Determination of Planck's Constant Using X-Ray Diffraction

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This paper analyses spectra of X-rays generated by an X-ray tube with a molybdenum anode to calculate Planck's constant (h) by measuring the minimum wavelength of X-rays emitted for different acceleration voltages. It is found that $h = (6.7 \pm 0.2) \times 10^{-34}\,Js$ which is in excellent agreement with the literature value. The validity of the experiment is checked by calculating K_α and K_β of Mo to be $(17.7 \pm 0.6)\,keV$ and $(19.9 \pm 0.7)\,keV$ respectively, which are also in excellent agreement with the literature values.

I Introduction

Planck's constant (h) is a physical constant defined as the ratio between the energy of a photon (in joules) and its frequency (in hertz) and has a numerical literature value of $h = 6.626070040(81) \times 10^{34}\,Js$ [1]. Accurately determining h is important as in 2019 it will define the kilogram [2] and it can be determined by analysing the diffraction of X-rays through a crystal. X-rays are high energy photons that can be produced when electrons (produced by thermionic emission in a cathode) are accelerated by a voltage to bombard a target (anode). This creates two distinct types of X-ray: characteristic X-rays and bremsstrahlung (braking radiation).

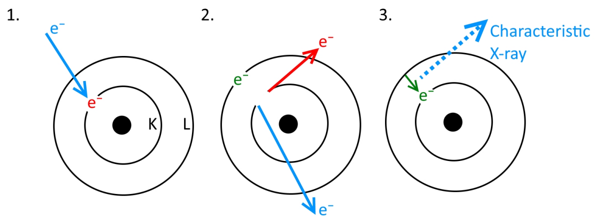

FIG. 1: (1) An electron emitted by thermionic emission (blue) strikes an electron in an anode atom (red) to knock it out of the atom (2). Characteristic X-rays are emitted when higher energy electrons (green) lose energy to fill this space. These X-rays make sharp peaks in spectra. K_α and K_β are names given to the characteristic X-rays produced when an electron drops to the 1s subshell from the 2p and 3p subshells respectively.

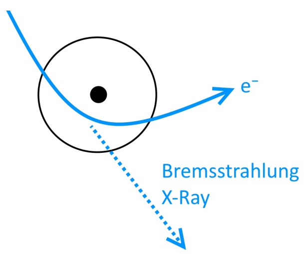

FIG. 2: Bremsstrahlung is emitted when electrons lose energy as they are decelerated by the electric field of an anode nucleus, it appears as a continuous curve on spectra.

However, the electrons have a maximum energy which depends on the acceleration voltage (V); there is therefore a maximum energy (minimum wavelength (λ_{min})) that the X-rays can have which corresponds to this voltage. This is described by the Duane-Hunt law:

$$\lambda_{min} = \frac{hc}{eV} \tag{1}$$

Where e is the elementary charge and c is the speed of light in a vacuum. Hence, h can be calculated by measuring the minimum wavelength of X-rays emitted for different voltages and plotting a graph of λ_{min} against $\frac{1}{V}$ such that the gradient is $\frac{hc}{e}$. To calculate the wavelength of the X-rays, they are diffracted through a rotating crystal so that Bragg's Law can be used:

$$n\lambda = 2dsin\beta_n \tag{2}$$

Where n is the principal order of maxima (1 in this case), $2d$ is the lattice constant (d is the atomic spacing) and β is the angle between the X-ray and crystal. In the following sections, the experimental setup is outlined and the results are presented and evaluated.

II Methods

Spectra were produced by measuring the rate of X-rays detected per second for varying crystal angles for different voltages. A Leybold X-ray apparatus [3] was used to produce the X-rays with an X-ray tube, rotate the crystal with a motor (and goniometer to measure β) and count the X-rays with a Geiger-Müller tube. It was found in preliminary experiments that a sensible range for the crystal angle was $2.0 < \beta < 10.0$ as it displayed both the minimum angle that X-rays were detected and characteristic peaks (for $n = 1$). A high range of voltages ($20.0 < V < 35.0$) was used as this minimised the uncertainty in the voltage and increased the intensity of the X-ray spectra relative to the background radiation (the maximum current of $1.00\,mA$ was used for the same reason). The time step for each measurement was $\Delta t = 4\,s$ as this interval was long enough to get sufficient data but short enough that the experiment finished within a reasonable time and allowed time for subsequent experiments. The anode and crystal were made from Mo and NaCl ($2d = 564.0\,pm$ [4]) respectively.

