Direct Methods for Solving Linear Systems.

Linear systems of equations

Equation 1 $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$ (E_1)

Equation 2 $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$ (E_2)

$\vdots$

Equation n $a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$ (E_n)

arise in many engineering and scientific applications.

Here the a_{ij} + b_i are given constants and the x_i are unknown quantities that we want to find.

We will first study methods that give an answer in a fixed number of steps, subject only to round off errors.

Three operations are permitted to simplify the system.

These three elementary operations are

1. Multiply equation i by a constant $\lambda \neq 0$. This is denoted by $\boxed{\lambda E_i \to E_i}$.
2. λ * Equation j can be added to Equation i.

$$\boxed{E_i + \lambda E_j \to E_i}$$

3. Equation j can be exchanged with Equation i

$$\boxed{E_i \leftrightarrow E_j}$$

We can rewrite the linear system as a matrix equation $Ax=b$

ie

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

Recall some definitions:

<u>Def</u>. An n by m matrix is a rectangular array of elements with n rows and m columns

<u>eg</u>

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{bmatrix}$$

$n \times m$ matrices denoted by capitals

entries are denoted by lowercase.

<u>Def</u>. An n-dimensional column vector is an $n \times 1$ matrix

<u>eg</u> $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

<u>Def</u> An n-dimensional row vector is a $1 \times n$ matrix

<u>eg</u> $y = [y_1 \; y_2 \; \cdots y_n]$

Also the matrix equation

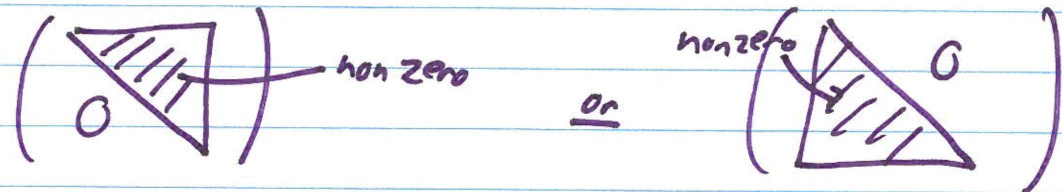
$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} \underset{=}{=} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

is often represented using an augmented matrix

$$[A, b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

the vertical line is used to separate the coefficients of the unknowns and the values of the rhs of the equations.

The idea behind <u>Gaussian elimination</u> is to use the three elementary operations to form a "triangular" equivalent matrix problem of the form



upper triangular   lower triangular

This is most easily seen with an example.

Consider the linear system

$$x_1 - x_2 + 2x_3 - x_4 = -8$$

$$2x_1 - 2x_2 + 3x_3 - 3x_4 = -20$$

$$x_1 + x_2 + x_3 = -2$$

$$x_1 - x_2 + 4x_3 + 3x_4 = 12.$$

This is compactly expressed using the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 12 \end{array}\right)$$

Call the first entry of E_1 the pivot element. The pivot will have to be a nonzero element.

Are there elementary operations that will produce zeros in all elements below the pivot element?

Using
$$E_2 - 2E_1 \to E_2$$
$$E_3 - E_1 \to E_3$$
$$E_4 - E_1 \to E_4$$

the system is converted into the equivalent problem

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 20 \end{array}\right)$$

We want an upper triangular matrix so we exchange E_2 and E_3 resulting in

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 20 \end{array}\right)$$

The pivot element is now the leftmost nonzero element in Row 3.

Are there elementary operations that will produce zeros in all elements below the pivot element?

