FGV EMAp

Fundamentos de Matemática

Lista 3 – 04/04/2024

A consulta é livre, mas você deve entregar suas soluções escritas de próprio punho e mencionar as fontes de consulta.

63. Em uma progressão aritmética de 17 termos, o sétimo termo é igual a 13 e o décimo primeiro termo é igual a 27. A soma dos termos dessa progressão é igual a

64. A soma $\sum_{k=1}^{2019} \frac{1}{k(k+1)}$ é igual a

65. A soma $\sum_{k=1}^{100} k(k-2)$ é igual a

66. A sequência (a_k) é tal que

$$\sum_{k=1}^{n} a_k = (n^2+n+1)3^n + c,$$

para todo inteiro positivo n, sendo c uma constante desconhecida. Então a_k é igual a

67. O somatório $\sum_{j=1}^{18}\sum_{k=1}^{j}(jk-k)$ é igual a

68. O somatório

$$\sum_{i=0}^{19}\sum_{j=2}^{16}\sum_{k=5}^{24} j$$

é igual a

69. Calcule as seguintes somas:

(a) $1+2+3+\cdots+n$
(b) $1^2+2^2+3^2+\cdots+n^2$
(c) $1^3+2^3+3^3+\cdots+n^3$
(d) $1+3+5+\cdots+2n-1$
(e) $1+x+x^2+x^3+\cdots+x^{n-1}$ (se x é diferente de 1)
(f) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}$
(g) $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}$
(h) $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2$
(i) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3n-2)(3n+1)}$
(j) $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\cdots+\frac{n^2}{(2n-1)(2n+1)}$

70. Calcule as seguintes somas:

(a) $\sum_{k\geq 0}\binom{n}{k}$
(b) $\sum_{k\geq 0}\binom{n}{k}2^k$
(c) $\sum_{k\geq 0}\binom{n}{k}(-1)^k$
(d) $\sum_{k\geq 0}\binom{n}{2k}$, Professor e mano Bruno
(e) $\sum_{k\geq 0}\binom{n}{2k}2^k$ Professor
(f) $\sum_{k\geq 0}\binom{n}{2k}(-1)^k$
(g) $\sum_{k\geq 0}\binom{n}{k}^2$ Mano Everten deu uma ajuda

71. Observe o padrão dos números dispostos nos quadriculados 3×3 e 4×4 a seguir. Seguindo o mesmo padrão, qual será a soma dos números no quadriculado 10×10? E em um quadrado $n \times n$?

1	2	3
1	2	2
1	1	1

1	2	3	4
1	2	3	3
1	2	2	2
1	1	1	1

72. Evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right] = \left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^k}{2^{k+1}}\right] + \cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)

73. Prove that the number $\sum_{k=0}^{n}\binom{2n+1}{2k+1}2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

74. Let m and n be positive integers such that Vi a ideia e consegui fazer

$$\frac{m}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that m is divisible by 1979.

75. Calcule

$$\sum_{k=1}^{n}\frac{k}{k^4+k^2+1}.$$

03 $PA(a_1, a_2, \dots, a_{17})$ Por definição: $a_n = a_1 + (n-1) \cdot r$ //

$\therefore a_7 = a_1 + (7-1) \cdot r = a_1 + 6r = 13$

$a_{11} = a_1 + (11-1) \cdot r = a_1 + 10r = 27$

$\begin{cases} 13 = a_1 + 6r \\ 27 = a_1 + 10r \end{cases}$ $4r = 14$ $\boxed{r = 7/2}$ // $\boxed{a_1 = -8}$ //

Chame de S a soma de todos os termos da PA

$\therefore S = a_1 + a_2 + \dots + a_{17}$

$S = a_{17} + a_{16} + \dots + a_1$

$2S = \underbrace{(a_1 + a_{17})}_{2a_1+16r} + \underbrace{(a_2 + a_{16})}_{2a_1+16r} + \dots + \underbrace{(a_8 + a_{10})}_{2a_1+16r} + 2a_9$

$2S = (2a_1 + 16r) \cdot 17$

$S = \dfrac{(2a_1 + 16r) \cdot 17}{2} \rightarrow \dfrac{(a_1 + a_{17}) \cdot 17}{2}$

• Geral: $\boxed{S_n = \dfrac{(a_1 + a_n) \cdot n}{2}}$ //

$S = \dfrac{(2 \cdot (-8) + \overset{8}{\cancel{16}} \cdot \frac{7}{\cancel{2}}) \cdot 17}{2}$

$S = \dfrac{(8 \cdot 7 - 2 \cdot 8) \cdot 17}{2} = \dfrac{5 \cdot \overset{4}{\cancel{8}} \cdot 17}{\cancel{2}} = \boxed{340}$

