24.711: Topics in Philosophical Logic

Alexander H. Patel
alexanderpatel@college.harvard.edu
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These are lecture notes for the Spring 2017 offering of MIT 24.711: Topics in Philosophical Logic. Pardon any mistakes or typos.

Contents

1 Wednesday, February 15, 2017

1.1 Zeno's Paradox of the Arrow

- An object is at rest during an interval iff it occupies only a single position during that interval (not during a moment of time). It's stationary during an instant only if the instant is contained in an interval during which it's at rest.
- Ancient Greeks could only answer the question qualitatively because they couldn't measure time (although they could measure distances). Use a compass to make circles and form multiples of a unit length on a line.
- Can use a straight edge and compass to get an estimate of length to any precision up to a rational number, but no further.

- Aristotelian physics gives no good way of measuring a length of time. Earth, air, wind, fire all have natural motion (up, down), but because of acceleration you can't figure out good lengths of time - you need something that moves at a uniform speed. The natural motion of things in the heavens is circular, so you can use star movement to measure lengths of time (not as simple as one would think because Aristotle's system was geo-centric where the planets orbited in a circle about the earth).
- Need to be able to measure time and distance in order to measure speed and so to get close to a solution for the core question of what it means to be stationary.
- Galileo look at pendulum movement - he saw a smooth continuous movement that happens at intervals (PE being converted into KE). The Aristotelian saw chaotic movement in the pendulum (you are displacing the bob and it's swinging is it trying to go downward but being contained by the string). Aristotle didn't have the principle of balancing of forces on the bob. Aristotle's answer to the question of why the bob doesn't stop is that there is some nonsense with a vacuum of air being generated behind the bob and pushing it forward. We can measure time by dividing the length of the pendulum rod and noting the period is inversely related to the square of the length of the rod.
- Speed during an interval is the quotient of change of distance over change and time. Zeno gave us the idea of 0 speed over an interval, now we have a quantitative idea of non-zero speed during an interval. But we still have to account for the idea that the ball is at rest only at its apogee but at no other point in its path.
- We can solve the ball problem by solving $x'(t) = 0$, where x' is the limit of the difference quotient of $(x(t+h) - x(t))/h$. Aristotle thought that bodies were the only things that were real and that bodies aren't made up of points or surfaces. So we can't just think of the ball but also need to consider the position in spacetime of the ball - we need to think of the trajectory (potential positions occupied by the ball) as existent, even when the ball isn't there. But Aristotle argued that a line isn't made up of points so this doesn't play out well under Aristotelian physics - i.e. we can talk about velocities but not of instantaneous velocities.
- Why does Aristotle preclude the solution of taking the derivative? Do we not just take the limit over smaller and smaller intervals about the instant? The Aristotelian post-Newton understood the limit as the continual dividing of the line into smaller pieces.
- Things change when we switch to dynamics - throw a ball up at some initial speed and the downward force of gravity will decelerate the ball until it reaches 0. But since acceleration is dv/dt there's no good way to define instantaneous acceleration out of velocity, which is also a function.
- You know there are times when the ball is going up and times when the ball is going down. We want to say there is a time when the ball reaches its highest point when its velocity was 0, how do we know there was such a time? We could solve this with the IVT but we want to do this constructively by constructing the cut between positive velocities and negative velocities. Can we get the LUB principle from numbers just from the movement of the ball?

If the LUB principle didn't hold for the numbers then there would points in the ball's flight where it didn't have any associated number for its height.

- Most of what we can use real numbers for (measurements of things) we can get away with just rational numbers. Dedekind had originally the bad idea that we could get away with just rational numbers, but that was thrown out with $\sqrt{2}$ and the Ancients. Also need cube roots to deal with the roots of cubic polynomials. So the real numbers fall out of solving polynomials with rational coefficients - but this doesn't solve the ball problem because you have no reason to believe that the place where the ball is stationary is the root of an equation.
- The Ancients don't allow for the idea that there is a point on a line that is defined by the all the points to the right of it or to the left of it. For the ball, there needs to be a top point to define the switch of signs.
- Is it true that there is a supremum of all the points achieved by the ball? Is the ball at rest at this point? What if the ball only achieves heights of rational numbers (so it skips between points) or if the ball has a hole at the maximum of its height curve?
- From Mathematica, you have a tire on a hubcap that each go through the same number of rotations - how does the hubcap not move much less distance than the tire? Galileo: the tire is just a polygon, the hubcap goes through places where there is no corner of the polygon instantaneously which is just a wacky idea.
- Conclusion: you can't explain what is going on if you just think in terms of bodies and don't think about space and lines as being made up of points.
- An alternative view says that only bodies are real, and the bodies can be in different places but the places aren't real, just the bodies are (and the distance relations they have with each other). Kant disputed this view - if God made a single marble hand before anything else, then He could've made either a right hand or a left hand (it has to be one or the other), but if all we have is bodies with spatial relations then there is nothing different between a right hand and a left hand; but there is a difference between right hand and left hhand so there has to be something to account for the difference - you need position.

2 Tuesday, February 21, 2017

2.1 Dedekind's Construction of $\mathbb{R}$

- Jill starts at the bottom of a hill when Jack starts and the top and the reach the top/bottom at the same time. Is there a point where they are at the same elevation? Yes - take the set of all times when the difference in height was positive and then take the least upper bound. Most people say that it comes from their motion being continuous.
- Really early on people learned how to use numbers to measure things - at least from having to re-establish property lines after the flooding of the Nile. This entailed measurement by yardstick which always yield results in fractional form. But can we get these same numerical properties without actually deferring to the use of numbers?

