Physics 212 Problem Set 3

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Problem 1

In co-moving units, the horizon scale at a given z is

$$\eta = c\int_{\infty}^{z} \frac{dz'}{H(z')}$$

Substituting into the Friedmann equation for $\Omega_m = 1$, $H = H_0(1+z)^{\frac{3}{2}}$. Then at $z = 1000$,

$$\begin{aligned}
\eta_1 000 &= c\int_{1000}^{\infty} \frac{dz'}{H_0(1+z')^{\frac{3}{2}}} = \frac{2c}{H_0\sqrt{1001}} \\
&= \frac{2(3\times 10^5\, km\, s^{-1}}{100\, h^{-1}\, km\, Mpc^{-1}\, s^{-1}\sqrt{1001}} \\
&= 189.6\, h^{-1}\, Mpc
\end{aligned}$$

Today, the distance to an object at $z = 1000$ is

$$\begin{aligned}
\eta_0 &= c\int_0^z \frac{dz'}{H_0(1+z')^{\frac{3}{2}}} \\
&= \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1001}}\right) \\
&= \frac{2(3\times 10^5\, km\, s^{-1})}{100\, km\, Mpc^{-1}\, s^{-1}}\left(1 - \frac{1}{\sqrt{1001}}\right) \\
&= 5810\, Mpc
\end{aligned}$$

This yields a subtended angle in degrees of

$$\theta = \frac{\eta_{1000}}{\eta_0} = \frac{189.6}{5810}\frac{180}{\pi} = 1.87^o$$

Problem 2

Assuming the monopoles are in thermal equilibrium with radiation at the GUT transition, we first compute the redshift corresponding to a temperature of $10^{15}\, GeV$. We then compute the Hubble volume and divide the energy of a single monopole by this volume to obtain the relic density.

From class, we have that the temperature dependence of radiation is $T(z) = (1+z)T_0$. Then

$$z_{GUT} = \frac{T_{GUT}}{T_0} - 1 = \frac{10^{24}\, eV}{2.34\times 10^{-4}\, eV} - 1 = 4.27\times 10^{27}$$

The Hubble radius at this redshift is given by

$$r = \frac{1}{1+z}\int \frac{c\, dz'}{H(z')}$$

As the GUT transition happens well within the radiation-dominated era, $H(z) \approx H_0(1+z)^2$. Then

$$\begin{aligned} r &= \frac{1}{1+z}\int_{z_{GUT}}^{\infty} \frac{c\,dz'}{H_0(1+z')^2} \\ &= \frac{c}{H_0(1+z_{GUT})^2} \end{aligned}$$

The Hubble volume is then given by $\frac{4}{3}\pi r^3$ so

$$V = \frac{4\pi c^3}{3H_0^3(1+z_{GUT})^6}$$

Then the density of monopoles should be

$$V = \frac{4\pi(3\times 10^{10}\,\frac{cm}{s})^3}{3(2.27\times 10^{-18}s^{-1})^3(1+4.27\times 10^{27})^6} = 1.60\times 10^{-81}\,cm^3$$

$$n = V^{-1} = 6.30\times 10^{80}\,cm^{-3}$$

This yields an energy density of

$$\begin{aligned} \rho = \frac{M}{V} &= \frac{10^{24}\,\frac{eV}{c^2}}{1.60\times 10^{-81}\,cm^3}\left(\frac{1.6\times 10^{-19}\,J}{1\,eV}\right)\left(\frac{c}{3\times 10^8\,\frac{m}{s}}\right)^2 \\ &= 1.11\times 10^{72}\,\frac{g}{cm^3} \end{aligned}$$

As density scales as a^{-3}, then today this density is

$$1.11\times 10^{-72}\left(\frac{1}{1+z_{GUT}}\right)^3 = 1.42\times 10^{-11}\,\frac{g}{cm^3}$$

The Parker bound induces a maximum monopole energy density of

$$\rho_{max} = 10^{-6}\left(\frac{3H_0^2}{8\pi G}\right) = 10^{-6}\left(\frac{3(2.27\times 10^{-18}s^{-1})^2}{8\pi(6.67\times 10^{-11}m^3\,kg^{-1}}\right) = 9.22\times 10^{-36}$$

Then since density scales as a^{-3}, the number of e-folds needed is

$$N = \frac{1}{3}\ln\left(\frac{\rho}{\rho_{max}}\right) = 18.6$$

Problem 3

a)

In the slow-roll regime, we have that $V \gg \dot{\phi}^2 \implies \rho = \frac{1}{2}\dot{\phi}^2 + V \approx V$.

$$\begin{aligned} 3H\dot{\phi} &= -V' \\ H^2 &= \frac{8\pi G}{3}V \end{aligned}$$

Next,

$$\epsilon = -\frac{1}{H^2}\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right)$$

From the Friedmann equations derived in class,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$
$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{3}\rho$$

Then

$$\begin{aligned}
\epsilon &= \frac{4\pi G}{H^2}(\rho + p) \\
&= \frac{4\pi G}{H^2}\dot{\phi}^2 \\
&= 4\pi G\left(\frac{3}{8\pi G V}\right)^2\left(\frac{V'^2}{9}\right) \\
&= \frac{1}{4\pi G\phi^2}
\end{aligned}$$

To calculate η we note that

$$|\dot{\epsilon}| = |\frac{1}{2\pi G\phi^3}\dot{\phi}| = \frac{V'}{6H\pi G\phi^3}$$

Then

$$\begin{aligned}
\eta &= \frac{|\dot{\epsilon}|}{H\epsilon} \\
&= \left(\frac{V'}{6\pi G\phi^3}\right)\left(4\pi G\phi^2\right)\left(\frac{3}{8\pi G V}\right) \\
&= \frac{1}{4\pi G\phi}\frac{V'}{V} \\
&= \frac{1}{2\pi G\phi^2}
\end{aligned}$$

b)

Slow-roll conditions end when $\eta \approx 1$ or $\epsilon \approx 1$. Since the two expressions are the same order of magnitude for the given potential, we use η as it is larger. Then

$$\frac{1}{2\pi G\phi_{end}^2} = 1 \implies \phi_{end} = \sqrt{2\pi G}$$

Problem 4

See handwritten sheet for diagram.

From the literature, several mechanisms proposed to realize a bounce. In Poplawski (2012), a spin interaction in Einstein-Cartan-Sciama-Kibble gravity causes gravitons to be inflationary rather than contracting at very small scales. Brandenberger (2012) proposes a matter bounce scenario in which quantum fluctuations in a matter-dominated era at small scales, especially those corresponding to very small k-modes, perturb the metric and induce a bounce.