Physics 212 Problem Set 1

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Problem 1

H_0

a)

Using natural units, we can convert between the $\frac{1}{time}$ units of H_0 to Mpc^{-1}.

$$H_0 = \frac{70km}{s \cdot Mpc} \cdot \frac{1000m}{km} \cdot \frac{1\,c}{3 \times 10^8 \frac{m}{s}} = 2.33 \times 10^{-4} Mpc^{-1}$$

b)

$$H_0 = \frac{70\,km}{s \cdot Mpc} \cdot \frac{1\,Mpc}{3.08 \times 10^{19}\,km} \cdot \frac{3.15 \times 10^{16}\,s}{1\,Gyr} = 7.13 \times 10^{-2}\,Gyr^{-1}$$

ρ_{crit}

a)

We use $G = 6.674 \times 10^{-8} \frac{cm^3}{g \cdot s^2}$. Since this already has units of $\frac{g}{cm^3}$ we simply need to re-express H_0 in $\frac{1}{s}$.

$$H_0 = \frac{70\,km}{s \cdot Mpc} \cdot \frac{1\,Mpc}{3.08 \times 10^{19}\,km} = 2.27 \times 10^{-18} s^{-1}$$

Therefore

$$\rho_{crit} = \frac{3(2.27 \times 10^{-18} s^{-1})^2}{8\pi \cdot 6.674 \times 10^{-8} \frac{cm^3}{g \cdot s^2}} = 9.21 \times 10^{-30} \frac{g}{cm^3}$$

b)

Since $1\,eV = q_e \times \frac{J}{C} = 1.6 \times 10^{-19} J$, we seek to convert the joules into units of $\frac{g}{c^2}$.

$$1\,\frac{eV}{c^2} = \frac{1.6 \times 10^{-19} J}{c^2} \left(\frac{c}{3 \times 10^8 \frac{m}{s}}\right)^2 \left(\frac{kg \cdot \frac{m^2}{s^2}}{J}\right) \left(\frac{1000\,g}{1\,kg}\right) 1.78 \times 10^{-33} g$$

Therefore, using our answer from part a),

$$\rho_{crit} = \frac{9.21 \times 10^{-30}\,g}{cm^3} \cdot \frac{1\frac{eV}{c^2}}{1.78 \times 10^{-33}\,g} = 5.18 \times 10^3 \frac{eV}{cm^3}$$

c)

We use the fact that in natural units, $\hbar = 6.582 \times 10^{-16}\,eV \cdot s$. Then

$$\rho_{crit} = \frac{5.18 \times 10^3 eV}{cm^3} \left(\frac{10^2\,cm}{1m}\right)^3 \left(\frac{3 \times 10^8 \frac{m}{s}}{1\,c}\right)^3 \left(\frac{6.582 \times 10^{-16}\,eV \cdot s}{\hbar}\right)^3 \left(\frac{1GeV}{10^9 eV}\right)^4 = 3.99 \times 10^{-47} GeV^4$$

Problem 2

a)

From class, we showed that

$$\left(\frac{H}{H_0}\right)^2 = \Omega_m(1+z)^3 + \Omega_\Lambda$$

Substituting in $H = \frac{\dot{a}}{a}$ we have

$$\frac{1}{a}\frac{d\,a}{d\,t} = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$$

$$\frac{da}{a\,H_0\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} = dt$$

We wish to integrate from $t = 0$ to now, t_age in terms of time and from $a = 0$ to $a = 1$ in terms of a. Then

$$t_age = \frac{1}{2.27\times10^{-18}s^{-1}}\int_0^1 \frac{da}{a\sqrt{.3a^{-3}+.7}} = 4.25\times10^{17}s = 13.5Gyr$$

b)

As stated in class, the current particle horizon is given by

$$\eta = \int_0^t \frac{c\,dt'}{a(t')}$$

As we do not a as an explicit function of t we change variables. In particular,

$$\left(\frac{H}{H_0}\right)^2 = \left(\frac{1}{a}\frac{d\,a}{d\,t}\right)^2 = \Omega_m\,a^{-3} + \Omega_\Lambda$$

$$dt = \frac{da}{H_0\,a\,\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}}$$

