MATH 240 Lecture 5.9
Markov Chains (Markov Matrices)

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October 02, 2024

1 Definitions

1.1 Probability Vector

A probability vector is a vector of probabilitys that sum to 1.
e.g. $\begin{pmatrix} 0.2 & 0.3 & 0.5 \end{pmatrix}$

1.2 Markov Matrix and Markov Chain

A Markov Matrix is a square matrix whose columns are probability vectors.
e.g.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

They are also called stochastic matrices.
Let x_0 be a probability vector and M be a Markov Matrix.
Let $x_1 = Mx_0$, $x_2 = Mx_2$ ... $x_k = Mx_k$ for $k \geq 1$.
Then x_k is a probability vector and the sequence $x_0, x_1, x_2, \ldots, x_k$ is called a **Markov Chain**.

Example

$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} \\ \frac{7}{12} \end{bmatrix}.$$

Example 2

In the land of Adanac, there are 3 political parties, C, L and N. Elections are held every 6 years. Voting preferences are as follows:

If $x_k = \begin{bmatrix} C \\ L \\ N \end{bmatrix}$ is the current vote (state), then,

$$x_{k+1} = \begin{bmatrix} 0.9C + 0.2L \\ 0.1C + 0.7L + 0.2N \\ 0.1L + 0.8N \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & 0.2 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} C \\ L \\ N \end{bmatrix} = \begin{bmatrix} 0.9C + 0.2L \\ 0.1C + 0.7L + 0.2N \\ 0.1L + 0.8N \end{bmatrix}$$

$$x_{k+1} = M \cdot x_k$$

This is a Linear Transformation.

If the current vote is $x_0 = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}$, what is x_1, x_2, $\ldots$?

$$x_1 = M \cdot x_0 = \begin{bmatrix} 0.9 & 0.2 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.36 + 0.8 = 0.44 \\ 0.04 + 0.28 + 0.04 = 0.36 \\ 0.04 + 0.16 = 0.20 \end{bmatrix}$$

If you continue to do this repeatedly,

$$x_{49} = \begin{bmatrix} 0.5714018 \\ 0.2857221 \\ 0.1428760 \end{bmatrix}$$

$$x_{50} = \begin{bmatrix} 0.5714061 \\ 0.2857208 \\ 0.1428730 \end{bmatrix}$$

$$\lim_{k \to \infty} x_k = \begin{bmatrix} \frac{4}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix}$$

Let M be a Markov Matrix.

A vector q is called a **steady state vector** if $Mq = q$. (i.e. the percentage of votes is not changing)

M is said to be regular if M^k has all non-zero positive entries for some $k \geq 1$.

If a node returns 100% of the votes, then it is a **absorbing state**.

Thm.

If M is regular, then M has a unique steady state probability vector q such that $Mq = q$. Moreover, if x_0 is **any** initial probability vector, then $\lim_{k \to \infty} x_k = q$. (i.e. the Markov Chain Sequence converges)

Assumption: The probabilities (m_{ij}) in the Markov Matrix **do not change**