Exercises

1. *(Influence on the solution of changing weights)* Solve the least squares problem given by

   a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   b) Change C to $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This corresponds to not using the third observation.

   c) Show that the solution in b) could be obtained from $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   d) Find the solution when $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \infty \end{bmatrix}$.

   Sketch the three lines given by $A\boldsymbol{x} = \boldsymbol{b}$ in a).

   Next plot the solutions in a), b), and d). Show that they lie on a straight line.

   In all the above exercises the third observation was given weights 1, 0, and ∞. Any other weight for the third observation will produce a solution lying on this line. Range of $c_3 = [-1, \infty)$.

2. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$, $C = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, and $\sigma_0^2 = 1$. Define W as `W = chol(C)`. Compute $A_1 = WA$, $\boldsymbol{b}_1 = W\boldsymbol{b}$ and find $\boldsymbol{x}$.

   MATLAB has a function `sqrtm` that computes the square root of a matrix. Use this function for an alternative definition of W and repeat the computations.

3. Let be given the normals

$$\begin{bmatrix} A & B \\ B^{\mathrm{T}} & D \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}.$$

   Use the formula

$$\boldsymbol{x}_2 = (D - B^{\mathrm{T}} A^{-1} B)^{-1} (\boldsymbol{b}_2 - B^{\mathrm{T}} A^{-1} \boldsymbol{b}_1)$$

   to estimate $\boldsymbol{x}_2$ and next by substitution find $\boldsymbol{x}_1$.

4. We assume given the same the normals as in Exercise 3. We split them into

$$A\boldsymbol{x} = B\boldsymbol{y} + G\boldsymbol{z}$$

or

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 8 \\ 26 \end{bmatrix} \boldsymbol{z}.$$

We compute the changed weight matrix (also a projector) $P = I - B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$ and find the unknown $\hat{\boldsymbol{z}}$ from the reduced normals

$$G^{\mathrm{T}}PG\hat{\boldsymbol{z}} = G^{\mathrm{T}}P\boldsymbol{b}.$$

Finally the unknown $\hat{\boldsymbol{y}}$ can be computed from

$$\hat{\boldsymbol{y}} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(\boldsymbol{b} - G\hat{\boldsymbol{z}}).$$

5. Let be given

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix} \qquad \text{and} \qquad C = I$$

and a constraint described by

$$B = \begin{bmatrix} 1 & 4 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{d} = \begin{bmatrix} 20 \end{bmatrix}.$$

Find the least-squares solution via QR decomposition and GSVD. Furthermore find the solution to the unconstrained system and compute the increase in the sum of squared residuals when constraining the problem.

6. Solve the problem described in Exercise 5 by allocating a big weight to the constraint.

7. A least-squares problem is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \qquad \text{and} \qquad C = I.$$

Fint the solution of this problem by means of recursive least squares.

8. A least squares problem is described by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad C = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}.$$

Solve this problem by means of a Kalman filter.

9. The M-file k_dd3 is described from the bottom of page 488 to the middle of page 490. Try to change the diagonal entries of $\Sigma_{e,k}$ and see if this affects the result.

10. Run the M-file smoother with the calls smoother(1,1) and smoother(.1,1).

11. Run the M-file kalclock with the call kalclock('pta.96o','pta.nav',1)

12. Run the M-file rec_cloc.