Hybrid Completeness

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Abstract

In this paper we discuss two *hybrid languages*, $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$, and provide them with complete axiomatizations. Both languages combine features of modal and classical logic. Like modal languages, they contain modal operators and have a Kripke semantics. Unlike modal languages, in these systems it is possible to 'label' states by using $\forall$ and $\downarrow$ to bind special *state variables*.

This paper explores the consequences of hybridization for completeness. As we shall show, the challenge is to blend the modal idea of *canonical models* with the classical idea of *witnessed* maximal consistent sets. The languages $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$ provide us with two extreme examples of the issues involved. In the case of $\mathcal{L}(\forall)$, we can combine these ideas relatively straightforwardly with the aid of analogs of the *Barcan* axioms coupled with a *modal theory of labeling*. In the case of $\mathcal{L}(\downarrow)$, on the other hand, although we can still formulate a theory of labeling, the Barcan analogs are not valid. We show how to overcome this difficulty by using COV^*, an infinite collection of additional rules of proof which has been used in a number of investigations of extended modal logic (see, for example, Passy and Tinchev [12] and Gargov and Goranko [7]).

1 Introduction

Propositional modal languages are simple and attractive formalisms that have been widely applied in computer science and other disciplines. However their very simplicity soon leads to expressivity problems. It is unusual for the basic modal language to be used. Rather, its expressivity is boosted by the addition of various (application dependent) new modalities, such as the universal modality, the *Until* operator, transitive closure operators, counting modalities, and so on. While many of the resulting systems of *extended modal logic* have proved interesting and important (Propositional Dynamic Logic (PDL) is a particularly noteworthy example) some seem rather *ad-hoc* and have proved difficult to axiomatize.

This paper explores the consequences of following a different route to enhanced modal expressivity: *hybridization*. Hybridization is an attempt to combine the key ideas of modal syntax and semantics with direct quantification over states. That is, hybrid languages retain the modal operators and Kripke semantics typical of modal logic. In addition, however, they contain variables over states and various (essentially classical) mechanisms for binding them. Hybridization has not been widely investigated. In fact, as far as we are aware, Bull [5], Passy and Tinchev [12], Goranko [8, 9, 10], Blackburn and Seligman [2, 3] and Seligman [15, 16] is a fairly exhaustive list of technical papers on the subject. Most of this work has dealt with very strong hybrid languages, namely hybrid languages enriched with the *universal modality* (dis-

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cussed below). For example Bull [5], Goranko [9, 10] and Passy and Tinchev [11, 12] discuss hybridizations of temporal logic, modal logic and PDL, and prove a number of completeness results, but only for hybrid languages containing the universal modality. The proof theoretical investigations of Seligman [15, 16] follow a rather different path; nonetheless, Seligman considers only systems in which the universal modality is definable. Hybrid languages *not* containing the universal modality are discussed in Blackburn and Seligman [2, 3], however neither paper addresses the issue of completeness. The present paper is an attempt to fill the gap: how should we go about proving completeness results for hybrid languages when we don't have the universal modality at our disposal?

We shall examine two hybrid extensions of the basic modal language, $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$, and provide them both with complete axiomatizations. In $\mathcal{L}(\forall)$ we will be able to build formulae such as the following:

$$\exists x(\Diamond(x \wedge \varphi) \wedge \Box(\Diamond x \rightarrow \psi)).$$

Here x is a *state variable* — a special sort of *formula* — and $\exists x$ should be read as "there is a state x". In $\mathcal{L}(\downarrow)$ we will be able to construct formulae such as:

$$\downarrow x(x \rightarrow \neg\Diamond x).$$

Here $\downarrow x$ should be read as "bind x to the current state", or "label the current state with x". As these examples suggest, hybrid languages have a rather novel syntax and semantics, and these are discussed in detail below.

What is involved in proving hybrid completeness theorems? Since hybrid extensions of modal languages possess features of both modal and classical logic, the basic challenge is to find ways of combining the modal idea of *canonical models* with the classical idea of *witnessed* maximal consistent sets. The languages $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$ provide us with two extreme examples of the issues involved. In the case of $\mathcal{L}(\forall)$ we combine these ideas with the aid of hybrid analogs of the famous *Barcan* axioms coupled with a *modal theory of labeling*. In the case of $\mathcal{L}(\downarrow)$, on the other hand, although we can still formulate a theory of labeling, the Barcan analogs are not valid, and there are no obvious substitutes. We show how this difficulty may be overcome by making use of COV^*, an infinite collection of additional rules of proof which has been used by the Sofia school in a number of investigations of extended modal logic (see, for example, Passy and Tinchev [12] and Gargov and Goranko [7]).

2 Two hybrid languages

We begin by reviewing the syntax and semantics of propositional modal logic. Given a denumerably infinite set PROP $= \{p, q, r, \ldots\}$ of *propositional symbols*, the well-formed formulae of propositional modal logic are defined as follows:

$$\text{WFF } \varphi \ := \ p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi.$$

The following notation is then introduced for the dual of the $\Box$ operator: $\Diamond\varphi := \neg\Box\neg\varphi$. Other Boolean operators (such as $\rightarrow$, $\vee$, $\top$, and $\bot$) are defined in the expected way.

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The usual semantics of propositional modal logic is Kripke semantics. Kripke semantics is a three-place relation $\models$ that can hold between a model, a state in that model, and a formula. A *model* $\mathcal{M}$ is a triple (S, R, V) such that S is a non-empty set of *states*, R is a binary relation on S (the *transition relation*), and $V : \text{PROP} \longrightarrow Pow(S)$, is the *valuation*, which tells us at which states (if any) each propositional symbol is true. The pair (S, R) is called the *frame* underlying the model.

The $\models$ relation is defined as follows. Let $\mathcal{M} = (S, R, V)$ and $s \in S$. Then:

$$\begin{array}{lll} \mathcal{M}, s \models p & \text{iff} & s \in V(p), \text{ where } p \in \text{PROP} \\ \mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi \\ \mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \ \&\ \mathcal{M}, s \models \psi \\ \mathcal{M}, s \models \Box\varphi & \text{iff} & \forall s'(sRs' \ \Rightarrow\ \mathcal{M}, s' \models \varphi). \end{array}$$

If $\mathcal{M}, s \models \varphi$ we say that φ is *satisfied* in $\mathcal{M}$ at s. The key intuition to note about Kripke semantics is its locality: formulae are evaluated in models at some particular state (called the *current state*), and the function of the $\Box$ operator is to scan the states accessible from the current state via the transition relation R. Note that

$$\mathcal{M}, s \models \Diamond\varphi \ \text{ iff } \ \exists s'(sRs' \ \&\ \mathcal{M}, s' \models \varphi).$$

We shall now hybridize propositional modal logic. The basic idea is to allow ourselves to quantify across states (in various ways) while staying as close to the syntax and semantics of the modal language as possible.

2.1 Hybrid syntax

We hybridize modal syntax by making two changes. The first is to *sort* the atomic symbols of the modal language: instead of having just one kind of atomic symbol (namely the symbols in PROP) we shall add two other kinds of atomic symbol: *state variables* and *nominals*. The second change is to add *binders*. Binders will be used to bind state variables (but not nominals or propositional symbols). In this paper, two different binders will be considered, namely $\forall$ and $\downarrow$.

Let's make these ideas precise. Assume we have a denumerably infinite set $\text{SVAR} = \{x, y, z, \ldots\}$, and a finite or denumerably infinite set NOM (if this set is not empty we typically write its elements as $i, j, k, \ldots$, and so on). Assume that SVAR, NOM and PROP are pairwise disjoint. We call SVAR the set of *state variables*, NOM the set of *nominals*, $\text{SVAR} \cup \text{NOM}$ the set of *state symbols*, and $\text{SVAR} \cup \text{NOM} \cup \text{PROP}$ the set of *atoms*. Both state variables and nominals (that is, both types of state symbol) will be used to 'label' states. The difference is simply that whereas state variables can be bound by binders, nominals cannot. In effect, nominals are like the 'parameters' used in proof theory: when we use them to make substitutions, we don't have to worry about accidental binding.

Let $B \in \{\forall, \downarrow\}$. We build the well-formed formulae of $\mathcal{L}(B)$, the hybrid language in B (over SVAR, NOM and PROP) as follows:

$$\text{WFF } \varphi \ := \ a \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid Bx\varphi.$$

(Here $a \in \text{ATOM}$ and $x \in \text{SVAR}$.) We define $\Diamond$ and other Booleans in the usual way. When working in a language $\mathcal{L}(\forall)$ (over some choice of SVAR, NOM and PROP)

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we define $\exists x\varphi := \neg\forall x\neg\varphi$. In $\mathcal{L}(\downarrow)$ no such definition is needed, for this binder will be self-dual. In what follows, we generally assume that some choice of SVAR, NOM and PROP has been fixed, and when we speak of hybrid languages, we mean the two languages $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$ defined over these sets. Sometimes, however, we will need to be more explicit about which nominals we have at our disposal. In particular, when we prove the completeness results we will need to expand our languages with a denumerably infinite set of new nominals (they will play a role analogous to the Henkin constants used in first-order completeness proofs).

Note that the syntactic definition of the language $\mathcal{L}(B)$ treats *all* atoms — whether state variables, nominals, or ordinary propositional symbols — as *formulae*. That is, although state symbols will allow us to 'label' or 'name' states, we can combine them with arbitrary formulae using the Boolean and modal operators, and when we do this we construct new formulae. For example, the following is a well-formed formula of $\mathcal{L}(\forall)$:

$$\Diamond(x \wedge p \wedge \Diamond(i \wedge q)) \wedge \forall x(x \rightarrow \neg\Diamond x).$$

In view of this, it should be clear that although we have introduced some sort of quantification over states, we have distorted the syntax of propositional modal logic as little as possible: the entities we bind are formulae, that is, the type of entity used in propositional modal languages. As a result of this, hybrid languages work in a rather novel way. Although the semantics will ensure that state variables and nominals perform the kind of labeling tasks carried out by terms in first-order languages, they are *not* segregated from the rest of the language (as terms are in first-order languages) but can be freely mixed with propositional information.

We need to draw a distinction between *free* and *bound* state variables and to perform *substitutions*. The intuition behind these is essentially classical. For example, in the above formula, the first occurrence of x is free and the last three are bound. We first define what it means for an occurrence of a state variable x to be free in a formula φ:

1. If $\varphi \in$ ATOM, then φ is a free occurrence of x iff $\varphi = x$.
2. An occurrence of x is free in $\neg\varphi$ or $\Box\varphi$ iff it is free in φ, and an occurrence of x is free in $\varphi \wedge \psi$ iff it is free in φ or in ψ.
3. An occurrence of x is free in $By\varphi$ iff it is free in φ and $x \neq y$. (Here $B \in \{\forall, \downarrow\}$.)

An occurrence of a state variable that is not free is called *bound*. The set of free state variables in a formula φ is the set of state variables that have at least one free occurrence in φ. A formula that contains no free state variables is called a *sentence*.

Let φ be a formula, $\mathbf{s}$ be a state symbol, and x a state variable. Then $\varphi[\mathbf{s}/x]$, the formula obtained by substituting $\mathbf{s}$ for all free occurrences of x in φ, is defined as follows:

1. If $\varphi \in$ ATOM, then $\varphi[\mathbf{s}/x]$ is $\mathbf{s}$ if $\varphi = x$, and φ otherwise.
2. $(\neg\varphi)[\mathbf{s}/x]$, $(\Box\varphi)[\mathbf{s}/x]$, and $(\varphi\wedge\psi)[\mathbf{s}/x]$ are defined to be $\neg(\varphi[\mathbf{s}/x])$, $\Box(\varphi[\mathbf{s}/x])$, and $\varphi[\mathbf{s}/x] \wedge \psi[\mathbf{s}/x]$ respectively.
3. $(By\varphi)[\mathbf{s}/x]$ is $By(\varphi[\mathbf{s}/x])$ if $x \neq y$, and $By\varphi$ otherwise.

As in classical logic, when we make substitutions for logical purposes we have to guard against accidental binding. That is, we need a definition of when a state symbol is *substitutable* for a state variable. Nominals, of course, are always substitutable, for

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as they cannot be bound, they cannot be accidentally bound. What about state variables? Let x and z be state variables and define:

1. If $\varphi \in \mathrm{ATOM}$, then z is substitutable for x in φ.
2. z is substitutable for x in $\neg\varphi$ or $\Box\varphi$ iff z is substitutable for x in φ, and z is substitutable for x in $\varphi \wedge \psi$ iff z is substitutable for x in both φ and ψ.
3. z is substitutable for x in $By\varphi$ iff x does not occur free in φ, or $y \neq z$ and z is substitutable for x in φ.