Substituting into the definition of η,

$$\begin{aligned}\eta &= \int_0^1 \frac{c\,da}{H_0\,a^2\,\sqrt{\Omega_m a^{-3}+\Omega_\Lambda}}\\ &= \frac{3\times10^8\frac{m}{s}}{2.27\times10^{-18}s^{-1}}\left(\frac{1\,ly}{9.46\times10^{15}m}\right)\int_0^1\frac{da}{a^2\sqrt{(.3)a^{-3}+.7}}\\ &= 4.62\times10^{10}\,ly\end{aligned}$$

In the limit that $a\to\infty$,

$$\eta = \int_0^\infty \frac{c\,da}{a^2 H_0\sqrt{\Omega_m a^{-3}+\Omega_\Lambda}} = 6.21\times10^{10}\,ly$$

so the particle horizon (in the present co-moving coordinate system) approaches this limit as the universe evolves.

Problem 3

From class we have that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. For a general FRW metric

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right)$$

Therefore we can read off the metric

$$g = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{bmatrix}$$

We now calculate Christoffel coefficients using the Lagrangian method. For a given metric

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2\dot{t}^2 + a^2(t)\left(\frac{\dot{r}^2}{1-kr^2} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right)$$

Where dots denote derivatives with respect to conformal time.
Starting with r, we have

$$\frac{\partial L}{\partial r} = \frac{2kra^2(t)\dot{r}^2}{(1-kr^2)^2} + 2a^2(t)r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{2a^2(t)\dot{r}}{1-kr^2}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{2a^2(t)\ddot{r}}{1-kr^2} + \frac{4a(t)\dot{a}\dot{t}\dot{r}}{1-kr^2} + \frac{2a^2(t)kr\dot{r}^2}{(1-kr^2)^2}$$

Using the chain rule, we have $\frac{da}{dt} = \frac{\dot{a}}{a}$. Setting our equations equal, we obtain,

$$\ddot{r} = r(1-kr^2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \frac{2\dot{a}\dot{t}\dot{r}}{a}$$

Therefore,

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a^2}$$

$$\Gamma^1_{22} = -r(1-kr^2)$$

$$\Gamma^1_{33} = -r\sin^2\theta(1-kr^2)$$

Similarly for θ,

$$\frac{\partial L}{\partial \theta} = 2a^2(t)r^2\sin\theta\cos\theta\dot{\phi}^2$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{ds}\left(2a^2(t)r^2\dot{\theta}\right) = 2a^2(t)r^2\ddot{\theta} + 4r^2a(t)\dot{a}\dot{t}\dot{\theta} + 4a^2(t)r\dot{r}\dot{\theta}$$

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - \frac{2\dot{a}\dot{t}\dot{\theta}}{a^2} - \frac{2\dot{r}\dot{\theta}}{r}$$

Then

$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{a}}{a}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

For ϕ,

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{ds}\left(2a^2(t)r^2\sin^2\theta\dot{\phi}\right) = 4r^2\sin^2\theta a(t)\dot{a}\dot{t}\dot{\phi} + 4a^2(t)r\sin^2\theta\dot{r}\dot{\phi} + 4a^2(t)r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2a^2(t)r^2\sin^2\theta\ddot{\phi}$$

$$\ddot{\phi} = -\frac{2\dot{a}\dot{t}\dot{\phi}}{a} - \frac{2\dot{r}\dot{\phi}}{r} - 2\cot\theta\dot{\theta}\dot{\phi}$$

Therefore

$$\Gamma^3_{03} = -\Gamma^3_{30} = \frac{\dot{a}}{a}$$

$$\Gamma^3_{13} = -\Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^3_{23} = -\Gamma^3_{32} = \cot\theta$$

