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EFFICIENT COMPUTATION OF COSMIC MICROWAVE BACKGROUND ANISOTROPIES IN CLOSED FRIEDMANN-ROBERTSON-WALKER MODELS

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ABSTRACT

We implement the efficient line-of-sight method to calculate the anisotropy and polarization of the cosmic microwave background for scalar and tensor modes in almost Friedmann-Robertson-Walker models with positive spatial curvature. We present new results for the polarization power spectra in such models.

Subject headings: cosmic microwave background — cosmology: theory

1. INTRODUCTION

The anisotropy of the cosmic microwave background (CMB) plays a key role in many areas of modern cosmology. Joint analyses of current CMB and Type Ia supernova data (e.g., Efstathiou et al. 1999; Turner 1999) suggest that the universe is within a factor of $\Omega_{\rm tot} \equiv \Omega_m + \Omega_\Lambda = 1 \pm 0.2$ of the critical density required for a flat geometry. Closed models (in which $\Omega_{\rm tot} > 1$) therefore account for an important sector of the possible parameter space. Moreover, maximum-likelihood searches require theoretical predictions over a much larger volume of parameter space to establish reliable error estimates on the parameters under consideration. It is therefore vital to have a fast and accurate method for calculating anisotropies for models at least within the range $0.4 < \Omega_{\rm tot} < 1.6$. Previous parameter determinations, such as those of Efstathiou et al. (1999), have had to rely on analytical approximations for rapid calculations of the CMB power spectrum in the closed region, since the current state-of-the-art codes, such as the widely used CMBFAST (Seljak & Zaldarriaga 1996), do not yet support closed models.

In this paper we describe a numerical implementation of the linearized equations of the 1 + 3 covariant approach to CMB anisotropies (Challinor & Lasenby 1999; Challinor 1999a, 1999b; Gebbie, Dunsby, & Ellis 1999) in almost Friedmann-Robertson-Walker (FRW) models with open, flat, and closed background geometries. Our code thus allows an efficient exploration of the full cosmological parameter space. We present new results for the polarization power spectra from scalar and tensor perturbations in closed models. The intensity power spectra in closed models have been calculated before: White & Scott (1996) integrated the full Boltzmann hierarchy directly for scalar perturbations, extending the earlier semianalytic predictions of Abbott & Schaefer (1986). In addition, Allen, Caldwell, & Koranda (1995) used the semianalytic approach (adequate on large scales) to calculate the tensor spectrum. None of these calculations included the effects of polarization.

The 1 + 3 covariant formalism provides a physically transparent, exact (fully nonlinear) description of both dynamics and radiative transfer in general cosmological models (Ellis & van Elst 1999; Maartens, Gebbie, & Ellis 1999). The full formalism admits a gauge-invariant linearization about FRW models, resulting in a linear perturbation theory that is arguably simpler and more physically transparent than other approaches (e.g., Ma & Bertschinger 1995; Hu et al. 1998). Our implementation of the 1 + 3 formalism is based on the field-tested CMBFAST code written by Seljak & Zaldarriaga (1996). Their code uses a line-of-sight integration method to achieve high efficiency without compromising accuracy.

2. IMPLEMENTATION

2.1. *Basic Formalism*

We employ the 1 + 3 covariant approach to perturbations in cosmology (e.g., Ellis 1998), in which departures from exact FRW symmetry are described by gauge-invariant variables derived from physical observables relative to some timelike 4-velocity field, u^a. The equations of radiative transfer can be recast as propagation equations along the integral curves of u^a for the multipoles of the total intensity, I_{A_l}, and the electric and magnetic components of the total linear polarization, $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ (Challinor 1999a). Here the multipoles are projected (relative to u^a) symmetric trace-free (PSTF) tensors (A_l represents the index string $a_1 \ldots a_l$), which provide a basis-free alternative to the more common scalar-valued multipole coefficients in spherical harmonic expansions of the intensity and polarization (Kamionkowski, Kosowsky, & Stebbins 1997; Seljak & Zaldarriaga 1997). We only consider linear polarization here, since circular polarization is not generated by Thomson scattering. For small departures from an FRW model, the intensity multipoles evolve as (Thorne 1981; Ellis, Matravers, & Treciokas 1983; Challinor & Lasenby 1999)

$$\dot{I}_{A_l} + \frac{4}{3}\Theta I_{A_l} + D^b I_{bA_l} - \frac{l}{(2l+1)} D_{\langle a_l} I_{A_{l-1}\rangle} + \frac{4}{3}\delta_l^1 I A_{a_1} - \frac{8}{15}\delta_l^2 I \sigma_{a_1 a_2} = -n_e \sigma_{\rm T}\left[I_{A_l} - \delta_l^0 I - \frac{4}{3}\delta_l^1 I v_{a_1} - \frac{1}{10}\delta_l^2 (I_{a_1 a_2} + 6\mathscr{E}_{a_1 a_2})\right], \quad (1)$$

