Estimating network-mediated causal effects via spectral embeddings

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Slides available at `https://tinyurl.com/ifds-alex`

This is joint work!

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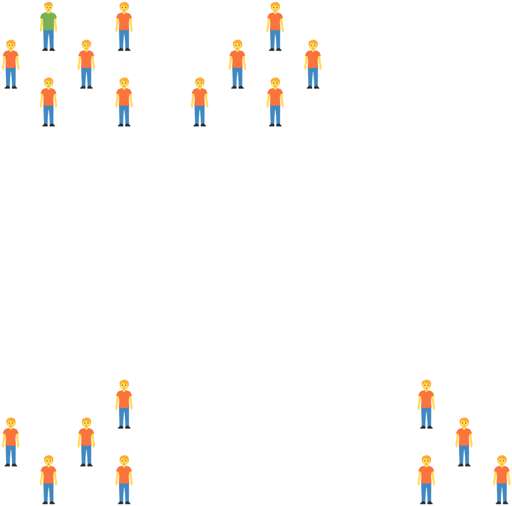
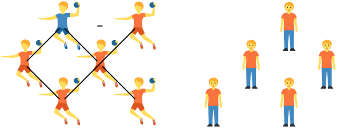
Two questions for the audience

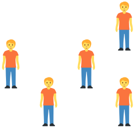
A short story about arriving in Madison & the Great Dane

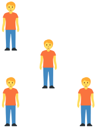
I didn’t know anyone when I first arrived here



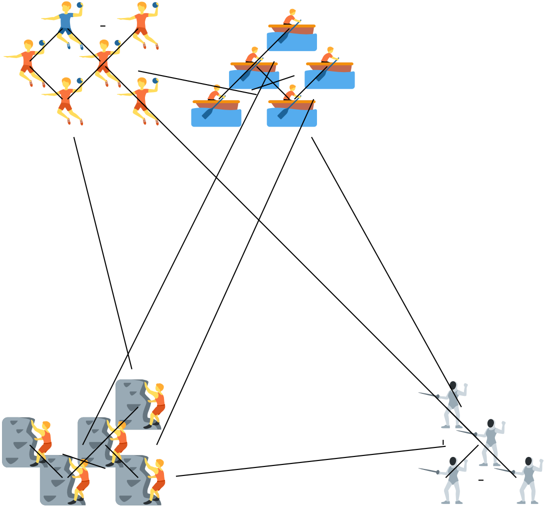
But! I like frisbee, so I joined a frisbee team!



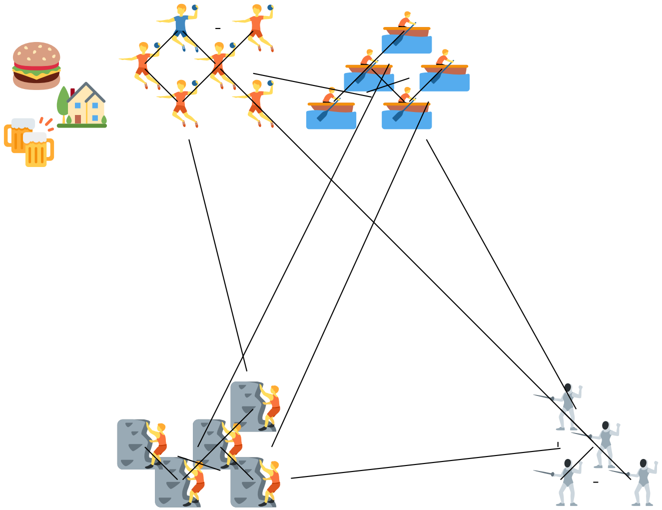




The Madison social network



Madison frisbee players go the Great Dane fairly often!

