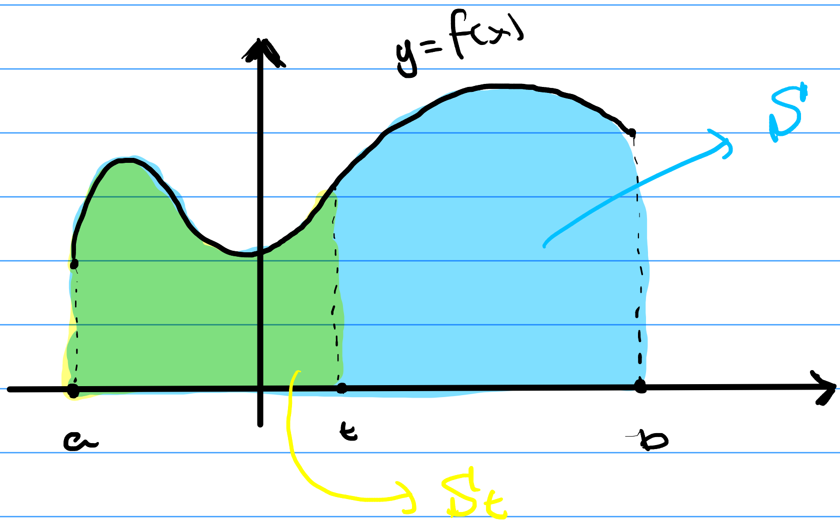
INTEGRAL

Áreas primitivas

Problema: Achar a área da região abaixo do gráfico de $f(x)$ e acima do eixo x, $a \leq x \leq b$, $f(x)$ contínua, $f(x) \geq 0$:



Queremos S.

Vamos definir uma função $g(t)$, $a \leq t \leq b$ em que $g(t)$ é a área de S_t (limitada por $y=0$, $y=f(t)$, $x=a$ e $x=t$.

$g: [a,b] \to \mathbb{R}$

Propriedades: 1) $g(a)=0$
$g(t) \geq 0$
g é crescente
g é derivável

• $g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}$

Fazendo $h>0$ $\therefore h\to 0^+$

O que é $g(t+h)-g(t)$? Região limitada por $y=f(x)$, $y=0$, $x=t$ e $x=t+h$.
Podemos dizer que ela tem um mínimo e um máximo

• Integral Definida:

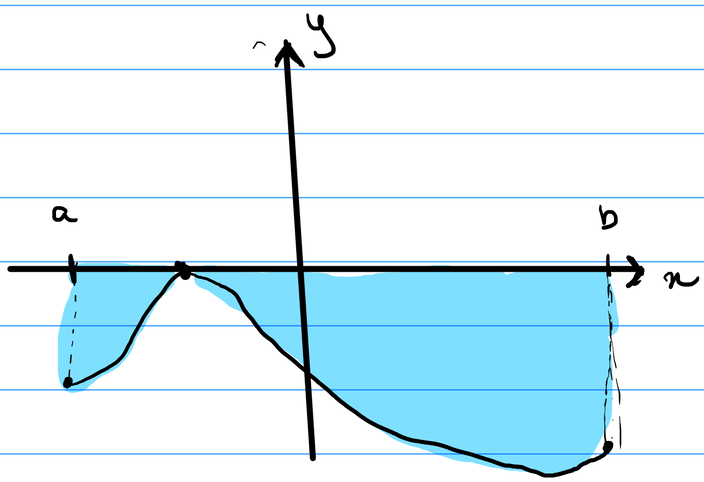
Para $f(x)$ contínua e $f(x)\geq 0$ em $[a,b]$, a integral definida $\int_a^b f(x)\,dx \overset{def}{=}$ área S

S é a região limitada por $y=f(x)$ e $y=0$, $a\leq x\leq b$.

