# ÁLGEBRA LINEAR

02/09/24

## RUSHADÃO TESTES 1

2018

**Questão 1** (valor 2,0)

Seja $A = \begin{bmatrix} 3 & 1 & -6 \\ 7 & 2 & 9 \\ 8 & 1 & -1 \end{bmatrix}$.

Escreva esta matriz como uma soma de uma matriz simétrica com outra antissimétrica.

$$\begin{bmatrix} a & e & f \\ e & b & g \\ f & g & c \end{bmatrix} + \begin{bmatrix} 0 & -h & -i \\ h & 0 & -j \\ i & j & 0 \end{bmatrix}$$

$$\begin{cases} e+h=7 \\ e-h=1 \end{cases} \rightarrow \begin{matrix} e=4 \\ h=3 \end{matrix}$$

$$\begin{cases} f+i=8 \\ f-i=-6 \end{cases}$$
$$2f=2 \rightarrow f=1 \quad i=7$$

$$\begin{cases} g+j=1 \\ g-j=9 \end{cases}$$
$$2g=10 \rightarrow g=5 \quad j=-4$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 4 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 4 \\ 7 & -4 & 0 \end{bmatrix}$$

**Questão 2** (valor 2,0)

Sejam $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & -3 & 2 \end{bmatrix}$ e $B = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 3 & 4 & 5 \end{bmatrix}$.

a) Calcule a matriz inversa de $A$.
b) Determine a matriz $X$ tal que $AX = B$.
Obs: o item b) pode ser resolvido com o resultado de a) ou não.

(a)

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\begin{matrix} L_2+L_1 \\ L_3-2L_1 \end{matrix}} \left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{L_3+L_2} \left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{\begin{matrix} L_1-2L_2 \\ L_2-L_3 \end{matrix}} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -1 & -2 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{L_1+L_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

(b) $\begin{bmatrix} -2 & -1 & 1 \\ -2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

**Questão 3** (valor 2,0)

Resolva o sistema $\begin{bmatrix} 1 & 4 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ sobre $Z_5$.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_3-2L_1]{L_2-L_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -7 & 3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{L_3-\frac{7}{2}L_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{7}{2} & 1 \end{array}\right]$$

$$(-2)\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 7 & -2 \end{array}\right] \xrightarrow[L_1+2L_2]{L_2-L_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -12 & 4 \\ 0 & -2 & 0 & 2 & -6 & 2 \\ 0 & 0 & 1 & -3 & 7 & -2 \end{array}\right] \rightarrow \begin{bmatrix} 5 & -12 & 4 \\ -1 & 3 & -1 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -12 & 4 \\ -1 & 3 & -1 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$$

**Questão 5** (valor 2,0)

Faça a decomposição LU da matriz $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 4 & 8 \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 4 & 0 & 0 & 1 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{L_3-L_1 \\ L_4-L_1}]{L_2-L_1} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 7 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[L_4-L_2]{L_3-L_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 6 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{L_4-L_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & -1 & 1 \end{array}\right]$$

U E

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{L_2+L_1 \\ L_3+L_2 \\ L_4+L_3}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$E \qquad I \qquad\qquad\qquad L$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

2023

**Questão 1** (4 pontos)

Considere a matriz:

$$A = \begin{bmatrix} 1 & -3 & 1 & -5 & -31 \\ 2 & -3 & 8 & -7 & -41 \\ -1 & 9 & 11 & 12 & 79 \\ 3 & -12 & -3 & -19 & -120 \end{bmatrix}.$$

(a) (1.5 ponto) Ache a forma escalonada reduzida de $A$.

(b) (1.5 ponto) Ache as soluções especiais de $A\mathbf{x} = 0$.

(c) (0.5 ponto) Ache a solução completa para $A\mathbf{x} = \begin{bmatrix} -5 \\ -7 \\ 12 \\ -19 \end{bmatrix}$, se existir.

(d) (0.5 ponto) Ache a solução completa para $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, se existir.

