Estimating network-mediated causal effects via spectral embeddings

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Motivating example: smoking in adolescent social networks

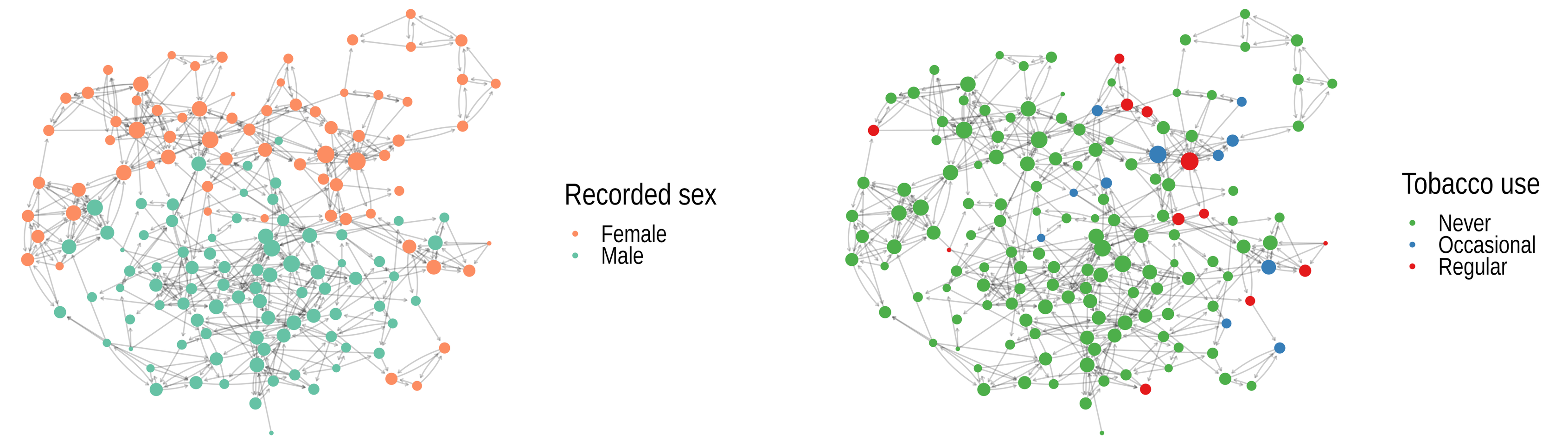

Figure 1. Directed friendships in a secondary school in Glasgow, reported in the Teenage Friends and Lifestyle Study (wave 1). Each node represents one student. An arrow from node i to node j indicates student i claimed student j as a friend. Node size is proportional to in-degree. On the left, the network is colored by sex. On the right, the network is colored by self-reported smoking frequency.

Notation & inferential targets

We assume we have a (symmetric) network with nodes $1, \ldots, n$.

Network	A	$\mathbb{R}^{n \times n}$
Treatment	T_i	$\{0,1\}$
Outcome	Y_i	$\mathbb{R}$
Confounders	$C_{i\cdot}$	$\mathbb{R}^p$
Friend group	$X_{i\cdot}$	$\mathbb{R}^d$

The *average treatment effect* Ψ_{ate} describes how much the outcome Y_i would change on average if the treatment T_i were changed from $T_i = t$ to $T_i = t^*$:

$$\Psi_{\text{ate}}(t, t^*) = \mathbb{E}[Y_i(t) - Y_i(t^*)].$$

The *natural direct effect* describes how much the outcome Y_i would change if the exposure T_i were set at level $T_i = t^*$ versus $T_i = t$ but for each individual the mediator $X_{i\cdot}$ were kept at the level it would have taken for that individual, had T_i been set to t^*:

$$\Psi_{\text{nde}}(t, t^*) = \mathbb{E}[Y_i(t, X_{i\cdot}(t^*)) - Y_i(t^*, X_{i\cdot}(t^*))],$$

The *natural indirect effect* describes how much the outcome Y_i would change on average if the exposure were fixed at level $T_i = t^*$ but the mediator $X_{i\cdot}$ were changed from the level it would take under $T_i = t$ to the level it would take under $T_i = t^*$

$$\Psi_{\text{nie}}(t, t^*) = \mathbb{E}[Y_i(t, X_{i\cdot}(t)) - Y_i(t, X_{i\cdot}(t^*))],$$

Structural causal model

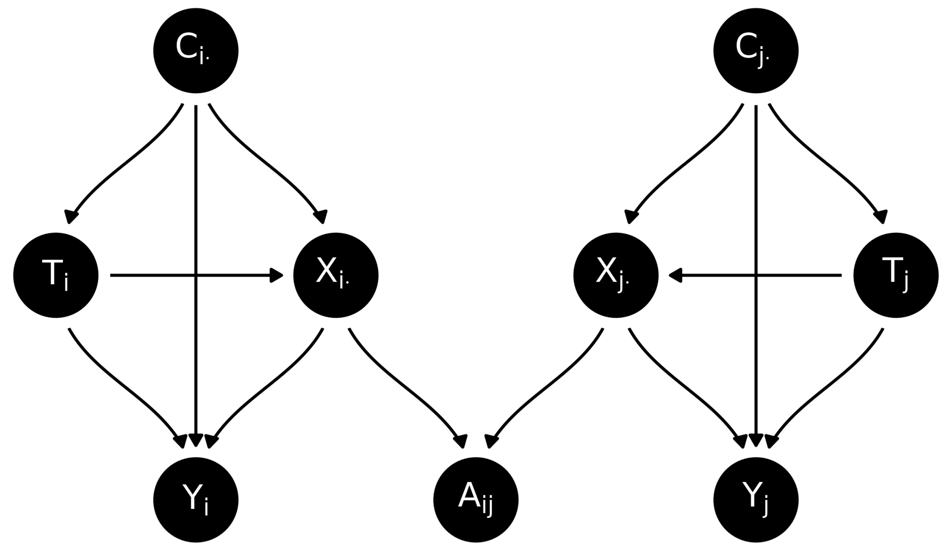

Figure 2. A directed acyclic graph (DAG) representing the causal pathways in a network with homophilous mediation, for node a network with two nodes called i and j. We are interested in the causal effect of T_i on Y_i as mediated by the latent position $X_{i\cdot}$.