2.2 Hybrid semantics

The basic idea underlying the semantics is straightforward. We want nominals to be formulae that 'name' states, and state variables to be formulae that act as 'variables ranging across states'. To achieve this we need merely stipulate that both state variables and nominals are interpreted by *singleton* subsets of models. That is, any state variable, and any nominal, will be satisfied at *exactly one* state in any model. Such formulae 'label' the unique state that satisfies them.

One other thing needs doing. As we wish to bind state variables (but not nominals or propositional symbols) we should be careful how we handle their interpretation. But there is a standard way of dealing with such issues: make use of the Tarskian idea of assignment functions. That is, while we will use valuations to handle the semantics of propositional symbols and nominals, we will handle the semantics of state variables separately, via assignment functions. This motivates the following definition.

Definition 2.1 *Let $\mathcal{L}(B)$ be a hybrid language over* PROP*,* NOM *and* SVAR *(where $B \in \{\forall, \downarrow\}$). A model $\mathcal{M}$ for $\mathcal{L}(B)$ is a triple (S, R, V) such that S is a non-empty set, R a binary relation on S, and $V : \mathrm{PROP} \cup \mathrm{NOM} \longrightarrow Pow(S)$. A valuation V is called* standard *iff for all nominals $i \in \mathrm{NOM}$, $V(i)$ is a singleton subset of S. A model $\mathcal{M}$ is called* standard *iff its valuation is standard. (That is: standard models treat nominals as labels.)*

An assignment for $\mathcal{L}(B)$ on $\mathcal{M}$ (an $\mathcal{M}$-assignment) is a mapping $g : \mathrm{SVAR} \longrightarrow Pow(S)$. An assignment is called standard *iff for all state variables $x \in \mathrm{SVAR}$, $g(x)$ is a singleton subset of S. (That is:* standard *assignments treat state variables as labels.)*

Now for the satisfaction definition. Obviously we should relativize the Kripke satisfaction definition to standard assignments (that is, we must turn $\models$ into a four-place relation). So, let $\mathcal{M} = (S, R, V)$ be a standard model, and g a standard assignment. For any atom a, let $[V, g](a) = g(a)$ if a is a state variable, and $V(a)$ otherwise. Then, for the binder-free fragment of our languages we have the following clauses:[1]

$$\begin{array}{lll}
\mathcal{M}, g, s \models a & \text{iff} & s \in [V, g](a), \text{ where } a \in \mathrm{ATOM} \\
\mathcal{M}, g, s \models \neg\varphi & \text{iff} & M, g, s \not\models \varphi \\
\mathcal{M}, g, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, g, s \models \varphi \ \&\ \mathcal{M}, g, s \models \psi \\
\mathcal{M}, g, s \models \Box\varphi & \text{iff} & \forall s'(sRs' \ \Rightarrow\ \mathcal{M}, g, s' \models \varphi).
\end{array}$$

Now for the binders. Here is the clause for $\forall$:

[1] Incidentally, modal languages enriched with state symbols but without binders have been investigated; see Gargov and Goranko [7] and Blackburn [1].

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$$\mathcal{M}, g, s \models \forall x\varphi \text{ iff } \forall g'(g' \overset{x}{\sim} g \Rightarrow \mathcal{M}, g', s \models \varphi).$$

The notation $g' \overset{x}{\sim} g$ (we say "g' is an x-variant of g") means that g' is a standard assignment (on $\mathcal{M}$) that agrees with g on all arguments save possibly x. That is, $\forall$ is essentially the classical universal quantifier in a modal setting. Note that it follows that the dual binder $\exists$ receives the expected interpretation, namely:

$$\mathcal{M}, g, s \models \exists x\varphi \text{ iff } \exists g'(g' \overset{x}{\sim} g \;\&\; \mathcal{M}, g', s \models \varphi).$$

Next, the clause for $\downarrow$:

$$\mathcal{M}, g, s \models \downarrow x\varphi \quad \text{iff} \quad \mathcal{M}, g', s \models \varphi, \text{ where } g' \overset{x}{\sim} g, \text{ and } g'(x) = \{s\}.$$

That is, $\downarrow$ binds state variables to the current state; it creates a label for the here-and-now. Given the importance of the current state to Kripke semantics, this is a natural choice of binder. Note that $\downarrow$ is self-dual. That is, $\mathcal{M}, g, s \models \downarrow x\varphi$ iff $\mathcal{M}, g, s \models \neg\downarrow x\neg\varphi$.

Let φ be any formula of $\mathcal{L}(B)$. If $\mathcal{M}, g, s \models \varphi$ then we say φ is *satisfied* in $\mathcal{M}$ at s under g. Note that, as in classical logic, whether or not a *sentence* is satisfied is independent of the choice of assignment. That is, if φ is a sentence, then there is an assignment g such that $\mathcal{M}, g, s \models \varphi$ iff for *every* assignment g, $\mathcal{M}, g, s \models \varphi$. A formula is *valid on a frame* (S, R) iff for all standard models $\mathcal{M} = (S, R, V)$ (that is, all standard models that have (S, R) as their underlying frame), all standard assignments g on $\mathcal{M}$, and all states $s \in S$, $\mathcal{M}, g, s \models \varphi$. A formula is *valid* iff it is valid on all frames.

To close this section, some historical remarks. The earliest discussions of $\forall$ (indeed, the earliest discussion of hybrid languages we know of) seem to be those of Prior [13] Chapter V.6 and and Bull [5]. These papers deal with tense logic enriched with the hybrid binder $\forall$ and the *universal modality* A.[2] However, this work seems to have lain dormant for about 15 years until Passy and Tinchev [11] introduced $\forall$ and A into propositional dynamic logic. (They remark that the idea of $\forall$ was suggested to them by Skordev, who in turn was inspired by certain investigations in recursion theory.) Passy and Tinchev [12] is an excellent overview of this line of work and its connections with extended modal logic. However, in spite of Passy and Tinchev's skillful defense of the importance of 'labels' to modal logic, the idea does not seem to have caught on. More recently, Seligman [15, 16] has proved a cut-elimination result for a system expressively equivalent to $\mathcal{L}(\forall)$ enriched by the universal modality.

The $\downarrow$ binder seems to have been independently invented on even more occasions than $\forall$. For example, Richards et. al. [14] introduce $\downarrow$ as part of an investigation into temporal semantics and temporal databases, Sellink [17] uses it to aid reasoning about automata, and Cresswell [6] uses it as part of his investigation of indexicality in natural language. Nonetheless, none of these systems is, strictly speaking, a hybrid language: they don't treat state variables as formulae. The earliest use of $\downarrow$ in a genuine hybrid language seems to be Goranko [8]. Other papers investigating $\downarrow$ in hybrid settings include Blackburn and Seligman [2, 3] (the first of these papers contains an example showing the failure of the finite model property for $\downarrow$, and proves undecidability) and Goranko [9, 10].

[2] The universal modality has as satisfaction definition $\mathcal{M}, s \models A\varphi$ iff for all states s' in $\mathcal{M}$, $\mathcal{M}, s' \models \varphi$. The consequences of adding the universal modality to hybrid languages are discussed below.

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2.3 Remarks on expressivity

Before turning to completeness, it will be helpful to explore the expressivity of $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$; for a more detailed discussion, see Blackburn and Seligman [2, 3].

First, both $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$ are more expressive than propositional modal logic. For example, it is well-known that no formula of propositional modal logic is valid on precisely those frames with an irreflexive transition relation. However there is a sentence of $\mathcal{L}(\downarrow)$ with this property, namely:

$$\downarrow x \neg \Diamond x.$$

Similarly, it is well-known that the *Until* operator is not definable in propositional modal logic. However, it is definable in $\mathcal{L}(\forall)$:

$$\mathit{Until}\,(\varphi, \psi) \;:=\; \exists x(\Diamond(x \wedge \varphi) \wedge \Box(\Diamond x \rightarrow \psi)).$$

Second, note that $\mathcal{L}(\forall)$ is strictly more expressive than $\mathcal{L}(\downarrow)$. To see this, note that we can define $\downarrow$ in $\mathcal{L}(\forall)$ by

$$\downarrow x \varphi \;:=\; \exists x(x \wedge \varphi).$$

Hence $\mathcal{L}(\forall)$ is at least as expressive as $\mathcal{L}(\downarrow)$. However, no sentence of $\mathcal{L}(\downarrow)$ defines $\forall$. To see this, note that sentences of $\mathcal{L}(\downarrow)$ are preserved under the formation of *generated submodels*.[3] That is, if φ is a sentence of $\mathcal{L}(\downarrow)$ and $\mathcal{M}, s \models \varphi$, then $\mathcal{M}^s, s \models \varphi$, where $\mathcal{M}^s$ is the submodel of $\mathcal{M}$ generated by s. This can be proved by induction on the structure of φ. The key point to note is that occurrences of $\downarrow$ in φ must bind state variables to *local* states (that is, to states in $\mathcal{M}^s$). In short, like propositional modal logic, $\mathcal{L}(\downarrow)$ is a truly local language.

On the other hand, state variable binding in $\mathcal{L}(\forall)$ is not local in this sense: a formula of the form $\exists x \varphi$ may well be true precisely because it is possible to bind x to a state *outside* the submodel generated by the current state. And indeed, it is easy to find sentences of $\mathcal{L}(\forall)$ that are *not* preserved under the formation of generated submodels.[4] It follows that no sentence of $\mathcal{L}(\downarrow)$ can define $\forall$.

Third, like propositional modal logic, both $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$ can be regarded as fragments of classical logic. To see this, note that we can extend the *standard translation* of propositional modal logic into the corresponding first-order language to both hybrid languages. Recall that the first-order language corresponding to a propositional modal language contains a binary relation symbol R, a denumerably infinite collection of one-place symbols P, Q, R, and so on (these correspond to the elements p, q, r of PROP) and a denumerably infinite collection of first-order variables. Any model $\mathcal{M} = (S, R, V)$ can be regarded as first-order model for the correspondence language: the relation R interprets the symbol R, and for all $p \in$ PROP, the subset $V(p)$ interprets the unary predicate symbol P. The standard translation for propositional

[3] Given a model $\mathcal{M} = (S, R, V)$ and a state s of S, the submodel of $\mathcal{M}$ generated by s is the smallest submodel of $\mathcal{M}$ that contains s and is R-closed. That is, the submodel generated by s contains exactly those states of $\mathcal{M}$ that are accessible from s by a finite number of transitions along R.

[4] For example, consider the sentence $\exists x \neg \Diamond x$. Let $\mathcal{M}$ be a model containing two states, s and s', such that s is reflexive, and neither s nor s' is related to the other. Then $\mathcal{M}, s \models \exists x \neg \Diamond x$ (because we can bind x to s') but clearly $\mathcal{M}^s, s \not\models \exists x \neg \Diamond x$ (because s' does not belong to $\mathcal{M}^s$ and we are forced to bind x to the reflexive state s).

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modal logic into this language is defined as follows:

$$\begin{array}{lcl} ST_x(p) & = & Px \\ ST_x(\neg\varphi) & = & \neg ST_x(\varphi) \\ ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi) \\ ST_x(\Box\varphi) & = & \forall y(xRy \rightarrow ST_y(\varphi)), \text{ where } y \text{ is a fresh variable.} \end{array}$$

Note that for any modal formula φ, $ST_x(\varphi)$ is a formula of the correspondence language containing exactly one free variable, namely x. It is clear that $\mathcal{M}, s \models \varphi$ iff $\mathcal{M} \models ST_x(\varphi)[s]$. (Here $\mathcal{M} \models ST_x(\varphi)[s]$ means that the model $\mathcal{M}$ satisfies the first-order formula $ST_x(\varphi)$ when s is assigned as the denotation of its single free variable x.)

To extend this translation to cover hybrid languages, we need merely add the nominals in NOM as constants to the correspondence language and define:

$$\begin{array}{lcl} ST_x(y) & = & x = y, \text{ for all state variables } y \\ ST_x(i) & = & x = i, \text{ for all nominals } i \\ ST_x(\forall y\varphi) & = & \forall y ST_x(\varphi) \\ ST_x(\downarrow y\varphi) & = & \exists y(x = y \wedge ST_x(\varphi)) \end{array}$$

(Note that the first clause implicitly assumes we are using the same set of symbols for state variables and first-order variables. This is to avoid needless notational clutter.) Using this extended translation we can translate every sentence from our hybrid languages to an equivalent one-free-variable formula of the (NOM enriched) correspondence language. (Incidentally, if we wanted to, we could translate nominals as free variables rather than enriching the correspondence language with constants.) So neither $\mathcal{L}(\forall)$ nor $\mathcal{L}(\downarrow)$ is stronger than the correspondence language.

