LISTA 8

Seção 15.3: 1, 3, 5, 7, 9, 11, 13, 15, 18, 19, 25, 29, 31, 33, 35, 37, 39, 41, 45, 47, 50.

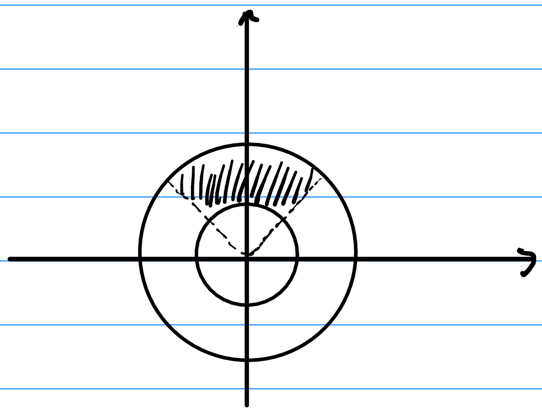
1) Coordenadas polares: $\int_0^{3\pi/2}\int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

3) Coordenadas polares: $\int_0^{\pi}\int_1^3 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

5) Coordenadas retangulares: $\int_0^1\int_{2y-2}^{2-2y} f(x,y)\, dx\, dy$

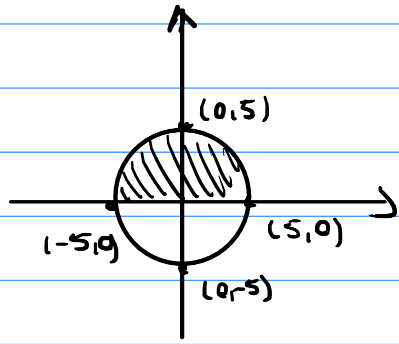
7) $\int_{\pi/4}^{3\pi/4}\int_1^2 r\, dr\, d\theta \Rightarrow (r^2/2)\Big|_1^2 = 2 - 1/2 = 3/2$

$\Rightarrow \frac{3}{2}\theta\Big|_{\pi/4}^{3\pi/4} = \frac{9\pi}{8} - \frac{3\pi}{8} = \boxed{\frac{3\pi}{4}}$



⑨ $$\iint_D x^2 y \, dA = \int_0^{\pi}\int_0^{5} r^4 \cos^2\theta \operatorname{sen}\theta \, dr \, d\theta$$

D:



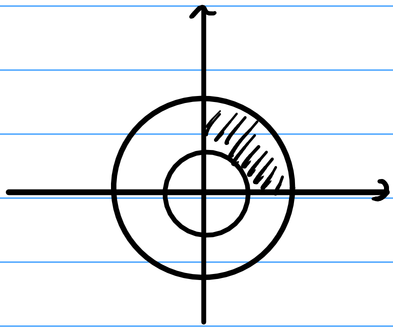
$$\Rightarrow \cos^2\theta \operatorname{sen}\theta \left.\frac{r^5}{5}\right|_0^5 = 625 \cos^2\theta \operatorname{sen}\theta$$

$$\Rightarrow 625 \int_0^{\pi} \cos^2\theta \operatorname{sen}\theta \, d\theta = -625 \left.\frac{\cos^3\theta}{3}\right|_0^{\pi}$$

$$= -625\left(-1/3 - (1/3)\right) = \frac{2 \cdot 625}{3} = \boxed{\frac{1250}{3}}$$ //

⑪ $$\iint_R \operatorname{sen}(x^2+y^2) \, dA$$

D:

