

Math 122: Midterm Review  
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1. What are all the class equations for the symmetric groups?

$$\begin{aligned}
 |D_n| &= 2n = 1 + \times \text{if } n \text{ is odd, otherwise.} \\
 |A_4| &\cong |T| = 12 = 1 + 3 + 4 + 4 \\
 |S_4| &\cong |O| = 24 = 1 + 3 + 6 + 6 + 8 \\
 |A_5| &\cong |I| = 60 = 1 + 12 + 12 + 15 + 20 \\
 |S_5| &= 120 = 1 + 10 + 15 + 20 + 20 + 24 + 30
 \end{aligned}$$

2. Find a composition series of  $S_3$ . The kernel of the sign homomorphism  $S_3 \rightarrow C_2$  is normal - it is  $A_3$ .  $S_2/A_3 = C_2$ .  $A_3 = \{1, (123), (132)\}$ , so  $A_3$  is just  $C_3$ .

If you don't have a homomorphism, then look at the class equation.  $S_3 = 1 + 2 + 3$ . 1 is the identity, 2 is the three cycles, and 3 is the transpositions.

Normal subgroups contain the identity and the size divides six. So the transpositions and the identity cannot form a normal subgroup because  $1 + 4$  doesn't divide 6. If their sum divides 6, then it is normal, but you still have to check that it is a subgroup (check closed).

3. Non-abelian group,  $|G| = 28$ , all Sylow 2-subgroups are cyclic. Prove this group is unique.  
 $28 = 2^2 \times 7$ , the number of 2-subgroups is 1 or 7 and 7-groups is 1. So the 7-subgroup is  $C_7$ , normal in  $G$ .

If there is 1 Sylow 2-subgroups (is also normal), then the group is abelian. So there are 7. Choose a generator  $C_7 = \langle a \rangle$  and  $C_4 = \langle b \rangle$ , What is  $bab^{-1}$ ?  $C_7$  is normal, so  $bab^{-1} = a^k$  for  $k \in \{1, 7\}$ .

There are 28 elements in form  $a^i b^j$ , but then it would be abelian. So  $bab^{-1} \neq a$ .

If you have  $bab^{-1}a^k$ , then  $b^2ab^{-1} = b(bab^{-1})b^{-1} = ba^k b^{-1} = (bab^{-1})^k = a^{k^2}$ . You continue this process and get  $b^4ab^{-4} = a^{k^4}$ .  $b$  has order 4, so  $k^4 = 1 \pmod{7}$ . The check all 0, 1, 2, 3, 4, 5, 6 by bringing to power of 4 and check mod 7.

The only power it works for is 6, so  $bab^{-1} = a^6 = a^{-1} \rightarrow ba = a^{-1}b$ , so this is relation of dihedral group.

4.  $X$  is a  $G$ -set. Can decompose the elements of  $X$  into orbits. For element  $x$ , there's the size of the orbit containing  $x$  and the size of the stabilizer of  $x$ .

Fixed point theorem