Finally for t,

$$\frac{\partial L}{\partial t} = 2a(t)\dot{a}\left(\frac{\dot{r}^2}{1-kr^2} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right)$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{d}{ds}\left(-2c^2\dot{t}\right) = -2c^2\ddot{t}$$

$$\ddot{t} = -\dot{a}\left(\frac{\dot{r}^2}{1-kr^2} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right)$$

where we have set $c = 1$. Then

$$\Gamma^0_{11} = \frac{a(t)\dot{a}}{1-kr^2}$$

$$\Gamma^0_{22} = a(t)\dot{a}r^2$$

$$\Gamma^0_{33} = a(t)\dot{a}r^2\sin^2\theta$$

We now calculate the components of the Ricci tensor and scalar.

$$R_{00} = \partial_\mu\Gamma^\mu_{00} - \partial_0\Gamma^\mu_{0\mu} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{00} - \Gamma^\mu_{\nu 0}\Gamma^\nu_{0\mu}$$

We immediately see that the first and third terms are 0 since no Christoffel coefficient with lower indices 00 has a non-zero component.

Then

$$\begin{aligned} R_{00} &= -\partial_t\left(\frac{3\dot{a}}{a}\right) - 2(\Gamma^1_{10}\Gamma^1_{01} - \Gamma^2_{02}\Gamma^2_{20} - \Gamma^3_{03}\Gamma^3_{30}) \\ &= -\frac{3\ddot{a}}{a} - \frac{6\dot{a}^2}{a^2} + \frac{6\dot{a}^2}{a^2} \\ &= \frac{-3\ddot{a}}{a} \end{aligned}$$

$$\begin{aligned} R_{11} &= \partial_\mu\Gamma^\mu_{11} - \partial_1\Gamma^\mu_{1\mu} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{11} - \Gamma^\mu_{\nu 1}\Gamma^\nu_{1\mu} \\ &= \partial_0\Gamma^0_{11} - \partial_1\Gamma^0_{10} - \Gamma^1_{01}\Gamma^1_{10} - \Gamma^2_{12}\Gamma^2_{21} - \Gamma^3_{13}\Gamma^3_{31} \\ &= \partial_t(\frac{a\dot{a}}{1-kr^2}) - \frac{1}{r^2} - \frac{1}{r^2} \\ &= \frac{2\dot{a}^2 + \ddot{a} + 2k}{1-kr^2} \end{aligned}$$

$$\begin{aligned} R_{22} &= \partial_\mu\Gamma^\mu_{22} - \partial_2\Gamma^\mu_{2\mu} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{22} - \Gamma^\mu_{\nu 2}\Gamma^\nu_{2\mu} \\ &= \partial_0\Gamma^0_{22} + \partial_1\Gamma^1_{22} - \Gamma^\mu_{0\mu}\Gamma^0_{22} - \Gamma^\mu_{1\mu}\Gamma^1_{22} - \Gamma^0_{22}\Gamma^2_{20} \\ &= \partial_t(a\dot{a}r^2) + \partial_r(-r(1-kr^2)) - \frac{3\dot{a}}{a}(a\dot{a}r^2) - \frac{2}{r}(-r(1-kr^2)) - (a\dot{a}r^2)(\frac{\dot{a}}{a}) \\ &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \end{aligned}$$

$$\begin{aligned}
R_{33} &= \partial_\mu \Gamma^\mu_{33} - \partial_3 \Gamma^\mu_{3\mu} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{33} - \Gamma^\mu_{\nu 3}\Gamma^\nu_{3\mu} \\
&= \partial_0 \Gamma^0_{33} + \partial_1 \Gamma^1_{33} + \partial_2 \Gamma^2_{33} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{33} - \Gamma^\mu_{\nu 3}\Gamma^\nu_{3\mu} \\
&= \partial_t (a\dot{a} r^2 \sin^2\theta) + \partial_r(-r\sin^2\theta(1-kr^2)) + \partial_\theta(-\sin\theta\cos\theta) \\
&+ \Gamma^\mu_{1\mu}\Gamma^1_{33} + \Gamma^\mu_{2\mu}\Gamma^2_{33} + \Gamma^\mu_{0\mu}\Gamma^0_{33} - \Gamma^3_{03}\Gamma^0_{33} - \Gamma^3_{13}\Gamma^1_{33} - \Gamma^3_{23}\Gamma^2_{33} \\
&= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta
\end{aligned}$$