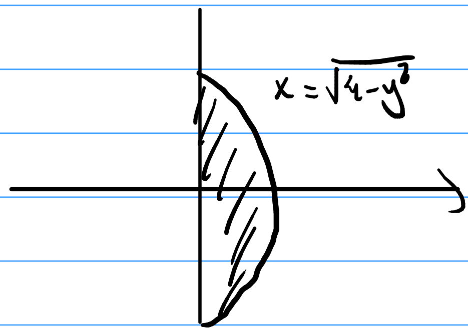


$$= \int_0^{\pi/2}\int_1^{3} r \operatorname{sen} r^2 \, dr \, d\theta$$

$$\Rightarrow -\frac{1}{2} \cos r^2 \Big|_1^3 = \frac{1}{2}(\cos 1 - \cos 9)$$

$$\frac{1}{2}(\cos 1 - \cos 9) \cdot \int_0^{\pi/2} d\theta = \boxed{\frac{\pi}{4}(\cos 1 - \cos 9)}$$ //

(13) $$\iint_D e^{-x^2-y^2}\,dA$$



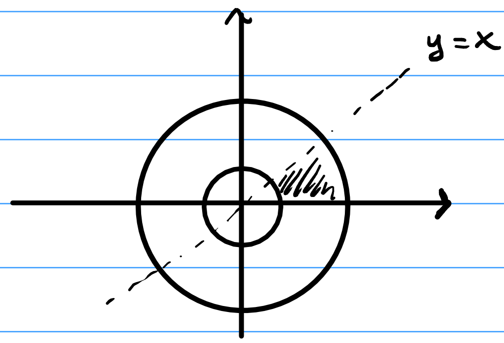
$$= \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\cdot r\,dr\,d\theta$$

$$\Rightarrow -\frac{1}{2}\left(e^{-r^2}\right)\Big|_0^2 = -\frac{1}{2}\left(e^{-4}-1\right) = \frac{1}{2}\left(1-e^{-4}\right).$$

$$\Rightarrow \frac{1}{2}\left(1-e^{-4}\right)\theta\Big|_{-\pi/2}^{\pi/2} = \boxed{\frac{\pi}{2}\left(1-e^{-4}\right)}.$$

(15) $$\iint_R \mathrm{Arctg}(y/x)$$

D:



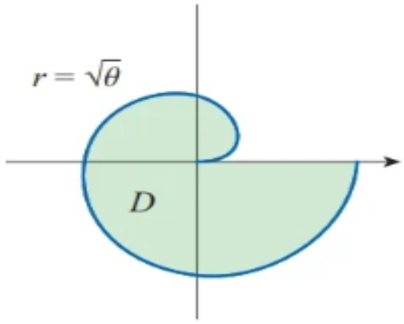
$$\Rightarrow \int_0^{\pi/4}\int_1^2 \mathrm{Arctg}(\mathrm{tg}\,\theta)\,r\,dr\,d\theta$$

$$\alpha = \mathrm{arctg}(\mathrm{tg}\,\theta) \Rightarrow \mathrm{tg}\,\alpha = \mathrm{tg}\,\theta \Rightarrow \text{"}\alpha=\theta\text{"}.$$

$$\Rightarrow \int_0^{\pi/4}\int_1^2 r\theta\,dr\,d\theta \Rightarrow \theta\cdot\frac{r^2}{2}\Big|_1^2 = \frac{\theta}{2}(4-1) = \frac{3\theta}{2}$$

$$\Rightarrow 3/4\cdot\theta^2\Big|_0^{\pi/4} = 3/4\cdot\pi^2/16 = \boxed{3\pi^2/64}$$

(18)

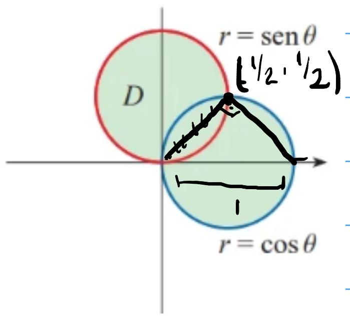


$$\int_0^{2\pi}\int_0^{\sqrt{\theta}} r \, dr\, d\theta \Rightarrow \int_0^{2\pi} \frac{\theta}{2}\, d\theta = \frac{1}{4}\theta^2 \Big|_0^{2\pi} = \boxed{\pi^2}$$

Jeito clássico: $A = \frac{1}{2}\int_0^{2\pi} (\sqrt{\theta})^2 \, d\theta = \pi^2.$

(19)

19.

