MACM 101 Chapter 1.3 - Logical Identities

Alexander Ng

Sunday, September 15, 2024

This document covers Rosen 1.3, Pearce 1.1 72-xx.

Summary

1. Tautologies, Contradictions and Contingencies
2. Logical Equivalence
   - Important Logical Equivalences
   - Showing Logical Equivalences
3. Logical Implication (not in Rosen)
4. Normal Forms
   - Distributive Normal Form (DNF)
   - Conjunctive Normal Form (CNF)

1 Tautologies, Contradictions and Contingencies

A Tautology (**T**) is a proposition that is always true.
Example: $p \vee \neg p$
A Contradiction (**F**) is a proposition that is always false.
Example: $p \wedge \neg p$
A contingency is a proposition that is neither a tautology nor a contradiction, such as p.

2 Logical Equivalences

Two statements, s_1 and s_2 are said to be logically equivalent if, when one is true, the other is also true, and conversely, when one is false, the other is also false.

In other words, two statements are logically equivalent if they have identical truth tables.

- We write $s_1 \Leftrightarrow s_2$ to denote that s_1 and s_2 are logically equivalent.
- Conversely, we write $s_1 \not\Leftrightarrow s_2$ to denote that s_1 and s_2 are not logically equivalent.
- The symbol $\equiv$ is also used to denote logical equivalence.

2.1 Implication and it's Cousins

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1
0	1	1	0	0	1
1	0	0	1	1	0
1	1	1	1	1	1

Clearly, $p \to q \Leftrightarrow \neg q \to \neg p$ and $q \to p \Leftrightarrow \neg p \to \neg q$.

- $p \to q$ is called **material implication**
- $q \to p$ is called the **converse** of material implication

- $\neg q \to \neg p$ is called the **contrapositive** of material implication
- $\neg p \to \neg q$ is called the **inverse** of material implication

2.2 The Formal Axiomatic System of Propositional Logic

Logical equivalence is the foundation of the formal axiomatic system of propositional logic.

The goal of this aximoatic system, and the following propositional identities, is to provide an algebraic foundation for simplifying expressions.

2.3 Propositional Identities

2.3.1 Identity 1 - Implication Reduction

$$p \to q \equiv \neg p \lor q \tag{1}$$

p	q	$p \to q$	$\neg p \lor q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

- $p \to q$ means $p \therefore q$
- If the argument is valid and p is true, then q is also true.
- If the argument is valid and p is false, then we can conclude nothing about q.
- Thus, $p \to q$ is logically equivalent to $\neg p \lor q$

**Implication Reduction will be one of the most important identities in propositional logic.

2.3.2 Identity 2 - De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{2}$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{3}$$

We can prove De Morgan's laws by using a truth table.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Figure 1: De Morgan's Second Law

Breaking down the truth table, column by column, we can see that De Morgan's laws do indeed hold. These will become fundamental for the algebraic manipulation of propositions.

TABLE 6 Logical Equivalences.	
Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Figure 2: Logical Equivalences

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Figure 3: Logical Equivalences Involving Biconditional Statements

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Figure 4: Logical Equivalences Involving Conditional Statements