Take $2E_3 + E_4 \to E_4$ to yield

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 12 \end{array}\right)$$

At this point we have an upper triangular matrix and back substitution starting with E_4 going back up to E_1 gives

$$2x_4 = 12 \Rightarrow x_4 = 6$$

$$-x_3 - x_4 = -4 \Rightarrow x_3 = -x_4 + 4 = -2$$

$$2x_2 - x_3 + x_4 = 6 \Rightarrow 2x_2 + 2 + 6 = 6 \Rightarrow x_2 = -1$$

$$x_1 - x_2 + 2x_3 - x_4 = -8 \Rightarrow x_1 + 1 - 4 - 6 = -8 \Rightarrow x_1 = 1.$$

In general, Gaussian elimination with backwards substitution for

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1\,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2\,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n\,n+1} \end{array}\right)$$

proceeds as follows:

Provided $a_{11} \neq 0$ the operations Corresponding to

$$E_j - \frac{a_{j1}}{a_{11}} E_1 \rightarrow E_j$$

are performed for each $j = 2, 3, \ldots, n$ to produce zeros in all the elements below the pivot element a_{11}.

(If $a_{11} = 0$ but some $a_{j1} \neq 0$ we exchange $E_1 \Leftrightarrow E_j$ and then carry out the previous steps)

This gives a matrix of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1\,n+1} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2\,n+1}^{(2)} \\ 0 & a_{32}^{(2)} & \cdots & a_{3n}^{(2)} & a_{3\,n+1}^{(2)} \\ \vdots & \vdots & & & \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} & a_{n\,n+1}^{(2)} \end{array}\right)$$

where $a_{jk}^{(2)}$ are (possibly) nonzero elements.

We follow a sequential procedure for $i = 2, 3, \ldots, n-1$ and perform the operation

$$E_j - \left(a_{ji}^{(i)}/a_{ii}^{(i)}\right) E_i \to E_j$$

for each $j = i+1, i+2, \ldots, n$ provided $a_{ii}^{(i)} \neq 0$ (we need to carry out an exchange first if $a_{ii}^{(i)} = 0$)

The final matrix has the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1\,n+1} \\ 0 & a_{22}^{(2)} & & a_{2n}^{(2)} & a_{2\,n+1}^{(2)} \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots\ 0 & a_{nn}^{(n)} & a_{n\,n+1}^{(n)} \end{array}\right),$$

This final matrix can be solved by back substitution.

<u>Q</u> What types of failures could we expect?

Here are two examples:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 2 & 6 \end{array}\right) \qquad B = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \end{array}\right)$$

$E_2 - 2E_1 \rightarrow E_2$, $E_3 - E_1 \rightarrow E_3$

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad \tilde{B} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$x_3 = 2$
$-x_3 = -2$
$\Rightarrow x_2 = 2 - x_1$
$\Rightarrow$ infinite # of solutions

$x_3 = 2$
$x_3 = 4$ #
no solutions

When the algorithm fails we can conclude there is no unique solution.

How does the number of operations change with the size of the matrices?

$$\#\text{ multiplications \& divisions} = \underbrace{\sum_{i=1}^{n-1}(n-i)(n-i+2)}_{\text{to make the system triangular}} + \underbrace{1+\sum_{i=1}^{n-1}((n-i)+1)}_{\text{back substitution.}}$$

$$= (n^2+2n)\sum_{i=1}^{n-1}1 - 2(n+1)\sum_{i=1}^{n-1}i + \sum_{i=1}^{n-1}i^2$$

$$+ 1 + \sum_{i=1}^{n-1}(n+1) + \sum_{i=1}^{n-1}(-i)$$

$$= \frac{2n^3+3n^2-5}{6} + \frac{n^2+n}{2} = \frac{n^3}{3} + n^2 - \frac{n}{3}$$

$$\#\text{ additions \& subtractions} = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

For large n the number of floating point operations is about $2n^3/3$ ($n^3/3$ multiplications & divisions and $n^3/3$ additions & subtractions)

The amount of computation & time required increase with n proportional to n^3.

Pivoting

In Gaussian Elimination we needed to perform a row interchange whenever one of the pivot elements was zero.

In practice, however, row interchanges are often necessary even when the pivot element is nonzero.