In fact, both hybrid languages are strictly *weaker* than the correspondence language. In particular, $\mathcal{L}(\forall)$ is *not* strong enough to capture the one-free-variable fragment of the correspondence language, as is shown in Blackburn and Seligman [2]. Nonetheless it is easy to gain full first-order expressivity: *simply add the universal modality*.

As we mentioned earlier, the universal modality A has the following satisfaction definition: $\mathcal{M}, s \models A\varphi$ iff for all states s' in $\mathcal{M}$, $\mathcal{M}, s' \models \varphi$. That is, $A\varphi$ means 'φ holds at all states'. Define $E\varphi := \neg A\neg\varphi$, and note that $E\varphi$ means that 'φ holds at some state'. Now, the key point to observe is that the universal modality gives us the power to inspect non-local states. In particular, note that $E(x \wedge \varphi)$ is essentially a 'test' which examines the state labeled by x and checks whether φ holds there. With this observed, it is easy to define a *hybrid translation* from the correspondence language into $\mathcal{L}(\forall) + A$.[5] Let $\mathcal{L}_0^x$ be the set of formulae of the correspondence language in which x is the only free variable, and x does not occur bound. Then (again assuming that the state variables in $\mathcal{L}(\forall) + A$ are identical with the first-order variables in $\mathcal{L}_0^x$) we

[5] The following translation dates back (at least) to Prior [13] (see in particular Chapter 5 and Appendix B). It extends to first-order languages of arbitrary signature; see Blackburn and Seligman [2] for further discussion.

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translate $\mathcal{L}_0^x$ into $\mathcal{L}(\forall) + A$ as follows:

$$\begin{array}{lcl} HT(y = z) & = & \downarrow xE(y \wedge z) \\ HT(Py) & = & \downarrow xE(y \wedge p) \\ HT(yRz) & = & \downarrow xE(y \wedge \Diamond z) \\ HT(\neg\varphi) & = & \neg HT(\varphi) \\ HT(\varphi \wedge \psi) & = & HT(\varphi) \wedge HT(\psi) \\ HT(\forall y\varphi) & = & \forall y HT(\varphi) \end{array}$$

Note that in the cases when either y or z is the special variable x, the hybrid translation produces formulae which are logically equivalent to much simpler formulae. For example, $HT(Px)$ is $\downarrow xE(x \wedge p)$, which is equivalent to p.

Indeed, adding the universal modality even to $\mathcal{L}(\downarrow)$ yields a language expressively equivalent to the correspondence language. To see this, it suffices to note that in such a language we can define

$$\forall x\varphi \ := \ \downarrow yA\downarrow xA(y \to \varphi), \quad \text{where } y \text{ is a variable not occurring in } \varphi$$

To sum up: both $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$ are genuine expressive extensions of propositional modal logic. $\mathcal{L}(\downarrow)$ is the weaker of the two, and retains more of the locality properties of the underlying modal language. On the other hand, while $\mathcal{L}(\forall)$ has obvious non-local properties, it is not a notational variant of the correspondence language; it is strictly weaker. Finally, boosting either $\mathcal{L}(\forall)$ or $\mathcal{L}(\downarrow)$ to full first-order strength is straightforward: just add A.

As we mentioned in the introduction, the only completeness results for hybrid languages we know of (in particular, Bull [5], Passy and Tinchev [12], Goranko [9, 10]) are for hybrid languages containing the universal modality. But what are we to do when we don't have access to A? This question dominates the rest of the paper.

3 The hybrid logic of ∀

Given any countable language $\mathcal{L}(\forall)$, we now axiomatize the set of valid $\mathcal{L}(\forall)$-formulae. The logic will be an extension of the usual axiomatization of the minimal modal logic K. In what follows, $\mathbf{v}$ is used as a metavariable over state variables, $\mathbf{s}$ as a metavariable over state symbols, and $\Diamond^k$ and $\Box^k$ denote k-length sequences of $\Diamond$s and $\Box$s respectively.

$\mathcal{H}(\forall)$, the *hybrid logic of* $\forall$, is defined to be the smallest set of $\mathcal{L}(\forall)$-formulae that is closed under the following conditions. First, it must contain the minimal modal logic K. That is, it contains all instances of *propositional tautologies*, all instances of the *distribution schema* $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and is closed under *modus ponens* (if $\{\varphi, \varphi \to \psi\} \subseteq \mathcal{H}(\forall)$ then $\psi \in \mathcal{H}(\forall)$) and *necessitation* (if $\varphi \in \mathcal{H}(\forall)$ then $\Box\varphi \in \mathcal{H}(\forall)$). In addition, it contains all instances of the five axiom schemas listed below and is closed under *generalization* (if $\varphi \in \mathcal{H}(\forall)$ then $\forall\mathbf{v}\varphi \in \mathcal{H}(\forall)$).

Here are the required axiom schemas:

Q1: $\forall\mathbf{v}(\varphi \to \psi) \to (\varphi \to \forall\mathbf{v}\psi)$, where φ contains no free occurrences of $\mathbf{v}$

Q2: $\forall\mathbf{v}\varphi \to \varphi[\mathbf{s}/\mathbf{v}]$, where $\mathbf{s}$ is substitutable for $\mathbf{v}$ in φ

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Name: $\exists \mathbf{v}\mathbf{v}$

Nom: $\forall \mathbf{v}[\Diamond^m(\mathbf{v} \wedge \varphi) \rightarrow \Box^n(\mathbf{v} \rightarrow \varphi)]$, for all $m, n \in \omega$

Barcan: $\forall \mathbf{v} \Box \varphi \rightarrow \Box \forall \mathbf{v} \varphi$

Q1 and *Q2* should be familiar. They are standard axiom schemas governing the universal quantifier $\forall$ found in first-order languages, and apply just as well to the hybrid universal quantifier. *Name* and *Nom* are probably unfamiliar. *Name* reflects the fact that it is always possible to bind state variable to the current state, while *Nom* reflects the fact that state variables are true at exactly one state. Together, *Name* and *Nom* are our *modal theory of labeling*. (Another way of thinking about them is to note that the theory of labeling they embody is analogous to something familiar from classical logic: the *theory of equality*.) Last, but certainly not least, we have analogs of the *Barcan* axioms, familiar from first-order modal logic. One important comment must be made here. In first-order modal logic, the status of Barcan is open to debate. This is because the quantifiers in first-order modal logic range over the points in some underlying collection of first-order models, and whether or not Barcan is valid depends on what assumptions we make about this collection. In hybrid languages, however, $\forall$ *ranges over the states themselves*. As a result, Barcan's logical status is fixed: it is a fundamental validity.

If a formula φ belongs to $\mathcal{H}(\forall)$ then we say that φ is a *theorem* of $\mathcal{H}(\forall)$ and write $\vdash \varphi$. A formula φ is *consistent* iff $\neg\varphi$ is *not* a theorem. By an $\mathcal{H}(\forall)$-*proof* in a language $\mathcal{L}(\forall)$ we mean a finite sequence of $\mathcal{L}(\forall)$-formula, each item of which is an axiom, or is obtained from earlier items in the sequence using the rules of proof. If Γ is a set of formulae, and φ a formula, then we say that φ is a *consequence* of Γ iff there is a formula χ such that χ is a conjunction of (finitely many) formulae in Γ and $\vdash \chi \rightarrow \varphi$; in such a case we write $\Gamma \vdash \varphi$. A set of sentences Γ is consistent iff it is not the case that $\Gamma \vdash \bot$. A set of $\mathcal{L}(\forall)$-formulae Γ is $\mathcal{L}(\forall)$ *maximal consistent* (an $\mathcal{L}(\forall)$-*MCS*) iff it is consistent and any set of $\mathcal{L}(\forall)$-formulae that properly extends it is inconsistent.

Our first goal is to show that $\mathcal{H}(\forall)$ is sound: that is, if φ is a theorem then φ is valid. We need two preliminary lemmas concerning variables and substitution.

Lemma 3.1 (Agreement Lemma) *Let $\mathcal{M}$ be a standard model. For all standard $\mathcal{M}$-assignments g and h, all states s in $\mathcal{M}$ and all formulae φ, if g and h agree on all state variables occurring freely in φ, then:*

$$\mathcal{M}, g, s \models \varphi \quad \textit{iff} \quad \mathcal{M}, h, s \models \varphi.$$

Proof. By induction on the complexity of φ. ∎

Lemma 3.2 (Substitution Lemma) *Let $\mathcal{M}$ be a standard model. For all standard $\mathcal{M}$-assignments g, all states s in $\mathcal{M}$ and all formulae φ, if y is a state variable that is substitutable for x in φ and i is a nominal then:*

1. $\mathcal{M}, g, s \models \varphi[y/x]$ *iff* $\mathcal{M}, g', s \models \varphi$, *where* $g' \overset{x}{\sim} g$ *and* $g'(x) = g(y)$.
2. $\mathcal{M}, g, s \models \varphi[i/x]$ *iff* $\mathcal{M}, g', s \models \varphi$, *where* $g' \overset{x}{\sim} g$ *and* $g'(x) = V(i)$.

Proof. By induction on the complexity of φ. ∎

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Theorem 3.3 (Soundness) *The logic $\mathcal{H}(\forall)$ is sound with respect to the class of all standard models.*

Proof. All instances of the minimal modal logic K in $\mathcal{H}(\forall)$ are valid, and modus ponens, necessitation and generalization preserve validity, so it only remains to check that all instances of the additional schemas are valid too. We give the required arguments for *Q2*, *Name*, and *Barcan* and leave *Q1* and *Nom* to the reader.

(*Q2*). Consider $\forall x\varphi \to \varphi[y/x]$, the instance of the *Q2* schema where $\mathbf{s}$ is the state variable y. Suppose that $\mathcal{M}, g, s \models \forall x\varphi$. Proving that $\mathcal{M}, g, s \models \varphi[y/x]$ is equivalent (by clause *1* of the Substitution Lemma) to showing that $\mathcal{M}, g', s \models \varphi$, where $g' \overset{x}{\sim} g$ and $g'(x) = g(y)$. But as $\mathcal{M}, g, s \models \forall x\varphi$, it is immediate that $\mathcal{M}, g', s \models \varphi$. Similarly, if $\forall x\varphi \to \varphi[i/x]$ is the instance of *Q2* where $\mathbf{s}$ is the nominal i, the result follows using clause *2* of the Substitution Lemma.

(*Name*). $\mathcal{M}, g, s \models \exists x x$ iff for some assignment g' such that $g' \overset{x}{\sim} g$, $\mathcal{M}, g', s \models x$. Clearly a suitable g' exists: we need merely stipulate that g' is to be the x-variant of g such that $g'(x) = \{s\}$.

(*Barcan*). Consider $\forall x\Box\varphi \to \Box\forall x\varphi$. Then $\mathcal{M}, g, s \models \forall x\Box\varphi$ iff for all g' such that $g' \overset{x}{\sim} g$ and all t such that sRt, $\mathcal{M}, g', t \models \varphi$. This is equivalent to: for all t such that sRt and all g' such that $g' \overset{x}{\sim} g$, $\mathcal{M}, g', t \models \varphi$, which is equivalent to $\mathcal{M}, g, s \models \Box\forall x\varphi$ as required. ∎

Familiarity with modal and classical logic is a reliable guide to the behavior of $\mathcal{H}(\forall)$. For example, α-conversion holds:

Lemma 3.4 *Suppose that y is substitutable for x in φ, and that φ has no free occurrences of y. Then $\vdash \forall x\varphi \leftrightarrow \forall y\varphi[y/x]$.*

Proof. $\forall x\varphi \to \varphi[y/x]$ is an instance of the *Q2* schema. Prefix $\forall y$ before it using generalization, and then distribute $\forall y$ over the implication using the *Q1* axiom; this proves the left to right implication. Next, note that under our assumptions concerning y, we have that x is substitutable for y in $\varphi[y/x]$, and x has no free occurrences in $\varphi[y/x]$. The right to left direction thus reduces to the previous case. ∎

Moreover, just as we can 'generalize on constants' in first-order logic, we can 'generalize on nominals' in $\mathcal{H}(\forall)$. More precisely:

Lemma 3.5 *Suppose $\vdash \varphi[i/x]$, where i is a nominal and x is a state variable. Then there is a state variable y that does not occur in φ such that $\vdash \forall y\varphi[y/x]$.*

Proof. If x does not occur free in φ the result is clear: $\varphi[i/x]$ is identical to $\varphi[y/x]$, hence as $\varphi[i/x]$ is provable, so is $\forall y\varphi[y/x]$ for any choice of y.

