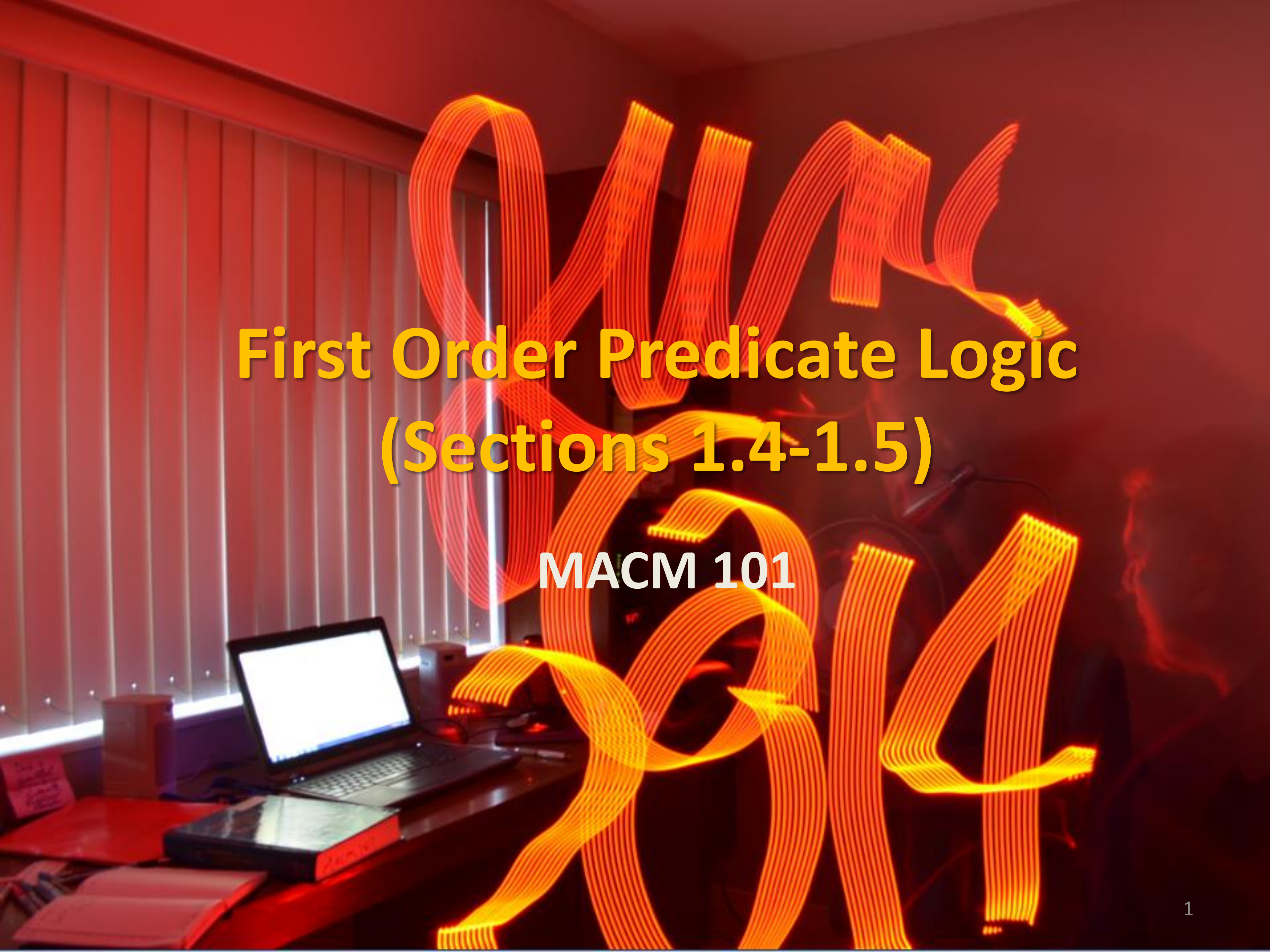

First Order Predicate Logic (Sections 1.4-1.5)

MACM 101

Last Time

- The main purpose of section is to provide a foundation for *proof theory* – a mathematical proof is purely deductive, *valid argument.*
- Review Tautologies, Contradictions, Logical Implication and the notion of a valid argument.
- Discuss the difference between *truth* and *validity.*
- Define the Rules of Inference and their application.