- The rationals as fractions seem very well understood - the gaps between them were less well understood. We know that all the rationals along the path will be reached by the ball - how can we justify whether there are non-rational points that are hit by the ball.
- Pythagoreans found that simple ratios of where you put your finger on the lyre string correspond to harmonic sounds, but then one of them discovered that there are length sthat aren't rationals. He didn't find it out by measurement but rather by reductio ad absurdum - finding the length of the diagonal of the square with side length 1. He showed that it can't be rational (and was thrown overboard for his discovery). So you have to have square roots if you want to do geometry like the Greeks (because of conic sections).
- There's a problem which is for any square find another square that has twice the area - it was known by Plato and was in the Meno - solve it by taking the diagonal of the first as the side length of the second. Can you do this with a cube? You can't do this with a compass and straight edge. The medeival Arabics pushed forward by going passed compass/straightedge and finding teh roots of a cubic equation without requiring that the length we're talking about can be one you can actually create with a compass and straightedge. That produced a whole new family of lengths never talked about by the Greeks. It was widely believed that you could create new numbers by constructing equations with irrational roots with only rational coefficients. For example $x^3 = 2$ you would introduce $\sqrt[3]{2}$. It was widely thought that all the real numbers were got by solving such equations, but for example π isn't.
- Cantor showed that there are many more real numbers that aren't solutions to equations than ones there are. The question at the end of the 19th century was how to get a picture of what the real line was like - how to fill in all the holes. Dedekind's answer was to introduce the cuts. To fill in all the gaps, we need a cut everywhere. He gave a criteria for what it would mean for $\mathbb{R}$ to be complete, and second to show that there really was such a thing as that complete system of real numbers. The algebraic properties of the real numbers are relatively easy to satisfy, and you need added properties in order to fill in all the gaps. The principle Dedekind proposed was the least-upper bound principle. He also wanted to convince people that this was final - that you couldn't go any further.
- Dedekind started by seeing if a geometric approach would suffice - he thought this was obvious - why? You also need a lot of (controversial) set theory to get Dedekind cuts, but the Dedekind cuts do reduce the reals to the rationals to the integers. Around this time, everyone considered the laws of geometry to be synthetic a priori (Kant), where we could only make sense of our sensory experience if we organized them into three-dimensional space. It seems dubious that the way we organize space should be the foundation of analysis because we have tons of applications of the real numbers that have nothing to do with space. Also, there's non-Euclidean geometry.
- Dedekind started by enumerating the rules needed for $\mathbb{R}$: field axioms, ordering, etc. But that doesn't distinguish from the rationals. You need something more: the completeness principle (LUB). Dedekind gives some axioms that the thought could answer the question of what the real numbers are, or at least he says the real numbers satisfy those structural features. Why is identifying the structural features enough to say that you've identified

the real numbers? There are going to be lots of other algebraic systems that satisfy those axioms. You have to also give the intended model for those axioms. The physical line satisfies those axioms, but we have no reason to believe that any particular one is not the real number system. Also any complete model of those axioms is going to be isomorphic to any other. Why is giving the structure not enough?

- At the same time, Frege was investigating the foundations of arithmetic. He thought that the only principle he needed was Hume's Principle (two collections have the same elements iff there's a one-to-one correspondence between the elements). He showed you could get PA axioms from Hume's Principle, but he still asked: how do I know whether Julius Ceasar is a number? We expect a numeral to pick out a definite thing, but we can never do this with only axioms. Dedekind introduced the idea that the numerals don't have to stand for definite objects - that mathematics is rather about identifying structural properties, and that identifying the structure is all you can and need to do.

3 Wednesday, February 22, 2017

3.1 Dedekind's Construction of $\mathbb{R}$

- Dedekind asked: what further properties than the normal ordering and field properties are necessary to specify the real numbers? Completeness. His proposal is to create cuts of the rationals into two non-empty classes so that everything in the lower portion is less than everything in the upper portion. Completeness means that anytime you have such a cut there is a real number defining it. This principle amounts to the least-upper bound principle. He showed that this condition characterizes the real number system (up to isomorphism). Historically, people had made systems that purported to be the real numbers but it turned out that they left some numbers out. But Dedekind showed that you can't add anything to his system and get anything that's properly larger.

- The isomorphism theorem says that if you have any two models of his axioms a and b, you first define the positive integers of a to be those that contain $0 \in a$ and are closed under succession (and define the positive integers of b the same way), and then define a map between the two sets of natural numbers. This map is injective - if it weren't, then the set of integers that had mappings would have a LUB b', and somewhere between b' and $b' - 1$ you would have an integer that gets mapped. This lets you map the rationals of a to the rationals of b, and so then we get an isomorphism of Dedekind cuts and so of the real numbers in a and b.

- So we've completely characterized the structure we're looking for - how do we know that there is something that exhibits that structure? Dedekind's response is to define the set of all cuts to be the real numbers and then show that this structure satisfies the axioms. So he takes the construction to be a sufficient answer to the question of what the real numbers are. But it feels like he's just given us something that is isomorphic to the real numbers.

- Frege equivalently gives structural axioms for the natural numbers, but wasn't satisfied because he couldn't answer the question of whether Julius Caesar was a natural number. Hume's Principle is a structural property that only characterizes the natural numbers up to isomorphism.

- All we need to construct the natural numbers is an infinite set from which we can generate a simply infinite set, but that's non-trivial because, starting from Zeno's Paradox, there is a long line of people saying that we can't have an infinite set, just a potentially infinite set. They say that all we can do is give methods for taking a finite sequence and extending it, but the idea that we can treat an infinite set as a whole was long treated as contradictory. Dedekind thinks he can build an infinite set by taking the totality of objects of his thought and take a member of that totality S and form the thought "S is an object of one's thought", and then recursively do that. That's a one-to-one function from the objects of one's thought to the objects of one's though, but it isn't onto because one's ego isn't of the form "X is the object of my thought". This skirts the question of potentially vs. actually infinite because the totality of objects of thought is something given to you, not something constructed.

- He says that the natural numbers are a "free created of the human mind" which he clarifies as being because they are a result of the process of abstraction specified above. This isn't completely new - Cantor does something similar when he wants to go beyond the natural numbers into the transfinite by looking at the different ways of ordering the collection and then defining an order isomorphism and then defining the order types as got from the orderings by the process of abstraction. You regard two orderings the same if their orderings are isomorphic (there's a bijection from one to the other that preserves the order). Or you can go even more abstract and say that two totalities are the same if there is a bijection between them, irrespective of ordering. That gives you the cardinal numbers. Aristotle also gets the objects of geometry similarly - he thought of them as ordinary solid bodies but we abstract away all the non-geometrical properties. Both Cantor and Dedekind were dipping into that tradition.

