Things to Remember

- a_{ij} is the ith row and jth column of A.

Floating-Point & Errors

- **Representation (IEEE 754):** Real numbers are discretized; rounding to nearest representable value.
- **Rounding vs. Chopping:** Rounding picks the nearest representable number; chopping just truncates bits.
- **Catastrophic Cancellation:** Occurs when subtracting nearly equal numbers, causing large relative error.
- **Condition Number (κ):** Measures sensitivity of output to small changes in input.
- **Stability:** An algorithm is stable if small input perturbations only cause proportionally small output changes.

Direct Methods for Linear Systems

- **Gaussian Elimination:** $O(n^3)$ operations. Pivoting (partial or complete) avoids large roundoff from tiny pivots.
- **LU Factorization:** $A = LU$. Do forward/back substitution for multiple RHS vectors. For SPD matrices, use Cholesky ($A = LL^\top$).
- **Pivoting:** *Partial pivoting* swaps rows to pick a large pivot; *complete pivoting* can swap rows/columns for further stability.
- **Band/Tridiagonal Matrices:** Exploit structure to reduce computational cost.

Iterative Methods for $Ax = b$

- **Jacobi:** $x_i^{(k+1)} = \frac{b_i - \sum_{j\neq i} a_{ij}\, x_j^{(k)}}{a_{ii}}, \quad$ for each i. Uses old values in each iteration.
- **Gauss-Seidel:** Similar formula but uses updated values immediately in iteration. Often converges faster.
- **SOR (Successive Over-Relaxation):** $x^{(k+1)} = x^{(k)} + \omega(\text{Gauss-Seidel update})$ with $1 < \omega < 2$ for faster convergence if well-chosen.
- **Convergence Criterion:** Typically $\rho(T) < 1$, where T is the iteration matrix.
- **Diagonally Dominant / SPD:** Guarantee convergence for Jacobi/Gauss-Seidel.

Nonlinear Equations (Root Finding)

- **Bisection Method:** Requires a sign change over $[a, b]$. Repeatedly halve interval. Guaranteed convergence (linear).
- **Fixed-Point Iteration:** $x_{k+1} = g(x_k)$. Converges if $|g'(p)| < 1$. Check iteration function carefully.
- **Newton's Method:** $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, Quadratic convergence near root if $f'(p) \neq 0$. Needs derivative f'.
- **Secant Method:** Derivative is approximated by $\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}$. Superlinear convergence.
- **Regula Falsi (False Position):** Combines bracketing with secant-like updates, maintaining bracket.

Polynomial Interpolation

- **Lagrange Form:** $P_n(x) = \sum_{j=0}^{n} f(x_j)\, L_j(x), \quad L_j(x) = \prod_{0\le m\le nm\neq j} \frac{x - x_m}{x_j - x_m}$.
- **Divided Differences (Newton Form):** Build polynomial incrementally. Good for reusing previous calculations if new points are added.
- **Error Term:** $f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$.
- **Hermite Interpolation:** Matches both f and f' at nodes (more conditions).
- **Cubic Splines:** Piecewise cubics ensuring $S(x_i) = f_i$, continuous first/second derivatives at interior nodes. Boundary conditions: natural ($S''(x_0) = S''(x_n) = 0$) or clamped ($S'(x_0)$, $S'(x_n)$ given).

Numerical Differentiation

- **Forward Diff:** $f'(x) \approx \frac{f(x+h) - f(x)}{h}, \ \mathcal{O}(h)$.
- **Centered Diff:** $f'(x) \approx \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}, \ \mathcal{O}(h^2)$.
- **Second Derivative:** $f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}, \ \mathcal{O}(h^2)$.
- **Richardson Extrapolation:** Combine approximations with different h to cancel leading error terms and boost accuracy.
- **Roundoff vs. Truncation:** Extremely small $h \Rightarrow$ roundoff error. Large $h \Rightarrow$ truncation error.

Numerical Integration

- **Trapezoid Rule (Basic):** $\int_a^b f(x)\, dx \approx \frac{b-a}{2}\big(f(a) + f(b)\big)$. Composite version: partition $[a, b]$ into n subintervals, sum trapezoids. Error $\mathcal{O}(h^2)$ for composite.
- **Simpson's Rule:** Fits parabolas through triples of points. Composite Simpson has error $\mathcal{O}(h^4)$.
- **Newton-Cotes Family:** General equally spaced formulas (e.g. Simpson, 3/8 rule). Degree of precision is higher if n is even.
- **Romberg Integration:** Trapezoid + Richardson extrapolation $\Rightarrow$ improved order systematically.
- **Adaptive Quadrature:** Subdivide intervals where function changes rapidly, ensuring error remains below tolerance.
- **Gaussian Quadrature:** Chooses nodes/weights (Legendre polynomials) to get exact results up to degree $2n - 1$ with n points.

Initial Value Problems (ODEs)

- **Existence & Uniqueness:** If $f(t, y)$ is continuous in t and Lipschitz in y, then the IVP $y'(t) = f(t, y)$, $y(t_0) = y_0$ has a unique solution.
- **Euler's Method:** $w_{k+1} = w_k + h\, f(t_k, w_k), \quad$ local error $\mathcal{O}(h^2)$, global $\mathcal{O}($
- **Taylor Methods:** Use derivatives of f up to nth order; local error $\mathcal{O}(h^{n+1})$, but can be cumbersome to compute derivatives.
- **Runge-Kutta Methods (RK2, RK4, etc.):** Achieve higher order without symbolic derivatives. E.g. RK4 has local error $\mathcal{O}(h^5)$, global $\mathcal{O}(h^4)$.
- **Stability in ODE Solvers:** Step size must be sufficiently small for stable integration, especially for stiff problems.

