MACM 316 Lecture 22

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1 Review

1.1 Polynomial Interpolation

We have data for a function f aat $x_0, x_1, \dots, x_n$

1.2 Lagrange Interpolation

$$\sum_{j=0}^{n} f(x_j) \underbrace{L_j(x)}_{0 \text{ at } x_i \text{ when } i \neq j}.$$

Reminder: a degree n polynomial interpolant for $n+1$ points is unique. This means any polynomial interpolation method will give the same polynomial, just in a different form.

1.3 Divided Differences

Divided differences is an easy way to add data points.

Assume we know $f(x)$ at several values for x.

The x_i points do <u>not</u> need to be evenly spaced, or in any particular order.

We choose to represent our degree n interpolating polynomial as follows:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\dots(x - x_{n-1}).$$

For every term we add, we interpolate another data point. We choose a_i such that $P_n(x) = f(x)$ at the points $x_0, x_1, \dots, x_n$.

The coefficients a_i are determined by divided differences

$$P_n(x_0) = f(x_0).$$

$$a_0 = f(x_0) = f[x_0].$$

Define the zeroth divided difference

$$f[x_j] = f(x_j).$$

$$a_0 + a_1(x_1 - x_0) = f(x_1).$$

$$f[x_0] + a_1(x_1 - x_0) = f[x_1].$$

$$a_1 = \frac{f[x_1] - f[x_0]}{x_1 - x_0}.$$

Define first divided difference

$$f[x_i, x_{i+1}] = f[x_{i+1}, x_i] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

$$a_0 = f[x_0].$$

$$a_1 = f[x_0, x_1].$$

$$a_2 = f[x_0, x_1, x_2].$$

Define the second divided difference

$$\begin{aligned} & f[x_i, x_{i+1}, x_{i+2}] \\ & = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} \end{aligned}$$

Define the k^{th} divided difference for $x_i, \ldots, x_{i+k}$

$$
\begin{aligned}
&f[x_i, x_{i+1}, \ldots, x_{i+k}] \\
&\quad = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}
\end{aligned}
$$

$$a_k = f[x_0, x_1, \ldots, x_k].$$

This is Newton's interpolating divided difference formula.

$$
\begin{aligned}
P(x) = {} & f[x_0] + f_[x_0, x_1](x - x_0) \\
& + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
& + \ldots \\
& + f[x_0, x_1, \ldots, x_n](x - x_0)\ldots(x - x_{n-1})
\end{aligned}
$$

To compute an extra datapoint, we only need to compute one more term:

$$P_n(x) = P_{n-1}(x) + f[x_0, x_1, \ldots, x_n](x - x_0)\ldots(x - x_{n-1}).$$

Note that while using this method, we do NOT need to have our x_i points in any particular order.

1.3.1 Example

$$(x_0, \ldots, x_4) = (0.3, 1.0, 0.7, 0.6, 1.9).$$

Note that our points are not in order, or evenly spaced.

We are going to interpolate

$$f(x) = 2x^3 - x^2 + x - 1.$$

We can use a table!

Divided differences table:

$$\begin{array}{llll}
x_1 = 0.3 & f[x_1] = -0.736 & & \\
 & & 2.48 & \\
x_2 = 1.0 & f[x_2] = 1.0 & & \\
 & & 3.68 & \\
x_3 = 0.7 & f[x_3] = -0.104 & & ? \\
 & & 2.24 & \\
x_4 = 0.6 & f[x_4] = -0.328 & & \\
 & & 8.72 & \\
x_5 = 1.9 & f[x_5] = 11.008 & &
\end{array}$$

The first entry in each column gives the coefficients

$$\begin{aligned}
P_4(x) = & -0.736 \\
& + 2.48(x - 0.3) \\
& + 3(x - 0.3)(x - 1.0) \\
& + 2(x - 0.3)(x - 1.0)(x - 0.7) \\
& + 0(x - 0.3)(x - 1.0)(x - 0.7)(x - 0.6) \\
& + 0(x - 0.3)(x - 1.0)(x - 0.7)(x - 0.6)(x - 1.9).
\end{aligned}$$

2 KEY TAKEAWAY

Basically, using the table, we can get all the constants we need to compute the next datapoint by just looking back and grabbing the appropriate column.

3 Midterm Review

Find the rate of convergence as $h \to 0$:

$$\lim_{h \to 0} (e^h + e^{-h}) = 2.$$

1. expand $e^h + e^{-h} - 2$ using taylor series

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4).$$

$$
\begin{aligned}
e^h + e^{-h} - 2 = &\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4)\right) \\
&+ \left(1 - h + \frac{h^2}{2!} - \frac{h^3}{3!} + O(h^4)\right) \\
&- 2 \\
&= h^2 + O(h^3) = O(h^2)
\end{aligned}
$$

For midterm, know:

- common taylor series ($e^x, \sin(x), \cos(x), \ln(1+x), (1+x)^{-p}$)
- Partial Pivoting w/ or w/0 scaling

3.1 Another Problem

Consider

$$
\begin{aligned}
x_1 + 30x_2 &= 50 \\
5x_1 - 10x_2 &= 3
\end{aligned}
$$

Use Gaussian Elimination w/ scaled partial pivoting to take the system to upper triangular form.

$\frac{1}{30} < \frac{5}{10} \implies$ we need to exchange rows

$$
\begin{aligned}
5x_1 - 10x_2 &= 3 \\
x_1 + 30x_2 &= 50 \\
\hline
5x_1 - 10x_2 &= 3 \\
32x_2 = 50 - \frac{3}{5}
\end{aligned}
$$