PHYS6017 Project C4

COMPUTER TECHNIQUES IN PHYSICS

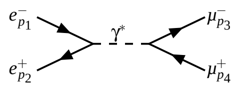
The $e^+e^- \to \mu^+\mu^-$ Cross Section in the Standard Model

Consider the annihilation of an electron e^- against a positron e^+ to produce a muon μ^- and antimuon μ^+, denoted by

$$e^+e^- \to \mu^+\mu^-,$$

at energies much higher than the fermion masses involved ($m_e = 0.511$ MeV and $m_\mu = 106$ MeV)[1].

This process is modelled in Quantum Electro-Dynamics (QED) through the exchange of a (virtual or off-mass-shell) photon (simetimes denoted by γ^*, with an asterisk) in the so-called s-channel, meaning that its squared invariant mass, $E_\gamma^2 - |\vec{p}_\gamma|^2$ is positive. This is sketched through a Feynman diagram,



where energy/momentum conservation is enforced at each vertex, i.e.,

$$E_\gamma = E_1 + E_2 = E_3 + E_4,$$

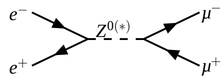
$$\vec{p}_\gamma = \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4.$$

In covariant (or Lorentz) notation we can write the above two equalities as

$$p_\gamma^\mu = p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu,$$

wherein $\mu = 0$ refers to the time-like component of a four-vector (so that $p_i^0 = E_i^0$) whereas $\mu = 1,2,3$ refer to the three space-like components of a four-vector (i.e., for $j = 1,2,3$).

In a similar way, it is possible to evaluate the weak contribution including the Z^0-boson s-channel diagram



Notice here the notation $Z^{0(*)}$, where $(*)$ signifies the fact that the Z^0 boson can be either real (on-mass-shell, Z^0) or virtual (off-mass-shell, Z^{0*}), depending on whether (again, in short-hand covariant notation)

$$s = (p_1^\mu + p_2^\mu) g_{\mu\nu} (p_1^\nu + p_2^\nu) \equiv (p_1 + p_2)^2 = M_{Z^0}^2$$

[1] Hereafter, we adopt natural units, whereby $\hbar = c = 1$ and we will take the electron to be massless, i.e., $m_e = 0$.

or otherwise, respectively[2]. Here, $g_{\mu\nu}$ is the so-called matrix tensor, which is a 4×4 diagonal matrix, such that

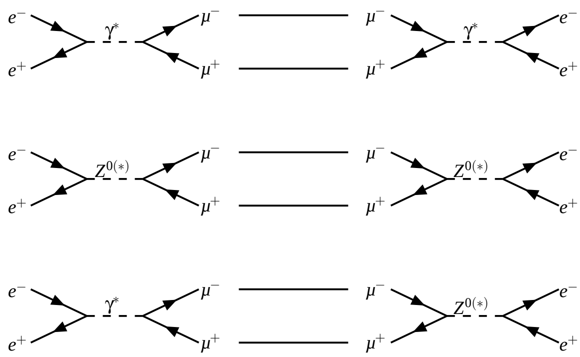
$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

and can be used to lower or raise Lorentz indices

$$p_\mu = g_{\mu\nu} p^\nu$$

for generic four-vectors.

The contributions to the observable cross section for $e^+e^- \to \mu^+\mu^-$ are due to photon exchange only (where the first diagram above is 'squared'), Z^0 exchange only (where the second diagram above is 'squared') and a combination of the two (where the first diagram is 'interfered' with the second one). Symbolically, these three contributions can be represented as below, respectively.



Adopting the so-called Feynman rule formalisms, the contributions of these three terms to the differential cross section $d\sigma/d\Omega$ for $e^+e^- \to \mu^+\mu^-$ can be be written as

$$\frac{d\sigma}{d\Omega}^{\gamma-\gamma} = \frac{g_e^4}{(8\pi)^2 s}\sqrt{1-4\varepsilon^2}(1+\cos^2\theta + 4\varepsilon^2\sin^2\theta),$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega}^{Z^0-Z^0} &= \frac{g_Z^4}{(32\pi)^2 s}\frac{\sqrt{1-4\varepsilon^2}}{(1-\lambda^2)^2+(\lambda\Gamma_{Z^0}/\sqrt{s})^2} \times \\
\times \quad & \Big\{[(C_V^e)^2+(C_A^e)^2](C_V^\mu)^2(1+\cos^2\theta+4\varepsilon^2\sin^2\theta) + \\
+ \quad & [(C_V^e)^2+(C_A^e)^2](C_A^\mu)^2(1+\cos^2\theta)(1-4\varepsilon^2) + \\
+ \quad & 8C_V^eC_A^eC_V^\mu C_A^\mu\sqrt{1-4\varepsilon^2}\cos\theta\Big\},
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega}^{\gamma-Z^0} &= \frac{2g_e^2g_Z^2}{(16\pi)^2 s}\frac{(1-\lambda^2)\sqrt{1-4\varepsilon^2}}{(1-\lambda^2)^2+(\lambda\Gamma_{Z^0}/\sqrt{s})^2} \times \\
\times \quad & \left[C_V^eC_V^\mu(1+\cos^2\theta+4\varepsilon^2\sin^2\theta)+2C_A^eC_A^\mu\sqrt{1-4\varepsilon^2}\cos\theta\right],
\end{aligned}$$