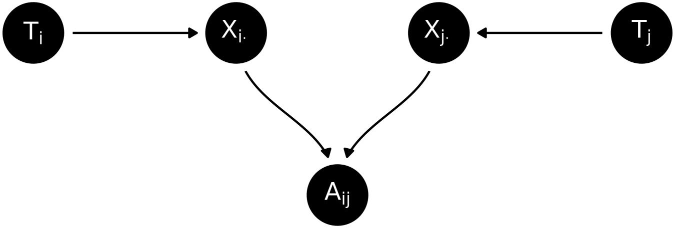


Idea: social groups mediate the causal effect of individual interests (e.g. frisbee) on visits to the Great Dane

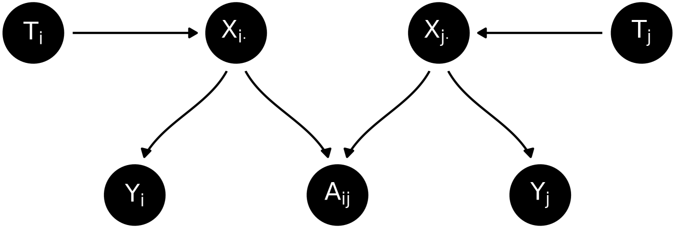
I like frisbee (T_i) so I join a frisbee team ($X_{i\cdot}$)

Assume there are only two people, person *i* and person *j*

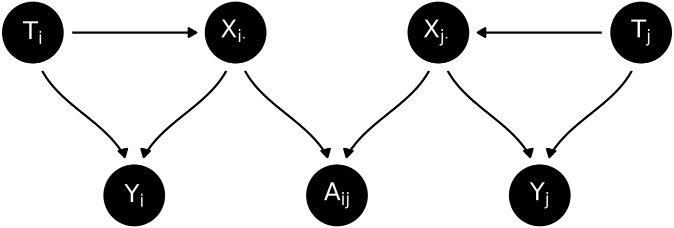
I’m on a frisbee team, so I form friendships (A_{ij}) with other people on my team



I'm on a frisbee team, so I go to the Great Dane (Y_i)

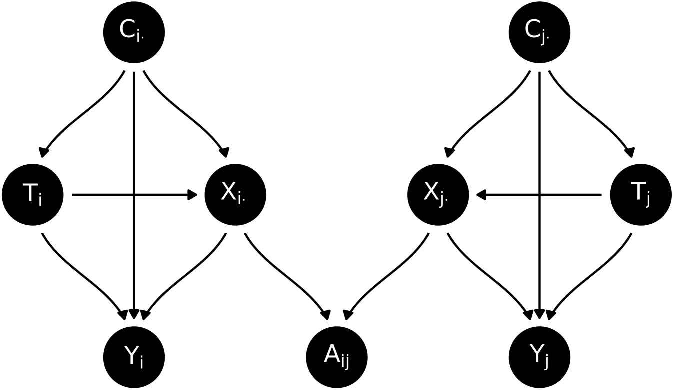


I like frisbee, and this might directly cause me to go to the Great Dane without my team



i.e. if I want to watch frisbee at the Great Dane

My individual choices might all be confounded



This is a good place to ask questions

Formalizing what I think happened with frisbee

First, a recap. I think:

1. enjoying frisbee caused me to visit the Great Dane more frequently than I would have otherwise, and
2. the mechanism proceeded in two stages: first I joined a frisbee team because I liked frisbee, and I went to the Great Dane with the team
3. I don't think liking frisbee caused me to go to the Great Dane independently of my frisbee team

Can disambiguate these causal effects using mediation analysis

Average treatment effects

- <u>Average treatment effect</u>: how much the outcome Y (Great Dane visits) would change on average if the treatment T were changed from $T = t$ (liking frisbee) to $T = t^*$ (not liking frisbee)

$$\Psi_{\text{ate}} = \mathbb{E}[Y_i(t) - Y_i(t^*)]$$

Note: $Y_i(t)$ is the counterfactual value of Y_i when T_i is set to t

I claim Ψ_{ate} is positive in my example

Natural indirect effects

- Natural indirect effect: how much the outcome Y (Great Dane visits) would change on average if the exposure were fixed at level $T = t^*$ (not liking frisbee) but the mediator X (friend group) were changed from the level it would take if $T = t$ (liking frisbee) to the level it would take if $T = t^*$ (not liking frisbee)

$$\Psi_{\text{nie}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t)) - Y_i(t, X_{i\cdot}(t^*))]$$

