EECS 240 Random Processes
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Computer Project # 2 Report

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1. Problem Statement:

Consider an autoregressive (AR) random process $Y_n = \alpha Y_{n-1} + X_n$ where X_n is a white Gaussian noise with zero mean and variance σ_x^2.

- Plot some sample realizations of the above AR process when $\sigma_x^2 = 1 - \alpha^2$, for example for $\alpha = 0.3$ and $\alpha = 0.95$.
- Simulate and plot the auto-correlation $R_Y(k)$ for $\alpha = 0.3$ and $\alpha = 0.95$.
- Simulate and plot the power spectral density $S_Y(f)$ for $\alpha = 0.3$ and $\alpha = 0.95$.

Background:

The first-order autoregressive (AR) process Y_n with zero mean is defined by

$$Y_n = \alpha Y_{n-1} + X_n$$

X_n is a zero-mean white noise input random process with average power σ_x^2. Y_n can be viewed as the output of the system in Figure 1 for an iid input X_n.

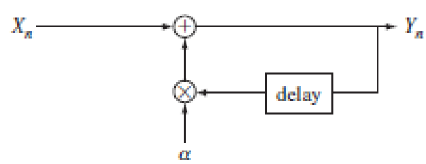


Figure 1: Generation of AR process

The unit-sample response can be determined by Y_n

$$h_\alpha = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ \alpha^n, & n > 0 \end{cases}$$

For the system to be stable, we require that $|\alpha| < 1$. Thus, the transfer function is as follows

$$H(f) = \sum_{n=0}^{\infty} \alpha^n e^{-j2\pi f n} = \frac{1}{1 - \alpha e^{-j2\pi f}}$$

The power spectral density of Y_n is the following:

$$S_Y(f) = \frac{\sigma_x^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}$$

The auto-correlation of Y_n is the following:

$$R_Y(k) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} h_j h_i \sigma_x^2 \delta_{k+j-i} = \sigma_x^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+k} = \frac{\sigma_x^2 \alpha^k}{1-\alpha^2}$$

Answer:

(a) Here, we are asked to look at two different cases for weighting constant α, which is multiplied to Y_{n-1} in the Y_n function. We consider $\alpha = 0.3$ and $\alpha = 0.95$. In Figure 2, two subplots show several sample realizations for these two α cases for the auto-regressive process model.

Notice in the top subplot for Figure 2, the AR sequences for $\alpha = 0.3$ have low correlation between the adjacent samples. This means the auto-regressive sequence will remain somewhat similar to the iid Gaussian sequence X_n. Notice in the bottom subplot for Figure 2, the AR sequence with $\alpha = 0.95$ has higher correlation correlation between adjacent samples. This high correlation tends to cause longer lasting trends which differ from the original iid Gaussian sequence X_n.

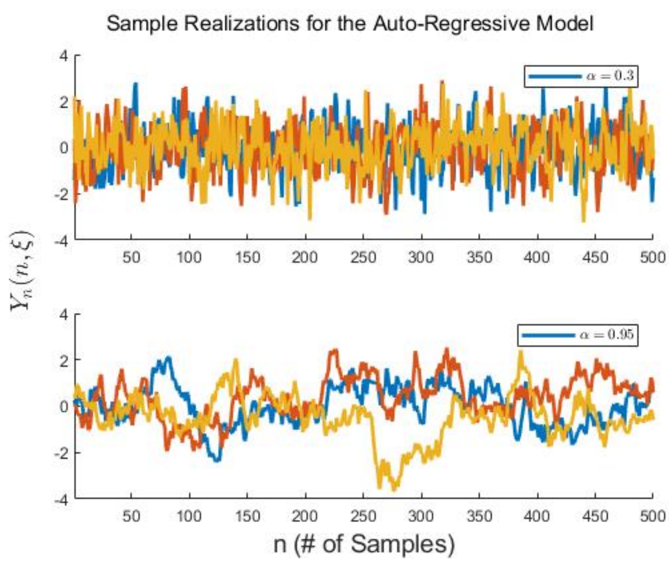


Figure 2: Different sample realizations for auto-regressive (AR) process with $\alpha = 0.3$ and $\alpha = 0.95$

(b) Next, we are asked to estimate and the auto-correlation for the auto-regression process. The auto-correlation, $R_Y(k)$, for a single sample realization was done using MATLAB's built-in function called `xcorr`. In particular, the 'biased' scaling option was used with `xcorr` to give proper scaling for the auto-correlation function.

