MACM 316 Lecture 7

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1 Scaled Partial Pivoting

We can improve the accuracy of the partial pivoting algorithm if we scale coefficients before deciding on row exchanges.

The scaling factor is the largest absolute value of any coefficient in the current row. The idea is to select the largest scaled value a_{ik}/S_i corresponding to elements that are below the pivot.

The extra work to apply the partial pivoting algorithm is some constant times $O(n^2)$.

In rare instances, complete pivoting may be needed, which is $O(n^3)$ extra work.

1.1 An excerpt on Time Complexity

Total cost of partial pivoting is $O(n^3) + O(n^2) = O(n^3)$

Complete pivoting is $O(n^3) + O(n^3) = O(n^3)$

In the end, the dominant term of complete pivoting is $(c_1 + c_2)O(n^3)$, which is technically more than $c_1 O(n^3)$ for partial pivoting, but it is the extra work is insignificant compared to the total work of the actual algorithm.

2 Some Review (Definitions)

1. Two matrices A and B are equal if they have the same size and if each element a_{ij} in A is equal to b_{ij} in B.

2. $A + B$ for two similarly sized matrices A and B is defined as the $n \times n$ matrix whose entries are $(a_{ij} + b_{ij})$ for all $i = 1..n$ and $j = 1..m$.

3. λA for a scalar λ and a matrix A is defined as the matrix whose entries are λa_{ij} for all $i = 1..n$ and $j = 1..m$.

3 Determinants

A very useful concept of linear algebra is the determinant of a matrix. The determinant of a matrix A is denoted by $\det(A)$ or $|A|$.

Determinants are important, in part, because of the following theorem:

3.1 Theorem

The following statements are equivalent for any $n \times n$ matrix A.

1. $\det(A) \neq 0$
2. The equatoin $Ax = 0$ has a unique solution $x = 0$.
3. The system $Ax = b$ has a unique solution for any n-dimensional column vector b.
4. The matrix A is nonsingular.
   i.e. A^{-1} exists.

3.2 Definition of the Determinant

The definition of the determinant is somewhat involved:

(a) If $A = [a]$, then $\det(A) = a$

(b) If A is some $n \times n$ matrix, the minor M_{ij} of A is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by removing the i^{th} row and j^{th} column.

(c) The cofactor A_{ij} of A associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} \det(M_{ij})$.

(d) The determinant of the $n \times n$ matrix A, when $n > 1$ is given either by

$\det(A) = \sum_{j=0}^{n} a_{ij} A_{ij}$ (Row cofactor expansion) or

$\det(A) = \sum_{j=0}^{n} a_{ij} A_{ij}$ (Column cofactor expansion)

3.3 Complexity of the Determinant Algorithm

Assume there are no free zero entries in A, and you must compute the fully expanded determinant of A.

3.3.1 Work (in general) for a 4x4 matrix

$$
\begin{aligned}
&= 4 \times (\text{work to compute } \det(3 \times 3)) \\
&= 4 \times 3 \times (\text{work to compute } \det(2 \times 2)) \\
&= 4 \times 3 \times 2 \times (\text{work to compute } \det(1 \times 1)) \\
&= 4 \times 3 \times 2 \times 1 \\
&= \boxed{4!}
\end{aligned}
$$

Which implies that the determinant of a $n \times n$ matrix can be computed in $O(n!)$ time.

4 More ways of computing the determinant

If $A = [a_{ij}]$ is an $n \times n$ matrix that is either upper or lower triangular form, then $\det(A) = \prod_{i=1}^{n} a_{ii}$.