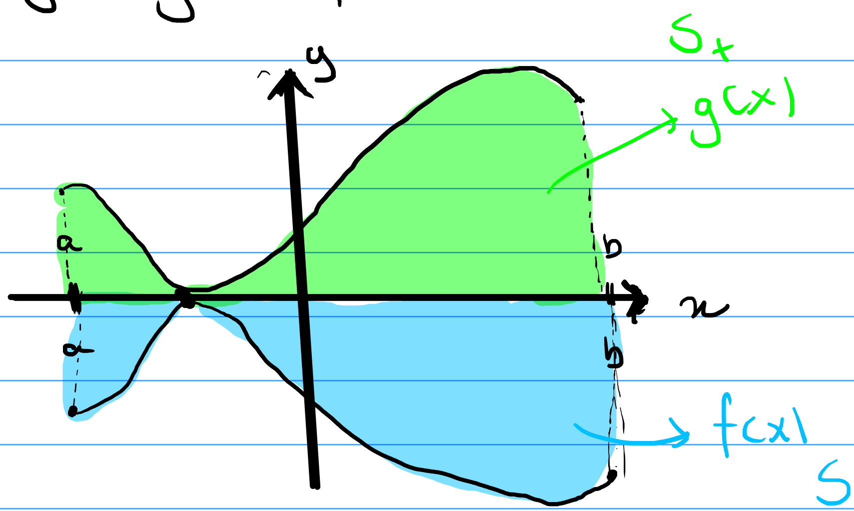
Face $H(x)$ | $H'(x)=f(x)$. Logo

$$\boxed{\int_a^b f(x)\,dx = H(b)-H(a).}$$

• Caso $f(x) \le 0$ em $[a,b]$, contínua



Agora, seja $g(x) = -f(x)$:



$$S_+ = \int_a^b g(x)\,dx = \int_a^b (-f(x))\,dx$$

Se $H(x)$ é primitiva de $f(x) \Rightarrow H'(x) = f(x)$
Portanto $-H'(x) = g(x)$.

$$S_+ = \int_a^b g(x)\,dx = -H(b) + H(a) = -(H(b) - H(a)).$$

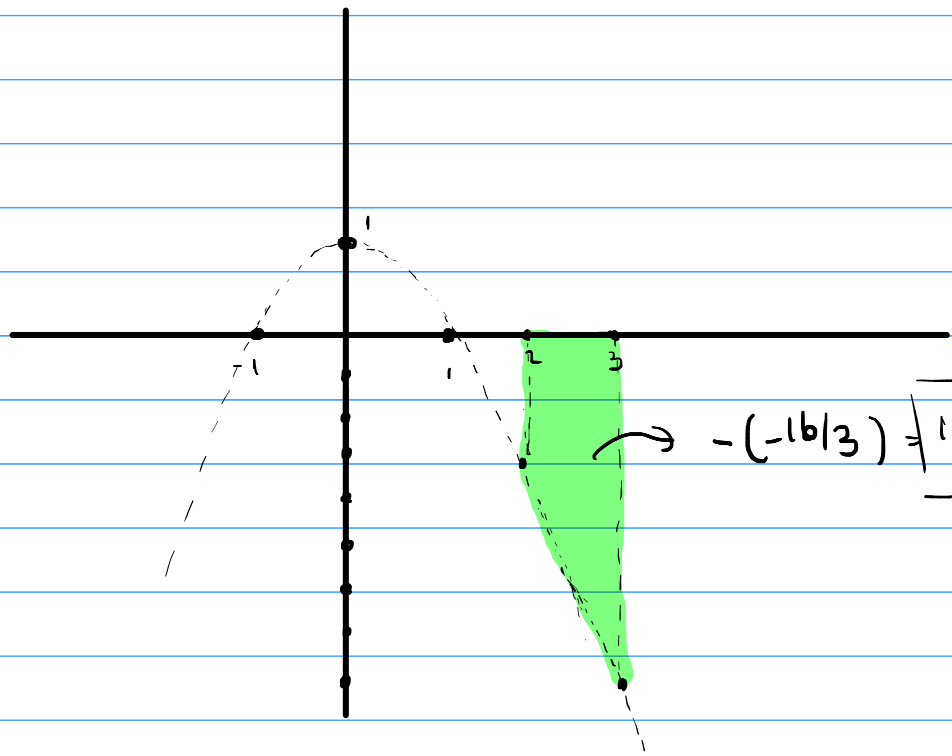
$\therefore$ $S = S_+$ e $S = -(H(b) - H(a))$

