

Geoid Undulations Computed from EGM 96

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Users of GPS repeatedly need to transform *ellipsoidal heights* h to *orthonormal heights* H which are heights identified as the elevation above mean sea level. These two quantities differ by the so-called *geoidal undulation* N . They are connected by the following relation

$$h = H + N. \quad (1)$$

The left side is known from the GPS positioning and in case we can compute N , the wanted height H can be determined.

This note describe how to compute N for any point on the surface of the Earth. The algorithms yield an accuracy of a few meters or better.

A general reference for the following material is Heiskanen & Moritz (1967), Chapters 1 and 2.

Let k denote the gravitational constant; let dv denote a volume element of the Earth and ρ its density; let l be the distance between the mass element ρdv and the attracted point Q ; then the gravitational potential V [m^2s^{-2}] of the Earth at point Q is given as a triple integral over the entire Earth

$$V = k \iiint_{\text{Earth}} \frac{\rho}{l} dv. \quad (2)$$

The actual potential is described via geopotential coefficients C_{nm} and S_{nm} of *degree* n and *order* m . These coefficients are determined empirically and they are coefficients in an orthonormal series expansion in $\cos m\lambda$, $\sin m\lambda$, and the *associated Legendre polynomials* P_{nm} :

$$V(\psi, \lambda, r) = \frac{kM}{r} \left(1 + \sum_{n=2}^{n_{\max}} \left(\frac{a}{r} \right)^n \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \psi) \right). \quad (3)$$

The total mass of the Earth is denoted M . The $\bar{}$ indicates sort of normalization to which we return later. The first term for $n = 0$ is nothing else than kM/r . There is no term with $n = 1$ if the origin is at the geocenter so the actual summation starts from $n = 2$.

The attracted point Q has geodetic coordinates (φ, λ, H) . Although Q can be any point outside the geoid, the present code is dedicated to small values of H . The series expansion in Legendre polynomials is based on spherical coordinates (ψ, λ, r) , so we need to convert the geodetical coordinates to spherical ones. This conversion is done via 3-D Cartesian coordinates (X, Y, Z) . We use the Matlab functions `frgeod` and `cart2sph`. The longitude λ is the same in both coordinate systems because of the rotational symmetry, while φ in general is different from ψ . The distance from Q to the origin is denoted by r .

The geopotential coefficients \bar{C}_{nm} and \bar{S}_{nm} in (3) correspond to an ellipsoid of revolution with semi-major axis $a = 6\,378\,136.46$ m and flattening $f = 1/298.25765$. These values correspond to a tide-free system.

Our Matlab implementation `geoidund.m` uses $n, m \leq 180$. The coefficients are read from the file `egm180.nor`. This file is downloaded from www.nima.mil/GandG/wgsegs/egm84.html. Today also files with coefficients $n, m = 360$ are available.

Our goal is to compute the *geoidal undulation* N at $Q = (\varphi, \lambda, H)$. We subtract the *normal gravity potential* $U + \Phi$ from the actual gravity potential $W = V + \Phi$, Φ being the potential due to the centrifugal force, and get the anomalous potential T :

$$T = W - (U + \Phi) = (V + \Phi) - (U + \Phi) = V - U.$$

The Normal Potential

The *normal gravitational potential* U may as well be described by a series expansion in associated Legendre polynomials. Because of the rotational symmetry there will be only zonal terms, and because of the symmetry with respect to the equatorial plane there will be only *even* zonal harmonics. The zonal harmonics of odd degree change sign for negative latitudes and must be absent. Accordingly, the series has the form

$$U = \frac{kM}{r} + A_2 \frac{P_2(\sin \psi)}{r^3} + A_4 \frac{P_4(\sin \psi)}{r^5} + \dots \quad (4)$$

Next we have to determine the coefficients A_2, A_4, \dots . The reference values A_n can be computed from the closed expression

$$A_n = (-1)^{1+n/2} \frac{3e^n}{(n+1)(n+3)} \left(1 - n/2 + \frac{5nA_2}{2e^2}\right), \quad n = 2, 4, 6, 8, \text{ and } 10 \quad (5)$$

where e is the ellipsoid eccentricity; A_2 is the value implied by the defined flattening of the reference ellipsoid. For $n = 2$ we recover the identity $A_2 = A_2$. For reasons of accuracy it is sufficient to include five terms.

