Mean Field Games

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1 A short introduction to mean field games

The theory of mean field games (*MFGs* in short), has been introduced in the pioneering works of J-M. Lasry and P-L. Lions [3, 4, 5], and aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of players tends to infinity. It supposes that the rational players are indistinguishable and individually have a negligible influence on the game, and that each individual strategy is influenced by some averages of quantities depending on the states (or the controls) of the other players. The applications of MFGs are numerous, from economics and finance to the study of crowd motion. On the other hand, very few MFG problems have explicit or semi-explicit solutions. Therefore, numerical simulations of MFGs play a crucial role in obtaining quantitative information from this class of models.

Let us start by supposing for simplicity that the state space is $\mathbb{R}^d$ so that no boundary conditions are needed. Boundary conditions will be discussed later. We fix a finite time horizon $T > 0$. Starting from Section 2 below, we will focus on the simpler situation in which $d = 1$.

Definition : differential operators in $\mathbb{R}^d$ Only for this introductory part, it is useful to define a few differential operators in $\mathbb{R}^d$. For a smooth function $\psi : \mathbb{R}^d \to \mathbb{R}$, let $\nabla\psi$ (the gradient of ψ) be the function $\nabla\psi : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\nabla\psi(x) = \left(\frac{\partial\psi}{\partial x_1}(x), \dots, \frac{\partial\psi}{\partial x_d}(x)\right)^T$ and let $\Delta\psi$ (the laplacian of ψ) be the function $\Delta\psi : \mathbb{R}^d \to \mathbb{R}$ defined by $\Delta\psi(x) = \sum_{i=1}^d \frac{\partial^2\psi}{\partial x_i^2}(x)$.
For a smooth function $V : \mathbb{R}^d \to \mathbb{R}^d$, let $\mathrm{div}(V)$ (the divergence of V) be the function $\mathrm{div}(V) : \mathbb{R}^d \to \mathbb{R}$ defined by $\mathrm{div}(V)(x) = \sum_{i=1}^d \frac{\partial V_i}{\partial x_i}(x)$.

Let $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(x, m, \gamma) \mapsto f(x, m, \gamma)$ and $\phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $(x, m) \mapsto \phi(x, m)$ be respectively a running cost and a terminal cost, on which assumptions will be made later on.

We consider the following MFG: find a flow of probability densities $\hat{m} : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and a feedback control $\hat{v} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the following two conditions:

1. $\hat{v}$ minimizes
$$J_{\hat{m}} : v \mapsto J_{\hat{m}}(v) = \mathbb{E}\left[\int_0^T f(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v))dt + \phi(X_T^v, \hat{m}(T, X_T^v))\right]$$
subject to the constraint that the process $X^v = (X_t^v)_{t\geq 0}$ solves the stochastic differential equation (SDE)
$$dX_t^v = b(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v))dt + \sigma dW_t, \qquad t \geq 0, \tag{1}$$
where σ is the volatility, b is a given function from $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, and X_0^v is an independent random variable in $\mathbb{R}^d$, distributed according to the law m_0;

2. For all $t \in [0,T]$, $\hat{m}(t,\cdot)$ is the law of $X_t^{\hat{v}}$.

It is useful to note that for a given feedback control v, the density m_t^v of the law of X_t^v following (1) solves the Kolmogorov-Fokker-Planck (KFP) equation:

$$\begin{cases} \dfrac{\partial m^v}{\partial t}(t,x) - \nu \Delta m^v(t,x) + \operatorname{div}\Big(m^v(t,\cdot) b(\cdot, \hat{m}(t,\cdot), v(t,\cdot))\Big)(x) = 0, & \text{in } (0,T] \times \mathbb{R}^d, \\ m^v(0,x) = m_0(x), & \text{in } \mathbb{R}^d, \end{cases} \tag{2}$$

where $\nu = \sigma^2/2$.