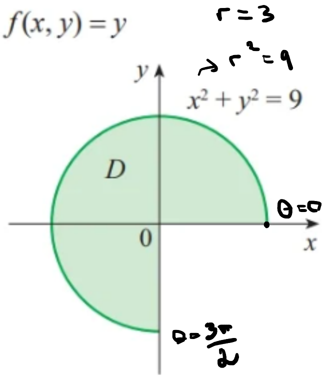


Por geometria:

$$2\cdot\pi\cdot\frac{1}{4} - \left(\pi\cdot\frac{1}{4}\cdot\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{8} + \frac{1}{4} = \frac{3\pi}{8} + \frac{1}{4}$$

(25) **25.** $f(x, y) = y$



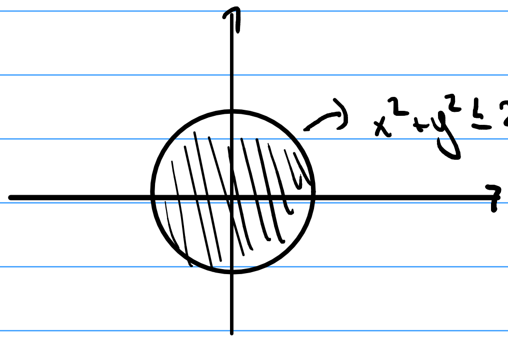
a) $\displaystyle\int_0^{3\pi/2}\int_0^{3} r\,\text{sen}\theta \cdot r\,dr\,d\theta$

b) $\displaystyle\int_0^{3} r^2\,\text{sen}\theta\,dr = \left(\frac{r^3}{3}\,\text{sen}\theta\right)\Big|_0^{3}$

$= 9\,\text{sen}\theta.$

$\displaystyle\rightarrow 9\cdot\int_0^{3\pi/2} \text{sen}\theta\,d\theta = -9\cdot\cos\theta\Big|_0^{3\pi/2} = -9(0-1) = 9$

(29) $z = x^2 + y^2$, $x^2 + y^2 \leq 25$

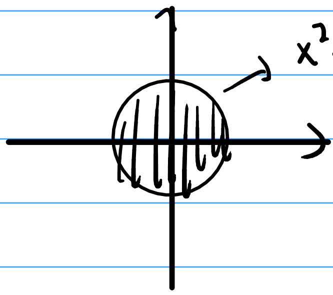


$\rightarrow x^2+y^2 \leq 25 \Rightarrow r \leq 5$

$\displaystyle V = \int_0^{2\pi}\int_0^{5} r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{5^4}{4}\,d\theta = \frac{5^4}{4}\,\theta\Big|_0^{2\pi} = \frac{2\pi\cdot 5^4}{4}$

$\displaystyle = \frac{625\pi}{2}$

(31) $z = 4 - 2x - y \quad , \quad x^2 + y^2 \leq 1$



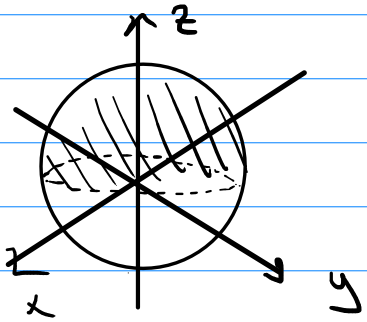
$$V = \int_0^{2\pi}\int_0^1 (4 - 2r\cos\theta - r\,\text{sen}\theta)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (4r - 2r^2\cos\theta - r^2\text{sen}\theta)\, dr\, d\theta$$

$$\Rightarrow \int_0^1 (4r - 2r^2\cos\theta - r^2\text{sen}\theta)\, dr = \left(2r^2 - 2r^3/3 \cdot \cos\theta - r^3/3\,\text{sen}\theta\right)\Big|_0^1$$
$$= 2 - 2/3\cos\theta - 1/3\,\text{sen}\theta.$$

