MACM 316 Lecture 4 - Numerical Approximation and Big-O Notation

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1 Linear Approximation of $\sqrt{16.1}$

We approximate $\sqrt{16.1}$ without using the square root algorithm.

1.1 Solution

Let $f(x) = \sqrt{x}$ and choose an expansion point $x_0 = 16$, since $\sqrt{16} = 4 \in \mathbb{Z}$ is easily computable. Using the first-order Taylor approximation:

$$f(x_0 + h) \approx f(x_0) + hf'(x_0),$$

where $h = 0.1$.

1.2 Computation

$$\begin{aligned} f(16) &= \sqrt{16} = 4, \\ f'(x) &= \frac{1}{2\sqrt{x}}, \quad f'(16) = \frac{1}{8}, \\ f(16.1) &\approx 4 + 0.1 \times \frac{1}{8} = 4.0125. \end{aligned}$$

The exact value is $4.01248052955\ldots$, with a small truncation error. Since the machine error is on the order of 10^{-14}, it is negligible compared to the Taylor approximation error.

2 Algorithm Quantification

Numerical methods construct a sequence of better approximations, converging to a solution α.

Given a sequence $\{\alpha_n\}$:

$$\lim_{n \to \infty} \alpha_n = \alpha.$$

We quantify convergence speed by analyzing $|\alpha - \alpha_n| \leq c$, where c is a target error.

2.1 Example: Convergence of $\sin(1/n)$

Consider $\alpha_n = \sin(1/n)$, which converges to $\alpha = 0$ as $n \to \infty$. We rewrite:

$$\lim_{n \to \infty} \sin(1/n) = \lim_{h \to 0} \sin(h),$$

which is easier to analyze. Expanding $\sin(h)$ in a Taylor series:

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} + \dots.$$

For small h, $\sin(h) \approx h$, implying:

$$|\alpha_n - \alpha| \leq \frac{1}{n}.$$

Thus, α_n converges to $\alpha = 0$ with rate of convergence $O(1/n)$.

3 Big-O Notation

For a sequence $\{A_n\}$, if:

$$|A_n - A| \leq k|B_n| \quad \text{for sufficiently large } n,$$

where k is a constant, then we say:

$$A_n = A + O(B_n).$$

3.1 Example: $\sin(1/n)$ Convergence

From before, $|\alpha_n - \alpha| \leq 1/n$, so:

$$A_n = \sin(1/n) \text{ converges to } A = 0 \text{ with rate of convergence } O(1/n).$$

3.2 Example: Convergence of $n \sin(1/n)$

We evaluate:

$$\lim_{n\to\infty} n \sin(1/n) = 1.$$

Changing variables, $h = 1/n$, we obtain:

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

Expanding $\sin(h)/h$ in Taylor form:

$$\frac{\sin(h)}{h} = 1 - \frac{h^2}{6} + O(h^4).$$

For small h:

$$\frac{\sin(h)}{h} - 1 \approx -\frac{h^2}{6},$$

so α_n converges to $\alpha = 1$ with rate $O(1/n^2)$.

4 Takeaways

Big-O notation quantifies algorithm efficiency by ignoring constants and focusing on convergence trends. Constants vary across systems, so we care about general convergence patterns rather than specific values.