Dynamic Models, Quality Control and Filtered GPS Receiver Position

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Dynamic Models

Constant speed, straight line

$$
\begin{aligned}
x_k &= x_{k-1} + v_{x,k-1}\Delta t \\
y_k &= y_{k-1} + v_{y,k-1}\Delta t \\
v_{x,k} &= v_{x,k-1} \\
v_{y,k} &= v_{y,k-1}
\end{aligned}
$$

Constant speed, straight line again

$$x_k = x_{k-1} + v_{k-1} \sin \alpha_{k-1} \Delta t$$

$$y_k = y_{k-1} + v_{k-1} \cos \alpha_{k-1} \Delta t$$

$$\alpha_k = \alpha_{k-1}$$

$$v_k = v_{k-1}$$

Constant speed, circle

$$x_k = x_{k-1} + v_{k-1}\sin\alpha_{k-1}\Delta t + \tfrac{1}{2}a_{cr,k-1}\cos\alpha_{k-1}\Delta t^2$$

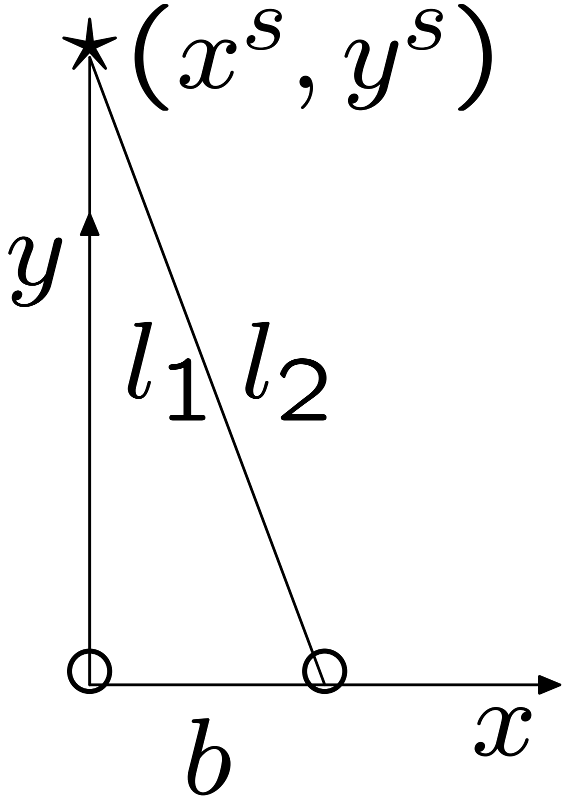
$$y_k = y_{k-1} + v_{k-1}\cos\alpha_{k-1}\Delta t - \tfrac{1}{2}a_{cr,k-1}\sin\alpha_{k-1}\Delta t^2$$

$$\alpha_k = \alpha_{k-1} + a_{cr,k-1}/v_{k-1}\Delta t$$

$$v_k = v_{k-1}$$

$$a_{cr,k} = a_{cr,k-1}$$

Tracking a GPS Satellite, 2D-Version



Observation equations

$$l_1 = (y^s)^0 + \Delta y^s$$

$$l_2 = \sqrt{b^2 + l^2} = \sqrt{b^2 + (l^2)^0}$$
$$+ \frac{b}{\sqrt{b^2 + (l^2)^0}} \Delta x^s + \frac{l}{\sqrt{b^2 + (l^2)^0}} \Delta y^s$$

