MATH 240 Lecture 3.1
Properties of Determinants

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Let A be an 3×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The $(1, 2)$ minor of A is

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

Definition

Let A be an $n \times n$ matrix. The determinant of A is defined as.

$$\det(A) = \begin{cases} n = 1 & a_{11} \\ n > 1 & a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + \cdots + (-1)^{n-1} \cdot a_{n1} \cdot \det(A_{1n}) \end{cases}$$

Let M be the number of multiplications in this formula.

$$M = \begin{cases} n = 1 & 0 \\ n = 2 & 2 \\ n > 2 & n \cdot M(n-1) + n.. \end{cases}.$$

So, the number of multiplications to compute the determinant grows exponentially.

$$M(3) = 3 \cdot M(2) + 3 = 3 \cdot 2 \cdot 3 = 9 \quad (1)$$
$$M(4) = 4 \cdot M(3) + 4 = 4 \cdot 3 \cdot 4 = 40 \quad (2)$$
$$M(20) = 4.1 \times 10^{18} \sim \text{ 132 years at } 10^9 \text{ multiplications per second} \quad (3)$$

1 Theorem.

(a) If $A \sim B$ via $R_i \leftarrow R_i + s \cdot R_j$, then $\det(B) = \det(A)$.

(b) If $A \sim B$ via $R_i \leftrightarrow R_j, i \neq j$, then $\det(B) = -\det(A)$.

(c) If $A \sim B$ via $R_i \leftarrow sR_i, s \neq 0$, then $\det(B) = s \cdot \det(A)$.

This means if we use row reduction (Gaussian Elimination),

$$A \sim A_n \text{ in REF}$$

then we can get $\det(A)$ from $\det(A_n)$.

1.1 Example

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$R_3 \leftarrow 3R_3$

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_1$

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$

$$D = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

A in REF. (Upper Triangular)
$\det(B) = 3\det(A)$
$\det(B) = \det(C) = \det(D)$

$$\begin{aligned} \det(D) &= \det(D^T) \\ &= \det(\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix}) \\ &= 3 \cdot 1 \cdot (-2) \\ &= -6 \end{aligned}$$

$$\begin{aligned} \det(B) &= -6 = 3 \cdot \det(A) \\ \det(A) &= \frac{-6}{3} = -2 \end{aligned}$$

Gaussian Elimination does $\sim \frac{1}{3}n^3$ multiplications.

$$\begin{cases} n = 20 & \frac{20^3}{3} = 2666 \\ n = 1000 & \frac{1000^3}{3} = 333,333,333 < 10^9 < \text{ 1 second} \end{cases}$$

Proof of (c)