Ex: Calcule $\int_2^3 f(x)\,dx$ onde $f(x) = 1 - x^2$.

$H(x)$ | $H'(x) = f(x)$

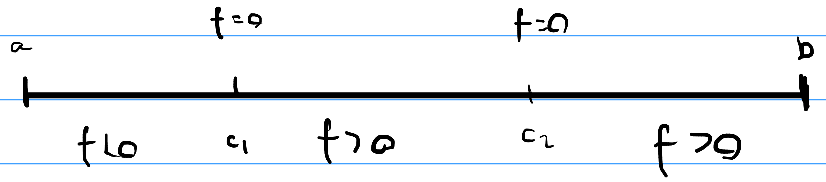
$$\boxed{H(x) = x - \frac{x^3}{3}}$$

$$\therefore \int_2^3 f(x)\,dx = 3 - \frac{3^3}{3} - 2 + \frac{2^3}{3} = 1 - \frac{19}{3} = \boxed{-\frac{16}{3}}$$



$\to -(-16/3) = \boxed{16/3}$

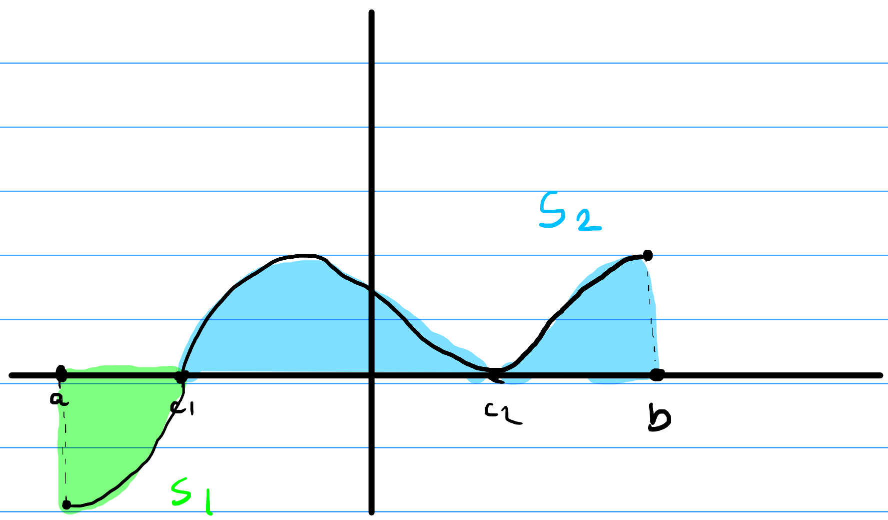
3º caso: $f(x)$ contínua em $[a,b]$ e muda de sinal em $[a,b]$.