Summary of Sections 1.4 – 1.5

Predicate Logic (First-Order Logic).

- The Language of Quantifiers.
- Logical Equivalences.
- Nested Quantifiers.
- Translation from Predicate Logic to English.
- Translation from English to Predicate Logic.

Goals

- To show that propositional logic is limited.

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- To extend propositional logic through the use of *predicates* and *quantifiers* (first order logic).

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- To extend propositional logic through the use of *predicates* and *quantifiers* (first order logic).
- To extend reasoning using the Rules of Inference to quantified statements (instantiation and generalization).
- We are covering sections 1.4 and 1.5.

Preamble

Limitations of Propositional Logic

- Propositional logic deals with propositions (declarative sentences possessing *truth value*).

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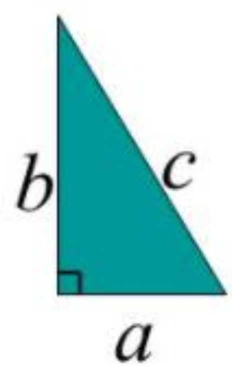
- This theory does not deal with anything smaller than a *sentence* (that is why propositional logic is also referred to as **sentential logic** as in sentence).
- As such, propositional logic is a formal axiomatic system in which formulae representing propositions can be formed by combining literals using logical connectives, to derive "theorems".
- As such, it works well with relationships like *not*, *and*, *or*, *if/then,* and *if and only if*.

Limitations of Propositional Logic

Propositional logic – logic of simple statements

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

How to formulate Pythagoreans' theorem using propositional logic?



How to formulate the statement that there are infinitely many primes?

http://slideplayer.com/slide/6088186/

Limitations of Propositional Logic

- There is no structure in propositional logic to describe all the elements of the problem domain.
- For example, when discussing the mathematical properties of integers, we use variables such as x and y to represent these numbers so that general assertions can be made about $\boldsymbol{Z}$.
- Then, one can say "for all integers x, y we have $x + y = y + x$" instead of listing all the examples of this assertion: ..., "$0 + 1 = 1 + 0$", "$1 + 1 = 1 + 1$", …

Limitations of Propositional Logic

Propositional logic cannot do the following:

- It cannot express statements possessing variables (*i.e.,* "*x* is an integer").
- It cannot express properties of objects.
- It cannot express relationships among objects.
- It cannot make a statement that is <u>universal</u> (*i.e.,* a general statement regarding even numbers).
- It can only describe events that are either true or false (it cannot facilitate uncertainty).

1.4 Predicates and Quantifiers

Section 1.4 Summary

Predicates.

Variables.

Quantifiers.

- Universal Quantifier.
- Existential Quantifier.

Negating Quantifiers.

- De Morgan's Laws for Quantifiers.

Translating English to Logic.

Logic Programming (*optional*).

Universal and Existential Statements

- Consider the pattern involved in the following logical equivalences, which cannot be expressed in propositional logic:
  - p: "Not all birds fly" and q: "Some birds don't fly"

Universal and Existential Statements

- Consider the pattern involved in the following logical equivalences, which cannot be expressed in propositional logic:
  - *p*: "Not all birds fly" and *q*: "Some birds don't fly"
- **They are logically equivalent, but there is no way to derive the fact that $p \Leftrightarrow q$** (*p* and *q* are independent statements).

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- Consider the pattern involved in the following logical equivalences, which cannot be expressed in propositional logic:
  - *p*: "Not all birds fly" and *q*: "Some birds don't fly"
- **They are logically equivalent, but there is no way to derive the fact that $p \Leftrightarrow q$** (*p* and *q* are independent statements).
- Statement *p* is **universal** (all birds), whereas statement *q* is **existential** (some birds).