So suppose x does occur free in φ. By assumption we have a proof of $\varphi[i/x]$. Choose any variable y that does not occur in this proof or in φ, and replace every occurrence of i in the proof of $\varphi[i/x]$ by y. It follows by induction on the length of proofs that this new sequence is a proof of $\varphi[y/x]$. Using generalization to prefix $\forall y$ to the last item in this proof yields a proof of $\forall y\varphi[y/x]$. ∎

Nor will the reader find it difficult to prove the following familiar looking schemas:

Lemma 3.6 *In $\mathcal{H}(\forall)$ we have that:*

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1. $\vdash (\varphi \to \exists x\psi) \to \exists x(\varphi \to \psi)$
2. $\vdash (\varphi \wedge \exists y\psi) \to \exists y(\varphi \wedge \psi)$, *for* y *not free in* φ
3. $\vdash \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$

Proof. As in first-order logic. ∎

And in fact, $\mathcal{H}(\forall)$ is complete: every consistent set of formulae has a model. We shall show this using a fairly even-handed mixture of modal and first-order techniques. In particular, from modal logic we shall borrow the idea of *canonical models*, and from classical logic the idea of *witnessed sets*. As we shall see, thanks to our theory of labeling and the presence of the Barcan analogs, these ideas work together smoothly.

Definition 3.7 (Canonical Models) *For any countable language* $\mathcal{L}(\forall)$*, the* canonical model $\mathcal{M}^c$ *is* (S^c, R^c, V^c)*, where* S^c *is the set of all* $\mathcal{L}(\forall)$*-MCSs;* R^c *is the binary relation (called the* canonical relation*) on* S^c *defined by* $\Gamma R^c \Delta$ *iff* $\Box\varphi \in \Gamma$ *implies* $\varphi \in \Delta$*, for all* $\mathcal{L}(\forall)$*-formulae* φ*; and* V^c *is the valuation defined by* $V^c(a) = \{\Gamma \mid a \in \Gamma\}$*, where* a *is a propositional symbol or nominal.*

Definition 3.8 (Witnessed Sets) *Let* $\mathcal{L}(\forall)$ *be some countable language and* Γ *an* $\mathcal{L}(\forall)$*-MCS.* Γ *is called* witnessed *iff for any* $\mathcal{L}(\forall)$*-formula of the form* $\exists x\varphi$*, there is a nominal* i *such that* $\exists x\varphi \to \varphi[i/x]$ *is in* Γ*.*

Note that any witnessed MCS Γ contains at least one nominal, as all instances of the *Name* axiom belong to Γ.

Witnessed sets are important because they provide the structure needed to handle the hybrid quantifiers in the manner familiar from Henkin-style completeness proofs for classical logic. That is, witnessed MCSs will be used for the inductive clause for the quantifiers in the Truth Lemma ("Truth = Membership in an MCS") we will eventually prove.

Roughly speaking, the model we shall eventually define will be made of witnessed MCSs related by the usual modal canonical relation, so the first thing we need to check is that any consistent set of sentences can be expanded to a witnessed MCS. In fact, this can be done, provided we expand the languages with countably many new nominals.

Lemma 3.9 (Extended Lindenbaum Lemma) *Let* $\mathcal{L}(\forall)$ *and* $\mathcal{L}(\forall)^+$ *be two countable languages such that* $\mathcal{L}(\forall)^+$ *is* $\mathcal{L}(\forall)$ *extended with a countably infinite set of new nominals. Then every consistent set of* $\mathcal{L}(\forall)$*-formulae* Γ *can be extended to a witnessed MCS* Γ^+ *in the language* $\mathcal{L}(\forall)^+$*.*

Proof. Let $E_n = \{i_1, i_2, i_3 ...\}$ be an enumeration of the set of all nominals that are contained in $\mathcal{L}(\forall)^+$ but not in $\mathcal{L}(\forall)$, and let $E_f = \{\varphi_1, \varphi_2, \varphi_3 ...\}$ be an enumeration of all $\mathcal{L}(\forall)^+$-formulae. We define the witnessed MCS Γ^+ we require inductively. Let $\Gamma^0 = \Gamma$. Note that Γ^0 contains no nominals from E_n (as it is a set of $\mathcal{L}(\forall)$-formulae) and that it is consistent when regarded as a set of $\mathcal{L}(\forall)^+$-formulae. (To see this, note that if we could prove $\perp$ by making use of nominals from E_n, then by replacing all the (finitely many) E_n nominals in such a proof with state variables from $\mathcal{L}(\forall)$, we could construct a proof of $\perp$ in $\mathcal{L}(\forall)$, which is impossible.) We define Γ^n as follows. If $\Gamma^n \cup \{\varphi_n\}$ is inconsistent, then $\Gamma^{n+1} = \Gamma^n$. Otherwise:

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1. $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\}$, if φ_n is not of the form $\exists x\psi$.
2. $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{\psi[i/x]\}$, if $\varphi_n = \exists x\psi$. (Here i is the first nominal in the enumeration E_n which is not used in the definitions of Γ^i for all $i \leq n$ and also does not appear in φ_n.)

Let $\Gamma^+ = \bigcup_{n\geq 0}\Gamma^n$. By construction it is maximal and witnessed; it remains to show it is consistent. Now, if Γ^+ is inconsistent, then for some $n \in \omega$, Γ^n is inconsistent, for all the (finitely many) formulae required to prove inconsistency belong to some Γ^n. But, as we shall now show by induction, all Γ^n are consistent, hence Γ^+ is too.

Clearly all we need to check is that expansions using clause 2 preserve consistency. To show this, argue by contrapositive. Suppose $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{\psi[i/x]\}$ is inconsistent. Then there is a formula χ which is a conjunction of a finite number of formulae from $\Gamma^n \cup \{\varphi_n\}$, such that $\vdash \chi \to \neg\psi[i/x]$. By generalization on nominals (see Lemma 3.5) we can prove $\vdash \forall y(\chi \to \neg\psi[y/x])$, for some state variable y that does not occur in $\vdash \chi \to \neg\psi[i/x]$. Hence by *Q1* we have $\vdash \chi \to \forall y\neg\psi[y/x]$. Hence $\Gamma^n \cup \{\varphi_n\} \vdash \forall y\neg\psi[y/x]$, and by Lemma 3.4 we obtain $\Gamma^n \cup \{\varphi_n\} \vdash \forall x\neg\psi$. But $\varphi_n = \exists x\psi$, and this contradicts the consistency of $\Gamma^n \cup \{\varphi_n\}$. ∎

We now set about defining the standard models (and standard assignments) needed to prove completeness. As a first step, we define the concept of *witnessed models*. Given a witnessed MCS Σ, we form the witnessed model generated by Σ by taking the submodel of the canonical model generated by Σ, and then throwing away any non-witnessed MCSs it contains. More precisely:

Definition 3.10 (Witnessed Models) *Let Σ be a witnessed MCS in some countable language $\mathcal{L}(\forall)$, let $\mathcal{M}^c = (S^c, R^c, V^c)$ be the canonical model in $\mathcal{L}(\forall)$, and let $Wit(\mathcal{M}^c)$ be the set of all witnessed MCSs in $\mathcal{M}^c$. The witnessed model $\mathcal{M}^w$ yielded by Σ is the triple (S^w, R^w, V^w), where $S^w = \{\Sigma\} \cup \{\Gamma \in Wit(\mathcal{M}^c) \mid$ there are $k > 0$ and $s_0, \ldots, s_k \in Wit(\mathcal{M}^c)$ such that $s_0 = \Sigma, s_k = \Gamma$ & $s_i R^c s_{i+1}$ for $0 \leq i \leq k-1\}$, and R^w and V^w are restrictions of R^c and V^c respectively to S^w.*

Lemma 3.11 *Let $\mathcal{L}(\forall)$ be some countable language and $\mathcal{M}^w = (S^w, R^w, V^w)$ the witnessed model yielded by some witnessed $\mathcal{L}(\forall)$-MCS Σ. Then, for all MCSs $\Gamma, \Delta \in \mathcal{M}^w$ and every state symbol $\mathbf{s}$, if $\mathbf{s} \in \Gamma$ and $\mathbf{s} \in \Delta$, then $\Gamma = \Delta$.*

Proof. Suppose Γ and Δ are different. Then there is a formula φ such that $\varphi \in \Gamma$ and $\neg\varphi \in \Delta$. The MCSs Γ and Δ are reachable from Σ in finitely many R^w-steps and hence there are $m, n \in \omega$ such that $\Diamond^m(\mathbf{s}\wedge\varphi) \in \Sigma$ and $\Diamond^n(\mathbf{s}\wedge\neg\varphi) \in \Sigma$. As Σ contains every instance of the *Nom* schema, for some state variable x that does not occur freely in φ, $\forall x[\Diamond^m(x\wedge\varphi) \to \Box^n(x \to \varphi)] \in \Sigma$. Hence, by *Q2*, $\Diamond^m(\mathbf{s}\wedge\varphi) \to \Box^n(\mathbf{s} \to \varphi) \in \Sigma$, and therefore $\Box^n(\mathbf{s} \to \varphi) \in \Sigma$. But because both $\Diamond^n(\mathbf{s}\wedge\neg\varphi) \in \Sigma$ and $\Box^n(\mathbf{s} \to \varphi) \in \Sigma$ it follows by easy modal reasoning that $\Diamond^n(\mathbf{s} \wedge \neg\varphi \wedge \varphi) \in \Sigma$, which contradicts the consistency of Σ. We conclude that Γ and Δ are identical. (Note that nothing in this proof trades on the fact that we are working with *witnessed* MCSs. In fact, the lemma holds for *any* submodel of a generated submodel of the canonical model.) ∎

Recall that a standard model is a model in which every nominal is true at *exactly* one state. From the previous lemma we know that nominals are contained in *at most one* MCS in a witnessed model, so it is clear that the natural definition of valuation on

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witnessed models (that is, that symbols are true at precisely the MCSs which contain them) almost provides us with a standard model. Moreover, it also follows from the previous lemma that the natural way of defining an assignment on witnessed models (namely, stipulating that $g(x)$ is to be the set of MCSs containing x) almost gives us the standard assignment we require. However we have no guarantee that every state symbol is contained in *at least one* MCS. Whenever we have a witnessed model $\mathcal{M}^w$ such that some state symbol occurs in no MCS in $\mathcal{M}^w$, we shall 'complete' the model by gluing on a new dummy state $*$. We will then stipulate that any state variable or nominal not occurring in any MCS in $\mathcal{M}^w$ will denote this new point. This motivates the following definition.

Definition 3.12 (Completed Models and Completed Assignments) *Let $\mathcal{M}^w = (S^w, R^w, V^w)$ is the witnessed model yielded by some witnessed MCS Σ. If every state symbol belongs to at least one MCS in S^w, then $\mathcal{M}$, the* completed model *of $\mathcal{M}^w$, is simply $\mathcal{M}^w$ itself. Otherwise, a* completed model *$\mathcal{M}$ of $\mathcal{M}^w$ is a triple (S, R, V), where $S = S^w \cup \{*\}$ ($*$ is an entity that is not an MCS); $R = R^w \cup \{(*, \Sigma)\}$; for all propositional symbols p, $V(p) = V^w(p)$; and for all nominals i, $V(i) = \{\Gamma \in \mathcal{M}^w \mid i \in \Gamma\}$ if this set is non-empty, and $V(i) = \{*\}$ otherwise.*

If $\mathcal{M} = (S, R, V)$ is a completed model of a witnessed model $\mathcal{M}^w$, then the completed assignment *g on $\mathcal{M}$ is defined as follows: for all state variables x, $g(x) = \{\Gamma \in \mathcal{M}^w \mid x \in \Gamma\}$ if this set is non-empty, and $g(x) = \{*\}$ otherwise.*

Clearly (by Lemma 3.11) completed models are standard models and completed assignments are standard assignments, thus (by Theorem 3.3) all theorems of the logic $\mathcal{H}(\forall)$ are true in completed models with respect to the relevant completed assignment. There is one other point about the previous definition that the reader should note: *we only glue on a dummy state $*$ when we are forced to*. As a consequence, every state in a completed model is labeled by some state symbol. This will shortly help us to give a smooth proof of the Truth Lemma.

