ELEMENTARY LOGIC

MACM 101

Chapter Summary

Propositional Logic

- The Language of Propositions.
- Applications.
- Logical Equivalences (and Logical Implication).
- The Laws of Propositional Logic

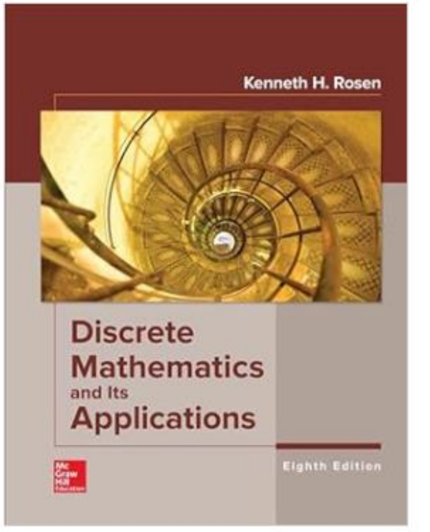
Predicate Logic

- The Language of Quantifiers.
- Nested Quantifiers.

Proofs

- Rules of Inference.
- Proof Methods.
- Proof Strategy.

Presentation Coverage:



Propositional Logic Summary

The Language of Propositions

- Connectives.
- Truth Values.
- Truth Tables.

Applications

- Translating English Sentences.
- System Specifications.
- Logic Puzzles.
- Logic Circuits.

Logical Equivalences

- Important Equivalences.
- Showing Equivalence.
- Normal Forms.
- Propositional Satisfiability

Objectives of this Presentation

- Introduce the fundamental concepts underlying Propositional Logic (§1.1) and applications (§1.2),
- Introduce Logical Equivalence and Logical Identities (*a.k.a.,* The Laws of Logic – §1.3),
- Introduce Logical Implication (not in textbook),
- CNF and DNF.

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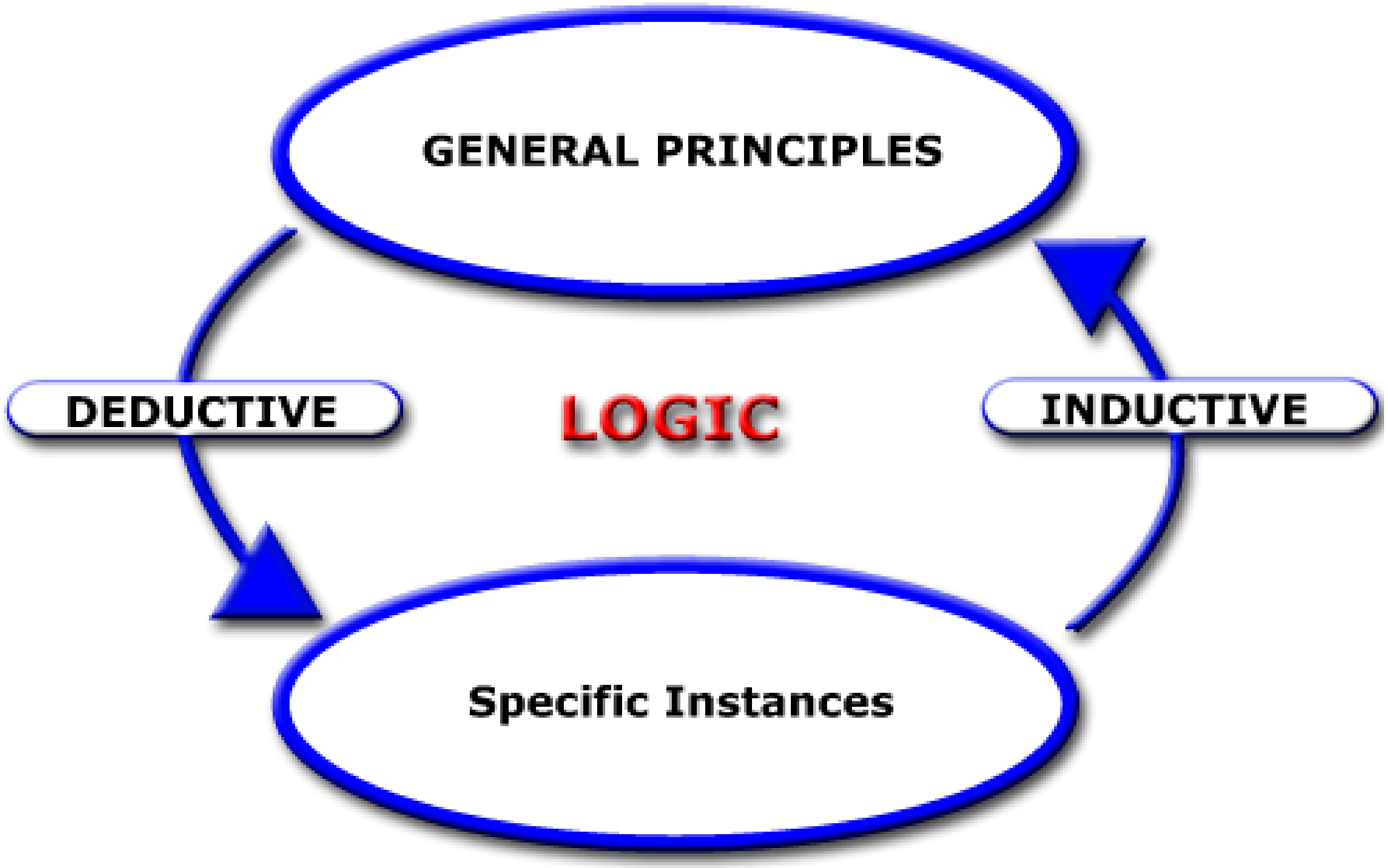
Induction vs Deduction



This course focuses on deductive logic.