The normalizing factors for A_n are $1/\sqrt{(n+1)}$. So we get

$$\bar{C}_n = -\frac{A_n}{\sqrt{n+1}}, \quad n = 2, 4, 6, 8, 10.$$

For EGM 96 we have $\bar{C}_2 = 108263 \times 10^{-8}$, $\bar{C}_4 = -237.1 \times 10^{-8}$, and $\bar{C}_6 = 0.6 \times 10^{-8}$.

The Normal Gravity

If we knew T we get from Bruns' formula the *height anomaly* ζ

$$\zeta = \frac{T}{\gamma}. \quad (6)$$

Here γ is the normal gravity and is computed by the Matlab function `gamma_h.m`:

$$\gamma(\varphi) = \frac{\gamma_e \cos^2 \varphi + (1-f)\gamma_p \sin^2 \varphi}{\sqrt{\cos^2 \varphi + (1-f)^2 \sin^2 \varphi}}$$

and γ at height h :

$$\gamma_h = \gamma(\varphi) \left(1 - 2(1+f+m-2f \sin^2 \varphi) \frac{h}{a} + 3\left(\frac{h}{a}\right)^2\right).$$

Finally for ζ we have

$$\zeta(\psi, \lambda, r) = \frac{kM}{\gamma(\varphi)r} \left(\sum_{n=2}^{n_{\max}} \left(\frac{a}{r}\right)^n \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \psi) \right). \quad (7)$$

The height anomaly ζ computed from this set of coefficients can only be considered a global approximation that neglects local irregularities. It is estimated that the computed value for ζ at most is off by 1–2 meters.

The *ellipsoidal height* h measured along the ellipsoidal normal is related to the geoid undulation N , the orthometric height H , the normal height H^* , and the height anomaly ζ as follows

$$h = H + N = H^* + \zeta. \quad (8)$$

To transform from height anomalies ζ to geoid undulations N , a zero degree undulation term of -0.53 m is applied: $N = -0.53 + \zeta$.

Recurrence Relations

Starting with $P_{00} = 1$, all associated Legendre polynomials P_{mm} up to the desired degree n_{\max} and order are first computed from

$$P_{mm}(t) = (2m - 1)\sqrt{1 - t^2} P_{m-1,m-1}(t) \quad (9)$$

where t and $\sqrt{1 - t^2}$ stand for $\sin(\psi)$ and $\cos(\psi)$, see Gradshteyn & Ryzhik (1965), formula 8.812. With these results the remaining values may be obtained from

$$P_{m+1,m}(t) = (2m + 1)t P_{mm}(t) \quad (10)$$

and from the recursion, see Gradshteyn & Ryzhik (1965), formula 8.731.2

$$P_{nm}(t) = \frac{2n - 1}{n - m} t P_{n-1,m}(t) - \frac{n + m - 1}{n - m} P_{n-2,m}(t) \quad (11)$$

for $n > m + 1$.

In order to make coefficients C_{nm} and S_{nm} more readily comparable in numerical work it is convenient to introduce a normalisation of P_{nm} defined as

$$\bar{P}_{nm}(t) = \sqrt{\frac{k(2n + 1)(n - m)!}{(n + m)!}} P_{nm}(t) \quad \text{with} \quad k = \begin{cases} 1 & \text{for } m = 0 \\ 2 & \text{for } m > 0. \end{cases}$$

Insertion into (11) yields

$$\begin{aligned} \sqrt{\frac{(n + m)!}{k(2n + 1)(n - m)!}} \bar{P}_{nm}(t) &= \frac{2n - 1}{n - m} \sqrt{\frac{(n + m - 1)!}{k(2n - 1)(n - m - 1)!}} t \bar{P}_{n-1,m}(t) \\ &\quad - \frac{(n + m - 1)}{n - m} \sqrt{\frac{(n + m - 2)!}{k(2n - 3)(n - m - 2)!}} \bar{P}_{n-2,m}(t) \end{aligned}$$

or

$$\begin{aligned}\bar{P}_{nm}(t) = & \sqrt{\frac{(2n-1)(2n+1)}{(n+m)(n-m)}} t \bar{P}_{n-1,m}(t) \\ & - \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n+m)(n-m)(2n-3)}} \bar{P}_{n-2,m}(t).\end{aligned}$$