- In retrospect, it seems like this is a big transition from thinking of mathematical terms like "natural/real/complex number" as denoting definite things (not things known from sense-experience, but still definite entities) to thinking about them as if we can talk about them as definite things once we've characterized them up to isomorphism. Modern mathematicians are pretty much invariably subscribers to the latter view - you talk about mathematical objects, but mathematical objects are what they are because of their properties and properties only characterize up to isomorphism. There was a change in the way mathematicians talked about mathematical objects that philosophy was really slow to notice.

- Cantor also had a way of creating the real numbers out of the rationals. Cantor himself was interested in Fourier Series and when they converge, for him the idea that the real numbers were complete was cashed out by the idea that series converge whenever it's possible for them to converge. For any convergent sequence, we have $\forall k, \exists N$ such that $n > N \Rightarrow |a_n - a_N| < \frac{1}{k}$ (Cauchy sequence). So one criterion for the completeness of the real numbers other than the LUB principle is the Cauchy criterion implies convergence for any sequence. You can derive one criterion from the other. Cantor then defined the equivalence relation between $\{x_n\}$ and $\{y_n\}$ as for any k there is a sequence term such that y_n and x_n are arbitrarily close passed the sequence term. You can show that Dedekind's and Cantor's principles yield isomorphic copies of the real numbers. But the question never arose as to which of these is the genuine construction of the real numbers. It didn't occur to them to ask: what's a real number - a Dedekind cut or an equivalence class of Cauchy sequences?

- Hilbert's axiomitization of geometry: he gave the set of axioms and wasn't initially concerned with the question of whether the system was complete. We know we get a model of the geometry axioms if you take a point to be an ordered triple of real numbers (identify a point with its coordinate in $\mathbb{R}^3$, which doesn't have anything to do with space and yet is still a perfectly good model). You also get a perfectly good model if you identify a point with an ordered triple of algebraic numbers - so the geometric axioms do not require completeness. Later on, Hilbert started to worry about completeness and so he added an axiom in the 9th edition so that the new axiom system is complete. One of the axioms of the original system was the Archimedean axiom: you can construct a line segment out of a sub-segment so that there is some multiple of the sub-segment that will surpass the end of the larger segment. Traditionally, geometric axioms were supposed to represent three-dimensional space, but Hilbert rather didn't deny that the axioms describe physical space but didn't assume that they were about physical space. He said the axioms were about systems that satisfy those axioms, and nothing more - there's nothing more to being an intended model for the geometric axioms. This is similar to group theory - there's no intended model for group theory other than a system that satisfies those axioms. This is a real departure from the idea that mathematics has a definite subject matter.

- You can have a name without a referent - for example, "Cerberus" or other names that occur within myths. You don't have that with names like "Cherry" - there's a definite referent to the name "Cherry". But we have these numerals in mathematics (e.g. $\sqrt{2}$) that don't have a uniquely determined thing that they refer to. The issue with idea of identifying the square root of 2 with the set of all things that when squared is 2 is problematic because you can't square a set. And you can't identify it with any one object whose square is two for similar reasons. There's a worry that Dedekind gave us a construction of what he called the real numbers and long before him the Egyptians were using numerals that purported to be real numbers, so did the Ancient Egyptians refer to a Dedekind cut when they wrote down one such numeral? Or, possibly, could the real number signs not refer to entities but instead to some action, like the act of taking the successor with the natural numbers? Well, the numerals do correspond in an objective way similarly to how people's names objectively refer to people. If it were purely made up, then there wouldn't be a "correct" answer for question with real-numbered answers.

- Dedekind described what he was doing as the reduction of analysis to arithmetic, which was understood to have secure foundations. But analysis has less secure foundations because people didn't have a clear understanding of continuity and limits and what was required for a sequence to converge - there was just a general lack of clarity. It's more accurate to say that he reduced analysis to set theory or set theory plus arithmetic because what he needs to form the cuts is a lot of set theory. Looking back, it seems that he really did do something important for our understanding of the foundations of analysis, but it was not reducing analysis to arithmetic but rather constructing analysis within this theory of sets. Next time let's look at the axiomatic theory of sets and consider Dedekind's construction as a construction of the real numbers within set theory.

4 Wednesday, March 1, 2017

4.1 Combinatorial Conception of Sets

- Last time we talked about Zermelo's axiom of set theory. One thing Zermelo wasn't addressing but were in the background were the paradoxes, which is why people responded to his proof of the well-ordering theorem with such uncertainty (their understanding of how sets work was shaken by the appearance of paradoxes). Formal set theory was still pretty new and their study only came to the foreground with Dedekind and Cantor and it seemed to fall immediately into contradiction.
- The most prominent of paradoxes was Russell's paradox - some sets are elements of themselves, so consider the set of sets that are not members of themselves. If you ask the question of whether that set is a member of itself, you get a contradiction. The paradox has a direct analogue about language - "dog" isn't a dog so it doesn't apply to itself, but the phrase "does not apply to itself" applies to itself iff it doesn't.
- Another paradox was with Cantor's study of ordinal numbers, he found that the ordinals were well ordered and for any well ordered set there is an ordinal that describes the order type of that set. Any ordinal is the order type of the ordinals less than it. But consider the collection of all ordinals, which is well ordered, and so there is a map from all the ordinals to all the ordinals less than some ordinal. Keep applying the map and then you get an infinite descending sequence which is a contradiction because the ordinals are supposedly well ordered. The semantic analogue is to take all the names of ordinals, there are only countably many names and so there are only countable many names of ordinals, but since there are uncountably many ordinals there are ordinals that aren't named. Since the ordinals are well-ordered there is some least ordinal is not named, but we have just specified a name for it. Also there is a least natural number nameable in fewer than 19 syllables.
- Cantor showed that there are uncountably many real numbers by assuming that there are only countably many real numbers and then disproving the bijection by constructing a new real number not in the enumerated set. You can make a bijection $\mathbb{R} \to \{q_1, q_2, q_3, \ldots \in \mathbb{Q}\}$. He generalized the result to show that for any set there are more subsets of S than the number of elements of S. But then take the set of all sets, the cardinal number of the power set of the set is larger than the cardinal number of the set itself. But the power set of all sets of sets is included in the set of all sets and so the cardinal number of the power set is less than or equal to the cardinal number of the set of all sets. There is no direct linguistic

analogue but it came from the generalization of Cantor's real number uncountability proof. If we look at that theorem linguistically, we see that there are only countably many names for the uncountable $\mathbb{R}$, and then put the names in a list an use Cantor's procedure to construct a real number that is nameable but not on the list of all possible names.