or

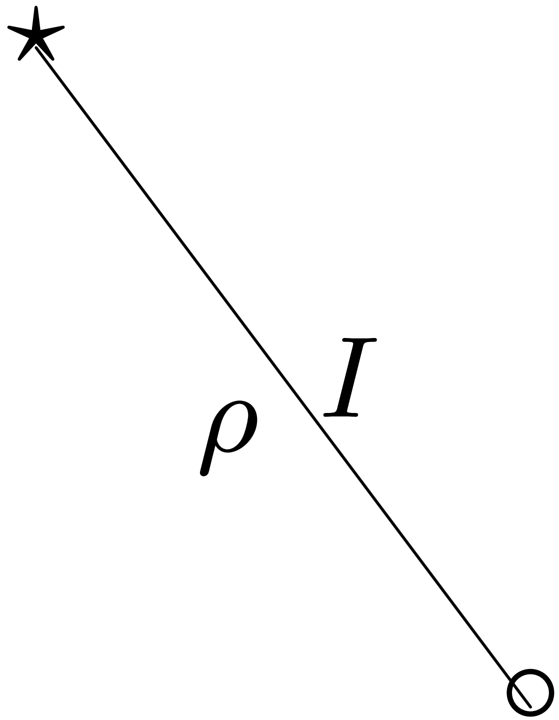
$$\Delta l_1 = l_1 - (y^s)^0 = \Delta y^s$$

$$\begin{aligned}\Delta l_2 &= l_2 - \sqrt{b^2 + (l^2)^0} \\ &= \frac{b}{\sqrt{b^2 + (l^2)^0}} \Delta x^s + \frac{l}{\sqrt{b^2 + (l^2)^0}} \Delta y^s\end{aligned}$$

or in matrix form

$$\begin{bmatrix} 0 & 1 \\ \frac{b}{\sqrt{b^2+(l^2)^0}} & \frac{l^0}{\sqrt{b^2+(l^2)^0}} \end{bmatrix} \begin{bmatrix} \Delta x^s \\ \Delta y^s \end{bmatrix} = \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \end{bmatrix}.$$

Tracking a GPS Satellite on L1 and L2



Observation equations

$$P_1 = \rho + I$$

$$P_2 = \rho + (f_1/f_2)^2 I$$

or in matrix form

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \alpha & 0 \end{bmatrix} \begin{bmatrix} \rho \\ \dot{\rho} \\ \ddot{\rho} \\ I \\ \dot{I} \end{bmatrix}.$$

Dynamical model

$$\begin{bmatrix} \rho_k \\ \dot{\rho}_k \\ \ddot{\rho}_k \\ I_k \\ \dot{I}_k \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & \frac{1}{2}(\Delta t)^2 & 0 & 0 \\ 0 & 1 & \Delta t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_{k-1} \\ \dot{\rho}_{k-1} \\ \ddot{\rho}_{k-1} \\ I_{k-1} \\ \dot{I}_{k-1} \end{bmatrix}.$$

Quality Control for Recursive Solution

Innovation

$$e_k = b_k - A_k \hat{x}_{k|k-1} \text{ with covariance matrix } \Sigma_{e,k}.$$

The overall model test for epoch k:

$$T = e_k^{\mathrm{T}} \Sigma_{e,k}^{-1} e_k \sim \chi^2.$$

Test for cycle slip: Let ∇ be the size of a (gross) error, and c_d and c_b denote (specified) vectors

$$\begin{aligned} d_{k,k-1} + c_d \nabla &= x_k - F_{k,k-1} x_{k-1} + \epsilon_k \\ b_k + c_b \nabla &= A_k x_k + e_k. \end{aligned}$$

Perform the tests

$$t_d = \frac{c_{d_k}^{\mathrm{T}} \Sigma_{d,k}^{-1} d_k}{\sqrt{c_{d_k}^{\mathrm{T}} \Sigma_{d,k}^{-1} c_{d_k}}} \sim N(0, 1)$$

