Math 155R: Combinatorics

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These are lecture notes for the Spring 2017 offering of Harvard Math 155R: Combinatorics. Pardon any mistakes or typos.

Contents

1 January 23, 2017

1.1 Enumerative Combinatorics

The subject matter of the course is enumerative combinatorics. In general, combinatorics deals with finite structures; enumerative combinatorics primarily deals with counting (the cardinality of a particular set).

Example 1.1. Take $S = \{1, \dots n\}$, $T = \{1, \dots m\}$. How many maps $f : S \to T$? There are n^m possible maps.

Example 1.2. Take $S = \{1, \dots n\}$. How many of the possible maps $f : S \to S$ are permutations? There are $n!$.

Example 1.3. How many of the $n!$ permutations in the last example are derangements (i.e. $f(i) \neq i \forall i$)? One way to solve is to find the probability that any particular permutation is a derangement.

Theorem 1 (Fermat's Little Theorem)**.** Let n be an integer and p be a prime. Then $n^p \equiv n \mod p$ (in other words, $n^p - n$ is divisible by p).

Proof. Let $S = \{1, \dots, p\}$ and $U = \{1, \dots n\}$. Let $X = \{f : S \to U\}$. Arrange every element of S in a cycle, and then compose any function from X with a rotation on the cycle. You then have the map $T : X \to X$, $T : f \to f \circ r$. If you compose with r p times, then you get back to where you started.
Divide X into 2 subsets. Let X_0 be the constant functions (all elements of S get sent to a single element in U). $\#X_0 = n$. Let X_1 be the non-constant functions. For $f \in X_1$, the maps $\{f, Tf, T^2 f, \dots T^{p-1} f\}$ are all distinct. X_1 is a disjoint union of subsets of size p. This means that p divides $\#X_1$, but we know that $\#X = n^p$, and $\#X_0 = n$, and so $\#X_1 = n^p - n$ and so $p | n^p - n$. $\square$

1.2 Graphs

Definition 1. A graph G is a set V ("vertices") and a subset E of the set of two-element subsets of V ("edges"). There are two extremes: say $\#V = n$. Could have $E = \emptyset$ or E consists of all two-element subsets of V.

Theorem 2 (Ramsey's Theorem)**.** Given $k > 0$, $\exists n$ such that $\forall$ graphs G, if $\#V = n$ then G contains either an L_k (k vertices with no edges) or a K_k (k vertices with edges between all of them).

Example 1.4. Given a $2 \times n$ array and n dominoes, where a domino is a 2×1 array. You want to cover the array with the dominoes: how many ways are there of tiling a $2 \times n$ with n dominoes?. Let T_n be the number of ways to tile. Note that $T_1 = 1, T_2 = 2, T_3 = 3$. In the n case, the uppermost left domino can either be horizontal or vertical. If vertical, then T_{n-1} ways. If horizontal, then T_{n-2}. So, $T_n = T_{n-1} + T_{n-2}$.

2 January 25, 2017

2.1 Generating Functions

Last time: domino-covering problem. We found that $T_n = T_{n-1} + T_{n-2}$ for $n \geq 2$. Can we express T_n in closed form? Yes, using the technique of a *generating function.* Introduce $F(x) = \sum_{n=0}^{\infty} T_n x^n \in \mathbb{Z}[[x]]$. Don't think about $F(x)$ as a function, think about it as a power series. It will converge for some values, but not all of them - so just do the calculations in the ring of formal power series.

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} T_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} T_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (T_{n-1} + T_{n-2}) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} T_{n-1} x^n + \sum_{n=2}^{\infty} T_{n-2} x^n \\
&= 1 + x + x(F(x) - 1) + x^2 F(x) \\
&= 1 + xF(x) + x^2 F(x) \\
(1 - x - x^2) F(x) &= 1 \\
F(x) &= \frac{1}{1 - x - x^2}
\end{aligned}$$

Next, write out the power series expansion of $F(x)$. Let $\phi = \frac{1+\sqrt{5}}{2}$. Then, the roots of the denominator of $F(x)$ are $-\phi, \phi - 1 = \frac{1}{\phi}$ - i.e. $(1 - x - x^2) = (1 - x/\phi)(1 + \phi x)$. So,

$$\begin{aligned}
F(x) &= \frac{1}{1-x-x^2} \\
&= \frac{\lambda}{1-x/\phi} + \frac{\mu}{1+\phi x} \\
1 &= \lambda(1-\phi x) + \mu(1+x/\phi) \\
\lambda &= \frac{1}{1+\phi^2}, \mu = \frac{\phi^2}{1+\phi^2} \\
\frac{1}{1-x-x^2} &= \frac{1}{1+\phi^2}\frac{1}{1-x/\phi} + \frac{\phi^2}{1+\phi^2}\frac{1}{1+\phi x}
\end{aligned}$$

Since $\frac{1}{1-x} = 1 + x^2 + \ldots$, we can re-write $\frac{1}{1-\phi x}$ as $\sum_{n=0}^{\infty} \phi^n x^n$ and $\frac{1}{1+x/\phi}$ as $\sum_{n=0}^{\infty}(-1\phi)^n x^n$. So, then just equate the coefficients from the two ways of expressing $F(x)$:

$$T_n = \frac{1}{1+\phi^2}(-1/\phi)^n + \frac{\phi^2}{1+\phi^2}\phi^n$$

2.2 Stirling Numbers

Definition 2. A **partition** of a set $\Sigma_n = \{1, \ldots n\}$ is an expression of Σ_n as a disjoint non-empty union of subsets. In other words, we have an equivalence relation $(a \equiv a, a \equiv b \Leftrightarrow b \equiv a, a \equiv b \& b \equiv c \Rightarrow a \equiv c)$ on Σ_n.

Question: How many partitions of $\{1, \ldots, n\}$ with k parts are there? These are denoted $\left\{ {n \atop k} \right\}$ and are called **Stirling numbers**.

- $\left\{ {0 \atop k} \right\}$ is 0 if $k > 0$ or 1 if $k = 0$ (mostly because of convention).
- $\left\{ {n \atop k} \right\} = 0$ whenever $k < n$
- $\left\{ {n \atop 0} \right\} = 0$ is 1 if $n = 0$ and 0 otherwise.
- $\left\{ {n \atop 1} \right\} = 1$ if $n > 0$ and 0 if $n = 0$.

Next, to find a closed-form expression of a given Stirling number for $\{1, \ldots, n\}$. Ask: is $\{n\}$ one of the subsets? Break the partitions in which $\{n\}$ is a subset and those partitions in which it is not. If "Yes", then just solve $\left\{ {n-1 \atop k-1} \right\}$. If "No", then $\left\{ {n-1 \atop k} \right\} * k$.

