Math 122: Final Review

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From the practice final: 5, 9, 10, 6

- Practice Final # 5: Prove group of order $1495 = 5 \times 13 \times 23$ is cyclic. Number of Sylow 5-subgroups divides 1, 13, 23, or 13×13, but neither of the final three is $1 \mod 5$, and so there is only one Sylow 5-subgroup, which is C_5 because 5 is prime, and this subgroup is normal.

  The same thing happens for Sylow 13-subgroup and Sylow 23 subgroup. So the Sylow subgroups are $C_5 = \langle a \rangle, C_13 = \langle b \rangle, C_23 = \langle c \rangle$, and all of them are normal.

  Take some element $a \in C_5$ and conjugate by $b \in C_13$. So, $bab^{-1} = a^m$. We want to prove that $m = 1$ to show that the group is abelian and so cyclic. Similarly, $a^{-1}ba = b^n$.

  So $ba = a^m b$ and substituting into second you get $a^{-1}(a^m b) = b^n$. So $a^{m-1} = b^{n-1}$. Since these are both equal, they must both be 1, so $m = n = 1$, and so a and b commute. The same argument applies to the other two pairs of Sylow subgroups, and so you get that all elements commute with one another.

  So you get generators a, b, c where $a^5 = b^13 = c^23 = 1$ and $ab = ba, bc = cb, ca = ac$. The fact that the generators commute with eachother shows that the group is $C_5 \times C_13 \times C_23$.

- Practice Final # 9: R is a ring with ideals I and J. $I + J = \{x + y | x \in I, y \in J\}$.

  Part (a) show $I + J$ is an ideal - omitted, just go from the definition.

  Part (b) Show if $I + J = R$ then $f : R \to R/J \times R/I$ that sends $a \to (a, a)$ is surjective.

  So take $(x, y) \in R/I \times R/J$. We want $f(a) = (x, y)$, so $x = a + I$ and $y = a + J$. So $a = x + I$ and $a = y + J$. If we can find an a so that $a = x + I$ and $a = y + J$ then we are done. So then subtracting the two equations we get $0 = x + I - (y + J) = x - y + (I = J) \Rightarrow y - x = I + J$. But this means that $\exists x \in I, j \in J$ such that $i + j = y - x \Rightarrow i + x = -j + y$.

- Practice Final # 10:

  $a, b \in R, \exists s, t \in R$ such that $as + bt = 1$.

  Part (a) show a^n, b are relatively prime. $a^2 s + abt = a$, plug in for a in first equation, you get $(a^2 s + abt)s + bt = 1 = a^2 s^2 + abts + bt = 1$ so $a^2 s^2 + b(ats + t) = 1$. Do the same for $a^3 s' + abt' = a$. Then prove by induction.

  Part (b) show a^n, b^n are relatively prime. Easy, just assume the result of (a).

  Part (c) Ideals I, J are relatively prime so that $I + J = R$ then I^n, J^m are relatively prime so that $I^n + J^m = p$. Because $I + J = R$, $\exists a \in I, b \in J$ such that $a + b = 1$. Also, if you know $a + b = 1$ for $a \in I, b \in J$ then you know that $I + J = R$ by just multiplying by any r.

  Assume $a + b = 1$. Re-write as $a(1) + b(1) = 1$. So $a^m(s) + b^n(t) = 1$. But then $a^m \in I^m$ and $b^n \in J^n$, so $a^m(s) \in I^m$ and $b^n(t) \in J^n$ and so I^m and J^n are relatively prime.

- Practice Final # 13

$$x^4 + x^2 + 1 = x(x^3 + 1) + x^2 + x + 1$$
$$x^3 + 1 = (x + 1)(x^2 + x + 1)$$

- Practice Final # 12

  Think of a free module of rank 3 as a vector space on three coordinates.

  $M \subset R$ is a submodule.

  (1) Show M is an ideal. Just go from the definition.

  (2) Show if R is PID then M is either the zero module or or else free of rank 1.

- Practice Final # 1

  There are either 1 or 11 Sylow 5-subgroups, but there is only 1 Sylow 11-subgroup. Take $a^5 = b^{11} = 1$, then $aba^{-1} = b^n$ so $a^5ba^{-5} = b^{n^5} = b$. So $n^5 \equiv 1 \mod 11$

- Practice Final # 6

  $R^\times$ is group of units, R_+ is R with abelian group addition.

  Part (a) take $\lambda \in R^x$, take $\phi(\lambda) : R_+ \to R_+$ that sends $x \to \lambda x$ is an automomorphism. Straightforward to show it is homomorphism. Next consider why it is an isomorphism.

  First, $x \neq y$ then $\lambda x \neq \lambda y$. True because $\lambda x - \lambda y = \lambda(x - y)$. Suppose this is equal to 0. Then since $\lambda isaunit$, $\exists \lambda'$ such that $\lambda'\lambda = 1$. So then $\lambda'\lambda(x - y) = 1(x - y) \Rightarrow x = y$ which is a contradiction.

  Part (b) show $\phi : R^\times \to Aut(R_+)$ is a group homomorphism. ϕ sends λ to $\phi(\lambda)$. Must show sends identity to identity and that it sends $\lambda\lambda' \to \phi(\lambda)\phi(\lambda') = \phi(\lambda\lambda')$. $\phi(\lambda')\phi(\lambda)(x) = \lambda'\lambda x$ but $\phi(\lambda'\lambda)(x) = \lambda'\lambda x$ so it is a group homomorphism.

  Part (c) Show when $R = \mathbb{Z}/n$ then $\phi : R^\times \to Aut(R_+)$ is an isomorphism of groups.