Types of Deductive Logical Systems

- Syllogistic Logic (syllogisms),
- **Propositional Logic,**
- **Predicate Logic (quantifiers),**
- Modal Logic (modified semantics).
- *etc.*

*We will concentrate only on the systems in **bold**.*

Abstraction

- In each system, you must only live in the "world" of said system.
- For example, Propositional Logic **only deals with declarative sentences that possess *truth value***.
- This is where we will begin with (**Section** §**1.1**).

1.1 Propositional Logic

Preliminary Definitions

- **Truth Value:** Being true or false, <u>but not both</u> (*principium tertii exclusi, Aristotle*).

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- **Truth Value:** Being true or false, but not both.
- **Proposition:** A declarative sentence that possesses truth value.

Remark: We use lower case letters to denote primitive propositions (that cannot be broken down into anything simpler).

e.g., p: 3+5=8,

q: It is raining.

Questions

Which of the following statements are propositions?

p: Sit down!

q: This statement that you are now reading is false.

r: The number *x* is an integer.

Questions

- Which of the following statements are propositions?
  - *p*: Sit down!
  - *q*: This statement that you are now reading is false.
  - *r*: The number *x* is an integer.
- **NONE are propositions. Why?**
- **Note:** Introducing a third state, *u*, for uncertain solves the problem with the second sentence.

From Philosophy

- **Syntactic Reasoning**: What can be shown.
  - Syntax = grammar (rules of sentence construction).
- **Semantic Reasoning**: What is true.
  - Semantics = sentence meaning.

From Philosophy

- **Syntactic Reasoning**: What can be shown.
  - Syntax = grammar (rules of sentence construction).

    The structure of propositions (grammar)
- **Semantic Reasoning**: What is true.
  - Semantics = meaning.

    The meaning of propositions (truth tables)

OPERATIONS ON PROPOSITIONS - SYNTAX

1. Negation, $\neg$

OPERATIONS ON PROPOSITIONS - SYNTAX

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   Example:

   q: It is raining.

   $\neg q$ reads "It is not raining."

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NOTE: The term *literal* is either a primitive proposition (atomic formula) or its negation. Some textbooks use the symbol ~.

OPERATIONS ON PROPOSITIONS - SYNTAX

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2. Logical Connectives:

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   a) Conjunction, $\wedge$

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      Example:

      p: 3+5=8,

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      $p \wedge q$ reads "3+5=8 AND it is raining."

OPERATIONS ON PROPOSITIONS - SYNTAX

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      Example:

      p: 3+5=8,

      q: It is raining

      $p \vee q$ reads "3+5=8 AND/OR it is raining."

NOTE: The symbol, $\vee$, is deriving from the Latin word *vel* ("either", "or").

OPERATIONS ON PROPOSITIONS - SYNTAX

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2. Logical Connectives:
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   b) Disjunction, $\vee$ (inclusive OR)
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      p: 3+5=8,

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      $p \oplus q$ reads "3+5=8 OR it is raining."

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2. Logical Connectives:
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      Example:

      p: 3+5=8,

      q: It is raining

      $p \oplus q$ reads "3+5=8 OR it is raining"

NOTE: Some textbooks use the symbol, $\underline{\vee}$.

OPERATIONS ON PROPOSITIONS - SYNTAX

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   b) Disjunction, $\vee$ (inclusive OR)
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   d) Implication, $\rightarrow$

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   d) Implication, $\rightarrow$

      Example:

      p: 3+5=8,

      q: It is raining

      $p \rightarrow q$ reads “IF 3+5=8, THEN it is raining.”

OPERATIONS ON PROPOSITIONS - SYNTAX

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   a) Conjunction, $\wedge$
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   e) Biconditional, $\leftrightarrow$

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Example:

p: 3+5=8,

q: It is raining

$p \leftrightarrow q$ reads "3+5=8 IF AND ONLY IF it is raining."

REMARKS

- All propositions formed with logical connectives are “compound propositions” as opposed to primitive propositions.

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- Compound propositions need not have causal relations between atomic components (they can sound nonsensical and still be valid) – **material implication** as opposed to **causal implication,** which lacks ***temporal ordering***.

REMARKS

- All propositions formed with logical connectives are "compound propositions" as opposed to primitive propositions.
- Compound propositions need not have causal relations between atomic components (they can sound nonsensical and still be valid) – **material implication** as opposed to causal implication.
- This is the syntax or grammar of propositions.

SEMANTICS

- Let us now discuss the meaning of each logical operator using truth tables.

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- We will use **0 for False** and **1 for true**.

OPERATIONS ON PROPOSITIONS - SEMANTICS

Negation, $\neg$

p	$\neg p$
0	1
1	0

OPERATIONS ON PROPOSITIONS - SEMANTICS

Conjunction, $\wedge$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

OPERATIONS ON PROPOSITIONS - SEMANTICS

Disjunction, $\vee$ (inclusive)

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

OPERATIONS ON PROPOSITIONS - SEMANTICS

Disjunction, $\oplus$ (exclusive)

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

OPERATIONS ON PROPOSITIONS - SEMANTICS

Disjunction, ⊕ (exclusive)

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

This says either *p* or *q* are true, but not both:
$p \oplus q$ **is the same as** $(p \vee q) \wedge \neg(p \wedge q)$.

