MACM 316 Lecture 11 - Chapter 7

Alexander Ng

Wednesday, January 29, 2025

1 Iterative Techniques in Matrix Algebra

We are interested in solving large linear systems $Ax = b$.

Suppose the matrix A has a high ($> 99.9\%$) sparsity, i.e., most of the entries are zeros. We would like to take advantage of this spare structure to reduce the amount of computational work required. Unfortunately, Gaussin Elimination is often unable to take advantage of the sparse structure. For this reason, we consider <u>iterative techniques</u>.

2 Vector Norms

To estimate how well a particular iterative technique approximates the true solution, we will need some measurement of distance. This motivates the notion of the vector norm.

Def. A vector norm on $\mathbb{R}^n$ is a function $||\cdot||$ from $\mathbb{R}^n \to \mathbb{R}^n$ satisfying the following properties:

- $||x|| \geq 0$ for all $x \in \mathbb{R}^n$
- $||x|| = 0 \iff x = \mathbf{0}$
- $||\alpha x|| = |\alpha|||x||$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$
- $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in \mathbb{R}^n$

Def. 2. The l_2 or Euclidean norm of the vector x is given by

$$||x||_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2}$$

This represents the usual notion of distance.
Def. 3. The infinity or max norm of a vector x is given by

$$||x||_\infty = \max_{i=1}^{n} |x_i|$$

Def. 4. If $x, y \in \mathbb{R}^2$, then the l_2 distance between x and y is given by

$$||x - y||_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2}.$$

and the l_∞ distance between x and y is given by

$$||x - y||_\infty = \max_{i=1}^{n} |x_i - y_i|.$$

3 Sequences?

Iterative techniques generate a sequence of vectors.

Def. A sequence $\left\{x^k\right\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to converge to x with respect to the norm $||\cdot||$ if, given any $\epsilon > 0$, there exists an integer $N('\epsilon)$ such that

$$||x^{(k)} - x|| < \epsilon \text{ for all } k \geq N$$

The notation $N('\epsilon)$ is used to emphasize that N is dependent on ϵ, however, N is not a function of ϵ.

3.1 Thm.

The sequence of vectors $\left\{x^k\right\}$ converges to x in $\mathbb{R}^n$ with respect to $||\cdot||_\infty$ if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$. for each i.

3.2 Thm. 2

For each $x \in \mathbb{R}^n$

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

This theorem relates the infinity norm to the Euclidean norm, which is very useful in the context of iterative techniques.