Análise Real _ Exercícios

① Considere a identidade

$$\frac{1}{1+s} = \sum_{j=1}^{n} (-1)^j s^j + \frac{(-1)^{n+1} s^{n+1}}{1+s} \quad , s \neq -1$$

Mostre que $\left| \log(1+x) - \sum_{j=1}^{n} \frac{(-1)^j x^{j+1}}{j+1} \right| \leq \frac{1}{n+2}$

para $0 \leq x \leq 1$.

Conclua que a série $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converge para $\log 2$

② Considere agora

$$\frac{1}{1+s^2} = \sum_{j=1}^{n} (-1)^j s^{2j} + (-1)^{n+1} \frac{s^{2(n+1)}}{1+s^2}$$

Mostre que $\left| \operatorname{arctg} x - \sum_{j=1}^{n} (-1)^j \frac{x^{2j+1}}{2j+1} \right| \leq \frac{1}{2n+3}$

para $0 \leq x \leq 1$

Conclua que a série $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ converge para $\frac{\pi}{4}$

③ Seja $g: \mathbb{R} \to \mathbb{R}$ derivável t.q.

$$g'(y) = a e^y + g(y) \quad \text{para algum } a \in \mathbb{R}$$

Se $g(0) = 0$, mostre que $g(y) = a y e^y$

Se $g(0) = c$, mostre que $g(y) = (ay + c) e^y$

④ Seja $g: \mathbb{R} \to \mathbb{R}$ derivável t.q.

$$g(x+y) = e^y g(x) + e^x g(y) \quad \text{para quaisquer } x, y$$

Calcule g.

⑤ Mostre que $\int_0^x t e^{-t} dt = e^{-x}(e^x - 1 - x)$

⑥ Mostre que a) $\frac{1}{x+\frac{1}{2}} \leq \log\left(1 + \frac{1}{x}\right) < \frac{1}{x}$ se $x > 0$

b) $x - \frac{x^3}{6} < \operatorname{sen} x < x$ se $x > 0$

⑦ Seja $f: [a,b] \to \mathbb{R}$ integrável. Mostre que f^2 $(= |f|^2)$ é integrável.

Deduza que se f, g são integráveis então $f \cdot g$ é integrável

(1) $\frac{1}{1-t} = \sum_{j=0}^{\infty} t^j = \sum_{j=0}^{n} t^j + \sum_{j=n+1}^{\infty} t^j \quad , t \neq 1$

Para $-t$, temos, $\frac{1}{1+t} = \sum_{j=0}^{n} (-1)^j t^j + \sum_{j=n+1}^{\infty} (-1)^j t^j \quad , t \neq -1$

e $\frac{1}{1+t} = \sum_{j=0}^{n} (-1)^j t^j + \frac{(-1)^{n+1} t^{n+1}}{1+t}$. Logo,

$$\int_0^x \frac{1}{1+t} \, dt = \int_0^x \left(\sum_{j=0}^{n} (-1)^j t^j + \frac{(-1)^{n+1} t^{n+1}}{1+t} \right) dt$$

$$\int_0^x \frac{1}{1+t} \, dt = \int_0^x \sum_{j=0}^{n} (-1)^j t^j \, dt + \int_0^x \frac{(-1)^{n+1} t^{n+1}}{1+t} \, dt$$

$$\ln(1+x) = \sum_{j=0}^{n} \frac{(-1)^j x^{j+1}}{j+1} + \int_0^x \frac{(-1)^{n+1} t^{n+1}}{1+t} \, dt$$

e $\left| \int_0^x \frac{(-1)^{n+1} t^{n+1}}{1+t} \, dt \right| = \left| \ln(1+x) - \sum_{j=0}^{n} \frac{(-1)^j x^{j+1}}{j+1} \right|$

Como $\left| \int_0^x \frac{(-1)^{n+1} t^{n+1}}{1+t} \, dt \right| \leq \int_0^x \left| \frac{(-1)^{n+1} t^{n+1}}{1+t} dt \right| dt = \int_0^x \frac{t^{n+1}}{1+t} \, dt$

Além disso, para $0 \leq t \leq 1$, temos que

$$\left|\int_0^x \frac{(-1)^{n+1}t^{n+1}}{1+t}\,dt\right| \leq \int_0^x \frac{t^{n+1}}{1+t}\,dt \leq \int_0^x t^{n+1}\,dt = \frac{x^{n+2}}{n+2} \leq \frac{1}{n+2}$$