Let $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \ni (x,m,p) \mapsto H(x,m,p) \in \mathbb{R}$ be the Hamiltonian of the control problem faced by an infinitesimal player. It is defined by

$$H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \ni (x,m,p) \mapsto H(x,m,p) = \max_{\gamma \in \mathbb{R}^d} \Big(-f(x,m,\gamma) - \langle b(x,m,\gamma), p \rangle \Big) \in \mathbb{R}.$$

In the sequel, we will assume that the running cost f and the drift b are such that H is well-defined, C^1 with respect to (x,p), and strictly convex with respect to p.

From standard optimal control theory, one can characterize the best strategy through the value function u of the above optimal control problem for a typical player, which satisfies a Hamilton-Jacobi-Bellman (HJB) equation. Together with the equilibrium condition on the distribution, we obtain that the equilibrium best response $\hat{v}$ is characterized by

$$\hat{v}(t,x) = \arg\max_{a \in \mathbb{R}^d} \Big\{ -f(x, m(t,x), a) - \langle b(x, m(t,x), a), \nabla u(t,x) \rangle \Big\},$$

and, denoting H_p the gradient of H with respect to p, that the drift at equilibrium is

$$b(x, m(t,x), \hat{v}(t,x)) = -H_p(x, m(t,x), \nabla u(t,x)),$$

where (u,m) solves the following forward-backward PDE system:

$$\begin{cases} -\dfrac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, m(t,x), \nabla u(t,x)) = 0, & \text{in } [0,T) \times \mathbb{R}^d, \quad \text{(3a)} \\ \dfrac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) - \operatorname{div}\big(m(t,\cdot) H_p(\cdot, m(t,\cdot), \nabla u(t,\cdot))\big)(x) = 0, & \text{in } (0,T] \times \mathbb{R}^d, \quad \text{(3b)} \\ u(T,x) = \phi(x, m(T,x)), \qquad m(0,x) = m_0(x), & \text{in } \mathbb{R}^d. \quad \text{(3c)} \end{cases}$$

Remark 1.1 *Note that the Hamilton-Jacobi equation* (3a) *is a backward parabolic nonlinear equation and is supplemented with a terminal condition, the left part of* (3c). *The Komogorov-Fokker-Planck equation* (3b) *is a forward parabolic linear equation and is supplemented with an initial condition, the right part of* (3c).

Example 1.1 (f depends separately on γ and m) *Consider the case where the drift is the control, i.e. $b(x,m,\gamma) = \gamma$, and the running cost is of the form $f(x,m,\gamma) = L_0(x,\gamma) + f_0(x,m)$ where $L_0(x,\cdot) : \mathbb{R}^d \ni \gamma \mapsto L_0(x,\gamma) \in \mathbb{R}$ is strictly convex and such that $\lim_{|\gamma|\to\infty} \min_{x\in\mathbb{R}^d} \frac{L_0(x,\gamma)}{|\gamma|} = +\infty$. We set $H_0(x,p) = \max_{\gamma\in\mathbb{R}^d} \langle -p, \gamma \rangle - L_0(x,\gamma)$, which is convex with respect to p. Then*

$$H(x,m,p) = \max_{\gamma \in \mathbb{R}^d} \{ -L_0(x,\gamma) - \langle \gamma, p \rangle \} - f_0(x,m) = H_0(x,p) - f_0(x,m).$$

In particular, for $1 < \beta' < \infty$ *and* $\beta = \beta'/(\beta'-1)$, $L_0(x,\gamma) = \frac{1}{\beta'}|\gamma|^{\beta'}$ *then* $H_0(x,p) = \frac{1}{\beta}|p|^\beta$ *and the maximizer in the above expression is* $\hat{\gamma}(p) = -|p|^{\beta-2}p^\beta$, *the Hamiltonian reads* $H(x,m,p) = \frac{1}{\beta}|p|^\beta - f_0(x,m)$ *and the equilibrium best response is* $\hat{v}(t,x) = -|\nabla u(t,x)|^{\beta-2}\nabla u(t,x)$ *where* (u,m) *solves the PDE system*