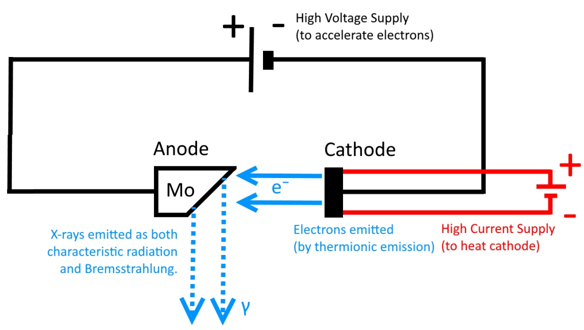

FIG. 3: Diagram of the X-ray tube.

From these spectra, the minimum angle was found by recording the value of β_{min} for which the rate of X-rays first went above $10\,s^{-1}$. The background radiation was of the order $1\,s^{-1}$ so $10\,s^{-1}$ was a clear indicator that the X-rays being detected were from the anode. β_{min} values were

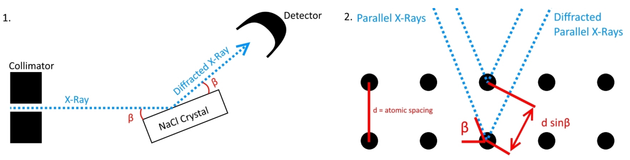

FIG. 4: Diagram of the X-ray diffraction.

converted to λ_{min} using Bragg's law. To check for systematic errors, characteristic X-ray energies K_α and K_β of Mo were calculated by taking β at which the peaks occur, Bragg's law was then used to convert them into wavelengths and $E = \frac{hc}{\lambda}$ was used to convert them into energies. To check the validity of the experiment; the value for h used will be the calculated value not the literature value.

III Results and Discussion

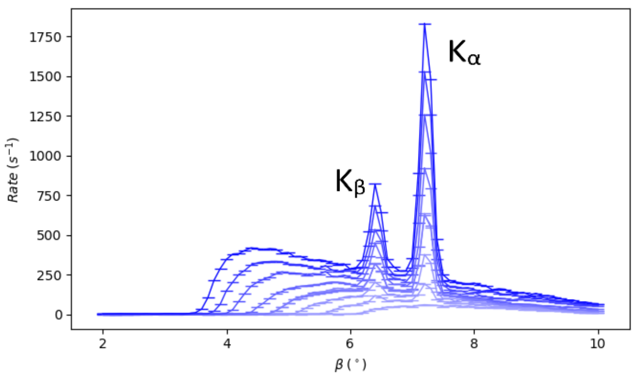

FIG. 5: X-ray spectra of the rate of X-rays against the angle of the crystal for varying voltages. The darker the line, the higher the voltage. The characteristic peaks and bremsstrahlung curve can be seen.

From Figure 5, λ_{min} was found for each voltage respectively as described previously and then plotted against $\frac{1}{V}$.

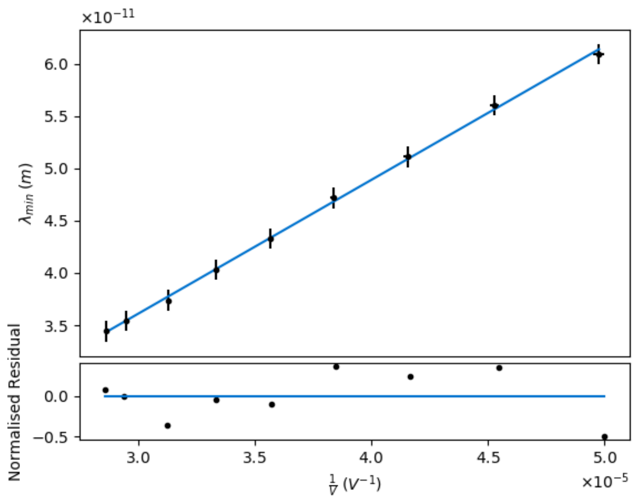

FIG. 6: Minimum wavelength against the reciprocal of voltage.

