MATH 240 Lecture 1.4 - The Matrix Vector Product Ax

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September 11, 2024

1 Matrices

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

1.1 Example

$$2 \times 3 \text{ matrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \tag{1}$$

1.2 Definition

Let A be an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Then the matrix product Av_1 is defined as

$$A \cdot x = x_1 \cdot v_1 + x_2 \cdot v_2 + \ldots + x_n \cdot v_n \in \mathbb{R}^m$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 6 \\ 14 \end{bmatrix} \tag{4}$$

Through some algebraic reasoning, we can show that this matrix product follows:

Letting v be the vector $\begin{pmatrix} x & y & z \end{pmatrix}$, we have

$$\begin{aligned}
\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \times v &= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} && (5) \\
&= x \cdot \begin{bmatrix} a \\ d \end{bmatrix} + y \cdot \begin{bmatrix} b \\ e \end{bmatrix} + z \cdot \begin{bmatrix} c \\ f \end{bmatrix} && (6) \\
&= \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix} && (7) \\
&= \begin{bmatrix} R_1 \cdot v \\ R_2 \cdot v \end{bmatrix} && (8)
\end{aligned}$$

Where $\cdot$ denotes the matrix dot product.

1.3 The Matrix Dot Product

The matrix dot product $u \cdot v$ is defined as

$$u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots + u_n \cdot v_n \tag{9}$$

1.3.1 Example

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 1 \cdot 5 = 9 \\ 1 \cdot 4 + 2 \cdot 5 = 14 \end{bmatrix} \tag{10}$$

1.4 Matrix Scalar Multiplication

If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, then

$$A \cdot x = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix} \tag{11}$$

1.5 Permutation Matrices

1.5.1 Definition

A permutation matrix is a square binary matrix that represents a permutation of elements. Each row and column of a permutation matrix contains exactly one 1, with all other entries being 0.

1.5.2 Basis Matrix in $\mathbb{R}^3$

This is the identity matrix in $\mathbb{R}^3$ (denoted by I_3). When you multiply any $v \in \mathbb{R}^3$ by this matrix, you get the same vector back.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{13}$$

This is the case because the first row of the matrix, corresponding to the first entry of the vector, is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. Taking the dot product of a vector with the first row of the matrix returns the first entry of the vector.

Following, the second row is $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, and the third row is $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

Therefore, the first entry of the vector is x, the second entry is y, and the third entry is z.

1.5.3 Permuting Rows of a Vector (Example)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \tag{14}$$

If you take the vector $\begin{pmatrix} x & y & z \end{pmatrix}$ and multiply it by the first row of the matrix according to the matrix dot product, you get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \cdot x + 0 \cdot y + 1 \cdot z \\ \vdots \end{bmatrix} = \begin{bmatrix} z \\ \vdots \end{bmatrix} \tag{15}$$

Following, you can see how the second row of the matrix leaves only the y and the third row leaves only the x components.