MATH 240 Lecture 2.2
The Inverse A^{-1} of a matrix A

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$A + B$, $A - B$, $r \cdot A$, $A \cdot B$, A^T, A^{-1}
Let I_n be the $n \times n$ identity matrix. (Matrix property 5);

$$\begin{bmatrix} 1 & ... & 0 \\ ... & 1 & ... \\ 0 & ... & 1 \end{bmatrix}$$

$A \cdot I_n = A$ and $I_n \cdot A = A$. (Matrix multiplication by the Identity Matrix is commutative).

1 Definition of the Inverse

Let A be an $n \times n$ matrix. (The definition of the inverse of a matrix is only for square matrices.)

A is **invertible** or **nonsingular** if there exists an $n \times n$ matrix C such that $AC = CA = I_n$.

C is unique. (Theorem)

1.1 Proof of the Uniqueness of C

Suppose C and B are inverses of A.

$$AC = I_n \text{ and } AB = I_n \text{ and } CA = I_n \text{ and } CB = I_n$$

Using the fact that $AC = I_n$, substitute in AC as I_n in $B = B \cdot I_n$.

$B = B \cdot I_n = B(AC) = (BA)C = I_n \cdot C = C$
$\square$

2 Forgot Name

In $\mathbb{R}$,

$$\begin{aligned} ax &= b \\ x &= b/a \\ x &= a^{-1}b\{a \neq 0\} \end{aligned}$$

in $\mathbb{R}^n$,

$$\begin{aligned} Ax &= b \\ x &= A^{-1}b\{A \text{ is invertible}\} \end{aligned}$$

If A is an invertible matrix, then the linear system $Ax = b$ has the unique solution $x = A^{-1}b$.

2.1 Proof

$$\begin{aligned} Ax &= b \\ \text{Is } A(A^{-1}b) &= b \end{aligned}$$

$$\begin{aligned} A(A^{-1}b) &= b \\ &= (AA^{-1})b \\ &= b \end{aligned}$$

Suppose $Au = b$ and $Av = b$. $\implies Au = Av$.
A is invertible $\implies A^{-1} \implies A^{-1}(Au) = A^{-1}(Av) = b$.
$A^{-1}(Au) = (A^{-1}A)u = u$.
$A^{-1}(Av) = (A^{-1}A)v = v$.
$\therefore u = v$.
$\square$

3 Theorem 6 (Properties of A^{-1})

(a) If A is invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

(b) If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is invertible, A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

3.1 Proof of (b)

Apply the Associative Law over and over again.

$(AB)(B^{-1}A^{-1}) = A(B(B^{-1}A^{-1})) = A((BB^{-1})A^{-1}) = A(I_nA^{-1}) = A(A^{-1}) = I_n$.

Prove $(B^{-1}A^{-1})(AB) = I_n$. (do it at home)

3.2 Exercise 2.2-25

Suppose A, B and C are invertible. So A^{-1}, B^{-1} and C^{-1} exist.

Find $A \cdot B)^{-1} = ((AB) \cdot C)^{-1}$.

4 How to find the inverse of a matrix

Is $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ invertible? If so, find A^{-1}.

Let's find C such that $AC = I_n$. Then check $CA = I_n$.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c = 1, d = 0 \\ a = 0, b = 1 \end{bmatrix} \implies C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Check (at home).

4.1 Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible.

And $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The thingie $ad - bc$ is called the **determinant** of A. (This will be part of Ch.3)

4.2 Theorem 7 (The most beautiful algorithm)

An $n \times n$ matrix A is invertible $\iff A \sim I_n$.

? Calculate A^{-1}. Let's find C such that $AC = I_n$. Then check $CA = I_n$.

Calculate the inverse of $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$
$R_2 \to -R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Calculate $A \cdot A^{-1}$ (at home).