OPERATIONS ON PROPOSITIONS - SEMANTICS

Implication, $\rightarrow$

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$
0	0	0	0	0	1
0	1	0	1	1	1
1	0	0	1	1	0
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OPERATIONS ON PROPOSITIONS - SEMANTICS

Implication, $\rightarrow$

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$
0	0	0	0	0	1
0	1	0	1	1	1
1	0	0	1	1	0
1	1	1	1	0	1

The statement is true when both *p* and *q* are true.

OPERATIONS ON PROPOSITIONS - SEMANTICS

Implication, $\rightarrow$

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$
0	0	0	0	0	1
0	1	0	1	1	1
1	0	0	1	1	0
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CRUX: But if *p* is true, *q* cannot possibly be false.

OPERATIONS ON PROPOSITIONS - SEMANTICS

Implication, $\rightarrow$

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$
0	0	0	0	0	1
0	1	0	1	1	1
1	0	0	1	1	0
1	1	1	1	0	1

Also, when *p* is false, *q* may be either true or false because the statement says nothing about the truth value of *q*.

Some “If” Indicator Words

- “If…” can be replaced by such phrases as:
  - “in case…”
  - “provided that…”
  - “given that…”
  - “on condition that…”
- Some indicator words for “then…” include:
  - “implies...”
  - “entails…”

Necessary Conditions

- For a normal car to run, it is necessary that there is fuel in its tank, its spark plugs properly adjusted, its oil pump working, etc.
  - If the car runs, then every one of the conditions necessary for its occurrence must be fulfilled

    "That there is fuel in its tank is a necessary condition for the car to run"

    = "The car runs only if there is fuel in its tank"

    = "If the car runs then there is fuel in its tank"
- All these are "$p \to q$"

Sufficient Conditions

- For a purse to contain over a dollar, it would be sufficient for it to contain 101 pennies, 21 nickels, 11 dimes, 5 quarters, etc.
  - If any one of these circumstances is obtained, the specified situation will be realized
- "That a purse contains 5 quarters is a sufficient condition for it to contain over a dollar"
  - "If the purse contains 5 quarters then it contains over a dollar"

Hints

- "p is a sufficient condition for q"

  $p \to q$ (Note: here q is a necessary condition for p)

- Compare: "p is a *necessary* condition for q"

  $q \to p$ (Note: here q is a sufficient condition for p)

- "Formula" to help you remember, given any sentence saying something is a necessary or sufficient condition:

  $p \to q$

  Necessary condition

  Sufficient condition

OPERATIONS ON PROPOSITIONS - SEMANTICS

$p \rightarrow q$ reads as any one of these phrases

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"

"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

OPERATIONS ON PROPOSITIONS - SEMANTICS

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- Material implication does not depend on any relationship between the *meanings* of the propositions, *p* and *q*, but only on their *truth values*.
- That is, do not be confused by the so-called *paradoxes of material implication*.

e.g., If the sky is red then ice is cold.

NOTE: The Four Types of Implication

1. If all humans are mortal and Socrates is a human, then Socrates is mortal.
   - **Logical Implication**: the consequent follows logically from its antecedent (*a.k.a.*, **material implication**).
2. If Leslie is a bachelor, then Leslie is unmarried.
   - **Definitional Implication**: the consequent follows the antecedent by definition.
3. If I put X in acid, then X will turn red.
   - **Causal Implication**: The connection between antecedent and consequent is discovered empirically (temporal ordering is implicit).
4. Is we lose the game, then I'll eat my hat.
   - **Decisional Implication**: no logical connection nor one by definition between the consequent and antecedent. This is a decision of the speaker to behave in the specified way under the specified circumstances.

NOTE: For Propositional Logic

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OPERATIONS ON PROPOSITIONS - SEMANTICS

Biconditional, $\leftrightarrow$

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

$p \leftrightarrow q$ is nothing more than $(p \rightarrow q) \wedge (q \rightarrow p)$

OPERATIONS ON PROPOSITIONS - SEMANTICS

Biconditional, $\leftrightarrow$

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

OPERATIONS ON PROPOSITIONS - SEMANTICS

$p \leftrightarrow q$ reads IF *AND* ONLY IF (***iff***)

"if p, then q"
"if p, q"
"p is sufficient for q"

$q \rightarrow p$ *reads* ***p*** **if** ***q*** → "q if p"

"q when p"
"a necessary condition for p is q"

AND "q unless $\neg p$"

"p implies q"
→ "p only if q"

$p \rightarrow q$ *reads* ***p*** **only if** ***q***

"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

OPERATIONS ON PROPOSITIONS – SEMANTICS
Order of Precedence

As a way of reducing the number of necessary parentheses, one may introduce precedence rules: $\neg$ has higher precedence than $\wedge$, $\wedge$ higher than $\vee$, and $\vee$ higher than $\rightarrow$. So for example, $P \vee Q \wedge \neg R \rightarrow S$ is short for $(P \vee (Q \wedge (\neg R))) \rightarrow S$.

Here is a table that shows a commonly used precedence of logical operators.

