Stability, Observability, and Controllability

Kai Borre

Stability of Discrete Filter

Remember the update equation:

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + K_k(\boldsymbol{b}_k - A_k\hat{\boldsymbol{x}}_{k|k-1}). \tag{1}$$

Prediction of the state equation is $\hat{\boldsymbol{x}}_{k|k-1} = F_{k-1}\hat{\boldsymbol{x}}_{k-1|k-1}$. Substitute to get

$$\hat{\boldsymbol{x}}_{k|k} = (F_{k-1} - K_k A_k F_{k-1})\hat{\boldsymbol{x}}_{k-1|k-1} + K_k\boldsymbol{b}_k. \tag{2}$$

We now take the z-transform on both sides. Note that retarding one step in the time domain (k to $k-1$) is the equivalent of multiplying by z^{-1} in the z-domain. Omitting most indices the z-transform becomes

$$\hat{X}(z) = (F - K_k AF)z^{-1}\hat{X}(z) + K_k B(z)$$

or $(zI - F + K_k AF)\hat{X}(z) = zK_k B(z)$. The *characteristic polynomial* is

$$\det(zI - F + KAF) = 0. \tag{3}$$

The roots of this polynomial tell the story about the ***stability of the filter***. The ***transfer function*** is the reciprocal $\left(zI - F + K_k AF\right)^{-1}$. In this particular case equation (3) becomes