But before we can prove the Truth Lemma we need to establish a crucial fact: that completed models contain all the information required to cope with the modalities. That is, we need an Existence Lemma which tells us that if $\Diamond\varphi$ belongs to an MCS Δ in a completed model then there is an R^c-successor MCS Γ, *which also belongs to the completed model*, and contains φ. This is not obvious. We formed the completed model by throwing away non-witnessed MCSs. How do we know that we did not throw away the φ-containing successor MCS Γ that we need?

In fact, by making use of the Barcan analogs, we can prove the required Existence Lemma. First a technical preliminary:

Lemma 3.13 *Let θ and χ be formulae and x and y state variables such that y is substitutable for x in χ, and y does not have free occurrences in either θ or χ. Then $\vdash \Diamond\theta \to \exists y \Diamond((\exists x\chi \to \chi[y/x]) \wedge \theta)$.*

Proof. It follows from Lemma 3.4 that $\vdash \exists x\chi \to \exists y\chi[y/x]$, hence by clause *1* of Lemma 3.6, $\vdash \exists y(\exists x\chi \to \chi[y/x])$, hence $\vdash \theta \to (\exists y(\exists x\chi \to \chi[y/x]) \wedge \theta)$. Applying clause *2* of Lemma 3.6 yields $\vdash \theta \to \exists y((\exists x\chi \to \chi[y/x]) \wedge \theta)$. Easy modal reasoning yields $\vdash \Diamond\theta \to \Diamond\exists y((\exists x\chi \to \chi[y/x]) \wedge \theta)$. Using the contrapositive of Barcan, we obtain $\vdash \Diamond\theta \to \exists y \Diamond((\exists x\chi \to \chi[y/x]) \wedge \theta)$. ∎

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Lemma 3.14 (Existence Lemma for Witnessed Models) *Let Δ be a witnessed MCS in some countable language $\mathcal{L}(\forall)$. If $\Diamond\varphi \in \Delta$ then there is a* witnessed *$\mathcal{L}(\forall)$-MCS Γ such that $\Delta R^c \Gamma$ and $\varphi \in \Gamma$.*

Proof. Define $\Psi = \{\psi \mid \Box\psi \in \Delta\}$ and $\Gamma^0 = \{\varphi\} \cup \Psi$. The proof that Γ^0 is consistent is standard. If we can expand Γ^0 to a witnessed MCS Ω, then Ω will be a suitable choice of Γ. We show that it is possible to make this expansion.

Enumerate all the $\mathcal{L}(\forall)$-formulae that are of the form $\exists v\varphi$, where v can be any state variable. We shall inductively expand Γ^0 by adding a suitable *witness conditional* for each formula in the enumeration. By $w(\exists v\chi, i)$, the witness conditional for $\exists v\chi$ in nominal i, we mean the formula $\exists v\chi \to \chi[i/v]$. We shall show that if ξ_{n+1} is the $n+1$-th formula in the enumeration, then it is always possible to choose a nominal i_{n+1} such that the set $\Gamma^{n+1} = \Gamma^n \cup \{w(\xi_{n+1}, i_{n+1})\}$ is consistent. Indeed, we will show that it is possible to choose i_{n+1} so that $\Diamond(\varphi \wedge w(\xi_1, i_1) \wedge \cdots \wedge w(\xi_n, i_n) \wedge w(\xi_{n+1}, i_{n+1})) \in \Delta$.

To construct the witness conditional for ξ_{n+1}, assume we have already constructed Γ^n containing witness conditionals $w(\xi_1, i_1), \ldots, w(\xi_n, i_n)$, such that $\Diamond(\varphi \wedge w(\xi_1, i_1) \wedge \cdots \wedge w(\xi_n, i_n)) \in \Delta$. (Here $\xi_1, \ldots, \xi_n$ are the first n items in the enumeration.) Let $\theta = \varphi \wedge w(\xi_1, i_1) \wedge \cdots \wedge w(\xi_n, i_n)$.

Now, suppose that ξ_{n+1} is $\exists v\chi$. By the previous lemma we have $\vdash \Diamond\theta \to \exists y\Diamond((\exists x\chi \to \chi[y/x]) \wedge \theta)$ where y is some state variable that does not appear in θ or χ. But $\Diamond\theta \in \Delta$ and so $\exists y\Diamond((\exists x\chi \to \chi[y/x]) \wedge \theta) \in \Delta$. Since Δ is a witnessed MCS, there is a nominal i_{n+1} such that $\Diamond((\exists x\chi \to \chi[i_{n+1}/x]) \wedge \theta) \in \Delta$. So we choose $\exists x\chi \to \chi[i_{n+1}/x]$ (that is, $w(\xi_{n+1}, i_{n+1})$) as our witness conditional and define $\Gamma^{n+1} = \Gamma^n \cup \{w(\xi_{n+1}, i_{n+1})\}$.

By construction, $\Diamond(\varphi \wedge w(\xi_1, i_1) \wedge \cdots \wedge w(\xi_{n+1}, i_{n+1})) \in \Delta$. But is Γ^{n+1} consistent? Suppose it is not. Then there is a conjunction τ of (finitely many) formulae in Ψ such that $\vdash \tau \to \neg(\varphi \wedge w_1 \wedge \cdots \wedge w_{n+1})$ (here we have abbreviated $w(\xi_i, i_i)$ to w_i). By easy modal reasoning we obtain $\vdash \Box\tau \to \neg\Diamond(\varphi \wedge w_1 \wedge \cdots \wedge w_{n+1}) \in \Delta$. But $\Box\tau \in \Delta$ and so $\neg\Diamond(\varphi \wedge w_1 \wedge \cdots \wedge w_{n+1}) \in \Delta$, which contradicts the consistency of Δ. $\bigcup_{n\geq 0} \Gamma^n$ is consistent since for every $n \in \omega$, Γ^n is. Now use the usual version of the Lindenbaum Lemma to expand $\bigcup_{n\geq 0} \Gamma^n$ to the required witnessed MCS Ω. ∎

Lemma 3.15 (Truth Lemma) *Let $\mathcal{M}$ be a completed model in some countable language $\mathcal{L}(\forall)$, g the completed $\mathcal{M}$-assignment, and Δ an $\mathcal{L}(\forall)$-MCS in $\mathcal{M}$. For every formula φ:*

$$\varphi \in \Delta \quad \textit{iff} \quad \mathcal{M}, g, \Delta \models \varphi.$$

Proof. The proof is by induction on the complexity of φ. If φ is a state symbol or a propositional symbol the required equivalence follows from the definition of the model $\mathcal{M}$ and the assignment g. The Boolean cases follow from obvious properties of MCSs. For the modal case, note that the Existence Lemma for Witnessed Models gives us precisely the information required to drive through the left to right direction. The right to left direction is more or less immediate, though there is a subtlety the reader should observe: if $\mathcal{M}, g, \Delta \models \Diamond\psi$, then there is a state s such that $\Delta R s$ and $\mathcal{M}, g, s \models \psi$. Since (by definition) no MCS precedes $*$, we conclude that $s \neq *$. Thus the successor to Δ that satisfies ψ is itself some MCS, and so we really can apply the inductive hypothesis.

Now for the quantifiers. Let φ be $\exists x\psi$. Suppose $\exists x\psi \in \Delta$. Since Δ is witnessed, there is a nominal i such that $\psi[i/x] \in \Delta$. By the inductive hypothesis $\mathcal{M}, g, \Delta \models$

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$\psi[i/x]$, hence by the contrapositive of the *Q2* axiom, $\mathcal{M}, g, \Delta \models \exists x\psi$.

For the other direction assume $\mathcal{M}, g, \Delta \models \exists x\psi$. This is, there exists an $s \in \mathcal{M}$ such that $\mathcal{M}, g', \Delta \models \psi$, where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$. Now, because of the way we defined completed models, we know that either a nominal i or a state variable y is true at s with respect to g (note that this is so even if $s = *$).

Suppose first that a nominal i is satisfied at s. That is $V(i) = \{s\}$. Then by clause *2* of the Substitution Lemma $\mathcal{M}, g, \Delta \models \psi[i/x]$ and by the inductive hypothesis $\psi[i/x] \in \Delta$. So, with the help of the contrapositive of the *Q2* axiom, $\exists x\psi$ is in Δ.

Suppose now that a state variable y is satisfied at s. That is $g(y) = \{s\}$. Since y may not be substitutable for x in ψ, we have to replace all bound occurrences of y in ψ by some state variable that does not occur in ψ at all; call the formula we obtain ψ'. It follows by Lemma 3.4 that $\psi \leftrightarrow \psi'$ is provable, hence by soundness it is valid, therefore $\mathcal{M}, g', \Delta \models \psi'$. Since y is now substitutable for x in ψ', by clause *1* of the Substitution Lemma $\mathcal{M}, g, \Delta \models \psi'[y/x]$. By the inductive hypothesis $\psi'[y/x] \in \Delta$, therefore, with the help of the contrapositive of the *Q2* axiom, $\exists x\psi' \in \Delta$. But it follows easily from clause *3* of Lemma 3.6 that $\exists x\psi \leftrightarrow \exists x\psi'$ is provable, and so $\exists x\psi \in \Delta$. ∎

Theorem 3.16 (Completeness) *Every consistent set of formulae in a countable language $\mathcal{L}(\forall)$ is satisfiable in a rooted and countable standard model with respect to a standard assignment function.*

Proof. Let Σ be a consistent set of $\mathcal{L}(\forall)$-formulae. By the Extended Lindenbaum Lemma we can expand Σ to a witnessed MCS Σ^+ in a countable language $\mathcal{L}(\forall)^+$. Let $\mathcal{M}$ be the completed model yielded by Σ^+ and g the completed $\mathcal{M}$-assignment on this model. It follows from the Truth Lemma that $\mathcal{M}, g, \Sigma^+ \models \Sigma^+$ and so $\mathcal{M}, g, \Sigma^+ \models \Sigma$. By the definition of completed models, either Σ^+ is a root of this model, or there is an additional state $*$ which is. Moreover, as every state in the model is named by one of the (countably many) state symbols in $\mathcal{L}(\forall)^+$, the model is countable. ∎

4 The hybrid logic of $\downarrow$

We now present an axiomatization $\mathcal{H}(\downarrow)$ of the set of valid $\mathcal{L}(\downarrow)$-formulae. Unless otherwise indicated, throughout this section, $\vdash \varphi$ will mean that φ is a theorem of $\mathcal{H}(\downarrow)$, and syntactic notions such as 'proof' and 'consistency' refer to $\mathcal{H}(\downarrow)$-proofs, $\mathcal{H}(\downarrow)$-consistency, and so on.

In certain respects, $\mathcal{H}(\downarrow)$ resembles $\mathcal{H}(\forall)$. For a start, $\mathcal{H}(\downarrow)$ is also an extension of the minimal modal logic K, and the axioms governing $\downarrow$ are analogs of those governing $\forall$. Moreover, $\mathcal{H}(\downarrow)$ is closed under the rules of modus ponens, necessitation, and an analog of generalization called *localization*, and contains a theory of labeling.

But there is an important difference. The Barcan analog for $\downarrow$ (that is, $\downarrow x\Box\varphi \rightarrow \Box\downarrow x\varphi$) is not valid. (Because $\downarrow$ binds to the current state, it cannot safely be permuted with $\Box$, as the reader can check.) Now, Barcan was crucial to the model building strategy of the previous section: it allowed us to prove Lemma 3.13 and hence to construct witness conditionals in the proof of the Existence Lemma. What are we to do without it?

We shall use a technique from extended modal logic: additional rules of proof. Although $\downarrow$ works too locally to validate the Barcan analogs, because there are 'labels'

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in the language we can make use of the COV^* rules. The COV^* rules were introduced by the Sofia school of modal logic as part of their investigation of various forms of modal and propositional dynamic logic with names (see, for example, Passy and Tinchev [11] and Gargov and Goranko [7]). Informally, COV^* will be useful because it gives us a way of pasting in all the required witness formulae 'by hand', thus enabling us to adapt our proof strategy to ↓.