Logo, $\boxed{\left|\ln(1+x) - \sum_{j=0}^{n} \frac{(-1)^j x^{j+1}}{j+1}\right| \leq \frac{1}{n+2}}$

Para $x=1$: $\left|\ln(2) - \sum_{j=0}^{n} \frac{(-1)^j}{j+1}\right| \leq \frac{1}{n+2}$

Fazendo $n \to \infty$, $1/_{n+2} \to 0$ e, pelo Teorema do confronto:

$$\lim_{n\to\infty}\left(\ln 2 - \sum_{j=0}^{n} \frac{(-1)^j}{j+1}\right) = 0$$

e $\ln 2 = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} = 1 - 1/2 + 1/3 - 1/4 + \dots$

(2) Substituindo por t^2 na mesma identidade, temos:

$$\frac{1}{1+t^2} = \sum_{j=0}^{n} (-1)^j t^{2j} + \frac{(-1)^{j+1} t^{2(j+1)}}{1+t^2}$$

logo, $\int_0^x \frac{1}{1+t^2}\,dt = \sum_{j=0}^{n}\left(\int_0^x (-1)^j t^{2j}\,dt\right) + \int_0^x \frac{(-1)^{j+1} \cdot t^{2j+2}}{1+t^2}\,dt$

$$\Rightarrow \operatorname{arctg} x = \sum_{j=0}^{n} \frac{(-1)^j x^{2j+1}}{2j+1} + \int_0^x \frac{(-1)^{n+1} \cdot t^{2n+2}}{1+t^2}\, dt$$

e $$\left| \int_0^x \frac{(-1)^{j+1} \cdot t^{2j+2}}{1+t^2}\, dt \right| = \left| \operatorname{arctg} x - \sum_{j=0}^{n} \frac{(-1)^j x^{2j+1}}{2j+1} \right|$$

Além disso, temos que:

$$\left| \int_0^x \frac{(-1)^{n+1} \cdot t^{2n+2}}{1+t^2}\, dt \right| \leq \int_0^x \left| \frac{(-1)^{n+1}\, t^{2n+2}}{1+t^2} \right| dt = \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt \leq \int_0^x t^{2n+2}\, dt$$

$$= \frac{x^{2n+3}}{2n+3} \leq \frac{1}{2n+3}$$ para $0 \leq x \leq 1$.

Fazendo $x=1$, temos:

$$\left| \operatorname{arctg} 1 - \sum_{j=0}^{n} \frac{(-1)^j}{2j+1} \right| \leq \frac{1}{2n+3}$$, fazendo $n \to \infty$,

$1/_{2n+3} \to 0$ e, pelo Teorema do Confronto, tem-se:

$$\lim_{n\to\infty} \left(\operatorname{arctg} 1 - \sum_{j=0}^{n} \frac{(-1)^j}{2j+1} \right) = 0$$ e $$\boxed{\operatorname{arctg} 1 = \pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

③ $g'(y) = ae^y + g(y)$, $g: \mathbb{R} \to \mathbb{R}$ derivável (∴ contínua) para algum $a \in \mathbb{R}$

a) $g(0) = 0$, $g'(0) = a$

Suponha $g(y) = p(y) \cdot q(y)$, logo
$g'(y) = p'(y)q(y) + p(y)q'(y) = ae^y + p(y)q(y)$.

Logo, s.p.g, suponha $\begin{cases} p'(y)q(y) = ae^y \\ p(y)q'(y) = p(y)q(y) \end{cases}$

$p'(y)q(y) = ae^y$
$p(y)(q(y) - q'(y)) = 0$

se $p(y) \equiv 0$, $p'(y) \equiv 0$ e $a = 0$ necessariamente, absurdo

Assim, $q(y) = q'(y)$ e $p'(y) \cdot q'(y) = ae^y$

Logo, $p'(y) = a$ e $q'(y) = e^y$. Logo $p(y) = ay + C_1$ e $q(y) = e^y + C_2$, C_1, C_2 constantes.

Contudo, como $g(0) = 0 \wedge g'(0) = a$, $C_1 = C_2 = 0$.
Logo, $g(y) = aye^y$.

b) Raciocínio análogo, mas $C_1 = c$ e $g(y) = (ay + c)e^y$.

4 $g: \mathbb{R} \to \mathbb{R}$ derivável ($\therefore$ contínua)

$g(x+y) = e^x g(y) + e^y g(x)$

$\frac{d}{dx} g(x+y) =$