Semi-parametric network model

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \mid X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\operatorname{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.

The outcome regression functional is linear in $T_i, C_{i\cdot}$, and $X_{i\cdot}$ and the mediator regression functional is linear in $T_i, C_{i\cdot}$, and $T_i \cdot C_{i\cdot}$:

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_{\text{t}}}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\beta_{\text{c}}}_{\mathbb{R}^p} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1\times d}} \underbrace{\beta_{\text{x}}}_{\mathbb{R}^d}, \qquad \text{(outcome model)}$$

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1\times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_{\text{t}}}_{\mathbb{R}^{1\times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{\text{c}}}_{\mathbb{R}^{p\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{\text{tc}}}_{\mathbb{R}^{p\times d}}. \qquad \text{(mediator model)}$$

Semi-parametric causal identification

Let μ_c denote the mean of $C_{i\cdot}$. Then,

$$\Psi_{\text{nde}}(t, t^*) = (t - t^*)\,\beta_{\text{t}}, \quad \text{and}$$
$$\Psi_{\text{nie}}(t, t^*) = (t - t^*)\,\theta_{\text{t}}\,\beta_{\text{x}} + (t - t^*)\,\mu_c\,\Theta_{\text{tc}}\,\beta_{\text{x}}.$$

Estimation challenge: friend groups X unknown!

Given a network with adjacency matrix A, the d-dimensional *adjacency spectral embedding* (ASE) of A is defined as

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2} \in \mathbb{R}^{n \times d},$$

where $\widehat{U}\widehat{S}\widehat{U}^T$ is the rank-d truncated singular value decomposition of A. That is, $\widehat{S} \in \mathbb{R}^{d \times d}$ is diagonal, with entries given by the d leading singular values of A, and $\widehat{U} \in \mathbb{R}^{n \times d}$ has the corresponding d orthonormal singular vectors as its columns.

Let $W = \begin{bmatrix} 1 & T & C \end{bmatrix} \in \mathbb{R}^{n \times (p+2)}$ and $L = \begin{bmatrix} W & T \cdot C \end{bmatrix} \in \mathbb{R}^{n \times (2p+2)}$.

Define $\widehat{D} = \begin{bmatrix} W & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$. We estimate β_{w} and β_{x} via ordinary least squares as follows

$$\begin{bmatrix} \widehat{\beta}_{\text{w}} \\ \widehat{\beta}_{\text{x}} \end{bmatrix} = \left(\widehat{D}^T \widehat{D}\right)^{-1} \widehat{D}^T Y.$$

Similarly, we estimate Θ via ordinary least squares as

$$\widehat{\Theta} = \left(L^T L\right)^{-1} L^T \widehat{X}.$$

Theory

$$\sqrt{n}\,\widehat{\Sigma}_{\text{vec}(\Theta)}^{-1/2} \left(\operatorname{vec}\left(\widehat{\Theta}\, Q_n^T\right) - \operatorname{vec}(\Theta)\right) \to \mathcal{N}(0, I_{pd}), \text{ and}$$

$$\sqrt{n}\,\widehat{\Sigma}_{\beta}^{-1/2} \begin{pmatrix} \widehat{\beta}_{\text{w}} - \beta_{\text{w}} \\ Q_n \widehat{\beta}_{\text{x}} - \beta_{\text{x}} \end{pmatrix} \to \mathcal{N}(0, I_d).$$

Results applied to Glasgow data

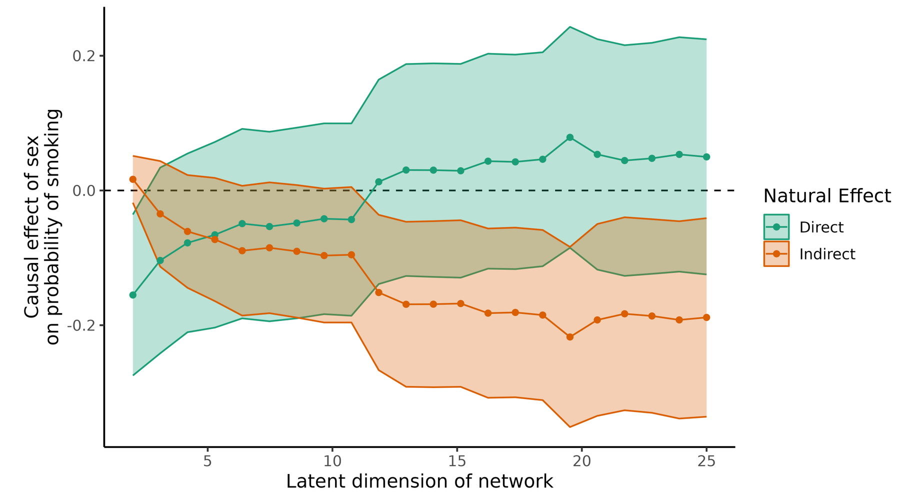

Figure 3. Estimated direct and indirect effects of sex on tobacco usage in the Glasgow social network. The estimated effects vary with the dimension d of the latent space, and are adjusted for possibly confounding by age and church attendance. Positive values indicate a greater propensity for adolescent boys to smoke, negative effects a greater propensity for adolescent girls to smoke.