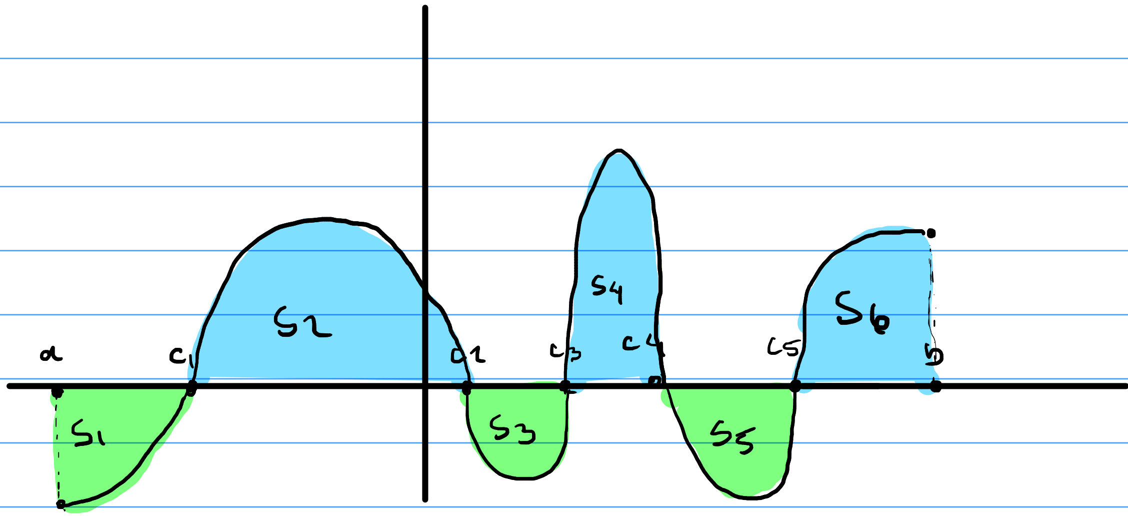
$$f \le 0 \text{ em } [a,c_1] \Rightarrow \int_a^{c_1} f(x)\,dx = -\text{área } S_1 = H(c_1) - H(a)$$

$$f \geq 0 \text{ em } [c_1,b] \Rightarrow \int_{c_1}^{b} f(x)\,dx = \text{área } S_2 = H(b) - H(c_1)$$

$$\Rightarrow \text{área } S_2 - \text{área } S_1 = H(b) - H(a) \boxed{= \int_a^b f(x)\,dx}$$



$$\therefore \int_a^b f(x)\,dx = -\left(\sum \begin{matrix}\text{Áreas abaixo}\\ \text{do eixo } x\end{matrix}\right) + \left(\sum \begin{matrix}\text{Áreas acima}\\ \text{do eixo } x\end{matrix}\right)$$



$$\int_a^b f(x)\,dx = \text{área}(S_2 \cup S_4 \cup S_6) - \text{área}(S_1 \cup S_3 \cup S_5)$$

Buscando Primitivas:

$f: I \to \mathbb{R}$ contínua, $H: I \to \mathbb{R}$, $H'(x) = f(x)$
$\forall x \in I$:

$(f(u(x)))' = f'(u(x)) \cdot u'(x) = H''(u(x)) \cdot u'(x)$

$(H(u(x)))' = H'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$

$\therefore$ Primitiva de $f(u) \cdot u'$ é $H(u)$

$\therefore \int_a^b f(u(x)) \cdot u'(x)\,dx = H(u(x))$

Fazendo uma substituição $\frac{du}{dx} = u'(x)$

$du = u'(x)\,dx$.

$\Rightarrow \int_a^b f(u)\,du \quad \{ u = u(x) \wedge du = u'(x)\,dx$

$$\Rightarrow \int f(u)\,du = \boxed{H(u(x)) + C} \ //.$$

Fórmulas Básicas:

- $u = u(x) \Rightarrow \dfrac{d}{dx}u(x) = u'(x) \therefore \boxed{du = u'(x)\,dx} \ //.$

