Physics 212 Problem Set 5

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Problem 1

a)

When X and $\bar{X}$ are in equilibrium, the annihilation reaction yields that $\mu_X = 0$. Moreover, if X decouples when $T > m_X c^2$, then X decouples in the ultra-relativistic regime. X is a massless boson with two states, X and $\bar{X} \implies X$ has the same statistics as photons. Using our derivation from class for bosons, (also repeated in Mukhanov Section 3.3),

$$\begin{aligned} n_X &= \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{\epsilon^2 - m^2}}{\exp(\frac{\epsilon-\mu}{T}) - 1} \\ &\approx \frac{g}{2\pi^2}\int_0^\infty \frac{\epsilon^2\, d\epsilon}{\exp(\frac{\epsilon}{T}) - 1} \end{aligned}$$

which looks exactly like the photon number density in the ultrarelativistic regime. We can therefore see how the cosmological catastrophe will arise. While n_X and therefore Ω_X is on the same order as the corresponding values for radiation in the radiation-dominated era, they decay as a^-3 rather than a^{-4}. Given how much smaller Ω_γ is than Ω_m today, this model results in dark matter whose energy density does not decay fast enough to produce the observed results.

Therefore up to the decoupling, $n_X = \frac{gT^3}{\pi^2}$ for $T \gg m_X c^2$. Let T^* denote the temperature of decoupling. As this occurs very early in the universe, it should occur during the radiation-dominated era. Therefore,

$$\rho_\gamma = \rho_{\gamma,0} a^{-4} = \frac{\pi^2 T^4}{15} \implies a^* = \left(\frac{15\rho_{\gamma,0}}{\pi^2 (T^*)^4}\right)^{\frac{1}{4}}$$

Finally, using the definition of Ω,

$$\begin{aligned} \Omega_X h^2 &= h^2 \frac{\rho_{X,0}}{\rho_{cr,0}} \\ &= \frac{h^2 m_X n_{X,0}}{\rho_{cr,0}} \\ &= \frac{h^2 m_X (a^*)^3 n_X^*}{\rho_{cr,0}} \\ &= \left(\frac{15^{\frac{3}{4}}}{\pi^{\frac{7}{2}}}\right)\left(\frac{\rho_{\gamma,0}^{\frac{3}{4}}}{\rho_{cr,0}}\right)(g m_X h^2) \\ &\approx \left(\frac{15^{\frac{3}{4}}}{\pi^{\frac{7}{2}}}\right)(\Omega_X h^2)(g m_X) \end{aligned}$$

We can use dimesional analysis to restore SI units from natural units using $\frac{c^2}{\hbar}$ so

$$\begin{aligned}\Omega_X h^2 &\approx \left(\frac{15^{\frac{3}{4}}}{\pi^{\frac{7}{2}}}\right)(4.2\times10^{-5})\frac{(2)(1.66\times10^{-27}\,kg)(3\times10^{8}\,\frac{m}{s})^2}{1.05\times10^{-34}J\cdot s}\\ &= 1.67\times10^{19}\end{aligned}$$

This is a cosmological catastrophe as X would contribute to the matter density, and the observed $\Omega_m h^2$ is much less than $\Omega_X h^2$, even though we used the lower bound on the mass–the proton mass.

b)

At termination for the scattering process,

$$\Gamma = n_X^* \sigma_a v = H$$

Therefore, the relic abundance is given by

$$n_{X,0} = (a^*)^3 n_X^* = \frac{(a^*)^3 H^*}{\sigma_a v}$$

From the assumptions in the problem, $T = \frac{.1 m_X c^2}{k}$ Using the equipartition theorem,

$$\begin{aligned}\frac{1}{2}m_X v^2 &= \frac{3}{2}k_B T\\ v = \sqrt{3k_B T}m_X &= c\sqrt{\frac{3k_B T}{m_X c^2}} = c\sqrt{\frac{3}{10}}\end{aligned}$$

Next, since this decoupling occurs during the radiation-dominated era, we use the Friedmann equations.