$$\begin{cases} -\dfrac{\partial u}{\partial t}(t,x) - \nu\Delta u(t,x) + \dfrac{1}{\beta}|\nabla u(t,x)|^\beta = f_0(x,m(t,x)), & \text{in } [0,T)\times\mathbb{R}^d, \\ \dfrac{\partial m}{\partial t}(t,x) - \nu\Delta m(t,x) - \operatorname{div}\left(m(t,\cdot)|\nabla u(t,\cdot)|^{\beta-2}\nabla u(t,\cdot)\right)(x) = 0, & \text{in } (0,T]\times\mathbb{R}^d, \\ u(T,x) = \phi(x,m(T,x)), \qquad m(0,x) = m_0(x), & \text{in } \mathbb{R}^d. \end{cases}$$

Remark 1.2 *Deterministic mean field games (i.e. for* $\nu = 0$*) are also quite meaningful. The numerical schemes discussed below can also be applied to these situations, even if the convergence results in the available literature are obtained under the hypothesis that* ν *is bounded from below by a positive constant.*

2 Finite difference schemes

We present a finite-difference scheme first introduced in [1]. We consider the special case described in Example 1.1, with $H(x,m,p) = H_0(x,p) - f_0(x,m)$, although similar methods have been proposed, applied and at least partially analyzed in situations when the Hamiltonian does not depend separately on m and p, for example models addressing congestion, see for example [2]. For simplicity, we restrict ourselves to Hamiltonians H_0 and coupling costs of the form

$$H_0(x,p) = \frac{1}{\beta}|p|^\beta - g(x), \qquad f_0(x,m) = \tilde{f}_0(m(x)). \tag{5}$$

For simplicity again, we focus on the one-dimensional setting, i.e. $d = 1$. We also suppose that the state space is the domain $\Omega =]0,1[$, i.e. the stochastic process involved in the dynamics of the players is reflected at $\partial\Omega$.

The boundary value problem becomes

$$\begin{cases} -\dfrac{\partial u}{\partial t}(t,x) - \nu\dfrac{\partial^2 u}{\partial x^2}(t,x) + \dfrac{1}{\beta}\left|\dfrac{\partial u}{\partial x}(t,x)\right|^\beta = g(x) + \tilde{f}_0(m(t,x)), & \text{in } [0,T)\times\Omega, \quad (6a) \\ \dfrac{\partial m}{\partial t}(t,x) - \nu\dfrac{\partial^2 m}{\partial x^2}(t,x) - \dfrac{\partial}{\partial x}\left(m(t,\cdot)\left|\dfrac{\partial u}{\partial x}(t,\cdot)\right|^{\beta-2}\dfrac{\partial}{\partial x}u(t,\cdot)\right)(x) = 0, & \text{in } (0,T]\times\Omega, \quad (6b) \\ \dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,1) = 0, & \text{on } (0,T), \quad (6c) \\ \dfrac{\partial m}{\partial x}(t,0) = \dfrac{\partial m}{\partial x}(t,1) = 0, & \text{on } (0,T), \quad (6d) \\ u(T,x) = \phi(x,m(T,x)), \qquad m(0,x) = m_0(x), & \text{in } \Omega. \quad (6e) \end{cases}$$

Discretization. Let N_T and N_h be two positive integers. We consider $N_T + 1$ and N_h points in time and space respectively. Set $\Delta t = T/N_T$, $h = 1/(N_h - 1)$, and $t_n = n \times \Delta t$, $x_i = i \times h$ for $(n,i) \in \{0,\ldots,N_T\} \times \{0,\ldots,N_h - 1\}$.

We approximate u and m respectively by vectors U and $M \in \mathbb{R}^{(N_T+1)\times N_h}$, that is, $u(t_n, x_i) \approx U_i^n$ and $m(t_n, x_i) \approx M_i^n$ for each $(n, i) \in \{0, \dots, N_T\} \times \{0, \dots, N_h - 1\}$. We use a superscript and a subscript respectively for the time and space indices.