We introduce the coefficients a_{nm} and b_{nm} :

$$a_{nm} = \sqrt{\frac{(2n-1)(2n+1)}{(n+m)(n-m)}} \quad \text{and} \quad b_{nm} = \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n+m)(n-m)(2n-3)}} \quad (12)$$

and get

$$\bar{P}_{nm}(t) = a_{nm} t \bar{P}_{n-1,m}(t) - b_{nm} \bar{P}_{n-2,m}(t). \quad (13)$$

This is the recurrence formula for the fully normalised associated Legendre functions with varying degree n and fixed order m .

The sectoral ($n = m$) $\bar{P}_{nm}(t)$ serve as seed values for the recursion in (13). They are computed using the initial values $\bar{P}_{00}(t) = 1$, $\bar{P}_{11}(t) = \sqrt{3} \sqrt{1-t^2}$. The higher degree and order values of $\bar{P}_{nm}(t)$ are then computed using the recursion

$$\bar{P}_{mm}(t) = \sqrt{1-t^2} \sqrt{\frac{2m+1}{2m}} \bar{P}_{m-1,m-1}(t) \quad \text{for } m > 1. \quad (14)$$

We quote the explicit associated Legendre functions and the normalised ones for $m = n = 0, 1, 2$:

$$\begin{aligned}P_{00}(t) &= (1-t^2)^{0/2} \frac{d^0}{dt^0} P_0(t) = 1 & \bar{P}_{00}(t) &= 1 \\ P_{11}(t) &= (1-t^2)^{1/2} \frac{d^1}{dt^1} P_1(t) = \sqrt{1-t^2} & \bar{P}_{11}(t) &= \sqrt{2 \cdot 3} P_{11}(t) = \sqrt{3} \sqrt{1-t^2} \\ P_{22}(t) &= (1-t^2)^{2/2} \frac{d^2}{dt^2} P_2(t) = 3(1-t^2) & \bar{P}_{22}(t) &= \sqrt{\frac{5}{4}} \sqrt{3} (1-t^2) = \frac{\sqrt{15}}{2} (1-t^2) \\ \dots & & \dots & \end{aligned}$$

Finally we need recurrence relations for $\sin(m\lambda)$ and $\cos(m\lambda)$ expressed in terms of sine and cosine of arguments $(m-1)\lambda$ and $(m-2)\lambda$.

It is not difficult to verify that the following relations hold

$$\sin m\lambda = 2 \cos \lambda \sin((m-1)\lambda) - \sin((m-2)\lambda) \quad (15)$$

$$\cos m\lambda = 2 \cos \lambda \cos((m-1)\lambda) - \cos((m-2)\lambda). \quad (16)$$

For $m = 0$ we get $\sin(0) = 0$ and $\cos(0) = 1$. For $m = 1$ we get the identities $\sin(\lambda) = \sin(\lambda)$ and $\cos(\lambda) = \cos(\lambda)$, etc.

Clenshaw Summation

There are many bad ways to evaluate the sum (7). A robust computational procedure is therefore needed. One key idea to a robust summation of (7) is to recognize that we are dealing with a sum of known coefficients c_k times functions $F_k(x)$:

$$f(x) = \sum_{k=0}^N c_k F_k(x) \quad (17)$$

and that $F_k(x)$ obeys a recurrence formula

$$F_{n+1}(x) = \alpha(n, x)F_n(x) + \beta(n, x)F_{n-1}(x).$$

We define

$$y_{N+2} = y_{N+1} = 0 \quad \text{and} \quad y_k = \alpha(k, x)y_{k+1} + \beta(k+1, x)y_{k+2} + c_k$$

for $k = N, N-1, \dots$ and solve backwards to obtain y_2 and y_1 :