Operator	***Precedence***
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Wikipedia

Bit Strings

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 12 101010011 is a bit string of length nine. ◀

We can extend bit operations to bit strings. We define the **bitwise** ***OR***, **bitwise** ***AND***, and **bitwise** ***XOR*** of two strings of the same length to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$ to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 13.

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EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

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EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

$$\begin{array}{ll} 01\ 1011\ 0110 & \\ \underline{11\ 0001\ 1101} & \\ 11\ 1011\ 1111 & \text{bitwise } OR \\ 01\ 0001\ 0100 & \text{bitwise } AND \\ 10\ 1010\ 1011 & \text{bitwise } XOR \end{array}$$

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

◀

1.2 Applications of Propositional Logic

Translating English Sentences

EXAMPLE 1 How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Translating English Sentences

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> "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Extra Examples

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p, this would not be useful when analyzing its meaning or reasoning with it.

Translating English Sentences

EXAMPLE 1 How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Extra Examples

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p, this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman," respectively. Noting that "only if" is one way a conditional statement can be expressed, this sentence can be represented as

$$a \to (c \vee \neg f).$$

◀

Translating English Sentences

EXAMPLE 2 How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Translating English Sentences

EXAMPLE 2 How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Let q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old," respectively. Then the sentence can be translated to

$$(r \land \neg s) \rightarrow \neg q.$$

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs. ◀

Logic Circuits

A **logic circuit** (or **digital circuit**) receives input signals $p_1, p_2, \ldots, p_n$, each a bit [either 0 (off) or 1 (on)], and produces output signals $s_1, s_2, \ldots, s_n$, each a bit. In this section we will restrict our attention to logic circuits with a single output signal; in general, digital circuits may have multiple outputs.

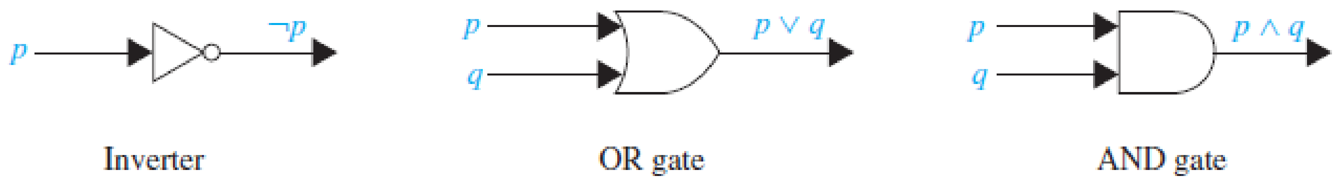

FIGURE 1 **Basic logic gates.**

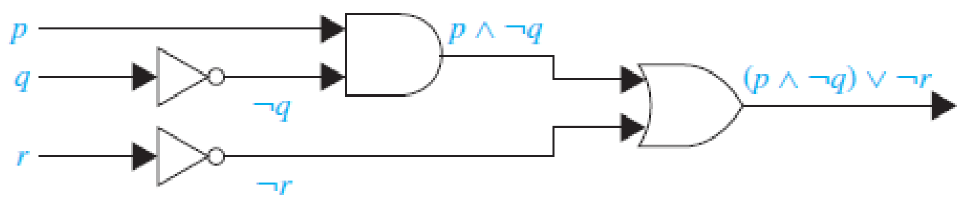

FIGURE 2 **A combinatorial circuit.**

Logic Circuits

EXAMPLE 9 Determine the output for the combinatorial circuit in Figure 2.

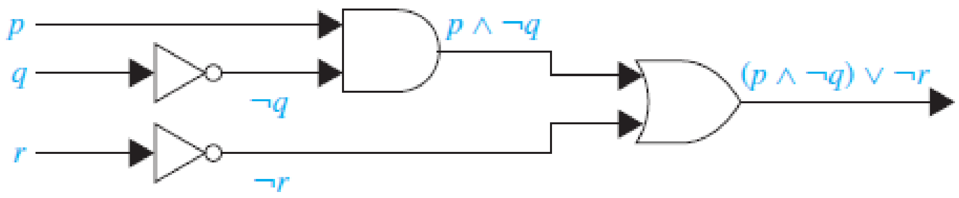

FIGURE 2 **A combinatorial circuit.**

Logic Circuits

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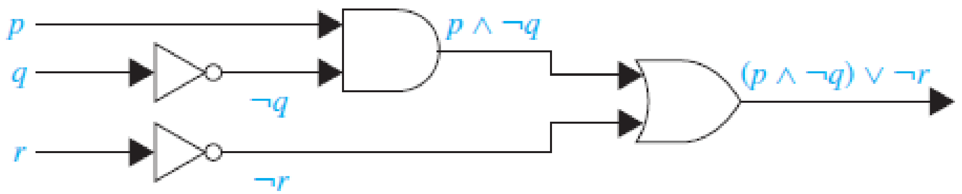

FIGURE 2 A combinatorial circuit.

Solution: In Figure 2 we display the output of each logic gate in the circuit. We see that the AND gate takes input of p and $\neg q$, the output of the inverter with input q, and produces $p \wedge \neg q$. Next, we note that the OR gate takes input $p \wedge \neg q$ and $\neg r$, the output of the inverter with input r, and produces the final output $(p \wedge \neg q) \vee \neg r$.

1.3 Logical Identities

Section 1.3 Summary

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences.
  - Showing Logical Equivalence.
- Logical Implication (not in Rosen)
- Normal Forms
  - DNF and CNF.
- Propositional Satisfiability
  - Sudoku Example.

