Physics 253a Problem Set 11

Alex Nie

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Problem 1

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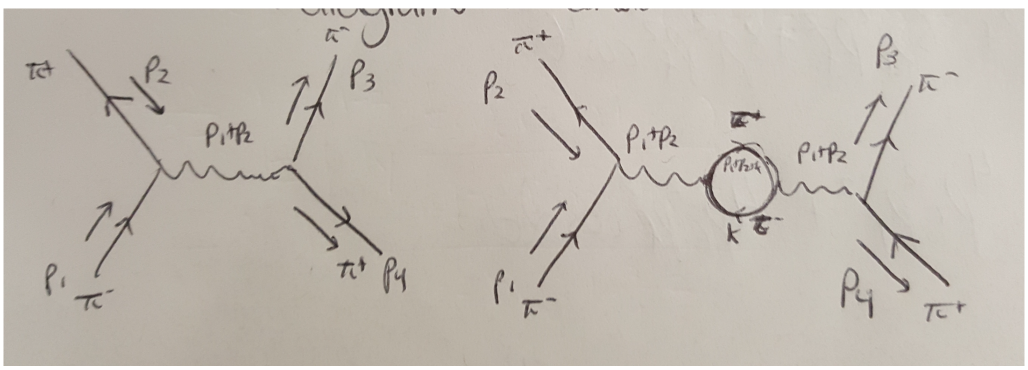
Problem 2

a)

We first calculate $|\mathcal{M}|^2$ using the following two diagrams:



Using the dressed propagator in the text, we simply need to add interaction terms for scalar QED. Then in the Feynman gauge,

$$\begin{aligned} i\mathcal{M} &= -ie^2(p_2-p_1)^\mu(p_4-p_3)^\nu \frac{(1-e^2\Pi_2(p^2))g^{\mu\nu}}{p^2} \\ &= \frac{-ie^2(4m_\pi^2-s)}{s}\left(1+\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(1-\frac{s}{m_\tau^2}x(1-x)\right)\right) \end{aligned}$$

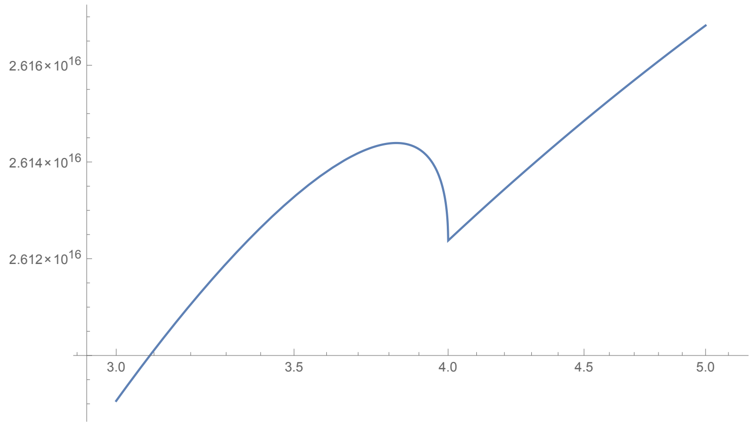
Where we have used the forward scattering condition $p_1 = p_3 \implies p_2 = p_4$ so $(p_2-p_1)^2 = 2m_\pi^2 - 2p_2\cdot p_1 = 4m_\pi^2 - s$. Then

$$|\mathcal{M}|^2 = \frac{e^4(4m_\pi^2-s)^2}{s^2}\left|1+\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(1-\frac{s}{m_\tau^2}x(1-x)\right)\right|^2$$

Allowing Mathematica to integrate this expression in $\frac{s}{m_\tau^2}$ we obtain a piecewise function which is split at $s = 4m_\tau^2$. Plotting, we see that this also corresponds to the kink desired. Physically, we can interpret this as the minimum energy scale at which the loop occurs. In particular, at $s < 4m_\tau^2$, the initial pions do not have enough energy to create the virtual tauons.

