

Estimating network-mediated causal effects via spectral embeddings

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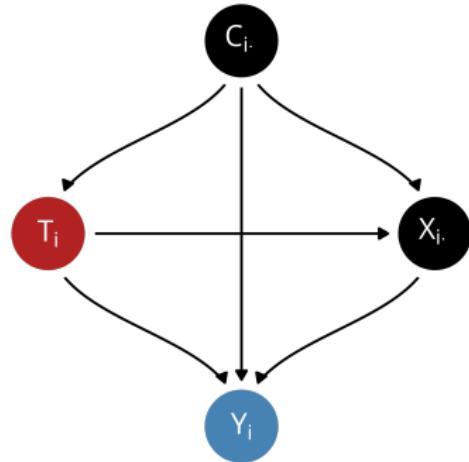
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Recent Developments in Causal Inference

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Causal mediation

Treatment	$T_i \in \{0, 1\}$
Outcome	$Y_i \in \mathbb{R}$
Mediators	$X_{i \cdot} \in \mathbb{R}^{1 \times d}$
Confounders	$C_{i \cdot} \in \mathbb{R}^{1 \times p}$



Decompose effect of T_i on Y_i :

1. Effect operating along $T_i \rightarrow Y_i$ path (direct)
2. Effect operating along $T_i \rightarrow X_{i \cdot} \rightarrow Y_i$ path (indirect)

Decomposing the average treatment effect

$Y_i(t)$: counterfactual value of Y_i when T_i is set to t . The average treatment effect is

$$\Psi_{\text{ate}} = \mathbb{E}[Y_i(t) - Y_i(t^*)] = \Psi_{\text{nde}} + \Psi_{\text{nie}},$$

which decomposes into the natural direct effect (not operating through $X_{i\cdot}$) and natural indirect effect (operating through $X_{i\cdot}$)

$$\Psi_{\text{nde}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t^*)) - Y_i(t^*, X_{i\cdot}(t^*))]$$

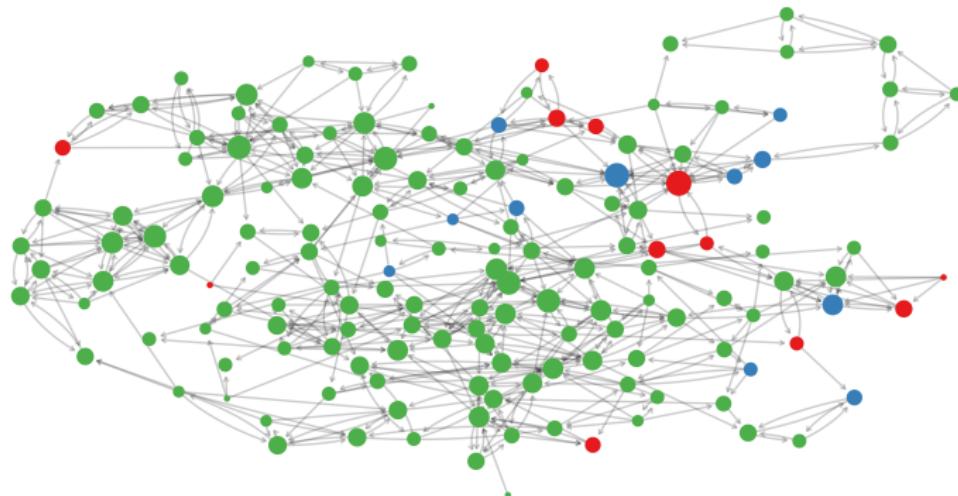
$$\Psi_{\text{nie}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t)) - Y_i(t, X_{i\cdot}(t^*))]$$

Effect of sex on smoking in adolescent social networks mediated by friend group!



Recorded sex • Female • Male

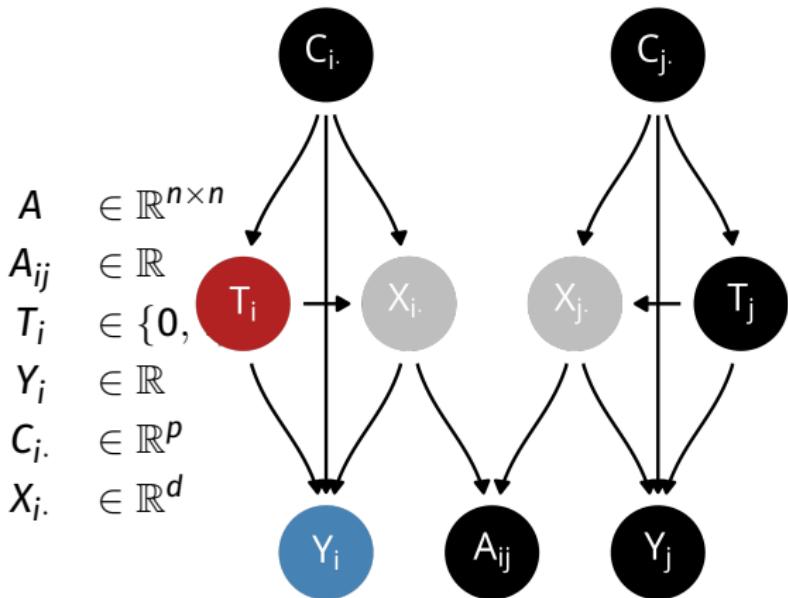
Effect of sex on smoking in adolescent social networks mediated by friend group!



