LISTA 6

Seção 14.7: 1, 3, 5, 7, 9, 11, 19, 21, 33, 35, 37, 39, 43, 47, 49, 51, 59

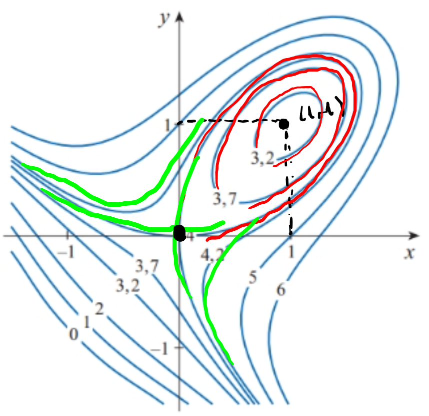
(1) a) $f_{xx}(1,1) = 4$
$f_{xy}(1,1) = 1$
$f_{yy}(1,1) = 2$

$D(1,1) = 4 \cdot 2 - 1^2 = 7 > 0$
$f_{xx} > 0$
$\therefore$ $(1,1)$ é ponto de mínimo.

b) $f_{xx}(1,1) = 4$
$f_{xy}(1,1) = 3$
$f_{yy}(1,1) = 2$

$D(1,1) = 4 \cdot 2 - 3^2 = -1 < 0$
$f_{xx} > 0$
$\therefore$ $(1,1)$ é ponto de máximo.

(3) $f(x,y) = 4 + x^3 + y^3 - 3xy$



O ponto $(1,1)$ seria ponto de mínimo local (curvas ovais em torno do ponto e se nos afastarmos, os valores vão aumentando)

O ponto $(0,0)$ seria ponto de sela (observe que temos formato de hipérboles, ou seja em uma direção os valores crescem e em outra os valores decrescem).

(5) $f(x,y) = x^2 + xy + y^2 + y$

$\dfrac{\partial f}{\partial x} = 2x + y$; $\dfrac{\partial f}{\partial y} = x + 2y + 1$

$\begin{cases} 2x+y = 0 & \Rightarrow \ y = -2x \\ x+2y+1 = 0 & \Rightarrow \ x - 4x + 1 = 0 \quad x = 1/3 \quad y = -2/3. \end{cases}$

$\dfrac{\partial^2 f}{\partial x^2} = 2$; $\dfrac{\partial^2 f}{\partial y^2} = 2$; $\dfrac{\partial^2 f}{\partial x \partial y} = 1$; $D(1/3, -2/3) = 2 \cdot 2 - 1^2 = 3 > 0$

Como $D(1/3, -2/3) > 0$ e $\dfrac{\partial^2 f}{\partial x^2}(1/3, -2/3) > 0$, então

$f(1/3, -2/3) = -1/3$ é um mínimo local.

(7) $f(x,y) = 2x^2 - 8xy + y^4 - 4y^3$

$\dfrac{\partial f}{\partial x} = 4x - 8y$; $\dfrac{\partial f}{\partial y} = -8x + 4y^3 - 12y^2$

$\begin{cases} 4x - 8y = 0 & \Rightarrow \ x = 2y \\ -8x + 4y^3 - 12y^2 = 0 & \Rightarrow \ -2x + y^3 - 3y^2 = 0 \end{cases}$

e $y^3 - 3y^2 - 4y = 0$

$y(y^2 - 3y - 4) = 0$

$y(y-4)(y+1) = 0$

$x = 0$	$x = 8$	$x = -2$
$y = 0$	$y = 4$	$y = -1$

$$\frac{\partial^2 f}{\partial x^2} = 4 \Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial^2 f}{\partial x^2}(8,4) = \frac{\partial^2 f}{\partial x^2}(-2,1) = 4$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 24y \; ; \; \frac{\partial^2 f}{\partial y^2}(0,0) = 0 \; ; \; \frac{\partial^2 f}{\partial y^2}(8,4) = 96 \; ; \; \frac{\partial^2 f}{\partial y^2}(-2,1) = 36$$

$$; \frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(8,4) = \frac{\partial^2 f}{\partial x \partial y}(-2,1) = -8.$$

$D(0,0) = 4 \cdot 0 - (-8)^2 < 0$, $D<0$ $\therefore$ $(0,0)$ é ponto de sela
$D(8,4) = 4 \cdot 96 - (-8)^2 > 0$, $D>0$ e $f_{xx}>0$ $\therefore$ $(8,4)$ é mínimo
$D(-2,1) = 4 \cdot (36) - (-8)^2 > 0$, $D>0$ $\therefore$ $(-2,1)$ é mínimo.