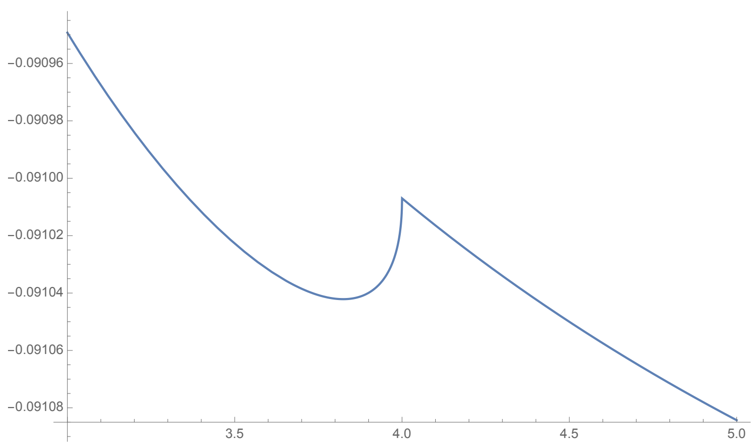
Plotting for the physical condition $s > 4m_\tau^2$, we obtain

$$|\mathcal{M}|^2 = \frac{e^4(4m_\pi^2 - Qm_\tau^2)^2}{(Qm_\tau)^2} \left(1 + \frac{e^2}{2\pi}\left(\frac{Q-4}{18(Q(4-Q))^{\frac{5}{2}}} \left(48Q\sqrt{(4-Q)Q} + 8\sqrt{(4-Q)Q^5} - 5\sqrt{(4-Q)Q^7} - 6(Q-4)^2Q(2+Q)\arctan\left(\sqrt{\frac{Q}{4-Q}}\right)\right)\right)\right)^2$$

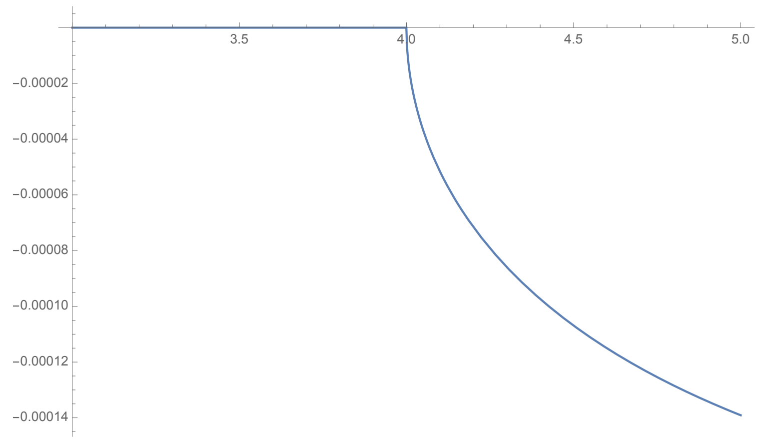


b)

Real part:



Imaginary part:

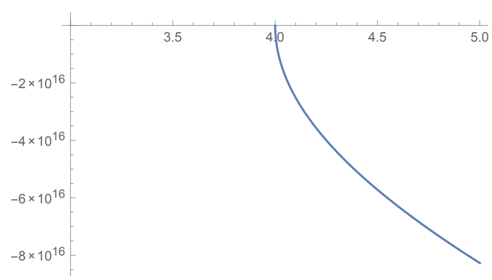


From the expression for $\mathcal{M}$ we see that the imaginary component comes from the argument of the logarithm. Specifically, the imaginary component contributes when $\frac{s}{m_\tau^2}x(1-x) > 1$. Since the resulting argument is again real, the imaginary component of the logarithm will always be $\pm\pi i$ depending on the branch cut. Integrating over this range,

Then

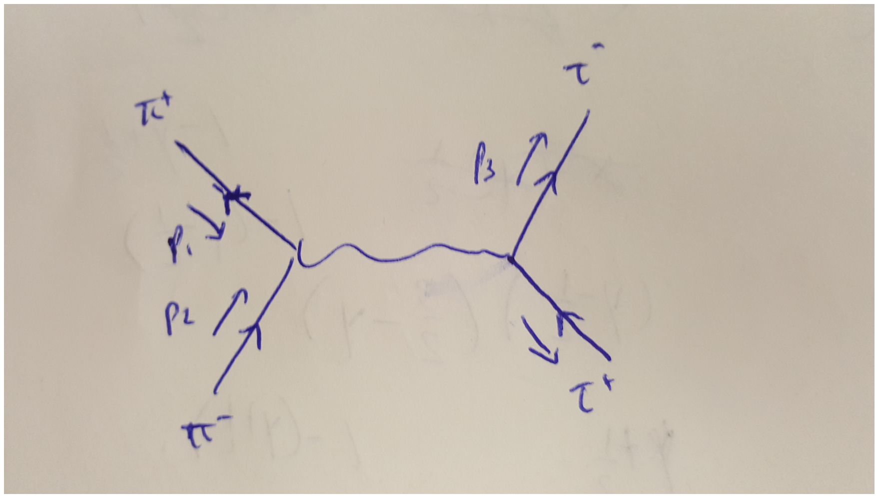
$$
\begin{aligned}
Im(\mathcal{M}) &= \frac{e^2(4m_\pi^2 - s)}{s}\left(\frac{e^2}{2\pi^2}(-\pi)\int_{\frac{1}{2}-\sqrt{\frac{s-4m_\tau^2}{4s}}}^{\frac{1}{2}+\sqrt{\frac{s-4m_\tau^2}{4s}}} dx\, x(1-x)\right) \\
&= \frac{e^4(s-4m_\pi^2)}{12\pi s}\sqrt{1+\frac{4m_\tau^2}{s}}(2m_\tau^2+s)
\end{aligned}
$$