- Cantor just rejected that there was such thing as the set of all sets. He thought there were theological implications. He wrote to the Pope (no response) and said that he was developing a theory of the actual infinite but there may be worries because the church teaches that only God is infinite. He says it isn't blasphemous because he says he has found an intermediate level between the finite and the absolutely infinite (transfinite) where the transfinite goes beyond what we ordinarely think of the finite, but it's not copmlete limitless. If you had something completely limitless then you can't number it or comprehend it mathematically - only God is at the level of the absolutely infinite which is above not just the finite but also the transfinite. So Cantor's work isn't bringing God down to the level of ordinary mathematicians, but rather he boosted God up to a level farther beyond the level of things that Earthly things are.
- So there's these analogues between set theoretic paradoxes and semantic paradoxes. Zermelo didn't seem interested in the semantic part. Russell had the exact opposite attitude - he thought this showed a deep discontinuity in human's thought, one that was so profound because it manifested in a lot of different ways. How important is it to get a unified solution to the set of two paradoxes? It really seems like they are the same phenomenon.
- This week we're going to look at a pardox that wasn't among the list of Russell's paradoxes - developed around 1960. You can't recursively pick sets as elements of other sets - there's no infinite descending chain. Say a set is well-founded if there aren't any infinite descending chains beginning with. Let W be the set of all well-founded sets. If there were an infinite descending chain ending with W then look at the chain the ends at the element which is an element of W and so that chain is infinite and the element is not well-founded. So W has to be well-founded. But then $W \in W$ and so there's an infinite chain ending at W by picking W for every link of the chain. Since no well-founded set is a member of itself, this paradox seems to reduce straight to Russell's Paradox. But it doesn't because you could have $A \in B \in A \in B \in \dots$. So no loops of any length around whereas Russell's paradox is just a loop of length one. This paradox turned people's attention to sets that are well-founded - but most sets that are useful for math are those that are well-founded. If you want to really think closely about set theory but your interest in set theory is just for finding a theory of sets that is useful for mathematics, then you can just stick with the well-founded sets.
- So there was an axiom proposed that said that all sets are well-founded. What amounts to the same thing is: for any non-empty set S, there is an element $s \in S$ such that $\forall t \in s, t \notin S$. Originally it was just adopted for convenience but it does yield a really pretty picture for what the universe of sets looks like. If we assume that the things that aren't sets form a set, then you get this picture of the theory of types. The crucial axiom for building out this structure is the axiom of ordered pairs - this allows you to put together two elements from different levels. This is the way that Zermelo was thinking about it but not the way

that Russell and Whitehead were thinking about it. Cantor's paradox entails forming a set at some particular level, but the set of all sets must be at a level before which the set of all sets is formed and so there is no level where the set of all sets is at. Since a set is always formed at a level after the level at which its elements are formed, so you solve the same set of paradoxes. For the ordinals, each level of the hierarchy introduces a new ordinal. Each level of the hierarchy introduces exactly one new ordinal. So in order to get the collection of all the ordinals it would have to be formed at a stage after which you'd already gotten all the ordinals and so there is no collection of all the ordinals. So there's a response to all the set-theoretic paradoxes that comes just from trying to form a set that violates the foundation axiom.

- So adding another axiom gave us the stronger system because it removes an inconsistency but does not introduce one. If you restrict the axioms of set theory to range only over well-founded sets. You can verify that each formula you get from an axiom of set theory by restricting the quantifier is a theorem of set theory. If you could derive a contradiction from the axioms of set theory including the axiom of foundation then you could get a contradiction in ZFC.
- The tradition, beginning with Cantor, is thinking about collections formed from contained elements and you need collections to be well-founded and so you get paradoxes if you don't restrict to well-founded collections. The opposite view is from Frege where you get sets by starting with concepts. Von Neumann thought that you get contradictions when you start forming sets that are really large ($>>> \mathbb{R}$) like the set of all the ordinals or the set of all the cardinals, etc. So forming sets that are way too big is impossible but formulating sets that are relatively small is okay. The range of a function on a set will be at most the same size as the domain set so then if the things in the domain aren't too many to form a set then the things in the range aren't too many to form a set, either. VN had the replacement principle and this more global principle that you can form sets unless they are too big, "too big" meaning things form a set unless those things are equinumerous with the entire universe. So if you have set that's soo big that if it's a set then the entire universe is a set then that set is not actually a set. So there's a different way of thinking about where the paradoxes come from by having sets that are just too big. Axel took this idea and build a whole set theory out of it.
- This is the combinatorial conception of sets as collections. Next time we will think about sets from a logical point of view by starting with concepts and then get to sets by a process of abstraction.

5 Monday, March 6, 2017

5.1 Logical Conception of Sets

- We've been talking about set theory, of which there are two conceptions. The combinatorial approach says that sets are collections built up from individual. The other idea is the logical conception of sets, where you gets sets by starting with concepts and forming sets by abstraction. This conception starts with Frege. Frege started out by looking at complex names like "the capital of France" and saw that the name was formed by "France" and the function sign "the capital of _". "France" names something, but "the capital of" only becomes complete by begin supplanted by a name. Concepts are unsaturated and are not individuals because it needs another object to be completed.

