Estimating network-mediated causal effects via spectral embeddings

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Two questions for the audience

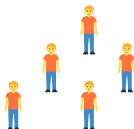# I didn't know anyone when I first arrived here
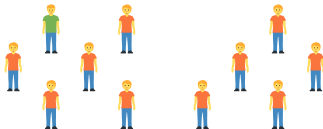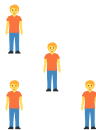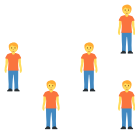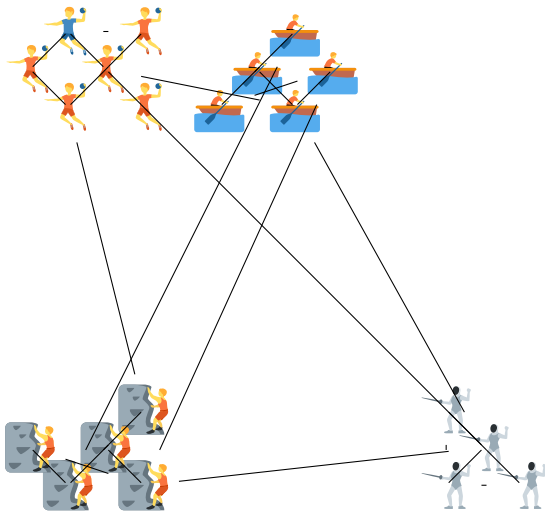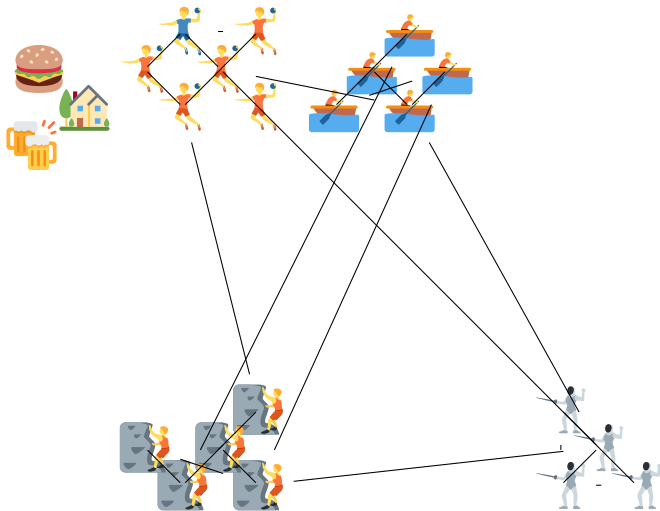
Idea: social groups _mediate_ the causal effect of individual interests (e.g. frisbee) on visits to the Great Dane

Assume there are only two people, person *i* and person *j*

I'm on a frisbee team, so I form friendships (A_{ij}) with other people on my team
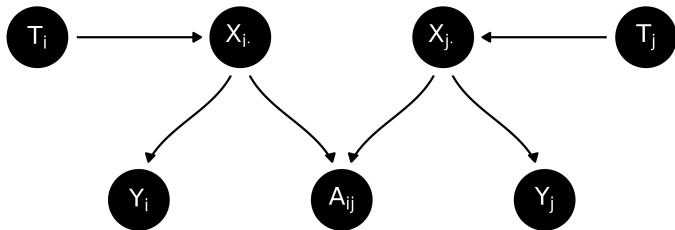
I like frisbee, and this might directly cause me to go to the Great Dane without my team
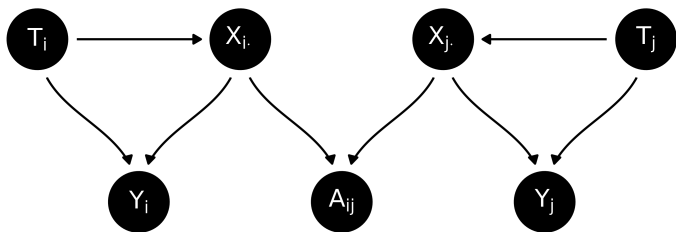
i.e. if I want to watch frisbee at the Great Dane
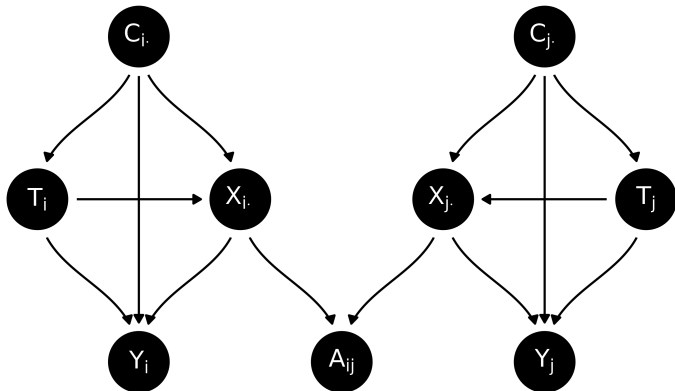
This is a good place to ask questions

