MACM 316 Lecture 20

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1 Approximation and Interpolation

It is often useful or necessary to approximate a complicated or expensive function, or a function only known at a discrete set of points, by a smpler function which can be computed or evaluated more easily over a whole interval. When the function in question is known accurately at a discrete set of points, we are inclined to use an interpolation procedure- where the graph of the approximating function runs exactly through the points of the discrete set.

If the dataset is expected to contain error, which is the case for measurements or observations in experimental studies, a better strategy is to allow the graph of the approximating function to stray from the data points.

A useful and well known class of functions for mapping the set of real numbers into itself is the class of algebraic polynomials.

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

where n is a non-negative integer and a_i are real constants. Polynomials have the desirable property that they can approximate <u>any</u> function over a closed, bounded interval.

This desired property is precisely captured by the **Weierstrass Approximation Theorem**

1.1 The Weierstrass Approximation Theorem

Suppose $f \in C[a,b]$.

$\forall \mathcal{E} > 0 \exists P \in \{\mathbb{P}_n, C[a,b]\}$ such that

$$|f(x) - P(x)| < \epsilon \forall x \in [a,b].$$

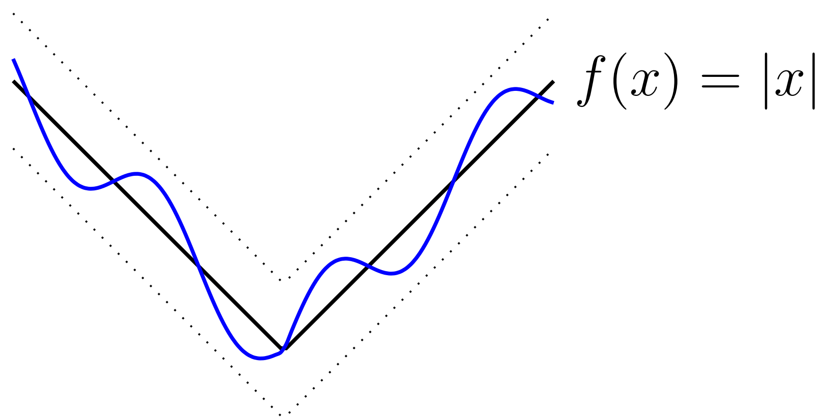


Figure 1: Polynomial approximation to $f(x) = |x|$.

This is a very strong theorem, as it only requires $f(x)$ to be continuous on the interval, and not necessarily differentiable.

Unfortunately, the Weierstrass Approximation Theorem does not tell us how to select such a polynomial. Many would immediately jump to using Taylor series polynomials for polynomial interpolation, however, Taylor series' concentrate their accuracy at the point a rather than over the entire interval, and are typically poorly suited for interpolation.

1.1.1 Taylor Series Polynomials

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A particularly clear demonstration of this drawback is seen for

$$f(x) = \frac{1}{x} \qquad \text{expanded about } x_0 = 1.$$

Then,

$$
\begin{aligned}
P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k \\
&= \sum_{k=0}^{n} (-1)^k (x-1)^k
\end{aligned}
$$

To approximate $f(3) = \frac{1}{3}$ by $P_n(3)$ for increasing values of n, we see a dramatic and catastrophic failure:

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

Table 1: Values of $P_n(3)$ for increasing n.

We will insetad focus on methods which use information throughout the entire interval to approximate f.

2 Polynomial Interpolation

We now assume that the given dataset is exact and represents values of some unknown function. We want to find the polynomial $P_n(x)$ of the smallest possible degree n such that

$$P_n(x_k) = f_k \qquad k = 1, 2, \ldots, N.$$

for $N+1$ distinct interpolation points $x_0, \ldots, x_N$ and $N+1$ values $\underbrace{f_0, \ldots, f_N}_{\text{data points}}$.

To solve this problem, we will first investigate the simpler problem where all the data equals zero, except at one point.