Universal and Existential Statements

Consider this famous argument:

- All human beings are mortal.
- Socrates is human.
- Therefore, Socrates is mortal.

Universal and Existential Statements

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Let us try to formalize this in propositional logic:

- If Socrates is human, then Socrates is mortal.
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Universal and Existential Statements

Consider this famous argument:

- **All** human beings are mortal.
- Socrates is human.
- Therefore, Socrates is mortal.

Propositional Logic lacks this structure.

Let us try to formalize this in propositional logic:

- If Socrates is human, then Socrates is mortal.
- Socrates is human.
- Therefore, Socrates is mortal.

These are NOT equivalent arguments.

See https://web.stanford.edu/~bobonich/glances%20ahead/III.logic.language.html

Universal and Existential Statements

Consider this famous argument:

- All human beings are mortal := p.
- Socrates is human := q.
- Socrates is mortal := r.

Let us try again to formalize this in propositional logic:

- p
- q
- Therefore, r

INVALID ARGUMENT.

See https://web.stanford.edu/~bobonich/glances%20ahead/III.logic.language.html

Problem

- The problem is:
  - We operate with constant objects only.
  - We have no domain context and no domain variables.
  - We cannot say “all”, “there exist”, etc.

- Solution?

http://www.powershow.com/view1/2532f4-ZDc1Z/Limitations_of_Propositional_Logic_powerpoint_ppt_presentation

Solution: Predicates

$$\underbrace{\overbrace{\text{LeftOf}}^{\textit{Predicate}}\ \overbrace{(\,x,\ y\,)}^{\textit{Arguments}}}_{\textit{Atomic Sentence}}$$

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Definition: A predicate is a *function* that maps variables to truth values, allowing one to go beyond atomic propositions.

Advancing to First-Order Logic

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  - We denote this statement by the **propositional function**, *P*(*x*), where *P* is the **predicate** “is greater than 3”.
- Predicates allows for a framework of so-called existential and universal **quantifiers** that generalize propositional logic enough to express a wider set of arguments occurring in a natural language, such as the issue encountered with our bird example.

Predicates: Examples

Given each propositional function determine its true/false value when variables are set as below.

- Prime(x) $=$ "x is a prime number."
  Prime(2) is true, since the only numbers that divide 2 are 1 and itself.
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Remark: The arguments of a propositional function are known as a **free variables** in what follows…

DEFINITION

Open Statement

– A declarative sentence is an open statement if
  - It contains one or more logical variables, and
  - It is not a statement, but
  - It becomes a statement when the logical variables in it are replaced by certain allowable choices (logical constants).

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  - It is not a statement, but
  - It becomes a statement when the logical variables in it are replaced by certain allowable choices (logical constants).
- **Remark**: These allowable choices constitute what is called the **universe**, or **universe of discourse** for the open statement.

Predicates and Propositions

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- Predicates are not propositions (they contain *free variables*).
- A predicate can be transformed into a proposition in two ways:
    1. Replacing the free variable with a logical constant, or
    2. The use of *quantifiers*.

Definition

Quantifier: One of two operators which transform free variables into **bound variables**.

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There are two logical quantifiers:

1. **The Universal Quantifier, $\forall$ ("for all")**
2. **The Existential Quantifier, $\exists$ ("there exists")**

Quantifiers

- They are operators (they operate on predicates, binding their free variables).

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  - Example, P(x) is a predicate containing a free variable and is, therefore, not a proposition.
  - $\forall x$P(x) is a proposition since the predicate now has a bound variable and reads "for all x, P(x) is true" (in an assumed universe of discourse).

Universe of Discourse, $\mathcal{U}$

Before deciding on the truth value of a quantified predicate, it is mandatory to specify the **domain** (also called domain of discourse or universe of discourse).

$P(x) =$ "x is an odd number"

$\forall x P(x)$ is **false** for the domain of **integer numbers**; but
$\forall x P(x)$ is **true** for the domain of **prime numbers greater than 2**.

