Applied Math 158: Feedback Control Systems
Alexander H. Patel
alexanderpatel@college.harvard.edu
September 16, 2017

Contents

1 September 5, 2017

- Reading: AM Sections 2.1 - 2.4.
- Agenda: (1) introduce "modeling" (ODE) and (2) state, inputs and outputs.
- Last class, we talked about speed control: $m\dot{v} = F - cv$ where $\dot{v}$ is speed, F is force, and cv is friction. In a spring-mass system, we would like to design the force f to use to maintain the spring at position q at location $q*$.
- How are we going to design the system to apply the force? If I already had a system in frontof me, I can just play around.
- But if we don't have the system, we need to do some modeling to see what we'd expect. We'd like to model how $f(t) \to q(t)$. This function is the mathematical model. We know tha t $F_{net} = ma(t)$ and that:
$$F - kq - cv = ma$$
But since $v = \frac{dq}{dt}$ and $a = \frac{d^2q}{dt^2}$, we can reduce the number of variables.
$$m\ddot{q} + c\dot{q} + kq = F = U(t)$$
q we call the **state**. Now we can put them together to get: input $F \to (m\ddot{q}+c\dot{q}+kq = F \to q$. In this example q is both state and output.
- A standard example of the state space model is $\dot{x} = f(x, u)$ and $y = h(x, u)$. Here x is a state, u is input, y is output, and f and h are just functions. So any model you come up with (for physics, economics, etc.) you can just translate into this standard form.
- To force the above exmple into the model, try $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ so that $\dot{x_1} = \dot{q} = x_2$ and $\dot{x_2} = \dot{q} = \frac{F-kq-cq}{m} = \frac{F-kx_1-cx_2}{m}$. So,
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-kx_2-cx_2+F}{m} \end{bmatrix} = f(x, u)$$
. Note that $f(x, u)$ is a linear function 9(can be written in the form $AX + BU$.
- Based on this example, what does "state", "input", "dynamics", and "output" mean?
  - State: memory, past -> future.
  - Input: control, disturbance - anything that is outside of the system itself
  - Output: measurement

- A system can have many models to answer different questions (often dependent on time scale and spatial scale). For example, consider the system of electricity where (1) New England vs. New York City, (2) Harvard facilities, (3) Maxwell Dworkin. In the previous example, if you just wanted to keep the spring at a constant speed then the input, dynamics, and state would be the same but the output would be different.
- **Epidemic Model (SIR Model)**. Initial population N, number of susceptible $S(t)$, number of infected $I(t)$, number of removed (recovered & immunized) $R(t)$. The controller is the vaccine $u(t)$. How to remove the epidemic?

$$S(t) = -\frac{\beta SI}{N} - U(t)$$
$$I(t) = \frac{\beta SI}{N} - \gamma I$$
$$R(t) = rI + U(t)$$

  SI are the most important parts of that fraction, $\frac{\beta}{N}$ is somewhat extra. To fit into the standard model, let $\dot{x} = f(x, u)$, $x_1 = s, x_2 = I, x_3 = R$.
- **Opinion dynamics in social networks:** Each node in the graph is a person. If I'm going to give out free iphones, who should I give it to so that I maximize the influence to market the product (spread good opinions of product)? How are we going to model this: let $x_i(t)$ be the opinion of person i about one product at time t.

$$\dot{x}_i(t) = \frac{W_{ii}x_i(t) + \sum_N W_{ij}x_j(t)}{\sum_N W_{ij}} + U_i(t)$$

- **Economy/market** - we have suppliers and consumers (demand) and want to model how the price is going to change. Let $s_1(t)$ and $s_2(t)$ be the amount available from suppliers 1 and 2 at time t, $d(t)$ be consumer demand, and $p(t)$ be the price. How is a rational company going to make a decision about setting $\dot{S}_1(t)$ (how much product should they produce)?.

$$\dot{S}_1(t) = \alpha_1 \left(-C_1(s_1(t)) + p(t)\right)$$
$$\dot{S}_2(t) = \alpha_2 \left(-C_2(s_2(t)) + p(t)\right)$$
$$d(\dot{t}) = \beta(B(d(t)) - p(t)$$
$$p(\dot{t}) = k(d - (s_1 + s_2))$$

  Where $B(d(t))$ is called "marginal utility". If $d > (s_1 + s_2)$ then the price increases, decreases otherwise.
- **Spring #2** We have two masses and three springs ordered such that wall$|k_1|m_1|k_2|m_2|k_3$ and $U(t)$ is the total length of the spring system. We want to pull spring k_3 such that we can control the speed and position of m_2. How do you design $U(t)$ so as to move m_1?

$$m_1 a_1 = F_1$$
$$m_1 r_! = -k_1 q_1 + k_2(q_2 - q_1)$$
$$m_2 a_2 = F_2$$

$$m_2\ddot{q}_2 = -k_2(q_2 - q_1) + k_2(U(t)q_2(t)) - C\dot{q}_2$$
$$\dot{x} = F(x,u) = AX + BU$$

Try to write this in linear form as an exercise - how do you write down F and how do you come up with A, B.