Tautologies, Contradictions, and Contingencies

A *tautology*, **T**, is a proposition which is always true.

- Example: $p \vee \neg p$.

A *contradiction*, **F**, is a proposition which is always false.

- Example: $p \wedge \neg p$.

A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Another Look at Implication

p	***q***	$\boldsymbol{p \rightarrow q}$
0	0	1
0	1	1
1	0	0
1	1	1

CRUX: But if *p* is true, *q* cannot possibly be false.

Another Look at Implication

p	***q***	***p → q***
0	0	1
0	1	1
1	0	0
1	1	1

CRUX: But if *p* is true, *q* cannot possibly be false.

Another way of looking at this is:
"If not *q*, then not *p*."

Another Look at Implication

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

CRUX: But if p is true, q cannot possibly be false.

Another way of looking at this is:
"If not q, then not p."

This suggests an equivalence of propositions.

Another Look at Implication

“If not q, then not p.”

p	$\neg p$	q	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
0	1	0	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	0	1	1

Identical Truth Tables

Definition

- **Logical Equivalence**: Two statements, s_1 and s_2 are said to be logically equivalent, and we write $s_1 \Leftrightarrow s_2$, when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

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- In other words, their truth tables are identical.
- We write $s_1 \not\Leftrightarrow s_2$ to indicate that s_1 is not logically equivalent to s_2.
- The symbol, $\equiv$ is also used instead of $\Leftrightarrow$.

Logical Equivalence

p	$\neg p$	q	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
0	1	0	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	1	0	1	1	1	1

Clearly, $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ and $(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$

Logical Equivalence

p	$\neg p$	q	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	1	0	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	1	0	1	1	1	1

Clearly, $(p \to q) \Leftrightarrow (\neg q \to \neg p)$ and $(q \to p) \Leftrightarrow (\neg p \to \neg q)$

Terminology:
$(p \to q)$ is called material implication,
$(q \to p)$ is called the **CONVERSE** of material implication,
$(\neg q \to \neg p)$ is called the **CONTRAPOSITIVE** of material implication, and
$(\neg p \to \neg q)$ is called the **INVERSE** of material implication.

Logical Equivalence

- Logical equivalence is the foundation of the formal axiomatic system of propositional logic.
- Let us derive our first identity:
  - $p \rightarrow q$ means $p \therefore q$.
  - If the argument is valid and p is true, the q is also true.
  - If the argument is valid and p is false, then we can conclude nothing about q.
  - Thus, $p \rightarrow q$ is the same as $\neg p \vee q$.
- This is known as **implication reduction**.

Logical Equivalence

p	$\neg p$	q	$\neg q$	$p \rightarrow q$	$\neg p \vee q$
0	1	0	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	0	1	1

- Implication reduction will turn out to be our most useful identity.

Logical Equivalence

- The goal of the following propositional identities is to be able to have system algebra where one can simplify expressions
  - *i.e.,* using the Distributive and associative laws,

$$3(2a + 3a + 2) + 7b$$
$$= 3(2a + 3a + 2) + 7b$$
$$= 6a + 9a + 6 + 7b$$
$$= 15a + 7b + 6$$

- In our case, we deal with logical expressions.
- Next, we introduce De Morgan's Laws.

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

Show that this truth table shows that De Morgan's Second Law holds.

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Show that this truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Propositional Identities

This forms the basis of identities (Rosen).

Table 6 contains some important equivalences. In these equivalences, **T** denotes the compound proposition that is always true and **F** denotes the compound proposition that is always

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Note that many identities appear in pairs – DUALITY

Principle of Duality (Grimaldi)

Definition 2.3 Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the *dual* of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

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If p is any primitive statement, then p^d is the same as p—that is, the dual of a primitive statement is simply the same primitive statement. And $(\neg p)^d$ is the same as $\neg p$. The statements $p \vee \neg p$ and $p \wedge \neg p$ are duals of each other whenever p is primitive—and so are the statements $p \vee T_0$ and $p \wedge F_0$.

Given the primitive statements p, q, r and the compound statement

$$s: \quad (p \wedge \neg q) \vee (r \wedge T_0),$$

we find that the dual of s is

$$s^d: \quad (p \vee \neg q) \wedge (r \vee F_0).$$

(Note that $\neg q$ is unchanged as we go from s to s^d.)

We now state and use a theorem without proving it. However, in Chapter 15 we shall justify the result that appears here.

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(Note that $\neg q$ is unchanged as we go from s to s^d.)

We now state and use a theorem without proving it. However, in Chapter 15 we shall justify the result that appears here.

THEOREM 2.1 (*The Principle of Duality*) Let s and t be statements as described in Definition 2.3. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Principle of Duality

- This is the case for De Morgan's Laws.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

- If you prove one law, you automatically prove its dual.