- Captures the effect of the exposure on the outcome that operates by changing the mediator

I claim Ψ_{nie} is positive in my example

Natural direct effects

- Natural direct effect: how much the outcome Y (Great Dane visits) would change if the exposure T were set at level $T = t^*$ (liking frisbee) versus $T = t$ (liking frisbee) but for each individual the mediator X (friend group) were kept at the level it would have taken, for that individual, if T had been set to t^* (not liking frisbee)

$$\Psi_{\mathrm{nde}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t^*)) - Y_i(t^*, X_{i\cdot}(t^*))]$$

- Captures the effect of the exposure on the outcome that would remain if we were to disable the pathway from the exposure to the mediator

I claim Ψ_{nde} is zero in my example.

Note $\Psi_{\mathrm{ate}} = \Psi_{\mathrm{nde}} + \Psi_{\mathrm{nie}}$.

If we knew the friend groups X, we could use standard tools for causal mediation analysis. But we don't know X! Instead, we observe the friendship network A

Semi-parametric network & network regression models

Intuition: stochastic blockmodels

d "blocks" or communities

$X_{i\cdot} \in \{0,1\}^d$ one-hot indicator of node i's block

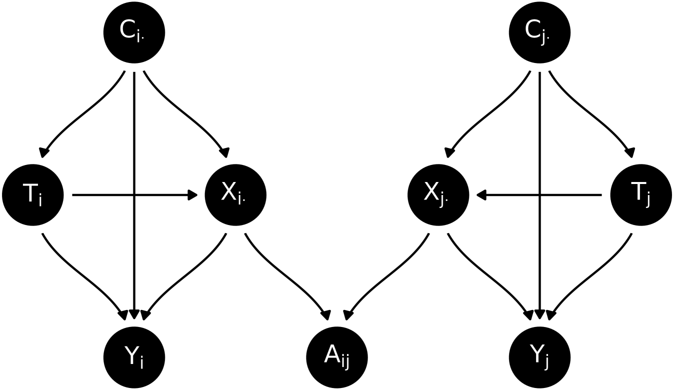
X is latent (i.e. unobserved)

$B \in [0,1]^{d \times d}$ inter-block edge probabilities

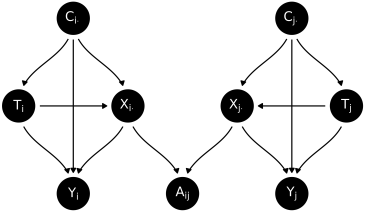
Friendships depend on group memberships and B

$\mathbb{P}\left(A_{ij} = 1 \,\middle|\, X\right) = X_{i\cdot} B X_{j\cdot}^T$

Returning to the structural causal model for a moment



A regression model for friend group membership

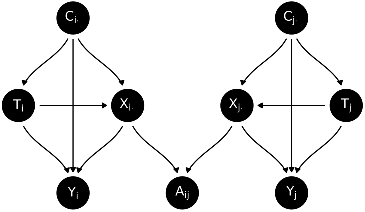


Idea: interventions T_i can cause community membership $X_{i\cdot}$

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1\times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_t}_{\mathbb{R}^{1\times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_c}_{\mathbb{R}^{p\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{tc}}_{\mathbb{R}^{p\times d}}.$$

Ex: I like frisbee so I joined an ultimate frisbee team (MUFA)

A regression model for outcomes



Idea: community membership $X_{i.}$ can cause outcomes Y_i

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i.}, X_{i.}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_{\mathsf{t}}}_{\mathbb{R}} + \underbrace{C_{i.}}_{\mathbb{R}^{1\times p}} \underbrace{\beta_{\mathsf{c}}}_{\mathbb{R}^p} + \underbrace{X_{i.}}_{\mathbb{R}^{1\times d}} \underbrace{\beta_{\mathsf{x}}}_{\mathbb{R}^d}$$

Ex: I'm on a frisbee team, and the frisbee team goes to the Great Dane together after each game

Semi-parametric causal identification

Recall the regression models:

