# MACM 316 Lecture 31-b

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Friday, March 28, 2025

## 1 Initial Value Problems for ordinary Differential Equations

Many natural, scientific and engineering problems can be described in terms of differential equations. Differential equations give us a way to mathematically expres rates of change. We will be considering methods for treating ordinary differential equations (ODEs) in this lecture. ODEs only consider derivatives with respect to one variable.

**Ex.** Let $y(t)$ denote the number of individuals in a certain population. If this population has a constant growth rate $\alpha$ (the difference between a constant birth rate and death rate), then the differential equation

$$y'(t) = \alpha y(t).$$

with initial condition $y(0) = y_0$ describes the population growth.

*Soln.*

$$\begin{aligned}
\frac{y'(t)}{y(t)} &= \alpha \\
D_t[\ln(y(t))] &= \alpha \\
\ln(y(t)) &= \alpha t + \text{const} \\
y(t) &= ce^{\alpha t} \qquad c = e^{\text{const}}
\end{aligned}$$

$$y(0) = y_0 \implies y(t) = y_0 e^{\alpha t}.$$

To include other effects such as overcrowding, competition for food, etc, one might introduce a second term into the equation

$$y'(t) = \alpha y(t) - \beta[y(t)]^2.$$

where $\beta > 0$ and $\beta$ is small. Introducing a nonlinear term makes the problem much more difficult to study analytically.

Indeed, few problems originating from the study of physical phenomena can be solved exactly. We begin by studying numerical methods for approximating the solution $y(t)$ to a problem.

$$\frac{dy}{dt} = f(t, y) \qquad \text{for } a \leq t \leq b.$$

subject to the initial condition

$$y(a) = \alpha.$$

## 1.1 The elementary theory of initial value problems

We want/need some theoretical results, in particular, we would like to show that solutions to equations exist and are unique.

**Def.** A function $f(t, y)$ satisfies a Lipschitz condition in the variable $y$ on a set $D \in \mathbb{R}^2$ if a constant $L > 0$ exists such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \qquad \text{for all } (t, y_1), (t, y_2) \in D.$$

The constant $L$ is called a <u>Lipschitz constant</u> for $f$.

### 1.1.1 Example

Does $f(t,y) = ty$ satisfy a Lipschitz condition on

$$D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}?$$

*Soln.*

$$|f(t,y_1) - f(t,y_2)| = | - ty_1 + \frac{4t}{y_1} + ty_2 - \frac{4t}{y_2}|.$$

Consider $y_1 = -1, t = 1,\ y_2 \to 0^+$. Under these conditions, we have $|f(t,y_1) - f(t,y_2)| \to \infty$.

RHS $= L|y_1 - y_2| \to L$

$L$ is finite.

We cannot have $|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$ for any finite $L$

$\therefore$ Lipschitz condition does not hold.