Conclusion: $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + \left\{ {n-1 \atop k} \right\} * k$

3 January 27, 2017

3.1 Sterling Numbers

Last time: $\left\{ n \atop k \right\}$ is the number of partitions of the set of n elements into k non-empty subsets (also the number of equivalence relations on the set with n elements with exactly k equivalence classes).

Example 3.1. $\left\{ 4 \atop 2 \right\} = 7$. You have $\{1,2\} \cup \{3,4\}, \{1,3\} \cup \{2,4\}, \{1,4\} \cup \{1,3\}, \{1\} \cup \{2,3,4\}, \{2\} \cup \{1,3,4\}, \{3\} \cup \{2,1,4\}, \{4\} \cup \{1,2,3\}$.

We want to solve the recursion relation derived at the end of the last lecture with a generating function. Fix k such that:

$$\begin{aligned}
F_k &= \sum_{n=0}^{\infty} \left\{ n \atop k \right\} x^n \\
&= \sum_{n=0}^{\infty} \left\{ n-1 \atop k-1 \right\} x^n + \sum_{n=0}^{\infty} k \left\{ n-1 \atop k \right\} x^n \\
&= xF_{k-1} + kxF_k \\
&= \frac{x}{1-kx} F_{k-1} \\
F_0 &= \sum \left\{ n \atop 0 \right\} x^n = 1 \\
F_1 &= \frac{x}{1-x} = x + x^2 + x^3 + \dots \\
F_2 &= \frac{x^2}{(1-x)(1-2x)}
\end{aligned}$$

Use partial fractions to solve F_2:

$$\begin{aligned}
\frac{1}{(1-x)(1-2x)} &= \frac{A}{1-x} + \frac{B}{1-2x} \\
1 &= A(1-2x) + B(1-x) \\
2A + B &= 0, B = -2A, A + B = 1 \\
A &= -1, B = 2 \\
F_2 &= x^2 \left(\frac{-1}{1-x} + \frac{2}{1-2x} \right) \\
&= x^2 \left(-\sum x^n + 2^{n+1} x^n \right)
\end{aligned}$$

$$\left\{ n \atop 2 \right\} = -1 + 2^{n-1} = 2^{n-1} - 1$$

Note that there are 2^n subsets of the set with n elements, but two (the empty set and the whole set) are not proper non-empty subsets. So then you get $\frac{2^n - 2}{2} = 2^{n-1} - 1$.

3.2 Binomial Coefficients

Definition 3. $\binom{n}{k}$ is the number of k-element subsets of the set with n elements. You could think of this also as the number of sequences of k distinct elements of the set $\{1, \dots, n\}$ divided by $k!$ (because of sequence ordering).

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Find the recursion relation: partition the set of subsets of the set with n elements with k elements into subsets containing n and subsets not containing n. A subset containing n amounts to a subset of k elements on the set with $n-1$ elements, The number of subsets not containing n is $\binom{n-1}{k}$. So $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

$$\begin{aligned}
F_n &= \sum_k \binom{n}{k} x^k \\
&= \sum \binom{n-1}{k} x^k + \sum \binom{n-1}{k-1} x^k \\
&= F_{n-1} + xFn - 1 \\
F_n &= (1+x)F_{n-1} \\
F_0 &= 1 \\
F_n &= (1+x)^n = \sum \binom{n}{k} x^k
\end{aligned}$$

Example 3.2. Evaluate $\sum_{k=0}^{n} k\binom{n}{k}$. One way to phrase this problem: how many chaired committees can be formed from n people?

- First choose k people for committee out of n people, then choose one of k as the chair.
- Choose chair first and then choose subset from remaining $n-1$. So the answer is $n2^{n-1}$.
- Alternatively, use a generating function (then take the derivative and plug in $x = 1$):

$$\begin{aligned}
(1+x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\
n(1+x)^{n-1} &= \sum k \binom{n}{k} x^{k-1} \\
n * 2^{n-1} &= \sum k \binom{n}{k}
\end{aligned}$$

3.3 Derangements

Definition 4 (Derangement)**.** Let $S = \{1, \dots n\}$. A permutation of S is a bijection $S \to S$. We have $n!$ permutations. A derangement is a bijection f such that $f(k) \neq k \forall k$. How many derangements are there? What is the probability that a randomly chosen permutation has a fixed point?

Let D_n be the number of derangements of $\{1,\dots,n\}$ and let X be the set of all permutations (of cardinality $n!$). We want to decompose X into subsets based on the number of fixed points per subset - so $X = \coprod X_k$ where X_k is the set of permutations with exactly k fixed points.
It follows that:

$$\begin{aligned}
n! = |X| = \sum_{k=0}^{n} |X_k| = \sum_{k=0}^{n} &= \binom{n}{k} D_{n-k} \\
1 = \sum_{k=0}^{n} \frac{D_{n-k}}{k!(n-k)!} \\
\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \sum_{x=0}^{n} \frac{D_{n-k}}{k!(n-k)!} x^n
\end{aligned}$$

4 January 30, 2017

There is a typo in Theorem 5 (L4p3). It should be $\sum \frac{m^n}{m!}$.

4.1 Derangements

Recall: D_n is the number of derangements on the set with n elements. We asked: what fraction of all permutations are derangements? The recursion relation is derived by letting $\{1,\dots,n\} = \coprod_{k=0}^{n} X_k$ where X_k is the number of permutations with exactly k fixed points. First specify the k elements that will be fixed, and then calculate the derangement on the remaining $n-k$ members. There are $\binom{n}{k}$ such fixed point combinations.

$$\begin{aligned}
n! &= \sum_{k=0}^{n} \binom{n}{k} D_{n-k} \\
\sum n! x^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} D_{n-k} x^n \\
\sum x^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{D_{n-k}}{k!(n-k)!} x^n \\
&= \sum_{k,l \geq 0} \frac{1}{k! l!} D_l x^{k+l} \\
&= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \Bigg) \sum_{l=0}^{\infty} \frac{D_l}{l!} x^l \Bigg) \\
&= e^x \left(\sum_{l=0}^{\infty} \frac{D_l}{l!} x^l \right)
\end{aligned}$$

We're summing all integers n and then all integers k in the range of 0 to n. This is the same thing as summing over all integers k and all integers $n-k$. So set $l = n-k$ so that $n = k+l$. Also, you can write x^{k+l} as the product of two series. The result is called the exponential generating function.
The point of the use of generating functions is to package the sequence in a compact form instead of having to take the recursion relations one at a time. If we want to package information, we

will often be able to do so in different ways, and the exponential generating function is often a helpful way of packaging. Often, the exponential generating function will be identifiable but not another generating function.
The ratio $\frac{D_l}{l!}$ is the probability of a permutation being a derangement.

Definition 5 (Exponential generating function)**.** If $c_0, c_1, c_2, \ldots$ is any infinite sequence, then the exponential generating function of the c_i is $\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$. So $\frac{1}{1-x} = e^x F \Rightarrow F = \frac{e^{-x}}{1-x}$ is the one for derangements.