More Generally

Let P(x) and Q(x) be open statements defined for some $\mathcal{U}$.

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when $P(a) \leftrightarrow Q(a)$ is true $\forall a \in \mathcal{U}$.

State the contrapositive, converse and inverse.

More Generally

For open statements $p(x), q(x)$ — defined for a prescribed universe — and the universall
quantified statement $\forall x\ [p(x) \rightarrow q(x)]$, we define:

1) The *contrapositive* of $\forall x\ [p(x) \rightarrow q(x)]$ to be $\forall x\ [\neg q(x) \rightarrow \neg p(x)]$.
2) The *converse* of $\forall x\ [p(x) \rightarrow q(x)]$ to be $\forall x\ [q(x) \rightarrow p(x)]$.
3) The *inverse* of $\forall x\ [p(x) \rightarrow q(x)]$ to be $\forall x\ [\neg p(x) \rightarrow \neg q(x)]$.

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The **universal quantification** of $P(x)$ is the statement:
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If the domain is finite $\{x_1, x_2, \ldots, x_n\}$, $\forall x P(x)$ is the same as

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n).$$

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$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

Grimaldi, p. 92

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$ True ($x = 2$)

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$
2. $\forall x R(x)$

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$
2. $\forall x R(x)$ False (e.g., $x = 1$)

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$
2. $\forall x R(x)$
3. $\exists x R(x) \wedge \exists x S(x)$

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$
2. $\forall x R(x)$
3. $\exists x R(x) \wedge \exists x S(x)$ True

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$
2. $\forall x R(x)$
3. $\exists x R(x) \wedge \exists x S(x)$
4. $\exists x[R(x) \wedge S(x)]$

Example

Consider R(x): $x + 3 = 5$ and S(x): $3x + 2 = 11$.

True or False?

1. $\exists x R(x)$
2. $\forall x R(x)$
3. $\exists x R(x) \wedge \exists x S(x)$
4. $\exists x[R(x) \wedge S(x)]$ False

Remark

- $\forall$ is usually paired with $\rightarrow$.
  - Sometimes paired with $\leftrightarrow$.
- $\exists$ is usually paired with $\wedge$.

Beyond ∀ and ∃

- **Other Quantifiers**
  The most important quantifiers are ∀ and ∃, but we could define many different quantifiers: "there is a unique", "there are exactly two", "there are no more than three", "there are at least 100", etc.
  A common one is the **uniqueness quantifier**, denoted by ∃!.
  $\exists! x P(x)$ states "There exists a unique x such that $P(x)$ is true."
  Advice: stick to the basic quantifiers. We can write $\exists! x P(x)$ as
  $\exists x (P(x) \wedge \forall y (P(y) \rightarrow y = x))$ or more compactly
  $\exists x \forall y (P(y) \leftrightarrow y = x)$

Beyond ∀ and ∃

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  $\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$ or more compactly
  $\exists x \forall y(P(y) \leftrightarrow y = x)$
- **Restricting the domain of a quantifier**
  Abbreviated notation is allowed, in order to restrict the domain of certain quantifiers.
  - $\forall x > 0(x^2 > 0)$ is the same as $\forall x(x > 0 \rightarrow x^2 > 0)$.
  - $\forall y \neq 0(y^3 \neq 0)$ is the same as $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.
  - $\exists z > 0(z^2 = 2)$ is the same as $\exists x(z > 0 \wedge z^2 = 2)$

Beyond $\forall$ and $\exists$

- Other Quantifiers
  The most important quantifiers are $\forall$ and $\exists$, but we could define many different quantifiers: "there is a unique", "there are exactly two", "there are no more than three", "there are at least 100", etc. A common one is the uniqueness quantifier, denoted by $\exists!$. $\exists! x P(x)$ states "There exists a unique x such that $P(x)$ is true".
  [illegible] $\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$, or more compactly $\exists x \forall y(P(y) \leftrightarrow y = x)$.