First, if the pivot $a_{kk}^{(k)}$ is small in magnitude compared to some $a_{jk}^{(k)}$ the multiplier

$$m_{jk} = a_{jk}^{(k)} / a_{kk}^{(k)}$$

will be very large. Round off error that was produced in the computation of $a_{kk}^{(k)}$ will be multiplied by the large factor m_{jk} when computing $a_{j\ell}^{(k+1)}$.

Roundoff can also be dramatically increased in the balckward substitution step

$$x_k = \frac{a_{k,n+1}^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)}}{a_{kk}^{(k)}}$$

when the pivot $a_{kk}^{(k)}$ is small...

Pivoting Strategies.

We have seen that roundoff error can be dramatically increased when small pivot elements are used.

To reduce roundoff errors we can use

partial pivoting
scaled partial pivoting
or complete pivoting

increasing expense & improved accuracy.

Partial Pivoting.

To illustrate this approach consider the linear system

$$E_1: \quad 0.003000x_1 + 59.14x_2 = 59.17$$
$$E_2: \quad 5.291\,x_1 - 6.130\,x_2 = 46.78$$

This has an exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Let's solve the system using Gaussian Elimination with four-digit rounding arithmetic...

$a_{11} = 0.00300$ is the pivot
(notice it is small)

The associated multiplier

$$m_{21} = \frac{5.291}{a_{11}}$$

$$= 1763.6\dot{6}$$

which rounds to 1764.

$(E_2 - m_{21} E_1) \to E_2$ gives

$$0.003000x_1 + 59.14x_2 \doteq 59.17$$
$$-104300\ x_2 \doteq -104400$$

instead of the precise values

$$0.003000x_1 + 59.14x_2 = 59.17$$
$$-104309.37\dot{6}x_2 = -104309.37\dot{6}$$

The disparity in the magnitudes of m_2, a_{13} and a_{23} has introduced round off error.