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_{\mathsf{t}}}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\beta_{\mathsf{c}}}_{\mathbb{R}^{p}} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1\times d}} \underbrace{\beta_{\mathsf{x}}}_{\mathbb{R}^{d}},$$

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1\times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_{\mathsf{t}}}_{\mathbb{R}^{1\times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{\mathsf{c}}}_{\mathbb{R}^{p\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{\mathsf{tc}}}_{\mathbb{R}^{p\times d}}.$$

Then:

$$\Psi_{\mathrm{nde}}(t, t^*) = (t - t^*)\, \beta_{\mathsf{t}}$$
$$\Psi_{\mathrm{nie}}(t, t^*) = (t - t^*)\, \theta_{\mathsf{t}}\, \beta_{\mathsf{x}} + (t - t^*)\, \mu_C\, \Theta_{\mathsf{tc}}\, \beta_{\mathsf{x}}.$$

Estimation

Regression estimators

Challenge: regression models depend on X, but we never see X. Luckily we can estimate it!

Definition (ASE)
Given a network A, the $\widehat{d}$-dimensional adjacency spectral embedding of A is

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2}$$

where $\widehat{U}\widehat{S}\widehat{U}^T$ is the rank-$\widehat{d}$ truncated singular value decomposition of A.

Note that the analyst must specify $\widehat{d}$

Uniform consistency of the adjacency spectral embedding

Well-known that $\widehat{X}$ is a good estimate of X

Lemma (Levin et al. (2022))
Under a suitably well-behaved network model, if $\widehat{d}$ is correctly specified or consistently estimated, there is some $d \times d$ orthogonal matrix Q such that

$$\max_{i \in [n]} \left\| \widehat{X}_{i \cdot} - X_{i \cdot} Q \right\| = o_p(1).$$

$\widehat{X}$ can be plugged in for X just fine

Let $\widehat{D} = \begin{bmatrix} 1 & T & C & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$ and
$L = \begin{bmatrix} 1 & T & C & T \cdot C \end{bmatrix} \in \mathbb{R}^{n \times (2p+2)}$.

We estimate β_{w} and β_{x} via ordinary least squares as follows

$$\begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_{\mathsf{t}} \\ \widehat{\beta}_{\mathsf{c}} \\ \widehat{\beta}_{\mathsf{x}} \end{bmatrix} = \left(\widehat{D}^T \widehat{D} \right)^{-1} \widehat{D}^T Y.$$

Similarly, we estimate Θ via ordinary least squares as

$$\widehat{\Theta} = \left(L^T L \right)^{-1} L^T \widehat{X}.$$

Causal estimators

To estimate Ψ_{nde} and Ψ_{nie} in our semi-parametric setting, we combine regression coefficients from the network regression models:

$$\widehat{\Psi}_{\mathrm{cde}} = \widehat{\Psi}_{\mathrm{nde}} = (t - t^*)\,\widehat{\beta}_{\mathsf{t}} \qquad \text{and}$$
$$\widehat{\Psi}_{\mathrm{nie}} = (t - t^*)\,\widehat{\theta}_{\mathsf{t}}\,\widehat{\beta}_{\mathsf{x}} + (t - t^*) \cdot \widehat{\mu}_{c} \cdot \widehat{\Theta}_{\mathsf{tc}}\,\widehat{\beta}_{\mathsf{x}}.$$

It's standard to fit two regressions and multiply coefficients to estimate an indirect effect like this (VanderWeele and Vansteelandt, 2014).

Theorem (Regression coefficients are asymptotically normal)

Under some mild assumptions, there is some unknown matrix Q such that

$$\sqrt{n}\,\widehat{\Sigma}_{\beta}^{-1/2}\begin{pmatrix}\widehat{\beta}_W - \beta_W \\ Q\,\widehat{\beta}_X - \beta_X\end{pmatrix} \to \mathcal{N}(\mathbf{0}, I_d), \textit{and}$$

$$\sqrt{n}\,\widehat{\Sigma}_{\mathrm{vec}(\Theta)}^{-1/2}\left(\mathrm{vec}\left(\widehat{\Theta}\,Q^T\right) - \mathrm{vec}(\Theta)\right) \to \mathcal{N}\left(\mathbf{0}, I_{pd}\right).$$

where $\widehat{\Sigma}_{\mathrm{vec}(\Theta)}^{-1/2}$ *and* $\widehat{\Sigma}_{\beta}^{-1/2}$ *are the typical heteroscedasticity robust covariance estimators, with* $\widehat{X}$ *plugged in for* X*.*