(9) $f(x,y) = (x-y)(1-xy) = x - x^2y - y + xy^2$

$$\frac{\partial f}{\partial x} = 1 - 2xy + y^2 \; ; \; \frac{\partial f}{\partial y} = -x^2 - 1 + 2xy.$$

$$\begin{cases} 1 - 2xy + y^2 = 0 \\ -x^2 - 1 + 2xy = 0 \end{cases} \Rightarrow x^2 = y^2 \quad \therefore \boxed{x = \pm y.}$$

I) $x = y$: $\quad 1 - 2y^2 + y^2 = 0 \Rightarrow y^2 = 1 \therefore y = \pm 1$
$x = 1, \; y = 1$
$x = -1, \; y = -1$

II) $x = -y$: $1 + 2y^2 + y^2 = 0$ (impossível).

$$\frac{\partial^2 f}{\partial x^2} = -2y \; ; \; \frac{\partial^2 f}{\partial x^2}(1,1) = -2 \; ; \; \frac{\partial^2 f}{\partial x^2}(-1,-1) = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2x \; ; \; \frac{\partial^2 f}{\partial y^2}(1,1) = 2 \; ; \; \frac{\partial^2 f}{\partial y^2}(-1,-1) = -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y - 2x \; ; \; \frac{\partial^2 f}{\partial x \partial y}(1,1) = 0 \; ; \; \frac{\partial^2 f}{\partial x \partial y}(-1,-1) = 0$$

$D(1,1) = -2 \cdot 2 - 0 < 0$

$D(-1,-1) = 2 \cdot (-2) - 0 < 0$

$\therefore$ $(1,1)$ e $(-1,-1)$ são pontos de sela.

(11) $f(x,y) = y\sqrt{x} - y^2 - 2x + 7y$

$$\frac{\partial f}{\partial x} = \frac{y}{2\sqrt{x}} - 2 \; ; \; \frac{\partial f}{\partial y} = \sqrt{x} - 2y + 7$$

$$\begin{cases} y/2\sqrt{x} - 2 = 0 \\ \sqrt{x} - 2y + 7 = 0 \end{cases} \quad \begin{aligned} &\Rightarrow \sqrt{x} = y/4 \\ &\Rightarrow 2y - y/4 = 7 \end{aligned}$$

$$\frac{7y}{4} = 7 \Rightarrow \boxed{\begin{aligned} y &= 4 \\ x &= 1. \end{aligned}}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{y \cdot x^{-3/2}}{4} \; ; \; \frac{\partial^2 f}{\partial x^2}(1,4) = -1$$

$$\frac{\partial^2 f}{\partial y^2} = -2 \; ; \; \frac{\partial^2 f}{\partial x^2}(1,4) = -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(1,4) = 1/2$$

$D(1,4) = -2(-1) - 1/2 > 0$ e $f_{xx} < 0$

$\therefore$ $(1,4)$ é ponto de máximo

(19) $f(x,y) = e^x \cos y$

$$\frac{\partial f}{\partial x} = e^x \cos y \qquad \frac{\partial f}{\partial y} = -e^x \operatorname{sen} y$$

$$\begin{cases} e^x \cos y = 0 \\ -e^x \operatorname{sen} y = 0 \end{cases}$$

Como $e^x > 0$, então é impossível $\cos y = \operatorname{sen} y = 0$

Portanto, $f(x,y)$ não possui máximos ou mínimos locais.

(21) $f(x,y) = y^2 - 2y\cos x \qquad -1 \leq x \leq 7$

$$\frac{\partial f}{\partial x} = 2y \operatorname{sen} x \; ; \; \frac{\partial f}{\partial y} = 2y - 2\cos x.$$

$$\begin{cases} 2y \operatorname{sen} x = 0 \\ 2(y - \cos x) = 0 \end{cases} \Rightarrow \begin{cases} y \operatorname{sen} x = 0 \\ \cos x = y \end{cases}$$

$\therefore$ $\operatorname{sen} 2x = 0$ e $x = \frac{k\pi}{2}$, $k \in \mathbb{Z}$

Como $-1 \leq x \leq 7$: $x = \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$
e $y = \{1, 0, -1, 0, 1\}$.

$\frac{\partial^2 f}{\partial x^2} = 2y\cos x \Rightarrow 2; 0; 2, 0; 2$

$\frac{\partial^2 f}{\partial y^2} = 2$

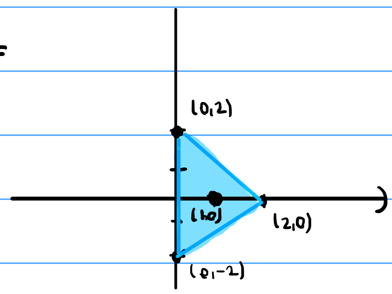
$\frac{\partial^2 f}{\partial x \partial y} : 2\operatorname{sen} x \Rightarrow 0; 2; 0; -2; 0$

$D \Rightarrow 4; -4; 4; -4; 4$

Ponto de mínimo : $(0,1), (\pi,-1), (2\pi,1)$
Ponto de sela : $(\pi/2, 0), (3\pi/2, 0)$ //.

(33) $f(x,y) = x^2 + y^2 - 2x$

D =



$\frac{\partial f}{\partial x} = 2x - 2$; $\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial x \partial y} = 0$

$\frac{\partial f}{\partial y} = 2y$; $\frac{\partial^2 f}{\partial y^2} = 2$

$\begin{cases} 2x - 2 = 0 \\ 2y = 0 \end{cases} \Rightarrow \boxed{x = 1, y = 0}$ //.