Backward substitution yields

$$x_2 \approx 1.001$$

which is close to the actual value

$$x_2 = 1.000$$

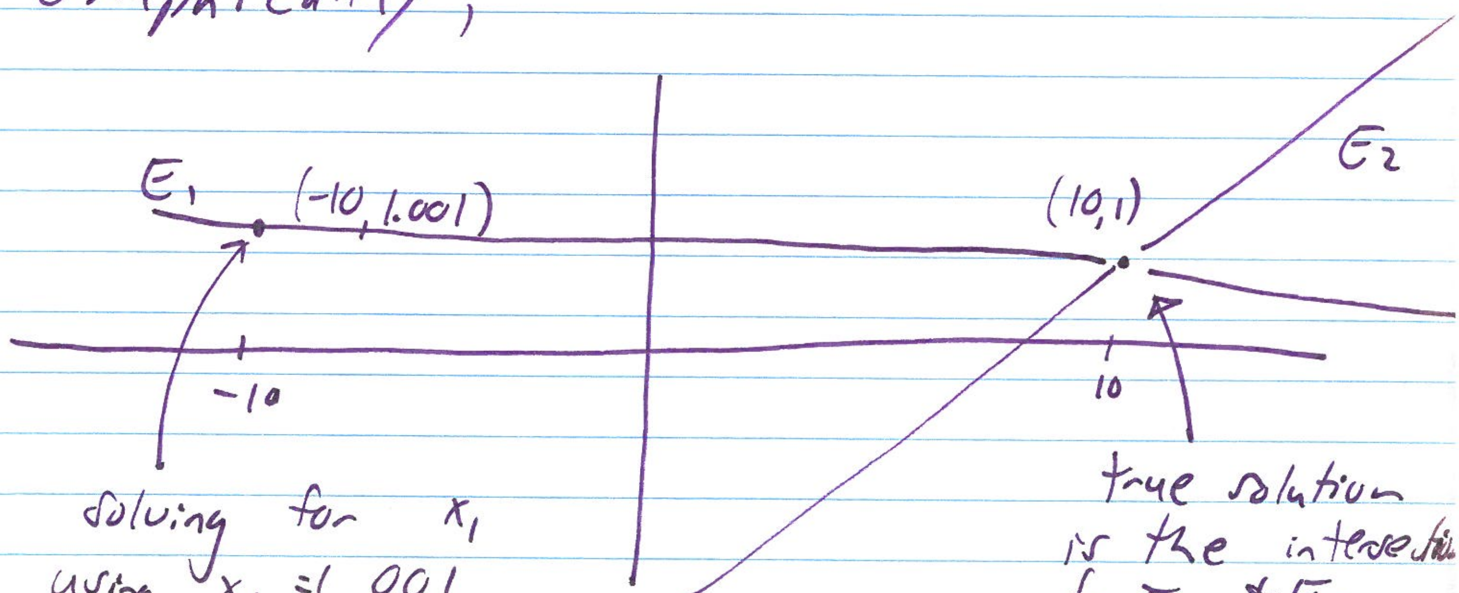
However,

$$x_1 \approx \frac{59.17 - (59.14)(1.001)}{0.00300} = -10.00$$

contains a small error of 0.001 multiplied by $59.14 / .00300$

which ruins the approximation to the actual value $x_1 = 10$.

Graphically,



solving for x_1 using $x_2 = 1.001$ gives $x_1 = -10$!!

true solution is the intersection of E_1 & E_2.

Simplest strategy to avoid these difficulties : Partial Pivoting

Idea is to select the largest element $a_{ik}^{(k)}$ that is below the pivot $a_{kk}^{(k)}$

If $|a_{ik}^{(k)}| > |a_{kk}^{(k)}|$

pivot element

then rows i and k are exchanged.

Let's reconsider the example using partial pivoting

$$E_1 : 0.003000x_1 + 59.14x_2 = 59.17$$
$$E_2 : 5.291x_1 - 6.130x_2 = 46.78$$

The largest element below the pivot 0.003000 is 5.291.

$$5.291 > 0.003000$$

So rows 2 and 1 are exchanged.

$E_1 \leftrightarrow E_2$ gives

$$5.291x_1 - 6.130x_2 = 46.78$$
$$0.003000x_1 + 59.14x_2 = 59.17$$

Now the multiplier is

$$m_{21} = a_{21}^{(1)}/a_{11}^{(1)} = 0.0005670$$

and the operation

$$(E_2 - m_{21}E_1) \rightarrow E_2$$

gives

$$5.291x_1 - 6.130x_2 = 46.78$$
$$59.14x_2 \approx 59.14.$$

The four digit solution to this system is the correct solution

$$x_1 = 10.00$$
$$x_2 = 1.0000.$$

Partial pivoting is sufficient for most linear systems. However, it can be inadequate for certain problems.

Example. Consider the linear system

$$E_1: \quad 30.00\,x_1 + 591400\,x_2 = 591700$$
$$E_2: \quad 5.291\,x_1 - 6.130\,x_2 = 46.78.$$

No row exchanges are carried out using partial pivoting.

Now the multiplier is $m_{21} = \dfrac{5.291}{30.000} = 0.1764$

and $(E_2 - m_{21} E_1) \to (E_2)$ gives the system

$$30.00\,x_1 + 591400\,x_2 \doteq 591700$$
$$-104300\,x_2 \doteq -104400$$

which has the same inaccurate solution $x_2 \approx 1.001$

$x_1 \approx -10.000$

as Gaussian elimination without pivoting.

This is not surprising because the system is essentially the same as our original example: we have only multiplied the first equation by 10 000.

We can obtain improved accuracy if we scale coefficients before deciding on row exchanges.

This is the idea behind <u>scaled partial pivoting</u>

Now we determine a scaling factor for the i^{th} row. This will be the magnitude of the largest element in the i^{th} row.

