24.711: Topics in Philosophical Logic
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Contents

1 Wednesday, February 15, 2017

1.1 Zeno's Paradox of the Arrow

- An object is at rest during an interval iff it occupies only a single position during that interval (not during a moment of time). It's stationary during an instant only if the instant is contained in an interval during which it's at rest.
- Ancient Greeks could only answer the question qualitatively because they couldn't measure time (although they could measure distances). Use a compass to make circles and form multiples of a unit length on a line.
- Can use a straight edge and compass to get an estimate of length to any precision up to a rational number, but no further.
- Aristotelian physics gives no good way of measuring a length of time. Earth, air, wind, fire all have natural motion (up, down), but because of acceleration you can't figure out good lengths of time - you need something that moves at a uniform speed. The natural motion of things in the heavens is circular, so you can use star movement to measure lengths of time (not as simple as one would think because Aristotle's system was geo-centric where the planets orbited in a circle about the earth).
- Need to be able to measure time and distance in order to measure speed and so to get close to a solution for the core question of what it means to be stationary.
- Galileo look at pendulum movement - he saw a smooth continuous movement that happens at intervals (PE being converted into KE). The Aristotelian saw chaotic movement in the pendulum (you are displacing the bob and it's swinging is it trying to go downward but

being contained by the string). Aristotle didn't have the principle of balancing of forces on the bob. Aristotle's answer to the question of why the bob doesn't stop is that there is some nonsense with a vacuum of air being generated behind the bob and pushing it forward. We can measure time by dividing the length of the pendulum rod and noting the period is inversely related to the square of the length of the rod.

- Speed during an interval is the quotient of change of distance over change and time. Zeno gave us the idea of 0 speed over an interval, now we have a quantitative idea of non-zero speed during an interval. But we still have to account for the idea that the ball is at rest only at its apogee but at no other point in its path.
- We can solve the ball problem by solving $x'(t) = 0$, where x' is the limit of the difference quotient of $(x(t+h) - x(t))/h$. Aristotle thought that bodies were the only things that were real and that bodies aren't made up of points or surfaces. So we can't just think of the ball but also need to consider the position in spacetime of the ball - we need to think of the trajectory (potential positions occupied by the ball) as existent, even when the ball isn't there. But Aristotle argued that a line isn't made up of points so this doesn't play out well under Aristotelian physics - i.e. we can talk about velocities but not of instantaneous velocities.
- Why does Aristotle preclude the solution of taking the derivative? Do we not just take the limit over smaller and smaller intervals about the instant? The Aristotelian post-Newton understood the limit as the continual dividing of the line into smaller pieces.
- Things change when we switch to dynamics - throw a ball up at some initial speed and the downward force of gravity will decelerate the ball until it reaches 0. But since acceleration is dv/dt there's no good way to define instantaneous acceleration out of velocity, which is also a function.
- You know there are times when the ball is going up and times when the ball is going down. We want to say there is a time when the ball reaches its highest point when its velocity was 0, how do we know there was such a time? We could solve this with the IVT but we want to do this constructively by constructing the cut between positive velocities and negative velocities. Can we get the LUB principle from numbers just from the movement of the ball? If the LUB principle didn't hold for the numbers then there would points in the ball's flight where it didn't have any associated number for its height.
- Most of what we can use real numbers for (measurements of things) we can get away with just rational numbers. Dedekind had originally the bad idea that we could get away with just rational numbers, but that was thrown out with $\sqrt{2}$ and the Ancients. Also need cube roots to deal with the roots of cubic polynomials. So the real numbers fall out of solving polynomials with rational coefficients - but this doesn't solve the ball problem because you have no reason to believe that the place where the ball is stationary is the root of an equation.
- The Ancients don't allow for the idea that there is a point on a line that is defined by the all the points to the right of it or to the left of it. For the ball, there needs to be a top point to define the switch of signs.

