Coordinate Changes From Datum Changes

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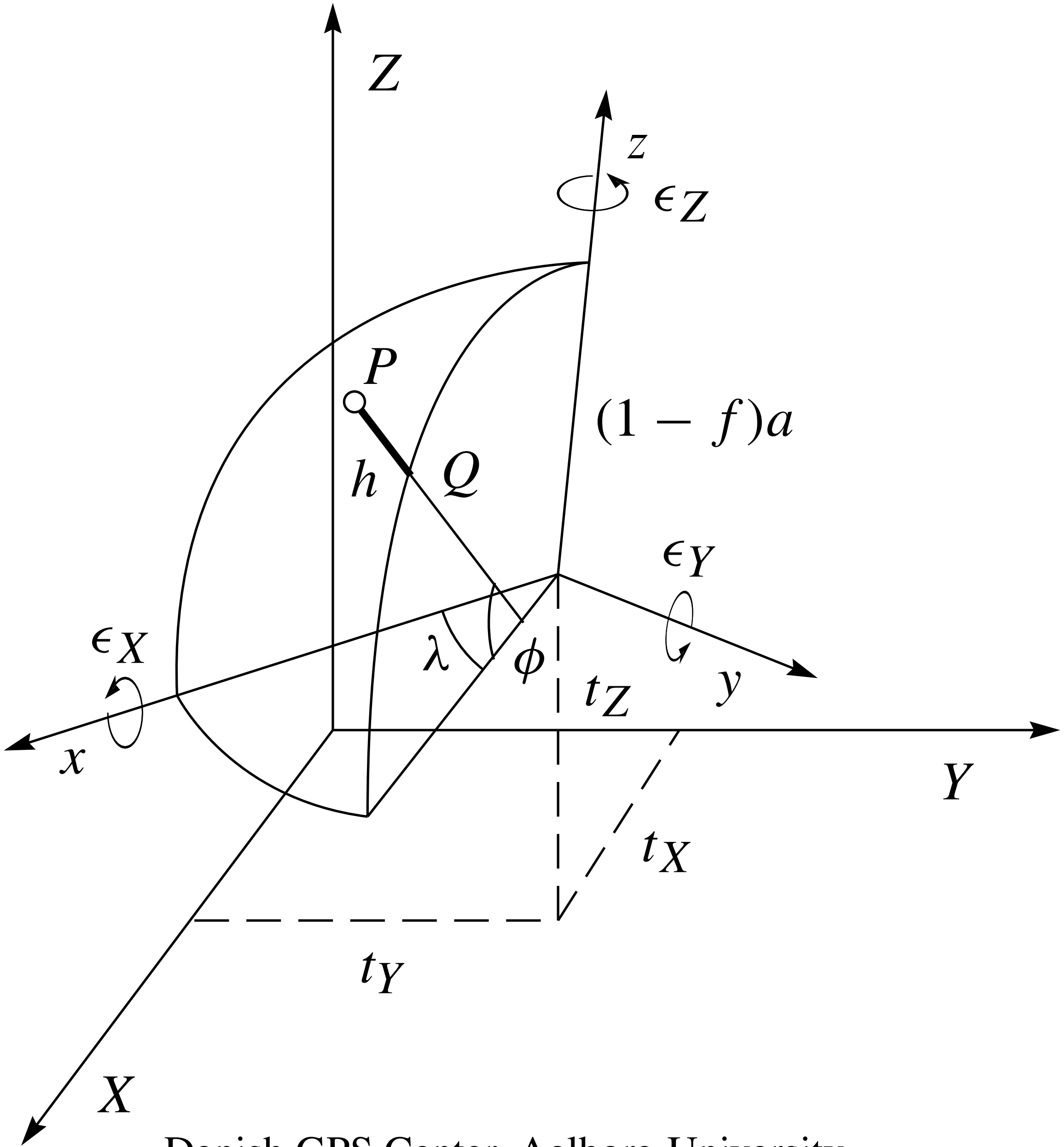
In geodetic practice it is today common to have coordinates derived from GPS and also from traditional terrestrial methods. For computations we convert the local *topocentric* coordinates to *geocentric* coordinates, which can be compared with the GPS derived coordinates. Thus physical control points may have two sets of coordinates, one derived from classical methods and one derived from GPS.

The connection between topocentric and geocentric coordinates is established by transformation formulas. The most general transformation includes rotations, translations, and a change of scale. *This type of transformation is only established between Cartesian systems*. It is described by seven parameters: three translations t_X, t_Y, t_Z, three rotations $\epsilon_X, \epsilon_Y, \epsilon_Z$, and a change of scale k. We assume that the transformation is infinitesimal.