<u>ie</u> $$s_i = \max_{j=1,2,\ldots,n} |a_{ij}|$$

Now the idea is to select the largest scaled value $a_{ik}^{(k)}/s_i$ corresponding to elements that are below the pivot.

<u>ie</u> we determine the row i so that

$$\frac{a_{ik}^{(k)}}{s_i} = \max_{j=1,\dots,n} \frac{|a_{j1}|}{s_j}$$

If $|a_{ik}^{(k)}/s_i| > |a_{kk}^{(k)}/s_k)$

then rows i and k are exchanged.

Let's reconsider the example with scaled partial pivoting:

$E_1 : 30.00\, x_1 + 591400\, x_2 = 591700$

$E_2 = \quad 5.291\, x_1 - 6.130\, x_2 = 46.78$

Calculate scaling factors.

$s_1 = \max\{|30.00|, |591400|\} = 591400$
$s_2 = \max\{|5.291|, |-6.130|\} = 6.130$

Since $\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 5.073 \times 10^{-5}$

$$\frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631$$

we have $\frac{|a_{11}|}{s_1} < \frac{|a_{21}|}{s_2}$

and the interchange is made $E_1 \leftrightarrow E_2$.

Applying Gaussian elimination to the new system

$$5.291x_1 - 6.130x_2 = 46.78$$

$$30.00x_1 + 591400x_2 = 591700$$

produces the correct result

$$x_1 = 10.00$$

$$x_2 = 1.000.$$

In rare instances complete pivoting may be needed.

Complete pivoting searches all the entries a_{ij}, $k \le i,j \le n$

to find the entry with the largest magnitude.

Both row and column interchanges are performed to bring this entry into the pivot position.

Complete pivoting is only recommended for systems where accuracy is essential since it requires an additional

$$\frac{n(n-1)(2n+5)}{6} = O(n^3)$$

comparisons over Gaussian Eliminations.

(Partial pivoting and scaled partial pivoting, however, only require an additional $O(n^2)$ operations.)

31.11

Linear Algebra and Matrix Inversion.

Some basic matrix properties.

<u>Def</u>. Two matrices A and B are equal if they have the same size and if each $a_{ij} = b_{ij}$.

<u>Def</u>. $A + B$ = the $n \times m$ matrix whose entries are $(a_{ij} + b_{ij})$ for $i = 1, \dots, n$; $j = 1, \dots, m$

A, B: $n \times m$ matrices

<u>Def</u>. λA = the $n \times m$ matrix whose entries are λa_{ij} $\quad i = 1, \dots, n$; $j = 1, \dots, m$

λ: real number; A: $n \times m$ matrix

<u>Thm</u>. Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. Then

a/ $A + B = B + A$
b/ $(A+B)+C = A+(B+C)$
c/ $A + 0 = A = 0 + A$
d/ $A + (-A) = 0 = (-A) + A$
e/ $\lambda(A+B) = \lambda A + \lambda B$
f/ $(\lambda + \mu) A = \lambda A + \mu A$
g/ $\lambda(\mu A) = (\lambda \mu) A$
h/ $1A = A$.

Note: 0 is the matrix whose entries are all zero.

31.12

Def. $\underbrace{\overset{n\times m}{A} \times \overset{m\times p}{B}}_{\text{matrix product of } A \text{ and } B} = \overset{n\times p}{C}$

where $C = [c_{ij}]$

and $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

eg

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 4 \\ -3 & 2 & 2 \\ 8 & 18 & 8 \end{bmatrix}$$

Note in particular that

$A \times B \neq B \times A$.

Def. A <u>square matrix</u> has the same number of rows and columns.

A <u>diagonal matrix</u> D is a square matrix with $d_{ij} = 0$ whenever $i \neq j$.

$$I_n = \underbrace{\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}}_{n} \Big\updownarrow n$$

identity matrix of order n

An <u>upper triangular matrix</u> U is an $n \times n$ matrix so that $u_{ij} = 0$, if $i > j$.