Corollary

Theorem (Causal estimators are asymptotically normal)

Under the same statistical assumptions as before, plus mediating homophily,

$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nde}}}\left(\widehat{\Psi}_{\mathrm{nde}} - \Psi_{\mathrm{nde}}\right) \to \mathcal{N}(0,1),\ \textit{and}$$
$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nie}}}\left(\widehat{\Psi}_{\mathrm{nie}} - \Psi_{\mathrm{nie}}\right) \to \mathcal{N}(0,1).$$

where $\widehat{\sigma}^2_{\mathrm{nde}}$ *and* $\widehat{\sigma}^2_{\mathrm{nie}}$ *are rather unfriendly variance estimators derived via the delta method and the previous theorem.*

Data example (if there is time)

Glasgow data

TODO

Thank you! Questions?

Read the manuscript at `https://arxiv.org/abs/2212.12041`

Slides available at `https://tinyurl.com/ifds-alex`

R package netmediate

Stay in touch

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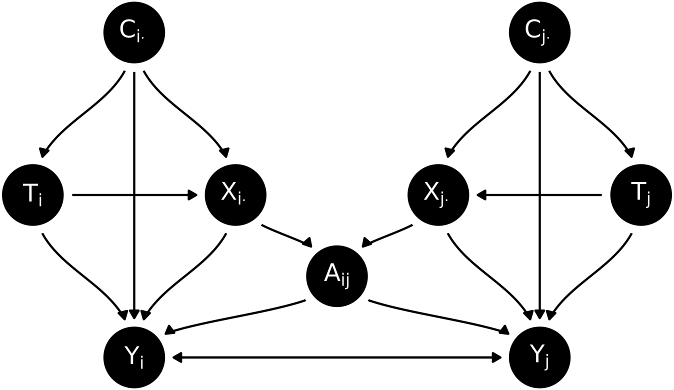
- `https://github.com/alexpghayes`

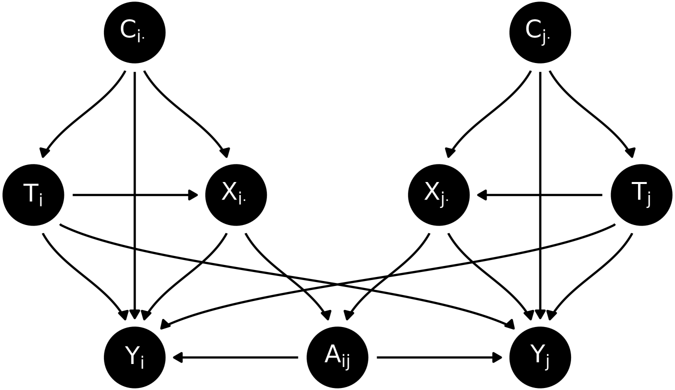
I'll be on the post-doc market in Fall 2023

Disambiguation: contagion is not allowed



Contagion ($Y_j \to Y_i$) is not allowed

Disambiguation: interference is not allowed



Interference ($T_j \rightarrow Y_i$) is not allowed

More on interference and contagion

Interference and contagion effects are allowed so long as they happen in the latent space. Suppose

$$\mathbb{E}[Y_i \mid W_{i\cdot}, X_{i\cdot}] = W_{i\cdot}\beta_{\mathsf{w}} + X_{i\cdot}\beta'_{\mathsf{x}} + \delta_{\mathsf{y}} \sum_j X_{i\cdot}^T X_{j\cdot} Y_j$$

This latent space contagion model is a special parametric case of the regression outcome model (take $\beta_{\mathsf{x}} = \beta'_{\mathsf{x}} + X^T Y \delta_{\mathsf{y}}$).