We have that:

$$\begin{aligned} \sum \frac{D_n}{n!} x^n &= \frac{e^{-x}}{1-x} \\ &= \left(\sum_{p=0}^{\infty} x^p \right) \left(\sum_{q=0}^{\infty} \frac{(-1)^q}{q!} x^q \right) \\ \frac{D_n}{n!} &= \sum_{p+q=n} \frac{(-1)^q}{q!} \\ &= \sum_{q=0}^{n} \frac{(-1)^q}{q!} \\ &= 1 - 1 + 1/2 - 1/6 + 1/24 + \ldots + (-1)^n/n! \\ D_n &= n! - n! + n!/2 - n!/6 + \ldots + (-1)^n \end{aligned}$$

$D_n/n!$ is e^x cut off after n, and note that the error between it and $1/e$ is very small in large n. D_n is the closest integer to $n!/e$. So the probability that a given permutation is a derangement approaches $\frac{n!}{e}$.

4.2 Bell Numbers

Definition 6 (The Bell Numbers)**.** Question: How many partitions are there of $\{1, \ldots, n\}$? Equivalently, how many equivalence relations are possible on $\{1, \ldots, n\}$? We call this number b_n.

Remark 1. The Bell Numbers are different than the Stirling Numbers because we are not considering how many subsets the partition should have. Clearly, $b_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.

Recursion relation: A partition of $\{1, \ldots, n+1\}$ gives rise to a decomposition of $\{1, \ldots, n\}$ into two set: the numbers i in the same part as $n+1$ and then the rest. Let's say that there are k elements in the first set and $n-k$ elements in the second part. What you get is: $b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}$.
Instead of waiting for the generating function to manifest, we just choose the exponential generating function (so, divide by $n!$, multiply by x^n, then sum).
Goal: find the exponential generating function for $F = \sum \frac{b_n}{n!} x^n$.

$$\frac{b_{n+1}}{n!} = \sum \frac{1}{k!(n-k)!} b_{n-k}$$

$$
\begin{aligned}
F' = \sum \frac{b_{n+1}}{n!}x^n &= \sum_n \sum_{k=0}^{n} \frac{1}{k!(n-k)!} b_{n-k} \\
&= \sum_{k,l} \frac{1}{k!l!} b_l x^{k+1} \\
e^x F &= \left(\sum \frac{1}{k!}x^k\right)\left(\sum \frac{1}{e!} b_l x^l\right) \\
F' &= e^x F
\end{aligned}
$$

4.3 Formal Power Series

We are working with the ring of formal power series. Fix a field K of characteristic 0 $K[[x]] = \{\sum_{n=0}^{\infty} a_n x^n : a_n \in K\}$. These are not functions, infinite sums do not have to exist or converge in fields. We just treat this as a formal entity (like a polynomial) just longer. We can carry out all the operations that we usually do with power series and polynomials:

1. add, multiply ($k[[x]]$ has structure of commutative ring with identity)
2. If $f = \sum a_n x^n$ and $a_o \neq 0$ then f has a reciprocal power series $g = \frac{1}{f}$. So you can divide power series under this condition.
3. composition: $f, g \in K[[x]]$ and $\sum b_n x^n, b_0 = 0$ then you can compose.

5 February 1, 2017

Today: Bell Numbers and the theory of species (through category theory). There are notes on categories and functors up on the course web page.

5.1 Bell Numbers

Definition 7. b_n is the number of partitions of the set $\{1, 2, \ldots, n\}$ into non-empty subsets. This is the same as the number of possible equivalence relations on the set of $\{1, \ldots, n\}$. $b_0 = 1, b_1 = 1, b_2 = 2, b_3 = 5$. $b_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.

In general, there are two extremes: every element is a different partition, or only one cell in partition with n elements.
Recursion relation: say the part containing $n+1$ also contains k elements of $\{1, \ldots, n\}$. The recursion relation is $b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}$.

$$
\begin{aligned}
b_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} b_{n-k} \\
\frac{b_{n+1}}{n!} &= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} b_{n-k}
\end{aligned}
$$

$$\frac{b_{n+1}}{n!}x^n = \sum_{k=0}^{n} \frac{1}{k!(n-k)!}b_{n-k}x^n$$
$$\sum \frac{b_{n+1}}{n!}x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}b_{n-k}x^n$$
$$\sum \frac{b_{n+1}}{n!}x^n = \sum_{k,l\geq 0}^{\infty} \frac{1}{k!l!}b_{n-k}x^{k+l}$$

Set $F = s\sum \frac{b_n}{n!}x^n$. It does not matter whether this converges (it does only for some small interval about the origin). However, convergence is relevant because if it does not converge than it is unlikely that the power series will be expressible with elementary functions. The relation between the expression we've derived and F is that our expression is F'. So,

$$\sum \frac{b_{n+1}}{n!}x^n = \sum_{k\geq 0}^{\infty} \frac{x^k}{k!}\sum_{l} \frac{b_l x^l}{l!}$$
$$F' = e^x F$$

Solutions to this equation are of the form $F = ce^{e^x}$. So now if you compare constant terms, then you get $c = \frac{1}{e}e^{e^x} = e^{e^x - 1}$. Subtracting the 1 makes sense for power series because it means there is no constant from the differential equation.

$$\begin{aligned} e^{e^x} &= \sum \frac{(e^x)^m}{m!} \\ &= \sum \frac{1}{m!}e^{xm} \\ &= \sum_m \sum_n \frac{1}{m!}\frac{(mx)^n}{n!} \\ &= \sum_{m,n} \frac{m^n}{m!n!}x^n \end{aligned}$$

Compare coefficients of x^n: $e\frac{b_n}{n!} = \sum_m \frac{m^n}{m!n!}$. So, $b_n = \frac{1}{e}\sum_m \frac{m^n}{m!n!}$.

5.2 Theory of Species

Definition 8 (Species)**.** A species S is a device that does 2 things:

- It associates to every finite set I another finite set $S(I)$.
- It associates to every bijection of sets $\pi : I \to J$ a corresponding bijection $S(\pi) : S(I) \to S(j)$.

Subject to the following conditions:

- I, J, K finite sets. $\pi : I \to J$, $\pi' : J \to K$ bijection. Then $S(I) \to_{S(\pi)} S(J) \to_{S(\pi')} S(K)$ and we require that $S(\pi' \circ \pi) = S(\pi') \circ S(\pi)$.
- If $\pi = id_I : I \to I$ then $S(\pi) = id_{S(I)}$. This follows from the first condition.

The relevant category is the category with objects being finite sets and whose morphisms are bijections amongst those sets. A species is just a functor from that category onto itself.