A <u>lower triangular matrix</u> L is an $n \times n$ matrix with the property that $l_{ij} = 0$ if $i < j$.

Thm. Let A be an $n \times m$ matrix, B be an $m \times K$ matrix, C a $K \times p$ matrix, D an $m \times K$ matrix & λ a real number.

a/ $A(BC) = (AB)C$
b/ $A(B+D) = AB + AD$
c/ $IB = B$ & $BI = B$
d/ $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Def. An $n \times n$ matrix A is said to be nonsingular (or invertible) if an $n \times n$ matrix A^{-1} exists with

$$A A^{-1} = A^{-1} A = I.$$

The matrix A^{-1} is called the inverse of A.

A singular matrix does not have an inverse.

Thm. For any nonsingular $n \times n$ matrix A

- A^{-1} is unique
- A^{-1} is nonsingular and $(A^{-1})^{-1} = A$
- If B is also nonsingular $n \times n$ then $(AB)^{-1} = B^{-1}A^{-1}$.

Determing Inverses.

eg. Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}.$$

we want to find the matrix $B = [b_{ij}]$ so that $AB = I$.

ie
$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so we want to solve the system

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \quad \text{for } b_{11}, b_{21}, b_{31}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{array}\right] \quad \text{for } b_{12}, b_{22}, b_{32}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 1 \end{array}\right] \quad \text{for } b_{13}, b_{23}, b_{33}$$

To save work and needless repetition, we can instead treat the larger augmented system

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Approach: Use elementary row operations to make the system upper triangular and then use back substitution to solve for the $[b_{ij}]$.

$(E_2 - 2E_1) \to (E_2)$, $(E_3 + E_1 \to E_3)$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right]$$

$(E_3 + E_2) \to (E_3)$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right]$$

Backward substitution is then performed on each of the three augmented matrices

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & 3 & -1 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

to give

$$B = [b_{ij}] = \begin{bmatrix} -2/9 & 5/9 & -1/9 \\ 4/9 & -1/9 & 2/9 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Some more key definitions:

<u>Def</u>. The transpose of an $n \times m$ matrix $A = (a_{ij})$ is the $m \times n$ matrix $A^t = (a_{ji})$

<u>eg</u>. $\begin{bmatrix} 3 & 2 \\ -1 & 7 \\ -5 & 4 \end{bmatrix}^t = \begin{bmatrix} 3 & -1 & -5 \\ 2 & 7 & 4 \end{bmatrix}$

<u>Def</u>. A square matrix A is symmetric if $A = A^t$.

<u>Thm</u>. a) $(A^t)^t = A$
b) $(A+B)^t = A^t + B^t$
c) $(AB)^t = B^t A^t$
d) if A^{-1} exists $(A^{-1})^t = (A^t)^{-1}$

Another very useful concept of linear algebra is the <u>determinant</u> of a matrix

The determinant of a matrix is denoted by $\det(A)$ or $|A|$.

Determinants are important, in part, because of the following theorem:

<u>Thm</u> The following statement are equivalent for any $n \times n$ matrix A.

a. $\det A \neq 0$

b. The equation $Ax = 0$ has the unique solution $x = 0$.

c. The system $Ax = b$ has a unique solution for any n-dimensional column vector b.

d. The matrix A is nonsingular <u>ie</u> A^{-1} exists.

