MACM 316 Lecture 32

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1 A continuation on the Elementary Theory of Initial Value Problems

1.1 Lipschitz Conditions

Def. A function $f(t, y)$ satisfies a Lipschitz condition in the variable y on a set $D \in \mathbb{R}^2$ if a constant $L > 0$ exists such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \qquad \text{for all } (t, y_1), (t, y_2) \in D.$$

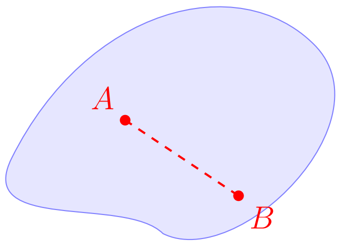
The constant L is called a Lipschitz Constant for f.
Examples can be found in the lecture notes for Lecture 31-b.

1.2 Convex Sets

Def. A set $D \in \mathbb{R}^2$ is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to D, the point

$$((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2).$$

also belongs to D for each $\lambda \in [0, 1]$. Geometrically, a set is convex if, for any two points in the set, a line segment connecting them lies entirely within the set. (**i.e.** every point in the set has line of sight to every other point within the set.)



1.2.1 Exercise 1 (22.5)

Show that the set

$$D = \{(t, y) : a \leq t \leq b, -\infty < y < \infty\}.$$

where a and b are constants, is convex.

To prove analytically, we can use the definition of convexity and show that each point falls within the set.

1.3 Theorem 1 (22.6)

Suppose $f(t, y)$ is defined on a convex set $D \in \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L$$

then for all $(t, y) \in D$ then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Proof. Let (t, y_1) and (t, y_2) be in D. Holding t fixed, define $g(y) = f(t, y)$.

Suppose $y_1 \leq y_2$. Since the line joining (t, y_1) to (t, y_2) lies in D and f is continuous on D we have $g \in C[y_1, y_2]$. Furthermore,

$$g'(y) = \frac{\partial f(t, y)}{\partial y}.$$

Using the Mean Value Theorem on g, a number ξ with $y_1 < \xi < y_2$ exists so that

$$\begin{aligned} g(y_2) - g(y_1) &= g'(\xi)(y_2 - y_1) \\ \implies f(t, y_2) - f(t, y_1) &= \frac{\partial f(t, y)}{\partial y}(y_2 - y_1) \\ \implies |f(t, y_2) - f(t, y_1)| &\leq L|y_2 - y_1| \end{aligned}$$

So f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L. □

The previous theorem in combination with the next is particularly fundamental for showing the existence and uniqueness of solutions to ODEs.

1.4 Theorem 2 (22.7)

Suppose that $D = \{(t, y) : a \leq t \leq b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D.

If f satisfies a Lipschitz condition on D in the variable y, then the initial value problem

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, y(a) = \alpha.$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

1.4.1 Example (22.8)

Ex. Show that the IVP

$$y' = y \cos t, 0 \leq t \leq 1, y(0) = 1.$$

has a unique solution.

Soln. Since $f(t, y) = y \cos t$ we have $\frac{\partial f}{\partial y} = \cos t$.

$\implies$ f satisfies a Lipschitz condition in y with $L = 1$ on

$$D = \{(t, y) : 0 \leq t \leq 1, -\infty < y < \infty\}..$$

Also, f is continuous on D — f is the product of continuous functions and is therefore continuous — so there exists a unique solution.

We also need to know if small changes in the statement of the problem introduce correspondingly small changes in the solution.

1.5 Theorem 3 (22.9)

Thm. The initial value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, y(a) = \alpha.$$

is said to be a well-posed problem if:

1. A unique solution, $y(t)$, to the problem exists
2. There exist constants $\mathcal{E}_0 \ge 0$ and $k > 0$ such that for any $\mathcal{E}$ with $\mathcal{E}_0 > \mathcal{E} > 0$, whenever $\delta(t)$ is continuous with

$$|\delta(t)| < \mathcal{E} \quad \text{for all } t \in [a, b]$$

and when $|\delta_0| < \mathcal{E}$, the initial value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\mathcal{E}$$

for all $t \in [a, b]$.

The perturbed problem assumes the possibility of an error $\delta(t)$ being introduced in the statement of the differential equation as well as an error δ_0 being present in the initial condition. Numerical methods also solve perturbed problems since roundoff errors perturb the original problem. $\Longrightarrow$ It only makes sense to approximate well-posed problems.

1.6 Theorem 4 (22.10)

Thm. Suppose $D = \{(t, y) : a \le t \le b, -\infty < y < \infty\}$

If f is continuous and satisfies a Lipschitz condition int he variable y on the set D, then the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, y(a) = \alpha.$$

is well-posed.

1.6.1 Example (22.11)

Ex. Show that the initial-value problem

$$y' = t^2 y + 1, \quad 0 \le t \le 1, y(0) = 1.$$

is well-posed.

Soln. Since

$$\left|\frac{\partial(t^2 y + 1)}{\partial y}\right| = \left|t^2\right| \le 1.$$

and $t^2 y + 1$ is continuous — it's a polynomial in (t, y) — we know that this problem is well-posed.

2 Euler's Method (23.1)

Our first numerical scheme for intial value problems will be Euler's Method — a very simple but low order method.

Consider the initial value problem

$$\textbf{IVP} \begin{cases} y' = f(t,y) & a \le t \le b \\ y(a) = y_0 \end{cases}$$

We will compute an approximation to the problem at the mesh points

$$t_k = a + kh, \quad k = 0, 1, \ldots, N.$$

where $h = \frac{(b-a)}{N}$ is called the step size. Here we have assumed h is a constant, although variable step sizes are also useful

Euler's Method can be derived using a Taylor series expansion:

$$\begin{aligned} y(t_{k+1}) = y(t_k + h) &= y(t_k) + hy'(t_k) + \frac{h^2}{2} y''(\xi_k) \\ &= y(t_k) + hf(t_k, y(t_k)) + \frac{h^2}{2} y''(\xi_k) \end{aligned}$$

Euler's Method constructs an approximation

$$w_k \approx y(t_k).$$

by dropping the remainder term.

$$
\begin{aligned}
w_0 &= y_0 \\
w_k &= w_{k-1} + hf(t_{k-1}, w_{k-1}) \quad 1 \le k \le N
\end{aligned}
$$