where Θ is the expansion of u^a, σ_{ab} is the shear, and A_a is the acceleration. Here D^a is the totally projected covariant derivative, an overdot denotes the covariant derivative along u^a, and angle brackets denote the PSTF part of the enclosed indices. The electron number density is n_e in its rest frame, which has relative velocity v^a, and the Thomson cross section is $\sigma_{\rm T}$. For the electric polarization, we have (Challinor 1999a)

$$\dot{\mathscr{E}}_{A_l} + \frac{4}{3}\Theta\mathscr{E}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2} D^b\mathscr{E}_{bA_l} - \frac{l}{(2l+1)} D_{\langle a_l}\mathscr{E}_{A_{l-1}\rangle} - \frac{2}{(l+1)}\,\mathrm{curl}\,\mathscr{B}_{A_l} = -n_e\sigma_{\rm T}\left[\mathscr{E}_{A_l} - \frac{1}{10}\delta_l^2(I_{a_1 a_2} + 6\mathscr{E}_{a_1 a_2})\right], \quad (2)$$

and for the magnetic polarization,

$$\dot{\mathscr{B}}_{A_l} + \frac{4}{3}\Theta\mathscr{B}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2} D^b\mathscr{B}_{bA_l} - \frac{l}{(2l+1)} \times D_{\langle a_l}\mathscr{B}_{A_{l-1}\rangle} + \frac{2}{(l+1)}\,\mathrm{curl}\,\mathscr{E}_{A_l} = -n_e\sigma_{\rm T}\mathscr{B}_{A_l}\,. \quad (3)$$

The evolution of the electric and magnetic multipoles are coupled through the curl terms, where $\mathrm{curl}\,\mathscr{E}_{A_l} \equiv \epsilon_{bc\langle a_l} D^b\mathscr{E}_{A_{l-1}\rangle}{}^c$, and ϵ_{abc} is the alternating tensor in the rest space of u^a. The multipole equations (1)–(3) hold for a general linear perturbation around an arbitrary FRW model, and for a general choice of u^a. The equations must be supplemented by the covariant hydrodynamic and gravitodynamic equations (e.g., Ellis & van Elst 1999) to determine the first-order source terms σ_{ab}, A_a, and v_a. For an alternative approach to polarized radiative transfer in general FRW geometries, see, e.g., Hu et al. (1998).

The dimensionless power spectrum of the intensity anisotropies is defined by the ensemble average (Gebbie & Ellis 1998; Challinor & Lasenby 1999),

$$C_l^{II} = \frac{4\pi}{(4I)^2}\frac{(2l)!}{(-2)^l(l!)^2}\langle I_{A_l} I^{A_l}\rangle\,. \quad (4)$$

Power spectra for the electric polarization multipoles, $C_l^{\mathscr{E}\mathscr{E}}$, the magnetic multipoles, $C_l^{\mathscr{B}\mathscr{B}}$, and the cross-correlation between electric polarization and the intensity, $C_l^{I\mathscr{E}}$, can be defined analogously (Challinor 1999a). (To conform with Seljak & Zaldarriaga 1997, we include a factor of $[(l+1)(l+2)/l(l-1)]^{1/2}$ on the right-hand side of eq. [4] for each factor of the polarization.)

We solve the multipole equations (1)–(3) by expanding the first-order variables in PSTF tensors derived from the appropriate scalar, vector, and tensor eigenfunctions of the comoving Laplacian $S^2 D^a D_a$, where S is the scale factor, with eigenvalue k^2. The different perturbation types decouple at linear order, with each giving rise to a set of coupled first-order differential equations. The Boltzmann multipole equations for scalar and tensor modes are given in detail in Challinor (1999c). The equations describing perturbations in the other matter components and the geometry, which determine the source terms in the Boltzmann hierarchies, can be found in Challinor & Lasenby (1999), Challinor (1999b), and Gebbie et al. (1999). The mode-expanded Boltzmann equations can be solved formally as integrals along the line of sight. Integral solutions of this type form the basis of the line-of-sight algorithm employed by CMBFAST. The solutions for the intensity are given in the 1 + 3 covariant formalism in Challinor & Lasenby (1999), Challinor (1999b), and Gebbie et al. (1999); solutions for the polarization are given in Challinor (1999c). Equivalent results in the total angular momentum formalism are given in Hu et al. (1998). The present-day multipoles are obtained by integrating the product of source functions, which are inexpensive to compute, with special functions (derived from the hyperspherical Bessel functions) that result from the projections of the eigenfunctions of the Laplacian onto directions on the sky.

