Math 122: Midterm Review

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1. What are all the class equations for the symmetric groups?

$$\begin{aligned}
|D_n| &= 2n = 1 + \times \text{if n is odd, otherwise.} \\
|A_4| \cong |T| &= 12 = 1 + 3 + 4 + 4 \\
|S_4| \cong |O| &= 24 = 1 + 3 + 6 + 6 + 8 \\
|A_5| \cong |I| &= 60 = 1 + 12 + 12 + 15 + 20 \\
|S_5| &= 120 = 1 + 10 + 15 + 20 + 20 + 24 + 30
\end{aligned}$$

2. Find a composition series of S_3. The kernel of the sign homomorphism $S_3 \to C_2$ is normal - it is A_3. $S_2/A_3 = C_2$. $A_3 = \{1, (123), (132)\}$, so A_3 is just C_3.

   If you don't have a homomorphism, then look at the class equation. $S_3 = 1 + 2 + 3$. 1 is the identity, 2 is the three cycles, and 3 is the transpositions.

   Normal subgroups contain the identity and the size divides six. So the transpositions and the identity cannot form a normal subgroup because 1 + 4 doesn't divide 6. If their sum divides 6, then it is normal, but you still have to check that it is a subgroup (check closed).

3. Non-abelian group, $|G| = 28$, all Sylow 2-subgroups are cyclic. Prove this group is unique.

   $28 - 2^2 \times 7$, the number of 2-subgroups is 1 or 7 and 7-groups is 1. So the 7-subgroup is C_7, normal in G.

   If there is 1 Sylow 2-subgroups (is also normal), then the group is abelian. So there are 7. Choose a generator $C_7 = \langle a \rangle$ and $C_4 = \langle b \rangle$, What is bab^{-1}? C_7 is normal, so $bab^{-1} = a^k$ for $k \in \{1, 7\}$.

   There are 28 elements in form $a^i b^j$, but then it would abelian. So $bab^{-1} \neq a$.

   If you have $bab^{-1}a^k$, then $b^2ab^{-1} = b(bab^{-1})b^{-1} = ba^kb^{-1} = (bab^{-1})^k = a^{k^2}$. You continue this process and get $b^4ab^{-4} = a^{k^4}$. b has order 4, so $k^4 = 1 \mod 7$. The check all 0, 1, 2, 3, 4, 5, 6 by bringing to power of 4 and check mod 7.

   The only power it works for is 6, so $bab^{-1} = a^6 = a^{-1} \to ba = a^{-1}b$, so this is relation of dihedral group.

4. X is a G-set. Can decompose the elements of X into orbits. For element x, there's the size of the orbit containing x and the the size of the stabilizer of x.

   Fixed point theorem