APLICAÇÕES DA INTEGRAL

Prop: Se $f(x) \leq g(x)$ $\forall x \in [a,b]$, f e g integráveis em $[a,b]$, então

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Dem: $S(f) = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$; $S(g) = \sum_{i=0}^{n-1} g(x_i)\,\Delta x$

Como $f(x_i) \leq g(x_i)$, então $f(x_i)\,\Delta x \leq g(x_i)\,\Delta x$ e

$$\sum_{i=0}^{n-1} f(x_i)\,\Delta x \leq \sum_{i=0}^{n-1} g(x_i)\,\Delta x$$

e $\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\,\Delta x \leq \lim_{n\to\infty} \sum_{i=0}^{n-1} g(x_i)\,\Delta x$

$$\boxed{\text{e } \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx}$$

■.

VALOR MÉDIO PARA INTEGRAIS

- $f(x)$ contínua em $[a,b]$.
- M = máx de $f(x)$
- m = mín de $f(x)$

$$\therefore \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M \Rightarrow \begin{cases} \operatorname{Im} f : [m, M] \\ f : [a,b] \to [m, M] \end{cases}$$

Como f é contínua, pelo Teorema do Valor Intermediário, $\exists\, c \in [a,b] \mid f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

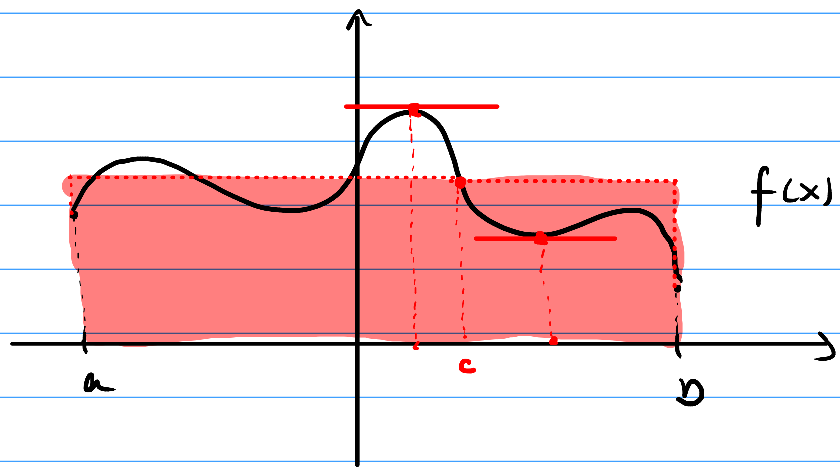
Logo, o valor médio é $\frac{1}{b-a}\int_a^b f(x)\,dx$!

Ex: $f(x) = x^3 \Rightarrow 0 \leq x \leq 1$. $\qquad \int_0^1 x^3\,dx = \frac{1}{4}$.

Encontrando $c \mid f(c)$ é o valor médio de f.

$$f(c)=c^3=\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{1-0}\int_0^1 x^3\,dx=\frac{1}{4}$$

$$\boxed{\therefore\ c=\frac{1}{\sqrt[3]{4}}}\ ///.$$



$$\boxed{\int_a^b f(x)\,dx=f(c)\,(b-a)}\ ///.$$

Ex: Use Integral:

$$S_n=\sum_{i=1}^{n}\frac{i^4}{n^5}$$

$$S=\sum_{i=1}^{n} f(c_i)\,\frac{b-a}{n}\Rightarrow f(c_i)\,\Delta x=\frac{i^4}{n^4}\cdot\frac{1}{n}=\left(\frac{i}{n}\right)^4\cdot\frac{1}{n}$$

$c_i = \frac{i}{n} \Rightarrow c_i \in [x_{i-1}, x_i] = \left[\frac{i-1}{n}, \frac{i}{n}\right]$

$\therefore f(c_i) = \left(\frac{i}{n}\right)^4 \Rightarrow \boxed{f(x) = x^4}$

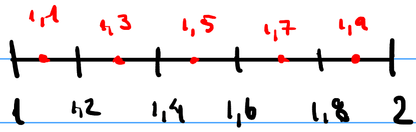
$\therefore S_n = \int_0^1 x^4 \, dx \boxed{= \frac{1}{5}}$

(é soma de Riemann)

Ex: Use integral para aproximar ln2.