04 $\displaystyle\sum_{k=1}^{2019} \frac{1}{k(k+1)} = \sum_{k=1}^{2019} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{2019} \frac{1}{k} - \sum_{k=1}^{2019} \frac{1}{k+1}$

$= \left(\frac{1}{1} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{2019}}\right) - \left(\cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \dots + \frac{1}{2020}\right) = 1 - \frac{1}{2020} = \boxed{\frac{2019}{2020}}$

(65) $\sum_{k=1}^{100} k(k-2) = \sum_{k=1}^{100} k^2 - 2\cdot\sum_{k=1}^{100} k$. Já sabemos da lista 2

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ e $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Logo, $\sum_{k=1}^{100} k(k-2) = \frac{100\cdot(100+1)(2\cdot 100+1)}{6} - \frac{2\cdot 100\cdot(100+1)}{2}$.

$= \frac{100\cdot 101}{2}\left(\frac{201}{3} - 2\right) = 50\cdot 101\cdot 65 = \boxed{328{,}250}$

(66) $\sum_{k=1}^{n} a_k = (n^2+n+1)\cdot 3^n + c$

Sabemos que $\sum_{k=1}^{n} a_k - \sum_{k=}^{n-1} a_k = a_k$,

$\therefore a_k = (n^2+n+1)\cdot 3^n + \cancel{c} - ((n-1)^2+(n-1)+1)\cdot 3^{n-1} - \cancel{c}$

$a_k = (n^2+n+1)\cdot 3\cdot 3^{n-1} - (n^2-2n+1+n-\cancel{1}+\cancel{1})\cdot 3^{n-1}$

$a_k = 3^{n-1}\left(3n^2+3n+3-n^2+n-1\right) = 3^{n-1}\cdot(2n^2+4n+2)$

$= \boxed{3^{n-1}\cdot 2(n+1)^2}$

(67) $\displaystyle\sum_{j=1}^{18}\sum_{k=1}^{j}(jk-k)$

- $\displaystyle\sum_{k=1}^{j}(jk-k) = j\sum_{k=1}^{j}k - \sum_{k=1}^{j}k = j\,\frac{(j)(j+1)}{2} - \frac{j(j+1)}{2} = \frac{j(j+1)}{2}(j-1)$

$\displaystyle = \frac{(j-1)j(j+1)}{2} = \frac{(j^2-1)j}{2} = \frac{j^3-j}{2}$

$\displaystyle\sum_{j=1}^{18}\sum_{k=1}^{j}(jk-k) = \sum_{j=1}^{18}\frac{j^3-j}{2} = \frac{1}{2}\left(\sum_{j=1}^{18}j^3 - \sum_{j=1}^{18}j\right).$

- $\displaystyle\sum_{k=1}^{n}k^3 = \left(\frac{n(n+1)}{2}\right)^2$ (será demonstrado no item 69 c))

- $\displaystyle\sum_{j=1}^{18}\sum_{k=1}^{j}(jk-k) = \frac{1}{2}\left(\left(\frac{18\cdot 19}{2}\right)^2 - \frac{18\cdot 19}{2}\right) = \frac{1}{2}\left(29241 - 171\right)$

$\displaystyle = \frac{29070}{2} = \boxed{14535}$

(68) $\displaystyle\sum_{i=0}^{19}\sum_{j=2}^{16}\sum_{k=5}^{24} j$

- $\displaystyle\sum_{k=5}^{24} j = 20j$

- $\displaystyle\sum_{j=2}^{16}\sum_{k=5}^{24} j = \sum_{j=2}^{16} 20j$

$\displaystyle = \left(\sum_{j=1}^{16} 20j\right) - 20\cdot 1 = 20\cdot\sum_{j=1}^{16} j - 20 = 20\cdot\frac{16\cdot 17}{2} - 20 = \underline{2700}$

- $\displaystyle\sum_{i=0}^{19}\sum_{j=2}^{16}\sum_{k=5}^{24} j = \sum_{i=0}^{19}\sum_{j=2}^{16} 20 = \sum_{i=0}^{19} 2700 = 20\cdot 2700 = \boxed{54000}$

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(69) a) $1+2+3+\dots+n = \boxed{\sum_{k=1}^{n} k}$

$$\sum_{k=1}^{n}(k+1)^2 = \cancel{\sum_{k=1}^{n} k^2} + 2\cdot\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \cancel{\sum_{k=1}^{n} k^2} + (n+1)^2 - 1$$

$$2\sum_{k=1}^{n} k = n^2 + 2n + \cancel{1} \cancel{-1} - \cancel{n} = n^2 + n$$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

b) $1^2+2^2+3^2+\dots+n^2 = \sum_{k=1}^{n} k^2$

$$\sum_{k=1}^{n}(k+1)^3 = \cancel{\sum_{k=1}^{n} k^3} + 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \cancel{\sum_{k=1}^{n} k^3} + (n+1)^3 - 1.$$