We then see that

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{00} + \frac{1-kr^2}{a^2}R_{11} + \frac{1}{a^2r^2}R_{22} + \frac{1}{a^2r^2\sin^2\theta}R_{33} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)$$

Substituting into the Einstein equation,

$$G^0_0 = R^0_0 - \frac{1}{2}g^0_0 R = -\frac{3\ddot{a}}{a} - \frac{1}{2}\left(6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)\right) = -\frac{3}{a^2}\left(\left(\frac{\dot{a}}{a}\right)^2 + k\right)$$

For $i \neq j$, $g^i_j = R^i_j = 0$ due to the symmetry of the problem. Therefore,

$$G^i_j = \left(R^i_j - \frac{1}{2}g^i_j R\right)\delta^j_i = -\frac{1}{a^2}\left(2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + k\right)\delta^i_j$$

Problem 4

a)

The comoving distance is given by

$$r = \int_t^{t_0} \frac{c\,dt'}{a(t')}$$

Since we would like to know r as a function of z, we substitute variables.

$$\begin{aligned}
a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} &= \frac{1}{1+z} \\
\frac{t}{t_0} &= (1+z)^{-\frac{3}{2}} \\
dt &= -\frac{3}{2}t_0(1+z)^{-\frac{5}{2}}dz
\end{aligned}$$

Substituting into our integral, we have at $t = t_0$, $z = 0$. Then

$$\begin{aligned}
r &= \int_0^z \frac{3ct_0\,dz}{2(1+z)^{\frac{3}{2}}} \\
&= 3ct_0\left(1 - \frac{1}{\sqrt{1+z}}\right)
\end{aligned}$$

b)

We can read off the differential co-moving volume from the spatial components of the metric. Letting $a(t) = 1$ in the co-moving coordinate system,

$$dV = \prod_{i=1}^{3}\sqrt{|g_{ii}|}dx^i = S(r)^2\sin\theta\,dr\,d\theta\,d\phi$$

For a flat universe, $S(r) = r$. Next, a unit steradian is defined as the unit solid angle. If we imagine a differential rectangular patch which is subtended by angles of $d\theta$ and $d\phi$, then its side lengths are given by $d\theta$ and $\sin\theta\, d\phi$ so $d\Omega = \sin\theta\, d\theta\, d\phi$. Therefore, the co-moving volume per unit steradian per unit redshift is

$$\frac{dV}{d\Omega\, dz} = r^2 \frac{d\,r}{d\,z}$$

Substituting in our expression from part a),

$$\frac{dV}{d\Omega\, dz} = \frac{27c^3 t_0^3}{2(1+z)^{\frac{3}{2}}} \left(1 - \frac{1}{\sqrt{1+z}}\right)^2$$

Problem 5

a)

Equating the expressions for densities at an arbitrary time, we obtain

$$\begin{aligned}
\rho_{m,0} &= \rho_{r,0} \\
(1+z)^3 \rho_{m,0} &= (1+z)^4 \rho_{r,0} \\
\Omega_m \rho_{crit} &= (1+z)\Omega_r \rho_{crit} \\
z &= \frac{\Omega_m}{\Omega_r} - 1 \\
&= \frac{.14}{4 \times 10^{-5}} - 1 \\
&= 3.499 \times 10^3
\end{aligned}$$

Since $a = \frac{1}{1+z}$ then

$$z = \frac{\Omega_m}{\Omega_r} - 1 \implies a_{eq} = \frac{\Omega_r}{\Omega_m}$$

b)