$$z \cdot I - 1 + K \cdot 1 \cdot 1 = 0 \qquad \text{or} \qquad z = 1 - K.$$

In order to determine K we run the M-file k_updatf

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[x, P, K, inn] = k_updatf(1,1,1,1.5,0.1,1.0,1); % x,P,A,b,Σe,Σϵ,F
[x, P, K, inn] = k_updatf(x,P,1,1.5,0.1,1.0,1);
[x, P, K, inn] = k_updatf(x,P,1,1.5,0.1,1.0,1);
...
```

The actual computation runs like this (notice P twice between updates of k):

$$
\begin{aligned}
P_{1|0} &= P_{0|0} + 1 = 1 + 1 = 2 \\
K_1 &= P_{1|0}(P_{1|0} + 0.1)^{-1} = 2/2.1 = 0.952\,380\,95 \\
P_{1|1} &= (1 - K_1)P_{1|0} = 0.095\,238\,095 \\
P_{2|1} &= P_{1|1} + 1 = 1.095\,238\,095 \\
K_2 &= 1.095\,238\,095/(1.095\,238\,095 + 0.1) = 0.916\,334\,66 \\
&\dots
\end{aligned}
$$

or

$$
\begin{aligned}
K = {} & 0.95\,238 \\
& 0.91\,633 \\
& 0.91\,608 \\
& 0.91\,608.
\end{aligned}
$$

Hence $z = 1 - K = 0.08\,392$ which is well within the unit circle in the z-plane. We realize the filter is highly stable.

Even though the input $\boldsymbol{b}$ is non-stationary, the filter itself is intrinsically stable. The location of the filter zero (pole of the transfer function) tells that any small perturbation (roundoff errors) from the steady-state condition will damp out quickly. Any perturbation will be attenuated by a factor of 0.084 with each step so the effect rapidly fades out.

We conclude that we can gain considerable insight into the filter operation just by looking at its characteristic poles in the steady-state condition, provided, of course, the steady-state condition exists.

We have just dealt with stability. Next we introduce the twin concepts of system ***observability***, and system ***controllability***.

Observability

A set of system and observation equations is said to be observable from a given set of observations $\boldsymbol{b}_k$ if the state vector $\boldsymbol{x}_k$ can be determined from those observations.

If the state vector $\boldsymbol{x}_k$ depends linearly on the observations this can be expressed as $\boldsymbol{b}_k = A_k \boldsymbol{x}_k$. If the square matrix $A_k^{\mathrm{T}} A_k$ is non-singular we can uniquely determine the state vector $\boldsymbol{x}_k$ from the observation vector $\boldsymbol{b}_k$. Then the system is ***observable***. However the system can be observable over an interval of time if $A_k^{\mathrm{T}} A_k$ is not invertible at any epoch. The latter only can happen in case A_k is varying over time. We define the matrix

$$\textbf{\textit{Observability}} \qquad M = \begin{bmatrix} A & AF & AF^2 & \dots & AF^{k-1} \end{bmatrix}. \tag{4}$$

The system is observable if and only if M has full rank. We note that observability does not depend on the observations $\boldsymbol{b}_k$. Remember $\operatorname{rank}(A) = \operatorname{rank}(A^{\mathrm{T}} A)$.

Example

Consider the system with $F_k = I$ (steady model):

$$\hat{\boldsymbol{x}}_{k|k-1} = \hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{\epsilon}_k.$$

Let $\alpha = (f_1/f_2)^2$, and let the observation equation include noise

$$\begin{bmatrix} P_1 \\ \Phi_1 \\ P_2 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & \lambda_1 & 0 \\ 1 & \alpha & 0 & 0 \\ 1 & -\alpha & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \rho^* \\ I \\ N_1 \\ N_2 \end{bmatrix} - \text{noise}$$

Next we compute

$$\mathcal{A}_0^{\mathrm{T}}\mathcal{A}_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} 4 & 0 & \lambda_1 & \lambda_2 \\ 0 & 2(1+\alpha^2) & -\lambda_1 & -\alpha\lambda_2 \\ \lambda_1 & \alpha\lambda_1 & \lambda_1^2 & 0 \\ \lambda_2 & -\alpha\lambda_2 & 0 & \lambda_2^2 \end{bmatrix}$$

This matrix surely is non-singular and the system is observable.

The following block normal equations have full rank

$$\mathcal{S}_1^{\mathrm{T}}\mathcal{S}_1 = \begin{bmatrix} A_0^{\mathrm{T}}A_0 + F_0^{\mathrm{T}}F_0 & -F_0^{\mathrm{T}} \\ -F_0 & I \end{bmatrix}$$

$$\mathcal{A}_1^{\mathrm{T}}\mathcal{A}_1 = \begin{bmatrix} A_0^{\mathrm{T}}A_0 + F_0^{\mathrm{T}}F_0 & -F_0^{\mathrm{T}} \\ -F_0 & A_1^{\mathrm{T}}A_1 + I \end{bmatrix}.$$

At all subsequent steps we continue to have full rank and observability.

Example

Pseudoranges on L_1 and L_2 are given as double differences P_1 and P_2 from two satellites and receivers:

$$P_1 = \rho + I$$
$$P_2 = \rho + \alpha I.$$

We put the variance of a double differenced pseudorange equal to $0.4\,\mathrm{m}^2$ so the covariance matrix is

$$\Sigma_{e,k} = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}.$$

