Solution to Exercise 8

Given

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}, \; \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \; \Sigma_{e,k} = \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix}, \; F_k = 1, \text{ and } \Sigma_{\epsilon,k} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We do not know the initial solution vector $\boldsymbol{x}_0$. We cannot do anything better than set $\boldsymbol{x}_0 = \boldsymbol{0}$. The initial covariance matrix P_0 for the solution $\boldsymbol{x}$ must be defined such that it allows for any values. So the variances are set to a big number, say 10^{12}. All components of $\boldsymbol{x}$ must be independent. Hence P_0 is diagonal.

The computations are shown below in all details. However, for typographical reasons we only reproduce the numbers to a certain numerical accuracy. For full detail run the M-file wc.

Kalman Version, confer page 565 in Strang & Borre (1997)

$k = 1$:

1. $\boldsymbol{x}_{1|0} = \boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. $P_{1|0} = P_0 = \begin{bmatrix} 10^{12} & 0 \\ 0 & 10^{12} \end{bmatrix}$

3. $K_1 = P_{1|0} A_1^{\mathrm{T}} \big(A_1 P_{1|0} A_1^{\mathrm{T}} + \Sigma_{e,1} \big)^{-1}$
$$= \begin{bmatrix} 10^{12} & 0 \\ 0 & 10^{12} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10^{12} & 0 \\ 0 & 10^{12} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \right)^{-1} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4. $\boldsymbol{x}_{1|1} = \boldsymbol{x}_{1|0} + K_1 (\boldsymbol{b}_1 - A_1 \boldsymbol{x}_{1|0})$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(2 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. $P_{1|1} = (I - K_1 A_1) P_{1|0}$
$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 10^{12} & 0 \\ 0 & 10^{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 10^{12} & -10^{12} \\ -10^{12} & 10^{12} \end{bmatrix}$$

$k = 2$:

1. $\boldsymbol{x}_{2|1} = \boldsymbol{x}_{1|1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $P_{2|1} = P_{1|1} = \frac{1}{2} \begin{bmatrix} 10^{12} & -10^{12} \\ -10^{12} & 10^{12} \end{bmatrix}$

3. $K_2 = P_{2|1} A_2^{\mathrm{T}} \big(A_2 P_{2|1} A_2^{\mathrm{T}} + \Sigma_{e,2} \big)^{-1}$
$$= \frac{1}{2} \begin{bmatrix} 10^{12} & -10^{12} \\ -10^{12} & 10^{12} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 10^{12} & -10^{12} \\ -10^{12} & 10^{12} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{2} \right)^{-1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

4. $\boldsymbol{x}_{2|2} = \boldsymbol{x}_{2|1} + K_2(\boldsymbol{b}_2 - A_2\boldsymbol{x}_{2|1})$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\left(1 - \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

5. $P_{2|2} = (I - K_2A_2)P_{2|1}$
$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 10^{12} & -10^{12} \\ -10^{12} & 10^{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}$$

$k = 3$:

1. $\boldsymbol{x}_{3|2} = \boldsymbol{x}_{2|2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

2. $P_{3|2} = P_{2|2} = \frac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}$

3. $K_3 = P_{3|2}A_3^{\mathrm{T}}\big(A_3P_{3|2}A_3^{\mathrm{T}} + \Sigma_{e,3}\big)^{-1}$
$$= \frac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1\right)^{-1} = \frac{1}{12}\begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

4. $\boldsymbol{x}_{3|3} = \boldsymbol{x}_{3|2} + K_3(\boldsymbol{b}_3 - A_3\boldsymbol{x}_{3|2})$
$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -7 \\ 4 \end{bmatrix}\left(0 - \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

5. $P_{3|3} = (I - K_3A_3)P_{3|2}$
$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{12}\begin{bmatrix} -7 \\ 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

Bayes Version, confer page 510 in Strang & Borre (1997)

$k = 1$:

1. $\boldsymbol{x}_{1|0} = \boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. $P_{1|0} = P_0 = \begin{bmatrix} 10^{12} & 0 \\ 0 & 10^{12} \end{bmatrix}$

3. $P_{1|1} = \big(P_{1|0}^{-1} + A_1^{\mathrm{T}}\Sigma_{e,1}^{-1}A_1\big)^{-1}$
$$= \left(\begin{bmatrix} 10^{-12} & 0 \\ 0 & 10^{-12} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 \begin{bmatrix} 1 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$$
where $a = 499\,955\,553\,660.135$ and $b = 499\,955\,553\,659.635$.

4. $K_1 = P_{1|1}A_1^{\mathrm{T}}\Sigma_{e,1}^{-1}$
$$= \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} 1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. $\boldsymbol{x}_{1|1} = \boldsymbol{x}_{1|0} + K_1(\boldsymbol{b}_1 - A_1\boldsymbol{x}_{1|0})$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(2 - \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$k = 2$:

1. $\boldsymbol{x}_{2|1} = \boldsymbol{x}_{1|1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $P_{2|1} = P_{1|1} = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$

3. $P_{2|2} = \left(P_{2|1}^{-1} + A_2^{\mathrm{T}}\Sigma_{e,2}^{-1}A_2\right)^{-1}$

$$= \left(\begin{bmatrix} 1+10^{-12} & 1 \\ 1 & 1+10^{-12} \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2 \begin{bmatrix} 1 & 2 \end{bmatrix}\right)^{-1} = \tfrac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}$$

4. $K_2 = P_{2|2}A_2^{\mathrm{T}}\Sigma_{e,2}^{-1}$

$$= \tfrac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} 2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

5. $\boldsymbol{x}_{2|2} = \boldsymbol{x}_{2|1} + K_2(\boldsymbol{b}_2 - A_2\boldsymbol{x}_{2|1})$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\left(1 - \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$k = 3$:

1. $\boldsymbol{x}_{3|2} = \boldsymbol{x}_{2|2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

2. $P_{3|2} = P_{2|2} = \tfrac{1}{2}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}$

3. $P_{3|3} = \left(P_{3|2}^{-1} + A_3^{\mathrm{T}}\Sigma_{e,3}^{-1}A_3\right)^{-1}$

$$= \left(\left(\tfrac{1}{2}\right)^{-1}\begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}^{-1} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} 1 \begin{bmatrix} -1 & 1 \end{bmatrix}\right)^{-1} = \tfrac{1}{12}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

4. $K_3 = P_{3|3}A_3^{\mathrm{T}}\Sigma_{e,3}^{-1}$

$$= \tfrac{1}{12}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} 1 = \tfrac{1}{12}\begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

5. $\boldsymbol{x}_{3|3} = \boldsymbol{x}_{3|2} + K_3(\boldsymbol{b}_3 - A_3\boldsymbol{x}_{3|2})$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \tfrac{1}{12}\begin{bmatrix} -7 \\ 4 \end{bmatrix}\left(0 - \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \tfrac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$