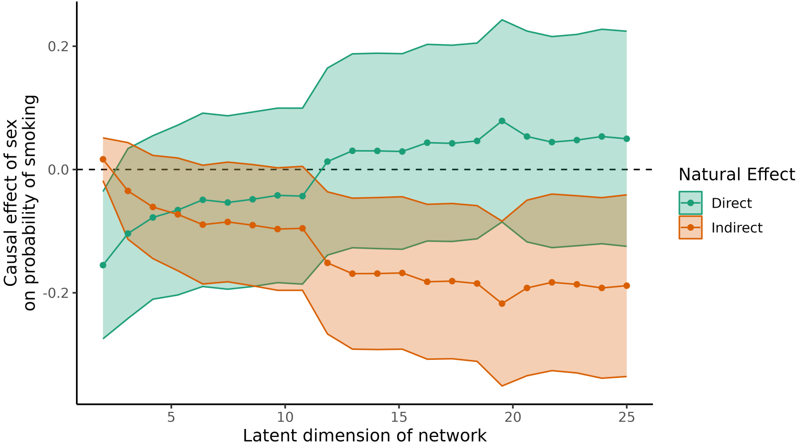
Semi-parametric network model

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \mid X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\mathrm{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.

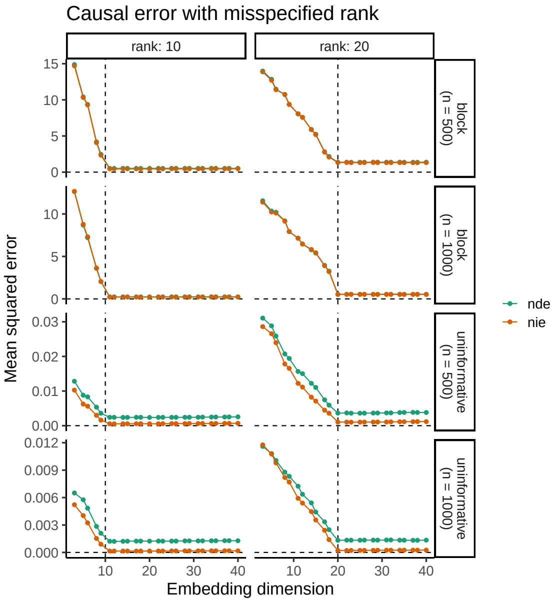
Semi-parametric network model: identification of X

$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q, the latent positions X are only identifiable up to an orthogonal transformation.

Choosing $\widehat{d}$: do a multiverse analysis



Choosing $\widehat{d}$: overestimating the embedding dimension is fine



Identifying assumptions

The random variables $(Y_i, Y_i(t,x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:
   if $T_i = t$, then $X_{i\cdot}(t) = X_{i\cdot}$ with probability 1, and
   if $T_i = t$ and $X_{i\cdot} = x$, then $Y_i(t,x) = Y_i$ with probability 1
2. Sequential ignorability:
   $$\{Y_i(t^*,x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot} \quad \text{and} \quad \{Y_i(t^*,x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$$
3. Positivity:
   $$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0 \text{ for each } x \in \mathrm{supp}(X_{i\cdot})$$
   $$\mathbb{P}(t \mid C_{i\cdot}) > 0 \text{ for each } t \in \mathrm{supp}(T_i)$$

Interventions allowed

Provided that controls $C_{i\cdot}$ are sufficiently informative about group membership $X_{i\cdot}$, treatment T_i is allowed to cause:

- Changes in popularity within a group
- Movement to a new friend group
- Becoming a member of a new friend group while remaining in current friend group
- Friendships becoming more or less likely between distinct friend groups
- Combinations of the above

See Appendix of manuscript for details.

References

Levin, K., A. Lodhia, and E. Levina (2022). Recovering shared structure from multiple networks with unknown edge distributions. Journal of Machine Learning Research 23, 1–48.

VanderWeele, T. and S. Vansteelandt (2014, January). Mediation Analysis with Multiple Mediators. Epidemiologic methods 2(1), 95–115.