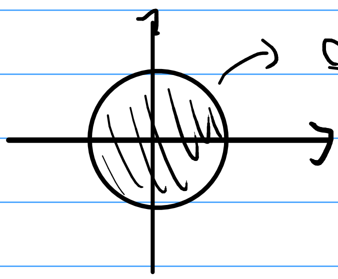
$$\Rightarrow \int_0^{2\pi} (2 - 2/3\cos\theta - 1/3\,\text{sen}\theta)\, d\theta = \left(2\theta - 2/3\,\text{sen}\theta + 1/3\cos\theta\right)\Big|_0^{2\pi}$$

$$= (4\pi - 0 + 1/3) - (0 - 0 + 1/3) = \boxed{4\pi}_{u.}$$

(33) Esfera de raio a:



Projeção em XY:



$z = \pm\sqrt{a^2-x^2-y^2}$ (parte de cima e de baixo são simétricos)

$$\therefore\ V = 2\cdot\int_0^{2\pi}\int_0^{a} \sqrt{a^2-r^2}\; r\,dr\,d\theta = \int_0^{2\pi}\int_0^{a} \sqrt{a^2-r^2}\;2r\,dr\,d\theta$$

$$\Rightarrow -\int_0^{a} -2r\sqrt{a^2-r^2}\;dr\,d\theta = -\left(\frac{2}{3}(a^2-r^2)^{3/2}\right)\Bigg|_0^{a}$$

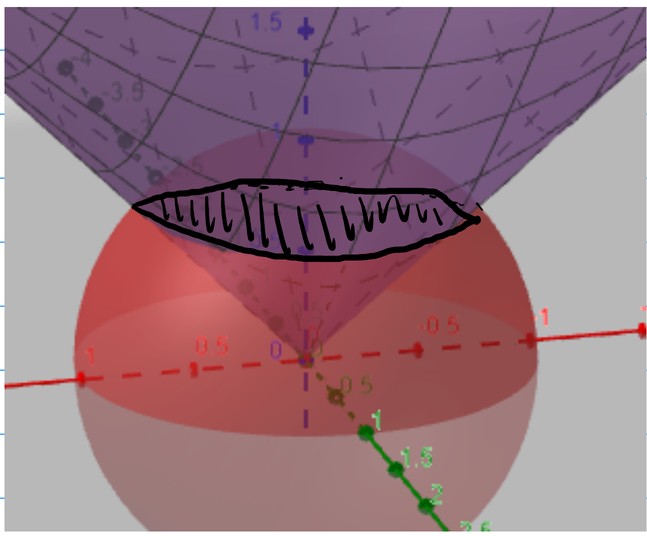
$$= -\frac{2}{3}\left[0 - (a^2)^{3/2}\right] = \frac{2}{3}\cdot a^3$$

$$\Rightarrow \int_0^{2\pi} 2/3\cdot a^3\,d\theta = 2/3\cdot a^3\,(\theta)\Big|_0^{2\pi} = \frac{4}{3}\pi a^3.$$

$z=r$

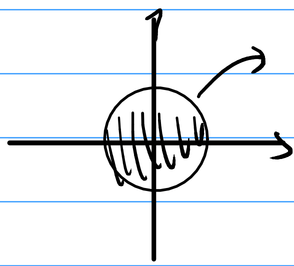
36) acima de $z=\sqrt{x^2+y^2}$ e abaixo de $z^2 = 1-x^2-y^2$.

