MATH 240 Lecture 1.7
Linear Independence

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1 Linear Independence

A set of vectors $v_1, v_2, \ldots, v_n$ is linearly independent (L.I.) if the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

has only the zero solution $c_1 = c_2 = \cdots = c_n = 0$.
Otherwise, the vectors are called linearly dependent (L.D.).

1.1 Example

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Question 1: Are they linearly independent?
Question 2: If not, find a non-zero solution for $c_1, c_2 \ldots c_n$.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} c_1 + 4c_2 + 2c_3 = 0 \\ 2c_1 + 5c_2 + c_3 = 0 \\ 3c_1 + 6c_2 + 0c_3 = 0 \end{aligned}$$

$$\left[\ A \mid b\ \right] = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$

R2:=R3-2R1

R3:=R3-3R1

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix}$$

R3:=R3-2R2
R2:=-1/3 R2

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c_3 is free, so there is one ore more non-zero (nontrivial) solutions, and therefore this system is not linearly independent.

In other words, there must be non-trivial solutions since c_3 is a free variable. So, v_1, v_2, v_3 are linearly dependent.

R1:=R1-4R2

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} c_1 - 2c_3 &= 0, \\ c_2 + c_3 &= 0 \end{aligned} \implies \begin{aligned} c_1 &= 2c_3, \\ c_2 &= -c_3, \\ c_3 &\text{ is free} \end{aligned}$$

Pick $c_3 = 1 \implies c_1 = 2, c_2 = -1$. Hence, $2v_1 - v_2 + v_3 = 0$

Check. (Exercise)

Let A be an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$

Then A can be written as

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ v_1 & v_2 & \ldots & v_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot v_1 + x_2 \cdot v_2 + \cdots + x_n \cdot v_n = 0$$

Thefore (Theorem), The columns of a matrix A are linearly independent if and only if the matrix equation $Ax = 0$ has only the trivial solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

1.2 Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Notice $v_1 + v_2 = v_3 \implies v_1 + v_2 - v_3 = 0$.
Therefore, v_1, v_2, v_3 are linearly dependent.
Therefore, the system $Ax = 0$ has non-trivial solutions.

1.3 Theorem

If two **non-zero** vectors, v_1 and v_2 are linearly dependent, then then, we can write v_1 as $v_1 = c \cdot v_2$ for some $c \in \mathbb{R}$. (i.e. v_1 is a scalar multiple of v_2)

1.3.1 E.g.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

This implies:

$$v_2 = 2 \cdot v_1$$

Thus:

$$v_2 - 2 \cdot v_1 = \mathbf{0}$$

1.3.2 Proof

Suppose $c_1 \cdot v_1 + c_2 \cdot v_2 = \mathbf{0}$ and v_1, v_2 are linearly dependent.
If $c_1 = 0$, then $0 \cdot v_1 + c_2 \cdot v_2 = \mathbf{0}$ and v_2 is non-zero, then $c_2 = 0$.
Therefore, $c_1 \neq 0$. And $\frac{1}{c_1}(c_1 v_1 + c_2 v_2) = \frac{1}{c_1}\mathbf{0}$
Recall $c(u + v) = cu + cv$.

So, $\frac{1}{c_1}(c_1v_1 + c_2v_2) = \frac{1}{c_1}(c_1v_1) + \frac{1}{c_1}(c_2v_2) = \mathbf{0}$
Recall $c(du) = (cd) \cdot u$.
$\frac{1}{c_1}c_1 \cdot v_1 + \frac{1}{c_1}c_2 \cdot v_2 = \mathbf{0}$
$1v_1 + \frac{c_2}{c_1}v_2 = \mathbf{0}$
$v_1 = -\frac{c_2}{c_1}v_2$
$\square$
Professor says to use smily face (☺) instead of $\square$.

1.3.3 What if 3 non-zero vectors are linearly dependent?

$c_1 \cdot u + c_2 \cdot v + c_3 \cdot w = \mathbf{0}$
Does this mean $u = d_1 \cdot v + d_2 \cdot w$ for some $d_1, d_2 \in \mathbb{R}$?
No, because u is not necessarly a scalar multiple of v or w.
Consider (proof by contradiction)
$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$
$0 \cdot u + 1 \cdot w - 2 \cdot v = \mathbf{0}$

Despite the fact that these vectors are linearly dependent, because u[0] is 1, and no other vector has a non-zero entry in the first row, we cannot write u as a scalar multiple of v or w.

1.4 Theorem 7

Suppose $S = v_1, v_2, \ldots, v_r, r \geq 2, and v_1, v_2, \ldots, v_r$ are non-zero and linearly dependent. Then, at least one of the vectors in S is a linear combination of the other vectors in S.

In our example, we have $w = 2v + 0u$.

Can the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

be linearly dependent?

We need to solve:

$$c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \mathbf{0}$$

This gives us the system of equations:

$$\begin{aligned} 2c_1 + c_2 + 3c_3 &= 0, \\ c_1 + 2c_2 + 5c_3 &= 0 \end{aligned}$$

Find the example in lecture notes and copy.

The vectors **must** be linearly dependent becasue there are 3 unknowns, c_1, c_2, c_3 but only 2 equations. Therefore, there **must** be at least one free variable, so the linear system has infinite non-trivial solutions.

1.5 Theorem 8

If $S = v_1, v_2, \ldots, v_r \in \mathbb{R}^n$ and $r > n$, then S is a linearly dependent set of vectors. (if you have more vectors than the space you're working in, they are linearly dependent)

Therefore, $\mathbb{R}^n$ can have at most n linearly independent vectors.