Making mistakes in proofs is part of the learning process. When you make a mistake that someone else finds, you should carefully analyze where you went wrong and make sure that you do not make the same mistake again. Even professional mathematicians make mistakes in proofs. More than a few incorrect proofs of important results have fooled people for many years before subtle errors in them were found.

1.7.9 Just a Beginning

We have now developed a basic arsenal of proof methods. In the next section we will introduce other important proof methods. We will also introduce several important proof techniques in Chapter 5, including mathematical induction, which can be used to prove results that hold for all positive integers. In Chapter 6 we will introduce the notion of combinatorial proofs.

In this section we introduced several methods for proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$, including direct proofs and proofs by contraposition. There are many theorems of this type whose proofs are easy to construct by directly working through the hypotheses and definitions of the terms of the theorem. However, it is often difficult to prove a theorem without resorting to a clever use of a proof by contraposition or a proof by contradiction, or some other proof technique. In Section 1.8 we will address proof strategy. We will describe various approaches that can be used to find proofs when straightforward approaches do not work. Constructing proofs is an art that can be learned only through experience, including writing proofs, having your proofs critiqued, and reading and analyzing other proofs.

Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

2. Use a direct proof to show that the sum of two even integers is even.

3. Show that the square of an even number is an even number using a direct proof.

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

5. Prove that if $m+n$ and $n+p$ are even integers, where $m, n,$ and p are integers, then $m+p$ is even. What kind of proof did you use?

6. Use a direct proof to show that the product of two odd numbers is odd.

7. Use a direct proof to show that every odd integer is the difference of two squares. [*Hint:* Find the difference of the squares of $k+1$ and k where k is a positive integer.]

8. Prove that if n is a perfect square, then $n+2$ is not a perfect square.

9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

10. Use a direct proof to show that the product of two rational numbers is rational.

11. Prove or disprove that the product of two irrational numbers is irrational.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

13. Prove that if x is irrational, then $1/x$ is irrational.

14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

15. Prove that if x is an irrational number and $x > 0$, then $\sqrt{x}$ is also irrational.

16. Prove that if x, y, and z are integers and $x+y+z$ is odd, then at least one of x, y, and z is odd.

17. Use a proof by contraposition to show that if $x+y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

☞ **18.** Prove that if m and n are integers and mn is even, then m is even or n is even.

19. Show that if n is an integer and n^3+5 is odd, then n is even using

a) a proof by contraposition.
b) a proof by contradiction.

20. Prove that if n is an integer and $3n+2$ is even, then n is even using

a) a proof by contraposition.
b) a proof by contradiction.

21. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

22. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?

23. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a+b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

24. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

25. Show that at least ten of any 64 days chosen must fall on the same day of the week.

26. Show that at least three of any 25 days chosen must fall in the same month of the year.

27. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]

28. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

29. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

30. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

31. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.

32. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.

33. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.

34. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, (*iii*) $3x - 1$ is rational.

35. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.

36. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (1) $\sqrt{2x^2 - 1} = x$ is given; (2) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (3) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (4) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (5) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.

37. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (1) $\sqrt{x + 3} = 3 - x$ is given; (2) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (3) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (4) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (5) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.

38. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.

39. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \to p_4$, $p_3 \to p_1$, $p_4 \to p_2$, $p_2 \to p_5$, and $p_5 \to p_3$ are true.

40. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

41. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

42. Use Exercise 41 to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

43. Prove that if n is an integer, these four statements are equivalent: (*i*) n is even, (*ii*) $n + 1$ is odd, (*iii*) $3n + 1$ is odd, (*iv*) $3n$ is even.

44. Prove that these four statements about the integer n are equivalent: (*i*) n^2 is odd, (*ii*) $1 - n$ is even, (*iii*) n^3 is odd, (*iv*) $n^2 + 1$ is even.

1.8 Proof Methods and Strategy

1.8.1 Introduction

Assessment

In Section 1.7 we introduced many methods of proof and illustrated how each method can be used. In this section we continue this effort. We will introduce several other commonly used proof methods, including the method of proving a theorem by considering different cases separately. We will also discuss proofs where we prove the existence of objects with desired properties.

In Section 1.7 we briefly discussed the strategy behind constructing proofs. This strategy includes selecting a proof method and then successfully constructing an argument step by step, based on this method. In this section, after we have developed a versatile arsenal of proof methods, we will study some aspects of the art and science of proofs. We will provide advice on how to find a proof of a theorem. We will describe some tricks of the trade, including how proofs can be found by working backward and by adapting existing proofs.