To take into account Neumann boundary conditions, we introduce ghost nodes $x_{-1} = -h$, $x_{N_h} = 1 + h$, and set

$$U_{-1}^n = U_0^n, \quad U_{N_h}^n = U_{N_h-1}^n, \quad M_{-1}^n = M_0^n, \quad M_{N_h}^n = M_{N_h-1}^n. \tag{7}$$

We introduce the finite difference operators

$$\begin{aligned}
(D_t W)^n &= \frac{1}{\Delta t}(W^{n+1} - W^n), && n \in \{0, \dots N_T - 1\}, && W \in \mathbb{R}^{N_T+1},\\
(DW)_i &= \frac{1}{h}(W_{i+1} - W_i), && i \in \{0, \dots N_h - 1\}, && W \in \mathbb{R}^{N_h},\\
(\Delta_h W)_i &= -\frac{1}{h^2}\left(2W_i - W_{i+1} - W_{i-1}\right), && i \in \{0, \dots N_h - 1\}, && W \in \mathbb{R}^{N_h},\\
[\nabla_h W]_i &= ((DW)_i, (DW)_{i-1}) \in \mathbb{R}^2, && i \in \{0, \dots N_h - 1\}, && W \in \mathbb{R}^{N_h},
\end{aligned}$$

in which the special cases $i = 0$ and $i = N_h - 1$ can be written thanks to the above mentioned discrete version of the Neumann boundary conditions, see (7).

Discrete Hamiltonian. Let $\tilde{H} : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, p_1, p_2) \mapsto \tilde{H}(x, p_1, p_2)$ be a discrete Hamiltonian, assumed to satisfy the following properties:

1. ($\tilde{\mathbf{H}}_\mathbf{1}$) Monotonicity: for every $x \in \overline{\Omega}$, $\tilde{H}$ is nonincreasing in p_1 and nondecreasing in p_2.
2. ($\tilde{\mathbf{H}}_\mathbf{2}$) Consistency: for every $x \in \overline{\Omega}$, $p \in \mathbb{R}$, $\tilde{H}(x, p, p) = H_0(x, p)$.
3. ($\tilde{\mathbf{H}}_\mathbf{3}$) Differentiability: for every $x \in \overline{\Omega}$, $\tilde{H}$ is of class C^1 in p_1, p_2
4. ($\tilde{\mathbf{H}}_\mathbf{4}$) Convexity: for every $x \in \overline{\Omega}$, $(p_1, p_2) \mapsto \tilde{H}(x, p_1, p_2)$ is convex.

Example 2.1 *If $H_0(x, p) = \frac{1}{\beta}|p|^\beta - g(x)$, then one can take $\tilde{H}(x, p_1, p_2) = \frac{1}{\beta}\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta}{2}} - g(x)$, where X_+, resp. X_- stand for the positive (resp. negative) part of X: $X = X_+ - X_-$ and $|X| = X_+ + X_-$, and where we set $X_+^2 = (X_+)^2$ and $X_-^2 = (X_-)^2$.*

The monotonicity of the discrete Hamiltonian guarantees uniqueness in the discrete HJB equations and discrete KFP equations below. It also guarantees that the solution of the discrete KFP equation below is non negative if its initial condition is non negative.

The consistency of the discrete Hamiltonian is a key ingredient for convergence of the discrete schemes.

The differentiability of the discrete Hamiltonian makes it possible to use Newton method for solving the discrete HJB equation below.

Discrete HJB equation. We consider the following discrete version of the HJB equation (6a), supplemented with the Neumann conditions (6c) and the terminal condition in (6e):

$$\begin{cases}
-(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}(x_i, [\nabla_h U^n]_i) = \tilde{f}_0(M_i^{n+1}), & 0 \le i < N_h,\ 0 \le n < N_T, \quad (8a)\\
U_{-1}^n = U_0^n, & 0 \le n < N_T, \quad (8b)\\
U_{N_h}^n = U_{N_h-1}^n, & 0 \le n < N_T, \quad (8c)\\
U_i^{N_T} = \phi(M_i^{N_T}), & 0 \le i < N_h. \quad (8d)
\end{cases}$$

Note that this is an implicit Euler scheme since the equation is backward in time.

