GPS models

Position computed from observations from one or more epochs

- *single point positioning*: one receiver tracks at least 4 satellites, $\sigma \approx 10\,\mathrm{m}$
- *differential positioning*: two receivers making code observations to at least 4 satellites, $\sigma \approx 0.1$–$1\,\mathrm{m}$, DGPS
- *precise differential positioning*: code and phase observations, fixed ambiguities, $\sigma \approx 1$–$10\,\mathrm{mm}$, surveying

One-way code observation on L_1:

$$P_{1,i}^{k} = \rho_i^k + c\,dt_i + T_i^k + I_i^k + \text{noise}$$

Single difference of code observations on L_1:

$$P_1 = P^k_{1,i} - P^k_{1,j}$$

Geodetic receivers additionally observe the phase of the carrier waves:

$$\Phi^k_{1,i} = \rho^k_i + T^k_i - I^k_i + \lambda_1 N_1 + \text{clock errors} + \text{multipath} + \text{noise}$$

Single difference of phase observations on L_1:

$$\Phi_1 = \Phi^k_{1,i} - \Phi^k_{1,j}$$

Standard deviation of P-code observation $\sigma_{\text{code}} = 0.3\,\text{m}$

Standard deviation of phase observation $\sigma_{\text{phase}} = 3\,\text{mm}$

Finally we make *double differences* of code observations

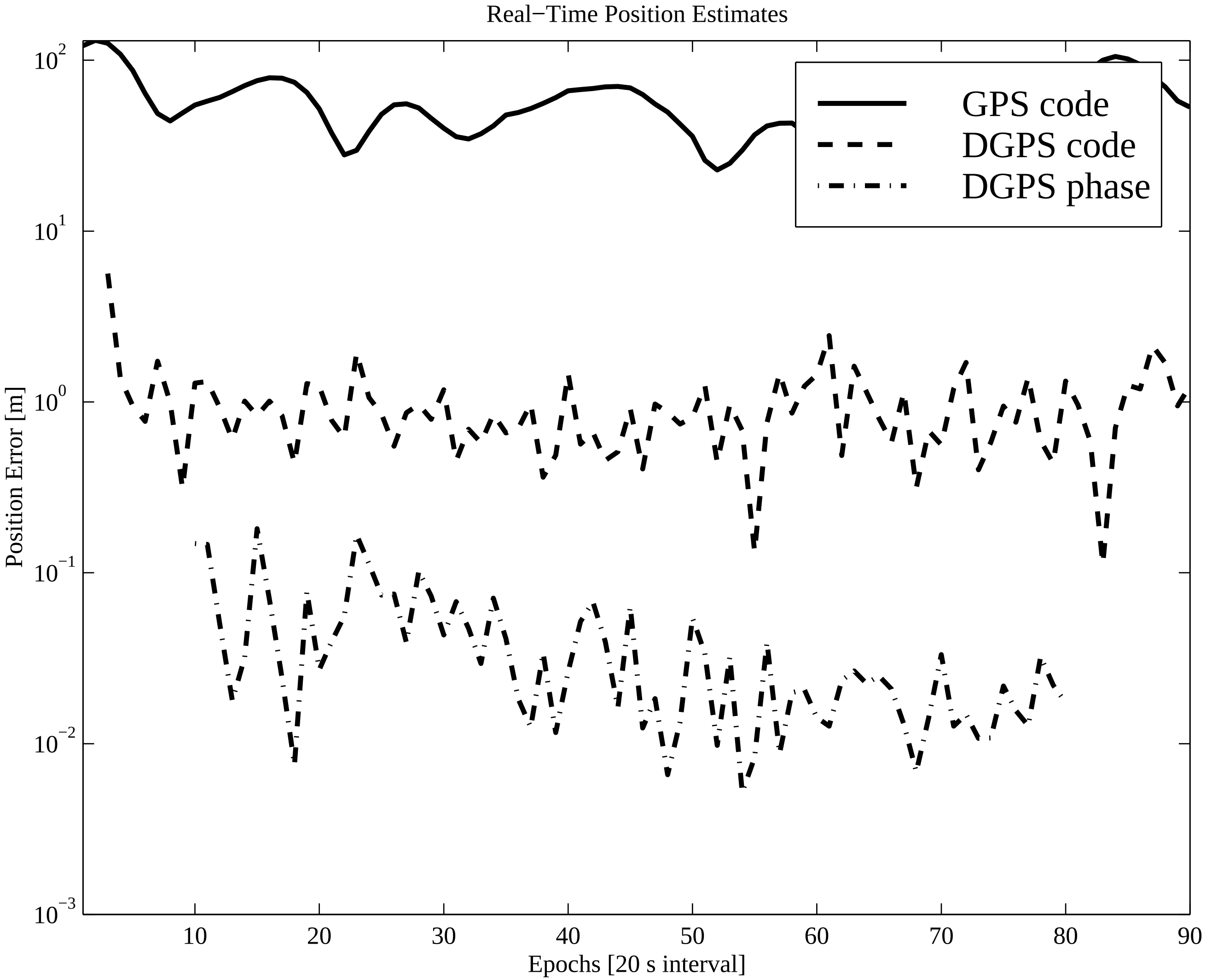
$$P^{k,l}_{1,i,j} = (P^{k}_{1,i} - P^{l}_{1,i}) - (P^{k}_{1,j} - P^{l}_{1,j}) + \text{noise}$$