This matches our previous calculation exactly:



c)

We first compute $|\mathcal{M}|^2$ for the following diagram:



Using the Feynman rules,

$$
\begin{aligned}
i\mathcal{M} &= e^2 \bar{u}_3 \gamma^\nu v_4 \frac{-i g_{\mu\nu}}{(p_1+p_2)^2 + i\epsilon} \bar{v}_2 \gamma^\mu u_1 \\
|\mathcal{M}|^2 &= \frac{e^4}{s^2} (p_2-p_1)^\mu (p_2-p_1)^\nu \, Tr[(\not{p}_3 + m\gamma^\mu(\not{p}_4 - m)\gamma^\mu] \\
&= \frac{4e^4}{s^2}(p_2-p_1)^\mu (p_2-p_1)^\nu \left(-m_\tau^2 g^{\mu\nu} + p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - (p_3\cdot p_4) g^{\mu\nu}\right) \\
&= \frac{4e^4}{s^2}\left(-(p_2-p_1)^2(p_3\cdot p_4 + m_\tau^2) + 2((p_3\cdot(p_2-p_1))(p_4\cdot(p_2-p_1))\right)
\end{aligned}
$$

In the zero momentum frame, $s = E_{CM}$. Using the coordinate system so that the initial momenta are aligned along the z axis, $p_2 - p_1 = (0,0,0,-2p_i)$. Let $p_3 = (\frac{1}{2}E_{CM}, p_f \sin\theta, 0, p_f\cos\theta)$ and $p_3 = (\frac{1}{2}E_{CM}, -p_f\sin\theta, 0, -p_f\cos\theta)$. Then

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{4e^4}{s^2}\left(4p_i^2(\frac{1}{4}E_{CM}^2 + p_f^2 + m_\tau^2) + 8p_i^2 p_f^2 \cos^2\theta\right) \\
&= \frac{32e^4}{E_{CM}^2}\left(p_i^2 p_f^2 \cos^2\theta + p_i^2 p_f^2 + p_i^2 m_\tau^2\right) \\
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 E_{CM}^2}\frac{p_f}{p_i}|\mathcal{M}|^2 \\
&= \frac{1}{2\pi^2 s^3}\frac{p_f}{p_i}\left(p_i^2 p_f^2\cos(2\theta) + \frac{1}{2}p_i^2(2m_\tau^2 + p_f^2)\right)
\end{aligned}
$$

Integrating over the solid angle, we obtain,

$$
\sigma_{tot} = \frac{e^4}{12\pi s}(2m_\tau^2 + 1)\sqrt{1 - \frac{4m_\pi^2}{s}}\sqrt{1 - \frac{4m_\tau^2}{s}}
$$

Thus we find the Optical theorem is

$$
\sigma_{tot} = \frac{1}{2E_{CM}|\vec{p}_f|} Im(\mathcal{M})
$$

Problem 3

a + b)

