MATH 240 Lecture 1.4
The Matrix Equation $Ax = b$

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**The equation $Ax = b$ is very important

Review

Let A be an $m \times n$ matrix.

Let x be a vector of length n, with entries $x_1, x_2, \ldots, x_n$.

The definition of how to multiply a matrix by a vector is

If $A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ v_1 & v_2 & \ldots & v_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ then, $Ax = \begin{bmatrix} x_1 \cdot v_1 + x_2 \cdot v_2 + \cdots + x_n \cdot v_n \end{bmatrix}$

If $A = \begin{bmatrix} r_1 \\ r_2 \\ \ldots \\ r_m \end{bmatrix}$ then $Ax = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \ldots \\ r_m \cdot x \end{bmatrix}$

1 Theorem 5: Properties of the matrix vector product

Let A and B be $m \times n$ matrices over $\mathbb{R}$, and $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$

1.1 The Distributive Law

$$A(u + v) = Au + Av \tag{1}$$

The Distributive Law implies that

$$\begin{aligned} A(u + v) &= A(u + v) + A(w) \\ &= Au + Av + Aw \end{aligned}$$

because addition is commutative.

1.2 The Associative Law

$$A(c \cdot v) = c \cdot A \cdot v \tag{2}$$

1.3 The Distributive Law

$$(A + B) \cdot u = Au + Bu \tag{3}$$

Proofs of the above properties will be tested on the exams.

1.4 Proving Property 1

2 Four ways to represent a linear system

1. Standard Form

$$\begin{aligned} 2x_1 + 1x_2 &= 5 \\ 1x_1 + 3x_3 &= 7 \end{aligned}$$

2. Augmented Matrix

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

3. Vector Equation

$x_1 \cdot v_1 + x_2 \cdot v_2 + \cdots + x_n \cdot v_n = b$

$$b = \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2x_1 + 1x_2 \\ 1x_1 + 3x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3x_3 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

4. Consider the following matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \\ 1 \cdot x_1 + 0 \cdot x_2 + 3 \cdot x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

This is the same as $Ax = b$ where A is the matrix above and x is the vector of unknowns to be solved for.

Compare this with $ax = b$ (the linear system). $x = \frac{b}{a}$.

What is the point of having 4 different ways to represent a linear system?

Method (1) is for building a linear system from scratch.

Method (2), the augmented matrix, is for solving a linear system that has already been built.

Method (3) is for proofs.

Method (4) is for reasoning and proofs, because it is concise.

3 Theorems

Let A be an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$ and x=$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$ and b be a vector in $\mathbb{R}^m$.

3.1 Theorem 3

$Ax = b$, $\begin{bmatrix} A|b \end{bmatrix}$ and $\begin{bmatrix} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \end{bmatrix}$ have the same solution set(s).

3.2 Theorem 4

If $B \sim A$ and B is in REF, then the following statements are equivalent.

(a) The linear system $Ax = b$ has a solution for every choice of $b \in \mathbb{R}^m$.

(b) Every $b \in \mathbb{R}^m$ is a linear combination of the columns of A.

(c) The span of $v_1, v_2, \ldots, v_n$ generates $\mathbb{R}^m$.
In other words, every vector in $\mathbb{R}^m$ can be obtained from the span of $v_1, v_2, \ldots, v_n$.

(d) The matrix B has a pivot (position) in every row.
This is our tool for testing if a..c are true.

What this theorem is saying is that these four statements are **all** either simultaneously true or simultaneously false.

3.2.1 Definition (Equivalence)

Two statements are equivalent if they are simultaneously true or simultaneously false.

3.2.2 Example

Is $Span(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}) = \mathbb{R}^3$?

Step 1. Apply (a) and (b).

$$A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 0 & 1\end{bmatrix}$$

Apply the row operations:

$$R_2 := R_2 - R_1; \quad R_3 := R_3 - R_1$$

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & -1\\0 & 2 & 0\end{bmatrix}$$

Next row operation:

$$R_3 := R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\implies \text{(d) is true.}$$

We can prove Theorem 4 by showing equivalence between (c) and (d), (b) and (a), and (a) and (d).

3.2.3 Proof that (d) and (a) are equivalent

(d) $\implies$ (a). If (d) is true, then $[A|b] \sim [B|\vdots]$ in REF has a pivot position in every row, meaning $Ax = b$ has a solution.

If (d) is false, then $A \sim B$ has at least one row of zeroes.

Consider the matrix

$$B = \begin{bmatrix} u & & & & \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

B is **inconsistent** because its augmented matrix has $0 = 1$ in the bottom row. Since row operations are reversible, the system $B \sim A$.

$$\therefore A \text{ is also inconsistent.}$$

$$\implies Ax = b \text{ is inconsistent.}$$

This means that statement (a) is false.

We have shown that if (d) is false, then there exists a vector b such that $Ax = b$ has no solutions, which implies that (a) is false.