Example 5.1 (Species of partitions)**.** Let $S(I)$ be the set of partitions of I and $\pi : I \to J$. $S(\pi)$ maps partitions of I onto partitions of J. So $S(\pi) : I = \coprod I_2 \to J = \coprod(\pi(I_2))$.

Example 5.2 (Species of subsets)**.** $S(I)$ is the subsets of I and the Specifies of k-element subsets $S(I)$ is subsets of I with k-elements.

Example 5.3 (Species of graphs)**.** $S(I)$ is the set of graphs with vertex set I. We can think of the set of edges as the symmetric product of $I \times I$ minus the diagonal. Think about how to formulate the species of connected graphs.

Example 5.4 (Species of undecorated sets)**.** $S(I)$ is the set of one element for all I.

Example 5.5 (Species of non-empty undecorated sets)**.** $S(I)$ is the set of one element if $I \neq \emptyset$ and $\emptyset$ if $I = \emptyset$.

6 February 6, 2017

6.1 Theory of Species

Last time: we introduced the notion of a species, which in its simplest form is something that associates finite sets to finite sets and bijections between finite sets. If you ahve category C whose objects are finite sets and whose morphisms are only bijections. A species S is a functor from $C \to C$.

Example 6.1. (species of partitions) $S[I]$ is the set of partitions of I, (species of subsets) $S[I]$ is the set of subsets of I, (linear orderings of I), the set of permutations of I, derangements, graphs with vertex set I, connected graphs with vertex set I.

There were two trivial species: $S[I]$ associates to every set a set with one element, $S[I]$ is the set with one element if I is non-empty and $\emptyset$ otherwise. These are called the species of decorated/non-decorated sets.
What are we going to do with species? Associated to a species S its **exponential generating function**. Let $a_n = |S[\{1, \ldots, n\}]|$ and set $F_S = \sum_{n=0}^{\infty} \frac{a_n}{n!}$. In the book this is written $F_S(x)$ to emphasize that it is a formal power series, but this wrongly suggests that this is a function.

Example 6.2. If S is the species of permutations, then $a_n = n!$ and the generating function $F_s = \sum_{n \geq 0} \frac{n!}{n!} x^n = \frac{1}{1-x}$.

Example 6.3. Let $S[I]$ associated to every finite set I a one-element set. Since $|S[I] = 1 \forall I$, then $F_s = \sum \frac{x^n}{n!} = e^x$.

What we will be trying to do is establish a dictionary that will allow us to move between species and power series. The basic idea is that if we have one or more species we can combine them in certain ways that are relatively natural and we can ask: if I know the generating function of two speices, can I describe the generating function that occurs when we combine them?

Operations on Species	Operations on Power Series
$S+T$	$F_{S+T} = F_S + F_T$
S'	$F_{S'} = (F_S)'$
ST	$F_{ST} = F_S F_T$
$S \circ T$	$F_{S\circ T} = F_S \circ F_T$

Example 6.4. Define the sum of two species S, T as $(S+T)[I] = S[I] \coprod T[I]$. Easy to note that the bijection is preserved and the cardinality is the sum of the species. So $F_{S+T} = F_S + F_T$.

Given S, can define new species S' by $S'[I] = S[I \cup \{*\}]$. Then $|S[< n >]| = |S[< n+1 >]|$. So $b_n = a_{n+1}$. But $F_S = \sum \frac{a_n}{n!} x^n$ and so $F_{S'} = \sum \frac{a_{n+1}}{n!} x^n = (F_S)'$

Definition 9 (Product Species)**.** Say S and T are each species. Define the product species ST as the species defined by $ST[I] = \coprod_{I=I_0 \cup I_1} S[I_0] \times T[I_1]$. This is the same thing as the set of triples (P, x, y) where P is a partition of I into two parts I_0 and I_1, $x \in S[I_0]$ and $y \in T[I_1]$.

Definition 10 (Power Series of Product Species)**.** Say $|S[< n >]| = a_n$ and $|T[< n >]| = b_n$ so $F_S = \sum \frac{a_n}{n!} x^n$ and $F_T = \sum \frac{b_n}{n!} x^n$ so $F_{ST} = \sum \frac{c_n}{n!} x^n$ where $c_n = |ST[< n >]|$.
A priori, $c_n = \sum_{I=I_0\cup I_1} \underset{=}{a_{|I_0|} b_{|I_1|}} \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$. So $\sum \frac{c_n}{n!} x^n = \sum_{n=0}^\infty \sum_{k=0}^n \frac{1}{k!(n-k)!} a_k b_{n-k} x^n$. Rearranging the summation and setting $l = n-k$, $F_{ST} = \sum_{k,l\geq 0} \frac{a_k b_l}{k! l!} x^{k+l} = F_S F_T$.

Example 6.5 (Species of Derangements)**.** Let $T[I]$ be the set of derangements of I. Our goal is to find the generating function for T. First, define $S[I] = \{*\}$ (assigns to every set a single element set, called in the notes the species of undecorated sets). If I have an arbitrary permutation of a set with n elements, I can specify the permutation by specifying the fixed points and then the derangement of the complimentary set. So you can specify the permutation as the fixed points and the derangement. So to specify the permutation you have to specify the fixed point set and the derangement of the complement. So the product ST is the species of permutations.
We know that $F_S = e^x$ and $F_{ST} = \frac{1}{1-x}$, so $F_T = \frac{1}{1-x}\frac{1}{e^x} = \frac{e^{-x}}{1-x}$.

Definition 11 (Species Composition)**.** Say S and T are species; assume that $T[\emptyset] = \emptyset$. Define $S \circ T$ by $S \circ T[I] = \coprod_{I=I_1\cup\ldots\cup I_K} S[\{I_1, \ldots, I_K\}] \times \prod_{\alpha=1}^k T[I_\alpha]$. Basically apply S to the set of equivalence classes and multiply it by T when applied to each individual part of the partition. In other words, $S \circ T$ is the set of triplets $\{(\ , x, y_{J\, J\in I/\ })\}$ where   is an equivalence relation on I, $x \in S[I/\ ]$, and $y_J \in T[J]$ where J is an equivalence class.

Example 6.6. Take $S[I]$ to be the set of with the single element $\{*\}$ if $I \neq \emptyset$ and $\emptyset$ if $I = \emptyset$. This is the same generating function $F_S = e^x - 1$ (note the lack of constant term).
If I have an arbitrary graph with n vertices and I draw edges between any symmetric subset of the pairs of vertices (edges), The first thing you get is a breakdown of the graph into connected components. If I want to count the connected graphs, relate it to generating functions of graphs that I do know.

7 February 13, 2017

7.1 Trees

Definition 12 (Tree)**.** A tree is a graph without cycles (or a minimal connected graph). Key fact: there exists only one simple path between any two vertices, so there are only $n-1$ edges in a tree..

Definition 13. If f^n is constant for some $n > 0$ the we say that f is *nilpotent*.