(a) $$\begin{bmatrix} 1 & -3 & 1 & -5 & -31 \\ 2 & -3 & 8 & -7 & -41 \\ -1 & 9 & 11 & 12 & 79 \\ 3 & -12 & -3 & -19 & -120 \end{bmatrix}$$

$$\xrightarrow[\substack{L_2-2L_1 \\ L_3+L_1 \\ L_4-3L_1}]{} \begin{bmatrix} 1 & -3 & 1 & -5 & -31 \\ 0 & 3 & 6 & 3 & 21 \\ 0 & 6 & 12 & 7 & 48 \\ 0 & -3 & -6 & -4 & -27 \end{bmatrix}$$

$$\xrightarrow[\substack{L_3-2L_2 \\ L_4+L_2}]{} \begin{bmatrix} 1 & -3 & 1 & -5 & -31 \\ 0 & 3 & 6 & 3 & 21 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 & -6 \end{bmatrix} \xrightarrow[L_4+L_3]{} \begin{bmatrix} 1 & -3 & 1 & -5 & -31 \\ 0 & 3 & 6 & 3 & 21 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{L_1+L_2 \\ L_2/3}]{} \begin{bmatrix} 1 & 0 & 7 & -2 & -10 \\ 0 & 1 & 2 & 1 & 7 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{L_2-L_3 \\ L_1+2L_3}]{} \begin{bmatrix} 1 & 0 & 7 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $$\begin{array}{c} \begin{bmatrix} 1 & 0 & 7 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

PIVÔS

$x_3 = 0$
$x_5 = 1$

$x_1 + 7x_3 + 2x_5$
$x_2 + 2x_3 + x_5$
$x_4 + 6x_5$

$x_4 + 6 = 0 \qquad x_2 + \cancel{2x_3}^{\,0} + 1 = 0 \qquad x_1 + 7\cdot 0 + 2\cdot 1 = 0$

$x_4 = -6 \qquad x_2 = -1 \qquad x_1 = -2$

$x_3 = 1$
$x_5 = 0$

$x_4 = 0 \quad x_2 = -2 \quad x_1 = -7$

SOLUÇÕES

$$v_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ -6 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -7 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(c) $b = \begin{bmatrix} -5 \\ -7 \\ 12 \\ -19 \end{bmatrix} \longrightarrow c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

SOLUÇÃO PARTICULAR

$$\begin{bmatrix} 1 & 0 & 7 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(d) COMO NA ELIMINAÇÃO EU NÃO USEI NENHUMA VEZ A LINHA 4, A MATRIZ FICA

$$\left[\begin{array}{ccccc:c} 1 & 0 & 7 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

NÃO HÁ SOLUÇÃO POIS A MATRIZ MOSTRA UM SISTEMA

$$0+0+0+0+0=1$$

ABSURDO!

**Questão 2** (3 pontos)

Como o núcleo $N(C)$ se relaciona aos núcleos $N(A)$ e $N(B)$, onde $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

$x \in N(A) \Leftrightarrow Ax = 0$

$x \in N(B) \Leftrightarrow Bx = 0$

$\rightarrow Cx = \begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0$

ZERAM A E B $\therefore$

$$N(C) = N(A) \cap N(B)$$

**Questão 3** (3 pontos)

Sejam $A$ uma matriz $n \times n$ inversível e $\mathbf{u}, \mathbf{w}$ vetores $n \times 1$ tais que $1 + \mathbf{w}^{\mathsf{T}} A^{-1} \mathbf{u} \neq 0$. Mostre que

$$(A + uw^{\mathsf{T}})^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{w}^{\mathsf{T}}A^{-1}}{1 + \mathbf{w}^{\mathsf{T}}A^{-1}\mathbf{u}}.$$

Para metade dos pontos, prove a fórmula acima com $A = I$. Isso pode ser útil para provar o caso geral.

$u_{n\times 1}$ $\quad w^T_{1\times n}$

$$(A + uw^T)\left(A^{-1} - \frac{A^{-1}uw^TA^{-1}}{1 + w^TA^{-1}u}\right)$$

$1 \times 1 \in \mathbb{R}$

$$I - \frac{uw^TA^{-1}}{1 + w^TA^{-1}u} + uw^TA^{-1} - \frac{u\boxed{w^TA^{-1}u}w^TA^{-1}}{1 + w^TA^{-1}u}$$

$$I - \cancel{uw^T A^{-1}} \left( \overset{1}{\cancel{\frac{1 + uw^T A^{-1}}{1 + w^T A^{-1} u}}} \right) + \cancel{uw^T A^{-1}}$$

$$\boxed{I}$$