MACM 316 Lecture 12

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Friday, January 31, 2025

Example 1

Prove that $x^{(k)} = \left(\frac{1}{k}, 1 + e^{1-k}, -\frac{2}{k^2}\right)^t$ is convergent w.r.t. $||\cdot||_2$.

We know $0 \leq ||x^{(k)} - x||_2 \leq \sqrt{3}||x^{(k)} - x||_\infty$

$$\lim_{k\to\infty} ||x^{(k)} - x||_\infty = 0 \implies \lim_{k\to\infty} ||x^{(k)} - x||_2 = 0$$

So, $\left\{x^{(k)}\right\}$ is convergent to x w.r.t. $||\cdot||_2$.

It can be shown that <u>all</u> norms on $\mathbb{R}^n$ are equivalent with respect to convergence.

i.e. If $||\cdot||_a$ and $||\cdot||_b$ are norms on $\mathbb{R}^n$, and $\left\{x^{(k)}\right\}_{k=1}^\infty$ has the limit x with respect to $||\cdot||_a$, then $\left\{x^{(k)}\right\}_{k=1}^\infty$ also has the limit x with respect to $||\cdot||_b$.

1 Matrix Norms

Def. A <u>matrix norm</u> on the set of all $n \times n$ matrices ($R^{n\times n}$) is a real-valued function $||\cdot||$ defined on this set satisfying for all $n \times n$ matries A and B and all real numbers α :

1. $||A|| \geq 0$
2. $||A|| = 0 \iff A = 0$
3. $||\alpha A|| = |\alpha| ||A||$
4. $||A + B|| \leq ||A|| + ||B||$

5. $||AB|| \leq ||A||||B||$

Def. A distance between two $n \times n$ matrices A and B is

$$||A - B||$$

1.1 Thm.

If $||\cdot||$ is a vector norm on $\mathbb{R}^n$, then

$$||A|| = \max_{||x||=1} ||Ax||.$$

is a matrix norm.

This is called the natural or induced matrix norm associated with the vector norm.

The following result gives a bound on the value of $||Ax||$:

1.2 Thm.

For any vector $x \neq 0$, matrix A, and any natural norm $||\cdot||$, we have

$$||Ax|| \leq ||A|| \cdot ||x||.$$

1.2.1 Notes on the Infinity Norm

The infinity norm is defined as

$$||x||_\infty = \max_i |x_i|$$

So, it's the maximum absolute value of every entry in x.

Example: $x = \begin{pmatrix} 1 \\ -2 \\ 1.5 \end{pmatrix}$

$||x||_\infty = 2$

Because the largest element (in magnitude) is -2.

2 Computing the Infinity Norm and the 1-Norm

Computing the ∞-norm of a matrix is straightforward:

2.1 Thm.

If $A = (a_{ij})$ is a $n \times n$ matrix, then

$$||A||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Example:
Find

$$\left|\left|\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\right|\right|_\infty$$

$$\begin{aligned} \sum_{j=1}^{n} |a_{1j}| &= |2| + |-1| + |0| = 3 \\ \sum_{j=1}^{n} |a_{2j}| &= |-1| + |2| + |-1| = 4 \\ \sum_{j=1}^{n} |a_{3j}| &= |0| + |-1| + |2| = 3 \\ \implies \; ||A||_\infty &= \max\{3, 4, 3\} = 4 \end{aligned}$$

2.2 Visualizing the Infinity Norm

Images courtesy of ChatGPT. I got really confused by the concept of the infinity norm, so I asked the bot for help.

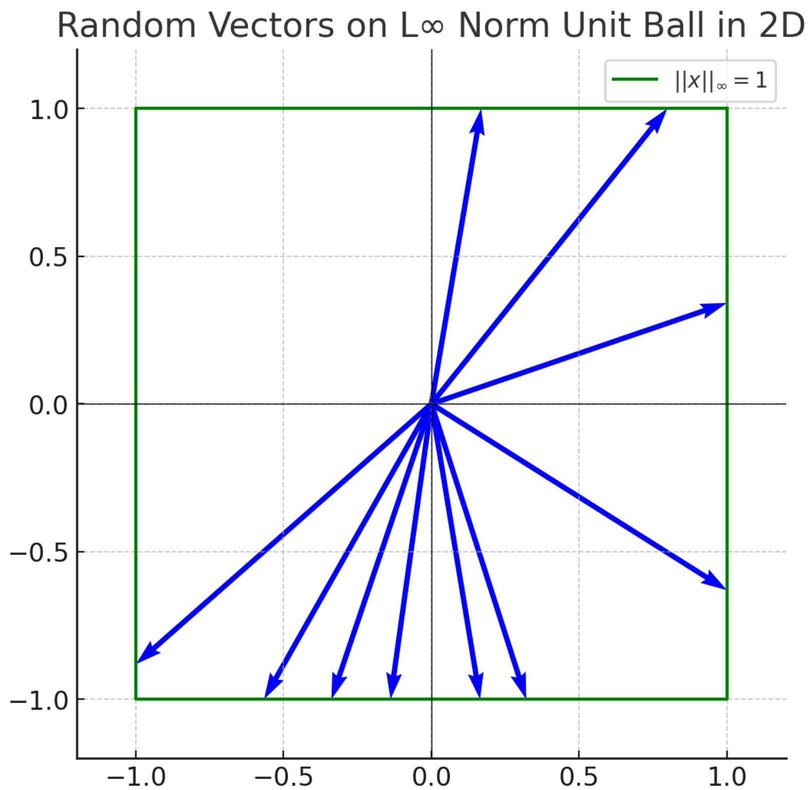

Figure 1: Visualizing the Infinity Norm

So the reason why the infinity norm, visualized this way, looks like a square, is because the equation $||x||_\infty = 1$ is equivalent to saying "the set of all vectors x such that the the largest component magnitude of the vector is 1". Meaning this is the set of all vectors that have $x = 1$ or $y = 1$

3 Eigenvalues and Eigenvectors

To calculate the l_∞-norm of a matrix, we did not need to directly appy the definition. This is also true for the l_2-norm, however, we will need to introduce eigenvalues and eigenvectors to apply this technique.

First we will need the following definition:

Def. If A is a square matrix, the polynomial defined by

$$p(\lambda) = \det(A - \lambda I)$$

is called the characteristic polynomial of A. It is easily shown that p is an n^{th} degree polynomial.

Now we can introduce eigenvalues and eigenvectors.

Def. If p is the characteristic polynomial of the matrix A, the zeros of p are called eigenvalues, or characteristic values of A. If λ is an eigenvalue of A and $x \neq 0$ has the property that $(A - \lambda I)x = 0$ then x is called an eigenvector, or characteristic vector, of A corresponding to the eigenvalue λ.

The professor does not specifically discuss eigenvectors and eigenvalues in the context of Numerical Analysis, but they will be important for future problems. He also does not mention how to compute them.

However, he did suggest that we review how to compute them by hand and study the related theorems.

4 Finding the l_2-norm of a matrix

Def. The spectral radius $\rho(A)$ of a matrix A is defined as

$$\rho(A) = \max|\lambda| \text{ where } \lambda \text{ is an eigenvalue of } A$$

Ex.

$$\rho(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}) = \max\{|3|, |3|, |1|\} = 3$$

And now, we can consider the following:

Thm. If A is a $n \times n$ matrix, then

1. $||A||_2 = \sqrt{\rho(A^T A)}$
2. $\rho(A) \leq ||A||$ for any natural norm $||\cdot||$

My editor broke in the middle so you should look at the Chapter 7 PDF notes for the proofs and examples. The end of this section is around page 19 of the PDF. (35.13)