

# MACM 316 Lecture 15

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## Bisection Method Example

He starts with an example on the bisection method. I will provide one later on.

Lowkey I missed notes from page 5.7 to 5.10 but I'll add them later. (page 5 of chapter 2 part 1 notes)

## 1 Fixed Point Iteration

We wish to find the roots of an equation  $f(p) = 0$ . We will be focusing on methods that iterate to find the root:

$$p_{n+1} = g(p_n)$$

We start by considering the fixed point problem.

**Def.** A fixed point  $p$  is the value of  $p$  such that  $g(p) = p$ .

We can form a fixed pt. problem from a root finding problem.

$$f(p) = 0$$

Find a fixed point problem.

e.g. set  $g(x) = f(x) + x$

$$g(p) = f(p) + p$$

$$p = g(p)$$

there are many choices of  $g$

$$\text{Try } g(x) = f^3(x) + x$$

Notice that fixed point problems and root finding problems are equivalent.

(???)

## 1.1 Example

We are given  $g(p) = p$ . Formulate a root finding problem.

$$f(x) = g(x) - x$$

Now we have  $f(p) = 0$ . We now have a root finding problem.

$$f(p) = 0 \implies g(p) = p$$

**i.e.**  $f$  has a root  $p$  implies  $g$  has a fixed point  $p$ .

$$g(p) = p \implies f(p) = 0.$$

**i.e.**  $g$  has a fixed point  $p$  implies  $f$  has a root  $p$ .

There are many possible choices for  $g$ : example  $g(x) - x = (f(x))^3$

Our ultimate goal is to find functions with fixed points.

## 2 Theorems

### Existence and Uniqueness

★ If  $g \in C[a, b]$  and  $g(x) \in [a, b]$  for all  $x \in [a, b]$  then  $g(x)$  has a fixed point in  $[a, b]$ .

★★ Suppose, in addition, that  $g'(x)$  exists on  $(a, b)$  and that a positive constant  $k < 1$  exists with

$$|g'(x)| \leq k < 1 \text{ for all } x \in (a, b).$$

then the fixed point in  $[a, b]$  is unique

*Proof.* (★): Existence

If  $g(a) = a$  or  $g(b) = b$  then  $g$  has a fixed point at an endpoint.

Suppose not, then it must be true that  $g(a) > a$  and  $g(b) < b$ .

Define  $h(x) = g(x) - x$ . Then  $h$  is continuous on  $[a, b]$  (adding two continuous functions yields a continuous function) and

$$h(a) = g(a) - a > 0 \text{ and } h(b) = g(b) - b < 0.$$

The **IVT** implies that there exists a  $p \in (a, b)$  for which  $h(p) = 0$

This  $g(p) - p = 0 \implies p$  is a fixed point of  $g$

*Proof.* (★★): Uniqueness

Suppose, in addition,

$$\forall x \in (a, b), |g'(x)| \leq k < 1.$$

and that  $p$  and  $q$  are both fixed points in  $[a, b]$  with  $p \neq q$ .

By the **MVT**, a number  $c$  exists between  $p$  and  $q$  such that

$$\frac{g(p) - g(q)}{p - q} = g'(c).$$

then

$$\begin{aligned} |p - q| &= |g(p) - g(q)| \\ &= |g'(c)| |p - q| \\ &\leq k |p - q| \\ &< |p - q| \end{aligned}$$

Which is a contradiction

This contradiction must come from the assumption that  $p \neq q$

$\therefore p = q$  and the fixed point is unique.

We want to approximate the fixed point of a function  $g$ .

**IDEA**

- choose an initial approximation  $p_0$
- generate a sequence  $\{p_n\}_{n=0}^{\infty}$  such that  $p_n = g(p_{n-1}); n \geq 1$

If the sequence converges to  $p$  and  $g$  is continuous;

$$\begin{aligned} p &\equiv \lim_{n \rightarrow \infty} p_n \\ &= \lim_{n \rightarrow \infty} g(p_{n-1}) \\ &= g\left(\lim_{n \rightarrow \infty} p_{n-1}\right) \\ &= g(p) \end{aligned}$$

This gives the Fixed Point Algorithm:

??? what the fuck is he yapping about

why are the notes on the board different than the notes on the canvas