$z=\sqrt{1-r^2}$



Interseção das curvas:

$x^2+y^2 = 1/2 = (1/\sqrt{2})^2$



$0 \le r \le 1/\sqrt{2}$

$0 \le \theta \le 2\pi$

O volume será acima de $x^2+y^2 \leq 1/2$ e abaixo de $\sqrt{1-x^2-y^2} = z$ (parte de cima):

$$V = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r \, dr \, d\theta$$

$$\int_0^{1/\sqrt{2}} \sqrt{1-r^2}\, r \, dr - \int_0^{1/\sqrt{2}} r^2 \, dr$$

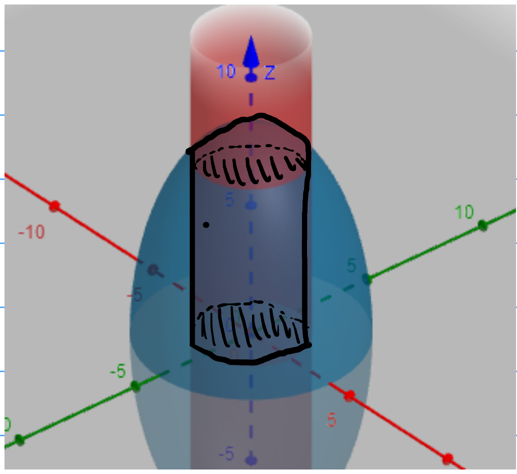
$$= -\frac{1}{\cancel{2}} (1-r^2)^{3/2} \cdot \frac{\cancel{2}}{3} \Bigg|_0^{1/\sqrt{2}} - \frac{r^3}{3} \Bigg|_0^{1/\sqrt{2}}$$

$$= \frac{1}{3}\left(1 - \frac{1}{\sqrt{8}}\right) - \frac{1}{3} \cdot \frac{1}{\sqrt{8}} = \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{\sqrt{8}} = \frac{1}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{3}\left(1 - \frac{\sqrt{2}}{2}\right) d\theta = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{3}(2-\sqrt{2})}$$

37 Dentro de $x^2+y^2=4$ e de $4x^2+4y^2+z^2=64$

$z^2 = 64-4(x^2+y^2)$



$$\text{Volume} = 2\cdot\int_0^{2\pi}\int_0^{2} \sqrt{64-4r^2}\cdot r\,dr\,d\theta$$

$$\Rightarrow -\frac{1}{8}\int_0^2 \sqrt{64-4r^2}\,(-8r)\,dr = -\frac{1}{8}\left(\frac{2}{3}(64-4r^2)^{3/2}\right)\Big|_0^2$$

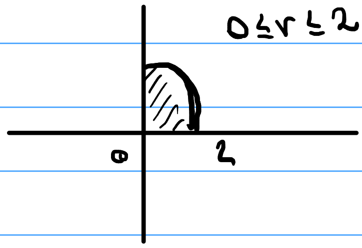
$$= -\frac{1}{12}\left(48^{3/2}-64^{3/2}\right) = \frac{1}{12}\left(64^{3/2}-48^{3/2}\right) = \frac{1}{12}\left(8^3-4^3\cdot 3\sqrt{3}\right)$$

$$\Rightarrow \int_0^{2\pi} 1/12\left(8^3-4^3\cdot 3\sqrt{3}\right)d\theta = 1/6\,\pi\left(512-192\sqrt{3}\right).$$

$$V = 2\cdot\frac{1}{6}\pi\left(512-192\sqrt{3}\right) = \frac{8}{3}\pi\left(64-24\sqrt{3}\right)$$