Formalizing what I think happened with frisbee

First, a recap. I think:

1. enjoying frisbee caused me to visit the Great Dane more frequently than I would have otherwise, and
2. the mechanism proceeded in two stages: first I joined a frisbee team because I liked frisbee, and I went to the Great Dane with the team
3. I don't think liking frisbee caused me to go to the Great Dane independently of my frisbee team

Can disambiguate these causal effects using mediation analysis

- Average treatment effect: how much the outcome Y would change on average if the treatment T were changed from $T = t$ to $T = t^*$

$$\Psi_{\mathrm{ate}} = \mathbb{E}[Y_i(t) - Y_i(t^*)]$$

Note: $Y(t)$ is the counterfactual value of Y_i when T_i is set to t

I claim Ψ_{ate} is positive in my example

- Natural indirect effect: how much the outcome Y would change on average if the exposure were fixed at level $T = t^*$ but the mediator X were changed from the level it would take if $T = t$ to the level it would take if $T = t^*$

$$\Psi_{\mathrm{nie}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t)) - Y_i(t, X_{i\cdot}(t^*))]$$

- Captures the effect of the exposure on the outcome that operates by changing the mediator

I claim Ψ_{nie} is positive in my example

Natural direct effects

- <u>Natural direct effect</u>: how much the outcome *Y* would change if the exposure *T* were set at level $T = t^*$ versus $T = t$ but for each individual the mediator *X* were kept at the level it would have taken, for that individual, if *T* had been set to t^*

$$\Psi_{\mathrm{nde}} = \mathbb{E}[Y_i(t, X_{i.}(t^*)) - Y_i(t^*, X_{i.}(t^*))]$$

- Captures the effect of the exposure on the outcome that would remain if we were to disable the pathway from the exposure to the mediator

I claim Ψ_{nde} is zero in my example.

Note $\Psi_{\mathrm{ate}} = \Psi_{\mathrm{nde}} + \Psi_{\mathrm{nie}}$.

What's new: X are latent friend group memberships in a social network

Identification and estimation for natural direct and indirect effects is well-established

What's new here is X, which I have been vague about so far

We assume that the friend groups X are <u>unobserved</u>

We will infer the friend groups X from a social network

Semi-parametric network & network regression models

d "blocks" or communities
$X_{i\cdot} \in \{0, 1\}^d$ one-hot indicator of
node i's block

X is latent (i.e. unobserved)

$B \in [0, 1]^{d \times d}$ inter-block edge
probabilities

Friendships depend on group
memberships and B
$\mathbb{P}(A_{ij} = 1 \,|\, X) = X_{i\cdot} B X_{j\cdot}^T$
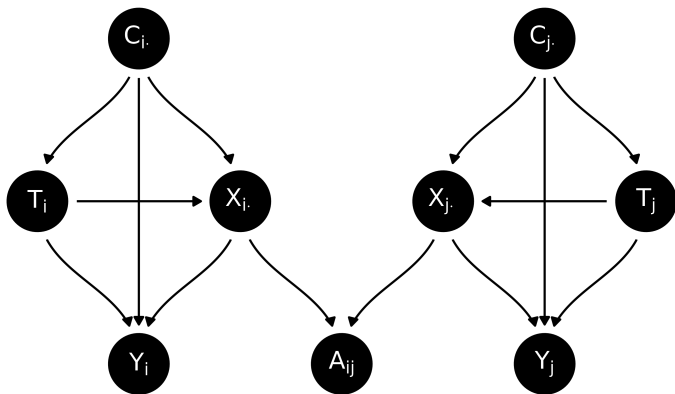
A regression model for friend group membership

Idea: interventions T_i can cause community membership X_i.