- We know what an incomplete building - there are still bricks and mortar, but just not enough to yield a whole. But what is it for a concept to be essentially incomplete? At the syntax level, you put a name into a concept-phrase; at the semantic level, you put an object into a concept to get a truth value. Someone might (incorrectly) think that Versailles was the capital of France and say "We're going to Paris and then going to the capital of France". But you can't substitute "Paris" for the latter reference because the person believes that the referrent of "the capital of France" is Versailles. Senses deal with things like this (i.e. belief contexts). Thinking about sentences as having the True/the False as their referrents turned out to be very fruitful because he could think as predicates as functions. This allowed standard operations with functions from basic algebra to be applied to concepts.
- Frege has individuals, functions, function signs with more than one argument, and second-level functions. Second-level functions: "someone is a wise philosopher" has a similar grammatical structure as "Socrates is a wise philosopher", so it looks like you have "_ is a wise philosopher" as a concept which can take "Socrates" and "someone" as arguments. But in the latter case the individual about which you're talking is only indefinitely specified. Frege saw that this wouldn't work because of things like "Someone is wise" and "Someone is a philosopher" does not yield "Someone is a wise philosopher" and so the logical structure of the two is really quite different. Frege's idea is that "someone" is not a name, but rather a function sign that takes as its argument a concept. So "someone" is the property that a property has if there is at least one person who falls under ther latter. So things like quantifiers or the definite integral sign are second-order.
- Frege wanted to use this logical framework to establish arithmetic on a purely logical foundation. The prevailing view was that the laws of arithmetic were synthetic a priori, where you become acquainted by arithmetic by experiencing the act of succession. He wanted to reject that view and show that the arithmetic truths are analytic. The principle he uses to establish this is called Hume's Principle, which says that two concepts have the same number just in case there is a one-to-one correspondence between the things that fall under them. He wanted to avoid postulating the existence of the numbers, but he has to go deeper than Hume's Principle in order to avoid this. He wants to be able to think of the laws of number as purely logical laws.
- Something deeper going on was the theory of classes. The motivating idea is that if you say Traveler falls into the class of horses, you aren't saying anything more or less than just that Traveler is a horse. So he wanted to do something like identify 5 with the class of all 5-element sets. But he was able to define the numbers so as to avoid the circularity.
- He had a theory of arithmetic based on counting as derivative from Hume's Principle, and then he set out on similar project with the real numbers. But then he got a letter from

Russell, who pointed out Russell's Paradox that comes out of the set of sets that aren't members of themselves. Frege (basically) gave up the project, but Russell took it over. Frege had gotten to this point by saying that for any concept there is a corresponding extension. The idea that you could match concepts and objects in a one-to-one fashion is exactly what Cantor showed you couldn't do with the real numbers. This is a very general argument that there are more concepts than extensions. Russel saw that and thought that the way to get around this is not to treat numbers as objects but rather as second-level concepts. So there is a concept that is true of all concepts under which exactly 5 things fall. The former concept is what Russell is going to treat as the number 5. Frege got into all this trouble by postulating classes, Russell tried to get away without classes. Russell wanted to postulate concepts, and he thought that you could guarantee the existence of concepts corresponding to the predicates of the language on purely logical grounds. Because from "Traveler is a horse" we can infer "$\exists x$ x is a horse" we can also get "$\exists F$ Traveler is F". He thought that would give you on purely logical grounds the existence of concepts.

- But you can still formulate Russell's Paradox by thinking about the concept of not falling under itself. Russell wanted to use concepts in all the places where Frege used classes, and even though you can still get arithmetic from both you can also still formulate the paradox in both. He wanted some principle to ensure that the expressions of the language are meaningful because he knew that you could take expressions that seemed reasonably and get contradictory results. He appealed to Poincare's Vicious Circle Principle, because then you can exclude notions like a concept applying to itself. Russell's idea was that you need to follow Poincare in order to avoid the paradoxes. He added to Frege's levels of concepts (the **type** of the concept, or the number of arguments) the **order** of the concept which pertains to which types of concepts are necessarily for formulating that concept. Later, he would replace concepts with propositional functions, which are functions that take an object as an argument and gives a proposition as its result.
- The Vicious Circle Principle gets in Russell's way. First, we want to be able to state the LUB principle of the real numbers. In terms of the vicious circle principle, you have this collection S that is bounded above, so you have a collection of upper bounds of S and the least member of that collection is the LUB of S, which violates the vicious circle principle. There's another example of violating the VCP with trying to define the natural numbers. To get $\mathbb{N}$, he needs to postulate the existence of an infinite set of individuals. Zermelo's theory was neutral in that it allowed individuals but didn't require individuals. Russell had to require the existence of infinitely many individuals, which already was a problem because General Relatively says that the universe is finite and QM says that at the level of the very small we can get indivisible pieces, so it was actually a question of whether there are infinite individuals (at least, the postulate is contingent and dubious). If you suppose there is a simply infinite set, then you show this by showing a one-to-one function whose range is a proper set of the domain and then generating a simply infinite set by taking something that is in the range but not in the domain and then following that cycle. Russell tried to use the axiom of infinity to get 0 and the successor operation from this simply infinite set. The natural numbers can't just be a set with 0 and everything accessible with

successor – you need to be talking about the *smallest* collection that containes 0 and closed under successor or else you don't get induction. But what is meant by "smallest"? The intersection of all collections with 0 and closed under successor. But then you are defining the natural numbers as the intersection of all collections that contain the natural numbers and your definition is circular.

6 Wednesday, March 8, 2017

6.1 Iterating on the Theory of Types

- Last time, we ended with a discussion on how Russell's theory of types is inadequate for doing basic mathematics (you can't get the real or rational numbers).The problem was that the viciuos circle principle was too restrictive and issues arose with only being able to define propositional functions on lower-order/type functions. As a patch, Whitehead and Russell proposed that for any propositional function there is a coextensive propositional function with the same type and lowest possible order (without talking about anything that isn't in the extension of the propositional function). This gave them the mathematics they wanted but completely gave up the possibility of mathematics being pure logic. As long as all the propositions you needed could come just from existential instantiation, then it seems that you could just use pure logic. But existential instantiation does not get you lower-order correlates of the propositional function. Assuming the axiom of reduciability, they really gave up the idea of reducing mathematics to logic (and admitted as much).
- It's pretty similar with the axiom of choice, that if you have a set of non-empty overlapping set there's another set that picks one element from each of them. To Vann, it seems very obvious if you're thinking about classes as collections (you can always form the product space and find a projection). If you're thinking not in terms of classes but in terms of propositional functions and classes as things that we construct by constructing predicates, then there's not really intuitive grounds for axiom of choice. If you have infinitely pairs of socks, then there's no reason you should be able to form a predicate of only one from each pair (??). If you think of sets as being created by some form of mental process, then the axiom of choice seems dubious.
- The Banach-Tarski paradox says that you can cut up a sphere of diameter 1 into pieces and re-assemble the pieces into two spheres of length 1. You need the axiom of choice in order to create the cut. Some people thin kit's a problem and other's don't. Why it seems like a problem is that you get 8x as much matter for free (it seems). Others say that it just seems like a problem because we expect volumes to be preserved, but if you think of how volume is defined (you get volume of a sphere by inscribing polyhedra inside and outside the square and taking the limit of many faces), but you can never really close the gap between the inside/outside and the sphere.
- Russell had two motives for adopting the vicious circle principle: he wanted to respond to the set-theoretic paradoxes and he watned to respond to the semantic paradoxes. In the introduction to *Principia Mathematica* there is some talk about the semantic paradoxes.