$$c_k = y_k - \alpha(k, x)y_{k+1} - \beta(k+1, x)y_{k+2}. \quad (18)$$

We substitute into (17) and get

$$\begin{aligned} f(x) &= \sum_{k=0}^N c_k F_k(x) \\ &= c_0 F_0(x) + (y_1 - \alpha(1, x)y_2 - \beta(2, x)y_3)F_1(x) + \\ &\quad (y_2 - \alpha(2, x)y_3 - \beta(3, x)y_4)F_2(x) + \\ &\quad (y_3 - \alpha(3, x)y_4 - \beta(4, x)y_5)F_3(x) + \\ &\quad (y_4 - \alpha(4, x)y_5 - \beta(5, x)y_6)F_4(x) + \dots \\ &= c_0 F_0(x) + y_1 F_1(x) + y_2 (F_2(x) - \alpha(1, x)F_1(x)) + \\ &\quad y_3 (F_3(x) - \alpha(2, x)F_2(x) - \beta(2, x)F_1(x)) + \\ &\quad y_4 (F_4(x) - \alpha(3, x)F_3(x) - \beta(3, x)F_2(x)) + \dots \\ &= c_0 F_0(x) + y_2 \left((\alpha(1, x)F_1(x) + \beta(1, x)F_0(x)) - \alpha(1, x)F_1(x) \right) + y_1 F_1(x) \\ &= c_0 F_0(x) + y_1 F_1(x) + \beta(1, x)F_0(x)y_2. \end{aligned}$$

This smart technique is named *Clenshaw summation* and is published in Clenshaw (1955).

Clenshaw Summation from a Matrix Point of View

Let f and c designate two vectors; their scalar product $f^T c$ is the sum

$$s = f^T c;$$

the components are

$$\begin{aligned} f^T &= (f_N, f_{N-1}, \dots, f_1) \\ c^T &= (c_N, c_{N-1}, \dots, c_1) \end{aligned}$$

and f is a function of t : $f_n = f_n(t)$, and c_n is a constant.

Furthermore, let f_n satisfy a three-term recurrence relation like (11):

$$(n - m + 2)P_{n+2,m}(t) - (2n + 3)tP_{n+1,m}(t) - (n + m + 1)P_{n,m}(t) = 0. \quad (19)$$

The algorithm for the sum s consists in finding a vector q :

$$q^T = (q_N, q_{N-1}, q_{N-2}, \dots, q_1)$$

which satisfy

$$c_n = (n - m + 2)q_{n+2} - (2n + 3)tq_{n+1} + (n + m + 1)q_n. \quad (20)$$

A recursive formula like (19) is related to a tri-diagonal matrix:

$$A = \begin{bmatrix} n + m + 1 & & & & & \\ -(2n + 3)t & n + m + 1 & & & & \\ n - m + 2 & -(2n + 3)t & n + m + 1 & & & \\ & n - m + 2 & -(2n + 3)t & n + m + 1 & & \\ & & \vdots & & & \\ & & & n - m + 2 & -(2n + 3)t & n + m + 1 \\ & & & & n - m + 2 & -(2n + 3)t & n + m + 1 \end{bmatrix}.$$

The unknown vector q can be found from the equation

$$Aq = c$$

combined with $q_{N+2} = q_{N+1} = 0$.

Now we have

$$s = f^T c = f^T Aq = (f^T A)q.$$

Due to the recurrence relation we get

$$f^T A = (0, 0, \dots, 0, (n - m + 2)f_2 - (2n + 3)tf_1, (n + m + 1)f_1).$$

Hence

$$s = (f^T A)q = ((n - m + 2)f_2 - (2n + 3)tf_1)q_2 + (n + m + 1)f_1q_1.$$

So knowing q_1 and q_2 we get the simple expression for s :

$$s = ((n - m + 2)f_2 - (2n + 3)tf_1)q_2 + (n + m + 1)f_1q_1. \quad (21)$$

Table 1: Nomenclature of Polynomials

Legendre polynomials	$P_n(t)$ with $t = \sin(\psi)$, $n = 0, 1, 2, \dots$
Recurrence relation	$P_n(t) = \frac{2n-1}{n}t P_{n-1}(t) - \frac{n-1}{n}P_{n-2}(t)$ with $P_0(t) = 1$ and $P_1(t) = t$
Associated Legendre polynomials	$P_{nm}(t) = (1-t)^{m/2} \frac{d^m}{dt^m} P_n(t)$, $m = 1, 2, \dots, n$
Fully normalized Legendre polynomials	$\bar{P}_{nm}(t) = \sqrt{2(2n+1) \frac{(n-m)!}{(n+m)!}} P_{nm}(t)$
Spherical harmonics	$Y_{nm}(\psi, \lambda) = \bar{P}_{nm}(t) \times \begin{cases} \cos m\lambda & \text{for } m \geq 0 \\ \sin m \lambda & \text{for } m < 0 \end{cases}$ $m = -n, -n+1, \dots, 0, 1, \dots, n-1, n$