The definition of the determinant is somewhat involved:

Def. a. If $A=[a]$, then $\det A = a$.

b. If A is an $n\times n$ matrix, the minor M_{ij} is the determinant of the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the i^{th} row and the j^{th} column of the matrix A.

eg if

$$A=\begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 7 & 0 \\ -3 & -4 & 1 & 5 \\ 6 & -6 & 8 & 0 \end{bmatrix}$$

then $$M_{42} = \det\begin{bmatrix} 2 & 3 & 0 \\ 4 & 7 & 0 \\ -3 & 1 & 5 \end{bmatrix}$$

c. The cofactor A_{ij} associated with M_{ij} is defined by $$A_{ij} = (-1)^{i+j} M_{ij}$$

eg The cofactor associated with M_{42} is $$A_{42} = (-1)^{4+2} M_{42} \overset{\det}{=} \begin{bmatrix} 2 & 3 & 0 \\ 4 & 7 & 0 \\ -3 & 1 & 5 \end{bmatrix}$$

(d.) The determinant of the $n \times n$ matrix A, when $n > 1$ is given either by

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} \quad \text{for any } i = 1, 2, \ldots, n$$

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} \quad \text{for any } j = 1, 2, \ldots, n$$

Example. Find $\det \begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 7 & 0 \\ -3 & -4 & 1 & 5 \\ 6 & -6 & 8 & 0 \end{bmatrix}$

expand about 4th column

$$\det A = a_{14} A_{14} + a_{24} A_{24} + a_{34} A_{34} + a_{44} A_{44}$$

$$= 5 A_{34}$$

$$= 5(-1)^{3+4} \det \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{bmatrix}$$

Expand along top row gives

$$\det A = -5\left[2(-1)^2 \det \begin{bmatrix} -2 & 7 \\ -6 & 8 \end{bmatrix} - (-1)^3 \det \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} + 3(-1)^4 \det \begin{bmatrix} 4 & -2 \\ 6 & -6 \end{bmatrix}\right]$$

$$= -30.$$

Notice that this computation requires $O(n!)$ multiplications and $O(n!)$ additions.

$\Rightarrow$ The definition is too expensive for practical computation when n is large.

Instead, we will use the following results:

Thm: Suppose A is an $n \times n$ matrix

a) If any row or column of A has only zero entries, then $\det A = 0$

b) If $\tilde{A}$ is obtained from A by $E_i \leftrightarrow E_j$ with $i \neq j$ then $\det \tilde{A} = -\det A$

c) If A has two rows or two columns the same, $\det A = 0$.

d) If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \to E_i$ then $\det \tilde{A} = \lambda \det A$

e) If $\tilde{A}$ is obtained from A by the operation $E_i + \lambda E_j \to E_i$ with $i \neq j$ then $\det \tilde{A} = \det A$.

f/ If B is an $n \times n$ matrix then $\det AB = \det A \det B$

g/ $\det A^t = \det A$

h/ When A^{-1} exists, $\det A^{-1} = (\det A)^{-1}$

__Thm__. If $A = [a_{ij}]$ is an $n \times n$ matrix that is either upper triangular or lower triangular then $\det A = \prod_{i=1}^{n} a_{ii}$

__Proof__ Follows from the definition.

Notice that by combining these results we can obtain the determinant of a matrix by reducing it to triangular form:

__Example.__ Find $\det A$

for $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

$E_2 - 2E_1 \to E_2$
$E_3 + E_1 \to E_3$ gives
$E_4 - 3E_1 \to E_4$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 3 & 3 & 2 \\ 0 & -4 & -1 & -7 \end{bmatrix} \quad \text{with } \det A = \det A_1$$

$E_3 + 3E_2 \to E_3$ gives
$E_4 - 4E_2 \to E_4$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 0 & -13 \\ 0 & 0 & 3 & 13 \end{bmatrix} \quad \text{with } \det A_1 = \det A_2$$

$E_3 \leftrightarrow E_4$ gives $A_3 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}$

with $\det A_3 = -\det A_2 = (1)(-1)(3)(-13)$
$= +39$

$\therefore \det A = \det A_1 = \det A_2 = -\det A_3$
$= -39$

LU-Decomposition

Suppose that we wanted to solve the system

$$\underset{n\times n}{A}x = b_k$$

for several vectors b_k.

This type of matrix problem arises frequently in initial value problems.

If we apply Gaussian elimination to the problem, then $O(n^3)$ operations are required each time x is found.