- Is it true that there is a supremum of all the points achieved by the ball? Is the ball at rest at this point? What if the ball only achieves heights of rational numbers (so it skips between points) or if the ball has a hole at the maximum of its height curve?
- From Mathematica, you have a tire on a hubcap that each go through the same number of rotations - how does the hubcap not move much less distance than the tire? Galileo: the tire is just a polygon, the hubcap goes through places where there is no corner of the polygon instantaneously which is just a wacky idea.
- Conclusion: you can't explain what is going on if you just think in terms of bodies and don't think about space and lines as being made up of points.
- An alternative view says that only bodies are real, and the bodies can be in different places but the places aren't real, just the bodies are (and the distance relations they have with each other). Kant disputed this view - if God made a single marble hand before anything else, then He could've made either a right hand or a left hand (it has to be one or the other), but if all we have is bodies with spatial relations then there is nothing different between a right hand and a left hand; but there is a difference between right hand and left hhand so there has to be something to account for the difference - you need position.

2 Tuesday, February 21, 2017

2.1 Dedekind's Construction of $\mathbb{R}$

- Jill starts at the bottom of a hill when Jack starts and the top and the reach the top/bottom at the same time. Is there a point where they are at the same elevation? Yes - take the set of all times when the difference in height was positive and then take the least upper bound. Most people say that it comes from their motion being continuous.
- Really early on people learned how to use numbers to measure things - at least from having to re-establish property lines after the flooding of the Nile. This entailed measurement by yardstick which always yield results in fractional form. But can we get these same numerical properties without actually deferring to the use of numbers?
- The rationals as fractions seem very well understood - the gaps between them were less well understood. We know that all the rationals along the path will be reached by the ball - how can we justify whether there are non-rational points that are hit by the ball.
- Pythagoreans found that simple ratios of where you put your finger on the lyre string correspond to harmonic sounds, but then one of them discovered that there are length sthat aren't rationals. He didn't find it out by measurement but rather by reductio ad absurdum - finding the length of the diagonal of the square with side length 1. He showed that it can't be rational (and was thrown overboard for his discovery). So you have to have square roots if you want to do geometry like the Greeks (because of conic sections).
- There's a problem which is for any square find another square that has twice the area - it was known by Plato and was in the Meno - solve it by taking the diagonal of the first as the side length of the second. Can you do this with a cube? You can't do this with a compass and

straight edge. The medeival Arabics pushed forward by going passed compass/straightedge and finding teh roots of a cubic equation without requiring that the length we're talking about can be one you can actually create with a compass and straightedge. That produced a whole new family of lengths never talked about by the Greeks. It was widely believed that you could create new numbers by constructing equations with irrational roots with only rational coefficients. For example $x^3 = 2$ you would introduce $\sqrt[3]{2}$. It was widely thought that all the real numbers were got by solving such equations, but for example π isn't.

