17.5 Correlated Process and Observation Noise

We shall follow a development from Brown & Hwang (1997), pages 296–298 and start by explicitly stating a somewhat changed version of the usual covariance conditions

$$E\{\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_j^{\mathrm{T}}\} = \begin{cases} \Sigma_{\epsilon,k} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$E\{\boldsymbol{e}_k\boldsymbol{e}_j^{\mathrm{T}}\} = \begin{cases} \Sigma_{e,k} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$E\{\boldsymbol{\epsilon}_k\boldsymbol{e}_j^{\mathrm{T}}\} = C_k \qquad \text{for all } j \text{ and } k.$$

The last equation states that process noise $\boldsymbol{\epsilon}_k$ and observation noise $\boldsymbol{e}_k$ are correlated. We want to study the modifications introduced by this change to the usual filter equations.

Filtering of the state vector:

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + K_k(\boldsymbol{b}_k - A_k\hat{\boldsymbol{x}}_{k|k-1}).$$

The system equation is

$$\boldsymbol{x}_k = F_{k-1}\boldsymbol{x}_{k-1} + \boldsymbol{\epsilon}_k$$

and hence the estimation error

$$\begin{aligned}
\boldsymbol{v}_{k|k} &= \boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k} \\
&= \boldsymbol{x}_k - \big(\hat{\boldsymbol{x}}_{k|k-1} + K_k(\boldsymbol{b}_k - A_k\hat{\boldsymbol{x}}_{k|k-1})\big) \\
&= (I - K_kA_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1}) - K_k\boldsymbol{e}_k \\
&= (I - K_kA_k)\boldsymbol{v}_{k|k-1} - K_k\boldsymbol{e}_k
\end{aligned}$$

with covariance matrix $P_{k|k} = E\{\boldsymbol{v}_{k|k}\boldsymbol{v}_{k|k}^{\mathrm{T}}\}$.

Now we compute the covariance between $\boldsymbol{v}_{k|k-1}$ and $\boldsymbol{e}_k$:

$$\begin{aligned}
E\{\boldsymbol{v}_{k|k-1}\boldsymbol{e}_k^{\mathrm{T}}\} &= E\{(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1})\boldsymbol{e}_k^{\mathrm{T}}\} \\
&= E\{(F_{k-1}\boldsymbol{x}_{k-1} + \boldsymbol{\epsilon}_k - F_{k-1}\hat{\boldsymbol{x}}_{k-1|k-1})\boldsymbol{e}_k^{\mathrm{T}}\}.
\end{aligned}$$

$\boldsymbol{e}_k$ is neither correlated with $\boldsymbol{x}_{k-1}$ nor $\hat{\boldsymbol{x}}_{k-1|k-1}$ due to their whiteness. Therefore we get

$$E\{\boldsymbol{v}_{k|k-1}\boldsymbol{e}_k^{\mathrm{T}}\} = E\{\boldsymbol{\epsilon}_k\boldsymbol{e}_k^{\mathrm{T}}\} = C_k.$$

We return to the main development. The covariance matrix for the estimation error is

$$\begin{aligned}
P_{k|k} &= E\{\boldsymbol{v}_{k|k}\boldsymbol{v}_{k|k}^{\mathrm{T}}\} \\
&= E\Big\{\big((I - K_kA_k)\boldsymbol{v}_{k|k-1} - K_k\boldsymbol{e}_k\big)\big((I - K_kA_k)\boldsymbol{v}_{k|k-1} - K_k\boldsymbol{e}_k\big)^{\mathrm{T}}\Big\} \\
&= (I - K_kA_k)P_{k|k-1}(I - K_kA_k)^{\mathrm{T}} + K_k\Sigma_{e,k}K_k^{\mathrm{T}} \\
&\qquad - (I - K_kA_k)C_kK_k^{\mathrm{T}} - K_kC_k^{\mathrm{T}}(I - K_kA_k)^{\mathrm{T}}.
\end{aligned} \tag{17.1}$$

This is a general expression for the error covariance matrix and valid for any gain matrix K_k. The two last terms are introduced due to the cross-correlation C_k. For $C_k = 0$ we recover the earlier Joseph form of the variance.