We will only concern ourselves with $\forall$ and $\exists$

- Restricting the domain of a quantifier
  Abbreviated notation is allowed, in order to restrict the domain of certain quantifiers.
  - $\forall x > 0 (x^2 > 0)$ is the same as $\forall x (x > 0 \rightarrow x^2 > 0)$.
  - $\forall y \neq 0 (y^3 \neq 0)$ is the same as $\forall y (y \neq 0 \rightarrow y^3 \neq 0)$.
  - $\exists z > 0 (z^2 = 2)$ is the same as $\exists z (z > 0 \wedge z^2 = 2)$.

Precedence

$\forall$ and $\exists$ have higher precedence than logical connectives:

Example:

$$\forall x P(x) \vee Q(x) \text{ means } (\forall x P(x)) \vee Q(x),$$

$$\text{and not } \forall x (P(x) \vee Q(x)).$$

Precedence

$\forall$ and $\exists$ have higher precedence than logical connectives:

Example:

$\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$,

and not $\forall x (P(x) \vee Q(x))$.

Note: This statement is not a proposition since there is a free variable.

Logical Equivalence and Logical Implication

Table 2.22 Logical Equivalences and Logical Implications for Quantified Statements in One Variable

For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x:

$$\exists x\,[p(x) \land q(x)] \Rightarrow [\exists x\; p(x) \land \exists x\, q(x)]$$

$$\exists x\,[p(x) \lor q(x)] \Leftrightarrow [\exists x\; p(x) \lor \exists x\, q(x)]$$

$$\forall x\,[p(x) \land q(x)] \Leftrightarrow [\forall x\; p(x) \land \forall x\, q(x)]$$

$$[\forall x\; p(x) \lor \forall x\, q(x)] \Rightarrow \forall x\,[p(x) \lor q(x)]$$

Grimaldi

Negation of Quantifiers

Consider the universal statement, “All men are mortal”.

- If it is not true, then *there exists a man that is not mortal*.

Similarly, consider the existential statement, “There exists a mortal man”.

- If it is not true, then all men are not mortal.

Negation of Quantifiers

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Rosen

Negation of Quantifiers

Table 2.23 **Rules for Negating Statements with One Quantifier**

$$\neg[\forall x\ p(x)] \Leftrightarrow \exists x\ \neg p(x)$$
$$\neg[\exists x\ p(x)] \Leftrightarrow \forall x\ \neg p(x)$$
$$\neg[\forall x\ \neg p(x)] \Leftrightarrow \exists x\ \neg\neg p(x) \Leftrightarrow \exists x\ p(x)$$
$$\neg[\exists x\ \neg p(x)] \Leftrightarrow \forall x\ \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$$

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Nested Quantifiers

Section 1.4

Section Summary

- Nested Quantifiers.
- Order of Quantifiers.
- Translating from Nested Quantifiers into English.
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other.

Example: "Every real number has an inverse" is

$\forall x \exists y(x + y = 0)$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Advancing to First-Order Logic

- $P(x)$ is a unary function,

Advancing to First-Order Logic

- *P*(*x*) is a unary function,
- A multi-variable example (binary function):

$$\underbrace{\overbrace{\text{LeftOf}}^{\textit{Predicate}}\ \overbrace{(\ x,\ y\ )}^{\textit{Arguments}}}_{\textit{Atomic Sentence}}$$

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Consider the following:

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- Clearly, order matters with mixed quantifiers.

Multivariable Quantifier Summary

statement	when true ?	when false ?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x, y.	There is a pair x, y for which $P(x,y)$ is false
$\forall x \exists y P(x,y)$	For every x there is y for which $P(x,y)$ is true	There is an x such that $P(x,y)$ is false for every y
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x, y.

Rosen

Arguing With Quantifiers

Arguing With Quantified Statements

Consider the following statements written both informally and with quantifiers:

1. There is a smallest positive integer.

2. There is no smallest positive real number.

3. The limit of a sequence, x_n, is written as $\lim(n \rightarrow \infty)\ x_n = L$ iff x_n becomes arbitrarily close to L as n gets larger without bound.

