# Mechanizing Many-Sorted Polyadic Hybrid Logic in the Lean 4 Prover

Andrei-Alexandru Oltean

University of Bucharest

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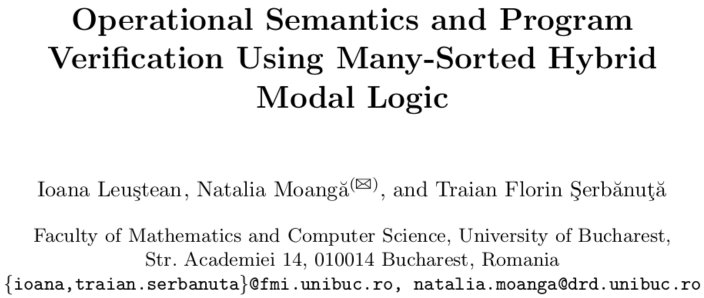
## Mechanizing what?!

- This 2019 paper [1]:



**Operational Semantics and Program Verification Using Many-Sorted Hybrid Modal Logic**

Ioana Leuştean, Natalia Moangă(✉), and Traian Florin Şerbănuţă

Faculty of Mathematics and Computer Science, University of Bucharest,
Str. Academiei 14, 010014 Bucharest, Romania
{ioana,traian.serbanuta}@fmi.unibuc.ro, natalia.moanga@drd.unibuc.ro

## Mechanizing what?!

### Many-Sorted Polyadic Hybrid Logic

```
s ::= 0;
i ::= 0;
while (++i <= n) do
    s ::= s + i
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$$
\begin{aligned}
\boldsymbol{S} &:= \{AExp,\ BExp,\ Var,\ Stmt\} \\
\Sigma_{Var,AExp} &:= \{++\} \\
\Sigma_{AExpAExp,AExp} &:= \{+\} \\
\Sigma_{AExpAExp,BExp} &:= \{<=\} \\
\Sigma_{VarAExp,Stmt} &:= \{::=\} \\
\Sigma_{StmtStmt,Stmt} &:= \{;\} \\
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$\boldsymbol{N}_{AExp} := \{0\}$ $\quad$ $\boldsymbol{N}_{Var} := \{s,\ i,\ n\}$

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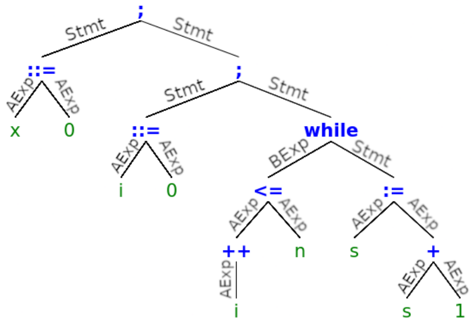
### Many-Sorted Polyadic Hybrid Logic

- Programming language AST? Or modal logic AST?



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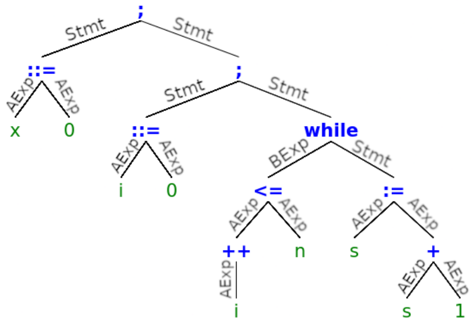

- **Both.**

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- Humans may err; maybe there are errors in the paper;
- The resulting formal artifact could be used as a foundation for verifying software and programming languages in the future.

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1. Implemented the language, its proof system, and its semantics;
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3. Proved soundness of the new system;
4. Work is in progress for the completeness proof.

## The syntax

**Definition (Signatures with constant nominals)**

A **signature with constant nominals** is a triple $(S, \Sigma, N)$, where:

- $S$ is a non-empty, countable set;
- $\Sigma$ is an $S^* \times S$-indexed family of countable sets;
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In code:

```
structure Signature (α : Type u) where
    S     : Set α
    «Σ»   : List S → S → Set α
    N     : S → Set α

    sortsCtbl : Encodable S
    opsCtbl (dom range) : Encodable («Σ» dom range)
    nomCtbl (s)         : Encodable (N s)
    sNonEmpty : Inhabited S
```

## The syntax

We further fix sorted sets of propositional variables (*PROP*), non-constant nominals (*NOM*) and state variables (*SVAR*):