Quick Error/Order Reference

- **Linear Systems:**
  - Gauss Elim: $O(n^3)$ ops
  - Jacobi/G-S: converge if $\rho(T) < 1$
- **Root Finding:**
  - Bisection: linear
  - Newton: quadratic
  - Secant: superlinear (≈ 1.618)
- **Interpolation Error:** $f(x) - P_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \prod (x - x_j)$
- **Num. Differentiation:**
  - Forward diff: $\mathcal{O}(h)$
  - Center diff: $\mathcal{O}(h^2)$
- **Num. Integration:**
  - Trapezoid (composite): $\mathcal{O}(h^2)$

- Simpson (composite): $\mathcal{O}(h^4)$
- Romberg: $\mathcal{O}(h^{2k})$ with extrapolation

- **ODE Solvers:**
  - Euler: local $\mathcal{O}(h^2)$, global $\mathcal{O}(h)$
  - RK4: local $\mathcal{O}(h^5)$, global $\mathcal{O}(h^4)$

Special Types of Matrices & Convergence Behavior

- **Diagonal Matrix:** Only non-zero entries are on the main diagonal. Easily invertible; iterative methods converge trivially.

- **Triangular Matrix:**
  - *Upper/Lower Triangular:* All entries below/above diagonal are zero.
  - Solvable via forward/backward substitution in $O(n^2)$ time.

- **Symmetric Matrix:** $A = A^\top$. Diagonalizable with real eigenvalues.

- **Positive Definite Matrix (SPD):**
  - $x^\top Ax > 0$ for all $x \neq 0$.
  - All eigenvalues are positive.
  - Allows Cholesky factorization: $A = LL^\top$.
  - Gauss-Seidel and Conjugate Gradient methods converge when A is SPD.

- **Diagonally Dominant Matrix:**

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i.$$

  - *Strictly diagonally dominant:* $>$ instead of $\geq$.
  - Guarantees convergence of Jacobi, Gauss-Seidel, and SOR methods.

- **Band Matrix:**
  - Nonzero entries confined to a diagonal band (e.g., tridiagonal).
  - Efficient to store and solve: $O(nb^2)$ where b is bandwidth.

- **Sparse Matrix:**
  - Majority of entries are zero.
  - Exploit sparsity for efficient storage and faster matrix-vector products.

- **Ill-Conditioned Matrix:**
  - Has large condition number $\kappa(A)$.
  - Small perturbations in input lead to large errors in output.
  - May cause instability in numerical methods (especially direct solvers).

- **Normal Matrix:** $A^\top A = AA^\top$. Includes symmetric and orthogonal matrices.

- **Convergence Summary for Iterative Methods:**
  - **Jacobi/Gauss-Seidel:** Converge if A is SPD or strictly diagonally dominant.
  - **SOR:** Converges if A is SPD and $0 < \omega < 2$.
  - **Spectral Radius Criterion:** Iteration matrix T satisfies $\rho(T) < 1$ for convergence.

Determinants and Eigenvalues

- **Determinant (Definition):**
  - Scalar value associated with a square matrix.
  - Denoted $\det(A)$ or $|A|$.
  - Indicates volume scaling factor of linear transformation and invertibility of matrix.
  - $\det(\mathbb{R}^{2\times2}) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

- **Properties of Determinants:**
  - $\det(I) = 1$
  - $\det(AB) = \det(A)\det(B)$
  - $\det(A^\top) = \det(A)$
  - $\det(A^{-1}) = \frac{1}{\det(A)}$ if A is invertible
  - Row swaps change sign of determinant.
  - Adding a multiple of one row to another does not change determinant.
  - If A has a row or column of zeros $\Rightarrow \det(A) = 0$

- **Cofactor Expansion (Laplace Expansion):**
  - Expand determinant along any row or column:

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij})$$

    where M_{ij} is the minor of A (matrix formed by deleting row i and column j).
  - Computationally expensive for large matrices (use LU for efficiency).

- **Triangular Matrix Determinant:**

$$\det(A) = \prod_{i=1}^{n} a_{ii} \quad \text{if } A \text{ is upper or lower triangular.}$$

- **Eigenvalues and Eigenvectors:**
  - For square matrix A, if $Ax = \lambda x$, then:
    * λ is an **eigenvalue**
    * x is a corresponding **eigenvector**
  - To find eigenvalues:

$$\det(A - \lambda I) = 0$$

    This is the characteristic polynomial.
  - Each eigenvalue has one or more associated eigenvectors, found by solving:

$$(A - \lambda I)x = 0$$

- **Properties of Eigenvalues:**
  - Sum of eigenvalues = $\mathrm{tr}(A)$
  - Product of eigenvalues = $\det(A)$
  - Eigenvalues of $A^\top = A$
  - If A is symmetric: all eigenvalues are real; eigenvectors are orthogonal.
  - If A is invertible: no eigenvalue equals 0.

- **Diagonalization:**
  - $A = PDP^{-1}$ if A has n linearly independent eigenvectors.
  - D is diagonal matrix of eigenvalues; P contains eigenvectors as columns.

- **Spectral Radius:**

$$\rho(A) = \max_i |\lambda_i|$$

  Determines convergence behavior of many iterative methods.