MATH 240 Lecture 3.1
The Determinant of a Matrix

Alexander Ng

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Definition - The Determinant of a Matrix

Let A be an $n \times n$ matrix. The ij^{th} minor of A, denoted A_{ij}, is the $(n-1) \times (n-i)$ matrix obtained from A by deleting the ith row and jth column.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}.$$

The determinant of A, denoted $\det(A)$ or $|A|$ is a number defined by

Example

$$\det \begin{bmatrix} 7 & 9 & 1 \\ 1 & 2 & 3 \\ 0 & 5 & 6 \end{bmatrix} = 7 \cdot \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} - 9 \cdot \det \begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

$$7 \cdot (12 - 15) + 9 \cdot (6 - 0) + 1 \cdot (5 - 0)$$

$$-21 - 54 + 5 = -70.$$

Example 2

$$\det \begin{bmatrix} 7 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 5 & 6 \end{bmatrix} = 7 \cdot \det \begin{bmatrix} 2 & 0 \\ 5 & 6 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}.$$

$$7 \cdot (2 \cdot 6 - 0 \cdot 5) - 0 \cdot (1 \cdot 6 - 0 \cdot 5) + 0 \cdot (1 \cdot 2 - 0 \cdot 5).$$

$$7 \cdot 2 \cdot 6 = 84.$$

Theorem 2

$$\text{Let } U = \begin{bmatrix} u_{11} & 0 & 0 & \dots & 0 \\ u_{21} & u_{22} & 0 & \dots & 0 \\ u_{31} & u_{32} & u_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & u_{n3} & \dots & u_{nn} \end{bmatrix}.$$

U is known as a **Lower Triangular Matrix**. When U is a lower triangular matrix, the determinant of U is the product of the diagonal elements of U.

There may be multiple matrices that have the same determinant.

Corollary

$$\text{Let } A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

$$\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn}$$

$$\det(I) = 1$$

Def.

Let A be an $n \times n$ matrix. The ij^{th} cofactor of A is $C_{ij} = (-1)^{i+j}\det(A_{ij})$.

Theorem 1

$\det(A)$ can be computed by expanding along row i or down col. j. as follows:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \rightarrow \text{row i}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \rightarrow \text{col j}$$

Example

$$\text{Let } A = \begin{bmatrix} 7 & 9 & 1 \\ 1 & 2 & 3 \\ 0 & 5 & 6 \end{bmatrix}.$$

We should choose the first column to expand along since there is a 0 entry and a 1 entry in the first column.

$$\det(A) = a_{11}\cdot(-1)^{1+1}\cdot\det(A_{11}) + a_{21}\cdot(-1)^{1+2}\cdot\det(A_{21}) + a_{31}\cdot(-1)^{1+3}\cdot\det(A_{31})$$

$$= 7 \cdot 1 \cdot \det(\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}) + 1 \cdot (-1) \cdot \det(\begin{bmatrix} 9 & 1 \\ 5 & 6 \end{bmatrix}) + \mathbf{0}$$

$$= 7 \cdot (12 - 15) + (54 - 5) + 0 = -70$$

Properties of the Determinant

Theorem 4 (Gun 13 IMT)

A is invertible $\iff \det(A) \neq 0$

Theorem 5

$\det(A^T) = \det(A)$

Theorem 6

$\det(AB) = \det(A) \cdot \det(B)$

Determinant of the Sum of Matrices

Is $\det(A + B) = \det(A) + \det(B)$? No!

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly, $\det(I) = 1$. However, $\det(A + B) = \det(A) + \det(B)$ is false by contradiction, since any matrix with a 0 row or column has $\det(A) = 0$.

$$1 \neq 0 + 0.$$

Proof of Theorem 5