$\mathcal{H}(\downarrow)$ is the smallest set of $\mathcal{L}(\downarrow)$-formulae containing the minimal modal logic K, and all instances of the five axiom schemas listed below, that is closed under modus ponens, necessitation, *localization* (if $\varphi \in \mathcal{H}(\downarrow)$ then $\downarrow\mathbf{v}\varphi \in \mathcal{H}(\downarrow)$) and COV^* (explained below). Note that localization is just the ↓ analog of the rule of generalization given for ∀ in the previous section.

Now for the five additional axiom schemas. As before, we use **v** and **s** as metavariables over state variables and state symbols respectively.

Q1: $\downarrow\mathbf{v}(\varphi \to \psi) \to (\varphi \to \downarrow\mathbf{v}\psi)$, where φ contains no free occurrences of $\mathbf{v}$

Q2: $\downarrow\mathbf{v}\varphi \to (\mathbf{s} \to \varphi[\mathbf{s}/\mathbf{v}])$, where $\mathbf{s}$ is substitutable for $\mathbf{v}$ in φ

Q3: $\downarrow\mathbf{v}(\mathbf{v} \to \varphi) \to \downarrow\mathbf{v}\varphi$

Self-dual: $\downarrow\mathbf{v}\varphi \leftrightarrow \neg\downarrow\mathbf{v}\neg\varphi$

Nom: $\Diamond^m(\mathbf{s} \wedge \varphi) \to \Box^n(\mathbf{s} \to \varphi)$, for all $m, n \in \omega$

Q1 is an exact analog of its $\mathcal{H}(\forall)$ counterpart. *Q2* is too, save that its consequent is an implication, whose antecedent **s** reflects the fact that ↓ binds state variables to the current state. This motivates the inclusion of *Q3*, which allows us to eliminate such 'antecedent labels'. *Self-dual* is self-explanatory. *Nom* is an analog of its $\mathcal{H}(\forall)$ counterpart. Moreover, although $\downarrow\mathbf{vv}$ (the analog of the *Name* schema for $\mathcal{H}(\forall)$) has not been included as an axiom schema, it is easily derivable with the aid of *Q3*, as we shall see. Thus $\mathcal{H}(\downarrow)$ contains a theory of labeling.

Conspicuous by its absence is any analog of Barcan. So let us now define the COV^* rules we shall use to replace it. As a first step we define:

Definition 4.1 (□-forms) *Let $\mathcal{L}(\downarrow)$ be a countable language, and # some symbol not belonging to $\mathcal{L}(\downarrow)$. We define the set of □-forms (for $\mathcal{L}(\downarrow)$) as follows: (1) # is a □-form, (2) if L is a □-form and φ is an $\mathcal{L}(\downarrow)$-formula then $\varphi \to L$ and $\Box L$ are □-forms, and (3) nothing else is a □-form.*

Note that every □-form L has exactly one occurrence of the symbol #. We use $L(\psi)$ to denote the formula obtained from L by replacing the unique occurrence of # by a formula ψ. We can now define the COV^* rules. For every □-form L, and every nominal i not occurring in L, we have the following rule:

$$L(\neg i) \in \mathcal{H}(\downarrow) \text{ implies } L(\bot) \in \mathcal{H}(\downarrow).$$

As we shall now see, these rules preserve validity. First, two preliminary lemmas.

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Lemma 4.2 *Let $\mathcal{M} = (S, R, V)$ and $\mathcal{M}' = (S, R, V')$ be two standard models. For all standard assignments g on S, all states s in S and all formulae φ, if $V(a) = V'(a)$ for all atoms a occurring in φ then $\mathcal{M}, g, s \models \varphi$ iff $\mathcal{M}', g, s \models \varphi$.*

Proof. By induction on the complexity of φ. ∎

We write $V' \overset{i}{\sim} V$ to indicate that V and V' are standard valuations on the same frame that agree on all arguments save possibly i.

Lemma 4.3 *Let $\mathcal{M} = (S, R, V)$ be a standard model, g a standard $\mathcal{M}$-assignment. For every state s in $\mathcal{M}$, every $\Box$-form L, and every nominal i not occurring in L, if $\mathcal{M}, g, s \models \neg L(\bot)$ then, there is a valuation V' such that $V' \overset{i}{\sim} V$ and $(S, R, V'), g, s \models \neg L(\neg i)$.*

Proof. By induction on the structure of L. The base case is when L is $\#$. In this case $\neg L(\neg i)$ is $\neg\neg i$. Let V' be a standard valuation such that $V' \overset{i}{\sim} V$ and $V'(i) = \{s\}$. Then $(S, R, V'), g, s \models i$ and the required result is immediate.

So consider the induction step for $L = \varphi \to L_1$, where L_1 is a $\Box$-form. Suppose $\mathcal{M}, g, s \models \neg(\varphi \to L_1(\bot))$. This means $\mathcal{M}, g, s \models \varphi$ and $\mathcal{M}, g, s \models \neg L_1(\bot)$. By the inductive hypothesis, there is a valuation V' such that $V' \overset{i}{\sim} V$ and $(S, R, V'), g, s \models \neg L_1(\neg i)$. Since i does not appear in φ, by Lemma 4.2 we have $(S, R, V'), g, s \models \varphi$. Therefore $(S, R, V'), g, s \models \neg(\varphi \to L_1(\neg i))$.

Now suppose $L = \Box L_1$. Assume that $\mathcal{M}, g, s \models \neg\Box L_1(\bot)$. Hence there is a state t with sRt and $\mathcal{M}, g, t \models \neg L_1(\bot)$. By the inductive hypothesis there is valuation V' such that $V' \overset{i}{\sim} V$ and $(S, R, V'), g, t \models \neg L_1(\neg i)$. Therefore $(S, R, V'), g, s \models \neg\Box L_1(\neg i)$. ∎

An immediate corollary is that if the premise of a COV^* rule is valid on a frame (S, R), then its conclusion is valid on (S, R) too. To see this, consider the contrapositive. Suppose we can falsify $L(\bot)$ on (S, R). That is, suppose there is a standard valuation V, a standard assignment g, and a state s such that $(S, R, V), g, s \not\models L(\bot)$. Since i does not occur in L, by the previous lemma, there is a valuation V' such that $(S, R, V'), g, s \not\models L(\neg i)$, and we have falsified the consequent on the same frame. Hence the COV^* rules are validity-preserving.

With this established, we are almost ready to prove the soundness of $\mathcal{H}(\downarrow)$. We first state analogs of Lemma 3.1 and Lemma 3.2.

Lemma 4.4 (Agreement Lemma) *Let $\mathcal{M}$ be a standard model. For all standard assignments g and h, all states s in $\mathcal{M}$ and all formulae φ, if g and h agree on all state variables occurring freely in φ, then $\mathcal{M}, g, s \models \varphi$ iff $\mathcal{M}, h, s \models \varphi$.*

Proof. By induction on the complexity of φ. ∎

Lemma 4.5 (Substitution Lemma) *Let $\mathcal{M}$ be a standard model. For all standard $\mathcal{M}$-assignments g, all states s in $\mathcal{M}$ and all formulae φ, if y is a state variable that is substitutable for x in φ and i is a nominal then:*

1. *$\mathcal{M}, g, s \models \varphi[y/x]$ iff $\mathcal{M}, g', s \models \varphi$, where $g' \overset{x}{\sim} g$ and $g'(x) = g(y)$.*
2. *$\mathcal{M}, g, s \models \varphi[i/x]$ iff $\mathcal{M}, g', s \models \varphi$, where $g' \overset{x}{\sim} g$ and $g'(x) = V(i)$.*

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PROOF. By induction on the complexity of φ. ∎

Theorem 4.6 (Soundness) *The logic $\mathcal{H}(\downarrow)$ is sound with respect to the class of all standard models.*

PROOF. Obviously all instances of the minimal modal logic K are valid. Moreover, modus ponens, necessitation, and COV^* preserve validity. Localization does too. To see this, consider the contrapositive. If $\downarrow x\varphi$ is *not* valid, we can falsify $\downarrow x\varphi$ in some model $\mathcal{M}$ at a state s. This means there is a standard assignment g such that $\mathcal{M}, g, s \not\models \downarrow x\varphi$. Hence $\mathcal{M}, g', s \not\models \varphi$, where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$, and φ is not valid either. Thus all rules of proof preserve validity.

So it only remains to check that all instances of the five additional schemas are valid too. We give the required arguments for *Q1*, *Q3*, and *Self-dual* and leave *Q2* and *Nom* to the reader.

(*Q1*). Consider $\downarrow x(\varphi \to \psi) \to (\varphi \to \downarrow x\psi)$, where φ does not contain free occurrences of x. Assume that $\mathcal{M}, g, s \models \downarrow x(\varphi \to \psi)$ and $\mathcal{M}, g, s \models \varphi$. Proving that $\mathcal{M}, g, s \models \downarrow x\psi$ is equivalent to showing that $\mathcal{M}, g', s \models \psi$ where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$. But as $\mathcal{M}, g, s \models \downarrow x(\varphi \to \psi)$ we have that $\mathcal{M}, g', s \models \varphi \to \psi$. Moreover, by the Agreement Lemma, $\mathcal{M}, g', s \models \varphi$, for $\mathcal{M}, g, s \models \varphi$ and φ contains no free occurrences of x. Hence, by modus ponens, $\mathcal{M}, g', s \models \psi$, and the desired result follows.

(*Q3*). Consider $\downarrow x(x \to \varphi) \to \downarrow x\varphi$. Suppose $\mathcal{M}, g, s \models \downarrow x(x \to \varphi)$. That is, $\mathcal{M}, g', s \models x \to \varphi$, where $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$. But then $\mathcal{M}, g', s \models x$, hence $\mathcal{M}, g', s \models \varphi$, and therefore $\mathcal{M}, g, s \models \downarrow x\varphi$.

(*Self-dual*). Consider $\downarrow x\varphi \leftrightarrow \neg\downarrow x\neg\varphi$. This is equivalent to $\neg\downarrow x\varphi \leftrightarrow \downarrow x\neg\varphi$. Now $\mathcal{M}, g, s \models \neg\downarrow x\varphi$ iff $\mathcal{M}, g, s \not\models \downarrow x\varphi$ iff $\mathcal{M}, g', s \not\models \varphi$ for $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$ iff $\mathcal{M}, g', s \models \neg\varphi$ for $g' \overset{x}{\sim} g$ and $g'(x) = \{s\}$ iff $\mathcal{M}, g, s \models \downarrow x\neg\varphi$. ∎

The $\downarrow$ binder will be new to most readers. So, before going any further, let us prove some $\mathcal{H}(\downarrow)$-theorems, and note some facts about $\mathcal{H}(\downarrow)$-provability.

Lemma 4.7 *In $\mathcal{H}(\downarrow)$ we have that:*

1. $\vdash \downarrow xx$
2. $\vdash \downarrow x(\varphi \to \psi) \to (\downarrow x\varphi \to \downarrow x\psi)$
3. $\vdash \downarrow x\varphi \to \downarrow x(x \wedge \varphi)$
4. $\vdash \varphi[y/x] \to (y \to \downarrow x\varphi)$, *where y is substitutable for x in φ.*

PROOF. (*1*). Note that for any state variable x we have $\vdash x \to x$, and hence (by localization) $\vdash \downarrow x(x \to x)$. But $\downarrow x(x \to x) \to \downarrow xx$ is an instance of *Q3*, thus $\downarrow xx$ follows by modus ponens.

(*2*). Note that $\downarrow x(\varphi \to \psi) \to (x \to (\varphi \to \psi))$ is an instance of *Q2*, as is $\downarrow x\varphi \to (x \to \varphi)$. Hence $\vdash (\downarrow x(\varphi \to \psi) \wedge \downarrow x\varphi)) \to (x \to \psi)$. Use localization to prefix this formula with $\downarrow x$, and then use *Q1* to distribute $\downarrow x$ over the main implication to get $\vdash (\downarrow x(\varphi \to \psi) \wedge \downarrow x\varphi) \to \downarrow x(x \to \psi)$. The result follows by applying *Q3* to the consequent of this last implication.

(*3*). The formula $\varphi \to (x \to (x \wedge \varphi))$ is a tautology. By localization and the previous clause we get $\vdash \downarrow x\varphi \to \downarrow x(x \to (x \wedge \varphi))$. Using *Q3* we get $\vdash \downarrow x\varphi \to \downarrow x(x \wedge \varphi)$.

