MACM 316 Lecture 34

Alexander Ng

Friday, April 4, 2025

1 The Taylor Method of Order n

If we drop the remainder term, we obtain the **Taylor Method of Order** n.

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hT^{(n)}(t_i, w_i) \quad i = 0, 1, \ldots, N-1. \end{cases}$$

where $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^n}{n!} f^{(n)}(t_i, w_i)$ is the n^{th} Taylor Polynomial of f about t_i.

Note: *Euler's Method is equivalent to Taylor's Method of Order* 1.

1.1 Example: Taylor's Method of Order 2 (24.1)

Use Taylor's Method of order Two to approximate the solution for the IVP

$$\begin{cases} y' = te^{3t} - 2y & 0 \leq t \leq 1 \\ y = 0 & t = 0. \end{cases}.$$

with $h = 0.5$.

Soln. The first approximation is

$$\begin{aligned} w_1 &= w_0 + h(t_0 e^{3t_0} - 2w + 0) + \frac{h^2}{2}(t_0 e^{3t_0} + 4w_0) \\ &= 0 + 0.5(0 - 0) + \frac{(0.5)^2}{2}(0 + 1 + 0) \\ &= 0.125 \end{aligned}$$

and the second is

$$
\begin{aligned}
w_2 &= w_1 + h\left(t_1 e^{3t_1 - 2w_1} + \frac{h^2}{2} f\left(t_1, e^{3t_1} + e^{3t_1} + f(w_1)\right)\right) \\
&= 0.125 + 0.5\left(0.5e^{1.5} - 2(0.125)\right) \\
&\quad + \frac{(0.5)^2}{2}\left(0.5e^{1.5} + e^{1.5} + 4(0.125)\right) \\
&= 2.02323897
\end{aligned}
$$

1.2 Intermediate Point Methods (24.2)

If we want to determine an intermediate point (**e.g.** for some $t \in (t_{i-1}, t_i)$), then **Cubic Hermite Interpolation** based on $(y(t_{i-1}), y'(t_{i-1}), y(t_i), y'(t_i))$ is a particularly natural choice for a Taylor Method of degree ≤ 4. Such an interpolation has the advantages that it can be constructed locally and that $y'(t) = f(t, y(t))$ is given.

To interpolate results from very high order Taylor Methods ($n > 4$), we will need higher order oscillating polynomials to preserve the overall accuracy of the results.

1.3 Error Analysis fo Taylor's Method (24.3)

The local truncation error for Taylor's Method of Order n is easily derivced:

$$
\begin{aligned}
& y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) \\
\text{gratuitous cancellations yield} \quad &= \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \\
\text{where} \quad & y_i \equiv y(t_i)
\end{aligned}
$$

Thus the local truncation error is

$$
\begin{aligned}
\tau_{i+1}(h) &= \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) - \frac{h}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) \\
&= \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))
\end{aligned}
$$

Thus, if $y \in C^{(n+1)}[a,b]$
$\implies y^{(n+1)}(t) = f^{(n)}(t, y(t))$ is bounded
and $\tau_i = \mathcal{O}(h^n)$ for each $i = 1, 2, \dots, N$.

2 Runge-Kutta Methods (24.4)

Taylor Methods are seldom used in practice because they require the computation and evaluation of the derivatives of $f(t, y)$. These evaluations can be complicated and expensive.

Runge-Kutta Methods have the high local truncation error of the Taylor Methods but do not need compute and evaluate the derivatives of $f(t, y)$. To give some idea of how Runge-Kutta methods are developed, we will now show the derivation of a simple second-order method. Here, the increment of w is a weighted average of two estimates of the increment which we will call k_1 and k_2.

$$\begin{cases} w_{n+1} = w_n + ak_1 + bk_2 \\ k_1 = hf(t_n, w_n) \\ k_2 = hf(t_n + \alpha h, w_n + \beta k_1). \end{cases}$$

we can think of k_1 and k_2 as estimates of the change in y when t advances by h because they are the product of the change in t and a value for the slope of the curve.

Runge-Kutta methods often use the simple Euler estimate as the first estimate of δy. Now our problem si to devise a scheme by choosing hte four parameters a, b, α, β. We do so by making the local truncation error of (2).

We re-write (2) as

$$w_{n+1} = w_n + ahf(t_n, w_n) + bhf(t_n + \alpha h, w_n + \beta h + f(t_n, w_n)).$$

The local truncation error is then

$$\tau_{n+1}(h) = \frac{y_{n+1} - y_n}{n} - af(t_n, y_n) - bf(t_n + \alpha h, y_n + \beta hf(t_n, y_n)).$$

Applying a Taylor Series of degree 2:

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2} \underbrace{f'(t_n, y_n)}_{f_t(t_n,y_n) + f_y(t_n,y_n) \cdot f(t_n,y_n)} + \mathcal{O}\Big(h^3\Big).$$

$$
\begin{aligned}
&f(t_n + \alpha h, y_n + \beta h f(t_n, y_n)) \\
&= f(t_n, y_n) + f_t(t_n, y_n)\alpha h + f_y(t_n, y_n) f(t_n, y_n)\beta h + \mathcal{O}\Big(h^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \tau_{n+1}(h) = &(1 - a - b) f(t_n, y_n) \\
&+ h(\frac{1}{2} - \alpha b) f_t(t_n, y_n) \\
&+ h(\frac{1}{2} - \beta b) f_y(t_n, y_n) f(t_n, y_n)
\end{aligned}
$$

Thus the local truncation error will be $\mathcal{O}(h^2)$ provided

$$
\begin{aligned}
a + b &= 1 \\
\alpha b &= \frac{1}{2} \\
\beta b &= \frac{1}{2}
\end{aligned}
$$

but there is not enough flexibility to obtain a third order method. (Proof is left as an exercise.)

2.1 Examples of Runge-Kutta Methods (24.7)

2.1.1 The Midpoint Method

$$
\boxed{\begin{cases} a = 0 \\ b = 1 \\ \alpha = \frac{1}{2} \\ \beta = \frac{1}{2} \end{cases} \implies \begin{cases} w_0 = y(t_0) \\ w_{n+1} = w_n + hf(t_n + \frac{h}{2}, w_n + \frac{h}{2} f(t_n, w_n)) \end{cases}}
$$

2.1.2 The Modified Euler Method

$$
\boxed{\begin{cases} a = \frac{1}{2} \\ b = \frac{1}{2} \\ \alpha = 1 \\ \beta = 1 \end{cases} \implies \begin{cases} w_0 = y(t_0) \\ w_{n+1} = w_n + \frac{h}{2}[f(t_n, w_n) + f(t_{n+1} w_n + hf(t_n w_n))] \end{cases}}.
$$

2.1.3 Heun's Method

$$\boxed{\begin{cases} a = \frac{1}{4} \\ b = \frac{3}{4} \\ \alpha = \frac{2}{3} \\ \beta = \frac{2}{3} \end{cases} \quad \Longrightarrow \begin{cases} w_0 = y(t_0) \\ w_{n+1} = w_n + \frac{h}{4}\Big[f(t_n, w_n) + 3f\Big(t_n + \frac{2}{3}h, w_n + \frac{2}{3}hf(t_n w_n)\Big)\Big] \end{cases}}.$$