The Actual Implementation

The double summation has limits $\sum_{n=2}^{n_{\max}} \sum_{m=0}^n$ which means that $m \leq n$. The expression for T is bounded so we may interchange the order of summation. We split into two levels: first summation over n with fixed m , and next a summation over m :

$$T = \frac{kM}{r} \left(\sum_{n=2}^{n_{\max}} \left(\frac{a}{r} \right)^n \sum_{m=0}^n (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \psi) \right) \quad (22)$$

$$= \frac{kM}{r} \sum_{m=0}^{n_{\max}} \left(\cos m\lambda \underbrace{\sum_{n=2}^{n_{\max}} \left(\frac{a}{r} \right)^n \bar{C}_{nm} \bar{P}_{nm}(\sin \psi)}_{v_{m1}} + \sin m\lambda \underbrace{\sum_{n=2}^{n_{\max}} \left(\frac{a}{r} \right)^n \bar{S}_{nm} \bar{P}_{nm}(\sin \psi)}_{v_{m2}} \right). \quad (23)$$

The double sum (22) can be arranged in several ways. We follow the one implemented in `clenqt.for` and which is identical to (23). Following Tscherning & Pöder (1982) the partial sums can be written as

$$v_{m1} = \sum_{n=m}^{n_{\max}} P_{nm}(t) q^n C_{nm}$$

$$v_{m2} = \sum_{n=m}^{n_{\max}} P_{nm}(t) q^n S_{nm}.$$

Remember that $t = \sin(\psi)$ and $u = \cos(\psi)$. We rewrite equations (11) and (9) for the associated Legendre functions:

$$P_{nm} - \frac{2n-1}{n-m} t P_{n-1,m} + \frac{n+m-1}{n-m} P_{n-2,m} = 0 \quad P_{m-1,m} = 0 \quad (24)$$

$$P_{mm} - (2m-1)u P_{m-1,m-1} = 0, \quad P_{00} = 1 \quad (25)$$

Table 2: The `loc.m` values

Order n	Degree m	Index s
2	0	4
	1	5
	2	6 = <code>loc</code> (4)
3	0	7
	1	8
	2	9
	3	10 = <code>loc</code> (5)
4	0	11
	1	12
	2	13
	3	14
	4	15 = <code>loc</code> (6)
...		

Multiplication with powers of $q = a_e/r$ introduces $p_{nm} = P_{nm}q^n$ and $p_{mm} = P_{mm}q^m$; or

$$p_{nm} - \frac{2n-1}{n-m}tqp_{n-1,m} + \frac{n+m-1}{n-m}q^2P_{n-2,m} = 0 \quad (26)$$

$$p_{mm} - (2m-1)uqp_{m-1,m-1} = 0. \quad (27)$$

With $w = kM/r$ the sum V

$$V = \sum_{m=0}^{n_{\max}} (v_{m1} \cos(m\lambda) + v_{m2} \sin(m\lambda)) = \sum_{m=0}^{n_{\max}} w (s_{m1} \cos(m\lambda) + s_{m2} \sin(m\lambda)) p_m$$

can then be computed using (18)

$$s_n = -a_{n+1}s_{n+1} - b_{n+2}s_{n+2} + y_n, \quad s_{n_{\max}+1} = s_{n_{\max}+2} = 0$$

with

$$a_m = -(2m-1)uq, \quad b_m = 0, \quad v_m = s_{m1} \cos(m\lambda) + s_{m2} \sin(m\lambda).$$

The normalized associated Legendre polynomials \bar{P}_{nm} and the normalized geopotential coefficients \bar{C}_{nm} and \bar{S}_{nm} can adequately be store in one-dimensional arrays at locations determined according to the scheme described in Table 2. The `loc`(i) = $i(i-1)/2$ array points to the entry where $n = m$.

References

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