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(*4*). Note that $\vdash \downarrow x\neg\varphi \to (y \to \neg\varphi[y/x])$ is an instance of *Q2*. Taking the contrapositive we obtain $\vdash (y \wedge \varphi[y/x]) \to \neg\downarrow x\neg\varphi$. Using *Self-dual* we get $\vdash (y \wedge \varphi[y/x]) \to \downarrow x\varphi$, and the result follows. ∎

Lemma 4.8 *Suppose that φ has no free occurrences of y, and that y is substitutable for x in φ. Then $\vdash \downarrow x\varphi \leftrightarrow \downarrow y\varphi[y/x]$.*

Proof. $\downarrow x\varphi \to (y \to \varphi[y/x])$ is an instance of *Q2*. Use localization to prefix $\downarrow y$, and *Q1* to distribute the quantifier $\downarrow y$ over the main implication (this is allowed because φ does not contain free occurrences of y) to obtain $\vdash \downarrow x\varphi \to \downarrow y(y \to \varphi[y/x])$. With the help of *Q3* we have $\vdash \downarrow x\varphi \to \downarrow y\varphi[y/x]$ and this completes the proof of the left to right implication. Next, note that by our assumptions for y, we have that x is substitutable for y, and x has no free occurrences in $\varphi[y/x]$. Hence the right to left direction reduces to the previous case. ∎

Lemma 4.9 *Let φ and ψ be two formulae such that $\vdash \varphi \leftrightarrow \psi$. Then for all formulae θ, $\vdash \theta \leftrightarrow \theta\{\psi/\varphi\}$, where $\theta\{\psi/\varphi\}$ is a formula obtained from θ by replacing some occurrences of φ in θ by ψ.*

Proof. Suppose $\vdash \varphi \leftrightarrow \psi$ is provable. The required result can be proved by induction on the structure of θ. We show the inductive step for $\theta = \downarrow x\chi$. By the inductive hypothesis $\vdash \chi \leftrightarrow \chi\{\psi/\varphi\}$. By localization, $\vdash \downarrow x(\chi \to \chi\{\psi/\varphi\})$, thus with the help of clause *2* of Lemma 4.7, $\vdash \downarrow x\chi \to \downarrow x\chi\{\psi/\varphi\}$. Similarly, $\vdash \downarrow x\chi\{\psi/\varphi\} \to \downarrow x\chi$. Hence $\vdash \downarrow x\chi \leftrightarrow \downarrow x\chi\{\psi/\varphi\}$. ∎

We now prove the completeness result. Once again, we shall do so by combining ideas from modal and classical logic, but this time there will be a bias towards modal ideas. The basic modal tool required is unchanged: as before we use canonical models.

Definition 4.10 (Canonical Models) *For any countable language $\mathcal{L}(\downarrow)$, the* canonical model *$\mathcal{M}^c$ is (S^c, R^c, V^c), where S^c is the set of all $\mathcal{L}(\downarrow)$-MCSs; R^c is the binary relation on S^c defined by $\Gamma R^c \Delta$ iff $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$, for all $\mathcal{L}(\downarrow)$-formulae φ; and V^c is the valuation defined by $V^c(a) = \{\Gamma \mid a \in \Gamma\}$, where a is a propositional symbol or a nominal.*

Now, the next step is to introduce a notion of witnessing for $\downarrow$:

Definition 4.11 ($\downarrow$-witnessed Sets) *An MCS Γ is* $\downarrow$-witnessed *iff for any formula of the form $\downarrow x\varphi$, there is a nominal i such that $\downarrow x\varphi \to (i \wedge \varphi[i/x])$ is in Γ.*

But now consider the following, somewhat simpler, notion:

Definition 4.12 (Named Sets) *An MCS Γ is* named *iff it contains at least one nominal. If $i \in \Gamma$ we say i* names *Γ.*

In fact these concepts are equivalent:

Lemma 4.13 *An MCS Γ is named iff it is $\downarrow$-witnessed.*

Proof. For the left to right direction, suppose $\downarrow x\varphi$ belongs to a named MCS Γ. If i is the nominal that names Γ, it follows by *Q2* that $i \wedge \varphi[i/x] \in \Gamma$, and thus Γ is $\downarrow$-*witnessed*. For the right to left direction, note that every $\downarrow$-witnessed set contains $\downarrow xx \to (i \wedge i)$ for some nominal i. But $\vdash \downarrow xx$ (see clause *1* of Lemma 4.7), thus every $\downarrow$-witnessed MCS is named. ∎

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As we have mentioned, the chief difficulty facing us is that, without Barcan at our disposal, it is not clear how to prove the required Existence Lemma. The COV^* rule gives us way around this difficulty. It does so by enabling us to build a special kind of named set:

Definition 4.14 (Closed Sets) *Let $\mathcal{L}(\downarrow)$ be a language and Γ be an $\mathcal{L}(\downarrow)$-MCS. Γ is called* closed *iff for all $\Box$-forms L we have: if $L(\neg i) \in \Gamma$ for all nominals $i \in \mathcal{L}(\downarrow)$, then $L(\bot) \in \Gamma$.*

Every closed MCS Γ is named. (To see this, suppose that for all nominals i, $\neg i \in \Gamma$. But since Γ is closed this means that $\bot \in \Gamma$, which contradicts the consistency of Γ.) Moreover, as we shall now show, by extending our language with new nominals and making use of the COV^* rule, we can build all the closed sets we need:

Lemma 4.15 (Extended Lindenbaum Lemma) *Let $\mathcal{L}(\downarrow)$ and $\mathcal{L}(\downarrow)^+$ be two countable languages such that $\mathcal{L}(\downarrow)^+$ is $\mathcal{L}(\downarrow)$ extended with a countably infinite set of new nominals. Then every consistent set of $\mathcal{L}(\downarrow)$-formulae Γ can be extended to a closed MCS Γ^+ in the language $\mathcal{L}(\downarrow)^+$.*

Proof. Let $E_n = \{i_1, i_2, i_3 ...\}$ be an enumeration of all nominals that are contained in $\mathcal{L}(\downarrow)^+$ but not in $\mathcal{L}(\downarrow)$, and let $E_f = \{\varphi_1, \varphi_2, \varphi_3 ...\}$ be an enumeration of all $\mathcal{L}(\downarrow)^+$-formulae. We define the required named MCS Γ^+ inductively. Let $\Gamma^0 = \Gamma$. Note that Γ^0 contains no nominals from E_n, and is consistent when regarded as a set of $\mathcal{L}(\downarrow)^+$-formulae.

Suppose we have defined Γ^k for $k \leq n$. If $\Gamma^n \cup \{\varphi_n\}$ is inconsistent, then $\Gamma^{n+1} = \Gamma^n$. Otherwise:

1. $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\}$ if φ_n is not of the form $\neg L(\bot)$, else:
2. $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{\neg L(\neg i)\}$ where $\varphi_n = \neg L(\bot)$ and i is the first nominal in the enumeration E_n which does not appear in Γ^k (for $0 \leq k \leq n$) nor in L. Clearly such a nominal exists, since only finitely many nominals from E_n are contained in Γ^k (for $0 \leq k \leq n$) and L.

Let $\Gamma^+ = \bigcup_{n \geq 0} \Gamma^n$. As proofs contain only finitely many formulae, to show that Γ^+ is consistent it suffices to show that Γ^n is consistent for all $n > 0$. Clearly this reduces to showing that if $\Gamma^n \cup \{\varphi_n\}$ is consistent, where $\varphi_n = \neg L(\bot)$, then $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{\neg L(\neg i)\}$ is consistent. So suppose for the sake of a contradiction that $\Gamma^{n+1} = \Gamma^n \cup \{\varphi_n\} \cup \{\neg L(\neg i)\}$ is inconsistent. Then there is a formula χ which is a conjunction of finitely many formulae from $\Gamma^n \cup \{\varphi_n\}$, such that $\vdash \chi \to L(\neg i)$. As $\chi \to L(\neg i)$ is a $\Box$-form and i does not occur in χ and L, using the COV^* rule we obtain $\vdash \chi \to L(\bot)$ and this contradicts the consistency of $\Gamma^n \cup \{\varphi_n\}$. So Γ^+ is consistent. Clearly Γ^+ is maximal. To see that Γ^+ is closed, suppose that $\neg L(\bot) \in \Gamma^+$, for some $\Box$-form L. The formula $\neg L(\bot)$ appears in the enumeration E_f; let it be φ_k. But then $\Gamma^k \cup \{\varphi_k\}$ is consistent as Γ^+ is consistent. Hence, by construction, Γ^{k+1} contains $\neg L(\neg i)$ for some nominal i, thus $\neg L(\neg i)$ is in Γ^+ and Γ^+ is closed. ∎

The crucial point to observe about the previous proof is this: *we used COV^* to paste names into $\Box$-forms of arbitrary depth.* (Intuitively, we built an MCS in which each possible sequence of transitions leads to a name.) It thus seems reasonable to hope that the names we have so carefully pasted in give us a precise blueprint for

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building a well-behaved model, that is, a model for which an Existence Lemma is provable. And this is precisely how things turn out, as we shall now see.

Definition 4.16 (Named Models) *Let Σ be a closed MCS in some countable language $\mathcal{L}(\downarrow)$, let $\mathcal{M}^c = (S^c, R^c, V^c)$ be the canonical model in $\mathcal{L}(\downarrow)$, and let $\mathcal{N}(S^c)$ be the set of all named MCSs in S^c. The named model $\mathcal{M}^n$ yielded by Σ is defined to be the triple (S^n, R^n, V^n), where $S^n = \{\Sigma\} \cup \{\Gamma \in \mathcal{N}(S^c) \mid$ there are $k > 0$ and $s_0, \ldots, s_k \in \mathcal{N}(S^c)$ such that $s_0 = \Sigma, s_k = \Gamma$ & $s_i R^c s_{i+1}$ for $0 \leq i \leq k-1\}$, and R^n and V^n are the restrictions of R^c and V^c, respectively, to S^n.*

Lemma 4.17 *Let $\mathcal{L}(\downarrow)$ be some countable language and $\mathcal{M}^n = (S^n, R^n, V^n)$ be the named model yielded by some closed $\mathcal{L}(\downarrow)$-MCS Σ. Then, for all MCSs $\Gamma, \Delta \in \mathcal{M}^n$, and every state symbol $\mathbf{s}$, if $\mathbf{s} \in \Gamma$ and $\mathbf{s} \in \Delta$, then $\Gamma = \Delta$.*

Proof. Suppose Γ and Δ are different MCSs in $\mathcal{M}^n = (S^n, R^n, V^n)$, both of which contain $\mathbf{s}$. Then there is a formula φ such that $\varphi \in \Gamma$ and $\neg\varphi \in \Delta$. Let Γ and Δ be reachable from Σ in $m \geq 0$ and $k \geq 0$ R^n-steps, respectively. We have $\Diamond^m(\mathbf{s} \wedge \varphi) \in \Sigma$. By the *Nom* schema, $\Box^k(\mathbf{s} \to \varphi) \in \Sigma$, therefore $\mathbf{s} \to \varphi \in \Delta$, hence $\varphi \in \Delta$. So both φ and $\neg\varphi$ are in Δ, which contradicts its consistency. ∎

Lemma 4.18 (Existence Lemma for Named Models) *Let $\mathcal{M}^n = (S^n, R^n, V^n)$ be a named model yielded by some closed MCS Σ, and let $\Gamma \in S^n$ be an MCS such that $\Diamond\varphi \in \Gamma$. Then there is an MCS $\Delta \in S^n$ such that $\Gamma R^n \Delta$ and $\varphi \in \Delta$.*

Proof. If we can find a nominal i such that $\Diamond(i \wedge \varphi)$ is in Γ, then the set $\Delta^0 = \{i \wedge \varphi\} \cup \{\psi \mid \Box\psi \in \Gamma\}$ is consistent. But then we can use the usual version of the Lindenbaum Lemma to extend Δ^0 to an MCS Δ. Clearly $\Gamma R \Delta$, $\Delta \in S^n$, and i names Δ, hence Δ is the required MCS.