Equivalence Proofs[1]

Example: Show that $\neg\left(p \vee \left(\neg p \wedge q\right)\right)$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg\left(p \vee \left(\neg p \wedge q\right)\right)$

Equivalence Proofs [1]

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is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg\left(p \vee \left(\neg p \wedge q\right)\right) \equiv \neg p \wedge \neg\left(\neg p \wedge q\right)$ by the second De Morgan law

$\equiv \neg p \wedge \left[\neg\left(\neg p\right) \vee \neg q\right]$ by the first De Morgan law

$\equiv \neg p \wedge \left(p \vee \neg q\right)$ by the double negation law

$\equiv \left(\neg p \wedge p\right) \vee \left(\neg p \wedge \neg q\right)$ by the second distributive law

$\equiv F \vee \left(\neg p \wedge \neg q\right)$ because $\neg p \wedge p \equiv F$

$\equiv \left(\neg p \wedge \neg q\right) \vee F$ by the commutative law for disjunction

$\equiv \left(\neg p \wedge \neg q\right)$ By the identity law for **F**

Equivalence Proofs 2

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$(p \wedge q) \rightarrow (p \vee q)$

Equivalence Proofs [2]

Example: Show that $(p \wedge q) \to (p \vee q)$

is a tautology.

Solution:

$(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$	by truth table for $\to$
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	by the first De Morgan law
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	by associative and commutative laws laws for disjunction
$\equiv T \vee T$	by truth tables
$\equiv T$	by the domination law

Exercise (Grimaldi)

Homework: Prove that XOR is the dual of *iff.*

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Summarizing from Grimaldi p. 60: Any logic law (stating the tautological equivalence of two statements) that involves only $\vee$ and $\wedge$ (and possibly T_0 and F_0) has a partner in which $\vee$ and $\wedge$ are interchanged (and T_0 and F_0 are interchanged). For example, $p \vee \neg p \iff T_0$ is dual to $p \wedge \neg p \iff F_0$.

In dualizing a statement, note carefully:

- If a statement s contains $\rightarrow$, $\longleftrightarrow$, or $\veebar$, those connectives can be reexpressed in terms of $\vee$ and $\wedge$, and then its dual s^{d} is defined by the prescription above.
- p is not interchanged with $\neg p$ (even though T_0 is interchanged with F_0).
- The $\iff$ in the logical law is not replaced by $\veebar$ (although $\veebar$ is the dual of $\longleftrightarrow$). The reason for this is that $s_1 \iff s_2$ is equivalent to $(s_1 \longleftrightarrow s_2) \iff T_0$, and the dual of the latter is ${s_1}^{\mathrm{d}} \veebar {s_2}^{\mathrm{d}} \iff F_0$, not ${s_1}^{\mathrm{d}} \veebar {s_2}^{\mathrm{d}} \iff T_0$.

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A **sketch of the proof of the duality principle** can be extracted from Sec. 15.4 of Grimaldi: All the logic laws follow from a list of 8 laws, which is (collectively) symmetric under the interchange of $\wedge$ with $\vee$ and T_0 with F_0. (A certain amount of notational translation is necessary to relate Sec. 15.4 to Chapter 2.)

Exercise (Grimaldi)

Homework: Prove that XOR is the dual of *iff.*

Now let's see why XOR is dual to IFF. The first step is to get rid of all connectives except $\vee$ and $\wedge$: $p \longleftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, which in turn should be rewritten as

$$(\neg p \vee q) \wedge (\neg q \vee p). \tag{27}$$

The dual of this statement is

$$(\neg p \wedge q) \vee (\neg q \wedge p). \tag{28}$$

Common sense indicates that (28) means the same thing as $p \veebar q$. The equivalence can be formally demonstrated by applying the distributive law and some other logic laws (in particular, "Inverse" and "Identity" from Grimaldi p. 59):

$$\begin{aligned}
(28) \iff & [\neg p \vee (\neg q \wedge p)] \wedge [q \vee (\neg q \wedge p)] \\
\iff & (\neg p \vee \neg q) \wedge (\neg p \vee p) \wedge (q \vee \neg q) \wedge (q \vee p) \\
\iff & (\neg p \vee \neg q) \wedge T_0 \wedge T_0 \wedge (p \vee q) \\
\iff & (p \vee q) \wedge (\neg p \vee \neg q) \\
\iff & (p \vee q) \wedge \neg(p \wedge q),
\end{aligned}$$

Logical Implication

Logical Implication

- Recall the relationship between implication and the biconditional:

  $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

Question

- Recall the relationship between implication and the biconditional:

  $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

- Is there a similar situation for logical equivalence?

  Is their a “logical implication, $\Rightarrow$?

Logical Implication and Valid Arguments

- **Recall Tautology**: A compound statement that is true for all truth value assignments for its component statements, denoted $\boldsymbol{T}$.

- **Logical Implication**: If p and q are arbitrary statements such that $p \to q \Leftrightarrow \boldsymbol{T}$, then we say that p logically implies q, denoted $p \Rightarrow q$.

Logical Implication and Valid Arguments

EXAMPLE 2.20

Let us now consider the truth table in Table 2.15. The results in the last column of this table show that for any primitive statements p, r, and s, the implication

$$[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$$

Table 2.15

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[(p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0			
0	0	1	0			
0	1	0	0			
0	1	1	0			
1	0	0	0			
1	0	1	0			
1	1	0	1			
1	1	1	1			

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Table 2.15

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[(p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

is a tautology. Consequently, for premises

$$p_1: \quad p \qquad p_2: \quad (p \wedge r) \rightarrow s$$

and conclusion q: $(r \rightarrow s)$, we know that $(p_1 \wedge p_2) \rightarrow q$ is a valid argument, and we may say that the truth of the conclusion q is *deduced* or *inferred* from the truth of the premises p_1, p_2.

Logical Implication and Valid Arguments

- We write $\not\Rightarrow$ if $p \to q$ is not a tautology.