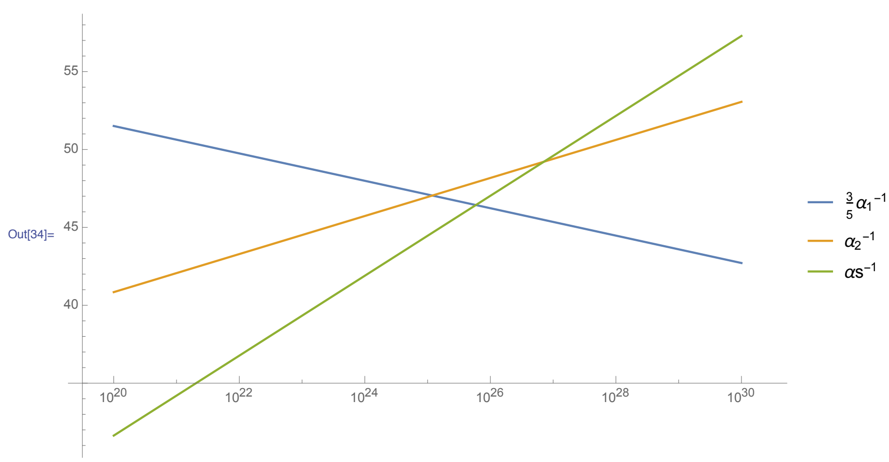
From the problem statement,

$$\begin{aligned}
\alpha_1(m_Z) &= (1-\sin^2\theta_W)\alpha_e(m_Z) \\
\frac{3}{5\alpha_1(\mu)} &= \frac{3}{5}\left(\frac{1}{(1-\sin^2\theta_W)\alpha_e(m_Z)} + \frac{\beta_{0,1}}{2\pi}\log\frac{m_Z}{\mu}\right) \\
\alpha_2(m_Z) &= \sin^2\theta_W\alpha_e(m_Z) \\
\frac{1}{\alpha_2(\mu)} &= \frac{1}{\sin^2\theta_W\alpha_e(m_Z)} + \frac{\beta_{0,2}}{2\pi}\log\frac{m_Z}{\mu} \\
\frac{1}{\alpha_s(\mu)} &= \frac{1}{\alpha_s(m_Z)} + \frac{\beta_{0,s}}{2\pi}\log\frac{m_Z}{\mu}
\end{aligned}$$

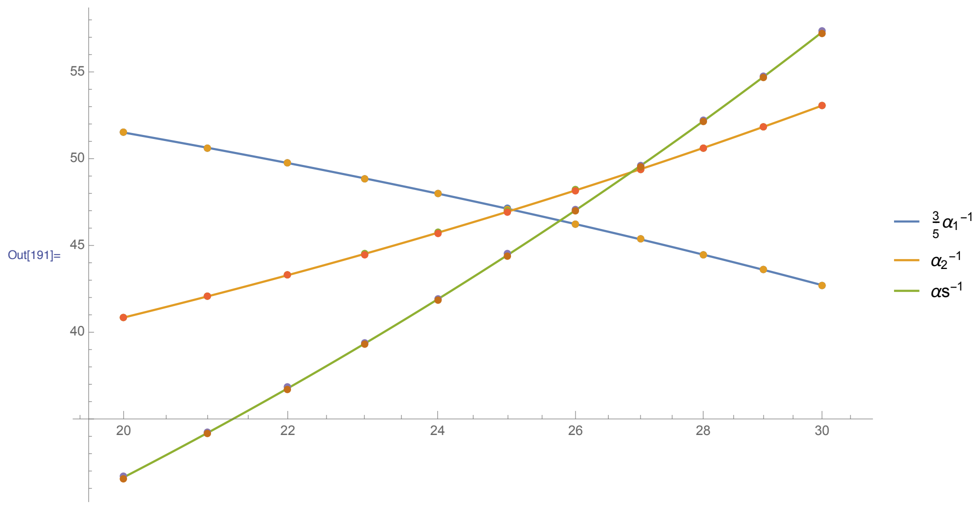
Substituting in

$$\begin{aligned}
\sin^2\theta_W &= .2312 \pm .0001 \\
\alpha_e^{-1}(m_Z) &= 128.9 \pm 10^{-5} \\
\beta_{0,1} &= 4 \\
\beta_{0,2} &= 4 - \frac{22}{3} \\
\beta_{0,s} &= -7
\end{aligned}$$

We have



We see that the forces nearly unify at about $10^{25}eV$.
Adding error bars, we see that convergence does not fall within experimental error:



c)

To calculate each Landau pole, we calculate the value of μ at which $\alpha^{-1} = 0$ so that the expansion is no longer perturbative and α blows up.

$$\frac{1}{\alpha_1^{-1}(m_Z)} + \frac{\beta_{0,1}}{2\pi} \log \frac{m_Z}{\mu} = 0$$

$$L_1 = 3.66 \times 10^{75}\, eV$$

Repeating the process for the other two terms, we find:

$$L_2 = 3.67 \times 10^{-14}\, eV$$

$$L_s = 4.65 \times 10^{7}\, eV$$

$$F_2(0) = \frac{\alpha m_e}{2\pi} \int_0^1 dx\, dy\, dz \delta(x+y+z-1) \frac{z(m_e(z-1) + m_{\tilde{\gamma}})}{(1-z)^2 m_{\tilde{e}}^2 + m_{\tilde{\gamma}}^2 z}$$

$$F_2(0) = \frac{\alpha m_2}{2\pi} \int_0^1 dz \frac{z(1-z)(m_e(z-1) + m_{\tilde{\gamma}})}{(1-z+z^2) m_{\tilde{e}}^2}$$