$$3\sum_{k=1}^{n} k^2 + 3\cdot\frac{n(n+1)}{2} + \cancel{n} = n^3 + 3n^2 + 3n$$

$$3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 6n^2 + 4n}{2} - \frac{3n^2 - 3n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

c) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \dots + k^3$

$$\sum_{k=1}^{n} (k+1)^4 = \cancel{\sum_{k=1}^{n} k^4} + 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \cancel{\sum_{k=1}^{n} k^4} + (n+1)^4 - 1.$$

$$4\sum_{k=1}^{n} k^3 + \cancel{6} \cdot \frac{\overbrace{n(n+1)(2n+1)}^{(n^2+n)(2n+1)}}{\cancel{6}} + \frac{\cancel{4}^{2} n(n+1)}{\cancel{2}} + n = k^4 + 4k^3 + 6k^2 + 4k + \cancel{1} - \cancel{1}$$

$$4 \cdot \sum_{k=1}^{n} k^3 + \cancel{2n^3} + n^2 + 2n^2 + \cancel{n} + 2n^2 + \cancel{2n} + \cancel{n} = n^4 + \cancel{4n^3} + \cancel{6n^2} + \cancel{4n}$$

$$4\sum_{k=1}^{n} k^3 = n^4 + 2n^3 + n^2 \Rightarrow \boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

d) $\sum_{k=1}^{n} 2k = I + P$ (soma dos pares + soma dos ímpares).

$$I = \sum_{k=1}^{n} 2k - P. \Rightarrow I = \frac{(2n)(2n+1)}{2} - \frac{2n(n+1)}{2}$$

$$I = 2n^2 + \cancel{n} - n^2 - \cancel{n} \boxed{= n^2}$$

e) $1 + x + x^2 + \dots + x^{n-1} = S$

$x + x^2 + x^3 + \dots + x^n = xS$

$xS - S = x^n - 1.$

$$\boxed{S = \frac{x^n - 1}{x - 1}} \ (x \neq 1)$$

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f) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$

• OBS: $\frac{1}{2k-1} - \frac{1}{2k+1} = \frac{2k+1-2k+1}{(2k-1)(2k+1)} = 2 \cdot \frac{1}{(2k-1)(2k+1)} \therefore \frac{1}{(2k-1)(2k+1)} =$

$\frac{1}{2} \cdot \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) \Rightarrow \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k+1}\right)$

$= \frac{1}{2}\left(\left(\frac{1}{1} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{5}} + \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{2n-1}}\right) - \left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{5}} + \cancel{\frac{1}{7}} + \dots + \frac{1}{2n+1}\right)\right)$

$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}\left(\frac{2n+1-1}{2n+1}\right) = \boxed{\frac{n}{2n+1}}$

g) Vamos usar frações parciais

$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{(n+1)} + \frac{C}{(n+2)} = \frac{A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)}{n(n+1)(n+2)}$

$= \frac{A(n^2+3n+2) + B(n^2+2n) + C(n^2+n)}{n(n+1)(n+2)} = \frac{n^2(A+B+C) + n(3A+2B+C) + 2A}{n(n+1)(n+2)}$

Daí, podemos montar um sistema: $\begin{cases} A+B+C=0 \\ 3A+2B+C=0 \\ 2A=1 \end{cases}$ Resolvendo

o sistema temos: $\boxed{(A,B,C) = \left(\frac{1}{2}, -1, \frac{1}{2}\right)}$

Logo, $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right)$

$= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+2}$

$= \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{k+1} - \sum_{k=1}^{n} \frac{1}{k+1} + \frac{1}{2}\sum_{k=2}^{n+1} \frac{1}{k+1}$

$= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right) + \frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k+1} - \left(\frac{1}{2} + \frac{1}{n+1} + \sum_{k=2}^{n-1} \frac{1}{k+1}\right) + \frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{n+1} + \sum_{k=2}^{n-1} \frac{1}{k+1}\right)$

$= \frac{1}{2}\cdot\frac{3}{2} + \cancel{\frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k+1}} - \frac{1}{2} - \frac{1}{n+1} - \cancel{\sum_{k=2}^{n-1} \frac{1}{k+1}} + \frac{1}{2(n+2)} + \frac{1}{2(n+1)} + \cancel{\frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k+1}}$

$= \frac{1}{4} - \frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{2(n+1)}$

$= \frac{(n+2)(n+1) + 2(n+1) - 2(n+2)}{4(n+1)(n+2)} = \frac{n^2+3n+\cancel{2}+\cancel{2n}+2-\cancel{2n}-\cancel{4}}{4(n+1)(n+2)}$

$= \frac{n(n+3)}{4(n+1)(n+2)}$

h) Para n par:

$S = 1^2 - 2^2 + 3^2 - 4^2 + \dots - n^2$

$-S = (n + n-1)(n-n+1) + \dots + (4+3).(4-3) + (2+1)(2-1)$

$-S = 2n-1 + \dots + 7 + 3$ $\quad (n=2K)$

$-S = 4K-1 + \dots + 7 + 3$

$-S = 4(1+2+3+\dots+K) - K$

$-S = 2K(K+1) - K$

$S = K - 2K(K+1)$

$S = -K(2K+1)$

$$\boxed{S = -\frac{n}{2}(n+1)}$$

Para n ímpar: O raciocínio é análogo, mas agora, o sinal se inverte $S = \frac{n}{2}(n+1)$

Para n: $\boxed{S = (-1)^{n-1} \frac{n}{2}(n+1)}$

i) $\sum_{K=1}^{n} \frac{1}{(3K-2)(3K+1)} = \frac{1}{3}\sum_{K=1}^{n}\left(\frac{1}{3K-2} - \frac{1}{3K+1}\right) = \frac{1}{3}\left(\sum_{K=1}^{n}\frac{1}{3K-2} - \sum_{K=1}^{n}\frac{1}{3K+1}\right)$

$= \frac{1}{3}\left(\frac{1}{1} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{3n-2}} - \left(\cancel{\frac{1}{4}} + \cancel{\frac{1}{7}} + \dots + \frac{1}{3n+1}\right)\right)$

$= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) = \boxed{\frac{n}{3n+1}}$

Logo, $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right)$

$$= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+2}$$

$$= \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{k+1} - \sum_{k=1}^{n} \frac{1}{k+1} + \frac{1}{2}\sum_{k=2}^{n+1} \frac{1}{k+1}$$

$$= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right) + \frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k+1} - \left(\frac{1}{2} + \frac{1}{n+1} + \sum_{k=2}^{n-1} \frac{1}{k+1}\right) + \frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{n+1} + \sum_{k=2}^{n-1} \frac{1}{k+1}\right)$$

$$= \frac{1}{2}\cdot\frac{3}{2} + \cancel{\frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k+1}} - \frac{1}{2} - \frac{1}{n+1} - \cancel{\sum_{k=2}^{n-1} \frac{1}{k+1}} + \frac{1}{2(n+2)} + \frac{1}{2(n+1)} + \cancel{\frac{1}{2}\sum_{k=2}^{n-1} \frac{1}{k+1}}$$

$$= \frac{1}{4} - \frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{2(n+1)}$$

$$= \frac{(n+2)(n+1) + 2(n+1) - 2(n+2)}{4(n+1)(n+2)} = \frac{n^2+3n+2+2n+2-2n-4}{4(n+1)(n+2)}$$

$$= \boxed{\frac{n(n+3)}{4(n+1)(n+2)}}$$

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j) $\sum_{K=1}^{n} \frac{K^2}{(2K-1)(2K+1)}$ Vamos fazer frações parciais:

$$\frac{KA}{2K-1} + \frac{KB}{2K+1} = \frac{K^2}{(2K-1)(2K+1)} \Rightarrow \frac{(2K+1)KA + (2K-1)KB}{(2K-1)(2K+1)}$$

$$\frac{(2K^2+K)A + 2K^2B - KB}{(2K-1)(2K+1)} = \frac{K^2(2A+2B) + K(A-B)}{(2K-1)(2K+1)}$$

$$\begin{cases} 2A+2B=1 & A+B=\frac{1}{2} \\ A-B=0 & A-B=0 \end{cases} \quad \boxed{A=\frac{1}{4}=B}$$

$$\therefore \sum_{K=1}^{n} \frac{K^2}{(2K-1)(2K+1)} = \frac{1}{4}\left(\sum_{K=1}^{n} \frac{K}{2K-1} + \sum_{K=1}^{n} \frac{K}{2K+1}\right)$$

$$\frac{1}{4}\left(\sum_{K=0}^{n-1} \frac{K+1}{2K+1} + \sum_{K=1}^{n} \frac{K}{2K+1}\right) = \frac{1}{4}\left(\frac{1}{1} + \frac{n}{2n+1} + \sum_{K=1}^{n-1} \frac{K+1}{2K+1} + \sum_{K=1}^{n-1} \frac{K}{2K+1}\right)$$

$$= \frac{1}{4}\left(1 + \frac{n}{2n+1} + \sum_{K=1}^{n-1} \frac{2K+1}{2K+1}\right) = \frac{1}{4}\left(1 + \frac{n}{2n+1} + (n-1)\right)$$

$$= \frac{1}{4}\left(\frac{2n+1+n+(2n+1)(n-1)}{2n+1}\right) = \frac{1}{4}\left(\frac{\cancel{2n}+\cancel{1}+n+2n^2-\cancel{2n}+\cancel{n}-\cancel{1}}{2n+1}\right)$$