Using the definition given in the problem statement,

$$\begin{aligned}
H^2 &= \frac{8\pi G\rho}{3} \\
&= \frac{8\pi G}{3}(\rho_m + \rho_r) \\
&= \frac{8\pi G}{3}(1+z)^3(\rho_{m,0} + (1+z)\rho_{r,0}) \\
&= \frac{8\pi G}{3}(1+z)^3 \rho_{crit}(\Omega_m + (1+z)\Omega_r) \\
&= (1+z)^3 H_0^2 (\Omega_m + (1+z)\Omega_r) \\
&= (1001)^3 \left(100 \frac{h \cdot km}{s \cdot Mpc}\right)^2 \left(\frac{.14}{h^2} + \frac{(1001)(4 \times 10^{-5})}{h^2}\right) \\
&= 1.80 \times 10^{12} (\frac{km}{s \cdot Mpc})^2 (\frac{1 Mpc}{3.08 \times 10^{19}\, km})^2 \\
&= 1.90 \times 10^{-27} s^{-2}
\end{aligned}$$

Then $H(z = 1000) = \sqrt{1.90 \times 10^{-27} s^{-2}} = 4.36 \times 10^{-14} s^{-1}$

The Hubble time is therefore

$$t_H(z = 1000) = \frac{1}{H} = 2.29 \times 10^1 3s = 7.27 \times 10^5 yr$$

c)

Starting at

$$\left(\frac{H}{H_0}\right)^2 = \Omega_m(1+z)^3 + \Omega_r(1+z)^4$$

We can substitute in $a = \frac{1}{1+z}$ and $H = \frac{\dot{a}}{a}$ to obtain

$$dt = \frac{da}{aH_0\sqrt{\Omega_m\, a^{-3} + \Omega_r a^{-4}}}$$

Since we would like the age of the universe, we want to integrate from $t = 0$ to t_age and from $a = 0$ to $a(z = 1000) = \frac{1}{1001}$. Then

$$t_{age} = \int_0^{\frac{1}{1001}} \frac{da}{100a\sqrt{.14a^{-3} + 4\times 10^{-5}a^{-4}}} = 4.45\times 10^{-7}\frac{s\cdot Mpc}{km}$$

Converting to years,

$$t_{age} = \frac{4.45\times 10^{-7}\, s\cdot Mpc}{km}\left(\frac{3.08\times 10^19\, km}{1\, Mpc}\right) = 1.37\times 10^13 s = 4.35\times 10^5 yr$$

d)

For a small period of physical time, a photon travels $\Delta x = c\,dt$ of physical distance, and $\frac{c\,dt}{a(t)}$ of co-moving distance. Therefore, the total co-moving distance is

$$r = \int_0^t \frac{c\,dt}{a(t)}$$

Using the chain rule,

$$dt\left(\frac{dt}{da}\frac{da}{dt}\right) = \frac{da}{aH}$$

Then

$$\begin{aligned} r &= \frac{c}{H_0}\int_0^{\frac{1}{1001}} \frac{da}{a^2\sqrt{\Omega_m a^{-3} + \Omega_r a^{-4}}} \\ &= \frac{3\times 10^8\,\frac{m}{s}}{100\frac{h\cdot km}{s\cdot Mpc}}\left(\frac{3.08\times 10^{19}\, km}{1\, Mpc}\right)\int_0^{\frac{1}{1001}} \frac{da}{\sqrt{\frac{.14}{h^2}a + \frac{4\times 10^{-5}}{h^2}}} \\ &= 9.35\times 10^{24} m\left(\frac{1\, Mpc}{3.08\times 10^{22}\, m}\right) \\ &= 303 Mpc \end{aligned}$$

e)

Repeating the integral with $a = 1$ at today,

$$\begin{aligned} r &= \frac{3\times 10^8\,\frac{m}{s}}{100\frac{h\cdot km}{s\cdot Mpc}}\left(\frac{3.08\times 10^{19}\, km}{1\, Mpc}\right)\int_{\frac{1}{1001}}^1 \frac{da}{\sqrt{\frac{.14}{h^2}a + \frac{4\times 10^{-5}}{h^2}}} \\ &= 4.76\times 10^{26} m\left(\frac{1\, Mpc}{3.08\times 10^{22} m}\right) \\ &= 1.54\times 10^4 Mpc \end{aligned}$$

The co-moving distance in d) is much smaller than the co-moving distance between $z = 1000$ and today.