

$$N_1\left(\frac{h}{4}\right) = 1.974232, \quad N_1\left(\frac{h}{8}\right) = 1.993570$$

Assuming

$$M = N_1(h) + K_1h^2 + K_2h^4 + k_3h^6 + k_4h^8 + O(h^{10}).$$

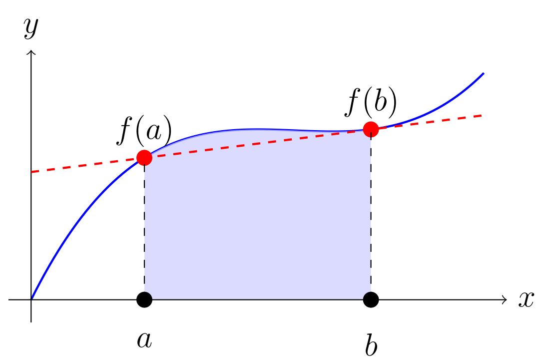
construct an extrapolation table to determine $N_4(h)$.
*Soln.* (from text)

# 3 Numerical Integration (18.4)

We often need to evaluate the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain. The usual strategy in developing formulas for numerical integration is similar to that for numerical differentiation. We pass a polynomial through points defined by the function and then integrate this polynomial approximation for the function. This permist us to use a function known only as a table of values. We get an expression for the error by integrating the error for our interpolating polynomial. As we progress, we will eventually turn to splitting our function into sub-intervals and using methods similar to spline interpolation to integrate each sub-interval.

## 3.1 2-Point Integration Formula (18.5)

Suppose we use a 2 point integration formula:




Let $x_0 = a$, $x_1 = b$, $h = b - 1$

The linear Lagrange polynomial passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$P_1(x) = \frac{(x - x_1)(x - x_0)}{f(x_0)} + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$

and

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_1} P_1(x)\,dx + \frac{1}{2}\int_{x_0}^{x} f''\left(\xi(x)\right)(x-x_0)(x-x_1)\,dx \\
&= \left.\frac{(x-x_1)^2}{2(x_0-x_1)} f(x_0) + \frac{(x-x_0)^2}{2(x_1-x_0)} f(x_1)\right|_{x_0}^{x_1} + \text{error} \\
&= \frac{h}{2}\left(f(x_0)+f(x_1)\right) + \text{error}.
\end{aligned}
$$

To evaluate the error we will need the **Weighted Mean Value Theorem for Integrals**.

### 3.1.1 Weighted Mean Value Theorem for Integrals (18.6)

If $f \in C[a,b]$, the Riemann Integral of $g$ exists on $[a,b]$ and $g(x)$ does not change sign on $[a,b]$ then there exists a number $c \in (a,b)$ such that

$$
\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x).
$$

---

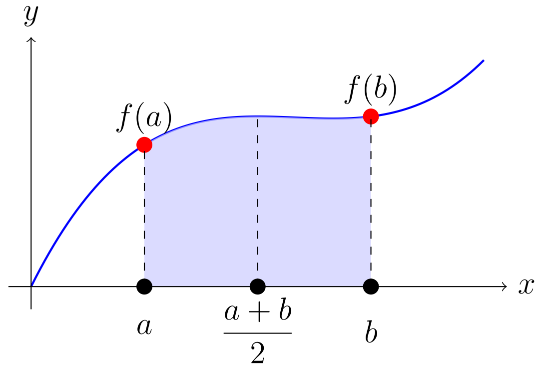
$$
\begin{aligned}
\text{error} &= \frac{1}{2}\int_{x_0}^{x_1} f''\left(\xi(x)\right)(x-x_0)(x-x_1)\,dx \\
&= \frac{1}{2} f''(\xi)\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx \quad [\xi \text{ some number in } (x_0,x_1)] \\
&= \frac{1}{2} f''(\xi)\left[\frac{x^3}{3} - \frac{(x_1+x_0)}{2}x^2 + x_0x_1x\right]_{x_0}^{x_1} \\
&= -\frac{h^3}{12} f''(\xi)
\end{aligned}
$$

$$
\text{Thus } \boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] - \frac{h^3}{12} f''(\xi)}
$$

This is known as the **Trapezoid Rule** since the integral is approximated by the area of a trapezoid.

## 3.2 Three Point Integration Formula (18.7)

We might also consider a 3 point integration formula based on equally spaced points:




If we use the usual strategy of integrating the error term for the Lagrange polynomial then we get an $O(h^4)$ error. A sharper estimate can be obtained using an alternative approach.

Expand $f$ about $x$, using the third Taylor polynomial:

$$
\begin{aligned}
f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)(x - x_1)^2}{2} \\
+ \frac{f'''(x_1)(x - x_1)^3}{6} + \frac{f^{(4)}(\xi(x))(x - x_1)^4}{24}
\end{aligned}
$$

$$
\begin{aligned}
\int_{x_0}^{x_2} P_3(f(x))\,dx = &\left[ f(x_1)(x - x_1) + \frac{f'(x_1)}{2}(x - x_1)^2 + \frac{f''(x_1)}{6}(x - x_1)^3 \right. \\
&\left. + \frac{f'''(x_1)}{24}(x - x_1)^4 \right]_{x_0}^{x_2} \\
&+ \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4\,dx
\end{aligned}
$$

$$
\begin{aligned}
\text{Consider} \quad & \frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(\xi(x))(x-x_1)^4\,dx \\
&= \frac{f^{(4)}(\xi_1)}{24}\int_{x_0}^{x_2}(x-x_1)^4\,dx \\
&= \frac{f^{(4)}(\xi_1)}{120}(x-x_1)^5\Big|_{x_0}^{x_2} \\
&= \frac{f^{(4)}(\xi_1)}{60}h^5
\end{aligned}
$$

$$
\therefore \int_{x_0}^{x_2} f(x)\,dx = 2hf(x_1) + \frac{h^3}{3}f''(x_1) + \frac{f^{(4)}(\xi_1)h^5}{60}.
$$

But, we know that $f''(x_1) = \dfrac{1}{h^2}\left[f(x_0) - 2f(x_1) + f(x_2)\right] + \dfrac{h^2}{12}f^{(4)}(\xi_1)$.

$$
\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &= 2hf(x_1) + \frac{h^3}{3}\left[\frac{1}{h^2}(f(x_0) - 2f(x_1) + f(x_2)) - \frac{h^2 f^{(4)}(\xi_2)}{12}\right] \\
&\quad + \frac{f^{(4)}(\xi_1)}{60}h^5 \\
&= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + O(h^5)
\end{aligned}
$$

This integration rule is known as **Simpson's Rule**:

$$
\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \underbrace{\frac{h^5}{90}f^{(4)}(\xi)}_{\text{part of assignment 4}}
$$