The auto-correlation for the auto-regressive process can be shown in Figure 3. Notice, the AR sequence with $\alpha = 0.3$ has a defined sharp peak. This was expected since the α term had a low correlation between adjacent samples. Due to this, the plot should be relatively flat aside from the initial peak at $R_Y(0)$, where $R_Y(0)$ means there is no lag between the random process Y_n and itself. On the other hand, the AR sequence with $\alpha = 0.95$ has a much higher correlation correlation between adjacent samples. Thus, we do not expect the plot to be flat with a giant spike, rather we expect there to be several peaks throughout the auto-correlation plot. Recall, this is for a single realization of both $\alpha = 0.3$ and $\alpha = 0.95$ but we expect the same results for each random process realization.

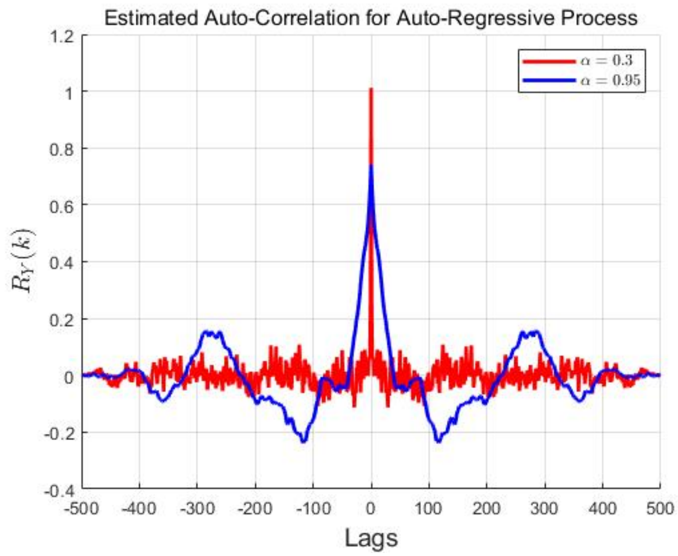


Figure 3: Auto-correlation for a single realization of $\alpha = 0.3$ and $\alpha = 0.95$

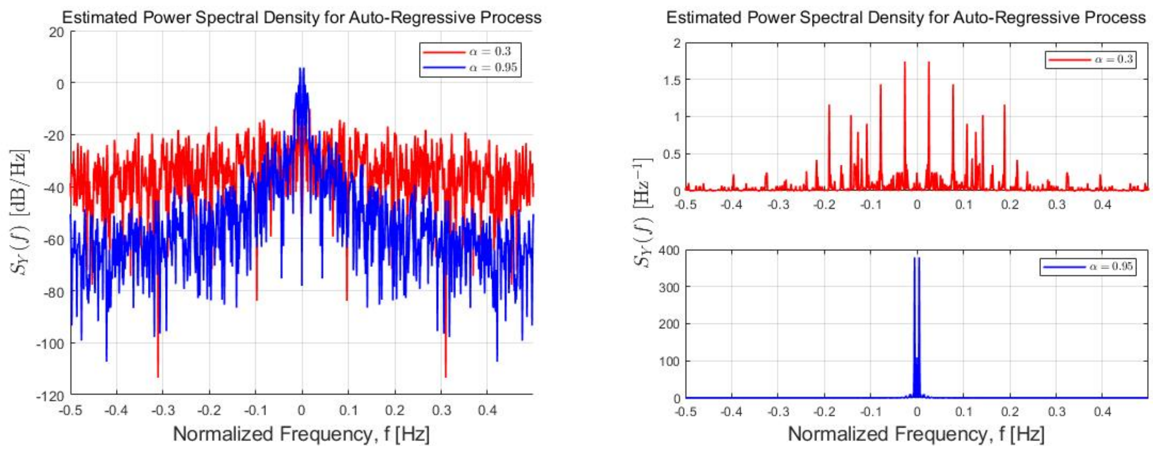
(c) Lastly, we are asked to estimate the power spectral density. Recall, the power spectral density is given by the following formula

$$S_Y(f) = \mathcal{F}\{R_Y(k)\} = \int_{-\infty}^{\infty} R_Y(k) e^{-j2\pi f t} dt$$

To perform the Fourier transform, we use the MATLAB built-in function `fft`. Furthermore, we will estimate the power spectral density (PSD) by using the normalized frequency [Hz] and power output in units of dB/Hz. First, we must perform a Fourier transform on our auto-correlation data using `fft`. Next, we must calculate the average power (i.e., $|x(t)|^2 = x^*(t) \cdot x(t)$) of our Fourier transformed data which will be scaled by a constant N. Finally, we shift the zero-frequency component to the center of the spectrum using the MATLAB built-in function `fftshift`.