```
structure Symbols (α : Type u) where
    signature   : Signature α

    prop : (s : signature.S) → Set α
    nom  : (s : signature.S) → Set α
    svar : (s : signature.S) → Set α

    propCtbl (s) : Encodable (prop s)
    nomCtbl  (s) : Encodable (nom s)
    svarCtbl (s) : Denumerable (svar s)
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### Definition (Formulas)

The set of formulas is given by the grammar:

$$\varphi_s := p \mid j \mid y \mid \neg\varphi_s \mid \varphi_s \vee \varphi_s \mid \sigma(\varphi_{s_1}, \ldots, \varphi_{s_n}) \mid @^s_k \varphi_t \mid \forall x \varphi_s,$$

$p \in PROP_s$, $j \in NOM_s \cup N_s$, $k \in NOM_t \cup N_t$, $y \in SVAR_s$, $x \in SVAR_t$, $\sigma \in \Sigma_{s_1 \ldots s_n, s}$

## The syntax

In code:

```
inductive FormL (symbs : Symbols α) :
    List symbs.signature.S → Type u
| prop : symbs.prop s → FormL symbs [s]
| nom  : symbs.nominal s → FormL symbs [s]
| svar : symbs.svar s → FormL symbs [s]
| appl : symbs.signature.«Σ» (h :: t) s →
            FormL symbs (h :: t) → FormL symbs [s]
| or   : FormL symbs [s] → FormL symbs [s] → FormL symbs [s]
| neg  : FormL symbs [s] → FormL symbs [s]
| at   : symbs.nominal t → FormL symbs [t] → FormL symbs [s]
| bind : symbs.svar t → FormL symbs [s] → FormL symbs [s]
| cons : FormL symbs [s₁] → FormL symbs (s₂ :: t) →
            FormL symbs (s₁ :: s₂ :: t)

abbrev Form (symbs : Symbols α) (s : symbs.signature.S) :=
    FormL symbs [s]
```

## The semantics

In code:

```
def Sat (M : Model symbs) (g : Assignment M)
    (w : WProd M.Fr.W sorts) : FormL symbs sorts → Prop
| .prop p         => w ∈ M.Vp p
| .nom n          => w = M.VNom n
| .svar x         => w = g x
| .appl σ arg     => ∃ w', Sat M g w' arg ∧ ⟨w, w'⟩ ∈ M.Fr.R σ
| .neg φ          => ¬ Sat M g w φ
| .or φ ψ         => Sat M g w φ ∨ Sat M g w ψ
| .at k φ         => let u := M.VNom k;  Sat M g u φ
| .bind x φ       => ∀ g', g'.variant g x → Sat M g' w φ
| .cons φ ψs      => Sat M g w.1 φ ∧ Sat M g w.2 ψs
```

## The proof system

Q: Can we take the following schema as an axiom?

$$\vdash^s \forall x \sigma^{\Box}(\varphi_1, \ldots, \varphi_n) \to \sigma^{\Box}(\varphi_1, \ldots, \forall x \varphi_i, \ldots, \varphi_n) \qquad \text{(Barcan)}$$

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No! We found a countermodel:

```
theorem BarcanAntecedentTrue  :
    ⟨countermodel, g, w₀⟩ ⊨ barcan_antecedent
theorem BarcanConsequentFalse :
    ⟨countermodel, g, w₀⟩ ⊭ barcan_consequent
```

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```

We need a restricted version:

$$\vdash^s \forall x \sigma^{\Box}(\varphi_1, \ldots, \varphi_n) \to \sigma^{\Box}(\varphi_1, \ldots, \forall x \varphi_i, \ldots, \varphi_n), \qquad \text{(Barcan)}$$

if $x$ does not occur free in $\varphi_j$ for $j \neq i$

## The proof system

Q: Can we use the following proof rule?

$$\text{If } \vdash^{s} @^{s}_{j}\varphi, \text{then } \vdash^{s'} \varphi, \quad \text{if } j \text{ does not occur in } \varphi \qquad \text{(Name@)}$$

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No! Using it on constant nominals leads to unsound derivations. We require:

$$\text{If } \vdash^{s} @_{j}^{s}\varphi, \text{then } \vdash^{s'} \varphi, \quad \text{if } j \notin N \text{ and } j \text{ does not occur in } \varphi \qquad (\text{Name@})$$

## The proof system

Q: Can we use the following proof rule?