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   $\forall \varepsilon > 0 \exists N > 0 |x_n - L| < \varepsilon \forall N \leq n$

Arguing With Quantified Statements

4. Simplify $(r^{k+1})r$, $r \in \boldsymbol{R}$, $k \in \boldsymbol{Z}$

   Consider the following facts (*universal truths*):

   a. $\forall x \in \boldsymbol{R},\ m,n \in \boldsymbol{Z}(x^m x^n = x^{m+n})$
   b. $\forall x \in \boldsymbol{R},\ (x^1 = x)$

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   So,

   a. $(r^{k+1})r = r^{k+1} \cdot r^{1}$
   b. $\phantom{(r^{k+1})r} = r^{(k+1)+1}$
   c. $\phantom{(r^{k+1})r} = r^{k+2}$

Arguing With Quantified Statements

a. $(r^{k+1})r = r^{k+1} \cdot r^{1}$

b. $\quad = r^{(k+1)+1}$

c. $\quad = r^{k+2}$

Let us examine this reasoning in more detail:

a.	-	$\forall x \in \boldsymbol{R}, (x^{1}=x)$	UNIVERSAL TRUTH
	-	$\underline{r \text{ is a particular element of } \boldsymbol{R}}$	PARTICULAR INSTANCE
	-	$\therefore\ r^{1}=r$	CONCLUSION
b.	-	$\forall x \in \boldsymbol{R}, m,n \in \boldsymbol{Z}(x^{m}x^{n}=x^{m+n})$	UNIVERSAL TRUTH
	-	$\underline{R \text{ is a particular element of } \boldsymbol{R}}$	PARTICULAR INSTANCE
	-	$\therefore (r^{k+1})r = r^{k+1} \cdot r^{1}$	CONCLUSION

Universal Specification

Definition: The Rule of Universal Specification (US) – p. 106

If some property is true of everything in a domain, then it is true of any particular element in the domain.

• All men are mortal	UNIVERSAL TRUTH
• Socrates is a man	PARTICULAR INSTANCE
• ∴ Socrates is mortal	CONCLUSION

NOTES:

- This is also referred to as the Rule of Universal Instantiation
- The Rule of Universal Specification is the fundamental tool of Deductive reasoning.
- Clearly, we can also reason in the opposite direction – The Rule of Universal Generalization (UG, p. 110), and the same principles can be extended to the existential quantifier (ES, EG, not in your textbook).

Quantified Rules of Inference

Let us extend these ideas to the Rules of Inference from section 2.3:

Universal Modus Ponens

1. $\forall x[P(x) \rightarrow Q(x)]$
2. $\underline{\exists a P(a)}$
3. $\therefore Q(a)$

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Universal Modus Tollens

1. $\forall x[P(x) \rightarrow Q(x)]$
2. $\underline{\exists \mathrm{a}[\neg Q(a)]}$
3. $\therefore \neg P(a)$

- All human beings are mortal
- Zeus is not mortal
- $\therefore$ Zeus is not human

Venn Diagrams and Nomenclature

Formally, H(x): x is human, M(x): x is mortal, Z: Zeus

- $\forall x[H(x)\rightarrow M(x)]$ MAJOR PREMISE
- $\neg\underline{M(Z)]}$ MINOR PREMISE
- $\therefore \neg H(Z)$ CONCLUSION

Human set, H, in Mortal set, A

Zeus

Mortals

Major Premise

Minor Premise

Question

How do we apply the Rules of Inference and Laws of Logic to quantified statements?

Rules of Inference AND Laws of Logic

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

Solution

Instantiation and Generalization.