For scalar modes, we define $\nu^2 \equiv (k^2 + K)/|K|$, where $6K/S^2$ is the curvature scalar of the spatial sections in the background FRW model. In the closed case, the regular scalar eigenfunctions of the comoving Laplacian are complete for ν an integer ≥ 1 (e.g., Tomita 1982; Abbott & Schaefer 1986). The mode with $\nu = 1$ does not contribute to the perturbations, while the modes with $\nu = 2$ (which can only represent isocurvature perturbations; Bardeen 1980) only contribute to the CMB dipole. For tensor modes, $\nu^2 \equiv (k^2 + 3K)/|K|$, and the regular eigentensors of the Laplacian are complete with ν an integer ≥ 3; explicit PSTF representations are given in Challinor (1999b).

2.2. *Numerical Evaluation*

The implementation strategy of CMBFAST (Seljak & Zaldarriaga 1996) requires only minor modifications for closed models. The modes are now discretized with wavenumber $\nu \geq 3$. A given ν only contributes to multipoles with $l < \nu$, so, unlike the open case, the Boltzmann hierarchies for a given ν truncate at finite l. For large ν, it is possible to terminate the hierarchies artificially (taking care to avoid spurious reflection of power) at some lower l without compromising accuracy in the evaluation of the source functions. The sources are calculated at approximately logarithmically spaced integer values of ν and interpolated as needed.

In closed models, a given linear scale at last scattering subtends a larger angle on the sky today than in open or flat models. This geometric effect shifts power in the CMB spectra to smaller l. Since CMBFAST only computes the power spectra at a few values of l, we adjust the l-sampling according to the curvature to maintain accurate interpolation in all cases.

For the line-of-sight integral over sources, we calculate the hyperspherical Bessel functions by integrating a second-order differential equation, as in the open CMBFAST code (Zaldarriaga, Seljak, & Bertschinger 1998). The starting values for the Bessel function and its derivative are found using a recursive evaluation or, for the cases in which it is accurate and faster, the WKB approximation (Kosowsky 1998). We assume that the development angle χ satisfies $\chi < \pi$, where $\chi = |K|^{1/2}\eta_0$, and η_0 is the conformal age of the universe. This permits calculations with models up to $\Omega_{\rm tot} \approx 1.7$ for a matter fraction $\Omega_m \approx 0.4$. For the case of $\chi \approx \pi$, all lines of sight converge to the antipodal point close to last scattering, and the power spectrum becomes featureless (White & Scott 1996).

It is important to maintain numerical stability in the differential equations for the dependent variables. This is especially true for scalar perturbations with isocurvature initial conditions, where a poor choice of dependent variables can lead to large violations of the Einstein constraint equations unless the initial conditions are specified to exquisite accuracy. (This problem is particularly acute in the Newtonian gauge, e.g., Ma & Bertschinger 1995.) We choose to work in a frame in which u^a coincides with the 4-velocity of the cold dark matter (CDM). For scalar perturbations, we determine the perturbations to the geometry by evolving the projected gradient of the 3-curvature on hypersurfaces orthogonal to u^a, which gives good numerical stability for all initial conditions. (With this choice of dependent variables, our equation set is equivalent to a subset of the widely used synchronous equations, gauge-fixed to the CDM.) We allow for adiabatic and two types of isocurvature initial conditions for scalar perturbations. We also provide the option of specifying the primordial power spectra in nonparametric form, which is useful for inverse problems such as initial power spectrum reconstruction.

We have verified our calculations against results obtained with CMBFAST version 2.4.1 for models supported by the latter (open and flat). We have also compared our results with a prerelease version of CMBFAST which Seljak & Zaldarriaga are developing to support closed models, with good agreement well into the damping tail.

3. RESULTS

In Figure 1 we plot the intensity and polarization power spectra in ΛCDM models assuming no reionization. One model is closed ($\Omega_{\rm tot} = 1.2$, $\Omega_\Lambda = 0.8$), while the other is flat ($\Omega_{\rm tot} = 1$, $\Omega_\Lambda = 0.6$). In both cases we take the matter fraction $\Omega_m = 0.4$, baryon fraction $\Omega_b = 0.045$, and Hubble constant $H_0 = 65$ km s^{-1} Mpc^{-1}. For the scalar modes, we assume adiabatic initial conditions with a scale-invariant primordial power spectrum $P_S(\nu) = $ const. Our conventions generalize those of Lyth & Woszczyna (1995), so that the gradient of the 3-Ricci scalar on comoving hypersurfaces $\tilde{D}_a \tilde{R}^{(3)}$ receives power

$$\langle |\tilde{D}_a \tilde{R}^{(3)}|^2 \rangle \propto \sum_{\nu \geq 3} \left(\frac{k}{S}\right)^6 \left(\frac{\nu^2 - 4}{\nu^2 - 1}\right)^2 \frac{\nu P_S(\nu)}{(\nu^2 - 1)}, \qquad (5)$$

from super-Hubble modes. The tildes denote that the quantity on the left is evaluated in the (energy) frame where the momentum density q_a vanishes. For the tensor modes, we assume a scale-invariant spectrum $P_T(\nu) = $ const. Our conventions are such that the power in the electric part of the Weyl tensor, E_{ab} (e.g., Ellis 1998), from super-Hubble modes is

$$\langle E_{ab} E^{ab} \rangle \propto \sum_{\nu \geq 3} \left(\frac{k}{S}\right)^4 \frac{(\nu^2 - 4)}{\nu^2} \left(\frac{\nu^2 - 1}{\nu^2 - 3}\right)^2 \frac{\nu P_T(\nu)}{(\nu^2 - 1)}. \qquad (6)$$