The tropospheric delays can be modeled, the receiver clock offsets cancel, and the satellite clock offsets can be computed

$$P^{k,l}_{1,i,j} = (\rho^{k}_{i} - \rho^{l}_{i}) - (\rho^{k}_{j} - \rho^{l}_{j}) + I^{k,l}_{i,j}$$

and similarly for the phase:

$$\Phi^{k,l}_{1,i,j} = (\rho^{k}_{i} - \rho^{l}_{i}) - (\rho^{k}_{j} - \rho^{l}_{j}) - I^{k.l}_{i,j} + \lambda_1\big((N^{k}_{1,i} - N^{l}_{1,i}) - (N^{k}_{1,j} - N^{l}_{1,j})\big) + \text{noise}$$



Real−Time Position Estimates
GPS code
DGPS code
DGPS phase
Position Error [m]
Epochs [20 s interval]
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10
20
30
40
50
60
70
80
90

Observation Equation

Pseudorange on L_1 frequency; signal travel time $\tau_i^k = t_i - t^k$:

$$P_i^k = \rho_i^k + c\,dt_i(t) - c\,dt^k(t - \tau_i^k) + I_i^k + T_i^k - e_i^k. \tag{1}$$

The geometric distance between satellite k and receiver i is $\rho_i^k = \sqrt{(X^k - X_i)^2 + (Y^k - Y_i)^2 + (Z^k - Z_i)^2}$. Neglecting satellite clock offset, ionospheric, and tropospheric delays we get for $k = 1, 2, 3, 4$

$$\begin{bmatrix} P_i^1 \\ P_i^2 \\ P_i^3 \\ P_i^4 \end{bmatrix} = \begin{bmatrix} \rho_i^1 + c\,dt_i \\ \rho_i^2 + c\,dt_i \\ \rho_i^3 + c\,dt_i \\ \rho_i^4 + c\,dt_i \end{bmatrix}.$$

The distance between satellite k and receiver i—corrected for Earth rotation, rotation rate of the Earth is ω_e—is defined by

$$\rho_i^k = \| R_3(\omega_e \tau_i^k) \boldsymbol{r}^k (t - \tau_i^k)_{\text{geo}} - \boldsymbol{r}_i(t) \|.$$

The matrix R_3 accounts for the rotation by the angle $\omega_e \tau_i^k$ corresponding to the signal travel time τ_i^k:

$$R_3(\omega_e \tau_i^k) = \begin{bmatrix} \cos(\omega_e \tau_i^k) & \sin(\omega_e \tau_i^k) & 0 \\ -\sin(\omega_e \tau_i^k) & \cos(\omega_e \tau_i^k) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rotation is necessary when using vectors referenced to an Earth centered and Earth fixed system (ECEF).

