MACM 316 Lecture 19

Monday, February 24, 2025

1 Continued from Lecture 18

Newton's Method can only be applied to simple zeros of a function. Identification fo the multiplicity of a zero is often made easier by the two following theorems.

Thm. 2.10

$f \in C'[a,b]$ has a simple zero at p in (a,b) if and only if $f(p) = 0$ but $f'(p) \neq 0$.

Thm. 2.11

The function $f \in C^m[a,b]$ has a zero of multiplicity m at p if and only if

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p).$$

but $f^{(m)}(p) \neq 0$.

We want to obtain quadratic convergence with Newton's Method for multiple roots.

One approach is to define a new function

$$\mu(x) = f\frac{(x)}{f'(x)}.$$

We assume p is a zero of multiplicity m and $f(x) = (x-p)^m q(x)$ where $q(p) \neq 0$. Then,

$$
\begin{aligned}
\mu(x) &= \frac{(x-p)^m}{m(x-p)^{m-1}q(x) + q'(x)(x-p)^m} \\
&= \frac{(x-p)q(x)}{mq(x) + q'(x)(x-p)} \\
&= (x-p)\frac{q(x)}{mq(x) + q'(x-p)}
\end{aligned}
$$

$q(p) \neq 0$ therefore, $\mu(p)$ has a simple root at $x = p$

so $\mu(p) = 0$, but $\dfrac{q(p)}{mq(p) + q'(p)(p-p)} = \dfrac{1}{m} \neq 0$. and p is a zero of multiplicity 1 of $\mu(x)$.

Good:

- Quadratic convergence for all roots

Bad:

- Need f''
- μ is more expensive to work with
- μ might give more roundoff error

We return to finding the roots of

$$p^3 - p^3 - p + 1 = 0.$$

Apply:

	Newton's Method	Modified Newton's Method
p_0	1.1	1.1
p_1	1.05116...	0.997735...
p_2	1.02589...	0.999999...
p_3	1.01303...	
p_4	1.00653...	
p_5	1.00327...	

This table shows that Newton's Method has linear convergence, and the Modified Newton's Method has quadratic convergence.

2 Congervence Speed and Acceleration

Suppose that we are given a linearly convergent sequence, and that we want to speed up the convergence. We want to analyze the behaviour of the error and use this knowledge to greatly reduce the error.

For example, we saw last day that the iterator method with

$$p_{n+1} = g(p_n)$$
$$\text{and } g(x) = \frac{1}{2}\sqrt{10 - x^3}$$

gives only linear convergence and the limit $p = 1.3652$

Iteration	$p - p_n$
$p - p_0$	$-0.13476\ldots$
$p - p_1$	$+0.07827\ldots$
$p - p_2$	$-0.03710\ldots$
$p - p_3$	$+0.01977\ldots$
$p - p_4$	$-0.00994\ldots$

Notice that the ratio of the errors is fairly constant- we can use this idea to accelerate the convergence of the method.

Suppose $\dfrac{p_{n+1} - p}{p_n - p} \approx \dfrac{p_{n+2} - p}{p_{n+1} - p}$

$$\therefore p \approx \frac{p_n p_{n+1} - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

or,

$$p \approx p_n - \frac{p_{n+1}^2 - 2p_n p_{n+1} + p_n^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

which is derived by adding and subtracting p_n to the right hand side.

The corresponding sequence

$$\hat{p}_{n+1} = p_n - \left[\frac{p_{n+1}^2 - 2p_n p_{n+1} + p_n^2}{p_{n+2} - 2p_{n+1} + p_n}\right].$$

is known as Aitken's Method.

Applying Aitken's method to our previous sequence,

Fixed Point Iteration		Aitken's Method	
Iteration	$p - p_n$	Iteration	$\hat{p} - p_n$
$p - p_0$	$-0.13476\ldots$	$\hat{p} - p_0$	0.00090088
$p - p_1$	$+0.07827\ldots$	$\hat{p} - p_1$	0.00023088
$p - p_2$	$-0.03731\ldots$	$\hat{p} - p_2$	0.00006107
$p - p_3$	$+0.01977\ldots$	$\hat{p} - p_3$	0.00001592
$p - p_4$	$-0.00994\ldots$	$\hat{p} - p_4$	0.00000418

we notice that Aitken's Method is much faster. It can be shown that the following theorem holds:

3 Theorem (2.13 of Text)

Suppose that $\{p_n\}$ is a sequence that converges linearly to the limit p and that for all sufficiently large values of n we have $(p_n - p)(p_{n+1} - p) > 0$.

