Question 14

Part a

$$\neg p \wedge (p \vee q) \to q$$

$p \to q$ is false when p is true and q is false.

Assume $\neg p \wedge (p \vee q)$ is true.

When $\neg p \wedge (p \vee q)$ is true, then both $\neg p$ and $p \vee q$ are true.

When $\neg p$ is true, then p is false.

When $p \vee q$ is true with p false, then we require that q is true, and thus $\neg p \wedge (p \vee q) \to q$ is always true (as we know that the conditional statement is also true whenever $\neg p \wedge (p \vee q)$ is false). This implies that $\neg p \wedge (p \vee q) \to q \equiv \mathbb{T}$.

Part b

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

$p \to q$ is false when p is true and q is false.

Assume $(p \to q) \wedge (q \to r)$ is true.

When $(p \to q) \wedge (q \to r)$ is true, then both $p \to q$ and $q \to r$ are true.

When p is true, because $p \to q$ is true, q must also be true, and because $q \to r$ is true, r must also be true. $p \to r$ is also true when both p and r are true.

When p is false, then $p \to r$ is true as well.

Therefore, $p \to r$ is true in each case, and thus $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is always true (as we know that the conditional statement is also true whenever $(p \to q) \wedge (q \to r)$ is false).

This implies that $[(p \to q) \wedge (q \to r)] \to (p \to r) \equiv \mathbb{T}$

Part c

$$p \wedge (p \to q) \to q$$

$p \to q$ is false when p is true and q is false.

Assume $p \wedge (p \to q)$ is true.

When $p \wedge (p \to q)$ is true, both p and $p \to q$ are true.

When $p \to q$ and p are both true, we also require that q is true. Thus, $p \wedge (p \to q) \to q$ is always true (since we know the conditional statement is also true whenever $p \wedge (p \to q)$ is false).

This implies that $p \wedge (p \to q) \to q \equiv \mathbb{T}$

Part d

$$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$$

$p \to q$ is false when p is true and q is false.

Assume $(p \vee q) \wedge (p \to r) \wedge (q \to r)$ is true.

When $(p \vee q) \wedge (p \to r) \wedge (q \to r)$ is true, then $p \vee q$, $p \to r$, and $q \to r$ are all true.

$p \vee q$ being true implies that either p or q is true.

When p is true, then r is true as well (since $p \to r$ is true).

When q is true, then r is true as well (since $q \to r$ is true).

However, r is true in each case, and thus $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is always true (as we know that the conditional statement is also true whenever $(p \vee q) \wedge (p \to r) \wedge (q \to r)$ is false). This then gives us that

$$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r \equiv \mathbb{T}$$