MATH 240 Lecture 1.5
Solution Sets of Linear Systems

Alexander Ng

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Every linear system can be written in the form $Ax = b$, where A is the coefficient matrix, x is the vector of unknowns, and b is the vector of constants.

In a linear system $Ax = 0$, there is always a solution of the form $x = \mathbf{0}$, where $\mathbf{0}$ is the zero vector. This is known as the **trivial solution**. Other solutions (there may be 0, 1, or many) are called **non-trivial solutions**.

A linear system is said to be **homogeneous** if it can be written in the form $Ax = 0$. Homogeneous systems are always consistent, becasue of the aformentioned trivial solution.

Theorem

The solutions of $Ax = 0$ can always be written as a

$$\text{span}(v_1, v_2, \ldots, v_n)$$

Example

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_4 = 0$$

$$x_2 + 2x_3 = 0;$$

$$
\begin{aligned}
x_1 &= -x_4 \\
x_2 &= -2x_3 \\
x_3 &= \text{free} \\
x_4 &= \text{free}
\end{aligned}
$$

Let $x_3 = s \in \mathbb{R}$
Let $x_4 = t \in \mathbb{R}$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ -2s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \text{span}(v_1, v_2).
$$

Theorem

Consider the linear systems $Ax = b$ and $Ax = 0$.
Suppose $Ap = b$ and $Aw = 0$.
• $p + w$ is a solution of $Ax = b$.
Check. $A(p + w) = Ap + Aw = b + 0 = b$.
If $Ax = b$ is consistent, then the solutions of $Ax = 0$ can be expressed as
$x = p + (\text{the solutions of } Ax = 0) = p + \text{span}(v_1, v_2, \ldots, v_n)$

Example

$A\vec{x} = b$

$$
\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}
$$

$x + y = 1$
$x = 1 - y$
$y = free$
let $y = t$
$x = 1 - t$
$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$t\begin{bmatrix}-1\\1\end{bmatrix}$ is span($\begin{bmatrix}-1\\1\end{bmatrix}$)
find example in lecture notes and solve

The Geometry of the solutions to $Ax = b$

Find the picture in lecture notes

Annotations:

- How can we describe the solutions of $x+y=1$ in terms of the solutions of $x+y=0$?
- When $t=1$, $w = \begin{bmatrix}-1\\1\end{bmatrix}$
- when you take any arbitrary solution to $x+y=0$, and add the vector p, then add the vector w, you get the solutions to $x+y=1$

Theorem 6

Let $Ax = b$ be a linear system with solution $x = p$, that is *if p is a solution to $Ax = b$, that means $Ap = b$.

Then,

1. If $Aw = 0$, i.e. w is a solution to $Ax = 0$ then, $A(p+w) = b$.
2. If $Az = b$, i.e. z is a solution of $Ax = b$, then $A(z-p) = b$.
   in other words, you can find a solution of $Ax = 0$

Proof of 6.1

- $Aw = 0$
- $A(p+w) = Ap + Aw = b + 0 = b$

Proof of 6.2

$A(z-p) \stackrel{?}{=} Az - Ap = b - b = 0$

Theorem 5, C1.4

1. $A(u+v) = Au + Av$
2. $A(c \cdot u) = c \cdot Au$

Proof of $A(z-p) = Az - Ap$

$A(z-p) \overset{\text{Prop. 9}}{=} A(z + (-1)\cdot p) \overset{\text{Th. 5, prop. 1}}{=} Az + A((-1)\cdot p) \overset{\text{Th. 5, prop. 2}}{=} Az + (-1)\cdot Ap$

Example

what

$$
\begin{aligned}
S &= \text{span}(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}) && (1)\\
&= \text{span}(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}) && (2)\\
&= \text{span}(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}) && (3)\\
&= \mathbb{R}^2 && (4)
\end{aligned}
$$

Every simplification of the above $span(\dots) \in \mathbb{R}^2$ has two vectors.
Every vector in $\mathbb{R}^2$ can be written as a linear combination of e_1 and e_2.