Discrete KFP equation. To define an appropriate discretization of the KFP equation (6b), together with (6d) and the initial condition in (6e), we consider the weak form. For a smooth test function $w \in C^\infty([0,T] \times \Omega)$, it involves, among other terms, the expression

$$- \int_\Omega \partial_x \big(H_p(x, \partial_x u(t,x)) m(t,x)\big) w(t,x) dx = \int_\Omega H_p(x, \partial_x u(t,x)) m(t,x)\, \partial_x w(t,x) dx\,, \tag{9}$$

where we used an integration by parts and the Neumann boundary conditions assuming that $H_p(x,0) = 0$. In view of what precedes, it is quite natural to propose the following discrete version of the right hand side of (9):

$$h \sum_{i=0}^{N_h - 1} M_i^{n+1} \left(\tilde{H}_{p_1}(x_i, [\nabla_h U^n]_i) \frac{W_{i+1}^n - W_i^n}{h} + \tilde{H}_{p_2}(x_i, [\nabla_h U^n]_i) \frac{W_i^n - W_{i-1}^n}{h} \right),$$

and performing a discrete integration by parts, we obtain the discrete counterpart of the left hand side of (9) as follows: $-h \sum_{i=0}^{N_h-1} \mathcal{T}_i(U^n, M^{n+1}) W_i^n$, where $\mathcal{T}_i$ is the following discrete transport operator:

$$\begin{aligned} \mathcal{T}_i(U, M) = \frac{1}{h} \Big(& M_i \tilde{H}_{p_1}(x_i, [\nabla_h U]_i) - M_{i-1} \tilde{H}_{p_1}(x_{i-1}, [\nabla_h U]_{i-1}) \\ & + M_{i+1} \tilde{H}_{p_2}(x_{i+1}, [\nabla_h U]_{i+1}) - M_i \tilde{H}_{p_2}(x_i, [\nabla_h U]_i) \Big)\,. \end{aligned}$$

Then, for the discrete version of (6b), together with (6d) and the initial condition in (6e), we consider

$$\begin{cases} (D_t M_i)^n - \nu (\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}) = 0\,, & 0 \le i < N_h\,,\ 0 \le n < N_T\,, \quad \text{(10a)} \\ M_{-1}^n = M_0^n\,, & 0 < n \le N_T\,, \quad \text{(10b)} \\ M_{N_h}^n = M_{N_h-1}^n\,, & 0 < n \le N_T\,, \quad \text{(10c)} \\ M_i^0 = \bar{m}_0(x_i)\,, & 0 \le i < N_h\,, \quad \text{(10d)} \end{cases}$$

where, for example,

$$\bar{m}_0(x_i) = \int_{|x - x_i| \le h/2} m_0(x) dx,$$

or simply

$$\bar{m}_0(x_i) = m_0(x_i). \tag{11}$$

Here again, the scheme is implicit since the equation is forward in time.

Remark 2.1 *For the implementation, it is important to realize that the matrix of the linear operator $M \mapsto -\mathcal{T}(U^n, M)$ is the conjugate of the Jacobian of the map from $\mathbb{R}^{N_h}$ to $\mathbb{R}^{N_h}$, $\mathbb{R}^{N_h} \ni U \mapsto (\tilde{H}(x_i, [\nabla_h U]_i)_{0 \le i < N_h}$, computed at $U = U^n$.*

3 Indications for writing the code

Solving the discrete HJB equation. Finding $(U^n)_n$ given $(M^n)_n$ amounts to solving a discrete version of a nonlinear parabolic equation posed backward in time with Neumann boundary conditions. This is much simpler than solving the complete forward-backward system for (U, M), because a backward time-marching procedure can be used. Since, as it was already observed above, the scheme is implicit, each time step consists of solving the discrete version of a nonlinear elliptic partial differential equation. Starting from the terminal time step N_T for which (8d) gives an explicit formula for U^{N_T}, the backward loop consists of computing U^n by solving (8a), (8b) (8c) given U^{n+1}, M^{n+1}. Let us rewrite this nonlinear system in the compact form $\mathcal{F}(U^n, U^{n+1}, M^{n+1}) = 0$. It is solved by means of Newton-Raphson iterations. The k-th Newton iteration reads

