MACM 316 Lecture 9

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1 LU Decomposition (contd.)

THM: If Gaussian elimination can be performed without row exchanges, then the matrix A has a unique LU factorization where L is lower triangular and with all diagonal entries equal to 1 and U is an upper triangular matrix.

Once the matrix factorization is complete, the solution to

$$LUx = b$$

is found by first setting

$$y = Ux$$

then determining the vector y from $Ly = b$ using forward substitution. The variable x is found from $Ux = y$ using backward substitution.

Notice: in the theorem, **we need to be able to perform Gaussian Elimination without row exchanges**. This means that we can't use row exchanges to solve the system.

2 Matrix Factorization

If A is any nonsingular matrix, $Ax = b$ can be solved by Gaussian Elimination with the possibility of row exchanges. If we can find the row interchanges required to solve the system by Gaussian Elimination, then we can re-arrange the equations in an order that would ensure that no row interchanges are required.

$\Longrightarrow$ There is a rearrangement of the equations that permits Gaussian Elimination without row exchanges.

But WHY does the rearrangement of the equations break LU decomposition?

3 Permutation Matrix

An $n \times n$ permutation matrix P is obtained by rearranging the rows of the identity matrix. This gives a matrix with precisely one nonzero entry in each row and column. The nonzero entries are all 1's.

On the other hand, multiplying A on the right by P will exchange the second and third **columns** of A.

We will be using the following two properties of permutation matrices:

1. If $K_1, ..., K_n$ is a permutation of the integers $1, ..., n$, and the permutation matrix $P = [p_{ij}]$ is defined by

   $$p_{ij} = \begin{cases} 1 & i = K_j \\ 0 & \text{otherwise} \end{cases}$$

   Then, PA permutes the rows of A according to (BIGASS MATRIX MISSING HERE) data from chapter6-part2.pdf page 3.

2. If P is a permutation matrix, then P^{-1} exists and $P^{-1} = P^T$

Now our approach will be to left multiply the system

$$Ax = b$$

by the appropriate permutiation matrix P_1 so that the system

$$(PA)x = Pb$$

can be solved without row exchanges. Then PA can be factorized into

$$PA = LU$$

This tells us that

$$
\begin{aligned}
P^{-1}LUx &= b \\
LUx &= Pb
\end{aligned}
$$

which can be solved rapidly for x.