Instantiation and Generalization

TABLE 2 Rules of Inference for Quantified Statements.	
Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization

Rosen

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$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Rosen

Example (Rosen)

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

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Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	
2. $C(a) \wedge \neg B(a)$	
3. $C(a)$	
4. $\forall x(C(x) \rightarrow P(x))$	
5. $C(a) \rightarrow P(a)$	
6. $P(a)$	
7. $\neg B(a)$	
8. $P(a) \wedge \neg B(a)$	
9. $\exists x(P(x) \wedge \neg B(x))$	

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Step	Reason
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

Exercise

$$\begin{array}{r} \forall x\,[p(x) \rightarrow q(x)] \\ \forall x\,[q(x) \rightarrow r(x)] \\ \hline \therefore \forall x\,[p(x) \rightarrow r(x)] \end{array}$$

is valid by considering the following.

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is valid by considering the following.

Steps	**Reasons**
1) $\forall x\,[(p(x) \rightarrow q(x)]$	Premise
2) $p(c) \rightarrow q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x\,[q(x) \rightarrow r(x)]$	Premise
4) $q(c) \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\therefore \forall x\,[p(x) \rightarrow r(x)]$	Step (5) and the Rule of Universal Generalization

Exercise (Grimaldi)

8. Let $p(x)$, $q(x)$, and $r(x)$ denote the following open statements.

$$p(x): \quad x^2 - 8x + 15 = 0$$

$$q(x): \quad x \text{ is odd}$$

$$r(x): \quad x > 0$$

For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.

a) $\forall x\, [p(x) \rightarrow q(x)]$
b) $\forall x\, [q(x) \rightarrow p(x)]$
c) $\exists x\, [p(x) \rightarrow q(x)]$
d) $\exists x\, [q(x) \rightarrow p(x)]$
e) $\exists x\, [r(x) \rightarrow p(x)]$
f) $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$
g) $\exists x\, [p(x) \rightarrow (q(x) \wedge r(x))]$
h) $\forall x\, [(p(x) \vee q(x)) \rightarrow r(x)]$

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a) $\forall x\, [p(x) \rightarrow q(x)]$ **b)** $\forall x\, [q(x) \rightarrow p(x)]$

c) $\exists x\, [p(x) \rightarrow q(x)]$ **d)** $\exists x\, [q(x) \rightarrow p(x)]$

e) $\exists x\, [r(x) \rightarrow p(x)]$ **f)** $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$

g) $\exists x\, [p(x) \rightarrow (q(x) \wedge r(x))]$

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(a) True (b) False: For $x = 1$, $q(x)$ is true while $p(x)$ is false.
(c) True (d) True (e) True (f) True
(g) True (h) False: For $x = -1$, $(p(x) \vee q(x))$ is true but $r(x)$ is false.

Exercise (Grimaldi)

10. For the following program segment, m and n are integer variables. The variable A is a two-dimensional array $A[1, 1]$, $A[1, 2]$, ..., $A[1, 20]$, ..., $A[10, 1]$, ..., $A[10, 20]$, with 10 rows (indexed from 1 to 10) and 20 columns (indexed from 1 to 20).

```
for m := 1 to 10 do
  for n := 1 to 20 do
    A[m,n] := m + 3 * n
```

Write the following statements in symbolic form. (The universe for the variable m contains only the integers from 1 to 10 inclusive; for n the universe consists of the integers from 1 to 20 inclusive.)

a) All entries of A are positive.

b) All entries of A are positive and less than or equal to 70.

c) Some of the entries of A exceed 60.

d) The entries in each row of A are sorted into (strictly) ascending order.

e) The entries in each column of A are sorted into (strictly) ascending order.

f) The entries in the first three rows of A are distinct.

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(a) $\forall m, n \;\; A[m,n] > 0$

(b) $\forall m, n \;\; 0 < A[m,n] \leq 70$

(c) $\exists m, n \;\; A[m,n] > 60$

(d) $\forall m \;\; [(1 \leq n < 19) \rightarrow (A[m,n] < A[m,n+1])]$

(e) $\forall n \;\; [(1 \leq m < 9) \rightarrow (A[m,n] < A[m+1,n])]$

(f) $\forall \; 1 \leq m, i \leq 3 \;\; \forall \; 1 \leq n, j \leq 20 \; [((m,n) \neq (i,j)) \rightarrow (A[m,n] \neq A[i,j])]$

Presentation Terminated