[2] The above equation can of course also be written with the replacements $1 \to 3$ and $2 \to 4$.

where $\Gamma_{Z^0} = 2.5$ GeV is the Z^0 width. (For the photon, $\Gamma_\gamma = 0$.) In the above three formulae, we have integrated over the final state phase space (i.e., all the degrees of freedom of the $\mu^+\mu^-$ pair) except the polar (also called 'scattering') and azimuthal angles θ and ϕ, respectively, so that

$$d\Omega = d\cos\theta d\phi.$$

Notice, however, that the integral over ϕ is trivial if the initial particles have no transverse polarisation (which is the case here). We have also included in the formulae a so-called flux factor, which takes into account the energy dependent probability of the two initial state particle to scatter and introduced the two dimension-less quantities

$$\begin{aligned} \varepsilon &= \frac{m_\mu}{\sqrt{s}}; \\ \lambda &= \frac{M_{Z^0}}{\sqrt{s}}. \end{aligned}$$

To uniquely identify the scattering angle, we introduce the four-vectors

$$\begin{aligned} p_1 &= (E, \vec{p}_i), \qquad & p_2 = (E, -\vec{p}_i), \\ p_3 &= (E, \vec{p}_f), \qquad & p_4 = (E, -\vec{p}_f), \end{aligned}$$

where

$$\begin{aligned} |\vec{p}_i| &= E, \\ |\vec{p}_f| &= \sqrt{E^2 - m_\mu^2}, \end{aligned}$$

and

$$\vec{p}_1 \cdot \vec{p}_3 = |\vec{p}_i||\vec{p}_f|\cos\theta$$

for the dot-product of three-vectors. Here, $E \equiv \sqrt{s}/2$, so that $\sqrt{s}$ is the collider energy and E the beam energy.

Finally, in the Standard Model (SM), we can express the strenght of all fundamental interactions in terms of 'couplings', as function of two variables: α and θ_w, the Electro-Magnetic (EM) constant and the so-called 'weak mixing angle', respectively. Their values are extracted by experiment and depend on the energy scale at which they are measured (e.g., they depend on s). For the purpose of this project, the following values can be adopted throughout though:

$$\alpha = \frac{1}{128}, \qquad \sin^2\theta_w = 0.23152.$$

In the preceeding cross section formulae then, we have:

$$\begin{aligned} g_e &= \sqrt{4\pi\alpha}, \\ g_Z &= \frac{\sqrt{4\pi\alpha}}{\cos\theta_w \sin\theta_w}. \end{aligned} \tag{1}$$

The so-called vectorial and axial couplings between electron/positrons or muon/anti-muon and the Z^0 boson are summarised in Table 1.

The aim of the project is to obtain an *accurate* value for the total cross section for $e^+e^- \to \mu^+\mu^-$ as a function of $\sqrt{s}$, first including the photon contribution alone (i.e.,

Fermions f	C_V^f	C_A^f
$e^\pm$, $\mu^\pm$	$-\frac{1}{2}+2\sin^2\theta_w$	$-\frac{1}{2}$

Table 1: The C_V and C_A couplings in the SM.

the QED contribution) and then also including the Z^0 one and the interference terms (i.e., the full SM result).

I recommend to integrate the three contributions separately, as the integrand functions (with varying $\cos\theta$) are very different. I suggest to perform the integration in each case with two different methods, trapezium rule and a Monte Carlo approach, as a cross check of the numerical results and in order to appreciate the different performances achievable with the two methods. Some comments are in order on this last point. First, integrate the three terms above separately for each algorithm. Then, integrate the sum of them. Again, compare the performance of the two algorithms in this case too.

Present the results for several values of $\sqrt{s}$ in graphic form. Scan over this variable over a wide range, ranging from a few GeVs up to 200 GeV or so. You'll notice that, as $\sqrt{s}$ approaches M_{Z^0}, a resonant behaviour will onset. Please comment on the origin of this feature and make sure the sampling on $\sqrt{s}$ is fine enough in this region, to actually see what happens in some detail. What is Γ_{Z^0} then ? What is the contribution of the interference term at $\sqrt{s}=M_{Z^0}$? Can you see why ? (Recall that the Z^0 is a massive unstable particle and the interference term is obtained by computing $2\mathrm{Real}(M_\gamma M^*_{Z^0})$, where $M_\gamma(M_{Z^0})$ is the scattering amplitude for $\gamma(Z^{0(*)})$ exchange and the symbol * represents the operation of complex conjugation.)

Remember you must always plot the integrand so you can be sure it is well behaved and that a numerical approach is sensible. Therefore, as a by-product, I suggest, for three values of energy (one well below $\sqrt{s}=M_{Z^0}$, one precisely at that point and another one above it), to show the differential distribution in $\cos\theta$, again, for the QED contribution separately and then for the full SM result. Comment on the modification of the curves that you will see as the energy changes. Try also plotting another kinematic variable, p_T, called the transverse momentum, and defined as

$$p_T = |\vec{p}_f|\sin\theta.$$

Recall to estimate the numerical error in all cases.