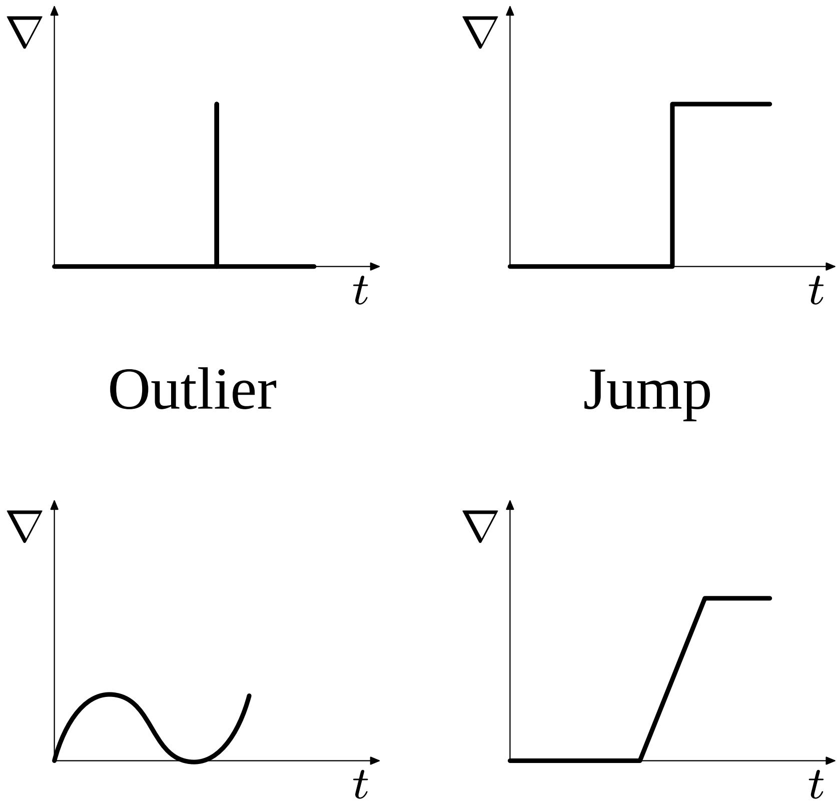
$$t_b = \frac{c_{b_k}^{\mathrm{T}} \Sigma_{b,k}^{-1} b_k}{\sqrt{c_{b_k}^{\mathrm{T}} \Sigma_{b,k}^{-1} c_{b_k}}} \sim N(0, 1).$$

If $\hat{\nabla}$ with variance σ_{∇}^2 and $\hat{x}_{k|k}$ have been identified the filter is corrected as follows

$$\hat{x}_{k|k} = \hat{x}_{k|k} - K_k c_b \hat{\nabla}$$

$$P_{k|k} = P_{k|k} + K_k c_b \sigma_{\nabla}^2 c_b^{\mathrm{T}} K_k^{\mathrm{T}}.$$

Error Types



Outlier   Jump

Periodic   Ramp

Filter Implementations

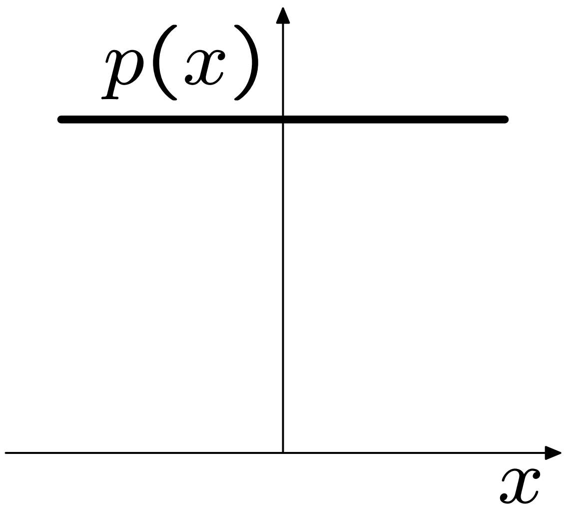
Kalman filter: P

- (Standard) Kalman Filter
- Square Root Filter
- U-D Decomposition

Bayes filter: P^{-1}

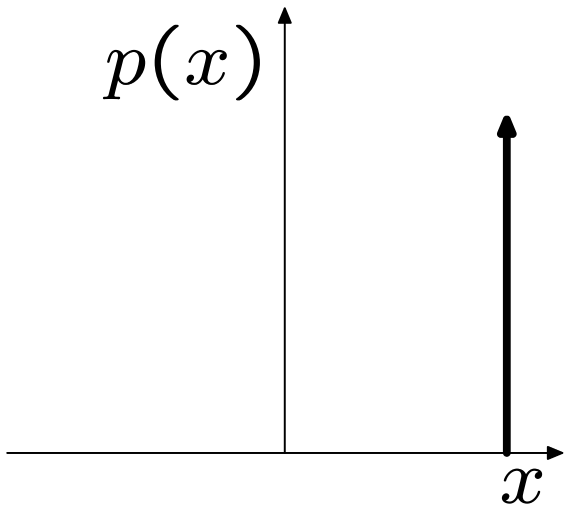
- Information Filter
- Square Root Information Filter (SRIF)

