MACM 101 Lecture 1.1

Alexander Ng

Friday, September 13, 2024

1 Chapter Summary

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences and Implication
  - The Laws of Propositional Logic
- Predicate Logic
  - The Language of Quantifiers
  - Nested Quantifiers
- Proofs
  - Rules of Inference
  - Proof Methods
  - Proof Strategy

This document covers everything from Rosen 1.0 to 1.3.

2 Definitions

2.1 Deduction/Deductive Logic

Deduction is the process of deriving a conclusion from a given set of axioms or premises. In Logic, we start from the ground (axioms) and work our way up to the conclusion.

2.2 Truth Value

A truth value can be either true or false, but not both. This comes from the *principium tertii eclusi* of Aristotle.

2.2.1 True and False

We will use 1 and 0 to denote true and false, respectively.

2.2.2 Unknown Truth Value

The proposition u is *unknown truth value*.

2.3 Proposition

A proposition is a declarative sentence (or statement) that possesses truth value.

2.3.1 Notation

Lowercase letters denote primitive propositions, and uppercase letters denote complex propositions.

Primitive propositions are:

- Propositions that cannot be decomposed into anything simpler
- $p : 3 + 5 = 8$
- q : It is raining

2.4 Examples of things that are not propositions

- p : Sit down! $\rightarrow$ not a proposition because it is not a declarative
- q : The statement you are reading is now false. $\rightarrow$ not a proposition because it is a contradiction.
- r : The number x is an integer. $\rightarrow$ not a proposition because it contains an unspecified variable, which means it's truth value cannot be definitively determined without additional information.

2.5 Syntactics and Semantics

Syntatic reasoning is what can be shown.

Syntax = grammar (rules of sentance construction), the structure of propositions

Semantics reasoning is what is true

Semantics = meaning (truth value), the truth value/tables of propositions

2.6 Literals

A *literal* is either a primitive proposition or its negation (some textbooks use to denote a literal)

3 Operator Syntax

1. Negation - $\neg$
   q : it is raining, $\neg q$: it is not raining
   Everything in this list other than $\neg$ is known as a *logical connective*
2. Conjunction - $\wedge$ - Logical and
   $p \wedge q$: it is raining and it is sunny
   $p \wedge \neg q$: it is raining and it is not sunny
3. Disjunction - $\vee$ - Inclusive Or
   $p \vee q$: it is raining or it is sunny
   $p \vee \neg q$: it is raining or it is not sunny

4. Disjunction - $\oplus$ - Exclusive Or
   $p \oplus q$: it is raining xor it is sunny
   $p \oplus \neg q$: it is raining xor it is not sunny
   XOR is generally what is meant in english sentences like "the meal comes with either soup or salad"

5. Implication - $\rightarrow$ - "If, then"

6. Biconditional - $\leftrightarrow$ - "If and only if"

Nobody knows why OR and XOR are both called Disjunction

All propositions formed with logical connectives are called *compound propositions*, as opposed to *primitive propositions*

Compound propositions need not have causal relations between atomic components (they can sound nonsensical and still be valid) – material implication as opposed to causal implication, which lacks temporal ordering. (straight from the slides, p. 34)

4 Logical Equivalences

TABLE 6 Logical Equivalences.	
Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Figure 1: Logical Equivalences

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Figure 2: Logical Equivalences Involving Biconditional Statements

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Figure 3: Logical Equivalences Involving Conditional Statements

5 Semantics

Basically, all semantics is is truth tables.

5.1 Negation, $\neg$

p	$\neg p$
T	F
F	T

5.2 Logical Connectives

1. Conjunction - $\wedge$ - Logical AND
2. Disjunction - $\vee$ - Logical OR

3. Exclusive Or - $\oplus$ - Exclusive OR
4. Implication - $\rightarrow$ - Material Implication
5. Biconditional - $\leftrightarrow$ - Biconditional

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
1	1	1	1	0	1	1
1	0	0	1	1	0	0
0	1	0	1	1	1	0
0	0	0	0	0	1	1

5.3 Extra Context for Material Implication

A wide range of terminology exists to describe material implication.

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Figure 4: Ways to Express $p \rightarrow q$

Note that Material Implication does not depend on any relationship between the meanings of the two propisitions involved, but only the *truth values* of the propositions.

5.3.1 Necessary Conditions (implication)

For a normal car to run, it is necessary that there is fuel in the tank, its spark plugs are properly adjusted, its oil pump is working, etc.

If the car runs, then every one of the conditions must be fulfilled.

All of these propositions are $p \rightarrow q$

Let p be the statement "the car runs"

Let q be the other statement

1. If the car runs, then fuel must be in the tank. ($p \to q$)

2. If the car runs, its spark plugs must be properly adjusted. ($p \to q$)

3. If the car runs, its oil pump must be working. ($p \to q$)

From this, you can see why material implication is defined by $\neg(p \wedge \neg q) \equiv \neg p \vee q$. I think that $\neg(p \wedge \neg q)$ makes a lot more sense than $\neg p \vee q$ when you read it out loud. With this definition, you can clearly visualize the truth table. Try thinking about (not (p land not q)) and look at the truth table.

5.3.2 Sufficient Conditions (implication)

For a purse to contain over a dollar, it would be sufficient for it to contain 101 pennies, 21 nickels, 11 dimes, 5 quarters, etc.

If any one of these circumstances is obtained, the specified situation will be realized.

5.3.3 Hints (wording)

1. "p is a **sufficient condition** for q"
   $p \to q$ (q is a necessary condition for p)

2. "p is a **necessary condition** for q"
   $q \to p$ (q is a sufficient condition for p)

3. Given any sentence saying something is *necessary* or *sufficient*, it is always:
   sufficient $\to$ necessary

$p \to q$ can read as any one of the phrases highlighted in Figure 4

5.3.4 The Four Types of Implication

There are 4 types of implication

1. **Logical Implication** - the consequent follows logically from its antecedent (a.k.a. material implication)

   If all humans are mortal and Socrates is a human, then Socrates is mortal.

2. **Definitional Implication** - the consequent follows the antecedent by definition.

   If Leslie is a Bachelor, then Leslie is unmarried.

3. **Causal Implication** The connection between antecedent and consequent is discovered empirically (temporal ordering is implicit)

   If I put X in acid, then X will turn red.

4. **Decisional Implication** - A decision of the speaker to behave in the specified way under the specified circumstances.

   If we lose the game, then I'll eat my hat.

For Propositional Logic, we will only be concerned with logical implication (material implication).

5.4 Order of Operations for Propositions

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

6 Bit Strings

A **bit string** is a sequence of zero or more bits.

We can extend our bit operations to bit strings. We define the **bitwise OR**, **bitwise AND**, and **bitwise XOR** of two bit strings of the same length to be the strings that have, as their bits, the **OR**, **AND**, and **XOR** of the corresponding bits in the two strings, respectively. (Pearce, 60)

Translated into plain English, the bitwise extension of the logical operations means to take each corresponding bit in the two strings and apply the logical operation to them.

Bitwise literally means by bit, so the bitwise extension of the logical operations in this way are called **bitwise logical operations**.

If you need examples of this, please refer to Pearce, 60 or Rosen, 12.