- $\displaystyle\int u^r\,du = \frac{u^{r+1}}{r+1} + C. \quad , \ r \neq -1$

- $\displaystyle\int \frac{du}{u} = \ln|u| + C. \quad , \ u \neq 0$

- $\displaystyle\int \cos u\,du = \operatorname{sen} u + C.$

- $\displaystyle\int \operatorname{sen} u\,du = -\cos u + C$

- $\displaystyle\int \sec^2 u\,du = \operatorname{tg} u + C$

- $\displaystyle\int \sec u \operatorname{tg} u\,du = \sec u + C$

- $\displaystyle\int \frac{1}{1+u^2}\,du = \operatorname{arctg} u + C$

- $\int e^u \, du = e^u + C$

- $\int a^u \, du = \dfrac{a^u}{\ln a} + C$

Ex:

a) Calcule $\int x e^{x^2} dx$ b) Calcule $\int_0^1 x e^{x^2} dx$

a) $\int \dfrac{2x}{2} \cdot e^{x^2} = \int \dfrac{e^u \cdot u'}{2} dx = \dfrac{1}{2} \int e^u du$

$= \dfrac{1}{2} \cdot \left(e^u + C\right) = \boxed{\dfrac{1}{2}\left(e^{x^2} + C\right).}$

b) $\dfrac{1}{2} e^{x^2} + \dfrac{1}{2} c \Big|_0^1 = \dfrac{1}{2} e^1 + \cancel{\dfrac{1}{2} c} - \dfrac{1}{2} e^0 - \cancel{\dfrac{1}{2} c} = \boxed{\dfrac{e-1}{2}}$

c) Calcule $\int \dfrac{x^2}{x^3 - 1} dx = \int \dfrac{3x^2}{3(x^3-1)} dx$

$d(x^3 - 1) = 3x^2$ $u = (x^3 - 1)$

$du = 3x^2 dx$

$\to \dfrac{1}{3} \int \dfrac{du}{u} = \dfrac{1}{3}\left(\ln|u| + c\right) = \boxed{\dfrac{1}{3}\left(\ln|x^3 - 1| + c\right)}$

1) $\int e^{2x}\,dx$

2) $\int \frac{\ln x}{x}\,dx$

3) $\int x\sqrt{2+3x^2}\,dx$

4) $\int_{-\pi/3}^{-\pi/4} \cos^3 x\,dx$

5) $\int_0^{\pi} x \operatorname{sen} x^2\,dx$

Mostre que

$(\ln|x|)' = 1/x,\ x \neq 0.$

1) $\int e^{2x}\,dx = \int \frac{2}{2}\cdot e^{2x}\,dx = \frac{1}{2}\int e^{u}\cdot u'\,dx =$

$= \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + C = \boxed{\frac{1}{2}e^{2x} + C}$

2) $\int \frac{\ln x}{x}\,dx = \int u\cdot u'\,dx = \int u\,du = \frac{u^2}{2} + C$

$\boxed{= \frac{(\ln x)^2}{2} + C}$

3) $\int x\sqrt{2+3x^2}\,dx = \frac{1}{9}\int 9x\sqrt{2+3x^2}\,dx$

$\left((2+3x^2)^{3/2}\right)' = \frac{3}{\cancel{2}}\sqrt{(2+3x^2)}\cdot \cancel{6}^{3}x = 9x\sqrt{2+3x^2}$

$$\boxed{= \frac{1}{9}(2+3x^2)^{3/2} + C}$$

4) $\int_{-\pi/3}^{-\pi/4} \cos^3 x\,dx = \int_{-\pi/3}^{-\pi/4} (1-\operatorname{sen}^2 x)\cos x\,dx$

$= \int_{-\pi/3}^{-\pi/4} (1-u^2).u'\,dx = \int_{-\pi/4}^{-\pi/4} (1-u^2)\,du$

$= \boxed{u - \frac{u^3}{3} + C} = \boxed{\operatorname{sen}x - \frac{\operatorname{sen}^3 x}{3} + C}$