$$= \frac{2n(n+1)}{4(2n+1)} = \boxed{\frac{n(n+1)}{2(2n+1)}}$$

70 a) $\sum_{k\geq 0}\binom{n}{k}$ sabemos da lista 2 que:

$$\boxed{(a+b)^n = \sum_{k\geq 0}\binom{n}{k}a^{n-k}b^k}$$

se fizermos $a=b=1$:

$$(1+1)^n = \sum_{k\geq 0}\binom{n}{k} = \boxed{2^n} \quad \text{(teorema das linhas)}.$$

b) $$(1+2)^n = \sum_{k\geq 0}\binom{n}{k}\cdot 1^{n-k}\cdot 2^k = \boxed{3^n}.$$

c) $$(1+(-1))^n = \sum_{k\geq 0}\binom{n}{k}\cdot 1^{n-k}(-1)^k = \boxed{\sum_{k\geq 0}\binom{n}{k}(-1)^k = 0}$$

Para $n\geq 1$

Para $n=0$ $\sum_{k\geq 0}\binom{0}{k}(-1)^k = \binom{0}{0}\cdot(-1)^0 = 1$. (Lembre-se, quando $k>n$, $\binom{n}{k}=0$ por definição)

d) $$\sum_{k\geq 0}\binom{n}{2k} = \sum_{k\geq 0}\left[\binom{n-1}{2k}+\binom{n-1}{2k-1}\right] = \sum_{k\geq 0}\binom{n-1}{2k} + \sum_{k\geq 0}\binom{n-1}{2k-1}$$

$$= \sum_{k\geq 0}\binom{n-1}{k} = 2^{n-1}.$$

Outra solução: sabemos que $\sum_{k\geq 0}\binom{n}{k} = 2^n$ e $\sum_{k\geq 0}\binom{n}{k}(-1)^k = 0$.

Logo $\sum_{k\geq 0}\binom{n}{2k} + \sum_{k\geq 0}\binom{n}{2k-1} = 2^n$ e $\sum_{k\geq 0}\binom{n}{2k} - \sum_{k\geq 0}\binom{n}{2k-1} = 0$

Por fim $\sum_{k\geq 0}\binom{n}{2k} = \frac{2^n}{2} = 2^{n-1}$.

e) $\sum_{k\geq 0}\binom{n}{2k}\cdot 2^k = \sum_{k\geq 0}\binom{n}{2k}(\sqrt{2})^{2k} = A$. Faça $B = \sum_{k\geq 0}\binom{n}{2k+1}(\sqrt{2})^{2k+1}$

Sabemos que $A+B = \sum\binom{n}{k}\cdot(\sqrt{2})^k = (1+\sqrt{2})^n$ e $A-B$ é

$A-B = \sum_{k\geq 0}\binom{n}{k}\cdot(-\sqrt{2})^k = (1-\sqrt{2})^n$. Portanto, $A = \dfrac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}$

f) $\sum_{k\geq 0}\binom{n}{2k}(-1)^k = \sum_{k\geq 0}\binom{n}{2k} i^{2k}$ $(i=\sqrt{-1}$ e $i^2=-1)$

$\sum_{k\geq 0}\binom{n}{k} i^k = (1+i)^n = \binom{n}{0}i^0+\binom{n}{1}i^1+\binom{n}{2}i^2+\dots+\binom{n}{n}i^n$

$\sum\binom{n}{k}(-i)^k = (1-i)^n = \binom{n}{0}i^0-\binom{n}{1}i^1+\binom{n}{2}i^2-\dots+\binom{n}{n}(-i)^n$

Para n par: $(1+i)^n = \binom{n}{0}i^0+\binom{n}{1}i^1+\binom{n}{2}i^2+\dots+\binom{n}{n}i^n$

$(1-i)^n = \binom{n}{0}i^0-\binom{n}{1}i^1+\binom{n}{2}i^2-\dots+\binom{n}{n}i^n$

n par $\boxed{\dfrac{(1+i)^n+(1-i)^n}{2} = \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\dots+\binom{n}{2k} = \sum\binom{n}{2k}(-1)^k}$

Para n ímpar: $(1+i)^n = \binom{n}{0}i^0+\binom{n}{1}i^1+\binom{n}{2}i^2+\dots+\binom{n}{n}i^n$

$(1-i)^n = \binom{n}{0}i^0-\binom{n}{1}i^1+\binom{n}{2}i^2-\dots-\binom{n}{n}i^n$

$\boxed{\dfrac{(1+i)^n+(1-i)^n}{2} = \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\dots+\binom{n}{n-1} = \sum\binom{n}{2k}(-1}$

9) $\sum_{k\geq 0} \binom{n}{k}^2$. Tome um conjunto de 2n pessoas.

