PROBABILISTIC ALGORITHMS FOR FINDING MATRIX DECOMPOSITIONS

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ABSTRACT. This paper is a sample prepared to illustrate the use of the American Mathematical Society's LaTeX document class `amsart` and publication-specific variants of that class for AMS-LaTeX version 2.

THEORY

1. FIXED RANK PROBLEM

PROTO-ALGORITHM: SOLVING THE FIXED-RANK PROBLEM

Given an $m \times n$ matrix $\boldsymbol{A}$, a target rank k, and an oversampling parameter p, this procedure computes an $m \times (k+p)$ matrix $\boldsymbol{Q}$ whose columns are orthonormal and whose range approximates the range of $\boldsymbol{A}$.

1. Draw a random $n \times (k+p)$ test matrix $\boldsymbol{\Omega}$.
2. Form the matrix product $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\Omega}$.
3. Construct a matrix $\boldsymbol{Q}$ whose columns form an orthonormal basis for the range of $\boldsymbol{Y}$.

Theorem 1.1. *Suppose that $\boldsymbol{A}$ is a real $m \times n$ matrix. Select a target rank $k \geq 2$ and an oversampling parameter $p \geq 2$, where $k+p \leq \min\{m,n\}$. Execute the proto-algorithm with a standard Gaussian test matrix to obtain an $m \times (k+p)$ matrix $\boldsymbol{Q}$ with orthonormal columns. Then*

$$\mathbb{E}\,\|\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{A}\| \leq \left[1 + \frac{4\sqrt{k+p}}{p-1} \cdot \sqrt{\min\{m,n\}}\right] \sigma_{k+1}, \tag{1.1}$$

where $\mathbb{E}$ denotes expectation with respect to the random test matrix and σ_{k+1} is the $(k+1)$th singular value of $\boldsymbol{A}$.

The probability that the error satisfies

$$\|\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{A}\| \leq \left[1 + 11\sqrt{k+p} \cdot \sqrt{\min\{m,n\}}\right] \sigma_{k+1} \tag{1.2}$$

is at least $1 - 6 \cdot p^{-p}$ under very mild assumptions on p.

2. Randomized SVD

The Randomized SVD procedure requires only $2(q+1)$ passes over the matrix, so it is efficient even for matrices stored out-of-core. The flop count satisfies

$$T_{\mathrm{randSVD}} = (2q+2)\,k\,T_{\mathrm{mult}} + \mathrm{O}(k^2(m+n)),$$

where T_{mult} is the flop count of a matrix–vector multiply with $\boldsymbol{A}$ or $\boldsymbol{A}^*$.

Prototype for Randomized SVD

Given an $m \times n$ matrix $\boldsymbol{A}$, a target number k of singular vectors, and an exponent q (say $q = 1$ or $q = 2$), this procedure computes an approximate rank-$2k$ factorization $\boldsymbol{U\Sigma V}^*$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthonormal, and $\boldsymbol{\Sigma}$ is nonnegative and diagonal.

Stage A:

1 Generate an $n \times 2k$ Gaussian test matrix $\boldsymbol{\Omega}$.
2 Form $\boldsymbol{Y} = (\boldsymbol{AA}^*)^q \boldsymbol{A\Omega}$ by multiplying alternately with $\boldsymbol{A}$ and $\boldsymbol{A}^*$.
3 Construct a matrix $\boldsymbol{Q}$ whose columns form an orthonormal basis for the range of $\boldsymbol{Y}$.

Stage B:

4 Form $\boldsymbol{B} = \boldsymbol{Q}^*\boldsymbol{A}$.
5 Compute an SVD of the small matrix: $\boldsymbol{B} = \widetilde{\boldsymbol{U}}\boldsymbol{\Sigma V}^*$.
6 Set $\boldsymbol{U} = \boldsymbol{Q}\widetilde{\boldsymbol{U}}$.

Note: The computation of $\boldsymbol{Y}$ in Step 2 is vulnerable to round-off errors. When high accuracy is required, we must incorporate an orthonormalization step between each application of $\boldsymbol{A}$ and $\boldsymbol{A}^*$; see Algorithm **??**.

Theorem 2.1. *Suppose that $\boldsymbol{A}$ is a real $m \times n$ matrix. Select an exponent q and a target number k of singular vectors, where $2 \le k \le 0.5\min\{m,n\}$. Execute the Randomized SVD algorithm to obtain a rank-$2k$ factorization $\boldsymbol{U\Sigma V}^*$. Then*

$$\mathbb{E}\,\|\boldsymbol{A} - \boldsymbol{U\Sigma V}^*\| \le \left[1 + 4\sqrt{\frac{2\min\{m,n\}}{k-1}}\right]^{1/(2q+1)} \sigma_{k+1}, \tag{2.1}$$

where $\mathbb{E}$ denotes expectation with respect to the random test matrix and σ_{k+1} is the $(k+1)$th singular value of $\boldsymbol{A}$.

In practice, we can truncate the approximate SVD, retaining only the first k singular values and vectors. Equivalently, we replace the diagonal factor $\boldsymbol{\Sigma}$ by the matrix $\boldsymbol{\Sigma}_{(k)}$ formed by zeroing out all but the largest k entries of $\boldsymbol{\Sigma}$. For this truncated SVD, we have the error bound

$$\mathbb{E}\,\left\|\boldsymbol{A} - \boldsymbol{U\Sigma}_{(k)}\boldsymbol{V}^*\right\| \le \sigma_{k+1} + \left[1 + 4\sqrt{\frac{2\min\{m,n\}}{k-1}}\right]^{1/(2q+1)} \sigma_{k+1}. \tag{2.2}$$

References

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