Question 42

Given $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, show that the statement is true if and only if p, q and r all have the same truth value.

p, q, r have the same truth value

1. Let $p = q = r$
2. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$
3. $(p \vee \neg p) \wedge (p \vee \neg p) \wedge (p \vee \neg p)$
4. $(p \vee \neg p) \equiv \mathbb{T}$
5. Apply the Idempotent Law, $p \wedge p \equiv p$
6. $\mathbb{T} \wedge \mathbb{T} \wedge \mathbb{T} \equiv \mathbb{T}$
7. $\therefore \mathbb{T}$.

p, q, r do not have the same truth value

1. Let $p = q = \neg r$
   It follows that $r = \neg p$
2. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$
3. $(p \vee \neg p) \wedge (p \vee \neg\neg p) \wedge (\neg p \vee \neg p)$
4. Apply the Double Negation Law
   $(p \vee \neg p) \wedge (p \vee p) \wedge (\neg p \vee \neg p)$
5. Apply the Idempotent Law
   $(p \vee \neg p) \wedge p \wedge \neg p$
6. $p \vee \neg p \equiv \mathbb{T}$
   $(\mathbb{T}) \wedge p \wedge \neg p$
7. $p \wedge \neg p \equiv \mathbb{F}$
   $\mathbb{T} \wedge \mathbb{F} \equiv \mathbb{F}$
8. $\therefore \mathbb{F}$