Para escolhermos n pessoas deste conjunto, temos $\binom{2n}{n}$ formas de fazer. Agora escolha k pessoas dos n.

Agora, sabemos que $\binom{2n}{n} = \sum_{k\geq 0} \binom{n}{k}\cdot\binom{n}{n-k} = \sum_{k\geq 0}\binom{n}{k}^2$

$$\boxed{\therefore \sum_{k\geq 0}\binom{n}{k}^2 = \binom{2n}{n}}$$

Bizurado do mano Everten.

$\sum_{k\geq 0}\binom{n}{k}^2 = \sum_{k\geq 0}\left[\binom{n}{k}\binom{n}{n-k}\right]$ Pense em um produto de

polinômios: $(1+x)^n\cdot(1+x)^n = \left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n}x^n\right]\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{n}x^n\right]$

$(1+x)^{2n} = \left\{\left[\binom{n}{0}+\binom{n}{1}\right]+\left[\binom{n}{0}\binom{n}{1}+\binom{n}{1}\binom{n}{0}\right]x+\dots+\left[\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\dots+\binom{n}{n}\binom{n}{0}\right]x^n\right\}$

Percebe que o coeficiente do x^n é exatamente $\sum_{k\geq 0}\left[\binom{n}{k}\binom{n}{n-k}\right]$.

Portanto, fazendo: $(1+x)^{2n} = \sum_{k\geq 0}\binom{2n}{k}\cdot x^k$. Quando $k=n$, temos que o coeficiente de x^n é $\binom{2n}{n}$.

método do Luciano

71) Vamos observar um padrão:

Para $n=1 \Rightarrow 1$

Para $n=2 \Rightarrow 1 \cdot 3 + 2 \cdot 1$

Para $n=3 \Rightarrow 1 \cdot 5 + 2 \cdot 3 + 3 \cdot 1$

$\vdots$

Para $n=10 \Rightarrow 10 \cdot 1 + 9 \cdot 3 + 8 \cdot 5 + 7 \cdot 7 + 6 \cdot 9 + 5 \cdot 11 + 4 \cdot 13 + 3 \cdot 15 + 2 \cdot 17 + 1 \cdot 19.$

Chamando de $a_n = 2n-1$, percebemos que para n:

$\boxed{1 \cdot a_n + 2 \cdot a_{n-1} + \dots + n \cdot a_1}$ (soma de um produto de duas P.A.s)

Podemos escrever tal soma deste modo: $\sum_{k=1}^{n} (k \cdot a_{n-k+1})$

$$= \sum_{k=1}^{n} [k \cdot (2n-2k+1)] = \sum_{k=1}^{n} (2kn - 2k^2 + k) = 2n \sum_{k=1}^{n} k - 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \frac{2n \cdot n(n+1)}{2} - \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{6n^2(n+1) - 2n(n+1)(2n+1) + 3n(n}{6}$$

$$= \frac{n(n+1)(6n - 2(2n+1) + 3)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

Logo a soma para $n=10$: $\frac{10 \cdot 11 \cdot 21}{6} \boxed{= 385}$

E para n: $\boxed{\frac{n(n+1)(2n+1)}{6}}$ (Perceba que é a mesma fórmula para $\sum_{k=1}^{n} k^2$)

(72) Sabemos que, a partir de um dado momento, $2^{k+1} > n+2^k$ $\forall n \in \mathbb{Z}$. Portanto, $\left\lfloor \frac{n+2^{k'}}{2^{k'+1}} \right\rfloor$ passará a ser 0 $\forall k' \geq k$.

Vamos provar por indução:

$n=1$: $$\sum_{k=0}^{\infty} \left\lfloor \frac{1+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{1+1}{2} \right\rfloor + \left\lfloor \frac{1+2}{4} \right\rfloor + \left\lfloor \frac{1+4}{8} \right\rfloor + \dots = 1+0+0+\dots = 1.$$

$n=2$: $$\sum_{k=0}^{\infty} \left\lfloor \frac{2+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{2+1}{2} \right\rfloor + \left\lfloor \frac{2+2}{4} \right\rfloor + \left\lfloor \frac{2+4}{8} \right\rfloor + \dots = 1+1+0+\dots = 2.$$

Suponha que $\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = n$ seja verdade.