$D(1,0) = 4 - 0 > 0$
$(1,0)$ é mínimo local de $f(x,y)$ em $D = \mathbb{R}^2$.

Candidatos para máximos e mínimos em D: $(1,0)$, $(0,\pm 2)$, $(2,0)$

I) $f(1,0) = -1$
$f(0,\pm 2) = 4$
$f(2,0) = 0$

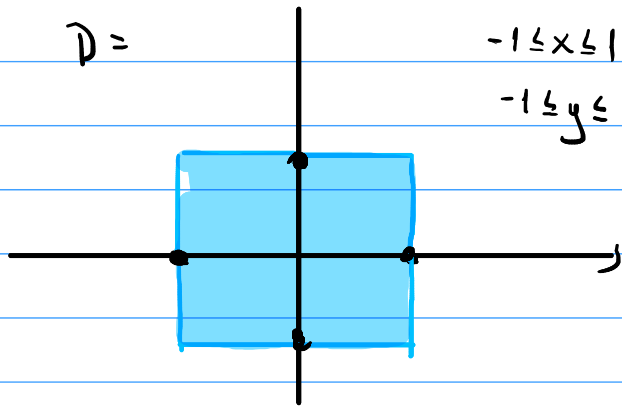
Mínimo local: $(1,0)$
Máximos locais: $(0,\pm 2)$

35) $f(x,y) = x^2 + y^2 + x^2 y + 4$ $\quad D = \{(x,y) \mid |x| \le 1 \text{ e } |y| \le 1\}$

$$\frac{\partial f}{\partial x} = 2x + 2xy$$

$$\frac{\partial f}{\partial y} = 2y + x^2$$

D =



$-1 \le x \le 1$
$-1 \le y \le 1$

$$\begin{cases} 2x + 2xy = 0 \\ 2y + x^2 = 0 \end{cases}$$

$\Rightarrow x = 0$ ou $x = -y$

I) $x = 0$: $y = 0$

II) $x = -y$: $2y + y^2 = 0 \Rightarrow y(y+2) = 0 \Rightarrow y = 0$ ou $y = -2$
$x = 0$ ou $x = 2$

$\boxed{\Rightarrow (0,0) \text{ ou } (2,-2)}$

$(2,-2) \notin D$, mas $(0,0) \in D$.

$$\frac{\partial^2 f}{\partial x^2} = 2 + 2y \Rightarrow 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \Rightarrow 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x \Rightarrow 0$$

$D = 2 \cdot 2 - 0 > 0$

$f_{xx}(0,0) > 0 \quad \therefore \quad (0,0)$ é ponto de mínimo.

candidatos para valores máximos e mínimos em D: $(1,1), (-1,1), (1,-1), (0,0)$

$f(1,1) = 7$

$f(-1,1) = 7$

$f(1,-1) = 5$

$f(0,0) = 4.$

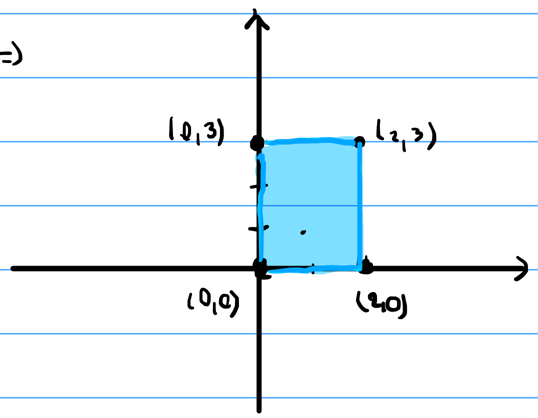
$$\boxed{\begin{array}{ll} \text{máx:} & f(\pm 1, 1) = 7 \\ \text{mín:} & f(0,0) = 4. \end{array}}$$

37) $f(x,y) = x^2 + 2y^2 - 2x - 4y + 1. \quad D = \{(x,y) \mid 0 \le x \le 2, \ 0 \le y \le 3\}$

$$\frac{\partial f}{\partial x} = 2x - 2 \quad \Rightarrow \quad \boxed{\begin{array}{l} x = 1 \\ y = 1 \end{array}}$$

$$\frac{\partial f}{\partial y} = 4y - 4$$

$D \Rightarrow$



$$\frac{\partial^2 f}{\partial x^2} = 2 \;;\; \frac{\partial^2 f}{\partial y^2} = 4 \;;\; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$D = 4 > 0$; $f_{xx} > 0 \Rightarrow (1,1)$ é ponto de mínimo

candidatos: $(1,1), (0,0), (2,0), (0,3), (2,3)$

$f(1,1) = -2$
$f(0,0) = 1$
$f(2,0) = 1$
$f(0,3) = 7$
$f(2,3) = 7$

$$\boxed{\begin{array}{l} \text{Máx: } f(0,3) = f(2,3) = 7 \\ \text{Mín: } f(1,1) = -2 \end{array}}$$