Ramsey proposed that we distinguish between the semantic (epistemic) paradoxes from the set-theoretic paradoxes. For the purposes of creating a foundation of mathematics you just focus on the set-theoretic paradoxes. Then you realize that you don't need the vicious circle principle to solve the set-theoretic paradoxes when you have the axiom of reducibility. Ramsey suggested that you scrap order altogether and just arrange the propositional functions so that a function just has to be a higher type than its arguments. Quine took that a step further: Frege wanted to talk about classes, Russell wanted to replace that with talk of propositional functions. Quine noticed that the talk of propositional functions was intended for the logical reduction project, but since that failed we might as well go back to talking about classes. So we had individuals, classes of individuals, relations (binary/ternary/so on) between individuals, binary relations relating classes to classes, individuals to classes, etc. So you end up with this structure that is still kid of complicated but still much simpler than Russell's solution.

- Norbert Weiner saw that even that picture could be simplified by instead of talking about relations, let's identify a relation with a class of ordered pairs/triples/etc. (identify $< a, b, c >$ with $< a, < b, c >>$. So you can just talk about individuals, classes of individuals, classes of classes of individuals, etc. Weiner's idea was that you didn't have to treat ordered pairs as primitive. The law of ordered pairs says that $< a, b > = < c, d >$ iff $a = c$ and $b = d$. Norbert noticed that you could take $\{\{\{a\}, \emptyset\}, \{\{b\}\}\}$, and you can verify that this satisfies the law of rodered pairs. There are more brackets than you'd think because he was working in the theory of types and need to be working with things of the same type. With this simplication, what started as a monstrously complicated system becomes very simple.
- To get mathematics, he needed infinitely many individuals. If you have only finitely many individuals, no matter how high up the hierarchy he went you'd still only have finitely many things. But if the individuals about which you're talking are physical objects than it is actually pretty dubious that are infinitely many. So they added the Axiom of Infinity and they didn't regard it as obviously true.
- There's still one feature of that project that is still kind of anomalous. You want to say that there are 8 people in the room, and how we're going to make sense of that is by saying that the number 8 is the collection of all 8-element classes of individuals. So a number is a class of classes of individuals such that every class of individuals that can be put in one-to-one correspondence with a member of the class is a member of the class. What is still peculiar about this theory of types is that you can't mix types which means you can't use the same numbers to count individuals as to count classes of individuals. So there's a different number system of each type and there are infinitely many systems of natural numbers. It's not fatal, but it's just a little fussy and seems kind of artificial.
- Gödel stepped in a suggested that there is no good reason not to make the classes cumulative instead of only allowing members of type $n-1$ in classes of type n. Originally, we wanted this hierarchy of classes in order to protect us from the paradoxes and the vicious circle principle, but we've given up the vicious circle principle. He noticed that we get the paradoxes when we ask whether a member of the class is at the same type as the class, and so we can lighten the restriction to allowing any type below the class, not just the one type just below it.

Originally, we get the type hierarchy from a grammatical restriction on concepts/objects, but once you allow cumulative classes you need to abandon the grammatical distinction - so you say that you won't make type distinctions among the things we're talking about as far as logic is concerned. Some of the things we will talk about are individuals, sets of individuals, and so forth, but we're going to be able to talk about those all at once without distinguishing between them (there's only one kind of variable for the logic). What gets us the hierarchical structure is not a grammatical restrictuion but rather an axiom that tells us that the sets are well-founded. so in Gödel's system you have a very simple logic (first-order predicate calculus) but you have a lot of not simple axioms guaranteeing the existence of sets.

- So it looks as if there is no hope for reducing mathematics to logic, it looks like Cantor and Dedekind won. The advantages of a logical conception of set over a combinatorial conception of set disappear once this project doesn't work. And ZF set theory is so much more powerful because it can extend into the transfinite. But this is also possibly bad because it can make it more vulnerable to contradiction. Still there are lot more points that have gone to the combinatorial conception rather than the logical conception. There is still a lot to be desired and it's a big loss to have mathematics be synthetic a priori rather than analytic.
- Gödel's incompleteness theorem throws a wrench in the works by showing that there are arithmetic truths that we can recognize as true that aren't derivable from the axioms, and that even if you add more axioms there are still more truths that aren't provable.
- More recently, the Neologicist program has been trying not to get us all of ZFC but rather to get us mathematics strong enough in order to serve or purposes will staying in an (arguably) logical framework.
- Dedekind proved that arithmetic is categorical, but if you think of arithmetic as a first-order theory the way we do know, then that is not categorical (it is not a second-order theory). You get induction from a quantification over properties and you get categoricity from induction. But when you do things in a first-order framework you have an axiom schema for induction and so you have infinitely many induction axioms instead of a single second-order axiom. But then this new set of axioms isn't categorical and by Gödel's work isn't complete.
- The next part of our story is going to be a revival of second-order logic as something we can do in teh context of modern mathematics when we're well beyond the *Principia* stage. That revival is due to George Boolos. Next time, we'll talk about Boolos and Quine and then we'll start talking about Benacerraf. Read Quine and Boolos articles.