We are looking for a polynomial $L_m(x)$ of degree $\leq N$ such that

$$L_m(x_k) = \delta_{mk} \qquad = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}.$$

Kronecker Delta

This is easy to find. Since the polynomial must vanish at the points $x_k, k \neq m$, it must contain the factors

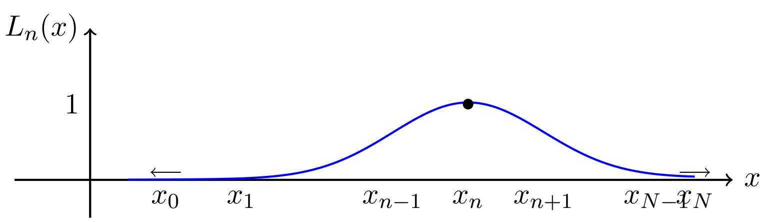
$$(x - x_k) \qquad \text{for } k \neq m.$$

$$\therefore L_m(x) = c \cdot \prod_{k=0;k\neq m}^{N} (x - x_k).$$

The constant is determined by the condition $L_m(x_m) = 1$.

$$\implies L_m(x) = \prod_{k=0;k\neq m}^{N} \frac{x - x_k}{x_m - x_k}.$$

Typically,



These polynomials are the building blocks for deriving a polynomial interpolating a general function. It is easy to see that

$$P(x) = \sum_{m=0}^{N} f_m < L_m(x).$$

is a polynomial of degree $\leq N$ satisfying the interpolation conditions.

This polynomial is called the n^{th} **Lagrange Interpolating Polynomial**

Recall that we wanted to find the polynomial $P_n(x)$ of smallest degree such that

$$P_n(x_k) = f_k \qquad k = 0, \ldots, N.$$

2.1 Uniqueness

Is this polynomial unique?

Yes.

Proof. Assume there are two different polynomials p and q of degree $\leq N$ which both satisfy the interpolation conditions. Their difference, $d = p - q$, is also a polynomial of degree $\leq N$ and vanishes at the $N+1$ distinct points $x_0, \ldots, x_N$.

However, a nonzero polynomial of degree $\leq N$ has at most N zeros, this

$$d = p - q = 0 \implies \begin{cases} p = q \\ \text{uniqueness} \end{cases}$$

$\therefore$ the n^{th} Lagrange Interpolating Polynomial is the unique interpolating polynomial satisfying the interpolation conditions.

2.2 Example

Fit a cubic through the first four points of the table

i	x^i	$f(x_i)$
0	3.2	22.0
1	2.7	17.8
2	1.0	14.2
3	4.8	38.3
4	5.6	51.7

and use it to find the interpolated value for $x = 3.0$.
Soln. The 3^{rd} Lagrange Interpolating Polynomial is given by

$$\begin{aligned} P(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 \\ & + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1 \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2 \\ & + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3 \end{aligned}$$

Substituting the values from the table and evaluating at $x = 3.0$ gives

$$
\begin{aligned}
P(3.0) = &\frac{(3.0-2.7)(3.0-1.0)(3.0-4.8)}{(3.2-2.7)(3.2-1.0)(3.2-4.8)}(22.0)\\
&+\frac{(3.0-3.2)(3.0-1.0)(3.0-4.8)}{(2.7-3.2)(2.7-1.0)(2.7-4.8)}(17.8)\\
&+\frac{(3.0-3.2)(3.0-2.7)(3.0-4.8)}{(1.0-3.2)(1.0-2.7)(1.0-4.8)}(14.2)\\
&+\frac{(3.0-3.2)(3.0-2.7)(3.0-1.0)}{(4.8-3.2)(4.8-2.7)(4.8-1.0)}(38.3)\\
&= 20.21
\end{aligned}
$$

Our next task is to develop estimates for the error. As it turns out, the form of the error (but not necessarily the magnitude) resembles that of the n^{th} Taylor Polynomial.

2.3 Error Estimates

The lecture ended just before this section

Thm. (3.3 of Text)

Suppose $x_0, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each $x \in [a, b]$, a number $\xi(x) \in (a, b)$ exists with the property

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

n-th Langrange Interpolating Polynomial

do you guys like my diagrams with arrows?