5) $\int_0^{\pi} x\operatorname{sen}x^2\,dx = \int \frac{\operatorname{sen}u}{2}\,du$ $\qquad u = x^2 \Rightarrow du = 2x$

$$= \frac{1}{2}\int \operatorname{sen} u \, du \boxed{= -\frac{1}{2}\cos x^2 + C}.$$

$$\int \cos^2 x \, dx = \int \frac{\cos 2x + 1}{2} \, dx$$

$$= \int \frac{1}{2} \, dx + \int \frac{\cos 2x}{2} \, dx$$

$$= \frac{1}{2}\left(\int dx + \int \frac{\cos 2x}{2} \, dx\right) = \frac{1}{2}\left(x + \frac{\operatorname{sen} 2x}{2}\right) + C$$

$$= \boxed{\frac{2x + \operatorname{sen} 2x}{4} + C}.$$

Integração por Partes

$$\int (uv)'\,dx = \int u.v'\,dx + \int v.u'\,dx = \int u\,dv + \int v\,du$$

$= uv$

$$\therefore \boxed{\int u\,dv + \int v\,du = uv}$$

Ex: $\int x e^x\,dx$ faça $x = u$ e $e^x\,dx = dv$

$\int u\,dv = uv - \int v\,du \Rightarrow v = e^x \therefore$

$$\int u\,dv = \boxed{x e^x - e^x + C}$$

$u.v - \int e^x\,dx$

Se temos $\int f(x)g(x)\,dx$ e temos relação de derivadas, posso aplicar regra da substituição.
Se não tiver, podemos tentar integração por partes.

Ex: $\int \ln x \, dx$ $\begin{cases} \ln x = u \Rightarrow du = \frac{1}{x} \cdot dx \\ dv = dx \Rightarrow v = x. \end{cases}$

$\therefore \int u\,dv = u.v - \int v\,du$

$= \int \ln x\,dx = \ln x . x - \int x \cdot \frac{1}{x}\,dx$

$\boxed{= x\ln x - x + C}$.

$\int e^x \cos x\,dx$

$u = \cos x \Rightarrow du = -\operatorname{sen} x\,dx$

$dv = e^x dx \Rightarrow v = e^x$

$\int e^x \cos x\,dx = e^x \cos x - \int e^x (-\operatorname{sen} x)\,dx$

$= e^x \cos x + \boxed{\int e^x \operatorname{sen} x\,dx} \rightarrow u = \operatorname{sen} x \Rightarrow du = \cos x\,dx$

$dv = e^x dx \Rightarrow v = e^x$

$\Rightarrow e^x \cos x + e^x \operatorname{sen} x - \int e^x \cos x\,dx = \int e^x \cos x\,dx$

$\therefore \int e^x \cos x\,dx \boxed{= \frac{e^x}{2}(\operatorname{sen} x + \cos x) + C}$.

Ex: $\int x^2 \operatorname{sen} x\,dx$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \operatorname{sen} x\,dx \Rightarrow v = -\cos x$

$\Rightarrow -x^2\cos x - \int -\cos x \cdot 2x\,dx = \int x^2 \operatorname{sen} x\,dx$

$\Rightarrow -x^2\cos x + 2\int x\cos x\,dx = \int x^2 \operatorname{sen} x\,dx$

$x = u \Rightarrow du = dx$
$dv = \cos x\,dx \Rightarrow v = \operatorname{sen} x$

$-x^2\cos x + 2\left(x\operatorname{sen} x - \int \operatorname{sen} x\,dx\right)$
$= -x^2\cos x + 2(x\operatorname{sen} x + \cos x)$
$\boxed{= 2(x\operatorname{sen} x + \cos x) - x^2\cos x + C}$

Ex: $\int \operatorname{arctg} x\,dx \Rightarrow du = \frac{1}{1+x^2}dx = u = \operatorname{arctg} x$
$dv = dx \Rightarrow v = x.$