7 Wednesday, March 15, 2017

7.1 Quine's Nominalism

- Quine really wants to be a thorough-going nominalist. He wants to deny that there are any abstract objects. But he doesn't think he can do it for reasons that we'll spend most of April talking about. Right now, we're going to talk about his more-limited version of nominalism.
- He's willing, albeit reluctantly, to concede that there are such things as classes, but he wants to deny the existence of universals and properties. His reasons for doing that are mostly having to do with the absucrity of the identity conditions for properties.
- You can tell whether two classes are the same, which is in the case where they have all the same elements. It is hard to tell whether two properties are the same. It's interesting that Russell went the other way, trying to get rid of classes and use only properties.
- It doesn't really matter to us that Quine wants to deny that they are properties. What is a more interesting question is how is that even an intelligbile thought. We see all the dogs and are able to distinguish all the thigns that are dogs from those that are not dogs. How can we classify that which is a dog and that which is not?
- The obvious answer is that there are some properties that dogs have in common and we can use those properties for our classification. Quine wants to deny that there are these properties that the dogs have in common.
- One approach is to take the nominalist view that the only thing dogs have in common is the name we give them. But this approach is bad because it isn't arbitrary how we classify the dogs but it is arbitrary how we name them. This is how Russell thought of it. To say, 'X is a dog', 'Y is a dog', and 'Z is not a dog' all logically entail that there is a property that X and Y share but Z lacks.
- There's this position that Quine wants to advocate that there are not any properties, but it is hard to formulate that in a way that a competent English speaker wouldn't regard as obviously false.
- The idea is that there is something that X and Y have that Z lacks. The thought is that there is this property that is shared between the former but not by the later. Question: why can't you not specify what they have in common but instead just say that they are both dogs.
- Or is there a difference between properties and propositional functions. Common doctrine is that 'that' clauses denote propositional functions. If you use 'that' clauses as direct objects of verbs (??) commit us to propositional functions.
- But Quine wants to deny that there are any propositional functions. Didn't we learn anything from going through this pointless complexity of going through the ramified theory of types. We really just need an inclusive view of what objects there are. In particular, we don't need a special logical category for propositional functions. Just put them in the same domain as your quantifiers. It's going to make our life more complicated without adding

much by build distinctions (like type distinctions) into the logic. So for type distinctions we just need some way to demarcate the differences within the universe of the domain of the quanitification instead of adding logical distinctions.

- Quine's answer to his own question: first, he has a nice slogan. "To be is to be the value of an variable", which tells us a method for answering the question of what the ontological commitments are of a given theory. That seems to be as far as it goes but really doesn't seem to go very far at all because anybody who doesn't agree that because X, Y, and Z all walk on four legs and share the property of being four-legged. Anyone who wants to deny that is not a competent speaker of English.
- Quine proposes that this criterion is right but in order to employ it you need to enter a process of translating English into a canonical form where you are trying to fully make the commitments of your theory as explicit as possible. In particular, Quine says that if you want to make it explicit what the ontological commitments of your theory are, then reformulate the theory within the first-order predicate calculus. Then, the things that have to serve in the variable place tell you what the ontological commitments of your theory are. He doesn't think this is obligatory but he thinks that if people refuse then it means that they are refusing to make their own logical/ontological commitments explicit.
- It seems like this is demanding too much. The machinations Carnap had to go through in order to talk about disposition terms (??) in the first-order predicate calculus. To talk about what it means for a chemical to be soluable: you know that water disolves sugar, soy ou can see which particular samples of sugar dissolve and you can predict which won't. You can predict that sawdust cannot dissolve. So a natural formulation is x is soluable iff x dissolves when put into water. But if you use this as your material conditional, you'll get that things that are never put into water are soluble. But you don't want that. Carnap went through this big effort to try and figure out a way to express solubility in the first-order predicate calculus by saying: if a theory of solubility contains two axioms:

  1. Disolves in water
  2. Two identical substances either both disolves or neither of them does. (??)

  But if something is never put into water and nothing else like it is ever put into water than you cannot say that is it not soluble.
- You get something that looks a lot more natural if you're allowed to understand the conditional that x is soluble if it dissolves when put into water - to understand this is a much more strong material conditional (if it were put into water, it would dissolve) - that seems like a use of disposition terms is deeply entrenched in our efforts to describe the world in science and without. And our use of counterfactual conditionals is also a very fruitful usage. It seems necessary restrictive to go with Quine and restrict to the lower-predicate calculus.
- But if you accept Quine's criteria you seem to be getting bogus ontological commitments inasmuch as you want to understand disposition terms in terms of counterfactuals. Well, you can't take counterfactual usage as primitive because you have to ultimately formulate things in terms of the predicate calculus, but you cant take such usage as defined by adopting

a possible world semantics. x is soluble if it would dissolve in the closest possible world. But we wanted to avoid talking about possible worlds because we wanted only to look at what is around us and speak about that.

- Vann wanted to return to the discussion we were having last time about the possible skeptical conclusions from the Löwenheim-Skolem theorem. Beyond our theorizing, there is a use of number talk in counting and measuring that doesn't really seem to get us significance beyond what we get from just the theorems. The fear is that if we accept Quine's mandate to restrict ourselves to first-order logic, we seem to be forced into some sort of skeptical position. Our mathematical theories don't have a unique intended model. That is something that we saw already from Dedekind and that's something we feel like we can cope with. What Dedekind did was to capture the real and rational numbers in a way that was unique up to isomorphism, but didn't go any farther than that to specify which of the isomorphic structures was the real number system. Look at how Dedekind develops the theory: that his terms doesn't have uniquely determined referrents doesn't seem to cause any problems for the way he does mathematics, even just for practice; classifying up to isomorphims it seems good enough. Take whatever our mathematical theory is, assumign infinite models, it is going to have wildly disimilar infinite models that don't resemble each other at all structurally.
- Something Putnam wants to insist on is that we don't appeal to occult powers of the mind to see how our terms latch on to their determinate referrents. But we're not we're just using number talk that leave it drastically undetermined what the referrents of our terms is. We get a kind of skepticism that is upstream from the kind of skepticism you get from thinking about whetehr theorems/axioms are really true. This is at a level of skepticism that beings even earlier. Nevermind how you know they're true, how are you going to have mathematical beliefs if supposedly beliefs are about these objects but there is nothing that you do that is even close to picking out those mathematical objects.
- Skolem's paradox was from the early paradox and was easily answered just by saying that you get the elementary submodel by neglecting some functions that we know are really there. So, yeah, if you neglect things you can get a theory that looks just like our theory. That looks like a sleepy little paradox. But then Putnam took it and made it look like a very serious paradox, both inside and outside of mathematics.
- There's a way to get around the paradox which is to do what Dedekind did and say that contemporary mathematics' describe teh natural numbers etc. as models of a first-order theory. Dedekind proceeded differently: wrote down his axioms as a second-order theory where he could range quantifiers over properties and was able to prove categoricity (that his axioms actually determined the real or natural numbers uniquely up to isomorphism). But his proof took place within this second-order theory, Can't we reapply the Löwenheim-Skolem argument to the higher theory. Unless we assume that we have some grasp of what properties there are that can't be explained in how we use property talk. So you just get back to using occult powers of the mind again.
- There's an answer to Skolem's problem that claims to be different than just kicking the