So it remains to show that there exists a nominal i such that $\Diamond(i \wedge \varphi) \in \Gamma$. For sake of a contradiction suppose that for each nominal i, $\neg\Diamond(i \wedge \varphi) \in \Gamma$. By definition, all MCSs in the named model $\mathcal{M}^n$ have names. So, let j be a name for Γ. Therefore we have $j \wedge \neg\Diamond(i \wedge \varphi) \in \Gamma$, for all nominals i. Since Γ is in $\mathcal{M}^n$, $\Diamond^m(j \wedge \neg\Diamond(i \wedge \varphi)) \in \Sigma$ for some $m \geq 0$. Using *Nom* we get $\Box^m(j \to \Box(\varphi \to \neg i)) \in \Sigma$. As this holds for all nominals i, and since Σ is closed, we get $\Box^m(j \to \Box(\varphi \to \bot)) \in \Sigma$. Equivalently, $\Box^m(j \to \Box\neg\varphi) \in \Sigma$. As Γ is reachable from Σ in m R^n-steps, $j \to \Box\neg\varphi \in \Gamma$ and therefore $\Box\neg\varphi \in \Gamma$. As $\Diamond\varphi \in \Gamma$, this contradicts the consistency of Γ. So for some nominal i, $\Diamond(i \wedge \varphi) \in \Gamma$. ∎

Note that this proof is intrinsically modal (or path-based) whereas the proof of the Existence Lemma for $\mathcal{H}(\forall)$ (that is, Lemma 3.14) was not. The proof just given makes *explicit* use of the MCS Σ that generates the named model; in fact, the proof hinges on the fact that there is a sequence of transitions leading from Σ to Γ so that *Nom* and COV^* can be exploited. However no appeal is made to a generating MCS in the proof of Lemma 3.14; rather than a path-based argument, the proof of Lemma 3.14 exploits the strong, global, concept of witnessing available in $\mathcal{H}(\forall)$.

Now we are ready to define the model and assignment. Although we have no guarantee that named models are standard, or that natural definition of assignment gives rise to a standard assignment, Lemma 4.17 tells us that they have most of the properties we require. And in fact we can obtain suitable standard models and assignments simply by adding an extra dummy root node:

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Definition 4.19 (Completed Models and Completed Assignments) *If $\mathcal{M}^n = (S^n, R^n, V^n)$ is a named model yielded by some closed $\mathcal{L}(\downarrow)$-MCS Σ, then we define a* completed model *$\mathcal{M}$ based on $\mathcal{M}^n$ to be a triple (S, R, V), where $S = S^n \cup \{*\}$ ($*$ is an entity that is not an MCS); $R = R^n \cup \{(*, \Sigma)\}$; $V(p) = V^n(p)$ for all propositional symbols p, and for all nominals i, $V(i) = \{\Gamma \in S^n \mid i \in \Gamma\}$ if this set is non-empty, and $V(i) = \{*\}$ otherwise.*

The completed assignment *g on $\mathcal{M}$ is defined as follows: for all state variables x, $g(x) = \{\Gamma \in S^n \mid x \in \Gamma\}$ if this set is non-empty, and $g(x) = \{*\}$ otherwise.*

It follows from Lemma 4.17 that completed models are standard. Moreover, completed assignments are standard too, thus (by the Soundness Theorem) all theorems of $\mathcal{H}(\downarrow)$ are true in completed models with respect to completed assignments.

One other thing is worth noting: Definition 4.19 is slightly simpler than the analogous definition for $\mathcal{H}(\forall)$. With $\mathcal{H}(\forall)$ we had to take care to glue on the dummy state $*$ only when it was required, for to prove the Truth Lemma we needed a guarantee that every state in the model had a label. With $\mathcal{H}(\downarrow)$ we don't need to bother. As $\downarrow$ binds variables to the current state, the presence of $*$ is irrelevant to the proof of the following Truth Lemma:

Lemma 4.20 (Truth Lemma) *Let $\mathcal{M}$ be a completed model in some countable language $\mathcal{L}(\downarrow)$, g the completed $\mathcal{M}$-assignment and Δ an $\mathcal{L}(\downarrow)$-MCS in $\mathcal{M}$. For every $\mathcal{L}(\downarrow)$-formula φ:*

$$\varphi \in \Delta \quad \textit{iff} \quad \mathcal{M}, g, \Delta \models \varphi.$$

Proof. The proof is by induction on the complexity of φ. If φ is a state symbol or a propositional symbol the required equivalence follows from the definition of the model $\mathcal{M}$ and the assignment g, and the Boolean cases are obvious. The modal case makes use of the definition of the canonical relation and the Existence Lemma.

So suppose $\downarrow x\psi \in \Delta$. Since Δ is named, it is $\downarrow$-witnessed (see Lemma 4.13) so there is a nominal i such that $i \wedge \psi[i/x] \in \Delta$. By the inductive hypothesis $\mathcal{M}, g, \Delta \models i \wedge \psi[i/x]$. Thus, by the contrapositive of the *Q2* axiom, $\mathcal{M}, g, \Delta \models \downarrow x\psi$.

For the other direction assume $\mathcal{M}, g, \Delta \models \downarrow x\psi$. That is, $\mathcal{M}, g', \Delta \models \psi$, where $g' \overset{x}{\sim} g$ such that $g'(x) = \{\Delta\}$. Now Δ contains a nominal, say i, so by clause 2 of the Substitution Lemma, $\mathcal{M}, g, \Delta \models \psi[i/x]$, hence by the inductive hypothesis $\psi[i/x] \in \Delta$. So, by the contrapositive of the *Q2* axiom, $\downarrow x\psi$ is in Δ as required. ∎

Theorem 4.21 (Completeness) *Every consistent set of formulae in a countable language $\mathcal{L}(\downarrow)$ is satisfiable in a rooted and countable standard model with respect to a standard assignment function.*

Proof. Let Σ be a consistent set of $\mathcal{L}(\downarrow)$-formulae. Using the Extended Lindenbaum Lemma we can expand Σ to a closed MCS Σ^+ in the countable language $\mathcal{L}(\downarrow)^+$. Let $\mathcal{M}$ be the completed model yielded by Σ^+ and g the completed $\mathcal{M}$-assignment. It follows from the Truth Lemma that $\mathcal{M}, g, \Sigma^+ \models \Sigma^+$ and so $\mathcal{M}, g, \Sigma^+ \models \Sigma$. By the definition of completed models, either Σ^+ is a root of this model, or there is an additional point $*$ which is. As every state in the model is named by one of the (countably many) state symbols in $\mathcal{L}(\downarrow)^+$, the model is countable. ∎

To close this section, let's deal with a matter that may be bothering some readers. Although we didn't use COV^* in our discussion of $\mathcal{H}(\forall)$, it should be clear that we

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could have done. (After all, the definition of a $\Box$-form is essentially modal: it makes no mention of $\downarrow$.) So surely the COV^* rules must be derivable in $\mathcal{H}(\forall)$? They are, and we can show this by making use of the fact that $\mathcal{H}(\forall)$ contains *Barcan* analogs.[6] In what follows, $\vdash$ means provable in $\mathcal{H}(\forall)$.

Lemma 4.22 *Given a language $\mathcal{L}(\forall)$, let L be any $\Box$-form in this language that contains no free occurrences of x. Then $\vdash \forall x L(\neg x) \to L(\bot)$.*

Proof. By induction on the structure of L. For the base case $\#$, note that we have $\vdash \forall x \neg x \to \bot$, for this is equivalent to $\exists x x$, an instance of the *Name* schema. So let L_1 be a $\Box$-form that contains no free occurrences of x and take $\vdash \forall x L_1(\neg x) \to L_1(\bot)$ as the inductive hypothesis.

Suppose L is $\varphi \to L_1$, where φ contains no free occurrences of x. By the inductive hypothesis and propositional calculus we have $\vdash (\varphi \to \forall x L_1(\neg x)) \to (\varphi \to L_1(\bot))$. Now, $\vdash \forall x(\varphi \to L_1(\neg x)) \to (\varphi \to \forall x L_1(\neg x))$ is an instance of *Q1*. Hence $\vdash \forall x(\varphi \to L_1(\neg x)) \to (\varphi \to L_1(\bot))$. That is, $\vdash \forall x L(\neg x)) \to L(\bot)$ as required.

Suppose L is $\Box L_1$. It follows from the inductive hypothesis, using simple modal reasoning, that $\vdash \Box \forall x L_1(\neg x) \to \Box L_1(\bot)$. As $\vdash \forall x \Box L_1(\neg x) \to \Box \forall x L_1(\neg x)$ (this is just an instance of *Barcan*) we have $\vdash \forall x \Box L_1(\neg x) \to \Box L_1(\bot)$. That is, $\vdash \forall x L(\neg x) \to L(\bot)$ as required. ∎

Proposition 4.23 *The COV^* rules are derivable in $\mathcal{H}(\forall)$.*

Proof. First we show that the simplest COV^* rules (that is, the rules of the form *if* $\vdash \neg i$*, for some nominal* i*, then* $\vdash \bot$) are derivable in $\mathcal{H}(\forall)$. In fact they are vacuously derivable, for we cannot prove $\neg i$ in $\mathcal{H}(\forall)$ for any nominal i. This is immediate by the soundness of $\mathcal{H}(\forall)$, for negations of nominals aren't valid.

So suppose $\vdash \varphi \to L(\neg i)$, where i does not occur in L or in φ. Applying generalization on nominals (see Lemma 3.5) we can prove $\vdash \forall x(\varphi \to L(\neg x))$, for some state variable x that does not occur in $\varphi \to L(\neg i)$. As x does not occur in φ or L the previous lemma is applicable and we have $\vdash \forall x(\varphi \to L(\neg x)) \to (\varphi \to L(\bot))$. Hence $\vdash \varphi \to L(\bot)$, as required.

So suppose $\vdash \Box L(\neg i)$, where i does not occur in L. By generalization on nominals we have $\vdash \forall x \Box L(\neg x)$, for some fresh variable x. By *Barcan*, $\vdash \Box \forall x L(\neg x)$. Now, by the previous lemma, $\vdash \forall x L(\neg x) \to L(\bot)$, hence by simple modal reasoning $\vdash \Box \forall x L(\neg x) \to \Box L(\bot)$. Hence $\vdash \Box L(\bot)$ as required. ∎

5 Concluding remarks

Hybridization is an interesting (and unusual) strategy for boosting modal expressivity. The basic mechanism — the use of state symbols to internalize labeling — is natural and can be developed in various directions. Nonetheless, hybrid completeness is a neglected topic. While there has been work on completeness for hybrid languages containing the universal modality, and for modal languages with nominals (essentially the free variable fragments of hybrid languages), this leaves a lot of unexplored territory in between.

[6] Passy and Tinchev [12] make a similar observation for the version of COV^* definable for hybridized PDL; their proof is more elegant than the one given below. This reflects the fact that the COV^* rules for PDL with nominals are simpler than the version used in this paper; in particular, the internal structure available in PDL modalities means that $\Box$-forms are not needed.

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In this paper we presented completeness results for two hybrid languages, $\mathcal{L}(\forall)$ and $\mathcal{L}(\downarrow)$, neither of which contains the universal modality. Both results were proved by combining techniques from modal and classical logic, but the balance of modal and classical ideas was very different. Intuitively, $\mathcal{L}(\forall)$ is more classical than $\mathcal{L}(\downarrow)$. This is borne out by its completeness proof, which uses a fairly even-handed blend of modal and classical model-building techniques. The weaker language $\mathcal{L}(\downarrow)$, on the other hand, is closer to the original modal language: in particular, it binds state variables locally. Because of its locality, we applied a technique from extended modal logic, the use of the COV^* rules of proof, and worked with named models. The completeness proof for $\mathcal{L}(\downarrow)$, and in particular, the proof of the Existence Lemma, had a strong modal bias.

Since completing this paper we have continued to investigate hybrid completeness. In one line of work (see Blackburn and Tzakova [4]) we investigate the hybrid binder $\downarrow^1$, an existential quantifier over *successor* states. That is:

$$\mathcal{M}, g, s \models \downarrow^1_x \varphi \quad \text{iff} \quad \mathcal{M}, g', s \models \varphi, \text{ for some } g' \overset{x}{\sim} g \text{ such that } g'(x) = \{t\} \text{ and } sRt$$

Given the fundamental importance of successor states to Kripke semantics, this is a natural choice of binder. Over transitive frames, the Barcan analogs for this binder are valid, and completeness can be proved without making use of COV^*. However the completeness proof is *not* similar to the proof given here for $\mathcal{H}(\forall)$; various non-classical features of the $\downarrow^1$ block straightforward adaptations of this approach. In fact, in spite of the fact that no use of COV^* is made, the completeness proof is much closer to the proof given for $\mathcal{H}(\downarrow)$. In particular, it hinges on the use of named models, and the Existence Lemma is proved using a path-based argument that makes explicit use of the generating MCS.

Acknowledgements

We would like to thank Hans Jürgen Ohlbach and both referees for their comments and suggestions.

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Received 30 October 1997. Revised 6 April 1998

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