Physics 212 Problem Set 2

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September 21, 2016

Problem 1

a)

For this problem we use the chain rule:

$$\frac{d\,n_0}{d\,v_0} = \frac{d\,n_0}{d\,n_d}\frac{d\,n_d}{d\,p_d}\frac{d\,p_d}{d\,p_0}\frac{d\,p_0}{d\,v_0}$$

In class we showed that for a given particle,

$$n = \int_0^\infty \frac{g}{h^3}\frac{4\pi p^2\,dp}{1+\exp(\frac{pc}{kT})}$$

Then taking the derivative and picking a particular time,

$$\frac{d\,n_d}{d\,p_d} = \frac{g}{h^3}\frac{4\pi p_d^2}{1+\exp(\frac{p_d}{kT_d})}$$

From the problem statement, we are given $p_d = p_0(1+z)$ and $T_d = \left(\frac{4}{11}\right)^{\frac{1}{3}}(1+z_d)T_0$. Moreover, as n is a density, if we assume that the same total number of neutrinos remains constant after decoupling, then $n_d = n_0(1+z)^3$. Then

$$\begin{aligned}
\frac{d\,n_d}{d\,p_d} &= \frac{g}{h^3}\frac{4\pi p_0^2(1+z_d)^2}{1+\exp\left(\frac{p_0(1+z_d)c}{k\left(\frac{4}{11}\right)^{\frac{1}{3}}(1+z_d)T_0}\right)} \\
\frac{d\,n_0}{d\,n_d} &= \frac{1}{(1+z_d)^3} \\
\frac{d\,p_d}{d\,p_0} &= 1+z_d \\
\frac{d\,p_0}{d\,v_0} &= m
\end{aligned}$$

Then multiplying our expressions together,

$$\frac{d\,n_0}{d\,v_0} = \frac{g}{h^3}\frac{4\pi m p_0^2}{1+\exp\left(\frac{p_0 c}{\left(\frac{4}{11}\right)^{\frac{1}{3}}kT_0}\right)}$$

Since the neutrinos are non-relativistic today, $p_0 = mv_0$. Substituting in,

$$\frac{d\,n_0}{d\,v_0} = \frac{g}{h^3}\frac{4\pi m_\nu^3 v_0^2}{1+\exp\left(\frac{m_\nu v_0 c}{\left(\frac{4}{11}\right)^{\frac{1}{3}}kT_0}\right)}$$

b)

For this problem we calculate the "velocity density" and divide by the number density. In particular,

$$\langle v \rangle = \frac{\int v\, dn}{\int dn}$$

(We can see that this produces the average velocity the terms in the numerator and denominator are densities and the term in the numerator is the first moment in a velocity distribution)

Then

$$\begin{aligned}
\int v\, dn &= \int v \left(\frac{d\,n}{d\,v}\right) dv \\
&= \int_0^\infty \frac{g}{h^3} \frac{4\pi m_\nu^3 v^3\, dv}{1 + \exp\left(\frac{m_\nu v c}{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}\right)} \\
&= \frac{4\pi m_v^3 g}{h^3} \left(\frac{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}{m_\nu c}\right)^4 \int_0^\infty \frac{u^3\, du}{1+\exp(u)} \\
&= \frac{4\pi m_\nu^3 g}{h^3} \left(\frac{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}{m_\nu c}\right)^4 3!\zeta(4)\left(\frac{7}{8}\right)
\end{aligned}$$

$$\begin{aligned}
\int dn &= \int \left(\frac{d\,n}{d\,v}\right) dv \\
&= \int_0^\infty \frac{g}{h^3} \frac{4\pi m_\nu^3 v^2\, dv}{1 + \exp\left(\frac{m_\nu v c}{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}\right)} \\
&= \frac{4\pi m_\nu^3 g}{h^3} \int_0^\infty \frac{v^2\, dv}{1 + \exp\left(\frac{m_\nu v c}{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}\right)} \\
&= \frac{4\pi m_\nu^3 g}{h^3} \left(\frac{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}{m_\nu c}\right)^3 2!\zeta(3)\left(\frac{3}{4}\right)
\end{aligned}$$

Then

$$\langle v \rangle = \frac{7\zeta(4)}{2\zeta(3)} \left(\frac{\left(\frac{4}{11}\right)^{\frac{1}{3}} kT_0}{m_\nu c}\right)$$

where $T_0 = 2.725\, K$

Problem 2

a)

In class, we showed that

$$Y_{He} = \frac{2\left(\frac{n_n}{n_p}\right)}{1 + \frac{n_n}{n_p}}$$

Therefore, we show how the extra neutrino species affects the $\frac{n_n}{n_p}$ ratio at neutrino freeze-out.