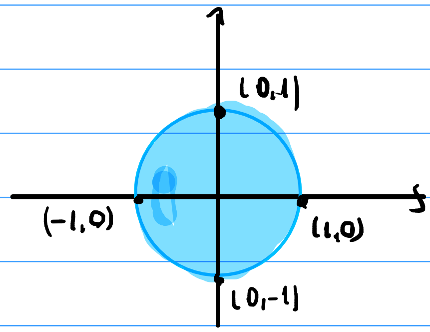
39) $f(x,y) = 2x^3 + y^4$, $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$

$\frac{\partial f}{\partial x} = 6x^2$

$\Rightarrow \boxed{\begin{array}{l} x = 0 \\ y = 0 \end{array}}$

$\frac{\partial f}{\partial y} = 4y^3$

D =



$\frac{\partial^2 f}{\partial x^2} = 12x \Rightarrow 0$

$D(0,0) = 0$ inconclusivo

$\frac{\partial^2 f}{\partial y^2} = 12x^2 \Rightarrow 0$

$\partial^2 f / \partial x \partial y = 0$

Candidatos: $(0,1), (0,-1), (1,0), (-1,0)$

$f(0,1) = 1$
$f(0,-1) = 1$
$f(1,0) = 2$
$f(-1,0) = -2$

$$\boxed{\begin{array}{l} \text{mín}: f(-1,0) = -2 \\ \text{máx}: f(1,0) = 2 \end{array}}$$

43) $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$

$d^2 = (x-2)^2 + y^2 + (z+3)^2$
$y = 1-x-z \Rightarrow d^2 = (x-2)^2 + (1-x-z)^2 + (z+3)^2$

$\frac{\partial f}{\partial x} = 2(x-2) + 2(1-x-z)(-1) = 2x-4 + 2(x+z-1)$
$= 4x + 2z - 6$

$\frac{\partial f}{\partial z} = 2(1-x-z)(-1) + 2(z+3) = 2(z+3) + 2(x+z-1)$
$= 4z + 2x + 4$

$\begin{cases} 4x + 2z = 6 \\ 2x + 4z = -4 \end{cases} \Rightarrow \begin{cases} 4x + 2z = 6 \\ 4x + 8z = -8 \end{cases}$

$6z = -14$

$$\boxed{z = -7/3 \quad x = 8/3 \quad y = 2/3}$$

$d_{min} = 2/\sqrt{3}$.

Só fazer $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$ e $D(x,y)$.

(47) $x+y+z=100$ $\qquad z=100-x-y$

xyz é máximo

$$f(x,y)=xy(100-x-y)=100xy-x^2y-xy^2$$

$$\frac{\partial f}{\partial x}=100y-2xy-y^2 \qquad \begin{cases} y(100-2x-y)=0 \\ x(100-2y-x)=0 \end{cases}$$

$$\frac{\partial f}{\partial y}=100x-x^2-2xy$$

I) $y=0 \Rightarrow x=0 \qquad (0,0)$

$x=100 \qquad (100,0)$

II) $y=100-2x \Rightarrow x=0 \qquad (0,100)$

$x=100/3 \qquad (100/3,100/3)$

$(100/3, 100/3)$ é de máximo

$$\boxed{\begin{aligned} x&=100/3 \\ y&=100/3 \\ z&=100/3 \end{aligned}}$$

(49) $V=abh$ $\qquad 2r=\sqrt{a^2+b^2+h^2}$

$a,b,h>0$ $\qquad 4r^2-a^2-b^2=h^2$

$$V^2=a^2b^2h^2 \Rightarrow a^2b^2(4r^2-a^2-b^2)=4a^2b^2r^2-a^4b^2-a^2b^4$$

$$f(a,b)=4a^2b^2r^2-a^4b^2-a^2b^4 \qquad \frac{\partial f}{\partial a}=8ab^2r^2-4a^3b^2-2ab^4$$

$$\frac{\partial f}{\partial b} = 8a^2br^2 - 2a^4b - 4a^2b^3$$

$$8ab^2r^2 - 4a^3b^2 - 2ab^4$$

$$\begin{cases} \cancel{2 \cdot ab^2}(4 \cdot r^2 - 2a^2 - b^2) = 0 \\ \cancel{2 \cdot a^2b}(4 \cdot r^2 - 2 \cdot b^2 - a^2) = 0 \end{cases}$$