$$\begin{aligned}H^2 = \frac{8\pi G\rho}{3} &= \frac{8\pi G}{3}\left(\frac{g_*\pi^2 k_B^4 T^4}{15\hbar^3 c^5}\right)\\ H &= \frac{m_X^2}{75}\left(\frac{\pi^3 g_* G c^3}{10\hbar^3}\right)^{\frac{1}{2}}\end{aligned}$$

where we have computed the energy density of photons and then multiplied by the total number of species as they are in thermal equilibrium. Finally, in the radiation dominated era, $T \propto a^{-1}$ so $a^* T^* = a_0 T_0 \implies T^* = \frac{a_0 T_0}{a^*} = \frac{m_X c^2}{10 k_B}$. Therefore

$$a^* = \frac{10 k_B T_0}{m_X c^2}$$

Substituting everything into the expression for n_X at $T_0 = 2.725\,K$, we have a relic abundance of

$$\begin{aligned}n_X &= \frac{40 k_B^3 T_0^3}{3\sigma_a m_X}\sqrt{\frac{g_* G\pi^3}{3c^{11}}}\\ &= \frac{4.11\times10^{-66}\,\frac{kg}{m}}{m_X \sigma_a}\end{aligned}$$

We can rewrite the rate equation in σ_a to give

$$\begin{aligned}\sigma_a &= \frac{H^*}{n_X^* v^*}\\ &= \frac{(a^*)^3 H^*}{n_X\sqrt{\frac{3}{10}}c}\end{aligned}$$

We have that $n_X = \frac{\Omega_X \rho_{cr}}{m_X} = \frac{3H_0^2 \Omega_X}{8\pi G m_X}$ so substituting in our expressions for a^* and H^* from earlier,

$$\begin{aligned}
\sigma_a &= \frac{320 k_B^3 T_0^3}{9c^7 H_0^2 \Omega_X} \left(\frac{\pi^5 c^3 g_* G^3}{3\hbar^3} \right)^{\frac{1}{2}} \\
&= \frac{320(1.38 \times 10^{-23} \frac{J}{K})^3 (2.725\, K)^3}{9(3 \times 10^8 \frac{m}{s})(100\, h \frac{km}{s \cdot Mpc})(\frac{.11}{h^2})} \left(\frac{\pi^5 (3 \times 10^8 \frac{m}{s})^3 (100)(6.67 \times 10^{-11} \frac{m^3}{kg \cdot s^2})^3}{3(1.05 \times 10^{-34} J \cdot s)^3} \right)^{\frac{1}{2}} \\
&= 1.99 \times 10^{-39} m^2
\end{aligned}$$

Problem 2

For the combined fluid,

$$\begin{aligned}
p \approx p_\gamma &= \frac{1}{3} \rho_{\gamma,0} a^{-4} c^2 \\
\rho &= \rho_{\gamma,0} a^{-4} + \rho_{b,0} a^{-3}
\end{aligned}$$

To insert cosmological parameters, we note that

$$\rho_{\gamma,0} = \rho_{cr,0} \Omega_{\gamma,0}$$

Then

$$\begin{aligned}
p &= \frac{1}{3} \left(\frac{3H_0^2}{8\pi G} \right) \Omega_{\gamma,0} a^{-4} c^2 \implies \\
\frac{dp}{da} &= -\frac{4H_0^2 \Omega_{\gamma,0} a^{-5} c^2}{8\pi G} \\
\rho &= \frac{3H_0^2}{8\pi G} \left(\Omega_{\gamma,0} a^{-4} + \Omega_{b,0} a^{-3} \right) \implies \\
\frac{d\rho}{da} &= -\frac{3H_0^2}{8\pi G} (4\Omega_{\gamma,0} a^{-5} + 3\Omega_{b,0} a^{-4})
\end{aligned}$$

Then

$$\begin{aligned}
c_s^2 &= \frac{\frac{dp}{da}}{\frac{d\rho}{da}} \\
&= \frac{1}{3} \left(\frac{1}{1 + \frac{3\Omega_{b,0}}{4\Omega_{\gamma,0}} a} \right) c^2
\end{aligned}$$

Then

$$c_s(z) = \frac{c}{\sqrt{3}} \left(\frac{1}{1 + \frac{3\Omega_{b,0}}{4\Omega_{\gamma,0}(1+z)}} \right)^{\frac{1}{2}}$$

In the limit that $z > 1000$, we have

$$\frac{3}{4} \frac{\Omega_{b,0}}{\Omega_{\gamma,0}} \frac{1}{1+z} = \frac{3}{4} \frac{2 \times 10^{-2}}{4.2 \times 10^{-5}} \frac{1}{1 + 1000 + \Delta z} \le .357$$