Logical Implication and Valid Arguments

- We write $\not\Rightarrow$ if $p \rightarrow q$ is not a tautology.
- If $p \Rightarrow q$ and $q \Rightarrow p$, then $p \Leftrightarrow q$.

Logical Implication and Valid Arguments

- We write $\not\Rightarrow$ if $p \to q$ is not a tautology.
- If $p \Rightarrow q$ and $q \Rightarrow p$, then $p \Leftrightarrow q$.
- We have introduced the notion of a valid argument – for now, that the statement

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$$

is a tautology.

NOTE: Vacuous Proof

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

If the hypothesis is always false, then $p \to q$ is always true.

NOTE: Vacuous Proof

PROPOSITION: If $(0 = 1)$, then $(1 = 1)$

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PROOF:

1. The hypothesis, (0 = 1), is always false.

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2. Hence, the proposition is vacuously true.

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PROOF:

1. The hypothesis, (0 = 1), is always false.
2. Hence, the proposition is vacuously true.

QED

Homework: Think of another immediate result, like vacuous proof, from the semantics of logical implication.

Let us examine the truth table for the compound statement "Margaret Mitchell wrote *Gone with the Wind*, and if $2 + 3 \neq 5$, then combinatorics is a required course for sophomores." In symbolic notation this statement is written as

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p	$\neg p$	q	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
0	1	0	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	1	0	1	1	1	1

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1		
0	0	1	0		
0	1	0	1		
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1	0	0	1		
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0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	1	0	1	1	1	1

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	
0	0	1	0	1	
0	1	0	1	0	
0	1	1	0	1	
1	0	0	1	1	
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0	1	0	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	1	0	1	1	1	1

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
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Example: Logic Puzzle (OMIT SECTION)

- An island has two kinds of inhabitants, ***knights*, who always tell the truth, and *knaves*, who always lie.**
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: **What are the types of A and B?**

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- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution: Let p and q be the statements that **A is a knight** and **B is a knight**, respectively. Then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight, and therefore, $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Normal Forms

Normal Forms

- A system in which words (expressions) of a formal language can be transformed according to a finite set of *rewrite rules* is called a *reduction system*.
- Term rewriting systems are reduction systems in which rewrite rules apply to terms.
- In mathematics, computer science and logic, rewriting covers a wide range of (potentially non-deterministic) methods of replacing subterms of a formula with other terms. In their most basic form, **they consist of a set of objects, plus relations on how to transform those objects**.
- In abstract rewriting, an object is in **normal form** if it cannot be rewritten any further. Depending on the rewriting system and the object, several normal forms may exist, or none at all.
- Stated formally, if $(A,\rightarrow)$ is an abstract rewriting system, some $x \in A$ is in **normal form** if no $y \in A$ exists such that $x \rightarrow y$.

CNF and DNF

Conjunctive Normal Form (CNF):

- In Boolean logic, a formula is in **conjunctive normal form (CNF)** if it is a conjunction of one or more clauses, where a clause is a disjunction of literals; otherwise put, it is **an AND of ORs**. It is useful in automated theorem proving. It is similar to the product of sums form used in circuit theory.
- Examples:
  - $\neg A \wedge (B \vee C)$
  - $(A \vee B) \wedge (\neg B \vee C \vee \neg D) \wedge (D \vee \neg E)$
  - $A \vee B$
  - $A \wedge B$

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CNF and DNF

Disjunctive Normal Form (DNF):

– In Boolean logic, a **disjunctive normal form** (**DNF**) is a standardization (or normalization) of a logical formula which is a disjunction of conjunctive clauses; it can also be described as an OR of ANDs, a sum of products, or (in philosophical logic) a *cluster concept*. As a normal form, it is useful in automated theorem proving.

– Examples:
- $(A \wedge \neg B \wedge \neg C) \vee (\neg D \wedge E \wedge F)$
- $(A \wedge B) \vee C$
- $A \wedge B$
- A

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Propositional Satisfiability (OMIT)

Sample Even Exercises

16. Let p, q, and r be the propositions

p: You get an A on the final exam.
q: You do every exercise in this book.
r: You get an A in this class.

Write these propositions using p, q, and r and logical connectives (including negations).

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final.

d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Sample Even Exercises

16. Let p, q, and r be the propositions

p: You get an A on the final exam.
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e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

16. **a)** $r \wedge \neg q$ **b)** $p \wedge q \wedge r$ **c)** $r \rightarrow p$ **d)** $p \wedge \neg q \wedge r$ **e)** $(p \wedge q) \rightarrow r$ **f)** $r \leftrightarrow (q \vee p)$

Sample Even Exercises

8. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone B has the most RAM of these three smartphones.

b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.

c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.

d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.

e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

Sample Even Exercises

8. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone B has the most RAM of these three smartphones.

b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.

c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.

d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.

e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

8. a) True, because $288 > 256$ and $288 > 128$.

b) True, because C has 5 MP resolution compared to B's 4 MP resolution. Note that only one of these conditions needs to be met because of the word *or*.

c) False, because its resolution is not higher (all of the statements would have to be true for the conjunction to be true).

d) False, because the hypothesis of this conditional statement is true and the conclusion is false.

e) False, because the first part of this biconditional statement is false and the second part is true.

Sample Even Exercises

12. Show that each of these conditional statements is a tautology by using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$
b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
c) $[p \wedge (p \rightarrow q)] \rightarrow q$
d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

Sample Even Exercises

12. We construct a truth table for each conditional statement and note that the relevant column contains only T's. For part (a) we have the following table.