Example 7.1. How many trees are there with a given vertex set I with n vertices?
The answer is n^{n-2}. We will use the theory of species to count the things that we want to count. The cast of characters that will appear in this argument are:

- $S_{tree}[I]$ is the set of trees with vertex set I.
- $S_{1-tree}[I]$ is the set of trees with vertex set I and a distinguished vertex $i \in I$ (the root). If $I = \emptyset$ then this returns the singleton set.
- $S_{2-tree}[I]$ is the set of trees with vertex set I and two distinguished vertices (and ordered pair of vertices, both can be the same).
- $S_{end}[I]$ (endomorphism) is the set of all maps $f : I \to I$.
- S_{perm} is the set of permutations of I.
- S_{lin} is the set of linear orderings of I.
- S_{nil} is the set of nilpotent maps $I \to I$.
- I_0 is the set of periodic elements for a map π. If π is a permutation then $I_0 = I$ and if π is nilpotent then $I_0 = \{i_0\}$.

The claim we are going to prove is that $|S_{2-tree}[< n >]| = n^n = |S_{end}[< n >]|$. But note that $S_{2-tree} \neq S_{end}$.

Lemma 3 ($S_{end} = S_{perm} * S_{nil}$)**.** We have to establish a bijection between these two species on any given set. Suppose we start with an arbitrary map $\pi : I \to I$. Say $i \in I$ is periodic for π if $\pi^n(i) = i$ for some $n > 0$.
$\pi : I_0 \to I_0$ is a permutation of I_0. For any $i \in I$, the sequence $i, \pi(i), \pi^2(i), \ldots$ eventually has to repeat. If it repeats, then it repeats after the first occurence of a periodic element.
Define a function $r : I \to I_0$ by saying that $r(i)$ is the first periodic element in the afformentioned sequence. If $i \in I_0$ then $r(i) = i$. r defines an equivalence relation on I so that $i \equiv j \Leftrightarrow r(i) = r(j)$. The equivalence classes J are indexed by elements in I_0, and an equivalence class consists of all the points i the gravitate towards a given periodic element. π is nilpotent on J with attractor $r(J)$. So we're grouping the elements I into subsets indexed by elements in I_0 where the map on the equivalence classes is nilpotent.
π gives me 3 things: an equivalence relation on I, a permutation of the set of equivalence classes, and a nilpotent map $J \to J$ on each equivalence class. But this is exactly what we want to prove: an endomorphism gives rise to a permutation and a nilpotent map.

Lemma 4 ($S_{nil} = S_{1-tree}$). The map $\pi : I \to i$ is nilpotent with attractor I_0 then we describe the tree by declaring that i is adjacent to $\pi(i) \forall i \neq i_0$. Conversely, if you have a tree with a root i_0, you can define a map by mapping a vertex to the next vertex in its simple path to the root.

Lemma 5 ($S_{2-tree} \equiv S_{lin} \circ S_{1-tree}$). Let $(T, v, v') \in S_{2-tree}[I]$. Let $I_0 = \{v_0, \ldots, v_n\}$ and Define map $r : I \to I_0$. $r(w)$ is the first point where the unique simple path from w to v hits I_0.
This gives rise to an equivalence relation on I, for each equivalence class J, a rooted tree with vertex set J, and a linear ordering of the set of equivalence classes.

Lemma 6. $F_{S_{end}} = F_{S_{perm}} \circ F_{s_{nil}} - F_{S_{perm}} \circ F_{S_{1-tree}} = F_{S_{lim}} \circ F_{S_{1-tree}} = F_{S_{2-trees}}$.

8 February 15, 2017

8.1 Groups

Definition 14 (Group action). Take a group G and a set X. By a (left)-action of G on X we mean a map $G \times X \to X$ for which $(g, x) \to g(x) = gx$ such that $ex = x \forall x \in X, g(hx) = (gh)x \forall g, h \in G, x \in X$. $\forall g \in G$, the map $\{g\} \times X \to X$ is a bijection, so we have a map $G \to Perm(X)$. The two conditions listed above amount to saying that this map is a group homomorphism.

Definition 15 (G-set). A G-set is a set X with an action of G.

Definition 16 (G-set isomorphism). An isomorphism of G-sets X, X' is a bijection $\phi : X \to X'$ such that $\phi(gx) = g(\phi x)$.

Definition 17 (Orbit). Say X is a G-set. For $x \in X$, the orbit Gx of x under G is the image of $G \times \{x\} \to X$. The orbits form a partition of X, so X is a disjoint union of orbits: $\forall x, y$ either $Gx = Gy$ or $Gx \cap Gy = \emptyset$. Being in some orbit is an equivalence relation. We define $G \setminus X$ to be the set of orbits.
The proto-question of this unit is going to be to describe the cardinality of $G \setminus X$.

Definition 18 (Fixed point set). X is a G-set. $\forall g \in G$, define $X^g = \{x \in X : gx = x\} \subset X$ is the fixed point set.

Definition 19 (Stabilizer). $\forall x \in X$, define $\text{stab}(x) = \{g \in G : gx = x\} \subset G$ is a subgroup.

Definition 20 (Transitive G-set). X is a transitive G-set if it has a unique orbit.

Example 8.1. Say $H \subset G$ is any subgroup. $X = G \setminus H$ is the set of left cosets gH which isn't a group unless H is normal. G acts on X in an obvious way: $g'(gH) = (g'g)H$. Observe that this is transitive: to get from gH to $g'H$ then multiply on the left by $g'g^{-1}$, so you can get from any coset to any other coset.

Proposition 1. Every transitive action of G is of the form $G \setminus H$ for some H. i.e. if X is any G-set, we have an isomorphism $X \equiv \coprod G \setminus H_i$ for some collection $\{H_i\}$ of subgroups of G.

Proof. The identity coset of $G \setminus H$ is H and the stabilizer of $id(G \setminus H)$ is the subgroup H itself. Check that we have an isomorphism $x \cong G \setminus H$. □

Example 8.2. Fix $t > 0$. How many ways are there of coloring the faces of a tetrahedron with t colors. X is the set of 4 faces of the tetrahedron, and $G = A_4$ is the set of 12 rotational symmetries of the tetrahedron. The action of G on X is of the form $\coprod G \setminus H_i$.

Example 8.3 (Burnside formula)**.** Let X be a G-set. Then $|G \setminus X| = \frac{1}{|G|}\sum_{g\in G}|X^g|$. So the trivial action is $G \times X \to X$ defined by $(g, x) \to x$, and the formula reads $|G \setminus X| = |X|$ and $|X^g| = |X|$ so the formula becomes trivial.

Definition 21 (Free action)**.** An action of G on X is free if the action carries no element to itself: $\forall g \neq e, X^g = \emptyset$. In other words, $gx = x \Rightarrow g = e$ for any x. What are the orbits? The map $G \times \{x\} \to X$ is an inclusion (the size of the orbit Gx has size $|G|$). And $|G \setminus X| = |X|/|G|$.