$a, b \neq 0$

$$b^2 = 4r^2 - 2a^2$$

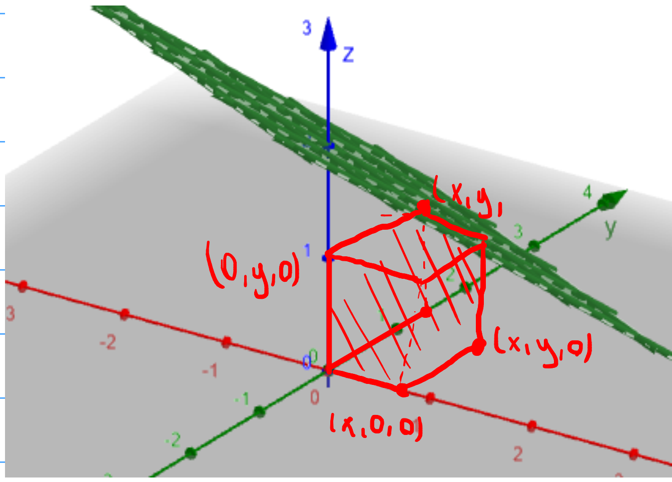
$$4r^2 - 8r^2 + 4a^2 - a^2 = 0$$

$$3a^2 = 4r^2$$

$$\boxed{a = \frac{2r}{\sqrt{3}}} \qquad \boxed{b = \frac{2r}{\sqrt{3}}}$$

$$h = \frac{2r}{\sqrt{3}} \qquad \boxed{V_{máx} = \frac{8r^3}{3\sqrt{3}}}$$

51



$$V = x \cdot y \cdot h$$

$$V = (6 - 2y - 3z)\, y\, z$$

$$V = 6yz - 2y^2z - 3yz^2$$

$$\frac{\partial V}{\partial y} = 6z - 4yz - 3z^2 = z(6 - 4y - 3z)$$

$$\frac{\partial V}{\partial z} = 6y - 2y^2 - 6yz = 2y(3 - y - 3z)$$

$$\begin{cases} z(6 - 4y - 3z) = 0 \\ 2y(3 - y - 3z) = 0 \end{cases} \qquad z, y \neq 0$$

$$6 - 4y = 3 - y \qquad \boxed{y = 1} \qquad \boxed{z = 2/3}$$

$$\boxed{x = 2}$$

$$V = 1 \cdot 2 \cdot 2/3 = \boxed{4/3}$$

59) $H = -(p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3)$

a) $p_1 + p_2 + p_3 = 1 \qquad p_3 = 1 - (p_1 + p_2)$

$H = -(p_1 \log p_1 + p_2 \log p_2 + (1 - (p_1 + p_2)) \log(1 - (p_1 + p_2)))$

b) $D = \{(p_1, p_2) \mid 0 < p_1 + p_2 < 1 \text{ e } 0 < p_1, p_2 < 1\}$

c) $H = -(p_1 \log p_1 + p_2 \log p_2 + (1 - (p_1 + p_2)) \log(1 - (p_1 + p_2)))$

Só fazer $\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2} = 0 \qquad H_{máx} = \ln 3$

$p_1 = p_2 = p_3 = 1/3.$

Seção 14.8: 5, 7, 9, 11, 13, 23, 27, 29, 35, 37, 41, 51, 61

5) $f(x,y) = xy \quad ; \quad 4x^2 + y^2 = 8$

$\nabla f(x,y) = (y, x)$
$\nabla g(x,y) = (8x, 2y).$

$\Rightarrow \begin{cases} y = \lambda \cdot 8x \\ x = \lambda \cdot 2y \\ 4x^2 + y^2 = 8 \end{cases}$

$y = \lambda \cdot 8 \cdot \lambda \cdot 2y$

$y = 16\lambda^2 y \Rightarrow y(1 - 16\lambda^2) = 0$

I) $y = 0 \Rightarrow x = 0$, não é possível.
II) $\lambda = + 1/4 \Rightarrow y = 2x \Rightarrow 4x^2 + 4x^2 = 8 \quad x = \pm 1$
$y = \pm 2$

III) $\lambda = -1/4 \Rightarrow y = -2x \Rightarrow 4x^2 + 4x^2 = 8 \quad x = \pm 1$
$y = \mp 2$

Pontos: $(1,2), (-1,-2), (1,-2), (-1,2)$

$f(1,2) = 2$
$f(-1,-2) = 2$
$f(1,-2) = -2$
$f(-1,2) = -2$

Máximos: $f(1,2) = f(-1,-2) = 2$
Mínimos: $f(1,-2) = f(-1,2) = -2$

⑦ $f(x,y) = 2x^2 + 6y^2$; $x^4 + 3y^4 = 1$

$$\nabla f(x,y) = (4x, 12y)$$
$$\nabla g(x,y) = (4x^3, 12y^3)$$
$$\Rightarrow \begin{cases} 4x = \lambda \cdot 4x^3 \\ 12y = \lambda \cdot 12y^3 \\ x^4 + 3y^4 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = \lambda x^3 \\ y = \lambda y^3 \\ x^4 + 3y^4 = 1 \end{cases} \Rightarrow \begin{cases} x(1 - \lambda x^2) = 0 \\ y(1 - \lambda y^2) = 0 \\ x^4 + 3y^4 = 1. \end{cases}$$

$x = 0 \Rightarrow y = \pm \sqrt[4]{1/3}$

$y = 0 \Rightarrow x = \pm 1.$

$$\begin{matrix} \lambda x^2 = 1 \\ \lambda y^2 = 1 \end{matrix} \Rightarrow x = \pm y. \Rightarrow \left. \begin{matrix} x = \pm 1/\sqrt{2} \\ y = \pm 1/\sqrt{2}. \end{matrix} \right\} \begin{matrix} x = \pm 1/\sqrt{2} \\ y = \mp 1/\sqrt{2} \end{matrix}$$

Puntos $(1,0)$, $(-1,0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$.