Queremos ver se $\sum_{k=0}^{\infty} \left\lfloor \frac{(n+1)+2^k}{2^{k+1}} \right\rfloor = n+1$.

$$\left\lfloor \frac{n+1+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{n}{2^k \cdot 2} + \frac{1}{2} + \frac{1}{2^k \cdot 2} \right\rfloor = \left\lfloor \frac{1}{2}\left(\frac{n+1}{2^k} + 1 \right) \right\rfloor$$

Suponha também que, dado um α, $\forall k > \alpha$ $\left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = 0$. Logo $\left\lfloor \frac{n+2^\alpha}{2^{\alpha+1}} \right\rfloor = 1$ e $\left\lfloor \frac{n+2^{\alpha+1}}{2^{\alpha+2}} \right\rfloor = 0$, se n par, então $\left\lfloor \frac{n+1}{2} \right\rfloor = p$ e $\left\lfloor \frac{n+1+1}{2} \right\rfloor = p+1$ e $2^{\alpha+1} = n+2^\alpha$ e $2^{\alpha+2} > n+2^{\alpha+1}$. Por isso, $2^{\alpha+1} < n+2^\alpha+1$. Portanto $\sum_{k=0}^{\infty} \left\lfloor \frac{n+1+2^k}{2^{k+1}} \right\rfloor = n+1$. O raciocínio é

72) Vamos usar uma propriedade forte:

$\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$.

Faço $x = a + b$ ($a \in \mathbb{Z}$, $b \geq 0$, $b \in \mathbb{R} - \mathbb{Z}^*$). Logo $\lfloor x \rfloor = a$; $\lfloor x + \frac{1}{2} \rfloor = \lfloor a + b + \frac{1}{2} \rfloor$
$= a + \lfloor b \rfloor$; $\lfloor 2x \rfloor = \lfloor 2a + 2b \rfloor = 2a + \lfloor b \rfloor$.

$\therefore \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$.

Analisando o somatório $\sum_{k=0}^{\infty} \left\lfloor \frac{n + 2^k}{2^{k+1}} \right\rfloor = \sum_{k=0}^{\infty} \left\lfloor \frac{n}{2^{k+1}} + \frac{1}{2} \right\rfloor$

$$= \sum_{k=0}^{\infty} \left[\left\lfloor \frac{n}{2^k} \right\rfloor - \left\lfloor \frac{n}{2^{k+1}} \right\rfloor \right] = \sum_{k=0}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor - \sum_{k=0}^{\infty} \left\lfloor \frac{n}{2^{k+1}} \right\rfloor$$

$$= \lfloor n \rfloor + \cancel{\left\lfloor \frac{n}{2} \right\rfloor} + \cancel{\left\lfloor \frac{n}{4} \right\rfloor} + \cancel{\ldots} + \cancel{\left\lfloor \frac{n}{2^k} \right\rfloor} + \cancel{\ldots} - \cancel{\left\lfloor \frac{n}{2} \right\rfloor} - \cancel{\left\lfloor \frac{n}{4} \right\rfloor} - \cancel{\left\lfloor \frac{n}{8} \right\rfloor} - \cancel{\ldots} - \cancel{\left\lfloor \frac{n}{2^{k+1}} \right\rfloor} - \cancel{\ldots}$$

$= \lfloor n \rfloor$. Como $n \in \mathbb{Z}$, $\lfloor n \rfloor = n$

$$\therefore \sum_{k=0}^{\infty} \left\lfloor \frac{n + 2^k}{2^{k+1}} \right\rfloor = n.$$

(73) $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} \frac{(\sqrt{8})^{2k+1}}{(\sqrt{8})} = \frac{1}{\sqrt{8}} \sum \binom{2n+1}{2k+1} (\sqrt{8})^{2k+1}$

Analisando $\sum_{k=0} \binom{2n+1}{k} (\sqrt{8})^k = (1+\sqrt{8})^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1}\sqrt{8} + \binom{2n+1}{2} 8 + \dots + \binom{2n+1}{2n+1}\sqrt{8}^{2n+1}$

$\sum_{k=0} \binom{2n+1}{k} (-\sqrt{8})^k = (1-\sqrt{8})^{2n+1} = \binom{2n+1}{0} - \binom{2n+1}{1}\sqrt{8} + \binom{2n+1}{2} 8 + \dots - \binom{2n+1}{2n+1}\sqrt{8}^{2n+1}$

Logo, $\frac{(1+\sqrt{8})^{2n+1} - (1-\sqrt{8})^{2n+1}}{2\sqrt{8}} = \frac{1}{\sqrt{8}} \sum \binom{2n+1}{2k+1} 2^{3k}$

Portanto, devemos provar que $5 \nmid \frac{(1+\sqrt{8})^{2n+1} - (1-\sqrt{8})^{2n+1}}{2\sqrt{8}}$

Vamos por absurdo e por indução

Faça $\frac{(1+\sqrt{8})^{2n+1} - (1-\sqrt{8})^{2n+1}}{2\sqrt{8}} = 5\alpha \ (\alpha \in \mathbb{Z}^*)$

$\therefore (1+\sqrt{8})^{2n+1} = 10\sqrt{8}\alpha + (1-\sqrt{8})^{2n+1}$.