Then

$$c_s(z) \ge \frac{c}{\sqrt{3}} \left(\frac{1}{1 + .357} \right)^{\frac{1}{2}} \ge (.858) \frac{c}{\sqrt{3}}$$

Therefore, for $z > 1000$, $c_s \approx \frac{c}{\sqrt{3}}$ is good to within 30% on either side.

b)

Using the definition $H = \frac{\dot{a}}{a}$, we can covert our integral over time to one over redshift.

$$\begin{aligned} s_* &= \int_0^{t_*} dt c_s(z)(1+z) \\ &= \int_0^{a_*} \frac{c_s(z)(1+z)da}{aH(a)} \\ &= \int_{z_*}^{\infty} \frac{c_s(z)\,dz}{H(z)} \end{aligned}$$

From the Friedmann equation with only matter and radiation,

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_{rad}(1+z)^4}$$

Then

$$s_* = \frac{2c}{\sqrt{3}H_0}\left(\sqrt{\frac{\Omega_m}{1+z_*} + \Omega_{rad}} - \sqrt{\Omega_{rad}}\right)$$

Substituting in $\Omega_m h^2 = .14$, $\Omega_{rad}h^2 = 4.2 \times 10^{-5}$, we have

$$s_* = 5.34 \times 10^{24} m = 173\,Mpc$$

We cannot neglect radiation in this calculation as $z = 1000$ as radiation makes a significant contribution to the Hubble parameter during this time. Scaling back, $\Omega_m h^2(1+1000)^3 = 1.4\times10^8$ and $\Omega_{rad}h^2(1+1000)^4 = 4.22 \times 10^7$ so even at the endpoint, the contribution to $H(z)$ is significant.. At even earlier times, $\Omega_{rad}h^2$ grows faster than $\Omega_m h^2$ due to the fourth power law and we enter the radiation-dominated era.

c)

The peaks in the acoustic oscillations are given by $l = ks_* = n\pi$ as the Bessel function $j_l(x)$ attains its maximum near $x = l$. We compare the size of variations in s_*.

Upon inspection, Ω_Λ should affect s_* less than Ω_K as Ω_Λ does not scale with redshift. Keeping terms to first order, a variation in Ω_Λ produces

$$s_* + \delta s_* = \frac{c}{\sqrt{3}H_0}\int_{z_*}^{\infty} \frac{dz}{\sqrt{\Omega_\Lambda + \delta\Omega_\Lambda + Q_\Lambda}}$$

where Q_Λ is the collection of all other terms in the integrand. Then

$$s_* + \delta s_* \approx \frac{c}{\sqrt{3}H_0}\int_{z_*}^{\infty} \frac{dz}{\sqrt{\Omega_\Lambda + Q_\Lambda}}\left(1 - \frac{\delta\Omega_\Lambda}{2(\Omega_\Lambda + Q_\Lambda)}\right) \delta s_* = -\frac{c}{2\sqrt{3}H_0}\int_{z_*}^{\infty} \frac{\delta\Omega_\Lambda\,dz}{(\Omega_\Lambda + Q_\Lambda)^{\frac{3}{2}}}$$

Repeating the procedure for Ω_K, we obtain

$$\delta s_* = -\frac{c}{2\sqrt{3}H_0}\int_{z_*}^{\infty} \frac{\delta\Omega_K(1+z)^2\,dz}{(\Omega_K + Q_K)^{\frac{3}{2}}}$$

We therefore have that $\delta_K s_* > 10^6 \delta_\Lambda s_*$ since $z > 1000$ in this regime.

Next we vary h.

$$\begin{aligned} s_* + \delta s_* &= \frac{c}{\sqrt{3}(h+\delta h)C_1}\int_{z_*}^{\infty} \frac{dz}{\sqrt{Q}} \\ &= \frac{c}{\sqrt{3}H_0}\left(1 - \frac{\delta h}{h}\right)\int_{z_*}^{\infty} \frac{dz}{\sqrt{Q}} \\ \delta s_* &= -\frac{c}{\sqrt{3}H_0}\frac{\delta h}{h}\int_{z_*}^{\infty} \frac{dz}{\sqrt{Q}} \\ &= -\frac{\delta h}{h}s_* \end{aligned}$$

This is essentially a second order term as $\frac{\delta h}{h}$ so it contributes marginally. Compared with $\delta_K s_*$, we see that the integrands differ by $\frac{(1+z)^2}{Q} > 1$ so $\delta_K s_* > \delta_h s_*$.