2 September 7, 2017

- Section Friday 11-12pm, location to be announced.
- Homework 1 due next Thursday in class.
- Recall: with x: state, u: input, y: output, then $\dot{x} = f(x,u)$ and $y = g(x,y)$, with the spring example we had $m\ddot{q}_1 = k_2(q_2 - q_1) - k_1 q_1$ and $m\ddot{q}_2 = k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_1$. We have two equations, but both equations have variables in the second order. How are we going to translate it into the standard form? Let $x_1 = q_1, x_2 = q_12, x_3 = \dot{q}_1, x_4 = \dot{w}_2$. So,

$$f_1(x,u) = \dot{x_1} = x_2$$
$$f_2(x,u) = \dot{x_2} = x_4$$
$$f_3(x,u) = \dot{x_2} = \frac{1}{m_1}(k_2(x_2 - x_1) - kx_1)$$
$$f_4(x,u) = \dot{x_4} = \frac{1}{m_2}(k_3(u - x_2) - k_2(x_2 - x_1) - cx_4)$$

Then the equation for $f(x,u)$ is $f(x,u) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$ - this is a time-invariant system, it does not depend on the current time. Notice that everythin gin this system is a linear function of x and u. We want to write this equation in the form $f(x,u) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + Bu$. So A is a 4x4 matrix and B is a 4x1 matrix - you know you need four rows because you have four functions, you have for variables x in the left term and just one term u in the right term. To not make a mistake, think about the dimensions of the matrices you need and then fill it in. A is easy because A is always going to be a matrix in this class. B is trickier because you can have multiple controllers.

$$f(x,u) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_2+k_1}{m1} & \frac{k_2}{m1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3}{m_2} \end{bmatrix} u$$

- Is the controller we have enough to control mass 1 in the system? Is it possible make your controller take on a value that would make q_1 be at any position you want? We'll find the answer in a couple of weeks.

- **Inverted Pendulum** - you have a pendulum of length l at angle θ with respect to the y-axis and a mass m at the end, and you have a controller u that can pull only in the horizontal direction. We need to come up with a model for how your force $u(t)$ affects your angle $\theta(t)$. We can use mechanics to do this. Specifically, we'll use the equation $T = J \cdot \alpha$, where T is torque, J is moment of intertia, and α is the angular acceleration.

$$\begin{aligned} T &= mgl\sin\theta - ul\cos\theta \\ J &= ml^2 \\ ml^2\ddot{\theta} &= mgl\sin\theta - ul\cos\theta \end{aligned}$$

  In standard form,

$$\begin{aligned} x_1(t) &= \theta \\ x_2(t) &= \dot{\theta} \\ \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} x_2(t) \\ \frac{mgl\cos x_1 - ul\sin x_1}{ml^2} \end{bmatrix} \\ y &= \theta(t), \theta(t) \to 0 \end{aligned}$$

  This is a non-linear system, in this class we're going to linearize it. We'll talk about when linearization will work and wehn it will not work.

- **RL Circuit** - Consider a circuit with a voltage V_s with an inductor L and a resistor R in series. We want to control V_{out}, the output voltage. If $u = V_s$, then how can you change u so as to control V_{out}?

  Using Kirchhoff's voltage lows, we can assign voltage V_R to R and V_L to L and have the equations:

$$\begin{aligned} V_S - V_R - V_L &= 0 \\ V_R &= RI \\ V_L &= L\dot{I} = V_S - V_R = V_S - RI \end{aligned}$$

  This last equation is a dynamical model which describes how the voltage source affects the current. Replacing I with x

$$\begin{aligned} \dot{x} &= \frac{1}{L}(U - Rx) \\ y = V_R &= Rx \end{aligned}$$

- $\dot{x} = \frac{dx}{dt}$. Instead of going to the differential equation, some models are written as **Difference Equations**. By this, she means: $x[k+1] = f(x[k], u[k])$ and $y[k] = h(x[k], u[k])$. This is also called a **discrete system** because time is discrete instead of continuous. All of the theory we've learned can be applied here.

