MATH 240 Lecture 1.2 - Row Reduction to Echelon Form

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1 Row Reduction to Echelon Form

1.1 Definition

A rectangular matrix is in **row echelon form** (REF) if

1. All zero rows are below non-zero rows.
2. Each leading nonzero entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading nonzero entry are zero

1.2 Theorem 1

Every Matrix can be row reduced to REF and to RREF. (Each) RREF is unique, that is every matrix is row equivalent to *one* RREF matrix only.

1.2.1 Example

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

This gives us $x + y = 0$ and $0 = 2$.
If we take $R_2 := \frac{1}{2}R_2$, we get the RREF matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This gives us $x + y = 0$ and $0 = 1$.

The leading entries of a row of a matrix in REF are called pivots.

1.3 Theorem 2

A linear system is inconsistent (no solutions) iff the rightmost column of an augmented matrix in REF has a pivot

1.4 Avoiding Fractions

Generally, we want to avoid fractions at all costs.

1.4.1 Example [Avoiding Fractions]

Given the matrix

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

We can reduce in two paths.

The first path begins as follows

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 := R_2 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{4}{3} & 2\frac{2}{3} \end{bmatrix}$$

This leaves us with some nasty fractions. Instead, if we first swap R_1 and R_2, we get

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 := R_2, R_2 := R_1} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 := R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix}$$
$$\xrightarrow{R_2 := -\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

and so on.