- Cantor showed that there are many more real numbers that aren't solutions to equations than ones there are. The question at the end of the 19th century was how to get a picture of what the real line was like - how to fill in all the holes. Dedekind's answer was to introduce the cuts. To fill in all the gaps, we need a cut everywhere. He gave a criteria for what it would mean for $\mathbb{R}$ to be complete, and second to show that there really was such a thing as that complete system of real numbers. The algebraic properties of the real numbers are relatively easy to satisfy, and you need added properties in order to fill in all the gaps. The principle Dedekind proposed was the least-upper bound principle. He also wanted to convince people that this was final - that you couldn't go any further.
- Dedekind started by seeing if a geometric approach would suffice - he thought this was obvious - why? You also need a lot of (controversial) set theory to get Dedekind cuts, but the Dedekind cuts do reduce the reals to the rationals to the integers. Around this time, everyone considered the laws of geometry to be synthetic a priori (Kant), where we could only make sense of our sensory experience if we organized them into three-dimensional space. It seems dubious that the way we organize space should be the foundation of analysis because we have tons of applications of the real numbers that have nothing to do with space. Also, there's non-Euclidean geometry.
- Dedekind started by enumerating the rules needed for $\mathbb{R}$: field axioms, ordering, etc. But that doesn't distinguish from the rationals. You need something more: the completeness principle (LUB). Dedekind gives some axioms that the thought could answer the question of what the real numbers are, or at least he says the real numbers satisfy those structural features. Why is identifying the structural features enough to say that you've identified the real numbers? There are going to be lots of other algebraic systems that satisfy those axioms. You have to also give the intended model for those axioms. The physical line satisfies those axioms, but we have no reason to believe that any particular one is not the real number system. Also any complete model of those axioms is going to be isomorphic to any other. Why is giving the structure not enough?
- At the same time, Frege was investigating the foundations of arithmetic. He thought that the only principle he needed was Hume's Principle (two collections have the same elements iff there's a one-to-one correspondence between the elements). He showed you could get PA axioms from Hume's Principle, but he still asked: how do I know whether Julius Ceasar is a number? We expect a numeral to pick out a definite thing, but we can never do this with only axioms. Dedekind introduced the idea that the numerals don't have to stand for definite objects - that mathematics is rather about identifying structural properties, and that identifying the structure is all you can and need to do.

3 Wednesday, February 22, 2017

3.1 Dedekind's Construction of $\mathbb{R}$

- Dedekind asked: what further properties than the normal ordering and field properties are necessary to specify the real numbers? Completeness. His proposal is to create cuts of the rationals into two non-empty classes so that everything in the lower portion is less than everything in the upper portion. Completeness means that anytime you have such a cut there is a real number defining it. This principle amounts to the least-upper bound principle. He showed that this condition characterizes the real number system (up to isomorphism). Historically, people had made systems that purported to be the real numbers but it turned out that they left some numbers out. But Dedekind showed that you can't add anything to his system and get anything that's properly larger.

- The isomorphism theorem says that if you have any two models of his axioms a and b, you first define the positive integers of a to be those that contain $0 \in a$ and are closed under succession (and define the positive integers of b the same way), and then define a map between the two sets of natural numbers. This map is injective - if it weren't, then the set of integers that had mappings would have a LUB b', and somewhere between b' and $b' - 1$ you would have an integer that gets mapped. This lets you map the rationals of a to the rationals of b, and so then we get an isomorphism of Dedekind cuts and so of the real numbers in a and b.

- So we've completely characterized the structure we're looking for - how do we know that there is something that exhibits that structure? Dedekind's response is to define the set of all cuts to be the real numbers and then show that this structure satisfies the axioms. So he takes the construction to be a sufficient answer to the question of what the real numbers are. But it feels like he's just given us something that is isomorphic to the real numbers.

- Frege equivalently gives structural axioms for the natural numbers, but wasn't satisfied because he couldn't answer the question of whether Julius Caesar was a natural number. Hume's Principle is a structural property that only characterizes the natural numbers up to isomorphism.

- All we need to construct the natural numbers is an infinite set from which we can generate a simply infinite set, but that's non-trivial because, starting from Zeno's Paradox, there is a long line of people saying that we can't have an infinite set, just a potentially infinite set. They say that all we can do is give methods for taking a finite sequence and extending it, but the idea that we can treat an infinite set as a whole was long treated as contradictory. Dedekind thinks he can build an infinite set by taking the totality of objects of his thought and take a member of that totality S and form the thought "S is an object of one's thought", and then recursively do that. That's a one-to-one function from the objects of one's thought to the objects of one's though, but it isn't onto because one's ego isn't of the form "X is the object of my thought". This skirts the question of potentially vs. actually infinite because the totality of objects of thought is something given to you, not something constructed.