$$\text{If } \vdash^{s} @_{j}\sigma(\ldots, k, \ldots) \land @_{k}\varphi \to \psi, \text{ then } \vdash^{s} @_{j}\sigma(\ldots, \varphi, \ldots) \to \psi, \tag{Paste}$$

for $k \neq j$, and $k$ does not occur in $\varphi$ or $\psi$

## The proof system

Q: Can we use the following proof rule?

$$\text{If } \vdash^s @_j\sigma(\ldots, k, \ldots) \wedge @_k\varphi \to \psi, \text{ then } \vdash^s @_j\sigma(\ldots, \varphi, \ldots) \to \psi, \qquad \text{(Paste)}$$

$$\text{for } k \neq j, \text{ and } k \text{ does not occur in } \varphi \text{ or } \psi$$

Corrected:

$$\text{If } \vdash^s @_j\sigma(\ldots, k, \ldots) \wedge @_k\varphi \to \psi, \text{ then } \vdash^s @_j\sigma(\ldots, \varphi, \ldots) \to \psi, \qquad \text{(Paste)}$$

$$\text{for } k \neq j, k \notin N, \text{ and } k \text{ does not occur in } \varphi, \psi, \text{ or the } \ldots \text{ formulas}$$

```
inductive Proof {symbs : Symbols α} (Λ : AxiomSet symbs) :
    (s : symbs.signature.S) → Form symbs s → Type u
  -- Λ:
  | ax     : (ψ : Λ s) → Proof Λ s ψ
  -- Propositional:
  | prop1 φ ψ   : Proof Λ s (φ → (ψ → φ))
  | prop2 φ ψ χ : Proof Λ s ((φ → (ψ → χ)) → (φ → ψ) → (φ → χ))
  | prop3 φ ψ   : Proof Λ s ((∼ψ → ∼φ) → (φ → ψ))
  -- K:
  | k φ ψ χ
      (σ : symbs.signature.«Σ» _ s)
      (C : (φ → ψ).Context χ):
          Proof Λ s (ℋ⟨σ⟩⍰ χ → (ℋ⟨σ⟩⍰ C[φ] → ℋ⟨σ⟩⍰ C[ψ]))
  | mp     : Proof Λ s (φ → ψ) → Proof Λ s φ → Proof Λ s ψ
  | ug {φ : Form symbs s₁}
       (C : φ.Context ψ):
          Proof Λ s₁ φ → Proof Λ s₂ (ℋ⟨σ⟩⍰ ψ)
  -- H(@, ∀):
  -- 1. Axioms about @
  | kAt j φ ψ      : Proof Λ s (ℋ@j (φ → ψ) → (ℋ@j φ → ℋ@j ψ))
  | agree j k φ    : Proof Λ s (ℋ@k (ℋ@j φ) ↔ ℋ@j φ)
  | selfDual j φ : Proof Λ s (ℋ@j φ ↔ ∼ ℋ@j (∼φ))
  | intro j φ      : Proof Λ s (ℋNom j → (φ ↔ ℋ@j φ))
  | back j φ ψ
```

## Soundness and completeness

```
theorem Soundness {Λ : AxiomSet symbs} : ⊢(Λ, s) ψ → ⊨Mod(Λ) ψ
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Can we do the same for completeness?

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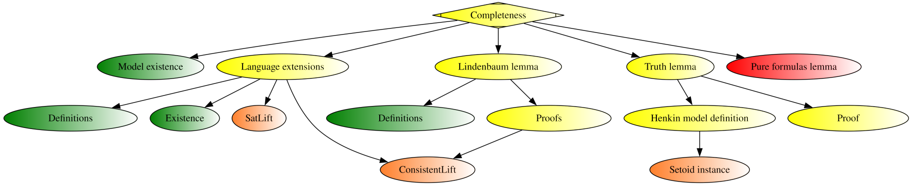
```

Can we do the same for completeness?

Yes, but it's more complex. We took a top-down approach.

# Soundness and completeness

State of the completeness formalization:



## Future work

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3. DSL for seamless specification of PL's.

Thank you!

## Bibliography I

[1] Ioana Leuștean, Natalia Moangă, and Traian Florin Șerbănuță. "Operational Semantics and Program Verification Using Many-Sorted Hybrid Modal Logic". In: *Automated Reasoning with Analytic Tableaux and Related Methods*. Ed. by Serenella Cerrito and Andrei Popescu. Cham: Springer International Publishing, 2019, pp. 446–476. DOI: 10.1007/978-3-030-29026-9_25.