Tangentes de curvas paramétricas

$r = f(\theta) \Rightarrow \begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\,\text{sen}\,\theta = f(\theta)\cdot\text{sen}\,\theta \end{cases}$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$, $dx/d\theta \neq 0$

$$\boxed{\text{Lembre-se: } \frac{d^2y}{dx^2} = \frac{\dfrac{d(dy/dx)}{d\theta}}{\dfrac{dx}{d\theta}}}$$

Ex 1: Ache a inclinação da reta tangente à curva $r = 1 + \cos\theta$ no ponto $\theta = \pi/3$.

$x = (1+\cos\theta)\cos\theta = \cos\theta + \cos^2\theta$

$y = (1+\cos\theta)\,\text{sen}\,\theta = \text{sen}\,\theta + \frac{1}{2}\cdot\text{sen}\,2\theta$

$dx/d\theta = -\text{sen}\,\theta - \text{sen}\,2\theta$

$dy/d\theta = \cos\theta + \cos 2\theta$

$\therefore dy/dx = \dfrac{\cos\theta + \cos 2\theta}{-(\text{sen}\,\theta + \text{sen}\,2\theta)} = \boxed{-\dfrac{\cos\theta + \cos 2\theta}{\text{sen}\,\theta + \text{sen}\,2\theta}}$

Logo, $\left.\dfrac{dy}{dx}\right|_{\theta=\pi/3} = -\dfrac{\cos\pi/3 + \cos 2\pi/3}{\text{sen}\,\pi/3 + \text{sen}\,2\pi/3} = -\dfrac{(1/2 - 1/2)}{\sqrt{3}/2 + \sqrt{3}/2} = \boxed{0}$.

Determine todos os pontos dessa curva onde a tangente é horizontal ($dy/d\theta = 0 \wedge dx/d\theta \neq 0$)

· $\cos\theta + \cos 2\theta = 0 \Rightarrow \cos\theta + \cos^2\theta - (1 - \cos^2\theta) = 0$

· $\sin\theta + \sin 2\theta \neq 0$

$\Rightarrow \cos\theta + 2\cos^2\theta - 1 = 0$

$$\cos\theta = \frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{4} = \frac{-1 \pm 3}{4} \Rightarrow \boxed{1/2 \text{ ou } -1}$$

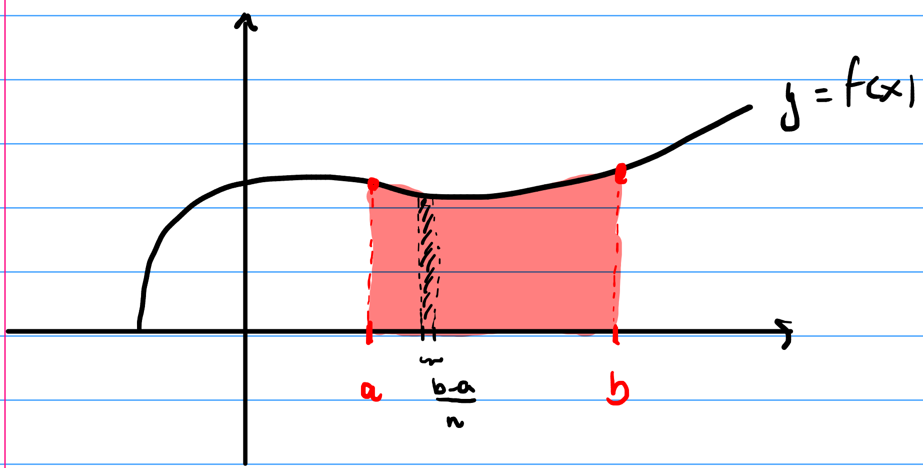
$\boxed{\therefore \theta = \{\pi/3, \pi, 5\pi/3\}}$. Mas $\sin\theta + \sin 2\theta \neq 0$