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1 \times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1 \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_t}_{\mathbb{R}^{1 \times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_c}_{\mathbb{R}^{p \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_{tc}}_{\mathbb{R}^{p \times d}}.$$

Example: I like frisbee so I joined an ultimate frisbee team (MUFA)

A regression model for outcomes

Idea: community membership $X_{i.}$ can cause outcomes Y_i

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i.}, X_{i.}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_t}_{\mathbb{R}} + \underbrace{C_{i.}}_{\mathbb{R}^{1\times p}} \underbrace{\beta_c}_{\mathbb{R}^p} + \underbrace{X_{i.}}_{\mathbb{R}^{1\times d}} \underbrace{\beta_x}_{\mathbb{R}^d}$$

Example: I'm on a frisbee team, and the frisbee team goes to the Great Dane together after each game

Semi-parametric causal identification

Recall the regression models:

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}}\underbrace{\beta_{\mathsf{t}}}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}}\underbrace{\beta_{\mathsf{c}}}_{\mathbb{R}^p} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1\times d}}\underbrace{\beta_{\mathsf{x}}}_{\mathbb{R}^d},$$

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1\times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1\times d}} + \underbrace{T_i}_{\{0,1\}}\underbrace{\theta_{\mathsf{t}}}_{\mathbb{R}^{1\times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}}\underbrace{\Theta_{\mathsf{c}}}_{\mathbb{R}^{p\times d}} + \underbrace{T_i}_{\{0,1\}}\underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}}\underbrace{\Theta_{\mathsf{tc}}}_{\mathbb{R}^{p\times d}}.$$

Then:

$$\Psi_{\mathrm{nde}}(t, t^*) = (t - t^*)\,\beta_{\mathsf{t}}$$
$$\Psi_{\mathrm{nie}}(t, t^*) = (t - t^*)\,\theta_{\mathsf{t}}\,\beta_{\mathsf{x}} + (t - t^*)\,\mu_c\,\Theta_{\mathsf{tc}}\,\beta_{\mathsf{x}}.$$

Estimation

Regression estimators

Challenge: regression models depend on X, but we never see X. Luckily we can estimate it!

Definition (ASE)
Given a network A, the $\widehat{d}$-dimensional <u>adjacency spectral embedding</u> of A is

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2}$$

where $\widehat{U}\widehat{S}\widehat{U}^T$ is the rank-$\widehat{d}$ truncated singular value decomposition of A.

Note the tuning parameter! The analyst must specify $\widehat{d}$

Well-known that $\widehat{X}$ is a good estimate of X

Lemma (Levin et al. (2022))
Under the sub-gamma random dot product model and some additional omitted conditions, if $\widehat{d} = d$, there is some $d \times d$ orthogonal matrix Q such that

$$\max_{i \in [n]} \left\| \widehat{X}_{i.} - X_{i.}Q \right\| = o_p(1).$$

$\widehat{X}$ can be plugged in for X just fine

Let $\widehat{D} = \begin{bmatrix} 1 & T & C & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$ and
$L = \begin{bmatrix} 1 & T & C & T \cdot C \end{bmatrix} \in \mathbb{R}^{n \times (2p+2)}$.

We estimate β_w and β_x via ordinary least squares as follows

$$\begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_t} \\ \widehat{\beta_c} \\ \widehat{\beta_x} \end{bmatrix} = \left(\widehat{D}^T \widehat{D} \right)^{-1} \widehat{D}^T Y.$$

Similarly, we estimate Θ via ordinary least squares as

$$\widehat{\Theta} = \left(L^T L \right)^{-1} L^T \widehat{X}.$$

To estimate Ψ_{nde} and Ψ_{nie} in our semi-parametric setting, we combine regression coefficients from the network regression models:

$$\widehat{\Psi}_{\text{cde}} = \widehat{\Psi}_{\text{nde}} = (t - t^*)\,\widehat{\beta}_{\text{t}} \qquad \text{and}$$
$$\widehat{\Psi}_{\text{nie}} = (t - t^*)\,\widehat{\theta}_{\text{t}}\,\widehat{\beta}_{\text{x}} + (t - t^*) \cdot \widehat{\mu}_c \cdot \widehat{\Theta}_{\text{tc}}\,\widehat{\beta}_{\text{x}}.$$

It's standard to fit two regressions and multiply coefficients to estimate an indirect effect like this (VanderWeele and Vansteelandt, 2014).