Problem 3

a)

From class we have that $\sigma_T = \frac{8\pi\alpha^2}{3m_e^2}$. Moreover,

$$n_e = \left(1 - \frac{Y_p}{2}\right)\chi_e n_b$$

Assuming all electrons are ionized, $\chi_e = 1$ and assuming all baryons are hydrogen, $Y_p = 0$ and $n_b = \frac{\rho_b}{m_H} = \frac{\Omega_b \rho_{cr}}{m_H}$
By definition,

$$\lambda(a) = \frac{c}{\Gamma(a)} = \frac{a^3 c}{n_e \sigma_T c} = \frac{a^3}{n_e \sigma_T}$$

in physical distance units. Substituting in,

$$\begin{aligned}\lambda(z) &= \frac{a^3}{\left(\frac{3H_0^2\Omega_b}{8\pi G m_H}\right)\left(\frac{8\pi\alpha^2}{3m_e^2}\right)} \\ &= \left(\frac{G m_e^2 m_H}{H_0^2 \Omega_b \alpha^2}\right)\frac{1}{(1+z)^3}\end{aligned}$$

We can also use the number for σ_T of an electron given in class:

$$\begin{aligned}\lambda(z) &= \frac{a^3}{\sigma_T n_e} \\ &= \frac{1}{(1+z)^3}\frac{1}{6.65\times 10^{-29} m^2}\frac{8\pi(6.67\times 10^{-11}\, m^3\, kg^{-1} s^{-2})(1.67\times 10^{-27}\, kg)}{3(.02)(100\frac{km}{s\cdot\, Mpc})^2} \\ &= \frac{6.68\times 10^{28}\, m}{(1+z)^3} \\ &= \frac{2.165\times 10^6 Mpc}{(1+z)^3}\end{aligned}$$

Using the Friedmann equation, the Hubble radius is given by

$$\begin{aligned}\frac{c}{H(z)} &= \frac{c}{H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda}} \\ &= \frac{9.24\times 10^{27} m}{\sqrt{\Omega_m h^2(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda}}\end{aligned}$$

We see that even if $\Omega_\gamma = 1$, the component which grows the fastest at high z, the denominator does not grow as fast as the denominator of $\lambda(z)$ for large z. Therefore, in the early universe, $\lambda < \frac{c}{H(z)}$.

We perform a sample calculation for $z = 2000$. Since the universe after $z = 3600$ is matter-dominated, we ignore the contributions from other components. Then

$$\lambda(z = 2000) = 8.38\times 10^{18} m = 2.72\times 10^{-4}\, Mpc$$

$$\frac{c}{H(z = 2000)} = \frac{9.24\times 10^{27}}{\sqrt{(.14)(1+2000)^3}} = 2.75\times 10^{23} m = 8.91\, Mpc$$

which matches our intuition previously.

b)

We reintroduce the reionization fraction, but retain the assumption that all baryons are hydrogen. Then $n_e = \chi_e n_b$ so

$$\lambda(z) = \frac{1}{\chi_e n_b \sigma_T} \frac{1}{(1+z)^3} = \frac{1}{\chi_e \sigma_T (1+z)^3} \frac{8\pi G m_H}{3 H_0^2 \Omega_b}$$

Setting this equal to the Hubble radius,

$$\begin{aligned}
\chi_e &= \frac{8\pi G m_H}{3 H_0 c \sqrt{\Omega_m} (1+z)^{\frac{3}{2}} \sigma_T} \\
&= \frac{8\pi (6.67 \times 10^{-11}\, kg\, m^{-3}\, s^{-2})(1.66 \times 10^{-27}\, kg)(3.09 \times 10^{19} \frac{Mpc}{km})(1\, Mpc)}{(3)(100\, km)(3 \times 10^8 \frac{m}{s})(\sqrt{.14})(1+1000)^{\frac{3}{2}}(6.65 \times 10^{-29} m^2)} \\
&= 1.21 \times 10^{-3}
\end{aligned}$$