$\boxed{\therefore \theta = \{\pi/3, 5\pi/3\}.}$

$\boxed{\text{Pontos: } (3/4, 3\sqrt{3}/4), (3/4, -3\sqrt{3}/4)}$

Áreas

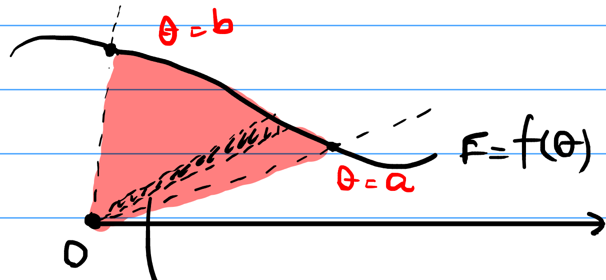
Lembrando do raciocínio utilizado em cálculo 1:



Temos intervalos (x_i, x_{i+1}) de comprimento $(b-a)/n$ e podemos eles serem tão pequenos quanto quisermos, além de escolher um x_i^* amostral em (x_i, x_{i+1}).

Logo, $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x \Rightarrow A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$.

Trazendo para o contexto de curvas paramétricas:



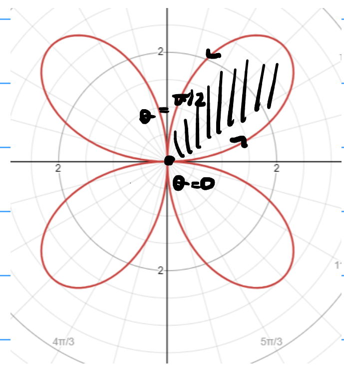
Lembrando da área de um setor: $\frac{1}{2}\theta r^2$

$$\Delta\theta = \frac{b-a}{n}$$

Podemos escolher um θ_i^* amostral no intervalo (θ_i, θ_{i+1}) de tamanho $(b-a)/n$.

$$\therefore A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\Delta\theta \cdot f(\theta_i^*)^2 = \boxed{\int_a^b \frac{1}{2} r^2\, d\theta} \quad //.$$

Ex: Calcule a área de uma pétala da rosácea $r = 3\,\text{sen}(2\theta)$:

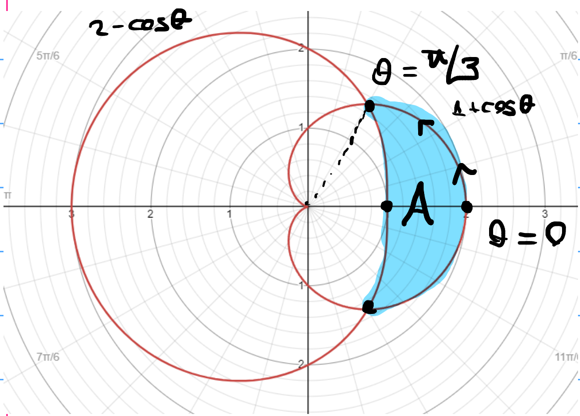


Primeiro: vemos como a curva é percorrida.

$$A = \frac{1}{2}\int_0^{\pi/2} 9\,\text{sen}^2 2\theta \, d\theta = \frac{9}{4}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{9}{4}\left(\theta + \frac{\text{sen}\,2\theta}{2}\right)\Big|_0^{\pi/2}$$

$$= \frac{9}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{9\pi}{8}} \;//.$$

Calcular a área dentro de $r = 1+\cos\theta$ (cardioide) e fora de $r = 2-\cos\theta$ (limaçon).