$$U^{n,k+1} = U^{n,k} - \mathcal{J}^{-1}(U^{n,k}, U^{n+1}, M^{n+1})\mathcal{F}(U^{n,k}, U^{n+1}, M^{n+1}),$$

where $U^{n,k}$ is the approximation of U^n at the k-th Newton iteration and where $\mathcal{J}(V, U^{n+1}, M^{n+1}) \in \mathcal{M}_{N_h}(\mathbb{R})$ is the Jacobian of the map $V \mapsto \mathcal{F}(V, U^{n+1}, M^{n+1})$.

Remark 3.1 *The drift contribution in the assembly of the Jacobian will also be useful for computing the matrix in the linear systems arising in the discrete Fokker-Planck equation, see Remark 2.1.*

Note that one may choose the initial guess $U^{n,0} = U^{n+1}$. The Newton iterations are stopped when the residual $\|\mathcal{F}(U^{n,k}, U^{n+1}, M^{n+1})\|$ is smaller than a given threshold, say 10^{-12}. It is well known that the Newton algorithm for (8a) is equivalent to an optimal policy iteration algorithm that it is convergent for any initial guess, and that the convergence is quadratic.

Solving the discrete Fokker-Planck equation. Finding $(M^n)_n$ given $(U^n)_n$ amounts to solving the discrete version of a linear parabolic equation posed forward in time with Neumann boundary conditions. Since the scheme is implicit, each time step consists of solving the discrete version of a linear elliptic partial differential equation. Starting from the initial time step 0 for which (10d) gives an explicit formula for M^0, the n-th step of the forward loop consists of computing M^{n+1} by solving (10a), (10b), (10c) given M^n and U^n. The matrix of the latter system of linear equations must be computed using Remark 2.1, see also the the previous paragraph.

Fixed point iterations for the whole forward-backward system Let $\mathcal{M}$ stand for the collection $(M^n)_{0,\dots,N_T}$ and $\mathcal{U}$ stand for the collection $(U^n)_{0,\dots,N_T}$.

The program consists of approximating $(\mathcal{M}, \mathcal{U})$ by Picard fixed point iterations.

Let θ be a parameter, $0 < \theta < 1$, $\theta = 0.01$ is often a sensible choice. Let $(\mathcal{M}^{(k)}, \mathcal{U}^{(k)})$ be the running approximation of $(\mathcal{M}, \mathcal{U})$. The next approximation $(\mathcal{M}^{(k+1)}, \mathcal{U}^{(k+1)})$ is computed as follows:

1. Solve the discrete HJB equation given $\mathcal{M}^{(k)}$. The solution is named $\widehat{\mathcal{U}}^{(k+1)}$.
2. Solve the discrete FP equation given $\widehat{\mathcal{U}}^{(k+1))}$. The solution is named $\widehat{\mathcal{M}}^{(k+1)}$.
3. Set $(\mathcal{M}^{(k+1)}, \mathcal{U}^{(k+1)}) = (1-\theta)(\mathcal{M}^{(k)}, \mathcal{U}^{(k)}) + \theta(\widehat{\mathcal{M}}^{(k+1)}, \widehat{\mathcal{U}}^{(k+1)})$.

These iterations are stopped when the norm of the increment $(\mathcal{M}^{(k+1)}, \mathcal{U}^{(k+1)}) - (\mathcal{M}^{(k)}, \mathcal{U}^{(k)})$ becomes smaller than a given threshold, say 10^{-7}.