We linearize (1) and let ^{0} denote a preliminary value:

$$
-\frac{X^k - X_i^0}{\left(\rho_i^k\right)^0} x_i - \frac{Y^k - Y_i^0}{\left(\rho_i^k\right)^0} y_i - \frac{Z^k - Z_i^0}{\left(\rho_i^k\right)^0} z_i + 1(c\,dt_i)
$$

$$
= P^k_{i\,\mathrm{obs}} - \left(P_i^k\right)^0 - e_i^k = b_i - e_i^k. \qquad (2)
$$

The unknowns are arranged as $\boldsymbol{x} = (x_i, y_i, z_i, c\,dt_i)$ and we get

$$
A\boldsymbol{x} = \begin{bmatrix}
-\frac{X^1 - X_i^0}{\rho_i^1} & -\frac{Y^1 - Y_i^0}{\rho_i^1} & -\frac{Z^1 - Z_i^0}{\rho_i^1} & 1 \\
-\frac{X^2 - X_i^0}{\rho_i^2} & -\frac{Y^2 - Y_i^0}{\rho_i^2} & -\frac{Z^2 - Z_i^0}{\rho_i^2} & 1 \\
 & \vdots & & \\
-\frac{X^m - X_i^0}{\rho_i^m} & -\frac{Y^m - Y_i^0}{\rho_i^m} & -\frac{Z^m - Z_i^0}{\rho_i^m} & 1
\end{bmatrix}
\begin{bmatrix} x_i \\ y_i \\ z_i \\ c\,dt_i \end{bmatrix} = \boldsymbol{b} - \boldsymbol{e}. \qquad (3)
$$

Introducing the unit direction vector from receiver i to satellite k

$$\left(\boldsymbol{u}_i^k\right)^0 = \left(\frac{X^k_{\text{ECEF}} - X_i^0}{\rho_i^k}, \frac{Y^k_{\text{ECEF}} - Y_i^0}{\rho_i^k}, \frac{Z^k_{\text{ECEF}} - Z_i^0}{\rho_i^k}\right)$$

we can rewrite (3) as

$$A\boldsymbol{x} = \begin{bmatrix} -\left(\boldsymbol{u}_i^1\right)^0 & 1 \\ -\left(\boldsymbol{u}_i^2\right)^0 & 1 \\ \vdots & \\ -\left(\boldsymbol{u}_i^m\right)^0 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b} - \boldsymbol{e}.$$

The least-squares solution becomes

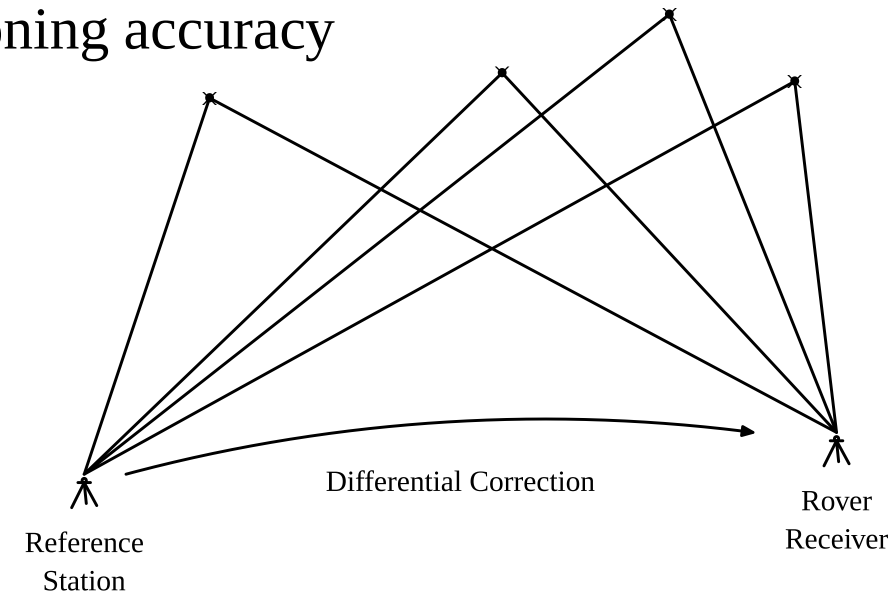
$$\begin{bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \\ \widehat{c\,dt_i} \end{bmatrix} = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}A^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}. \tag{4}$$

Often one uses $\Sigma = \sigma^2 I$.