Partes

$I = x\operatorname{arctg} x - \int x \frac{1}{1+x^2}\,dx$

$u = x^2+1$
$du = 2x\,dx$
$x\,dx = du \cdot \frac{1}{2}$

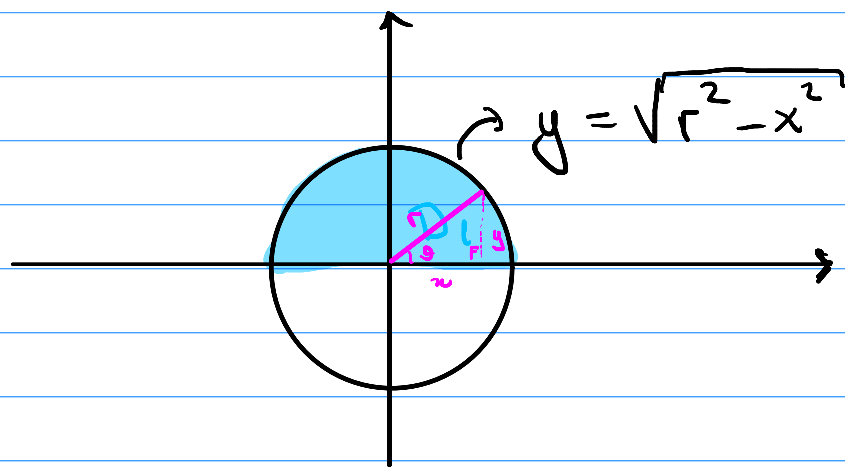
$= x\operatorname{arctg} x - \int \frac{1}{u} \cdot \frac{1}{2}\,du$

Subs.
$u = x^2+1$

$x\operatorname{arctg} x - \frac{1}{2} \cdot \ln|u| = \boxed{x\operatorname{arctg} x - \frac{\ln(x^2+1)}{2} + C}$

Substituição trigonométricas

a) Calcular a área do disco D:
$x^2+y^2 \leq r^2$



$$\text{Área total} = 2 \cdot D_1 = 2 \cdot \int_{-r}^{r} \sqrt{r^2-x^2}\, dx$$

$$= 4 \cdot \int_0^r \sqrt{r^2-x^2}\, dx$$

da $\sqrt{r^2-x^2} \Rightarrow |x| \leq r \Rightarrow \frac{|x|}{r} \leq 1.$

$\boxed{-1 \leq \frac{x}{r} \leq 1}$, mas $-1 \leq \operatorname{sen}\theta \leq 1$

$\frac{x}{r} = \operatorname{sen}\theta$ para algum $\theta \in [-\pi/2, \pi/2]$

$\operatorname{sen}\theta$ é crescente para $-\pi/2 \leq \theta \leq \pi/2$ e
$(\operatorname{sen}\theta)' = \cos\theta > 0$ para $-\pi/2 < \theta < \pi/2$.

Logo $x = r\,\text{sen}\theta$
$dx = r\cos\theta\, d\theta.$

$$E \int \sqrt{r^2-x^2}\,dx = \int \left(\sqrt{r^2-r^2\text{sen}^2\theta}\right)(r\cos\theta)\,d\theta$$

$$= \int r\sqrt{1-\text{sen}^2\theta}\,.\,r\cos\theta\,d\theta = r^2\int \cos\theta\sqrt{\cos^2\theta}\,d\theta$$

$$= r^2\int \cos^2\theta\,d\theta. \quad (-\pi/2 \leq \theta \leq \pi/2) \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{r^2}{2}\int 1+\cos 2\theta\,d\theta = \frac{r^2}{2}\left[\int d\theta + \int \cos 2\theta\,d\theta\right]$$

$$\boxed{= \frac{r^2}{2}\cdot\left[\theta + \frac{\text{sen}2\theta}{2}\right]. + C}$$

$$\frac{r^2}{2}\left[\theta + \text{sen}\theta\cos\theta\right] \Rightarrow \text{sen}\theta = \frac{x}{r}\ ; \quad \cos\theta = \frac{\sqrt{r^2-x^2}}{r}$$

$$\boxed{\therefore\ \frac{r^2}{2}\left[\text{arcsen}(x/r).+ \frac{x\sqrt{r^2-x^2}}{r^2}\right]+C}$$

