Tutorial 4B: Beta-reduction

Computation in the lambda-calculus is by **beta-reduction**, defined by the following **rewrite rule**, the **beta-rule**.

$$(\lambda x.N)M \longrightarrow_\beta N[M/x]$$

As a rewrite rule, it may be applied anywhere in a term, not just at the top level. Within a term, a sub-term that is of the form $(\lambda x.N)M$, where the beta-rule can be applied, is called a **redex**, short for "reducible expression".

The **substitution** $[M/x]$ in the definition of beta-reduction is a **capture-avoiding** substitution. With a regular substitution, **variable capture** is when a substitution introduces a free variable x into a term $\lambda x.N$. This causes the variable to be **captured**: to become bound by the abstraction λx. In most cases, this gives an incorrect solution.

Capture-avoiding substitution avoids capture by **renaming** bound variables where necessary. That is, before carrying out a substitution under an abstraction λx, the variable x is given a new name z that does not occur anywhere in the term yet:

$$(\lambda x.N)[M/y] \;=\; \lambda z.\,(N\,[z/x]\,[M/y])$$

For these exercises, you do not need to follow the definition of capture-avoidance **exactly** — it is very mechanical, and thus tedious. You only need to rename an abstraction λx that would otherwise capture a variable (in the substitution rule above, when x is a free variable in M). Renaming an abstraction, from $\lambda x.M$ to $\lambda z.(M[z/x])$, simply means renaming all variables x bound by λx, and λx itself, to z.

Exercise 1: Perform the following (capture-avoiding) substitutions.

a) $(\lambda x.xy)[\lambda z.z/y]$

b) $(\lambda x.xy)[\lambda z.zx/y]$

c) $(f(\lambda x.yx)yx)[fy/x]$

d) $(\lambda f.f(\lambda x.yx)yx)[fy/x]$

Exercise 2: Identify all the redexes in the following terms, and compute the λ-terms that result from β-reducing each redex.

a) $(\lambda x.\lambda y.x)yx$

b) $(\lambda f.f(\lambda x.x))(\lambda y.z)$

c) $(\lambda x.\lambda y.yx)(\lambda x.xy)$

d) $(\lambda x.xx)((\lambda y.y)(\lambda x.x))$

e) $(\lambda x.xx)(\lambda y.y)(\lambda x.x)$

f) $(\lambda x.xx)(\lambda x.xx)((\lambda y.y)(\lambda x.x))$

Exercise 3:

a) Find distinct terms M, N, P such that $M \to_\beta P$ and $N \to_\beta P$.

b) Find distinct terms M, N, P, Q such that $M \to_\beta N$, $M \to_\beta P$ and $M \to_\beta Q$.

Exercise 4: The **combinators** S, K and I are defined to be the following λ-terms:

$$\begin{aligned} S &= \lambda xyz.xz(yz) \\ K &= \lambda xy.x \\ I &= \lambda x.x \end{aligned}$$

Reduce the following terms to normal form.

a) SKK

b) SIK

c) SSS

Exercise 5: Let W be the term

$$\lambda x.\lambda y.xyy.$$

a) Reduce the term WW to normal form.

b) Give a complete analysis of the sequences of beta-reductions that the term WWW can perform.