AMBIGUITY ESTIMATION

The Two-Frequency Case

We consider ***double differenced*** observations. We omit the subscripts and superscripts related to the receivers and satellites, since there are exactly two of each:

$$
\begin{aligned}
P_1 &= \rho^* + I - e_1 \\
\Phi_1 &= \rho^* - I + \lambda_1 N_1 - \epsilon_1 \\
P_2 &= \rho^* + (f_1/f_2)^2 I - e_2 \\
\Phi_2 &= \rho^* - (f_1/f_2)^2 I + \lambda_2 N_2 - \epsilon_2.
\end{aligned}
\tag{1}
$$

Actually, we have $\alpha = f_1/f_2 = 154/120 = 1.283\,333\ldots$. Equation (1) is transformed into the elegant matrix equation

$$\begin{bmatrix} P_1 \\ \Phi_1 \\ P_2 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & \lambda_1 & 0 \\ 1 & \alpha^2 & 0 & 0 \\ 1 & -\alpha^2 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \rho^* \\ I \\ N_1 \\ N_2 \end{bmatrix} - \begin{bmatrix} e_1 \\ \epsilon_1 \\ e_2 \\ \epsilon_2 \end{bmatrix}. \tag{2}$$

When all e and ϵ values are set to zero, we can solve the four equations to find the four unknowns. This determines the ideal pseudorange ρ^*, the instantaneous ionospheric delay I, and the ambiguities N_1 and N_2.

AALBORG UNIVERSITY

The inverse coefficient matrix is

$$\begin{bmatrix} \frac{\alpha}{\alpha-1} & 0 & -\frac{1}{\alpha-1} & 0 \\ -\frac{1}{\alpha-1} & 0 & \frac{1}{\alpha-1} & 0 \\ -\frac{\alpha+1}{\lambda_1(\alpha-1)} & \frac{1}{\lambda_1} & \frac{2}{\lambda_1(\alpha-1)} & 0 \\ -\frac{2\alpha}{\lambda_2(\alpha-1)} & 0 & -\frac{1}{\lambda_2} & \frac{1}{\lambda_2} \end{bmatrix}.$$

In one epoch of observations we estimate the *reals* N_1 and N_2 from (2). Next, the following Matlab code estimates the *integers* N_1 and N_2 from the real values just computed (the algorithm is due to Clyde C. Goad):

```
K1 = round(N1 − N2);
K2 = round(60 ∗ N1 − 77 ∗ N2);
trueN2 = round((60 ∗ K1 − K2)/17);
trueN1 = round(trueN2 + K1);
```

Observation Equations

Let the vector $\boldsymbol{b}$ contain the three components of the baseline, the vector $\boldsymbol{a}$ contains ambiguities for L_1 and possibly for L_2. The double differenced observations are collected in the vector $\boldsymbol{y}$:

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{bmatrix} = \boldsymbol{y} + \text{errors}. \tag{3}$$

The normal equations are

$$\begin{bmatrix} B^{\mathrm{T}}B & B^{\mathrm{T}}A \\ A^{\mathrm{T}}B & A^{\mathrm{T}}A \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{b}} \\ \hat{\boldsymbol{a}} \end{bmatrix} = \begin{bmatrix} B^{\mathrm{T}} \\ A^{\mathrm{T}} \end{bmatrix} \boldsymbol{y} \tag{4}$$

and the solution is

$$\begin{bmatrix} \hat{\boldsymbol{b}} \\ \hat{\boldsymbol{a}} \end{bmatrix} = \begin{bmatrix} B^{\mathrm{T}}B & B^{\mathrm{T}}A \\ A^{\mathrm{T}}B & A^{\mathrm{T}}A \end{bmatrix}^{-1} \begin{bmatrix} B^{\mathrm{T}} \\ A^{\mathrm{T}} \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} Q_{\hat{b}} & Q_{\hat{b}\hat{a}} \\ Q_{\hat{b}\hat{a}}^{\mathrm{T}} & Q_{\hat{a}} \end{bmatrix} \begin{bmatrix} B^{\mathrm{T}} \\ A^{\mathrm{T}} \end{bmatrix} \boldsymbol{y}. \qquad (5)$$

The components of the solution vector $\hat{\boldsymbol{a}}$ are reals; however, we want a solution $\check{\boldsymbol{a}}$ of *integers!*

Hence we find an integer vector $\boldsymbol{a}$ such that

$$(\hat{\boldsymbol{a}} - \boldsymbol{a})^{\mathrm{T}} Q_{\hat{a}}^{-1} (\hat{\boldsymbol{a}} - \boldsymbol{a}) = \min . \qquad \text{over integer vectors } \boldsymbol{a}. \qquad (6)$$

After the integer solution $\check{\boldsymbol{a}}$ is found we substitute it for $\hat{\boldsymbol{a}}$. Consequently the solution for $\hat{\boldsymbol{b}}$ changes to $\check{\boldsymbol{b}}$.

In order to determine $\check{\boldsymbol{b}}$ we multiply the lower block row in (5) by $Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}$ and subtract from the upper block row:

$$\begin{bmatrix} \hat{\boldsymbol{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \hat{\boldsymbol{a}} \\ \hat{\boldsymbol{a}} \end{bmatrix} = \begin{bmatrix} Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} & 0 \\ Q_{\hat{b}\hat{a}}^{\mathrm{T}} & Q_{\hat{a}} \end{bmatrix} \begin{bmatrix} B^{\mathrm{T}} \\ A^{\mathrm{T}} \end{bmatrix} \boldsymbol{y}. \tag{7}$$

The upper block row gives

$$\hat{\boldsymbol{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \hat{\boldsymbol{a}} = \left(Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \right) B^{\mathrm{T}} \boldsymbol{y}.$$

The right side is known and constant. If we change $\hat{\boldsymbol{a}}$ to $\check{\boldsymbol{a}}$ then $\hat{\boldsymbol{b}}$ changes to $\check{\boldsymbol{b}}$ and we have:

$$\hat{\boldsymbol{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \hat{\boldsymbol{a}} = \check{\boldsymbol{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \check{\boldsymbol{a}}$$

or

$$\check{\boldsymbol{b}} = \hat{\boldsymbol{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{\boldsymbol{a}} - \check{\boldsymbol{a}}). \tag{8}$$

The right side is known and $\check{\boldsymbol{b}}$ is quickly found.

