# MATH 240 Lecture 1.4
# The Matrix Equation $Ax = b$

Alexander Ng

September 13, 2024

**The equation $Ax = b$ is very important

## Review

Let $A$ be an $m \times n$ matrix.

Let $x$ be a vector of length $n$, with entries $x_1, x_2, \ldots, x_n$.

The definition of how to multiply a matrix by a vector is

If $A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ v_1 & v_2 & \ldots & v_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

Then $Ax = \begin{bmatrix} x_1 \cdot v_1 + x_2 \cdot v_2 + \cdots + x_n \cdot v_n \end{bmatrix}$

If $A = \begin{bmatrix} r_1 \\ r_2 \\ \ldots \\ r_m \end{bmatrix}$

Then $Ax = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \ldots \\ r_m \cdot x \end{bmatrix}$

# 1 Th. 5: Properties of the matrix vector product

Let $A$ and $B$ be $m \times n$ matrices over $\mathbb{R}$, and $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$

$$A(u+v) = Au + Av \rightarrow \text{Distributive law} \tag{1}$$

This implies that $A(u+v+w) = A(u+v) + A(w) = Au + Av + Aw$ because addition is commutative.

$$A(c \cdot v) = c \cdot A \cdot v \rightarrow \text{Associative law} \tag{2}$$

$$(A+B) \cdot u = Au + Bu \rightarrow \text{Distributive law} \tag{3}$$

Proofs of the above properties will be tested on the exams.

## 1.1 Proving Property 1

# 2 Four ways to represent a linear system

1. Standard Form

$$\begin{aligned} 2x_1 + 1x_2 &= 5 \\ 1x_1 + 3x_3 &= 7 \end{aligned}$$

2. Augmented Matrix

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

3. Vector Equation
$x_1 \cdot v_1 + x_2 \cdot v_2 + \cdots + x_n \cdot v_n = b$

$$b = \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2x_1 + 1x_2 \\ 1x_1 + 3x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3x_3 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

4. Consider the following matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \\ 1 \cdot x_1 + 0 \cdot x_2 + 3 \cdot x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

This is the same as $Ax = b$ where $A$ is the matrix above and $x$ is the vector of unknowns to be solved for.

Compare this with $ax = b$ (the linear system). $x = \frac{b}{a}$.

*What is the point of having 4 different ways to represent a linear system?*

Method (1) is for building a linear system from scratch.

Method (2), the augmented matrix, is for solving a linear system that has already been built.

Method (3) is for proofs.

Method (4) is for reasoning and proofs, because it is concise.

# 3 Theorems

Let $A$ be an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$ and x=$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$ and $b$ be a vector in $\mathbb{R}^m$.

## 3.1 Theorem 3

$Ax = b$, $\begin{bmatrix} A|b \end{bmatrix}$ and $\begin{bmatrix} x_1v_1 + x_2v_2 + \cdots + x_nv_n \end{bmatrix}$ have the same solution set(s).

## 3.2 Theorem 4

If $B \sim A$ and $B$ is in REF, then the following statements are equivalent.

1. The linear system $Ax = b$ has a solution for every choice of $b \in \mathbb{R}^m$.
2. Every $b \in \mathbb{R}^m$ is a linear combination of the columns of $A$.
3. The span of $v_1, v_2, \ldots, v_n$ generates $\mathbb{R}^m$.
   In other words, every vector in $\mathbb{R}^m$ can be obtained from the span of $v_1, v_2, \ldots, v_n$.

4. The matrix $B$ has a pivot (position) in every row.
   This is our tool for testing if 1..3 are true.

What does it mean for the statement to be equivalent? What is the theorem saying?

What this theorem is saying is that these four statements are **all** either simultaneously true or simultaneously false.

### 3.2.1 Definition (Equivalence)

Two statements are equivalent if they are simultaneously true or simultaneously false.

### 3.2.2 Example

Is $Span(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}) = \mathbb{R}^3$?

Apply (3) and (4). (Please clean this up)

$A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 0 & 1\end{bmatrix}$

R2:=R2-R1; R3:=R3=R1;

$\begin{bmatrix}1 & 1 & 1\\0 & 1 & -1\\0 & 2 & 0\end{bmatrix}$

R3:=R3-2R2;

$\begin{bmatrix}1 & 1 & 1\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix}$

$\implies$ (4) is true.

We can prove Theorem 4 by showing equivalence between (3) and (4), (2) and (1), and (2) and (4).

### 3.2.3 Proof that (4) and (1) are equivalent

(4) $\implies$ (1). If (4) is true, then $[A|b] \sim [B|\vdots] in REF$ has a pivot position in every row. ($Ax = b$ has a solution);

If (d) is false, $A \sim B$ has at least one row of zeroes.

Consider the matrix $B = \begin{bmatrix} u & & & & \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$.

This system is inconsistent because it's augmented matrix has $0 = 1$ in the bottom row.

But, since row operations are reversible, this system is equivalent to $A$. $\therefore A$ is also inconsistent.

$\implies Ax = b$ is inconsistent.

This means that the statement (a) is not true.

We have shown that if (d) is false, then there exists a vector $b$ such that $Ax = b$ has no solutions, which implies that $a$ is false.