MATH 240 Lecture 2.2
The Inverse A^{-1} of a matrix A

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Let A be an $n \times n$ matrix. A is invertible if there is an $n \times n$ matrix A^{-1}, called the inverse of A, such that $AA^{-1} = A^{-1}A = I_n$.

Theorem 7 - C2.2 A is invertible $\iff A \sim I_n$.

$(\implies)$ If A is invertible, then $A \sim I_n$.

$(\impliedby)$ If $A \sim I_n$, then A is invertible.

Proof of Theorem 7-2.2

Suppose A is invertible. Then A^{-1} exists. Let $A \sim B$ where B is in REF.

(Th5-2.2) If A is invertible, then $Ax = b$ has a unique solution. (Th4-1.4) B in REF has a pivot position in every row

We know A is square $\implies B = \begin{bmatrix} p_1 & x & x \\ 0 & p_2 & x \\ 0 & 0 & p_3 \end{bmatrix}$ where p are pivots.

We can then apply row operations to B to get B' in RREF. B' in RREF $= I_n$

Elementary Matrices

Every elementary row operation corresponds to a multiplication of A by an $n \times n$ elemtary matrix E, i.e. if $A \sim B$, then $B = EA$. Since row operations are invertible, E is invertible.

Example

For $n = 3$ and $R_2 \leftarrow R_2 - 2R_1$, find E and E^{-1}.

Start with the I_n, I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \leftarrow R_2 - 2R_1$ to I_3.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ -2a + u & -2b + v & -2c + w \\ x & y & z \end{bmatrix}$$

Find the inverse.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply the **Inverse** of the row operation to I_3.

$R_2 \leftarrow R_2 + 2R_1$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check $EE^{-1} = I_3$.
Check $E^{-1}E = I_3$.

Proof of Part 2 of Theorem 7-2.2

Suppose $A \sim I$. $\quad A \sim A_1 \sim A_2 \dots A_{m-1} \sim I$
$= E_1 A_1 \dots E_m A_m = I$

Let $E_1 = E_2 \dots E_m$ be the elementary matrices corresponding to the row operations in $A \sim I$.

So,

$$
\begin{aligned}
A_1 &= E_1 \cdot A \\
A_2 &= E_2 \cdot A_1 \\
&\vdots \\
I &= E_m \cdot A_{m-1}
\end{aligned}
$$

Because $A_2 = E_2 \cdot A_1$, we can write $A_2 = E_2 \cdot (E_1 \cdot A)$.

$$I = E_m(\dots(E_2 \cdot (E_1 \cdot A))) = (E_m \dots E_2 \cdot E_1) \cdot A.$$

We can call the matrix product $E_m \dots E_2 \cdot E_1$ the equivalent matrix C.
We now have $CA = I$. Is $AC = I$?
We know that $E_1, E_2, \dots, E_m$ are invertible.
Theorem 6b-2.2 $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
$C^{-1} = E_1^{-1} \cdot E_2^{-1} \dots E_m^{-1}$
$CA = I$
$C^{-1}(CA) = C^{-1}I$
$(C^{-1}C)A = C^{-1}I \implies A = C^{-1} \implies AC = C^{-1}C = I$

Invertible Linear Transformations

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. T is invertible if there exists another linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that.

1. $S(T(x)) = x$
2. $T(S(y)) = y$

Theorem 9

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A so $T(x) = Ax$.

T is invertible $\iff$ A is invertible.

Proof of Theorem 9

Part 1

($\Longleftarrow$) If T is invertible, then A is invertible.

Assume A is invertible. So A^{-1} exists.

Consider $S(y) = A^{-1}y$. Then,

$$S(T(X)) = S(Ax) = A^{-1}(Ax) = (A^{-1}A)x = Ix = x \tag{1}$$

$$T(S(X)) = T(A^{-1}x) = A(A^{-1}x) = (A^{-1}A)x = Ix = x \tag{2}$$

Part 2

($\Longrightarrow$) If A is invertible, then T is invertible.

Assume $T(x) = Ax$ is invertible.

Let S be the inverse of T. S is a Linear Transformation. So, let $S(y) = Cy$, where C is the standard matrix of S.

$S(T(x)) = x \implies C(Ax) = x \implies (CA)x = x.\forall x \in \mathbb{R}^n$

$\implies (CA)e_i = e_i$

$$CA = (CA)\begin{bmatrix} \vdots & \vdots & & \vdots \\ e_i & e_2 & \dots & e_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} (CA)e_1 & (CA)e_2 & \dots & (CA)e_n \end{bmatrix}$$

$$= \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix} = I_n$$

We have $CA = I_n$. Is $AC = I_n$?

Yes, by IMT.