(39) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$

$0 \le r \le 2, \quad 0 \le \theta \le \pi/2$

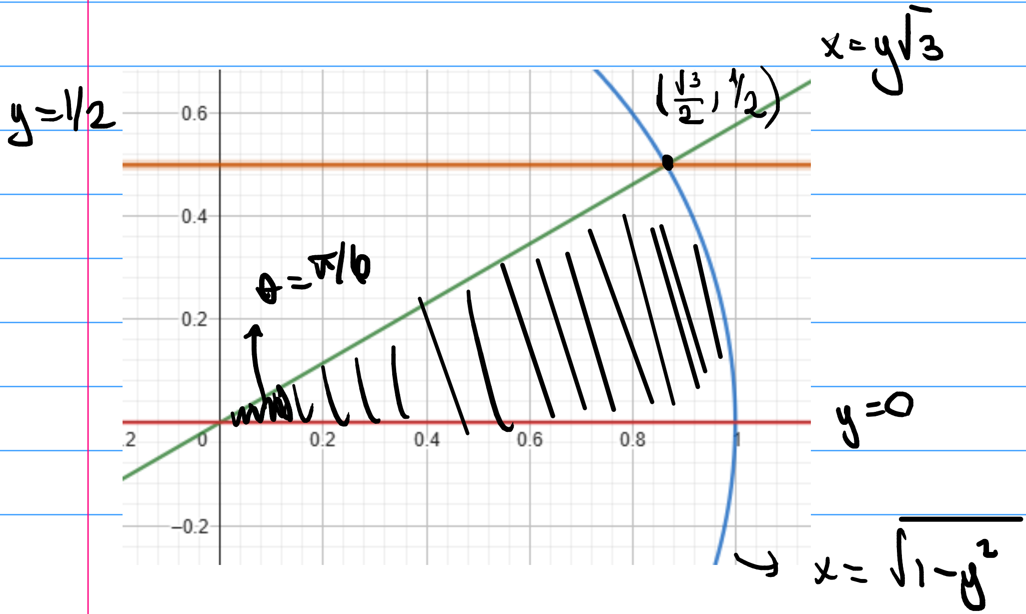


$$\int_0^{\pi/2}\int_0^2 e^{-r^2}\cdot r\,dr\,d\theta \Rightarrow -\frac{1}{2}\int_0^2 e^{-r^2}(-2r)\,dr\,d\theta = -1/2\,\left(e^{-r^2}\right)\Big|_0^2$$

$$= -1/2\left(e^{-4}-1\right) = 1/2\left(1-e^{-4}\right)$$

$$\int_0^{\pi/2} 1/2\left(1-e^{-4}\right)d\theta = \boxed{\pi/4\left(1-e^{-4}\right)}$$

(41) $\displaystyle\int_0^{1/2}\int_{y\sqrt{3}}^{\sqrt{1-y^2}} xy^2\,dx\,dy$



$$\int_0^{\pi/6}\int_0^1 r\cos\theta\, r^2 \sin^2\theta\, r\, dr\, d\theta = \int_0^{\pi/6}\int_0^1 r^4\cos\theta\,\mathrm{sen}^2\theta\, dr\, d\theta$$

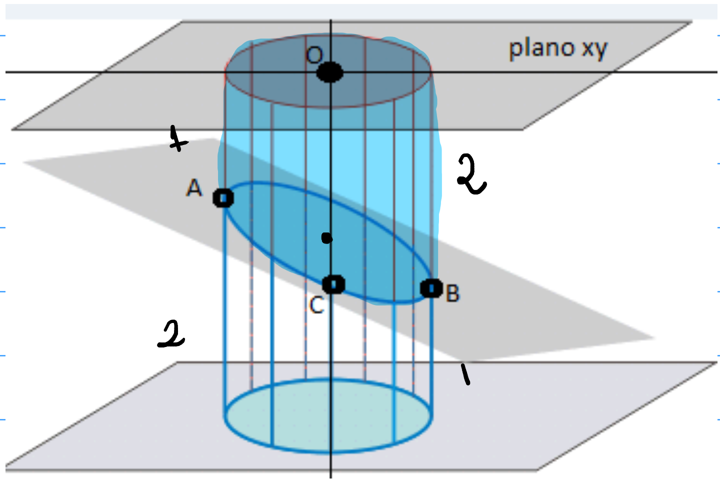
$$\Rightarrow \int_0^1 r^4 \cos\theta\, \mathrm{sen}^2\theta\, dr = \frac{1}{5}\cos\theta\,\mathrm{sen}^2\theta.$$