After a few iterations the final receiver coordinates become $\widehat{X}_i = X_i^0 + \hat{x}_i$, $\widehat{Y}_i = Y_i^0 + \hat{y}_i$, and $\widehat{Z}_i = Z_i^0 + \hat{z}_i$.

Geometry for Differential GPS

- occupy a station with known coordinates
- compute range corrections
- transmit range corrections to rover
- corrections applied at rover
- improved positioning accuracy



Differential GPS

Assume the position of receiver i is ***known***. P_i^k is related to $c\,dt_i$ and can be applied as a range correction to P_j^k of the rover. Combining the models at i and j leads to the principle of ***differential GPS***:

$$\begin{bmatrix} \left(P^1_{j\,\mathrm{obs}} - (P^1_j)^0\right) - \left(P^1_{i\,\mathrm{obs}} - (P^1_i)^0\right) \\ \left(P^2_{j\,\mathrm{obs}} - (P^2_j)^0\right) - \left(P^2_{i\,\mathrm{obs}} - (P^2_i)^0\right) \\ \vdots \\ \left(P^m_{j\,\mathrm{obs}} - (P^m_j)^0\right) - \left(P^m_{i\,\mathrm{obs}} - (P^m_i)^0\right) \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{u}^1_j)^0 & 1 \\ -(\boldsymbol{u}^2_j)^0 & 1 \\ \vdots & \\ -(\boldsymbol{u}^m_j)^0 & 1 \end{bmatrix} \begin{bmatrix} x_j \\ y_j \\ z_j \\ c\,dt_{ij} \end{bmatrix}.$$

Here $c\,dt_{ij} = c\,dt_j - c\,dt_i$ is the difference of receiver clock offsets.

Code and Phase Observations

We repeat the pseudorange observation and let E^k denote the combination of ephemeris error and the a priori coordinate error of base station

$$P^k = \rho^k + c\,dt(t) - c\,dt^k(t-\tau^k) + E^k + I^k + T^k - e^k, \text{[length]}. \tag{5}$$

The geometric distance between satellite k and the receiver is

$$\rho^k = \sqrt{(X^k - X)^2 + (Y^k - Y)^2 + (Z^k - Z)^2}.$$

Similarly a phase observation looks like

$$\Phi^k = \rho^k + c\,dt(t) - c\,dt^k(t-\tau^k) + E^k - I^k + T^k + \lambda N - \epsilon^k, \text{[length]} \tag{6}$$

with carrier ambiguity N [cycles], and changed sign for ionospheric delay I^k.

The RTCM standard makes the following recommendations:

1. The range and range rate corrections be the best available estimates at the instant identified by the time tag. They should not be predicted forward in time
2. A base station should not attempt to remove the effects of ionospheric error from the broadcast corrections
3. A base station should not attempt to remove the effects of tropospheric error from the broadcast corrections
4. Except for time-transfer applications, the effect of base receiver clock errors should be removed from the broadcast corrections
5. The base station antenna should be located and the corrections processed to minimize the effects of multipath
6. The effects of satellite clock errors should be removed from broadcast corrections
7. The pseudorange observations should be simultaneous.

We combine receiver clock bias, ephemeris error, and tropospheric delay into

$$S^k = c\,dt + E^k + T^k.$$

Futhermore we combine all systematic errors in the code observation into a bias parameter b^k. The four observations to satellite k are:

$$\begin{bmatrix} P_1^k - \rho^k + c\,dt \\ \Phi_1^k - \rho^k + c\,dt \\ P_2^k - \rho^k + c\,dt \\ \Phi_2^k - \rho^k + c\,dt \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & \alpha & 0 & 0 & 0 & 1 \\ 1 & -\alpha & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S^k \\ I^k \\ \lambda_1 N_1 \\ \lambda_2 N_2 \\ b_1^k \\ b_2^k \end{bmatrix} - \begin{bmatrix} e_1^k \\ \epsilon_1^k \\ e_2^k \\ \epsilon_2^k \end{bmatrix}.$$