The gradient and intercept of Figure 6 were calculated using a weighted regression line. The gradient of Figure 6 was shown earlier to be $\frac{hc}{e}$ so $h = \frac{Gradient \times e}{c} = (6.7 \pm 0.2) \times 10^{-34}\, Js$. This is in excellent agreement with the literature value of $6.626070040(81) \times 10^{34}\, Js$. The result has not been affected significantly by systematic error for 3 reasons:

1. 100% of the normalised residuals are within $\pm$ 1 standard error of the regression line, which surpasses the 65% required for a good fit [5, pg. 72].

2. The intercept of Figure 6 is $(-2 \pm 2) \times 10^{-12}$ which is within one standard error of zero. A sign of a systematic error is a non-zero intercept [5, pg. 63].

3. The characteristic peaks of Figure 5 at $(7.2 \pm 0.1)°$ and $(6.4 \pm 0.1)°$ are K_α and K_β for Mo (these are the same for all voltages). These angles correspond to energies of $(17.7 \pm 0.6)\, keV$ and $(19.9 \pm 0.7)\, keV$ for K_α and K_β respectively. These are in excellent agreement with the literature values of $17.479372(10)\, keV$ and $19.60834(42)\, keV$ [6].

The biggest source of error in this experiment was the angular resolution of the X-ray apparatus of $\pm 0.1°$. It was not small enough to see the two separate K_α peaks that exist in reality and it was much greater than the uncertainty in the voltage of $\pm 0.1\, kV$ (as evidenced by the relative size of the error bars in Figure 6).

IV Conclusions

Planck's constant has been determined as $h = (6.7 \pm 0.2) \times 10^{-34}\, Js$ and K_α and K_β for Mo have been determined as $(17.7 \pm 0.6)\, keV$ and $(19.9 \pm 0.7)\, keV$ respectively, all of these values are accurate as they are in excellent agreement with the literature values. Any systematic errors present had little to no affect on the experiment. This implies that this method of calculating Planck's constant and X-ray transition energies has credence and could be used to calculate more precise values in the future if the angular resolution of the X-ray apparatus was increased (as this was the primary error source).

V Acknowledgements

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VI Error Appendix

Unless stated otherwise, all following equations [5, pg. 44] and propagations [5, Ch. 4] come from *Measurements and their Uncertainties*.
The error from the gradient was propagated through into Planck's constant using Equation 3. The values for the elementary charge and the speed of light used were the literature values [1].

$$\alpha_Z = k\alpha_A \;\forall\; Z = kA \tag{3}$$

The error of β was propagated through into λ (for calculating λ_{min} and K_α and K_β) using the functional approach described by Equations 4 and 5, as was the error from λ into E for K_α and K_β.

$$\alpha_Z^2 = (\alpha_Z^A)^2 + (\alpha_Z^B)^2 + \dots \;\forall\; Z = f(A, B, \dots) \tag{4}$$

$$\alpha_Z^A = |f(\bar{A} + \alpha_A, \bar{B}, \dots) - f(\bar{A}, \bar{B}, \dots)| \tag{5}$$

The gradients and intercept of Figure 5 were calculated using a weighted regression line described by the following equations:

$$c = \frac{\Sigma_i w_i x_i^2 \Sigma_i w_i y_i - \Sigma_i w_i x_i \Sigma_i w_i x_i y_i}{\Delta'} \tag{6}$$

$$m = \frac{\Sigma_i w_i \Sigma_i w_i x_i y_i - \Sigma_i w_i x_i \Sigma_i w_i y_i}{\Delta'} \tag{7}$$

$$\alpha_c = \sqrt{\frac{\Sigma_i w_i x_i^2}{\Delta'}} \tag{8}$$

$$\alpha_m = \sqrt{\frac{\Sigma_i w_i}{\Delta'}} \tag{9}$$

$$\Delta' = \Sigma_i w_i \Sigma_i w_i x_i^2 - (\Sigma_i w_i x_i)^2 \tag{10}$$