$$\Rightarrow \int_0^{\pi/6} 1/5 \cos\theta\, \mathrm{sen}^2\theta\, d\theta = \frac{1}{5}\, \frac{\mathrm{sen}^3\theta}{3}\Big|_0^{\pi/6} = \frac{1}{5}\cdot\frac{1}{3}\cdot\frac{1}{8} = \boxed{\frac{1}{120}}$$

(45) Função do crescimento da profundidade

$$z = \frac{1}{2} + \frac{1}{10}y.$$

Desenho esquemático da piscina:



$$V = \int_0^{2\pi}\int_0^5 \left(\frac{1}{2} + \frac{1}{10} r\,\mathrm{sen}\theta\right) r\, dr\, d\theta = 37{,}5\pi.$$

47 $f_{méd} = \dfrac{1}{A(D)} \iint_D f(x,y)\,dx\,dy$

$A(D) = \pi(b-a)(b+a)$

$\int_0^{2\pi}\int_a^b \dfrac{1}{r}\, r\,dr\,d\theta = 2\pi(b-a)$

$f_{méd} = \dfrac{1}{\cancel{\pi}(b-a)(b+a)} \cdot 2\cancel{\pi}\cancel{(b-a)} \boxed{= \dfrac{2}{a+b}}$

50 a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}$

$\Rightarrow \iint_{D_a} e^{-(x^2+y^2)} = \int_0^{2\pi}\int_0^{a} e^{-r^2}\, r\,dr\,d\theta$

$\Rightarrow -1/2 \int_0^a e^{-r^2}(-2r\,dr) = -1/2\,(e^{-r^2})\Big|_0^a = 1/2(1-e^{-a^2})$

$\Rightarrow \int_0^{2\pi} 1/2(1-e^{-a^2})\,d\theta = \pi(1-e^{-a^2}) = \pi\left(1-\dfrac{1}{e^{a^2}}\right)$

$$\lim_{a\to\infty} \pi\left(1-\frac{1}{e^{a^2}}\right) = \pi\,(1-0) = \pi$$

b) $$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA$$

$$= \lim_{a\to\infty} \left[\int_{-a}^{a} e^{-x^2}\, dx \cdot \int_{-a}^{a} e^{-y^2}\, dy\right]$$

↳ Teorema de Fubini, já que S_a é uma região retangular, junto com propriedade: $\int_c^d\int_a^b f(x)g(y)\,dx\,dy = \int_a^b f(x)dx \cdot \int_c^d g(y)\,dy$

e $$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA =$$

$$\lim_{a\to\infty} \left[\int_{-a}^{a} e^{-x^2}\, dx \cdot \int_{-a}^{a} e^{-y^2}\, dy\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\,dy$$

$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)} = \pi$$

c) Como $\int_{-\infty}^{\infty} e^{-x^2} dx = \lim_{a \to \infty} \int_{-a}^{a} e^{-x^2} dX$

e $\int_{-\infty}^{\infty} e^{-y^2} dy = \lim_{a \to \infty} \int_{-a}^{a} e^{-y^2} dx$ variam

nos mesmos intervalos, então

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy \quad e \quad \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi$$

Como $e^{-x^2} > 0 \ \forall x \in \mathbb{R}$, então $\int_{-\infty}^{\infty} e^{-x^2} dx > 0$ e

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

d) $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \lim_{a \to \infty} \int e^{-t^2} dt =$

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2} \frac{dt}{\sqrt{2}} \Rightarrow \sqrt{\pi}$$

$t = \sqrt{2} x$

$dt = \sqrt{2} dx$

$dx = dt / \sqrt{2}$

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi}$$

$$e \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi}.$$