- **Predator-Prey** If we have a Lynx and a hare, how do we model the relation between the two and how they affect each other? Let:

- H be the population of hares
- L be the population of lynxes
- k is the discrete time index (e.g. the month number)
- (controller) u that is the food supply for the hares.

We're trying to model $U \rightarrow \begin{bmatrix} L \\ H \end{bmatrix}$. Our model can be:

$$L[k+1] = L[k] + cL[k]H[k] - d_f[L[k]$$

So the number of lynxes next month depends on: the number of lynxes this month, the *production rate*, and the last term is the death rate of the Lynx. Similarly, to model the hares (the third term is the hares getting eaten by the Lynx):

$$H[k+1] = H[k] + b_r(u)H[k] - aL[k]H[k] - d_nH[k]$$

This model is a good starting point based on intuition. But there are some terms that are pretty hand-wavy, the real data can be helpful, here.

- **Block Diagram** - How can you model complex systems? Use block diagrams, draw small blocks and ten put them together. We'll cover on Friday how to use Simulink to do this.
- **Dynamic Behavior (and stability)** - consider $\dot{v} = -av$, if you have $v(0) = v_0$ then the dynamical behavior is what is $v(t)$? How is the speed going to evolve over time? The answer is $v(t) = e^{-at} \cdot v_0$. As $t \rightarrow \infty, v(t) \rightarrow 0$. Now you have a controller $u = bv$ so you have $\dot{v} = (b-a)v$. If $b > a$, then $v(t) = e^{(b-a)t}V_0$ and you accelerated ($v(t) \rightarrow \infty$). That $v(t)$ converges in the first case means that it is **stable**; that it diverges in the second case means that it is **unstable**.
- If we have $\dot{x} = f(x,u)$, then in this class u is always going to be a function of x and so $f(x,u) = f(x,u(x)) = F(x)$ (system *). So now we look for the solution of $\dot{x} = F(x)$. Formally, $x(t)$ is a solution of the system * on time interval $[t_0, t_f]$ if $\frac{dx(t)}{dt} = F(x(t)) \forall t_0 < t < t_f$. For example, the linear system $\dot{x} = Ax + Bu$, we have a form for this solution; but for the non-linear system we do not have such a closed-form solution.

3 September 14, 2017

- Agenda: dynamic system colution, phase portraits, equilibrium, stability.
- Recall: $\dot{x} = f(x,u)$ (system control input), $u = k(x)$ (state-feedback control, $\dot{x} = f(x, k(x) = F(x)$ (closed-loop system). Good: stability, response time, etc. The equation $u = k(x)$ is call state feedback contorl, so like with autonomous driving we dont' have to put in any extra input to the system, that's why we can reduce it down to $F(x)$. This is what the **closed-loop** part of the system is, you close the loop—this means that how you control is based on your state. Today we're going to talk about what is meant by stability in the context of this.

Definition 1 (Solution). $x(t)$ is a solution to $\dot{x} = F(x)$ with initial value $x_0 \in \mathbb{R}^n$ at $t_0 \in \mathbb{R}$ if when $x(t_0) = x_0$,

$$\frac{dx(t)}{dt} = F(x(t) \forall t_0 < t < t_f$$

- In order to understand the dynamical behavior for the system is to solve the equation $\dot{x} = F(x)$. But not every system can give you a closed-form solution; what this definition gets you is that if you want to check if $x(t)$ is a solution to the euqation, it's easy to check but potentially hard to come up with. You can also check the derviative; if the derivative satisfies this equation, then it works as well.
- For a n example of this, consider the spring-mass system with k spring, mass m, and input u. Remember that we worked through the dynamics:

$$m\ddot{q} + c\dot{q} + kq = u$$

$$m\ddot{q} + c\dot{q} + kq = 0$$

If we have two states $x_1 = q$ and $x_2 = \frac{\dot{q}}{\omega_0}$. Remember for this one that the state variable is not unique, you cannot just use define the state variable because you also have a factor of ω_0.

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \omega_0 x_2 \\ -\omega_0 x_1 - 2\zeta\omega_0 x_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Consider if $\zeta > 1$, then:

$$\begin{aligned} x_1(t) &= \frac{\beta x_1(0) + x_2(0)}{\beta - \alpha} e^{-\alpha t} - \frac{\alpha x_1(0) + x_2(0)}{\beta - \alpha} e^{-\beta t} \\ (\alpha = \omega_0(\zeta + \sqrt{\zeta^2 - 1}), &\beta = \omega_0(\zeta - \sqrt{\zeta^2 - 1})) \end{aligned}$$