Portanto, $\frac{(1+\sqrt{8})^{2n+3} - (1-\sqrt{8})^{2n+3}}{2\sqrt{8}} = 5\beta \ (\beta \in \mathbb{Z}^*)$

Faça $\varepsilon = (1-\sqrt{8})^{2n+1}$, temos que $\frac{(9+2\sqrt{8})[10\sqrt{8}\alpha + \varepsilon] - \varepsilon(9-2\sqrt{8})}{2\sqrt{8}} = 5\beta$

Simplificando: $5\alpha(9+2\sqrt{8}) + 2\varepsilon = 5\beta$. Substituindo os valores de 5α e ε, temos que:

$$5 \cdot 9\alpha + 10\alpha\sqrt{8} + 2(1-\sqrt{8})^{2n+1} = 5\beta$$

$$5 \cdot 9\alpha + \underbrace{(1+\sqrt{8})^{2n+1} + (1-\sqrt{8})^{2n+1}} = 5\beta.$$

Agora, se provarmos que $(1+\sqrt{8})^{2n+1} + (1-\sqrt{8})^{2n+1}$ é irracional, finalizaremos por aqui, pois β não seria inteiro e, consequentemente

$$5 \nmid \frac{(1+\sqrt{8})^{2n+1} - (1-\sqrt{8})^{2n+1}}{2\sqrt{8}}$$

Provemos por absurdo.

$$(1+\sqrt{8})^{2n+1} + (1-\sqrt{8})^{2n+1} = \frac{p}{q} \quad (p,q,p',q' \in \mathbb{Z} \text{ e } q,q' \neq 0)$$

$$(1+\sqrt{8})^{2n+3} + (1-\sqrt{8})^{2n+3} = \frac{p'}{q'}$$

$$(9+2\sqrt{8})\left(\frac{p}{q} - (1-\sqrt{8})^{2n+1}\right) + (9-2\sqrt{8}) \cdot (1-\sqrt{8})^{2n+1} = \frac{p'}{q'}$$

$$(1-\sqrt{8})^{2n+1}(9-2\sqrt{8} - 9 - 2\sqrt{8}) + \frac{p}{q}(9+2\sqrt{8}) = \frac{p'}{q'}$$

$$\frac{p}{q}(9+2\sqrt{8}) - 4\sqrt{8}(1-\sqrt{8})^{2n+1} = \frac{p'}{q'}.$$

$$\frac{p}{q}(9+2\sqrt{8}) - 4\sqrt{8}(1-\sqrt{8})(9-2\sqrt{8})^{n} = \frac{p'}{q'}$$

$$\underbrace{\frac{p}{q}(9+2\sqrt{8})}_{\neq 0} + \underbrace{4\sqrt{8}(\sqrt{8}-1)(9-2\sqrt{8})^{n}}_{>0} = \frac{p'}{q'}$$

$\therefore \frac{p'}{q'}$ seria irracional, o que é um absurdo.

(74) $\frac{m}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{1318} + \frac{1}{1319}$

$\frac{m}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1319} - 2\cdot\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{1318}\right)$

$\frac{m}{n} = \cancel{1} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \dots + \frac{1}{1319} - \cancel{1} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \dots - \cancel{\frac{1}{659}}$

$\frac{m}{n} = \underbrace{\frac{1}{660} + \frac{1}{661} + \dots + \frac{1}{1319}}_{660 \text{ termos}}$

observe que $1319 + 660 = 1979$
$1318 + 661 = 1979$
$\vdots$
$989 + 990 = 1979$

$\therefore \frac{m}{n} = 1979 \cdot \alpha$

Portanto $m = 1979 \cdot \alpha \cdot n$

$\boxed{\text{Por fim } 1979 \mid m}$

Ajuda do AoPS Online.

(75) $\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} = \sum_{k=1}^{n} \frac{k}{(k^2+k+1)(k^2-k+1)}$

Usando o método das frações parciais chegamos em

$\frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k^2-k+1} - \sum_{k=1}^{n} \frac{1}{k^2+k+1}\right)$

$\frac{1}{2}\left(1 + \cancel{\frac{1}{3}} + \cancel{\frac{1}{7}} + \cancel{\frac{1}{13}} + \dots + \cancel{\frac{1}{n^2-n+1}} - \left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{7}} + \cancel{\frac{1}{13}} + \dots + \cancel{\frac{1}{n^2-n+1}} + \frac{1}{n^2+n+1}\right)\right)$

$\frac{1}{2}\left(1 - \frac{1}{n^2+n+1}\right) = \frac{n^2+n}{2(n^2+n+1)} = \boxed{\frac{n(n+1)}{2(n^2+n+1)}}$

- $k^4+k^2+1 = \boxed{(k^2+k+1)(k^2-k+1)}$