9 February 17, 2017

Chapter 2 of Artin's Algebra book is on the course website in the files section.

9.1 Groups

It is useful to think of groups as symmetries of an object. A square has four 90-degree symmetries, a non-square rectangle has two 180-degree symmetries. But it's not jsut the number of symmetries that concerns us - it's not just the cardinality of the number of symmetries, but the fact that the symmetries satisfies the group axioms. For many applications, you need to know the group structure in addition to the cardinality of things.

Example 9.1. G is a group acting on a set X. If $H \subset G$ is a subgroup, then we can form the set of cosets of H $gH \forall g \in G$. G acts on X by $g' : gH \to g'gH$.

Proposition 2. Every transitive G-set is of the form G/H. Any G-set is of the form $\coprod G/H$.

Remark 2 (Burnside formula)**.** X is a G-set, then $|G/X| = \frac{1}{|G|}\sum_{g\in G}|X^g|$ where $X^g = \{x \in X : gx = x\}$.

Proof. We can assume $X = G/H$ and then we have to show in this case that $|G| = \sum_{g\in G}|X^g|$. So $X^g = \{g'H \in G/H : gg'H = g'H\} = \{g'H : g'^{-1}gg' \in H\} = \{g'H : g^{'-1}gg' \in H\}$.
In particular, $|X^g|$ is going to be the number of cosets such that $g^{'-1}gg' \in H\}$. Every coset of H has the same cardinality, so instead of counting cosets we can count elements of G and then divide by teh cardinality of H. So, $|X^g| = |\{g' \in G : g^{'-1}gg' \in H\}|/|H|$. The key step is sum the previous expression over all $g \in G$. Note that $|g'Hg^{'-1}| = |H|$.

$$
\begin{aligned}
\sum_{g\in G}|X^g| &= \frac{1}{|H|}|\{(g, g') \in G \times G : g^{'-1}gg' \in H\}| \\
&= \frac{1}{|H|}|\{(g, g') : g \in g'Hg^{'-1}\}| \\
&= \frac{|G|}{|H|}|H| = |G|
\end{aligned}
$$

$\square$

Example 9.2. How many ways are there to color the faces of a tetrahedron if t colors are available (we don't have to use them all) up to rotational symmetry?
Let F be the set of faces ($|F| = 4$). We want to consider X, the colourings of F with colors in T where $|T| = T$ so that $X = T^F$ where T^F is the set of maps from faces to colors. The question we ask is: if G is the group of rotational symmetries of the tetrahedron, then G acts on F by definition and so G acts on the set of maps $T^F : F \to T$. So the question is how many orbits of there on the latter map: what is the cardinality of G/T^F.
Claim: there are three types of symmetries. The identity, for which the orbits are just the faces and so there are 4; rotation by 120 degrees about the center of a face, there are 8 elements like this and each one has two orbits; finally, midpoints of the edges with rotations of 180-degrees, of which there are three.
In current situation, $|G| = 12$. For a given g, what is $|X^g|$? If $g = e$, then every coloring is fixed and so we have a choice of any one of t colors and so $|X^e| = t^4$. If g is a 120-degree rotation, then $|X^g| = t^2$. If g is a 180-degree rotation, then $|X^g| = t^2$.
Conclusion: $|G\ T^F| = \frac{1}{12}(t^4 + 8t^2 + 3t^2) = \frac{1}{12}(t^4 + 11t^2)$. Check that this is always an integer.

In general, if G acts on a set X, for $g \in G$ we can consider the number of g-orbits $= o(g)$. The Burnside formula implies that $|G\ T^X| = \frac{1}{|G|}\sum_{g\in G} t^{o(g)}$.

10 February 22, 2017

10.1 Unlabelled Graphs

How many graphs are there with vertex set $\{1, \ldots, n\}$? How many graphs with n vertices are there, up to isomorphism (modulo some kind of symmetry)? For the first question, we look at all possible sets of edges. An edge is a two-element subset, so there are $2^{\binom{n}{2}}$. For the second question, the answer is much less clear. Basic tool: Burnside formula: for a finite group G which acts on a set X, then the number of orbits in X $|G/X| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 10.1 (Polya's Enumeration Theorem)**.** How many ways are there to color X with a finite set T of t colors? This is to say, how many maps T^X (up to symmetries of G)?
For a given $g \in G$, let $o(g)$ be the number of g-orbits in X. For a coloring to be preserved under g, you have to choose a color for each orbit of g on X. Burnside says: $|G/T^X| = \frac{1}{|G|}\sum_{g\in G} t^{o(g)}$. This is Polya's enumeration theorem.

Example 10.2 (Number of unlabelled graphs on n vertices)**.** Polya's theorem as expressed seems specific to coloring, but it's not. It can be used for the unlabelled graph problem. Let $< n >= \{1, \ldots, n\}$ and associate to this set $E = E_n$ which is the set of unordered pairs of distinct elements of $< n >$ (or, equivalently, subsets of $< n >$ with cardinality 2.) So $|E| = \binom{n}{2}$. A graph on $< n >$ is an arbitrary subset of E, so the set of graphs with vertex set $< n >$ is exactly the set of all subsets of E. Since Σ_n acts on $< n >$ it also acts on E and so it acts on the set of subsets of E (graphs on $< n >$).
How do we set this up as a coloring problem? To specify a subset of a given set, you can think of it as a coloring of the set by two colors (in and out). Color it 0 if its in the subset and 1 if its not. So the subsets of $|E|$ are going to be exactly the colorings of E with two elements.

We're trying to find $|\Sigma_n/\{0,1\}^E| = \frac{1}{n!}\sum_{g\in\Sigma_n} 2^{o(g)}$ where $o(g)$ is the number of orbits of g acting on E.

Example 10.3 (How many graphs on 4 vertices?)**.** In all these situations, you just have to consider the conjugacy classes of the symmetric group. There are 5 types (conj. classes) of permutations $\sigma \in \Sigma_4$. $|E| = \binom{4}{2} = 6$.

1. e - there's 1, there are 6 orbits each with 1 element.

2. (ab) - there are 6, 4 orbits (2 fixed points, 2 orbits of size 2).

3. (abc) - there are 8, 2 orbits (each of size 3).

4. $(abcd)$ - there are 6, 2 orbits (sizes 2 and 4).

5. $(ab)(cd)$ - there are 3, 4 orbits (2 fixed opints, 2 orbits of size 2).

So the answer is $\frac{1}{24}(2^6 + 6(2^4) + 8(2^2) + 6(2^2) + 3(2^4)) = 11$.
Let's organized them by number of edges: there's 1 graph with no edges, there's 1 graph with 1 edge, etc. You'll notice there's a symmetry between a graph and the complementary graph by taking the edges on vertices that aren't connected in the first graph.
We haven't talked about the number of conjugacy classes of Σ_n, but that grows fairly fast. There isn't a closed-form formula for this.