Z
z
ϵ_Z
P
$(1 - f)a$
h
Q
ϵ_Y
ϵ_X
λ
ϕ
t_Z
y
x
Y
t_X
t_Y
X

Our starting point is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ \big((1-f)^2N+h\big)\sin\phi \end{bmatrix} \tag{1}$$

which for small changes becomes

$$\boldsymbol{T} = \begin{bmatrix} 1 & \epsilon_Z & -\epsilon_Y \\ -\epsilon_Z & 1 & \epsilon_X \\ \epsilon_Y & -\epsilon_X & 1 \end{bmatrix} \begin{bmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ \big((1-f)^2N+h\big)\sin\phi \end{bmatrix} + k \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}. \tag{2}$$

A first order Taylor expansion of the radius of curvature N in the prime vertical (that is the direction orthogonal to the meridian)

$$N = a \Big/ \sqrt{1 - f(2-f)\sin^2\phi} \approx a(1 + f\sin^2\phi)$$

yields

$$(1-f)^2 N \approx (1-2f+f^2)a(1+f\sin^2\phi) \approx a(1+f\sin^2\phi - 2f).$$

We substitute into equation (2) and introduce $\boldsymbol{t} = (t_X, t_Y, t_Z)$:

$$\boldsymbol{T} = \begin{bmatrix} 1 & \epsilon_Z & -\epsilon_Y \\ -\epsilon_Z & 1 & \epsilon_X \\ \epsilon_Y & -\epsilon_X & 1 \end{bmatrix} \begin{bmatrix} \big(a(1+f\sin^2\phi)+h\big)\cos\phi\cos\lambda \\ \big(a(1+f\sin^2\phi)+h\big)\cos\phi\sin\lambda \\ \big(a(1+f\sin^2\phi-2f)+h\big)\sin\phi \end{bmatrix} + k \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \boldsymbol{t}$$

$$= R\,\boldsymbol{X} + k\,\boldsymbol{X} + \boldsymbol{t}. \qquad (3)$$

To find the linear dependency between the variables we differentiate $\boldsymbol{T}$:

$$d\boldsymbol{T} = R\,d\boldsymbol{X} + dR\,\boldsymbol{X} + dk\,\boldsymbol{X} + k\,d\boldsymbol{X} + d\boldsymbol{t}.$$

The product $k\,d\boldsymbol{X}$ is of second order. To first order we have $R\,d\boldsymbol{X} \approx d\boldsymbol{X}$ and thus

$$dT = dX + dR\,X + dk\,X + dt.$$

This yields the corrections $d\lambda$, $d\phi$, dh from changes in the size a and shape f of the ellipsoid, rotation by the small angles $d\epsilon_X$, $d\epsilon_Y$, $d\epsilon_Z$, and translations by dt_X, dt_Y, dt_Z. This can be done by setting $dT = \mathbf{0}$. Then a point before and after the transformation remains physically the same:

$$dX = -dR\,X - dk\,X - dt. \tag{4}$$

The left side dX depends on λ, ϕ, h, a, and f. The *total differential* of X in (3) is

$$dX = \frac{\partial X}{\begin{bmatrix} \partial\lambda & \partial\phi & \partial h \end{bmatrix}} \begin{bmatrix} d\lambda \\ d\phi \\ dh \end{bmatrix} + \frac{\partial X}{\begin{bmatrix} \partial a & \partial f \end{bmatrix}} \begin{bmatrix} da \\ df \end{bmatrix} =$$

$$
\begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} a\cos\phi\,d\lambda \\ a\,d\phi \\ dh \end{bmatrix}
$$

$$
+ \begin{bmatrix} \cos\phi\cos\lambda & \sin^2\phi\cos\phi\cos\lambda \\ \cos\phi\sin\lambda & \sin^2\phi\cos\phi\sin\lambda \\ \sin\phi & (\sin^2\phi - 2)\sin\phi \end{bmatrix} \begin{bmatrix} da \\ a\,df \end{bmatrix} = F\boldsymbol{\alpha} + G\boldsymbol{\beta}. \qquad (5)
$$

Substitution into (4) gives

$$
F\boldsymbol{\alpha} + G\boldsymbol{\beta} = -dR\,\boldsymbol{X} - dk\,\boldsymbol{X} - d\boldsymbol{t}.
$$

We look for $\boldsymbol{\alpha}$ and isolate it—remembering that orthogonality implies $F^{-1} = F^{\mathrm{T}}$:

$$
\boldsymbol{\alpha} = -F^{\mathrm{T}} dR\,\boldsymbol{X} - dk\,F^{\mathrm{T}}\boldsymbol{X} - F^{\mathrm{T}}d\boldsymbol{t} - F^{\mathrm{T}}G\boldsymbol{\beta}. \qquad (6)
$$