We consider the extra neutrino species as adding to the radiation density in the universe. As weak decoupling occurred during the radiation-dominated era, then this extra neutrino species would influence the expansion of the universe. Therefore, we examine how this affects the solution to $\Gamma_w = H$, the time of decoupling and the $\frac{n_n}{n_p}$ ratio.

In a radiation-dominated universe, $a(t) \propto t^{\frac{1}{2}} \implies$

$$\Gamma_w = n_0 c \sigma_w \propto a^{-5} \propto t^{-\frac{5}{2}}$$

and from the Friedmann equation, assuming a flat universe,

$$H = \frac{H_0}{a^2}\sqrt{\Omega_\gamma} \propto \frac{H_0\sqrt{\Omega_\gamma}}{t}$$

Then let $t_d =$ time of decoupling.

$$Ct^{-\frac{3}{2}} = H_0\sqrt{\Omega_\gamma}$$

where C is some positive constant. Therefore, increasing Ω_γ with an extra neutrino species would cause t_d to decrease. This would cause the temperature at decoupling to increase, and since

$$\frac{n_n}{n_p} = \left(\frac{n_n}{n_p}\right)^{\frac{3}{2}} e^{\frac{m_p - m_n}{kT}} \propto e^{-\frac{1}{T}}$$

Then $\frac{n_n}{n_p}$ increases as well.

As Y_{He} is a monotonic function in $\frac{n_n}{n_p}$ for positive values, then this would cause Y_{He} to increase.

b)

As stated in part a), $\frac{n_n}{n_p} \propto e^{-\frac{1}{kT}}$ so decreasing T will decrease the $\frac{n_n}{n_p}$ ratio and Y_{He}. We can also verify this by comparing against the decoupling value stated in class, $T_d = 9 \times 10^9 K = .776 MeV/k$. We see that the new ratio should be $E^{-\frac{1.3\times10^6}{.25\times10^6}+\frac{1.3\times10^6}{.776\times10^6}} = .029$ times the original $\frac{n_n}{n_p}$ ratio and therefore around a similar scale of difference between the new Y_{He} and the original from the standard model.

c)

Let $Q = m_p - m_n$. From class we have that $\frac{n_n}{n_p} \propto E^{-\frac{q}{kT}}$ up to the time of neutrino decoupling. Therefore, increasing q will decrease $\frac{n_n}{n_p}$ and also decrease Y_{He} since Y_{He} is monotonic in $\frac{n_n}{n_p}$ at freeze-out.

Problem 3

For a radiation dominated universe, $H = H_0 a^{-2}$. Then

$$t = H_0 \int_0^a a'\, da' \implies a(t) = \sqrt{2H_0 t}$$

As demonstrated in class, for the radiation dominated universe,

$$\rho_\gamma \propto a^{-4} \propto T^4 \implies T \propto a^{-4}$$

Therefore, we can can scale back from the observed $2.725\,K$ to $t = 300\,s$. That is,

$$T_{300} = T_{now} a_{300}^{-1} = \frac{T_{now}}{\sqrt{2H_0 t}} = \frac{2.725\,K(\sqrt{3.086\times10^{19}\frac{km}{Mpc}})}{\sqrt{(2)(70\frac{km}{Mpc\cdot s})(300\,s)}} = 7.39\times10^7\,K = 6.36 keV$$

Since the protons, neutrons, and deuterium are in equilibrium, we can use the Saha equation to find an expression for

$$\frac{n_D}{n_p n_n} = \frac{g_D}{g_p g_n}\left(\frac{m_D}{m_p m_n}\right)^{\frac{3}{2}}\left(\frac{T}{2\pi}\right)^{-\frac{3}{2}} e^{Q_D}T$$

Let $m_D \approx 2m_p$ and $m_n \approx m_p$. Moreover, let n_b be the number density of baryons so $n_p = .86n_b$, $n_n = .14n_b$. Then

$$\frac{n_D}{n_n} = .84n_b\frac{g_D}{g_p g_n}\left(\frac{4\pi}{m_p T}\right)^{\frac{3}{2}} e^{\frac{Q_D}{T}}$$

From class we also have

$$n_b = \eta n_\gamma \approx \eta(.2)T^3$$

$$\eta \approx 5\times 10^{-10}\frac{\Omega_b h^2}{.02} = 5\times 10^{-10}$$

Therefore, since we have 1 neutron per deuterium,

$$\frac{n_D}{n_n} = (.86)(.2)\eta\frac{g_D}{g_p g_n}\left(\frac{4\pi T}{m_p}\right)^{\frac{3}{2}} e^{\frac{Q_D}{T}} = (.84)(.2)(5\times 10^{-10})\frac{3}{(2)(2)}\left(\frac{4\pi(6.36keV)}{938\,MeV}\right)^{\frac{3}{2}} e^{\frac{-2.2MeV}{6.36keV}} = 3.00\times 10^{-167}$$