MACM 316 Lecture 14

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Lemma If the spectral radius $\rho(T)$ satisfies $\rho(T) < 1$ then $(I - T)^{-1} = I + T + T^2 + \dots$

And we will prove the following theorem:

Thm. For any $x^{(0)} \in \mathbb{R}^n$, $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ the sequence defined by

$$x^{(k)} = Tx^{(k-1)} + c$$

converges to the unique solution of

$$x = Tx + c \text{ if and only if } \rho(T) < 1$$

Proof ($\Longleftarrow$): assume $\rho(T) < 1$

$$\begin{aligned} x^{(k)} &= Tx^{(k-1)} + c \\ &= T(Tx^{(k-2)} + c) + c \\ &= T^2 x^{(k-2)} + (T + I)c \\ &\vdots \\ &= T^k x^{(0)} + (T^{k-1} + \dots + T + I)c \end{aligned}$$

Since $\rho(T) < 1$, the matrix T is convergent and

$$\lim_{k \to \infty} T^k x^{(0)} = 0$$

Proof ($\Longrightarrow$)

HAS NOT BEEN WRITTEN DOWN YET

The **Lemma** implies that

$$\lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} T^k x^{(0)} + \lim_{k\to\infty}\left(\sum_{j=0}^{k-1} T^j\right) c = 0 + (1-T)^{-1}c$$

$\implies \left\{x^{(k)}\right\}$ converges to the unique solution of $x = Tx + c$
i.e. $(I-T)x = c \implies x = (I-T)^{-1}c$
This allows us to derive some related results on the rates of convergence.
Corollary: If $||T|| < 1$ for any natural matrix norm and c is a given vector, then the sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ defined by

$$x^{(k)} = Tx^{(k-1)} + c$$

converges for any $x^{(0)} \in \mathbb{R}^n$ to a vector $x \in \mathbb{R}^n$ and the following error bounds hold:

(i) $||x - x^{(k)}|| \leq ||T|^k||x^{(0)} - x||$

(ii) $||x - x^{(k)}|| \leq \frac{||T||^k}{-||T||}||x^{(1)} - x^{(0)}||$

Note, however, that $\rho(A) \leq ||A||$ for <u>any</u> natural norm. In practice,

$$||x - x^{(k)}|| \approx \rho(T)^k ||x^{(0)} - x||$$

so it is desirable to have $\rho(T)$ as small as possible.
Some results for Jacobi's and Gauss-Seidel methods:
Thm. If A is strictly diagonally dominant, then for any choice of $x^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ that converge to the unique solution of $Ax = b$.
No general results exist tot tell which of the two methods will converge more quickly, but the following result applies in a variety of examples:
Thm. Stein Rosenberge
If $a_{ij} \leq 0$ for each $i \neq j$ and $a_{ii} > 0$ for each $i = 1, 2, \ldots, n$, then exactly one of the following holds.

(a) $0 \leq \rho(T_g) < \rho(T_j) < 1$

1 Successive Over-Relaxation (SOR)

To define, suppose $\tilde{x}^{(k+1)}$ is the iterate from Gauss-Seidel using $x^{(k)}$ as the initial guess. The $(k+1)^{st}$ iterate of SOR is defined by

$$x^{(k+1)} = \omega\tilde{x}^{(k)} + (1-\omega)x^{(k)}$$

where $1 < \omega < 2$. It can be difficult to select ω optimally. Indeed, the answer to this question is not known for general $n \times n$ linear systems.

However, we do have the following results:

Thm. (kahan): If $a_{ii} \neq 0$ for each i, then

$$\rho(T_{SOR}) \geq |\omega - 1|$$

$\implies$ SOR can converge only if $0 < \omega < 2$

Thm. (ostrowski-reich): If A is a positive definite matrix and $0 < \omega < 2$, then the SOR method converges for any choice of initial approximate vector $x^{(0)}$

Thm.. If A is positive definite and tridiagonal, then

$$\rho(T_g) = \rho(T_j)$$

$^2 < 1$

and the optimal choice of ω for the SOR method is

$$\omega = \frac{2}{1 + \sqrt{1 - \rho(T_j)^2}}$$

with this choice of ω, we have $\rho(T_{SOR}) = \omega - 1$