Zero knowledge



$P = \infty$ versus $P^{-1} = 0$

Perfect knowledge



$P = 0$ versus $P^{-1} = \infty$

Matlab file `abs_pos.m`

Computation of absolute receiver position:

1. Given navigation data
2. Computation of satellite positions at given time
3. Given pseudoranges
4. Computation of satellite clock offset, earth rotation correction of satellite coordinates, and tropospheric correction
5. Bayes filer, but first computation of preliminary position by means of Bancroft's algorithm
6. The state vector contains $(X, Y, Z, c\,dt)$
7. Kalman filtering

Output: omc, PDOP, change in state vector, and $P_{k|k}$

```
>> abs_pos
pos =
    596899.62
  −4847828.94
   4088206.16
       −5.49
```

omc for satellite 23: -5.52 m

PDOP: 1443.1

Change in position [m] $3.26 \quad -3.35 \quad 1.96 \quad -1.52$

P =

$$
\begin{matrix}
620597.38 & 390198.83 & -227845.91 & 176881.42 \\
390198.83 & 598697.74 & 234329.45 & -181914.73 \\
-227845.91 & 234329.45 & 863169.74 & 106224.12 \\
176881.42 & -181914.73 & 106224.12 & 917536.05
\end{matrix}
$$

omc for satellite 9: -13.17 m

PDOP: 1046.6

Change in position [m] 6.39 5.41 10.60 -2.98

P =

$$\begin{matrix} 560358.03 & 222028.39 & -393816.14 & 204863.17 \\ 222028.39 & 129215.63 & -229010.31 & -103797.95 \\ -393816.14 & -229010.31 & 405891.98 & 183318.87 \\ 204863.17 & -103797.95 & 183318.87 & 904538.26 \end{matrix}$$

omc for satellite 5: 1.94 m

PDOP: 311.7

Change in position [m] $-6.88 \quad -1.11 \quad 22.15 \quad -2.20$

P =

$$
\begin{matrix}
60701.49 & -23192.42 & 40935.96 & 234096.41 \\
-23192.42 & 8866.47 & -15643.22 & -89450.90 \\
40935.96 & -15643.22 & 27613.36 & 157882.97 \\
234096.41 & -89450.90 & 157882.97 & 902827.92
\end{matrix}
$$

omc for satellite 1: 3.76 m

PDOP: 4.8

Change in position [m] 17.97 $-$10.60 38.90 93.62

$$P = \begin{matrix} 3.32 & -3.97 & 4.87 & 17.99 \\ -3.97 & 7.81 & -6.35 & -26.15 \\ 4.87 & -6.35 & 11.97 & 34.96 \\ 17.99 & -26.15 & 34.96 & 120.04 \end{matrix}$$

omc for satellite 21: -87.87 m

PDOP: 4.1

Change in position [m] 53.67 -86.61 144.04 385.60

$$P = \begin{matrix} 2.86 & -2.98 & 3.51 & 14.20 \\ -2.98 & 5.71 & -3.45 & -18.09 \\ 3.51 & -3.45 & 7.95 & 23.80 \\ 14.20 & -18.09 & 23.80 & 89.05 \end{matrix}$$

omc for satellite 17: -1.66 m

PDOP: 2.9

Change in position [m] 14.91 -59.88 46.25 112.47

$$P = \begin{matrix} 1.81 & -2.26 & 0.87 & 6.82 \\ -2.26 & 5.21 & -1.63 & -13.00 \\ 0.87 & -1.63 & 1.29 & 5.19 \\ 6.82 & -13.00 & 5.19 & 37.06 \end{matrix}$$

