MACM 316 Lecture 13

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Monday, February 3, 2025

When we use iterative matrix techniques, we want to know when powers of a matrix become small.

Def. We call an $n \times n$ matrix A convergent if

$$\lim_{k\to\infty} \left(A^k\right)_{ij} = 0, \text{ for all } i, j$$

Ex. Consider $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 16 & \frac{1}{2} \end{bmatrix}$

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 16 & \frac{1}{4} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ 12 & \frac{1}{8} \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ 8 & \frac{1}{16} \end{bmatrix}$$

$$A^k = \begin{bmatrix} \frac{1}{2^k} & 0 \\ P_k & \frac{1}{2^k} \end{bmatrix}$$

where $P_k = \begin{cases} 16 & k = 1 \\ \frac{16}{2^{k-1}} + \frac{1}{2}P_{k-1} & k > 1. \end{cases}$

Since $\lim_{k\to\infty} P_k = 0$, we also know that $\lim_{k\to\infty} P_k = 0$. $\therefore$ A is a convergent matrix.

Notice that this convergent matrix has a spectral radius (see Lecture 12 notes , page 5) less than 1.

This generalizes:

Thm. The following statements are equivalent:

1. A is a convergent matrix.
2. $\rho(A) < 1$
3. $\lim_{n\to\infty} A^n x = 0$ for every x
4. $\lim_{n\to\infty} ||A^n|| = 0$ for all natural norms $||\cdot||$

Iterative techniques convert the system $Ax = b$ into an equivalent system of the form $x = Tx + c$ where T is a fixed matrix and c is a vector. An initialv vector $x^{(0)}$ is chosen, and then a sequence of approximate solution vectors is generated:

$$x^{(k)} = Tx^{(k-1)} + c$$

Iterative techniques are rarely used in very small systems (i.e. when n^3 is small). In these cases, iterative techniques may be slower since they require several iterations to obtain the desired accuracy.

<u>**IDEA:**</u> It is possible to "split" the matrix A :

$$\begin{aligned} Ax &= b \\ [M + (A - M]\, x &= b \\ Mx &= b + (M - A)x \\ x &= (I - M^{-1}A)x + M^{-1}b \end{aligned}$$

Iteration becomes

$$x^{(k+1)} = (I - M^{-1}A)x^{(k)} + M^{-1}b$$

We set $T \cong I - M^{-1}A$ (the amplification matrix) and $c \cong M^{-1}b$.

$$x^{(k+1)} = Tx^{(k)} + c$$

How do we choose M?
We want:

1. M easy to "invert"

2. M "close to A" in the sense that $\rho(T)$ is small.

Ex. Let $M = D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_{nn} \end{bmatrix}$

This gives the Jacobi Iterative Method.

In the text's notation,

$$A = D - L - U$$

Where D is diagonal, L is lower triangular, and U is upper triangular.

$$\begin{aligned} Ax &= b \\ (D - L - U)x &= b \\ Dx &= (L + U)x + b \\ x &= D^{-1}(L + U)x + D^{-1}b \end{aligned}$$

Which results in the iteration

$$x^{(k+1)} = D^{-1}(L + U)x^{(k)} + D^{-1}b$$

Let $T = D^{-1}(L + U)$ and $c = D^{-1}b$.

$$x^{(k+1)} = Tx^{(k)} + c$$

See example in the notes, there are too many matrices to type out in LaTeX. See "Chapter 7.pdf" page 27 (7-36.7)

0.0.1 Comments on Jacobi's Method

$$x^{(k+1)} = D^{-1}(L + U)x^{(k)} + D^{-1}b$$

1. The algorithm requirs $a_{ii} \neq 0$ for $i = 1, \ldots, n$ If one of the $a_{ii} = 0$, and the system is nonsingular, then a reordering of the equations can be performed so that no $a_{ii} = 0$.

2. To speed convergence, the equations should be arranged such that $|a_{ii}$ is as large as possible.

3. A possible stopping criterion is to iterate until $\frac{||x^{(k)}-x^{(k-1)}||}{||x^{(k-1)}||} \leq \epsilon$

If we write out Jacobi's Method

$$x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$$

we find that

$$x_i^{(k+1)} = \frac{\sum_{j=1;j\neq i}^{n}(-a_{ij}x_j^{(k)}) + b_i}{a_{ii}}$$

Notice that to compute $x_i^{(k+1)}$, the components $x_i^{(k)}$ are used. But, for $i > 1$, $x_1^{(k+1}, x_2^{(k+1)}, \ldots, x_n^{(k+1)}$ have already been computed and are likely better approximations to the actual solutions than

$$x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$$

So it seems reasonable to compute with these most recently computed values.

i.e.:

$$x_i^{(k+1)} = \frac{-\sum_{j=1}^{i=1}(a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k)} + b_i}{a_{ii}}$$

This is called the Gauss-Seidel iterative technique, and it also has a matrix formulation with $M \cong (D-L)$:

$$\begin{aligned} Ax &= b \\ (D-L-U)x &= b \\ (D-L)x &= Ux + b \\ x &= (D-L)^{-1}Ux + (D-L)^{-1}b \end{aligned}$$

$\Longrightarrow$ iteration becomes

$$x^{(k+1)} = (D-L)^{-1}Ux^{(k)} + (D-L)^{-1}b$$

*Notice that $D-L$ is lower triangular. It is invertible $\iff$ each $a_{ii} \neq 0$