Now follows a calculation of each component in $\boldsymbol{\alpha}$:

$$-F^{\mathrm{T}} dR\, \boldsymbol{X} = -F^{\mathrm{T}} \begin{bmatrix} d\epsilon_Z\, Y - d\epsilon_Y\, Z \\ -d\epsilon_Z\, X + d\epsilon_X\, Z \\ d\epsilon_Y\, X - d\epsilon_X\, Y \end{bmatrix} = -F^{\mathrm{T}} \begin{bmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{bmatrix} \begin{bmatrix} d\epsilon_X \\ d\epsilon_Y \\ d\epsilon_Z \end{bmatrix}$$

$$= \begin{bmatrix} -a \sin\phi \cos\lambda & -a \sin\phi \sin\lambda & a\cos\phi \\ a\sin\lambda & -a\cos\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d\epsilon_X \\ d\epsilon_Y \\ d\epsilon_Z \end{bmatrix},$$

$$-F^{\mathrm{T}} G = \begin{bmatrix} 0 & 0 \\ 0 & -\sin\phi\cos^3\phi \\ -1 & \sin^2\phi \end{bmatrix}, \qquad \text{and} \qquad -F^{\mathrm{T}} \boldsymbol{X} = -\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}.$$

This allows us to give an explicit expression for $\boldsymbol{\alpha}$:

$$
\begin{bmatrix} a\cos\phi\, d\lambda \\ a\, d\phi \\ dh \end{bmatrix} =
\begin{bmatrix} \sin\lambda & -\cos\lambda & 0 \\ \sin\phi\cos\lambda & \sin\phi\sin\lambda & -\cos\phi \\ -\cos\phi\cos\lambda & -\cos\phi\sin\lambda & -\sin\phi \end{bmatrix}
\begin{bmatrix} dt_X \\ dt_Y \\ dt_Z \end{bmatrix}
$$

$$
+ \begin{bmatrix} -a\sin\phi\cos\lambda & -a\sin\phi\sin\lambda & a\cos\phi \\ a\sin\lambda & -a\cos\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} d\epsilon_X \\ d\epsilon_Y \\ d\epsilon_Z \end{bmatrix}
$$

$$
- \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} dk - \begin{bmatrix} 0 & 0 \\ 0 & a\sin\phi\cos^3\phi \\ 1 & -a\sin^2\phi \end{bmatrix}
\begin{bmatrix} da \\ df \end{bmatrix}. \qquad (7)
$$

The number of datum parameters has been augmented by two, namely the changes in the ellipsoidal parameters a and f. The total number of parameters is thus nine. How does that agree with the fact that a datum is defined by five parameters? Well, we have increased this number by three rotations and one change of scale. By that we again come to the number nine and our bookkeeping is in balance again.

Example An example of a transformation of the type decribed by (2) is the transformation between European Datum 1950 ED 50 and WGS 84:

$$\begin{bmatrix} X_{\mathrm{WGS\,84}} \\ Y_{\mathrm{WGS\,84}} \\ Z_{\mathrm{WGS\,84}} \end{bmatrix} = \begin{bmatrix} 0.0\,\mathrm{m} \\ 0.0\,\mathrm{m} \\ 4.5\,\mathrm{m} \end{bmatrix} + k \begin{bmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\mathrm{ED\,50}} - 89.5\,\mathrm{m} \\ Y_{\mathrm{ED\,50}} - 93.8\,\mathrm{m} \\ Z_{\mathrm{ED\,50}} - 127.6\,\mathrm{m} \end{bmatrix}.$$

The inverse transformation back to the European Datum is

$$\begin{bmatrix} X_{\mathrm{ED\,50}} \\ Y_{\mathrm{ED\,50}} \\ Z_{\mathrm{ED\,50}} \end{bmatrix} = \frac{1}{k} \begin{bmatrix} 1 & -\alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\mathrm{WGS\,84}} \\ Y_{\mathrm{WGS\,84}} \\ Z_{\mathrm{WGS\,84}} - 4.5\,\mathrm{m} \end{bmatrix} + \begin{bmatrix} 89.5\,\mathrm{m} \\ 93.8\,\mathrm{m} \\ 127.6\,\mathrm{m} \end{bmatrix}.$$

The variables have the following values: $\alpha = 0.156'' = 0.756 \times 10^{-6}$ rad and $k = 1 + 1.2 \times 10^{-6}$ or $1/k = 0.999\,998\,8$. The transformation looks a little more involved than earlier expressions. This is due to inclusion of some minor differences between WGS 84 and the original Doppler datum NSWC 9Z-2.