Theorem (Regression coefficients are asymptotically normal)

Under some mild assumptions, there is some unknown matrix Q such that

$$\sqrt{n}\,\widehat{\Sigma}_\beta^{-1/2} \begin{pmatrix} \widehat{\beta}_w - \beta_w \\ Q\,\widehat{\beta}_x - \beta_x \end{pmatrix} \to \mathcal{N}(0, I_d), \text{ and}$$

$$\sqrt{n}\,\widehat{\Sigma}_{\mathsf{vec}(\Theta)}^{-1/2} \left(\mathsf{vec}\left(\widehat{\Theta}\,Q^T\right) - \mathsf{vec}(\Theta) \right) \to \mathcal{N}(0, I_{pd}).$$

where $\widehat{\Sigma}_{\mathsf{vec}(\Theta)}^{-1/2}$ and $\widehat{\Sigma}_\beta^{-1/2}$ are the typical heteroscedasticity robust covariance estimators, with $\widehat{X}$ plugged in for X.

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Theorem (Causal estimators are asymptotically normal)

Under the same statistical assumptions as before, plus mediating homophily,

$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nde}}}\Big(\widehat{\Psi}_{\mathrm{nde}} - \Psi_{\mathrm{nde}}\Big) \to \mathcal{N}(0, 1), \ and$$
$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nie}}}\Big(\widehat{\Psi}_{\mathrm{nie}} - \Psi_{\mathrm{nie}}\Big) \to \mathcal{N}(0, 1).$$

where $\widehat{\sigma}^2_{\mathrm{nde}}$ and $\widehat{\sigma}^2_{\mathrm{nie}}$ are rather unfriendly variance estimators derived via the delta method and the previous theorem.

Read the manuscript at https://arxiv.org/abs/2212.12041

Slides available at https://tinyurl.com/ifds-alex

R package netmediate

Stay in touch

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I'll be on the post-doc market in Fall 2023
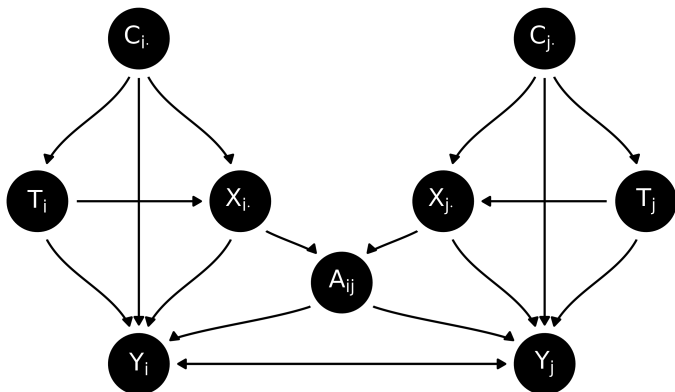
Contagion ($Y_j \rightarrow Y_i$) is not allowed
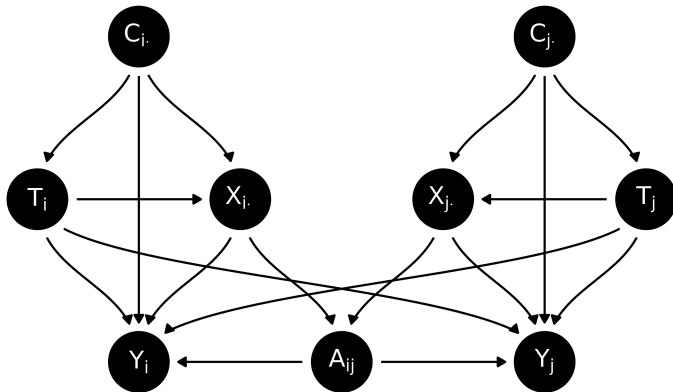
Interference ($T_j \rightarrow Y_i$) is not allowed

Interference and contagion effects are allowed <u>so long as they happen in the latent space</u>. Suppose

$$\mathbb{E}[Y_i \mid W_{i\cdot}, X_{i\cdot}] = W_{i\cdot}\beta_{\mathsf{w}} + X_{i\cdot}\beta'_{\mathsf{x}} + \delta_{\mathsf{y}} \sum_j X_{i\cdot}^T X_{j\cdot} Y_j$$

This latent space contagion model is a special parametric case of the regression outcome model (take $\beta_{\mathsf{x}} = \beta'_{\mathsf{x}} + X^T Y \delta_{\mathsf{y}}$).