Consider if $\zeta = 1$, then:

$$x_1(t) = e^{-\zeta\omega_0 t}(x_1(0) + x_2(0) + \zeta\omega_0 x_1(t))$$

Consider if $0 < \zeta < 1$, then:

$$\begin{aligned} x_1(t) &= e^{-(\zeta\omega_0 t)}(x_1(0)\cos\omega_\alpha t + \frac{1}{\omega_\alpha}(\omega_0\zeta x_1(0) + x_2(0))\sin(\omega_\alpha t))x_2(0) + \zeta\omega_0 x_1(t)) \\ \omega_\alpha &= \omega_0\sqrt{1 - \zeta^2} \end{aligned}$$

We have the solution to the equation $\dot{x} = ax \Rightarrow x(t) = e^{-at}x_0$, somehow the exponent of e in the equation where $\zeta = 1$ is a similar sort of situation. If we have the equation $x_1(t) = (x_1(0)\cos\omega_\alpha t + \frac{x_2(0)}{\omega_\alpha}\sin\omega_0 t)$. This is just going to oscillate when $\zeta = 0$. So any equation you have that I guess fits in this form is going to be such that when $\zeta = 0$ then it is go9ing to converge to zero in the limit of large t.

So we can say that the system is stable because it's eventually going to converge to a point. For the special case when $\zeta = 0$, it's never going to go outside of the range of $|x_1(t)|$.

- **Phase Portraits (2-dimension)**

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_1(x_2, x_2) \\ F_2(x_1, x_2) \end{pmatrix}$$

  To draw this graph, just put x_1 as the x axis and x_2 as the y-axis and then draw a vector value that is given the function: so your point would be $F_1(x_1 = 1, x_2 = 3), F_2(x_1 = 1, x_2 = 3))$. So for each point if you draw the vector for every point, then you look at the vectors and it tells you how the system is going to change. So if you have a bunch of arrows that are pointing down, then you can tell what your dynamics are going to be.

  If you've got some sort of cycles that don't converge to a point but the arrows are such that they yield a circle then you have a *limit cycle*. So you can have it such that there's a circle and then if you are outsie the circle then eventually you'll get in the circle, and then if you're in the circle then you stay in the circle. And then there's potentially one special point (on the radius) that will just stay in that radious perpetually.

- Now she's going to make the discussion more "rigorous" (I don't know what that means in the context of applied math).
- **Equilibrium Point** - an equilibrium point represents "stationary conditions" for dynamics. For example, say that x^{eq} is an equilibrium point. This means that if $x(t_0) = x^{eq} \Rightarrow x(t) = X^{eq}$. To guarantee this one, the way you're going to calculate is by finding $\dot{x} = F(x)$ such that $F(X^{eq}) = 0$.

  Now if we come back to the spring-mass system, if we want to find the equilibrium point we'll have $\dot{x} = Ax$ and we want to find $Ax = 0 \Leftrightarrow x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

- Another example is with the inverted pendulum. So we have **no** u and also:

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{pmatrix} \dot{\theta} \\ \frac{mgl}{J}\sin\theta - \frac{r}{J}\dot{\theta} \end{pmatrix}$$

  With $\theta = 0, \theta = k\pi, \dot{\theta} = 0, \dot{\theta} = 0$. For the integral control, we have $\dot{x} = -ax + q\int_{t_0}^{t} q dt$ and $\dot{q} = a(x)$.

- Now back to the discussion of stability, for an unstable pendulum (inverted) we have $\theta = 0, \dot{\theta} = 0$, but for a regular pendulum hanging downwards has $\theta = k\pi, \dot{theta} = 0$.

  How are we going to define stabilitiy? This means that $\forall\epsilon > 0, \exists\delta$ such that $\forall x_0$, if $|x_0 - x^{eq}| < \delta$ then $|x(t) - x^{eq}| < \epsilon, \forall t \geq 0$ (marginal stable). This doesn't require global stability, you just have to find a delta that is going to work (local stability).

  In order to be **Asymptotically Stable**, you must have:

  1. Stable in the sense of Layman.
  2. $\lim_{t\to\infty} x(t) = x^{eq}$.

- **Linear-Time-Invariant Systems**

  You have an nxn matrix A, n x p matrix B, q x n matrix C, q x p matrix D, all with values in $\mathbb{R}$.

WIth u_1, u_2,

$$\dot{x}(t) = Ax(t) + Bu(t) = F_1(x, u)$$
$$y(t) = cx(t) + Du(t) = G(x, u)$$
$$F_1(0, \alpha u_1 + \beta u_2) = F_1(0, \alpha u_1 + \beta u_2) = F_1(0, \alpha u_1) + F_2(0, \beta u_2)$$

The last is the statement of linearity. Let $x(0) = 0$, we have $u_1(t)$