Results from the Bayes Filter

Change in position [m] 14.91 $-$59.88 46.25 112.47

omc for satellite 23: 27.90 m

omc for satellite 9: $-$2.31 m

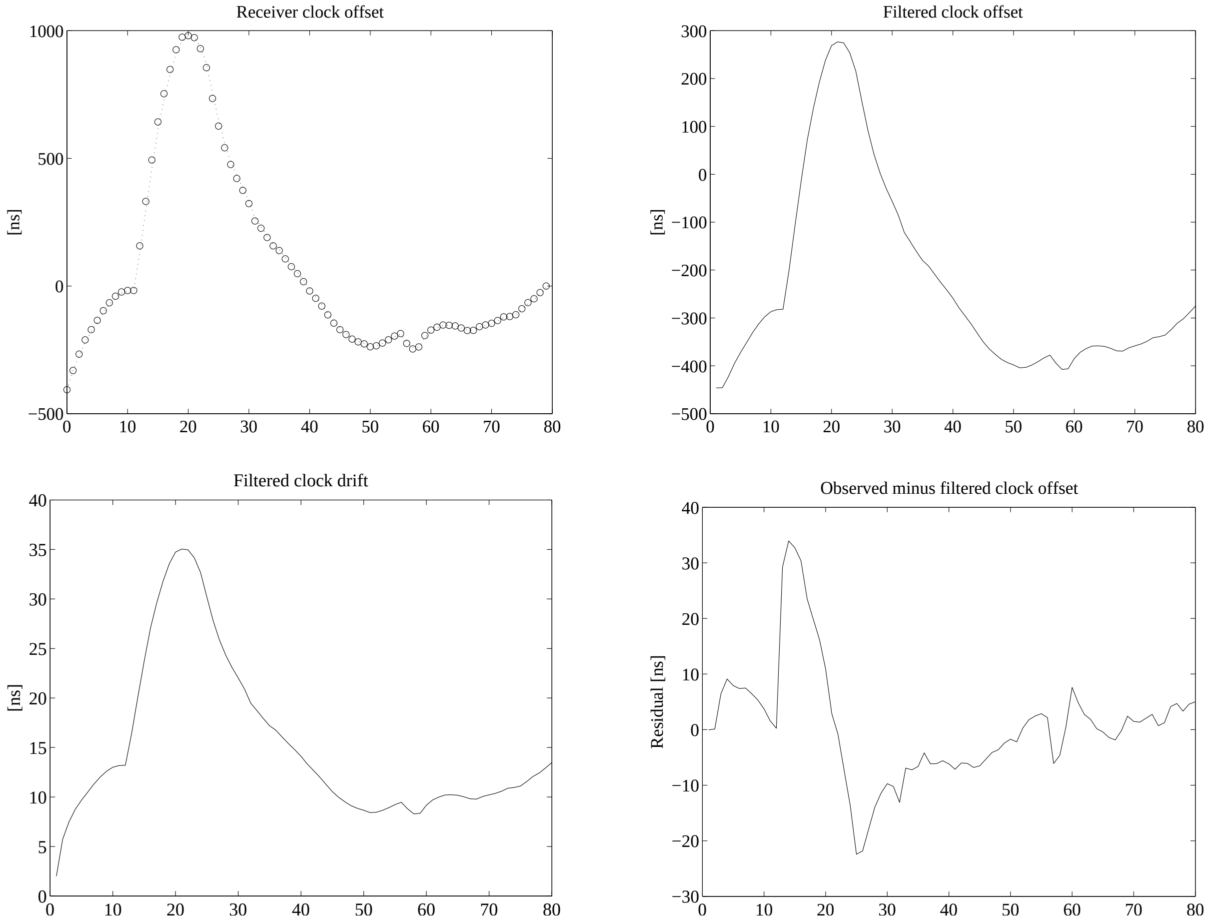
omc for satellite 5: 2.23 m

omc for satellite 1: 31.85 m

omc for satellite 21: $-$46.54 m

omc for satellite 17: $-$11.97 m

RMS error of absolute position: 26.18 m


Receiver clock offset
[ns]
Filtered clock offset
[ns]
Filtered clock drift
[ns]
Observed minus filtered clock offset
Residual [ns]