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

For part (b) we have the following table.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Sample Even Exercises

12. For part (c) we have the following table.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

For part (d) we have the following table. We have omitted some intermediate steps to make the table fit.

p	q	r	$(p \vee q) \wedge (p \to r) \wedge (p \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (p \to r)] \to r$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Sample Even Exercises

16. Show that each conditional statement in Exercise 12 is a tautology by applying a chain of logical identities as in Example 8. (Do not use truth tables.)

Sample Even Exercises

16. The solutions provided here use the conditional-disjunction equivalence as the initial step, but other approaches can be equally effective. Relevant equivalences are listed at each step, although commutativity and associativity are frequently used without comment.

a)
$$
\begin{aligned}
[\neg p \wedge (p \vee q)] \to q &\equiv \neg[\neg p \wedge (p \vee q)] \vee q && \text{by the conditional-disjunction equivalence} \\
&\equiv p \vee \neg(p \vee q) \vee q && \text{by a De Morgan's law} \\
&\equiv (p \vee q) \vee \neg(p \vee q) && \text{by commutativity and associativity} \\
&\equiv \mathbf{T} && \text{by a negation law}
\end{aligned}
$$

b) $[(p \to q) \wedge (q \to r)] \to (p \to r)$
$$
\begin{aligned}
&\equiv \neg[(p \to q) \wedge (q \to r)] \vee (p \to r) && \text{by the conditional-disjunction equivalence} \\
&\equiv \neg(p \to q) \vee \neg(q \to r) \vee (p \to r) && \text{by a De Morgan's law} \\
&\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee \neg p \vee r && \text{by the negation of conditionals and} \\
& && \quad \text{the conditional-disjunction equivalences} \\
&\equiv [\neg p \vee (p \wedge \neg q)] \vee [r \vee (q \wedge \neg r)] && \text{by associativity} \\
&\equiv [(\neg p \vee p) \wedge (\neg p \vee \neg q)] \vee [(r \vee q) \wedge (r \vee \neg r)] && \text{by a distributive law} \\
&\equiv [\mathbf{T} \wedge (\neg p \vee \neg q)] \vee [(r \vee q) \wedge \mathbf{T}] && \text{by a negation law} \\
&\equiv (\neg p \vee \neg q) \vee (r \vee q) && \text{by an identity law} \\
&\equiv (\neg p \vee r) \vee (\neg q \vee q) && \text{by associativity} \\
&\equiv (\neg p \vee r) \vee \mathbf{T} && \text{by a negation law} \\
&\equiv \mathbf{T} && \text{by a domination law}
\end{aligned}
$$

Sample Even Exercises

16. **c)** $[p \wedge (p \to q)] \to q \equiv \neg[p \wedge (p \to q)] \vee q$ by the conditional-disjunction equivalence

$\equiv \neg p \vee \neg(p \to q) \vee q$ by a De Morgan's law

$\equiv (\neg p \vee q) \vee \neg(p \to q)$ by commutativity and associativity

$\equiv (p \to q) \vee \neg(p \to q)$ by the conditional-disjunction equivalence

$\equiv \mathbf{T}$ by a negation law

d) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

$\equiv \neg[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \vee r$ by the conditional-disjunction equivalence

$\equiv \neg(p \vee q) \vee \neg(p \to r) \vee \neg(q \to r) \vee r$ by a De Morgan's law

$\equiv \neg(p \vee q) \vee [(p \wedge \neg r) \vee r] \vee [(q \wedge \neg r) \vee r]$ by an idempotent law, commutativity, and associativity

$\equiv \neg(p \vee q) \vee [(p \vee r) \wedge (\neg r \vee r)] \vee [(q \vee r) \wedge (\neg r \vee r)]$ by a distributive law

$\equiv \neg(p \vee q) \vee [(p \vee r) \wedge \mathbf{T}] \vee [(q \vee r) \wedge \mathbf{T}]$ by a negation law

$\equiv \neg(p \vee q) \vee (p \vee r) \vee (q \vee r)$ by an identity law

$\equiv \neg(p \vee q) \vee (p \vee q) \vee r$ by associativity and an idempotent law

$\equiv \mathbf{T} \vee r$ by a negation law

$\equiv \mathbf{T}$ by a domination law

Sample Even Exercises

36. Construct a truth table for each of these compound propositions.

a) $p \oplus p$

b) $p \oplus \neg p$

c) $p \oplus \neg q$

d) $\neg p \oplus \neg q$

e) $(p \oplus q) \vee (p \oplus \neg q)$

f) $(p \oplus q) \wedge (p \oplus \neg q)$

Sample Even Exercises

36. For parts (a) and (b) we have the following table (column two for part (a), column four for part (b)).

p	$p \oplus p$	$\neg p$	$p \oplus \neg p$
T	F	F	T
F	F	T	T

For parts (c) and (d) we have the following table (columns five and six).

p	q	$\neg p$	$\neg q$	$p \oplus \neg q$	$\neg p \oplus \neg q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

For parts (e) and (f) we have the following table (columns five and six). This time we have omitted the column explicitly showing the negation of q. Note that the first is a tautology and the second is a contradiction (see definitions in Section 1.3).

p	q	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \vee (p \oplus \neg q)$	$(p \oplus q) \wedge (p \oplus \neg q)$
T	T	F	T	T	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	F	T	T	F

Presentation

Terminated