We introduce a transformation matrix[a]

$$
T = \begin{bmatrix}
1 & & & 1 & & & & & & & & 1 & \\
& 1 & & & 1 & & & & & & & & \\
& & 1 & & & & & & & & & & \\
& & & 1 & & \frac{1}{2} & & & & & & \frac{1}{2} & \\
& & & & 1 & & & & & & & & \\
& & & 1 & & & \frac{1}{\alpha+1} & & & & & \frac{1}{\alpha+1} & \\
& & & 1 & & & & & & & & -\frac{1}{\alpha-1} & \frac{1}{\alpha-1} \\
& & & & & & & 1 & & & & -1 & \\
& & & & & & & & 1 & & & & \\
& & & & & & & & & 1 & & & -1 \\
& & & & & & & & & & 1 & &
\end{bmatrix}.
$$

[a] See Xin-Xiang Jin, Hans van der Marel, and Cees de Jong: Computation and Quality Control of Differential GPS Corrections. Proceedings of ION '95, pages 1071–1079.

Using the transformation matrix T we get

$$\begin{bmatrix} P_1^k - \rho^k + c\,dt \\ \Phi_1^k - \rho^k + c\,dt \\ P_2^k - \rho^k + c\,dt \\ \Phi_2^k - \rho^k + c\,dt \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \alpha - 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -(\alpha+1) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla^k + b_1^0 \\ \dot{\nabla}^k \\ \ddot{S}^k \\ I^k + \frac{\lambda_1 N_1 + b_1^0}{2} \\ I^k + \frac{\lambda_2 N_2 + b_1^0}{\alpha+1} \\ \dot{I}^k \\ I^k - \frac{b_1^0 - b_2^0}{\alpha - 1} \\ b_1^k - b_1^0 \\ \dot{b}_1^k \\ b_2^k - b_2^0 \\ \dot{b}_2^k \end{bmatrix} - \begin{bmatrix} e_1^k \\ \epsilon_1^k \\ e_2^k \\ \epsilon_2^k \end{bmatrix}.$$

The dynamical model for updating the state vector x from epoch $i-1$ to i is

$$
\begin{aligned}
x_i &= \begin{bmatrix}
1 & \Delta t_i & (\Delta t_i)^2/2 & & & & & & & & \\
 & 1 & \Delta t_i & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 1 & \Delta t_i & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & \Delta t_i & 1 & & & & & \\
 & & & & \Delta t_i & & 1 & & & & \\
 & & & & & & & 1 & \Delta t_i & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & 1 & \Delta t_i \\
 & & & & & & & & & & 1
\end{bmatrix} x_{i-1} \\
&= F_{i,i-1} x_{i-1}.
\end{aligned}
$$

We assume the process noise in mean is zero and has covariance

$$
Q = T \begin{bmatrix}
q_{\dddot{S}} M_{33} & & & & & \\
& q_{\ddot{I}} M_{22} & & & & \\
& & 0_{2\times 2} & & & \\
& & & q_{\ddot{b}} M_{22} & & \\
& & & & q_{\ddot{b}} M_{22} & \\
& & & & & 0_{2\times 2}
\end{bmatrix} T^{\mathrm{T}}
$$