$\int_1^2 \frac{1}{x} \, dx = \ln 2 - \ln 1 = \boxed{\ln 2}$

$\int_1^2 \frac{1}{x} \, dx = \sum_{i=1}^{5} f(c_i) \Delta x$

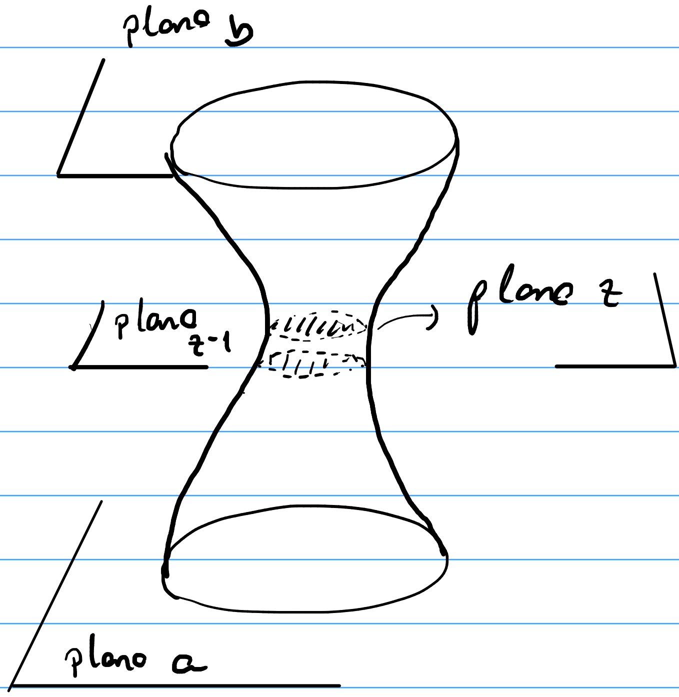


c_i = ponto médio do intervalo $[x_{i-1}, x_i]$.

$S = [f(1,1) + f(1,3) + f(1,5) + f(1,7) + f(1,9)] \cdot 0,2 \boxed{\approx 0,69}$

CÁLCULO DE VOLUMES:

Cilindro: Princípio de Cavalieri.



$a \leq z \leq b$

$\Delta z = \frac{b-a}{n}$

$W = \bigcup_{i=1}^{n} W_i$

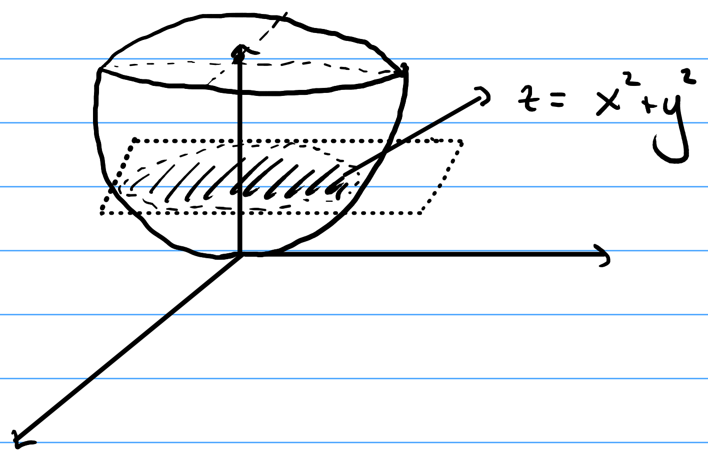
(W_i sólido limitado pelas seções A_{i-1} e A_i)

volume $W_i = A(z_i) . \Delta z$

volume $W = \sum_{i=1}^{n}$ volume $W_i = \sum_{i=1}^{n} A(\hat{z}_i) \Delta z \quad \hat{z}_i \in [z_{i-1}, z_i]$

volume $W = \int_a^b A(z) dz \Rightarrow A(z) =$ área da seção z (contínua)

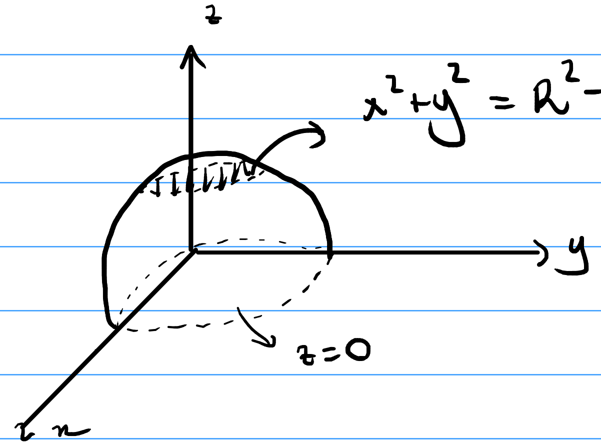
Ex:



círculo de raio $\sqrt{z}$ $\Rightarrow$ $A(z) = \pi z$

Volume = $\displaystyle\int_0^b \pi z \, dz = \boxed{\frac{\pi b^2}{2}}$ //.

Ex: Volume da semi-esfera. $x^2+y^2+z^2=1$, $z \geq 0$



$0 \leq z \leq R$.

$$A(z) = (R^2 - z^2)\pi$$

$$\text{Volume} : \int_0^R (R^2 - z^2)\pi \, dz$$

$$= \pi \left[\int_0^R R^2 \, dz - \int_0^R z^2 \, dz \right]$$

$$\pi \left(R^3 - \frac{R^3}{3} \right) = \boxed{\frac{2}{3}\pi R^3}$$