2-Dimensional Linear Interpolation

Let a scaler quantity be given at all three corners of a triangle. You may think of the scalar as the height. We want to interpolate this quantity z linearly at any point in the interior of the tringle.

Geometrically we determine the plane through the three points 1, 2, and 3 in a 3-D space and next compute z as function of x and y.

Analytical geometry tells us that the equation for the plane through 3 points is given by the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x & y & z & 1 \end{vmatrix} = 0.$$

We develop this determinant after the last column and get

$$\begin{aligned} &z(x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1) \\ &\quad + z_1(-x_3y + xy_3 + x_2y - xy_2 - x_2y_3 + x_3y_2) \\ &\quad + z_2(x_3y - xy_3 - x_1y + xy_1 + x_1y_3 - x_3y_1) \\ &\quad + z_3(-x_2y + xy_2 + x_1y - xy_1 - x_1y_2 + x_2y_1) = 0. \end{aligned}$$

This expression can rewritten into the following symmetric form

$$\begin{aligned} z = &\frac{(x_2 - x)(y_3 - y_2) - (y_2 - y)(x_3 - x_2)}{(x_2 - x_1)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2)} z_1 \\ &+ \frac{(x_3 - x)(y_1 - y_3) - (y_3 - y)(x_1 - x_3)}{(x_3 - x_2)(y_1 - y_3) - (y_3 - y_2)(x_1 - x_3)} z_2 \\ &+ \frac{(x_1 - x)(y_2 - y_1) - (y_1 - y)(x_2 - x_1)}{(x_1 - x_3)(y_2 - y_1) - (y_1 - y_3)(x_2 - x_1)} z_3. \end{aligned}$$

The denominator in all three terms is the double of the area of the triangle. For given z at all three points 1, 2, and 3 this expression interpolates z at any point (x, y).

The same expression may be derived by means of *barycentric coordinates*, see Strang & Fix (1973). The result is the following

$$z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Let triangle I be given by (0, 0), (0, 1), and (1, 1):

$$\begin{aligned} z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} &\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\ &\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 1 - y \\ y - x \\ x \end{bmatrix}. \end{aligned}$$

Verify that we obtain the original z-values at the corners.

Another triangle II is given by the points $(0,0)$, $(1,0)$, and $(1,1)$. Verify that the following expression is valid for triangle II:

$$z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 1-y \\ x+y-1 \\ 1-x \end{bmatrix}.$$