problem upstairs to higher-order logic. Though whether or not it really is different is controversial. The alternative answer started with Peter Geach with the project of formulating ordinary English sentences into ordinary language. He was particularly interested in the use of plural noun phrases. Take the sentence: "there are some critics that only admire one another". The sentence "there are some critics who just admire impressionists" can be formulated easily in a first-order idiom. But "there are some critics that only admire one another" can only be expressed in a first-order idiom if we can appeal to the use of classes.

$$(\exists S)((\exists x)x \in S \,.\, (\forall y)(y \in S \Rightarrow \text{Critic}(y))\forall x \forall y(x \in S \,.\, \text{Admire}(x,y) \Rightarrow y \in S))$$

But then:

$$(\exists S)(\exists x Sx \,.\, \forall y Sy \Rightarrow \text{Critic}(y)\forall x \forall y Sx \,.\, \text{Admires}(x,y))$$

But if I say that there are critics who admire each other it doesn't seem like I am telling you anything about classes and you could imagine someone who is a strict nominalist and certainly believe that there are critics, you can think anything you like about their group relations, and you can belief that there are critics who admire only one another, but you can't be a nominalist and only belief this sentence. If you accept the commonly held believes about sentences then you learn that there are the same truth conditions for the first and second sentences but you may belief the things about critics without believing the reformulation.

- You can't formulate the statement about classes just in terms of first-order logic; take the sentence: "there are some critics who only admire one another". That sentence has the same form as "there are some non-zero natural numbers that are adjacent only to one another" but the latter sentence is only going to be true in a non-standard model of arithmetic. If you could successfully formulate in first-order logic that there are some critics who only admire one another then you could successfully formulate that there are some non-zero natural numbers that are adjacent to one another, then you could construct a sentence that is true in the standard models but not in the non-standard models. But then it is not possible to give a categorical description of the natural numbers system.
- Boolos proposed that the correct way to understand that kind of use of plurals we see in the last sentence is by taking a different kind of the use of the quantifiers. We have quantifiers that range over individuals, second-order quantification where they range over properties. Boolos is proposing the notion of plural quantification where the variables range over individuals and the difference is that in singular quantification the variables range over individuals one at a time while with plural quantification the quantifiers range over individuals many at a time. So there is in the semantics for first-order predicate calculus a variable assignment is a function assigning a value with each of the variables. Then you inductively define what it is for a variable assignment to satisfy an open sentence.
- There are critics who only admire one another with:

$$\exists xx((\forall y)y \in xx \rightarrow (\text{Critic}(y) \,.\, (\forall y \forall z)(y \in xx \,.\, \text{Admire}(y,z)) \Rightarrow z \in xx))$$

Note the $\exists xx$ which is the plural quanitifier. This theory doesn't posulate any new things and doesn't add any new attributes or properties to get the range. Both quantifiers range over individuals. The plural variables are as restrictive because you can have multiple individuals instead of just one: you have to associate at least one individual but you can associate more than one. Once we have plural quantification we get our categorial descriptions of the objects of classical mathematics.

- The key to dscribing the natural numbers if: for any set of natural numbers, there is one among them that is less than all the others. If you want to classify the real numbers, you have to know: for any real numbers, if there is a number that is greater or equal to all of them then there is a least number that is greater or equal to all of them. Zermelo formulated the separation principle in terms of definite properties, people weren't happy with brining the metaphysics of properties into set theory so Z said that we should formulate it istead as a first-order theory, with the difficulty that the first-order schemata admits non-standard models. Now that we have plural quantification, we can say that for any set X, $\forall x \in X$, $\exists Y$ such that only the x such taht $x \in X$ are members of Y. W ecan similarly give a plural quantification version with the replacement axiom schema. Vann will embarass himself if he tries and says it, but there is a second-order formulation of the replacement axiom, if you insert the replacements into ZF set theory then you do not get a categorical representation but you get instead ZF axioms and their plurally quantified versions and the fact that for any two models of the axioms either they are isomorphic or one of them is isomorphic to an initial segment of the other, which you get by clipping off the taller model. So that means we don't get a categorical generalization of the universe of set theorem.
- The continuum problem comes from Cantor's proof that the number of real numbers is greater than the number of natural numbers. Are there any infinite sets with cardinality between the two? That question is going to have a definite answer, especially if we're using plural quantification. The axioms of set theory do not have a uniquely determined model, but they are just alike to the part of the model that is smaller, and the real numbers and all the functions defined on them are in the part of the universe below the first inaccessible, and so if the continuum hypothesis is true an one model of ZFC reformulated in terms of plural quantification, then it is true in all the models and so the continuum hypothesis is going to have a definite answer. It would take a bit of patience to verify or disprove the continuum hypothesis because you have to check equinumerosity with all possible subsets of the real numbers, of which there are one or two.
- We have lots of results about things being undecidable in first-order ZFC. The only things we have that are undecidable in second-order ZFC will be if it implies some large-cardinal axiom. If you can prove that this sentence proves the large-cardinal axiom and you think it's consistent (but you can't prove it), then that will tell you that it is independent of plural quantification of ZFC. There also aren't that many second-order models just like with first-order models. It's so hard to construct models that there is hardly anything to say about second-order model theory.

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