$\Rightarrow\; 1+\cos\theta = 2-\cos\theta$

$\cos\theta = 1/2$

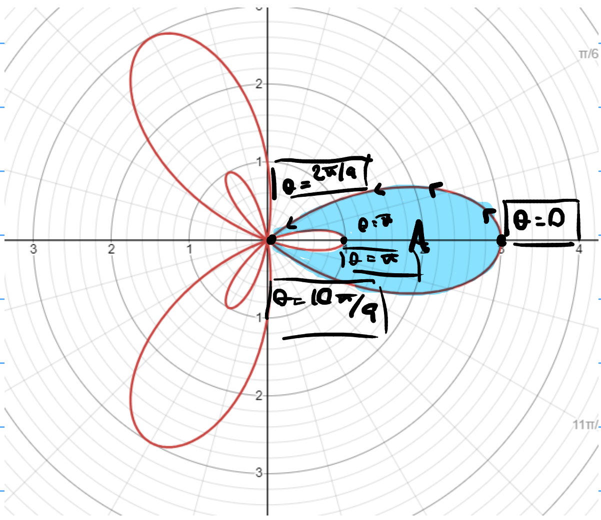
$\theta = \{\pi/6, -\pi/6\}$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/3}\left((1+\cos\theta)^2 - (2-\cos\theta)^2\right)d\theta = \int_0^{\pi/3}(6\cos\theta - 3)\,d\theta$$

$$= \left(6\,\text{sen}\,\theta - 3\theta\right)\Big|_0^{\pi/3} = \boxed{3\sqrt{3} - \pi} \;//$$

$$\boxed{= 3\sqrt{3} - \pi} \;//.$$

36. Encontre a área entre o laço maior e o laço menor da curva $r = 1 + 2\cos 3\theta$.



$$\begin{cases} x = (1+2\cos 3\theta)\cos\theta \\ y = (1+2\cos 3\theta)\,\text{sen}\,\theta \end{cases}$$

$$A = 2 \cdot \frac{1}{2}\left(\int_0^{2\pi/9} (1+2\cos 3\theta)^2\, d\theta - \int_{\pi}^{10\pi/9} (1+2\cos 3\theta)^2\, d\theta\right)$$

$$= \int_0^{2\pi/9} (1 + 4\cos 3\theta + 4\cos^2 3\theta)\, d\theta - \int_{\pi}^{10\pi/9} (1 + 4\cos 3\theta + 4\cos^2 3\theta)\, d\theta$$

$$\int_0^{2\pi/9} (3 + 4\cos 3\theta \; 2 \cdot \cos 6\theta)\, d\theta - \int_{\pi}^{10\pi/9} (3 + 4\cos 3\theta \cdot 2\cos 6\theta)\, d\theta$$