Minimizing a Quadratic Expression over Integer Variables

If Q_a^{-1} in (6) is diagonal, the best vector comes from rounding each component of $\hat{\boldsymbol{a}}$ to the nearest integer. The components are uncoupled when Q_a^{-1} is diagonal. The quadratic is purely a sum of squares $\sum Q_{jj}^{-1}(a_j - \hat{a}_j)^2$. The minimum comes by making each term as small as possible. So the best $\check{a}_j$ is the the integer nearest to $\hat{a}_j$.

In practice the individual $\hat{a}_j$ components are highly correlated. Recalling that we often deal with 20 ambiguities we realize that a search procedure to find the minimum is unrealistic. Therefore an idea about decorrelating the ambiguities best possible, before starting the search, should lead to a good procedure. This is just what the Lambda method does.

The first step consists in transforming Q_a^{-1} such that its off-diagonal entries become numerically smaller; these entries measure the correlation.

We start by computing the LDL^{T} decomposition of the given covariance matrix

$$Q_a^{-1} = LDL^{\mathrm{T}}.$$

Next we construct a matrix of integers Z from L by a sequence of *integer Gauss transformations* and *permutations* such that

$$Q_z = Z^{\mathrm{T}} Q_a Z$$

is as "diagonal" as possible and

$$\boldsymbol{z} = Z^{\mathrm{T}} \boldsymbol{a}. \tag{9}$$

Now the search defined by (6) is substituted by a search over integers $\boldsymbol{z}$:

$$(\hat{\boldsymbol{z}} - \boldsymbol{z})^{\mathrm{T}} Q_{\hat{z}}^{-1} (\hat{\boldsymbol{z}} - \boldsymbol{z}) = \min . \tag{10}$$

The choice of Z is based on integer elimination starting with the first row of L. There will certainly be row exchanges and therefore Z will not be triangular. The essential idea was given by Lenstra, Lenstra, and Lovász in 1982, and the algorithm is sometimes called L^3. For futher details see Strang and Borre (1997), pages 495–499.

The result of the search is the integer vector $\check{\boldsymbol{z}}$. Finally $\check{\boldsymbol{z}}$ is transformed back to the ambiguity space according to

$$\check{\boldsymbol{a}} = \left(Z^{-1}\right)^{\mathrm{T}} \check{\boldsymbol{z}}. \tag{11}$$

Note that both Z and Z^{-1} must have integer entries. A consequence of this is that $\det(Z) = 1$ as seen from Cramer's rule. The condition on the determinant means that the Z transformation preserves the search volume. We have transformed a highly correlated space (elongated ellipses) by a sphere-like search space and this deminishes the search time tremendously. The M-file l_detail illustrates the computational steps including several numerical details.

Details of the implementation can be found in the report
http://www.geo.tudelft.nl/mgp/papers/pdf/lgr12.pdf

The Three-Frequency Case

The modernized GPS will have a third frequency $f_5 = 1176.45\,\text{MHz} = 115\,f_0$. The basic frequency is $f_0 = 10.23\,\text{MHz}$. Remember that the L_2 signal is at $f_2 = 1227.60\,\text{MHz} = 120\,f_0$ and the L_1 signal is at $f_1 = 1575.42\,\text{MHz} = 154\,f_0$.

We suppose given the three pseudoranges P_1, P_2, and P_5 and the observed phases ϕ_1, ϕ_2, and ϕ_5 (in cycles) from one epoch. We seek N_1, N_2, and N_5.

Ron Hatch *et al.* have suggested the subsequent algorithm:

```
v_light = 299792458;   % vacuum speed of light m/s
f1 = 154*10.23E6;      % L1 frequency Hz
f2 = 120*10.23E6;      % L2 frequency Hz
f5 = 115*10.23E6;      % L5 frequency Hz
```

```
lambda1 = v_light/f1;  % wavelength on L1:  .19029367 m
lambda2 = v_light/f2;  % wavelength on L2:  .244210213 m
lambda5 = v_light/f5;  % wavelength on L5:  .2528280488 m

%We compute combinations of wavelengths
l5 = 1/(1/lambda2 – 1/lambda5);  % 5.861 m
l8 = 1/(1/lambda1 – 1/lambda2);  % 0.862 m
l11 = 1/(1/lambda1 + 1/lambda2); % 0.107 m

%Given observations
P1 = ...
P2 = ...
P5 = ...
```

$$phi1 = \ldots$$

$$phi2 = \ldots$$

$$phi5 = \ldots$$

$$N58 = (f2*P2+f5*P5)/(l5*(f2-f5))-(phi2-phi5);$$

$$N86 = (f1*P1+f2*P2)/(l8*(f1+f2))-(phi1-phi2);$$

$$N11 = (f1*P1-f2*P2)/(l11*(f1-f2))-(phi1+phi2)/2;$$

$$N1 = (N86+N11)/2;$$

$$N2 = (N86-N11)/2;$$

$$N5 = N2-N58;$$

Hatch, Ron & Jaewoo Jung & Per Enge & Boris Pervan (2000):*Civilian GPS: The Benefits of Three Frequencies*