We want to rewrite (17.1), and collect a factor from the first and fourth terms and manipulate the other terms:

$$P_{k|k} = P_{k|k-1} - K_k(A_k P_{k|k-1} + C_k^{\mathrm{T}}) + K_k(A_k P_{k|k-1} A_k^{\mathrm{T}} + \Sigma_{e,k} + A_k C_k + C_k^{\mathrm{T}} A_k^{\mathrm{T}}) K_k^{\mathrm{T}} \\ - (P_{k|k-1} A_k^{\mathrm{T}} + C_k)^{\mathrm{T}} K_k^{\mathrm{T}} \quad (17.2)$$

The Kalman matrix associated with a minimum estimation error is determined by setting $\frac{\partial\, \mathrm{tr}(P_{k|k})}{\partial K_k} = 0$ or

$$-2(A_k P_{k|k-1})^{\mathrm{T}} + 2K_k A_k P_{k|k-1} A_k^{\mathrm{T}} + 2K_k \Sigma_{e,k} - 2C_k + 2K_k A_k C_k + 2K_k C_k^{\mathrm{T}} A_k^{\mathrm{T}} = 0.$$

This yields

$$K_k = \big(P_{k|k-1} A_k^{\mathrm{T}} + C_k\big)\big(A_k P_{k|k-1} A_k^{\mathrm{T}} + \Sigma_{e,k} + A_k C_k + C_k^{\mathrm{T}} A_k^{\mathrm{T}}\big)^{-1}. \quad (17.3)$$

Let $C_k \to 0$ and this equation reduces to the gain in the zero cross-correlation model

$$K_k = P_{k|k-1} A_k^{\mathrm{T}} \big(A_k P_{k|k-1} A_k^{\mathrm{T}} + \Sigma_{e,k}\big)^{-1} \quad (17.4)$$

as it should be. Finally we insert (17.3) into (17.2) and get

$$P_{k|k} = (I - K_k A_k) P_{k|k-1} - K_k C_k^{\mathrm{T}}. \quad (17.5)$$

The prediction equations are not affected by the cross-correlation between $\boldsymbol{\epsilon}_k$ and $\boldsymbol{e}_k$ because of the whiteness property of each.

A possible correlation between the process noise $\boldsymbol{\epsilon}_k$ and the observation noise $\boldsymbol{e}_k$ leads to modifications of the Kalman gain matrix K_k and the expression for the filter covariance matrix $P_{k|k}$. The modifications involving terms with C_k are in (17.3) and (17.5).

Useful partial derivatives of traces and the determinant:

$$
\begin{aligned}
\frac{\partial \operatorname{tr}(A)}{\partial A} &= I \\
\frac{\partial \operatorname{tr}(AB)}{\partial A} &= \begin{cases} B^{\mathrm{T}} & \text{if } A \text{ is asymmetric} \\ B + B^{\mathrm{T}} - \operatorname{diag}(B), & \text{if } A \text{ is symmetric} \end{cases} \\
\frac{\partial \operatorname{tr}(BAC)}{\partial A} &= B^{\mathrm{T}} C^{\mathrm{T}} \\
\frac{\partial \operatorname{tr}(ABA^{\mathrm{T}})}{\partial A} &= A(B + B^{\mathrm{T}}) \\
\frac{\partial \operatorname{tr}(e^{A})}{\partial A} &= e^{A^{\mathrm{T}}} \\
\frac{\partial |BAC|}{\partial A} &= |BAC| \big(A^{-1}\big)^{\mathrm{T}}
\end{aligned}
$$

REFERENCES

Brown, Robert Grover & Hwang, Patrick Y. C. (1997). *Introduction to Random Signals and Applied Kalman Filtering*. John Wiley & Sons, Inc., New York, 3rd edition.