Double differenced pseudoranges may well be described by a first order model (a random ramp) for both ρ and I. The state vector $\boldsymbol{x}_k$ then has the four components $(\rho, \dot{\rho}, I, \dot{I})$. The observation equations are

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = A_k \boldsymbol{x}_k = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & \alpha & 0 \end{bmatrix} \begin{bmatrix} \rho \\ \dot{\rho} \\ I \\ \dot{I} \end{bmatrix}.$$

We introduce $t_k - t_{k-1} = \Delta t$ and the system equation becomes

$$\boldsymbol{x}_k = F_{k-1}\boldsymbol{x}_{k-1} + \boldsymbol{\epsilon}_k = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ \dot{\rho} \\ I \\ \dot{I} \end{bmatrix} + \int_{t_{k-1}}^{t_k} \begin{bmatrix} (t_{k-1} - t)\ddot{\rho}(t) \\ \ddot{\rho}(t) \\ (t_{k-1} - t)\ddot{I}(t) \\ \ddot{I}(t) \end{bmatrix} dt.$$

The second order time derivatives $\ddot{\rho}$ and $\ddot{I}$ are assumed to come from zero mean white noise processes with constant spectral densities s_ρ and s_I, unit $\mathrm{m}^2/\mathrm{s}^3$. Then we have $E\{\boldsymbol{\epsilon}_k\} = 0$, and $E\{\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_l^{\mathrm{T}}\} = 0$ for $k \neq l$. The covariance matrix is

$$\Sigma_{\epsilon,k} = E\{\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^{\mathrm{T}}\} = \left[\begin{array}{cc|cc} s_\rho(\Delta t)^3/3 & s_\rho(\Delta t)^2/2 & 0 & 0 \\ s_\rho(\Delta t)^2/2 & s_\rho \Delta t & 0 & 0 \\ \hline 0 & 0 & s_I(\Delta t)^3/3 & s_I(\Delta t)^2/2 \\ 0 & 0 & s_I(\Delta t)^2/2 & s_I \Delta t \end{array}\right]. \tag{5}$$

Reasonable values for the spectral densities are $s_\rho = 0.36 \times 10^{-8}$ and $s_I = 4 \times 10^{-8}$.

For a set of given observations the initial covariance matrix $P_{0|0}$ is

$$P_{0|0} = \begin{bmatrix} 0.89 & & & \\ & 0.00 & & \\ & & 0.48 & \\ & & & 0.00 \end{bmatrix}.$$

The value $\sigma_I = 0.7\,\mathrm{m}$ allows for a reasonable variability of the ionospheric delay I at the beginning.

We have $\operatorname{rank}(A) = 2$ and $\operatorname{rank}(F) = 4$. So $\operatorname{rank}(F^{\mathrm{T}} A^{\mathrm{T}}) = 2$ and this system is not directly observable with one epoch. However, after 2 epochs the system is observable. The M-file observa verifies this fact.

Controllability

Next we focus on the conditions for controlling the state of a deterministic linear dynamic system. Controllability allows us to select an input so that the state $\boldsymbol{x}$ takes any desired value after k steps. The vector $\boldsymbol{u}$ denotes the deterministic control input:

$$\boldsymbol{x}_k = F\boldsymbol{x}_{k-1} + L\boldsymbol{u}_{k-1}.$$

The derivation of controllability is analogous to the derivation of observability:

$$\textbf{\textit{Controllability}} \qquad S = \begin{bmatrix} L & FL & \dots F^{k-1}L \end{bmatrix}.$$

The system is controllable if and only if S has full rank.

Example

Let a discrete-time system be described by

$$\boldsymbol{x}_k = F\boldsymbol{x}_{k-1} + L\boldsymbol{u}_{k-1} = \begin{bmatrix} -\beta & 0 \\ 0 & -\gamma \end{bmatrix} \boldsymbol{x}_{k-1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \boldsymbol{u}_{k-1}.$$

In this case the controllability matrix is

$$S = \begin{bmatrix} L & FL \end{bmatrix} = \begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix}.$$

The characteristic equation then is

$$\det(S) = \beta - \gamma.$$

The controllability condition requires a nonzero determinant for S: $\beta \neq \gamma$. If β and γ are equal, the two first order systems are identical and there is no way that an input $\boldsymbol{u}_k$ could reach a state $\boldsymbol{x}_k = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ with different values of x_1 and x_2.

The test applied to the block matrix S only establishes controllability. It does not provide a means to determine the input $\boldsymbol{u}$ required to control the system. The control signals $\boldsymbol{u}_k$ cannot bring the system from an initial state $\hat{\boldsymbol{x}}_0$ to a final state $\hat{\boldsymbol{x}}_N$.

A useful interpretation of the new conditions is expressed by variances:

- Observability of the state guarantees a steady flow of information about each component. This prevents the covariance from becoming unbounded.
- Controllability guarantees that the process noise enters each state component. This prevents the covariance of the state estimate from converging to zero.

Thus observability yields the existence of the steady-state solution and controllability causes it to be positive definite.