- He says that the natural numbers are a "free created of the human mind" which he clarifies as being because they are a result of the process of abstraction specified above. This isn't completely new - Cantor does something similar when he wants to go beyond the natural numbers into the transfinite by looking at the different ways of ordering the collection and then defining an order isomorphism and then defining the order types as got from the orderings by the process of abstraction. You regard two orderings the same if their orderings are isomorphic (there's a bijection from one to the other that preserves the order). Or you can go even more abstract and say that two totalities are the same if there is a bijection between them, irrespective of ordering. That gives you the cardinal numbers. Aristotle also gets the objects of geometry similarly - he thought of them as ordinary solid bodies but we abstract away all the non-geometrical properties. Both Cantor and Dedekind were dipping into that tradition.

- In retrospect, it seems like this is a big transition from thinking of mathematical terms like "natural/real/complex number" as denoting definite things (not things known from sense-experience, but still definite entities) to thinking about them as if we can talk about them as definite things once we've characterized them up to isomorphism. Modern mathematicians are pretty much invariably subscribers to the latter view - you talk about mathematical objects, but mathematical objects are what they are because of their properties and properties only characterize up to isomorphism. There was a change in the way mathematicians talked about mathematical objects that philosophy was really slow to notice.

- Cantor also had a way of creating the real numbers out of the rationals. Cantor himself was interested in Fourier Series and when they converge, for him the idea that the real numbers were complete was cashed out by the idea that series converge whenever it's possible for them to converge. For any convergent sequence, we have $\forall k, \exists N$ such that $n > N \Rightarrow |a_n - a_N| < \frac{1}{k}$ (Cauchy sequence). So one criterion for the completeness of the real numbers other than the LUB principle is the Cauchy criterion implies convergence for any sequence. You can derive one criterion from the other. Cantor then defined the equivalence relation between $\{x_n\}$ and $\{y_n\}$ as for any k there is a sequence term such that y_n and x_n are arbitrarily close passed the sequence term. You can show that Dedekind's and Cantor's principles yield isomorphic copies of the real numbers. But the question never arose as to which of these is the genuine construction of the real numbers. It didn't occur to them to ask: what's a real number - a Dedekind cut or an equivalence class of Cauchy sequences?

- Hilbert's axiomitization of geometry: he gave the set of axioms and wasn't initially concerned with the question of whether the system was complete. We know we get a model of the geometry axioms if you take a point to be an ordered triple of real numbers (identify a point with its coordinate in $\mathbb{R}^3$, which doesn't have anything to do with space and yet is still a perfectly good model). You also get a perfectly good model if you identify a point with an ordered triple of algebraic numbers - so the geometric axioms do not require completeness. Later on, Hilbert started to worry about completeness and so he added an axiom in the 9th edition so that the new axiom system is complete. One of the axioms of the original system was the Archimedean axiom: you can construct a line segment out of a sub-segment so that there is some multiple of the sub-segment that will surpass the end of the larger segment. Traditionally, geometric axioms were supposed to represent three-dimensional space, but Hilbert rather didn't deny that the axioms describe physical space but didn't assume that they were about physical space. He said the axioms were about systems that satisfy those axioms, and nothing more - there's nothing more to being an intended model for the geometric axioms. This is similar to group theory - there's no intended model for group theory other than a system that satisfies those axioms. This is a real departure from the idea that mathematics has a definite subject matter.

- You can have a name without a referent - for example, "Cerberus" or other names that occur within myths. You don't have that with names like "Cherry" - there's a definite referent to the name "Cherry". But we have these numerals in mathematics (e.g. $\sqrt{2}$) that don't have a uniquely determined thing that they refer to. The issue with idea of identifying the square root of 2 with the set of all things that when squared is 2 is problematic because you can't square a set. And you can't identify it with any one object whose square is two for similar reasons. There's a worry that Dedekind gave us a construction of what he called the real numbers and long before him the Egyptians were using numerals that purported to be real numbers, so did the Ancient Egyptians refer to a Dedekind cut when they wrote down one such numeral? Or, possibly, could the real number signs not refer to entities but instead to some action, like the act of taking the successor with the natural numbers? Well, the numerals do correspond in an objective way similarly to how people's names objectively refer to people. If it were purely made up, then there wouldn't be a "correct" answer for question with real-numbered answers.