The next question is: how many graphs are there with n vertices and k edges? We'll call this number c_k.
Let $F(t) = \sum_{k=0}^{\binom{n}{2}} c_k t^k$. We know that $F(1) = \sum c_k = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} 2^{o(\sigma)}$. Let E be the set of unordered pairs of elements of $< n >$. Let X_k be the set of graphs with vertex set $< n >$ with k edges which is the same thing as the set of subsets of E having cardinality k. So $|X_k| = \binom{\binom{n}{2}}{k}$.
By Burnside formula, $c_k = |\Sigma_n \ X_k| = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} |X_k^\sigma|$. For each $\sigma \in \Sigma_n$, look at action of σ on E. Say this has orbits $E_1, E_{@}, \ldots, E_{o(\sigma)}$. We want a collection of orbits whose sizes add up to k. $|X_k^\sigma|$ is the set of subsets of E of size k consisting of a union of orbits. How many of these are there? One way to keep track is to consider the following polynomial: $(1+t^{|E_1|})(1+t^{E_2|})\ldots(1+t^{|E_{o(\sigma)}|})$. What is the coefficient of t^k in this product? This corresponds exactly to collections of orbits whose total cardinality adds up to k.

11 February 22, 2017

11.1 Wreath Products

Basic construction: say for composite n that $n = kl$. Take set $S =< n >$ and express $S = S_1 \coprod S_2 \ldots \coprod S_k$ where $|S_i| = l$. Consuder the group K of permutations of S that preserve this decomposition into subsets. We choose an equivalence relation with k equivalence classes, each of cardinality l. We want to consider the group of permutations that preserve the equivalence relation.

So we have a map $K \to \Sigma_k$. The kernel of this map comprises permutations that contain any of the S_i to itself. So $\cong (\Sigma_l)^k$; in particular, $|K| = k!(l!)^k$. As a set: $K \cong \Sigma_k \times (\Sigma_l^k)$ (not as a group). K is called the wreath product of Σ_l and Σ_k.
Now say $G \subset \Sigma_k$. Equivalently say X, Y are sets of size k, l and G acts on X and H acts on Y. Now we consider permutations of $X \times Y \to X$.

12 March 20, 2017

12.1 Inclusion/Exclusion Principle

This week: inclusion/exclusion principle and partially ordered sets. In the notes, this starts with lecture 18. We're now three lectures behind the notes.
Three questions:

1. How many derangements of $< n >$ are there?
2. How many integers between 1 and n inclusive are relatively prime to n?
3. How many surjective maps $f :< m > \to < n >$ are there?

In each, if you don't specify a condition then the answer is pretty immediate (e.g. remove "relatively prime" or "surjective"). Then we just exclude the non-surjective or non-relatively prime elements.

12.2 Cardinality of set unions

Starting point: if X, Y are any sets, what is the $|X \cup Y|$? If they are disjoint, then $|X \cup Y| = |X| + |Y|$; otherwise, it is strictly less than this sum, so $|X \cup Y| = |X| + |Y| - |X \cap Y|$. If X, Y, Z any sets, then $|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$. You have to add in the last term because you are removing elements in all three sets three times and adding them three times without it. General formula: if $X_1, \ldots, X_n$ are any sets, what is $|\cup X_i|$? Let $X_J = \cap_{i \in J} X_i$. For $x \in X$, let $I_x = \{i : x \in X_i$. Note that the empty set is included in the second formulation (using the complement equivalence).

$$\begin{aligned}
|\cup X_i| &= \sum_{J \subset \{1,\ldots,n\}, J \neq \emptyset} (-1)^{|J|+1} |X_j| \\
&= \sum_{J \subset \{1,\ldots n\}} (-1)^{|J|} |X_J| \\
&= \sum_J \sum_{x \in X} (-1)^{|J|} \text{ if } x \in X_J \text{ and } 0 \text{ otherwise} \\
&= \sum_{x \in X} \left(\sum_{J \subset \{1,\ldots,n\}} (-1)^{|J|} \text{ if } x \in X_J \text{ and } 0 \text{ otherwise} \right) \\
&= \sum_{x \in X} \left(\sum_{J \subset I_x} (-1)^{|J|} \right)
\end{aligned}$$

$$= |\{x \in X : I_x = \emptyset\}| = |X - \cup X_i|$$

For a two element set, two subsets have odd elements and two have even cardinality. So the expression sums to 0 except with the emptyset. For $\emptyset$, then $(-1)^{|J|} = 1$, and so $|\cup X_i| = 1$.

12.3 Derangements

Let $X = \text{Perm}(< n >)$ and $X_i = \{\sigma \in I : \sigma(i) = i\}$. Then, Derangements$(< n >) = X - \cup X_i$. If $|J| = k$, then $|X_J| = (n-k)!$. The conclusion is that the number of derangments is $|X - \cup X_i|$:

$$\begin{aligned}
|X - \cup X_i| &= \sum_{J \subset \{1,\ldots,n\}} X_J \\
&X_J\{\sigma \in X : \sigma(i) = i \forall I \subset J\} \\
&= \text{Perm}(< n > - J) \\
&|X_J| = \sum_{k=0}^{n} \sum_{|J|=k} (-1)^k (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k \geq 0} (-1)^k \frac{n!}{k!} \\
&= n! \sum (-1)^k / k! \\
&= n!(1 - \frac{1}{1!} + \frac{1}{2!} \ldots) \\
&\sim \frac{n!}{e}
\end{aligned}$$

12.4 Surjective maps

How many surjective maps $f :< m > \to < n >$. Let X be the set of all maps so that $|X| = n^m$. The way to test whether a map is surjective is to see if there is an element in the target set not in the image of the function. Let $X_i = \{f : i \in Im(f)\}$ for $i = 1, \ldots, n$. So $|X_i| = (n-1)^m$ because you're just excluding i from the target set and then collecting all other maps. So $|X_J| = (n - |J|)^m$. So the number of surjective maps is:

$$\begin{aligned}
|X - \cup X_i| &= \sum_{J \in < n >} (-1)^{|J|} (n - |J|)^m \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m
\end{aligned}$$

12.5 Euler ϕ-function

The question is how many integers between 1 and n are relatively primte to n? The answer is denoted as $\phi(n)$ and is classically denoted as the Euler ϕ-function.

