MACM 316 Lecture 8

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1   LU-Decomposition

Suppose that we wanted to solve the system

$$Ax = b_k \qquad A \in \mathbb{R}^{n \times n}$$

for several different $b_k \in \mathbb{R}^n$.

This type of matrix problem arises frequently in initial value problems. If we apply Gaussian Elimination to solve the system, then $O(n^3)$ operations are required for each b_k. On the other hand, suppose we could factor A into the form $A = LU$ where L is lower triangular and U is upper triangular.

Then, we can let $y = Ux \implies Ly = b_k$, and solving for y via forward substitution is $O(n^2)$ operations.

Next, we have to solve for $Ux = y$ via backward substitution, which is also $O(n^2)$ operations.

The total number of operations is $O(n^2) + O(n^2) = O(n^2)$, which is much better than the $O(n^3)$ operations required for Gaussian Elimination.

Finding L and U is $O(n^3)$ requires $O(n^3) + O(n^3)$ operations **HOW-EVER**, by doing this expensive operation once, we can save a lot of operations when there are many b_k.

1.1   LU-Factorization

We will proceed to derive this LU-factorization using Gaussian Elimination.

Example:

$$\begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 39 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix}$$

We want to zero out entries below the pivot element a_{11} in the first row. So we want to take $E_2 - 3E_1 \rightarrow E_2$. This same effect can be obtained by multiplying A by the elementary matrix E_{21}.

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 39 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{bmatrix}$$

An elementary matrix is equal to the edentity matrix except for one nonzero entry off of the main diagonal.

Now we want to zero out the remaining entry below the pivot.

This can be done by taking $E_3 - 2E_1 \rightarrow E_3$ or by left-multiplying A by the elementary matrix E_{31}.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix}$$

To obtain an upper triangular system, we would finally zero out the remaining entry below the pivot in E_2, taking $E_3 - 4E_2 \rightarrow E_3$. Or, we can left multiply by the elementary matrix E_{32}.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

Notice that $E_{32}E_{31}E_{21}A$ is upper triangular.

Set $U = E_{32}E_{31}E_{21}$ and $L = A$.

Now $E_{32}^{-1}E_{31}^{-1}E_{21}^{-1}U = L$.

To find the inverses of elementary matrices, we have to recall their meaning.

For example, $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtracts 3 times the first row

from the second.

The inverse will need to add 3 times the first row to the second.

$$\implies (E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which costs $O(1)$ operations to flip the sign of the single entry.
Furthermore, notice that
$L \cong E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

So the matrix multiplication can be performed by putting the nonzero offdiagonal elements of the elementary matrices intot he appropriate positions in the matrix L.

This means that the matrix L is easily constructed during the Gaussian Elimination process just by storing the multipliers.

We conclude that

$$\underset{A}{\begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix}}$$