Máximos: $f(1/\sqrt{2}, \pm 1/\sqrt{2}) = f(-1/\sqrt{2}, \pm 1/\sqrt{2}) = 4$
Mínimos: $f(\pm 1, 0) = 2$

⑨ $f(x,y,z) = 2x + 2y + z$; $x^2 + y^2 + z^2 = 9$

$$\nabla f(x,y,z) = (2, 2, 1)$$
$$\nabla g(x,y,z) = (2x, 2y, 2z)$$
$$\Rightarrow \begin{cases} 2 = \lambda \cdot 2x \\ 2 = \lambda \cdot 2y \\ 1 = \lambda \cdot 2z \\ x^2 + y^2 + z^2 = 9 \end{cases}$$

$$\Rightarrow \begin{cases} x = 1/\lambda \\ y = 1/\lambda \\ z = 1/2\lambda \\ x^2+y^2+z^2 = 9 \end{cases} \Rightarrow \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 9$$

$$\frac{\cancel{9}}{4\lambda^2} = \cancel{9} \qquad \lambda = \pm 1/2$$

Pontos: $(2,2,1)$ e $(-2,-2,-1)$

$$\boxed{\begin{array}{l} \text{Máximos: } f(2,2,1) = 9 \\ \text{Mínimos: } f(-2,-2,-1) = -9 \end{array}}$$

4) $f(x,y,z) = x y^2 z \qquad x^2+y^2+z^2 = 4$

$$\nabla f(x,y,z) = (y^2 z, 2xyz, xy^2)$$
$$\nabla g(x,y,z) = (2x, 2y, 2z)$$

$$\Rightarrow \begin{array}{l} y^2 z = \lambda \cdot 2x \\ 2xyz = \lambda \cdot 2y \\ xy^2 = \lambda \cdot 2z \\ x^2+y^2+z^2 = 4 \end{array}$$

$$\Rightarrow \begin{array}{l} xyz \cdot y = 2\lambda x^2 \\ xyz \cdot y = \lambda y^2 \\ xyz \cdot y = 2\lambda z^2 \end{array}$$

$$\begin{array}{l} x^2 = z^2 \\ y^2 = 2x^2 \end{array} \Rightarrow 4x^2 = 4$$

$$x = \pm 1 \; ; \; y = \pm\sqrt{2} \; ; \; z = \pm 1$$

Pontos: $(1,\sqrt{2},1)$, $(1,\sqrt{2},-1)$, $(1,-\sqrt{2},1)$, $(1,-\sqrt{2},-1)$
$(-1,\sqrt{2},1)$, $(-1,\sqrt{2},-1)$, $(-1,-\sqrt{2},1)$, $(-1,-\sqrt{2},-1)$.

$$\boxed{\begin{array}{l}\text{Máximos: } f(1,\pm\sqrt{2},1) = f(-1,\pm\sqrt{2},-1) = 2 \\ \text{mínimos: } f(1,\pm\sqrt{2},-1) = f(-1,\pm\sqrt{2},1) = -2\end{array}}$$

(13) $f(x,y,z) = x^2+y^2+z^2$; $x^4+y^4+z^4=1$.

$$\begin{aligned}\nabla f &= (2x,2y,2z) \\ \nabla g &= (4x^3,4y^3,4z^3)\end{aligned} \Rightarrow \left\{\begin{array}{l} 2x = \lambda\cdot 4x^3 \\ 2y = \lambda \cdot 4y^3 \\ 2z = \lambda\cdot 4z^3 \\ x^4+y^4+z^4=1 \end{array}\right.$$

$$\left\{\begin{array}{l} x = 2\lambda x^3 \\ y = 2\lambda y^3 \\ z = 2\lambda z^3 \\ x^4+y^4+z^4=1 \end{array}\right. \qquad x^2+y^2+z^2 = 2\lambda = p$$

$$\boxed{(x^2+y^2+z^2)^2 = x^4+y^4+z^2+2(x^2y^2+z^2y^2+x^2z^2)}$$

$$\left\{\begin{array}{l} x = 2\lambda x^3 \\ y = 2\lambda y^3 \\ z = 2\lambda z^3 \end{array}\right. \Rightarrow \left\{\begin{array}{l} xy(1-4\lambda^2x^2y^2) = 0 \\ yz(1-4\lambda^2y^2z^2) = 0 \\ xz(1-4\lambda^2x^2z^2) = 0 \end{array}\right.$$

$\Rightarrow$ Ou pelo menos um 0 ou pelo um $\frac{1}{(2\lambda)^2} = \frac{1}{p^2}$
(x,y,z) (x^2y^2, y^2z^2, x^2z^2).

I) $x=0$ ou $y=0$ ou $z=0$:

Inconclusivo pois $(x^2+y^2)^2 = x^4+y^4+2x^2y^2$.

