Physics 212 Problem Set 4

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Problem 1

a)

From class we have that

$$\frac{\Delta_T^2}{\Delta_R^2} = \frac{\frac{2H_*^2}{\pi^2 M_{Pl}^2}}{\frac{H_*^4}{(2\pi)^2\dot{\Phi}^2}} = \frac{8\dot{\Phi}^2}{M_{Pl}^2 H_*^2}$$

In the slow-roll approximation, we have

$$\begin{aligned} 3H\dot{\Phi} &= V' \\ H^2 &= -\frac{8\pi G}{3}V \\ \epsilon &= -\frac{\dot{H}}{H^2} \end{aligned}$$

Differentiating the second equation,

$$\begin{aligned} 2H\dot{H} &= -\frac{8\pi G}{3}V'\dot{\Phi} \\ &= -\left(\frac{8\pi G}{3}\right)(3H\dot{\Phi})\dot{\Phi} \\ \dot{H} &= -4\pi G\dot{\Phi}^2 \end{aligned}$$

Then substituting into ϵ,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G\dot{\Phi}^2}{H^2}$$

Then

$$\frac{\Delta_T^2}{\Delta_R^2} = \frac{2\epsilon}{\pi}$$

b)

By definition, $N = \int H\,dt$ so $dN = H\,dt = \frac{H}{\dot{H}}\,dH$. Then

$$\frac{d\ln H}{dN} = \frac{1}{H}\frac{dH}{dN} = \frac{\dot{H}}{H^2} = -\epsilon$$

We use the chain rule through N to rewrite $d\ln k$. We then have

$$\ln k = -\ln(aH) = -\ln(a_* e^N H) = -\ln a_* - N - \ln H$$

$$\frac{d\ln k}{dN} = -(1 + \frac{d\ln H}{dN}) = \epsilon - 1$$

Then the implicit function theorem,

$$\frac{dN}{d\ln k} = -\frac{1}{\epsilon - 1}$$

As shown in part 1,

$$\dot{H} = -4\pi G\dot{\phi}^2$$

Then

$$\Delta_R^2 = \frac{H_*^4}{(2\pi)^2}\left(\frac{-4\pi G}{\dot{H}}\right) = C_2\frac{H_*^2}{\epsilon}$$

for some constant C_2. Therefore,

$$\begin{aligned} n_s - 1 &= \frac{d\ln\Delta_R^2}{d\ln k} \\ &= 2\frac{d\,lnH_*}{d\ln k} - \frac{d\ln\epsilon}{d\ln k} \\ &= 2\epsilon - \frac{1}{\epsilon}\frac{d\epsilon}{dN}\frac{dN}{d\ln k} \\ &= 2\epsilon - \eta \end{aligned}$$

Where we have used

$$\begin{aligned} \frac{1}{\epsilon}\frac{d\epsilon}{dN} &= \frac{1}{\epsilon}\frac{d\epsilon}{dt}\frac{dt}{dH}\frac{dH}{dN} \\ &= \frac{\dot{\epsilon}}{\epsilon H} \\ &= \eta \end{aligned}$$

c)

From class,

$$\ln\Delta_t^2 = \ln\left(\frac{2H_*^2}{\pi M_{Pl}^2}\right) = 2\ln H_* + C_1$$

for some constant C_1. Then similar to part b), we have

$$n_t = \frac{d\ln\Delta_t^2}{d\ln k} = \frac{d\ln\Delta_t^2}{dN}\frac{dN}{d\ln k} = \frac{2\epsilon}{\epsilon - 1} \approx -2\epsilon$$

d)

We have that $r = \frac{2\epsilon}{\pi}$ and $n_t = 2\epsilon$ so $r = -\frac{n_t}{\pi}$.