- Dedekind described what he was doing as the reduction of analysis to arithmetic, which was understood to have secure foundations. But analysis has less secure foundations because people didn't have a clear understanding of continuity and limits and what was required for a sequence to converge - there was just a general lack of clarity. It's more accurate to say that he reduced analysis to set theory or set theory plus arithmetic because what he needs to form the cuts is a lot of set theory. Looking back, it seems that he really did do something important for our understanding of the foundations of analysis, but it was not reducing analysis to arithmetic but rather constructing analysis within this theory of sets. Next time let's look at the axiomatic theory of sets and consider Dedekind's construction as a construction of the real numbers within set theory.

4 Wednesday, March 1, 2017

4.1 Paradoxes

- Last time we talked about Zermelo's axiom of set theory. One thing Zermelo wasn't addressing but were in the background were the paradoxes, which is why people responded to his proof of the well-ordering theorem with such uncertainty (their understanding of how sets work was shaken by the appearance of paradoxes). Formal set theory was still pretty new and their study only came to the foreground with Dedekind and Cantor and it seemed to fall immediately into contradiction.

- The most prominent of paradoxes was Russell's paradox - some sets are elements of themselves, so consider the set of sets that are not members of themselves. If you ask the question of whether that set is a member of itself, you get a contradiction. The paradox has a direct analogue about language - "dog" isn't a dog so it doesn't apply to itself, but the phrase "does not apply to itself" applies to itself iff it doesn't.
- Another paradox was with Cantor's study of ordinal numbers, he found that the ordinals were well ordered and for any well ordered set there is an ordinal that describes the order type of that set. Any ordinal is the order type of the ordinals less than it. But consider the collection of all ordinals, which is well ordered, and so there is a map from all the ordinals to all the ordinals less than some ordinal. Keep applying the map and then you get an infinite descending sequence which is a contradiction because the ordinals are supposedly well ordered. The semantic analogue is to take all the names of ordinals, there are only countably many names and so there are only countable many names of ordinals, but since there are uncountably many ordinals there are ordinals that aren't named. Since the ordinals are well-ordered there is some least ordinal is not named, but we have just specified a name for it. Also there is a least natural number nameable in fewer than 19 syllables.
- Cantor showed that there are uncountably many real numbers by assuming that there are only countably many real numbers and then disproving the bijection by constructing a new real number not in the enumerated set. You can make a bijection $\mathbb{R} \to \{q_1, q_2, q_3, \ldots \in \mathbb{Q}\}$. He generalized the result to show that for any set there are more subsets of S than the number of elements of S. But then take the set of all sets, the cardinal number of the power set of the set is larger than the cardinal number of the set itself. But the power set of all sets of sets is included in the set of all sets and so the cardinal number of the power set is less than or equal to the cardinal number of the set of all sets. There is no direct linguistic analogue but it came from the generalization of Cantor's real number uncountability proof. If we look at that theorem linguistically, we see that there are only countably many names for the uncountable $\mathbb{R}$, and then put the names in a list an use Cantor's procedure to construct a real number that is nameable but not on the list of all possible names.
- Cantor just rejected that there was such thing as the set of all sets. He thought there were theological implications. He wrote to the Pope (no response) and said that he was developing a theory of the actual infinite but there may be worries because the church teaches that only God is infinite. He says it isn't blasphemous because he says he has found an intermediate level between the finite and the absolutely infinite (transfinite) where the transfinite goes beyond what we ordinarely think of the finite, but it's not copmlete limitless. If you had something completely limitless then you can't number it or comprehend it mathematically - only God is at the level of the absolutely infinite which is above not just the finite but also the transfinite. So Cantor's work isn't bringing God down to the level of ordinary mathematicians, but rather he boosted God up to a level farther beyond the level of things that Earthly things are.