II) $x=y=0$, $x=z=0$, $z=y=0$

$p^2 = 1 \Rightarrow \boxed{p=\pm 1} \Rightarrow \boxed{p=1}$ ($f \geq 0$)

III) $x, y, z \neq 0$

$p^2 = 1 + 2.3.\frac{1}{p^2} \Rightarrow p^4 - p^2 - 6 = 0 \Rightarrow \boxed{p = \pm\sqrt{3}, \text{ outras } \in \mathbb{C}}$

$\therefore \boxed{p=\sqrt{3}}$

$$\boxed{\begin{array}{l} \text{Máximo}: \sqrt{3} \\ \text{mínimo}: 1 \end{array}}$$

23 $f(x,y) = x^2+y^2$, $xy=1$.

Pelo método dos multiplicadores de Lagrange:

$\nabla f(x,y) = (2x, 2y)$
$\nabla g(x,y) = (y, x)$ $\Rightarrow \begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 1 \end{cases}$ $(x, y \neq 0)$

$$\frac{2x}{y} = \frac{2y}{x} \Rightarrow x^2 = y^2 \Rightarrow x = \pm y.$$

$$x^2 = 1 \Rightarrow x = \pm 1, \quad y = \pm 1$$

Pontos $(1,1)$, $(-1,-1)$
mínimos: $f(1,1) = f(-1,-1) = 2$.

Mas, fazendo do jeito convencional:

$$\frac{\partial f}{\partial x} = 2x$$
$$\Rightarrow x = y = 0 \text{ (único ponto crítico)}$$
$$\frac{\partial f}{\partial y} = 2y$$

Portanto, o único ponto que pode ser de máximo ou mínimo é $(0,0) \notin \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$.

27 $f(x,y) = x^2 + y^2 + 4x - 4y \; ; \; x^2 + y^2 \leq 9$

$$\nabla f = (2x+4, 2y-4) \quad \Rightarrow \quad \begin{cases} 2(x+2) = \lambda \cdot 2x \\ 2(y-2) = \lambda \cdot 2y \\ x^2 + y^2 = 9 \end{cases}$$
$$\nabla g = (2x, 2y)$$

$$\Rightarrow \begin{cases} x + 2 = \lambda x \\ y - 2 = \lambda y \\ x^2 + y^2 = 9 \end{cases} \Rightarrow \quad x = \frac{-2}{1-\lambda} = \frac{2}{\lambda - 1}$$

$$y = \frac{2}{1-\lambda} = -\frac{2}{\lambda - 1}$$

$$\frac{4}{(\lambda-1)^2} + \frac{4}{(\lambda-1)^2} = 9$$

$$(\lambda-1)^2 = 8/9 \Rightarrow \boxed{\lambda = 1 \pm 2\sqrt{2}/3}$$

$$x = \frac{2}{\pm 2\sqrt{2}/3} = \pm\frac{3}{\sqrt{2}}$$

Pontos: $(3/\sqrt{2}, -3/\sqrt{2}), (-3/\sqrt{2}, 3/\sqrt{2})$.

$$y = \frac{2}{\mp 2\sqrt{2}/3} = \mp 3/\sqrt{2}.$$

Pelo método tradicional

$\nabla f = (2x+4, 2y-4)$ e $(x,y)_{\text{crítico}} =$

Pontos da fronteira: $(\pm 3, 0)$ e $(0, \pm 3)$.

Candidatos: $(\pm 3, 0)$, $(0, \pm 3)$, $(-2, 2)$, $(3/\sqrt{2}, -3/\sqrt{2})$ e $(-3/\sqrt{2}, 3/\sqrt{2})$.

$$\boxed{\begin{array}{ll} \text{Máximos:} & f(3/\sqrt{2}, -3/\sqrt{2}) = 9+12\sqrt{2}. \\ \text{mínimos:} & f(-2,2) = -8 \end{array}}$$

Pelo método tradicional, o único ponto possível de ser crítico é $(-2,2)$, mas $(-2,2) \notin \{(x,y) \in \mathbb{R} \mid x^2+y^2 \leq 9\}$.

29 $f(x,y) = e^{-xy}$; $x^2+4y^2 \leq 1.$

$\nabla f = (-ye^{-xy}, -xe^{-xy})$
$\nabla g = (2x, 8y)$

$\Rightarrow \begin{cases} -ye^{-xy} = \lambda \cdot 2x \\ -xe^{-xy} = \lambda \cdot 8y \\ x^2+4y^2 = 1, \quad x,y \neq 0 \end{cases}$

$\dfrac{-\cancel{\lambda}\cdot\cancel{2}x}{y} = \dfrac{-\cancel{\lambda}\cdot \cancel{8}^{4} y}{x} \qquad x^2 = 4y^2$

$8y^2 = 1 \Rightarrow y = \pm 1/2\sqrt{2}$

$y = 1/2\sqrt{2}, \quad x = \pm 1/\sqrt{2}$
$y = -1/2\sqrt{2}, \quad x = \pm 1/\sqrt{2}.$