Then the sequence $\{\hat{p}_{n=0}^{\infty}\}$ converges faster than $\{p_n\}_{n=0}^{\infty}$. to p in the sense that $\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$

The theorem does not apply to alternating sequences, but as we saw from our example it is often very useful even there for accelerating convergence.

Finally, we remark that this method is often written using difference operations to simplify the notation.

... sorry i missed a bunch here

4 Zeros of Polynomials

We want to compute the zeros (roots) of polynomials. A polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

where $a_i, i = 0, \ldots, n$ are constants and $a_n \neq 0$.

4.1 Fundamental Theorem of Algebra

If P is a polynomial of degree $n \geq 1$, then $P(x)$ has at least one (possibly complex) root.

Corollary:

If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist unique constants $x_1, x_2, \ldots, x_k$ (possibly complex), and positive integers $m_1, m_2, \ldots, m_k$ such that

$$\sum_{i=1}^{k} m_i = n$$

and

$$P(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \ldots (x - x_k)^{m_k}.$$

Corollary:

Let P and Q be polynomials of degree at most n. If $x_1, x_2, \ldots, x_{n+1}$ with $n + 1 > n$ are distinct numbers with

$$P(x_i) = Q(x_i) \quad \text{for } i = 1, 2, \ldots, n+1,$$

then

$$P(x) = Q(x) \quad \text{for all } x.$$

We want to use Newton's Method to locate the approximate roots of P. It will be necessary to evaluate P and its derivative at specified values. We now direct our attention to efficient methods for this task. The idea is to use nesting to evaluate an arbitrary n^{th} degree polynomial using only

$$n \text{ multiplications and } n \text{ additions.}$$

For illustration, consider $n = 4$. To evaluate $P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$, we write:

$$P(x) = ((((a_4 x + a_3)\, x + a_2)\, x + a_1)\, x + a_0.$$

Algorithmically,

1. set $b_4 = a_4$
2. set $b_3 = b_4 x + a_3$
3. set $b_2 = b_3 x + a_2$
4. set $b_1 = b_2 x + a_1$
5. set $b_0 = b_1 x + a_0$

Now b_0 gives the value $P(x)$.
For general polynomials of degree n :

$$\begin{aligned} b_n &= a_n \\ b_k &= a_k + b_{k+1} x \qquad\qquad 0 \leq k \leq n-1 \\ \text{and } b_0 &= P(x) \end{aligned}$$

we also want $P'(x)$. Differentiate each of the steps listed above, keeping in mind that b_i is a function of x. We get:

$$\begin{aligned} b_n' &= 0 \\ b_k' &= b_{k+1}' x + b_{k+1} \\ \text{and } b_0' &= P'(x) \end{aligned}$$

Relabel: $c_{n+1} = b_n'$. Then an efficient method for computing $P'(x)$ is

$$\begin{aligned} c_n &= a_n \\ c_k &= c_{k+1} x + b_k \\ \text{and then } c_1 &= P'(x) \end{aligned}$$

This is called Horner's Method for evaluating a polynomial.

5 Horner's Method

Honer's Method has another useful property.

Consider $P(x)$

$$
\begin{aligned}
&= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
&= b_n x^n + (b_{n-1} - b_n x_0)x^{n-1} + \cdots + (b_1 - b_2 x_0)x + (b_0 - b_1 x_0) \\
&= (b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1)x \\
&\quad - (b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1)x_0 \\
&\quad + b_0 \\
&= (b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1)(x - x_0) + b_0
\end{aligned}
$$

This property is useful because it gives us a way to find the next approximate zero after we have found our first zeros.

i.e. If x_0 is a root $P(x_0) = b_0 = 0 \implies Q(x)(x - x_0) = P(x)$

so we can find the next root by examining roots of $Q(x)$.

Suppose we want additional roots of P. If x_0 is an approximate root of P,

$$P(x) \approx Q(x)(x - x_0).$$

and the next step is toa pply Newton's Method to

$$Q(X) = \frac{P(x)}{(x - x_0)}.$$

this process is called deflation.