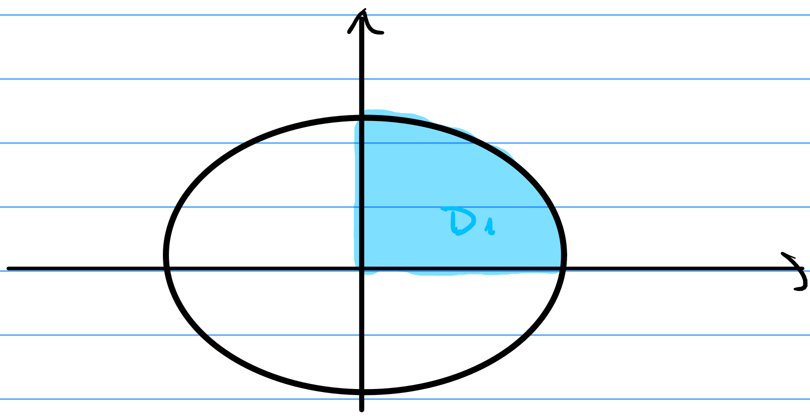
$$\Rightarrow \int_0^r \sqrt{r^2-x^2}\,dx = \frac{r^2}{2}\left[\text{Arcsen}1 + \frac{1}{r}.0\right] - \frac{r^2}{2}\left[\text{arcsen}0 + 0\right]$$

$$= \frac{r^2}{2}\left[\text{arcsen}1 - \text{arcsen}0\right]. = \frac{r^2}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi r^2}{4}}$$

$\therefore$ Área total $= 4 \cdot \frac{\pi r^2}{4} = \boxed{\pi r^2}$

. Determine a área limitada por

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Área total $= 4 \cdot D_1$.

Exercício: $\int \frac{dx}{(x^2+9)^{3/2}}$; $\int (\sec\theta - tg\theta)\, d\theta$.

Decomposição por frações parciais

Ex: $\int \frac{x^2+x-2}{x+1}\,dx = \int \frac{x(x+1)-2}{x+1}\,dx$

$= \int x - \frac{2}{x+1}\,dx = \int x\,dx - 2\int \frac{1}{x+1}\,dx$

$\boxed{= \frac{x^2}{2} - 2\ln|x+1| + C}$.

Ex: $\int \frac{x^2+2x+5}{x^3-x^2}\,dx = \int \frac{ax+b}{x^2} + \frac{c}{x-1}\,dx$

$= \int \frac{ax}{x^2} + \frac{b}{x^2} + \frac{c}{x-1}\,dx = \int\left(\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}\right)dx$

$= a\int \frac{1}{x}\,dx + b\int \frac{1}{x^2}\,dx + c\int \frac{1}{x-1}\,dx$

$\boxed{= a\ln|x| - b\cdot\frac{1}{x} + c\ln|x-1| + K}$

$\boxed{\Rightarrow -7\ln|x| + 5\cdot\frac{1}{x} + 8\ln|x-1| + K}$

$$\frac{ax+b}{x^2} + \frac{c}{x-1} \Rightarrow (ax+b)(x-1) + cx^2$$

$$\Rightarrow x^2(a+c) + x(b-a) - b.$$

$$\begin{cases} a+c = 1 \\ b-a = 2 \\ -b = 5 \end{cases}$$

$$\boxed{\begin{array}{l} b = -5. \\ a = -7 \\ c = 8 \end{array}}$$

Igualdade de polinômios.

Ex 3: $\int \frac{x}{x^2+x+1}\,dx$

Ex 4: $\int \frac{1}{x^2-1}\,dx = \int \frac{1}{x+1} \cdot \frac{1}{x-1}\,dx = \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x+1}\,dx$

$$\boxed{= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C}$$

Ex 5: $\int \frac{x+4}{x^3+4x}\,dx = \int \frac{x+4}{x(x^2+4)}\,dx$

$=$