Pelo método tradicional:
$\nabla f = (-ye^{-xy}, -xe^{-xy})$ e $\Rightarrow x = y = 0$

Pontos da fronteira: $(0, \pm 1/2), (\pm 1, 0)$

candidatos: $(0, \pm 1/2), (\pm 1, 0), (0,0), (1/\sqrt{2}, \pm 1/2\sqrt{2}), (-1/\sqrt{2}, \pm 1/2\sqrt{2})$

Máximos: $f(\pm 1/\sqrt{2}, \mp 1/2\sqrt{2}) = e^{1/4}$
Mínimos: $f(\pm 1/\sqrt{2}, \pm 1/2\sqrt{2}) = e^{-1/4}$

Pelo método tradicional, o único ponto possível de ser crítico é $(0,0)$, mas $(0,0) \notin \{(x,y) \in \mathbb{R} \mid x^2+4y^2 \leq 1\}$

35 ?

37) $P(L,K) = bL^{\alpha} K^{1-\alpha}$ $\quad b, \alpha > 0$ e $\alpha < 1$

Restrição: $mL + nK = p$

$$\nabla P = (\alpha \cdot b \cdot K^{1-\alpha} \cdot L^{\alpha-1}, (1-\alpha) \cdot bL^{\alpha} \cdot K^{-\alpha})$$
$$\nabla g = (m, n)$$

$$\begin{cases} \alpha b K^{1-\alpha} \cdot L^{\alpha-1} = \lambda \cdot m \\ (1-\alpha) b L^{\alpha} \cdot K^{-\alpha} = \lambda \cdot n \\ mL + nK = p. \end{cases}$$

$$\Rightarrow \begin{cases} \alpha b (K/L)^{1-\alpha} = \lambda m \\ (1-\alpha) b (K/L)^{-\alpha} = \lambda n \\ mL + nK = p \end{cases}$$

$$\frac{b}{\lambda} = \frac{m}{\alpha (K/L)^{1-\alpha}} = \frac{n}{(1-\alpha)(K/L)^{-\alpha}}$$

$$\frac{m}{\alpha (K/L) \cdot \cancel{(K/L)^{-\alpha}}} = \frac{n}{(1-\alpha) \cancel{(K/L)^{-\alpha}}}$$

$$\boxed{L \cdot \frac{m(1-\alpha)}{n\alpha} = K}$$ (maximizando utilizando λ).

$$mL + \cancel{n} \cdot \frac{mL(1-\alpha)}{\cancel{n}\alpha} = p$$

$$\cancel{\alpha mL} + mL - \cancel{\alpha mL} = \alpha p$$

$$\boxed{L = \frac{\alpha p}{m}} \; ; \; \boxed{K = \frac{(1-\alpha) p}{n}}$$

(41) $d^2 = (x-2)^2 + y^2 + (z+3)^2$; $x+y+z=1$

$\nabla f = (2(x-2), 2y, 2(z+3))$
$\nabla g = (1, 1, 1)$

$$\Rightarrow \begin{cases} 2(x-2) = \lambda \cdot 1 \\ 2y = \lambda \cdot 1 \\ 2(z+3) = \lambda \cdot 1 \\ x+y+z=1 \end{cases}$$

$2(x+y+z) - 4 + 6 = 3\lambda$
$2 + 2 = 3\lambda$
$\lambda = 4/3$

$\Rightarrow$ $x = 8/3$ $y = 2/3$ $z = -7/3$

$d^2 = (2/3)^2 + (2/3)^2 + (2/3)^2$

$d^2 = 3 \cdot \frac{4}{9} = \frac{4}{3}$ $\boxed{d = 2/\sqrt{3}.}$

(5) $V = xyz$ $\quad 4(x+y+z) = c$

$\nabla V = (yz, xz, xy)$
$\nabla g = (4, 4, 4)$

$$\begin{cases} yz = 4\lambda \\ xz = 4\lambda \\ xy = 4\lambda \\ 4(x+y+z) = c \end{cases}$$

Mas $x = y = z$.
$12x = c$
$\boxed{x = c/12}$.

Cubo de aresta $c/12$.

(6) $f(x,y) = 3x^2 + y^2$ $\quad x^2 + y^2 - 4y = 0$

$\nabla f = (6x, 2y)$
$\nabla g = (2x, 2y-4)$ $\Rightarrow$ $\begin{cases} 6x = \lambda \cdot 2x \\ 2y = \lambda \cdot 2(y-2) \\ x^2 + y^2 - 4y = 0 \end{cases}$

$$\begin{cases} x(3-\lambda) = 0 \\ y - \lambda(y-2) = 0 \\ x^2 + y^2 - 4y = 0 \end{cases}$$

I) $x = 0$: $y = 0$ ou $y = 4$
II) $\lambda = 3$: $y = 3$ $\quad x = \pm\sqrt{3}$

Pontos: $(0,0)$, $(0,4)$, $(\pm\sqrt{3}, 3)$

Máximos: $f(\pm\sqrt{3}, 3) = 18$
Mínimos: $f(0,0) = 0$