Note that in closed models, the infrared divergence in the tensor power spectrum seen in open models with $P_T(\nu) = $ const is avoided because of the cutoff at $\nu = 3$.

The models in Figure 1 have equal physical densities, $\Omega_m H_0^2$ and $\Omega_b H_0^2$, so the sound horizon at last scattering and the early-time dynamics are approximately equal in the two models. On small angular scales, where the polarization and scalar anisotropies are projections of effects at last scattering, restricting parameter changes to Ω_Λ and the curvature leads to approximate scaling of the CMB power spectra, $C_l \rightarrow C_{\alpha l}$, where α is the ratio of the angular-diameter distances to last scattering in the original and final models. Approximations of this sort were used by Efstathiou et al. (1999) to include closed models in their recent joint analysis of CMB and supernova data. On large angular scales (small l), the approximate scaling is broken for scalar anisotropies by the (late-time) integrated Sachs-Wolfe effect (e.g., Hu & Sugiyama 1995), which is responsible for the enhancement at low l seen in the closed model in Figure 1 (Abbott & Schaefer 1986; White & Scott 1996). The effects of curvature terms in the primordial power spectrum are also potentially observable on large scales (Hu et al. 1998). The presence of the curvature scale in the primordial power spectrum, combined with the cutoff at $\nu = 3$, suppresses the intensity quadrupole for tensor modes in the closed model relative to the flat model in Figure 1 (cf. the case of open models, e.g., Hu & White 1997).

4. CONCLUSION

We have presented the first calculation of the CMB power spectra, including the effects of polarization, in closed FRW models. We have implemented the efficient line-of-sight algorithm for general geometries, using covariantly defined, gauge-invariant variables, thus allowing accurate and rapid modeling over the full volume of parameter space of FRW models.[1]

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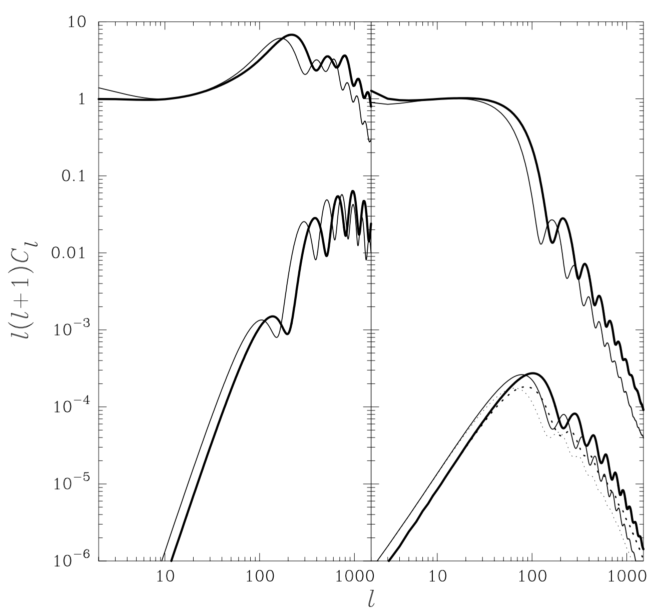

FIG. 1.—Scalar (*left*) and tensor (*right*) intensity and polarization power spectra in a closed CDM model ($\Omega_{\rm tot} = 1.2$, $\Omega_\Lambda = 0.8$; *thin lines*), and a flat model ($\Omega_{\rm tot} = 1$, $\Omega_\Lambda = 0.6$; *thick lines*). In both cases, $\Omega_m = 0.4$, $\Omega_b = 0.045$, and $H_0 = 65$ km s^{-1} Mpc^{-1}. The upper solid lines show the intensity and the lower ones the electric component of the polarization, $C_l^{\mathcal{E}\mathcal{E}}$; dashed lines show the magnetic component, $C_l^{\mathcal{B}\mathcal{B}}$. (Note that scalar modes produce no magnetic polarization.) The scalar and tensor intensities are normalized to unity at $l = 10$.

[1] Our FORTRAN 90 code, based on CMBFAST version 2.4.1, is available at http://www.mrao.cam.ac.uk/~aml1005/cmb.

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