where

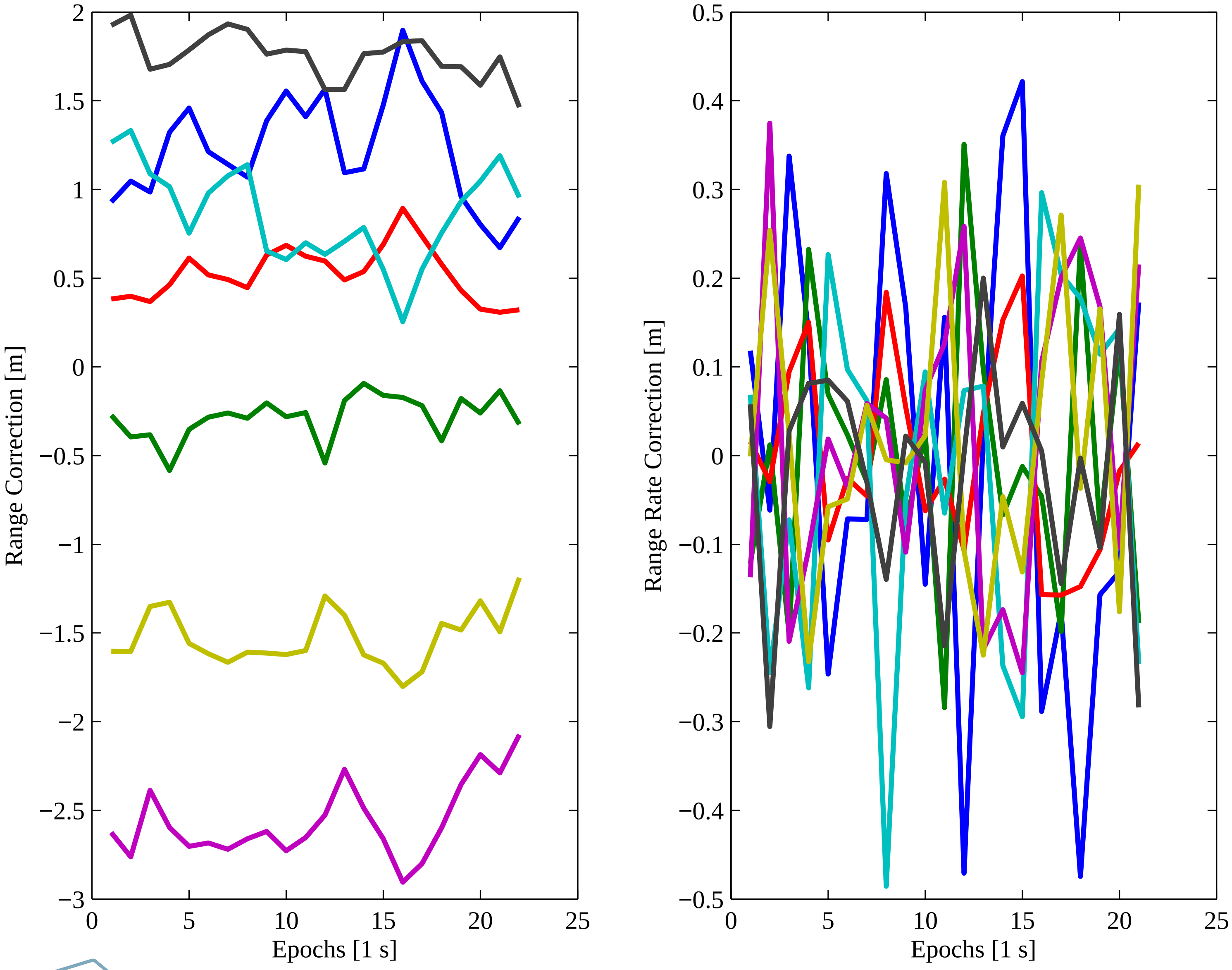
$$
M_{33} = \begin{bmatrix}
\frac{(\Delta t_i)^5}{20} & \frac{(\Delta t_i)^4}{8} & \frac{(\Delta t_i)^3}{6} \\
\frac{(\Delta t_i)^4}{8} & \frac{(\Delta t_i)^3}{3} & \frac{(\Delta t_i)^2}{2} \\
\frac{(\Delta t_i)^3}{6} & \frac{(\Delta t_i)^2}{2} & \Delta t_i
\end{bmatrix}, \qquad
M_{22} = \begin{bmatrix}
\frac{(\Delta t_i)^3}{3} & \frac{(\Delta t_i)^2}{2} \\
\frac{(\Delta t_i)^2}{2} & \Delta t_i
\end{bmatrix}.
$$

The initial values of the filter state and its covariance matrix can be determined by solving the first three epochs simultaneously by least squares.

The filter defined above is for a single satellite k. Identical, but independent, filters have to be implemented for each satellite in view of the base station.

We assume that the errors have zero mean and are uncorrelated among observations and epochs, but have constant variances $\sigma^2_{e_1}$, $\sigma^2_{\epsilon_1}$, $\sigma^2_{e_2}$, and $\sigma^2_{\epsilon_2}$.

If only single frequency data are available we cannot estimate the rates $\dot{\nabla}^k$, $\dot{I}^k$, and $\dot{b}^k$.


Range Correction [m]
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
0
5
10
15
20
25
Epochs [1 s]
Range Rate Correction [m]
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
0
5
10
15
20
25
Epochs [1 s]