The above algorithm yields a the power spectral density results in Figure 4(a). Notice, the AR sequence with $\alpha = 0.3$ has a power spectral density which resembles the power spectral density of white noise which is constant for all frequencies (i.e., $S_x(f) = N_0/2 \forall f$). This similarity is, again, due to the low correlation between the adjacent samples which yields a random process similar to iid Gaussian white noise. Furthermore, the AR sequence with $\alpha = 0.95$ has a much more defined power spectral density which is "peak-like". This phenomenon is because this AR sequence has a higher correlation between the adjacent samples.

Aside, but since $\alpha = 0.3$ had a "sharp" peak in the time-domain it had a "flat" peak in the frequency-domain. Also, since $\alpha = 0.95$ had a "flat" peak in time-domain while the frequency-domain had a "sharp" peak. The frequency-domain comparison can be seen most clearly in Figure 4(b). This happens because "poor" performance in either the time- or frequency-domain gives "good" performance in the other domain.



(a) (b)

Figure 4: Power Spectral Density (PSD) for a single realization of $\alpha = 0.3$ and $\alpha = 0.95$. a) PSD with units of [dB/Hz]. b) PSD with units of [Hz^{-1}1]

2. Problem Statement:
Consider the random process $X_n = \cos(0.2\pi n + \Omega)$ where Ω is a uniform random variable between $(-\pi, \pi)$. Draw one plot that contains 100 different realizations of X_n versus n (all in one plot). Use a dot to represent every (n, X_n) pair in the 2D plane.

Background:
Let ξ be selected at random uniformly from the interval $(-\pi, \pi)$. Furthermore, let $X_n(n, \xi) = \cos(2\pi f n + \xi)$. The different realizations of $X_n(n, \xi)$ are phase-shifted versions of $\cos(2\pi f n)$ where the sinusoid shifts with different values of phase with $\xi \sim \mathcal{U}(-\pi, \pi)$.

Answer:
Here, we are asked to consider the random phase sinusoid model of $X_n = \cos(0.2\pi n + \xi)$ where ξ is a uniform random variable between $(-\pi, \pi)$. Figure 5(a) displays the 100 different realizations of the random phase process for X_n. Notice, many (n, X_n) pairs seem to cover a wide range of sample paths. See in Figure 5(b), the sinusoid plots vary with different phase-shifted versions of $\cos(0.2\pi n)$. This is matches our expectation since phase-shifted sinusoids will produce different sample paths for a random phase sinusoid process.

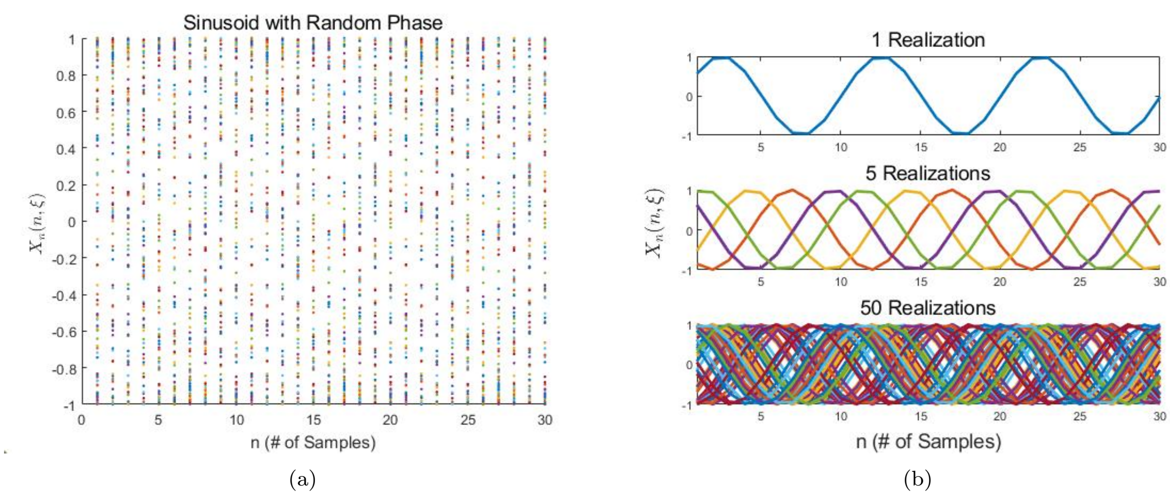


Figure 5: (a) Random phase sinusoid with 100 realizations of X_n vs n where a dot is used to represent ever (n, X_n) pair in the 2D plane. (b) Subplots with a different number of realizations for the random phase process.