- So there's these analogues between set theoretic paradoxes and semantic paradoxes. Zermelo didn't seem interested in the semantic part. Russell had the exact opposite attitude - he thought this showed a deep discontinuity in human's thought, one that was so profound because it manifested in a lot of different ways. How important is it to get a unified solution to the set of two paradoxes? It really seems like they are the same phenomenon.
- This week we're going to look at a pardox that wasn't among the list of Russell's paradoxes - developed around 1960. You can't recursively pick sets as elements of other sets - there's no infinite descending chain. Say a set is well-founded if there aren't any infinite descending chains beginning with. Let W be the set of all well-founded sets. If there were an infinite descending chain ending with W then look at the chain the ends at the element which is an element of W and so that chain is infinite and the element is not well-founded. So W has to be well-founded. But then $W \in W$ and so there's an infinite chain ending at W by picking W for every link of the chain. Since no well-founded set is a member of itself, this paradox seems to reduce straight to Russell's Paradox. But it doesn't because you could have $A \in B \in A \in B \in \ldots$. So no loops of any length around whereas Russell's paradox is just a loop of length one. This paradox turned people's attention to sets that are well-founded - but most sets that are useful for math are those that are well-founded. If you want to really think closely about set theory but your interest in set theory is just for finding a theory of sets that is useful for mathematics, then you can just stick with the well-founded sets.
- So there was an axiom proposed that said that all sets are well-founded. What amounts to the same thing is: for any non-empty set S, there is an element $s \in S$ such that $\forall t \in s, t \notin S$. Originally it was just adopted for convenience but it does yield a really pretty picture for what the universe of sets looks like. If we assume that the things that aren't sets form a set, then you get this picture of the theory of types. The crucial axiom for building out this structure is the axiom of ordered pairs - this allows you to put together two elements from different levels. This is the way that Zermelo was thinking about it but not the way that Russell and Whitehead were thinking about it. Cantor's paradox entails forming a set at some particular level, but the set of all sets must be at a level before which the set of all sets is formed and so there is no level where the set of all sets is at. Since a set is always formed at a level after the level at which its elements are formed, so you solve the same set of paradoxes. For the ordinals, each level of the hierarchy introduces a new ordinal. Each level of the hierarchy introduces exactly one new ordinal. So in order to get the collection of all the ordinals it would have to be formed at a stage after which you'd already gotten all the ordinals and so there is no collection of all the ordinals. So there's a response to all the set-theoretic paradoxes that comes just from trying to form a set that violates the foundation axiom.
- So adding another axiom gave us the stronger system because it removes an inconsistency but does not introduce one. If you restrict the axioms of set theory to range only over well-founded sets. You can verify that each formula you get from an axiom of set theory by restricting the quantifier is a theorem of set theory. If you could derive a contradiction from the axioms of set theory including the axiom of foundation then you could get a

contradiction in ZFC.

- The tradition, beginning with Cantor, is thinking about collections formed from contained elements and you need collections to be well-founded and so you get paradoxes if you don't restrict to well-founded collections. The opposite view is from Frege where you get sets by starting with concepts. Von Neumann thought that you get contradictions when you start forming sets that are really large ($>>> \mathbb{R}$) like the set of all the ordinals or the set of all the cardinals, etc. So forming sets that are way too big is impossible but formulating sets that are relatively small is okay. The range of a function on a set will be at most the same size as the domain set so then if the things in the domain aren't too many to form a set then the things in the range aren't too many to form a set, either. VN had the replacement principle and this more global principle that you can form sets unless they are too big, "too big" meaning things form a set unless those things are equinumerous with the entire universe. So if you have set that's soo big that if it's a set then the entire universe is a set then that set is not actually a set. So there's a different way of thinking about where the paradoxes come from by having sets that are just too big. Axel took this idea and build a whole set theory out of it.
- This is the combinatorial conception of sets as collections. Next time we will think about sets from a logical point of view by starting with concepts and then get to sets by a process of abstraction.