Let $X = \{1, \ldots, n\}$. Let S be the set of primes p such that $p|n$. $\forall p \in S$, let $x_p\{i \in X : p|i\}$. Let $J = \{, p_1, p_2, \ldots p_k\} \subset S$. So,

$$
\begin{aligned}
|X_J| &= \{i \in X : p_1 p_2 \ldots p_k | x\} \\
|X_J| &= \frac{n}{p_1 p_2 \ldots p_k} \\
\frac{\phi(n)}{n} &= \sum_{J \subset S} (-1)^{|J|} \frac{1}{p_1 p_2 \ldots p_k} \\
&= \sum_{J \subset S} \prod_{p \in J} \frac{-1}{p} \\
&= \sum_{J \subset S} \prod_{p \in S} \left(\frac{-1}{p} \text{ if } p \in J \text{ and } 1 \text{ if } p \notin J \right) \\
&= \prod_{p \in S} \left(1 - \frac{1}{p}\right)
\end{aligned}
$$

So if $n = p_1^{a_1} p_2^{a_2} \ldots$ written in prime factor form, then $\phi(n) = (p_1 - 1)p_1^{a_1 - 1}(p_2 - 1)p_2^{a_2 - 1} \ldots$.

12.6 Partially Ordered Sets

Definition 22. A partially ordered set is a set A with a binary relation $\leq$ with the following properties:

- Reflexivity: $a \leq a \forall a \in A$
- Transitivity: $a \leq b \,.\, b \leq c \Rightarrow a \leq c \forall a, b, c, \in A$
- Anti-symmetry: $a \leq b$ and $b \leq a$ then $b = a$.

You can think of a partially ordered set as a subset of $A \times A$ that contains the diagonal and is anti-symmetric (harder to express transitivity).

We're going to hold off on assuming that A is finite at this point. But we are still going to think of A as basically a finite set, but we are now in Axiom of Choice territory so watch out. Joe has told us that when you counter a new notion you ought to look at the extreme cases. So how big can this subset of $A \times A$ be?

Example 12.1. Let $A = P(S)$ where A is the powerset of S. We can say that $T \leq U \Leftrightarrow T \subset U$. Another case is where $A = \mathbb{Z}$ and $x \leq y$ if $x|y$. The extreme examples: define $a \leq b$ iff $a = b$ (the **discrete/trivial ordering**). The opposite extreme is where either $a \leq b$ or $a \leq b$ for any pair (a, b). This is called a **total/linear ordering**. So if $A \subset \mathbb{R}$ and you define $a \leq b$ if $b - a \geq 0$.

The reason to look at these extreme examples is because if I give you an arbitrary partially ordered set, to what extent can you find subsets of the extremes?

13 March 22, 2017

13.1 Partially ordered sets

A lot of this material applies to infinite sets and even uncountable sets, but for the most part the things we are going to be doing involve finite sets.

Definition 23 (Partially ordered set). A partially ordered set is a set A and a relation $\leq$ such that $a \leq a$, $a \geq b \,.\, b \geq c \to a \geq c$, and $a \geq b \,.\, b \geq a \to a = b$. You can think of this as a subset $\Sigma \subset A \times A$. Also any subset $B \subset A$ inherits the structure of the partially ordered set.

Example 13.1 (Extreme examples). The extreme examples are $\Sigma = \Delta$ i.e. $a \leq b \Rightarrow a = b$ (discrete partial ordering), called an **antichain**. If $\forall a, b \in A$, either $a \leq b$ or $b \leq a$ is called a **totally ordered** (linearly ordered). If A is finite and totally ordered then there exists a unique bijection for $A \to < n >$ which pres totally ordered set is called a **chain**.

If you take an arbitrary partially ordered set, what can you say about the existence of chains and anti-chains within that set? We want to express something about the cardinality of the longest possible chain or anti-chain.

Example 13.2 (Power set). The other example is if S is any finite set and let $P(S)$ be the power set of S, then we have a partial ordering on $P(S)$ by $A \leq B \Leftrightarrow A \subset B$ for $A, B \in P(S)$.

Definition 24 (Least element). $a \in A$ is a least element if $a \leq b$ for all $b \in A$.

Definition 25 (Minimal element). $a \in A$ is a minimal element if $b \leq a \Rightarrow b = a$.

Also note the equivalent definitions for greatest and maximal elements.

Lemma 7. Every partially ordered set has minimal elements

Proof. Start with a_0, choose $a_1 \leq a_0, a_1 \neq a_0$; keep on choosing $a_0 \geq a_1 \geq a_2 \geq a_3$. Eventually we have to repeat in this sequence, and so for some $k < l$ we have $a_k \geq a_{k+1} \ldots > a_l = a_k$. But this isn't possible by anti-symmetry, so you have to find a minimal element. Note this only works for finite sets (the integers have no minimal element). □

Definition 26 (Height). For a PO set A, $a \in A$, define $\text{ht}(a)$ as the maximum length of a chain $a = a_1 \geq a_2 \geq a_3 \ldots \geq a_k$. So $\text{ht}(a) = 0$ means that a is a minimal element. The height of A is simply maximum of the height of its elements: $\max\{\text{ht}(a) : a \in A\}$. This is the maximal possible length of a chain $C \subset A$.

Definition 27 (Width). Define $\text{width}(A)$ to be the maximum cardinality of an anti-chain in A.

Definition 28 (Order-preserving map). We say any map $f : A \to B$ between two PO sets is order-preserving if $a \leq a' \Rightarrow f(a) \leq f(a')$. We say f is an isomorphism if $a \leq a' \Leftrightarrow f(a) \leq f(a')$. That f is an isomorphism implies that f is injective. If you have two distinct elements of that map to the same element is contradictory.

Proposition 3. Given any A, $|A| = n$, there exists and order-preserving map $A \to \{1, \ldots, n\}$. Think of a partial ordering as a subset of $A \times A$, we're saying that this subset can be enlarged to a subset that contains a total ordering.

Proof. By induction on n. Chose maximal element $a \in A$, by induction we have an order-preserving $f : A \setminus \{a\} \to \{1 \ldots (n-1)\}$, so just setnt $a \to n$ and we have $\bar{f} : A \to \{1, \ldots, n\}$. Since a is maximal and sent to the largest of n we have a total ordering. $\square$

Proposition 4. If A is isomorphic to a subset of $P(A)$, then $\phi : A \to P(A)$ will send a to a subset $\phi(a)$ which is $\{b \in A : b \leq a\}$.

Proposition 5. With $|A| = n$, the height of $P(A)$ is n. The width of $P(A)$ is $\binom{n}{n/2}$. If we look at the collection of subset $\{S \subset A : |S| = k\}$ is antichain of size $\binom{n}{k}$. This is largest when k is $n/2$ or $(n-1)/2$. We claim that this is the width of the power set.

Proof. Let X be the set of all bijections from $A \to \{1, \ldots, n\}$ (A is just an arbitrary set, no ordering). Assume $K, K_{@}, \ldots K_q$ is an antichain in $P(A)$. Claim that $q \leq C$. Consider $X_i \subset X$ where $X_i = \{\phi : \phi(k_i) = \{1, 2, \ldots, |k_i|\}\}$. $\square$