4 Work program

1. Simulate the MFG corresponding to the following data:
   - $\Omega =]0, 1[$, $T = 1$
   - $\sigma = 0.2$

- $H_0(x,p) = \frac{1}{\beta}|p|^\beta - g(x)$ with $g(x) = -\exp\left(-40(x-1/2)^2\right)$. Try $\beta = 2$, $\beta = 1.1$, $\beta = 4$. See Example 2.1 for the discrete Hamiltonian.
- $\tilde{f}(m(x)) = 10m(x)$
- $\phi(x,m) = -\exp\left(-40(x-0.7)^2\right)$
- $m_0(x) = \exp\left(-3000(x-0.2)^2\right)$ (this is not a probability law because the integral is not 1, but this does not matter).
- Choose for example $N_h = 201$, $N_T = 100$, $\theta = 0.01$
- Stopping criteria in the Newton method: 10^{-12}
- Stopping criteria in the Picard fixed point method: 10^{-6} (with norms normalized so that $\|(1,\dots,1)\| = 1$)

The outputs will be

(a) The contour lines of u and m in the plane (x,t)

(b) Animations (in the mp4 format) showing the evolution of u and m with respect to time.

2. Consider also $g(x) = -\exp\left(-40(x-1/3)^2\right) - \exp\left(-40(x-2/3)^2\right)$.

3. Understand why the mass of M^n, i.e. $\sum_{i=0}^{N_h-1} M_i^n$, does not depend on n

4. Try to prove uniqueness in the discrete HJB equation, i.e. that given $(M^n)_n$, $(U^n)_n$ is unique. Hint: take two solutions $(U^n)_n$ and $(V^n)_n$, and consider n_0, i_0 such that $\max_{(n,i)}(U_i^n - V_i^n)$ is achieved at (n_0, i_0), and use the monotonicity of the discrete Hamiltonian.

5. Try to prove uniqueness in the discrete KFP equation, i.e. that given $(U^n)_n$, $(M^n)_n$ is unique. Hint: prove that the matrices of the linear systems arising in the discrete KFP are the conjugate of M-matrices.

6. Try to prove that if M^0 is positive, then M^n is positive for all n.

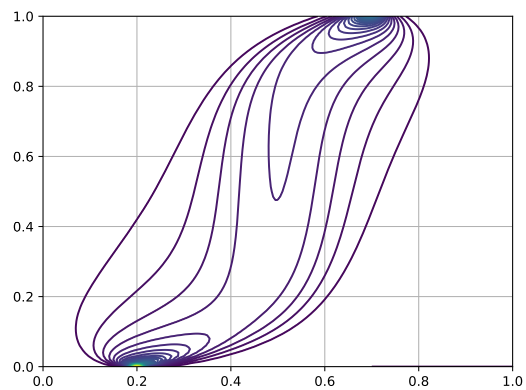

Figure 1: Contour lines of m in the example described above with $g(x) = -\exp\left(-40(x-1/2)^2\right)$, $\tilde{f}(m) = 10m$ and $\beta = 2$ (x in abcissa, t in ordinate)

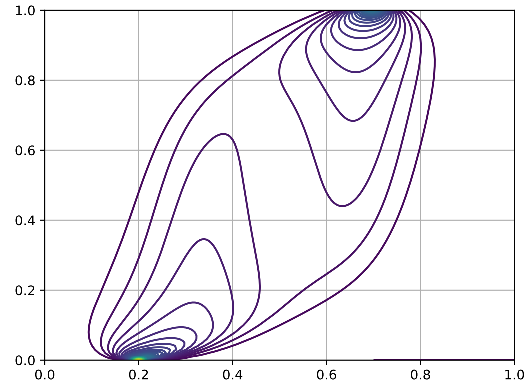

Figure 2: Contour lines of m in the example described above with $g(x) = -\exp\left(-40(x-1/3)^2\right) - \exp\left(-40(x-2/3)^2\right)$, $\tilde{f}(m) = 10m$ and $\beta = 2$ (x in abcissa, t in ordinate)

How to check that the code is correct

- The Newton method in the HJB solver should converge fast (the number of iteration is of the order of 10) If it does not converge, it usually means that there are errors in the computation of the Jacobian.
- M should be positive. If not there is an error in the code.
- Check that the total mass of m is conserved, i.e $\sum_{i=0}^{N_h-1} M_i^n$ should not depend on n
- There may be a large number of fixed point iterations before convergence. If the fixed point method does not converge, decrease θ.

References

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