On the other hand, suppose we could factor A into the form

$$A = \underbrace{L}_{\text{Lower triangular matrix}}\ \underbrace{U}_{\text{Upper Triangular Matrix.}}$$

Then we can let $y = Ux$

$$\Rightarrow Ly = b_k.$$

Since L is triangular, determining y from this equation requires only $O(n^2)$ operations.

Once y is known, the triangular system

$$Ux = y$$

requires only an additional $O(n^2)$ operations to determine the solution x.

$\Rightarrow$ The number of operations needed to solve the system $Ax = b_k$ is reduced from $O(n^3)$ to $O(n^2)$. (Of course, determining L and U in the first place will require $O(n^3)$ operations, but this is only done once.)

We will proceed to derive this LU-factorization using Gaussian Elimination.

Consider the example

$$\underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 5 \\ 14 \end{pmatrix}}_{b}$$

We want to zero out entries below the pivot element a_{11}.

So we want to take $E_2 - 3E_1 \to E_2$.

This same effect can be obtained by multiplying A by the elementary matrix E_{21}

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_{21}} \underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{pmatrix}}_{A}$$

$$= \begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{pmatrix}$$

which is nonzero.

An elementary matrix is equal to the identity matrix except for one off diagonal element

Now we want to zero out the remaining entry below the pivot.

This can be done by taking
$E_3 - 2E_1 \to E_3$
or by left multiplying by the elementary matrix

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{E_{31}} \underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{pmatrix}}_{E_{21}A} = \underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{pmatrix}}_{E_{31}E_{21}A}.$$

To obtain an upper triangular system we would take
$E_3 - 4E_2 \to E_3$
or we can left multiply by the elementary matrix E_{32}

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}}_{E_{32}} \underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{pmatrix}}_{E_{31}E_{21}A} = \underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{pmatrix}}_{E_{32}E_{31}E_{21}A}$$

Notice that $E_{32} E_{31} E_{21} A$ is upper triangular.

Set $U = E_{32} E_{31} E_{21} A$

Now $E_{32}^{-1} U = E_{31} E_{21} A$

$E_{31}^{-1} E_{32}^{-1} U = E_{21} A$

$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U = A.$

To find the inverses of elementary matrices, we have to recall their meaning.

For example $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

subtracts 3 times the first row from the second.

The inverse will need to add 3 times the first row to the second

$$\Rightarrow \quad (E_{21})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So inverses of elementary matrices are simply obtained by changing the sign of the nonzero off diagonal element.

Furthermore notice that

$$L \equiv E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & \underset{1}{\cancel{0}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

So the matrix multiplication can be performed by putting the nonzero offdiagonal elements of the elementary matrices into the appropriate positions in the matrix L.

This means that the matrix L is easily constructed during the Gaussian elimination process just by storing the multipliers.

We conclude that

$$\underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}}_{\substack{L \\ \uparrow \\ \text{lower triangular}}} \underbrace{\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{pmatrix}}_{\substack{U \\ \uparrow \\ \text{upper triangular}}}$$

In fact this LU factorization process is quite general:

__Thm__. If Gaussian elimination can be performed without row exchanges then the matrix A has a unique LU factorization where L is a lower triangular matrix with all diagonal entries equal to 1 and U is an upper triangular matrix.

Once the matrix factorization is complete, the solution to

$$LUx = b$$

is found by first setting

$$y = Ux.$$

The vector y is determined from $Ly = b$ using forward substitution.

The variable x is found from $Ux = y$ via backward substitution.

Notice in the theorem that we need to be able to perform Gaussian Elimination without row exchanges.

<u>ex</u> Gaussian Elimination applied to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ requires a row exchange.

Can it be factored into

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ lu_1 & lu_2 + u_3 \end{pmatrix}$$

<u>Ans</u> No $u_1 = 0 \Rightarrow lu_1 = 0$ #.