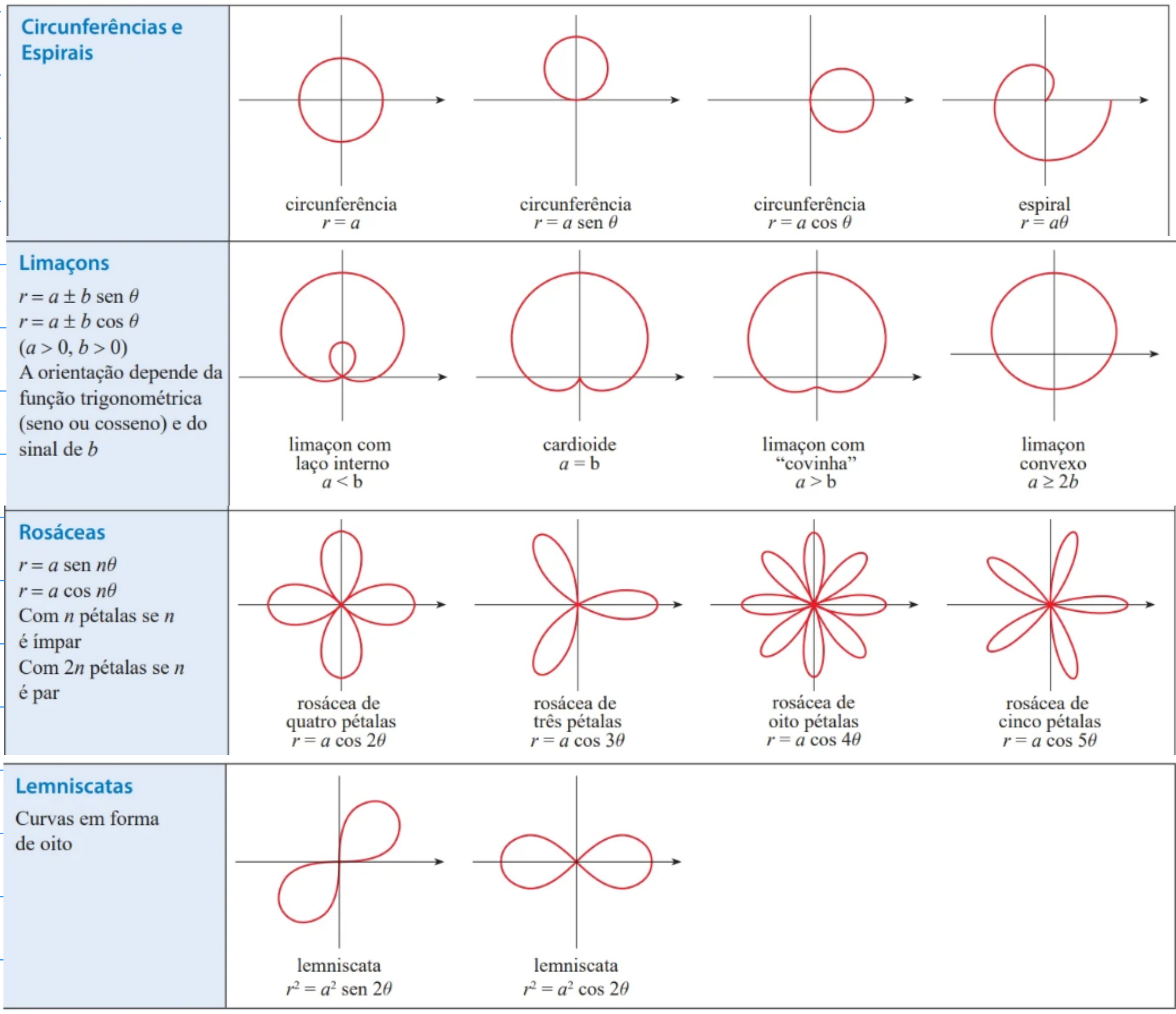
$$\left(3\theta + \frac{4}{3}\,\text{sen}\,3\theta + \frac{1}{3} \cdot \text{sen}\,6\theta\right)\Bigg|_0^{2\pi/9} - \left(3\theta + \frac{4}{3}\,\text{sen}\,3\theta + \frac{1}{3} \cdot \text{sen}\,6\theta\right)\Bigg|_{\pi}^{10\pi/9}$$

$$\left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} - \frac{1}{3} \cdot \frac{\sqrt{3}}{2}\right) - \left(\frac{10\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{\sqrt{3}}{2} - (3\pi)\right)$$

$$= \frac{2\pi}{3} + \frac{4\sqrt{3}}{6} - \frac{\sqrt{3}}{6} - \frac{10\pi}{3} + \frac{4\sqrt{3}}{6} - \frac{\sqrt{3}}{6} + 3\pi$$

$$3\pi + \frac{2\pi}{3} - \frac{10\pi}{3} + \frac{4\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = \frac{\pi}{3} + \sqrt{3}$$

$$\boxed{= \frac{\pi}{3} + \sqrt{3}}$$



Circunferências e Espirais	circunferência $r = a$	circunferência $r = a \text{ sen } \theta$	circunferência $r = a \cos \theta$	espiral $r = a\theta$
Limaçons $r = a \pm b \text{ sen } \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ A orientação depende da função trigonométrica (seno ou cosseno) e do sinal de b	limaçon com laço interno $a < b$	cardioide $a = b$	limaçon com "covinha" $a > b$	limaçon convexo $a \geq 2b$
Rosáceas $r = a \text{ sen } n\theta$ $r = a \cos n\theta$ Com n pétalas se n é ímpar Com $2n$ pétalas se n é par	rosácea de quatro pétalas $r = a \cos 2\theta$	rosácea de três pétalas $r = a \cos 3\theta$	rosácea de oito pétalas $r = a \cos 4\theta$	rosácea de cinco pétalas $r = a \cos 5\theta$
Lemniscatas Curvas em forma de oito	lemniscata $r^2 = a^2 \text{ sen } 2\theta$	lemniscata $r^2 = a^2 \cos 2\theta$		