Predicate Logic, Syntax, and Semantics

Exercise 1. Translate the following sentences to WFFs in Predicate Logic. Define the predicates, functions, and constant symbols as appropriate. **(21 marks)**

1. Everybody has a mother and a father.
2. All users must enter passwords that include special characters.
3. Every prize was won by a girl.
4. No lunatics is fit to serve on a jury.
5. Every sane person can do logic.
6. No student has more than one student number.
7. Whoever contributes to the Times is a writer.

Exercise 2. Use induction to define the following for an arbitrary WFF ϕ: **(10 marks)**

- *ForAll*(ϕ): The set of universally quantified variables in ϕ.
- *Predicates*(ϕ): The set of predicates.

For example, if $\phi_1 = \forall x \exists y R(x, y)$ and $\phi_2 = (\forall x \exists y R(x, y) \wedge \forall y P(y))$, then *ForAll*$(\phi_1) = \{x\}$, *ForAll*$(\phi_2) = \{x, y\}$, *Predicate*$(\phi_1) = \{R\}$, and *Predicate*$(\phi_2) = \{R, P\}$.

Exercise 3. Consider a sentence $\phi = \forall x\ (Q(x) \vee \exists z \forall y\ ((P(f(x), z) \wedge Q(a)) \vee \forall x\ R(y, z, g(x))))$.

1. Draw its syntax tree. **(5 marks)**
2. List all the sub-formulas, terms, predicates, functions, and free variables in ϕ. **(15 marks)**
3. Specify a model for ϕ (a structure that satisfies ϕ). This consists of a universe and interpretations of the predicates and the functions in the signature of the model. **(15 marks)**

Exercise 4. Decide if the following sentences hold in the structure of natural numbers $\mathbb{N}$, the structure of integers $\mathbb{Z}$, and the structure of real numbers $\mathbb{R}$. **(20 marks)**
Note: Traditionally, natural numbers start with 1 ($\{1, 2, 3, ...\}$). In computer science, they often start from 0 ($\{0, 1, 2, ...\}$). Assume natural numbers start from 0 for this exercise.

1. $\forall x \forall y\ (x + y = x \to y = 0)$.
2. $\forall x \forall y\ (x \times y = x \to y = 1)$.
3. $\exists x \forall y\ (x \times y = x + y \to y = x)$.
4. $\forall x \exists y\ (x \times y = x + y \to y = x)$.

Theories in Predicate Logic

Exercise 5. Answer the following questions about theories in Predicate Logic. Explain your answers. **(18 marks)**

- Group Theory is (finitely) axiomatizable.
- Group Theory is complete.
- $\{\}$ (the empty set) is a theory.
- $\forall x P(x)$ is in *Cons*$(\{\})$.
- $\forall x (P(x) \vee \neg P(x))$ is in *Cons*$(\{\})$.
- $\forall x P(x)$ is in *Cons*$(\exists x (P(x) \wedge \neg P(x)))$.

Natural Deduction and Resolution

Exercise 6. Prove the following entailments using natural deduction and resolution. **(30 marks)**

1. $\forall x \forall y (R(x,y) \lor R(y,x)) \vdash \forall x R(x,x)$.
2. $\forall x (P(x) \lor Q(x)), \exists x (\neg P(x)) \vdash \exists x Q(x)$.