Semi-parametric network model

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \mid X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\mathrm{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.
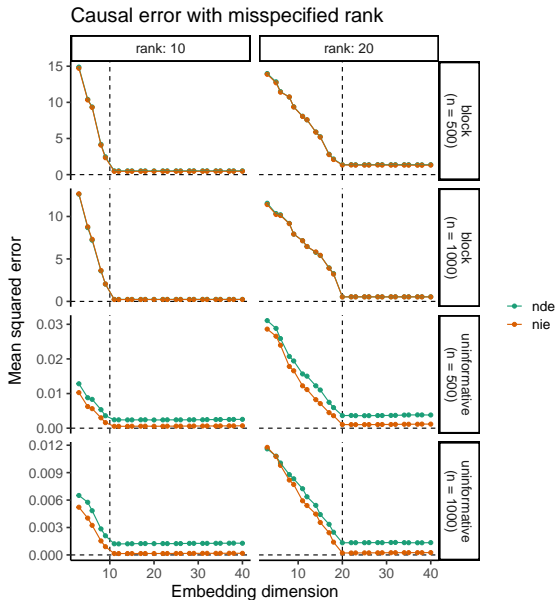
$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q, the latent positions X are only identifiable up to an orthogonal transformation.

Choosing $\widehat{d}$: do a multiverse analysis
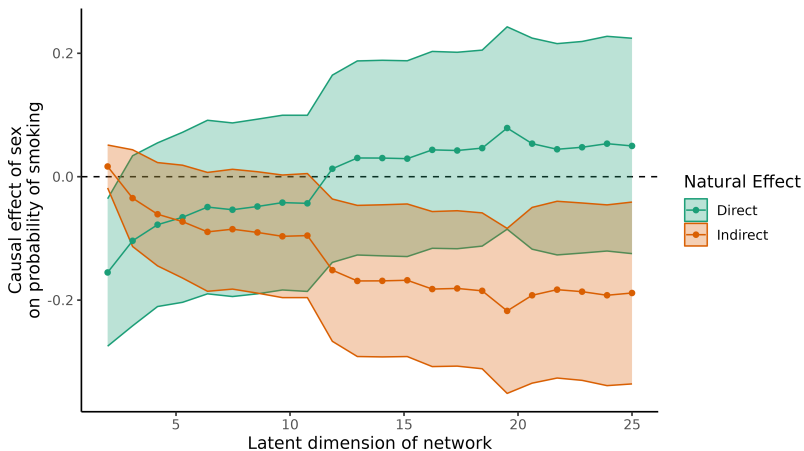
Choosing $\widehat{d}$: overestimating the embedding dimension is fine

Causal error with misspecified rank

Identifying assumptions

The random variables $(Y_i, Y_i(t,x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:
   if $T_i = t$, then $X_{i\cdot}(t) = X_{i\cdot}$ with probability 1, and

   if $T_i = t$ and $X_{i\cdot} = x$, then $Y_i(t,x) = Y_i$ with probability 1

2. Sequential ignorability:

   $\{Y_i(t^*,x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot}$ and $\{Y_i(t^*,x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$

3. Positivity:

   $$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0 \text{ for each } x \in \mathrm{supp}(X_{i\cdot})$$
   $$\mathbb{P}(t \mid C_{i\cdot}) > 0 \text{ for each } t \in \mathrm{supp}(T_i)$$

Interventions allowed

Provided that controls $C_{i\cdot}$ are sufficiently informative about group membership $X_{i\cdot}$, treatment T_i is allowed to cause:

- Changes in popularity within a group
- Movement to a new friend group
- Becoming a member of a new friend group while remaining in current friend group
- Friendships becoming more or less likely between distinct friend groups
- Combinations of the above

See Appendix of manuscript for details.

References

Levin, K., A. Lodhia, and E. Levina (2022). Recovering shared structure from multiple networks with unknown edge distributions. *Journal of Machine Learning Research* 23, 1–48.

VanderWeele, T. and S. Vansteelandt (2014, January). Mediation Analysis with Multiple Mediators. *Epidemiologic methods* 2(1), 95–115.