Tobacco use • Never • Occasional • Regular

Mediation on a social network

Adjacency matrix	$A \in \mathbb{R}^{n \times n}$
Edge $i \sim j$	$A_{ij} \in \mathbb{R}$
Treatment	$T_i \in \{0, 1\}$
Outcome	$Y_i \in \mathbb{R}$
Confounding	$C_{i \cdot} \in \mathbb{R}^p$
Friend group	$X_{i \cdot} \in \mathbb{R}^d$



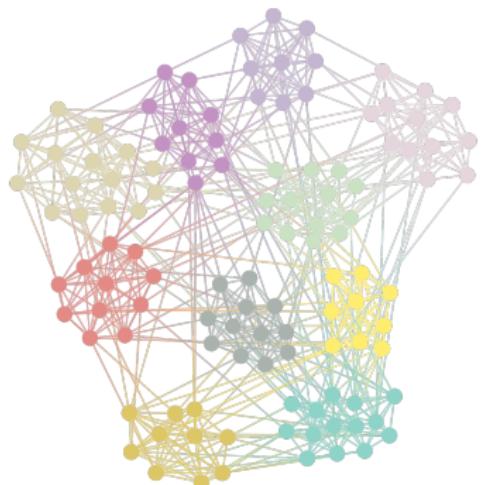
A directed acyclic graph (DAG) representing the causal pathways in a network with homophilous mediation, for a network with two nodes called i and j .

The challenge: the friend groups X_i are not observed!

Solution: estimate friend groups X_i using community models for networks!

Semi-parametric network & network regression models

Intuition: stochastic blockmodels



d “blocks” or communities

$X_{i \cdot} \in \{0, 1\}^d$ one-hot indicator of node i 's block

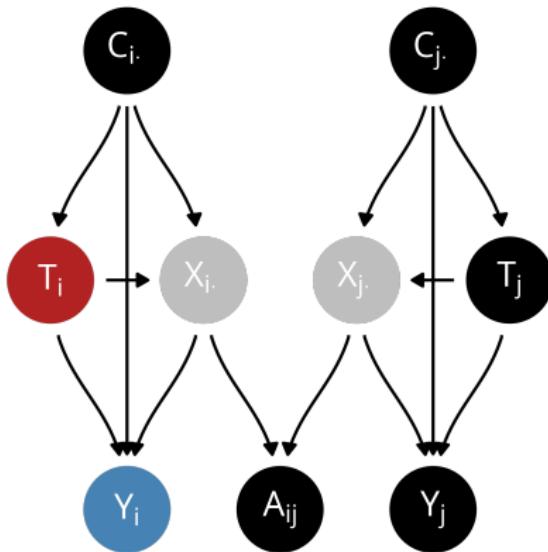
X is latent (i.e. unobserved)

$B \in [0, 1]^{d \times d}$ inter-block edge probabilities

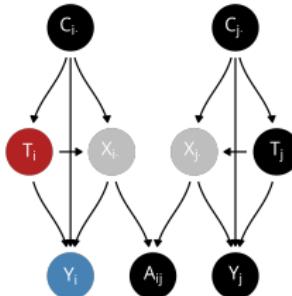
Friendships depend on group memberships and B

$$\mathbb{P}(A_{ij} = 1 \mid X) = X_{i \cdot} B X_{j \cdot}^T$$

Returning to the structural causal model for a moment



A regression model for friend group membership

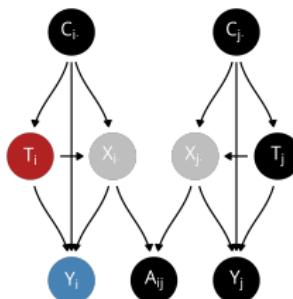


Idea: interventions T_i can cause community membership X_i .

$$\underbrace{\mathbb{E}[X_{i \cdot} | T_i, C_{i \cdot}]}_{\mathbb{R}^{1 \times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1 \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_t}_{\mathbb{R}^{1 \times d}} + \underbrace{C_{i \cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_c}_{\mathbb{R}^{p \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i \cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_{tc}}_{\mathbb{R}^{p \times d}}.$$

Ex: I like frisbee so I joined an ultimate frisbee team (MUFA)

A regression model for outcomes



Idea: community membership $X_{i\cdot}$ can cause outcomes Y_i

$$\underbrace{\mathbb{E}[Y_i | T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_t}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\beta_c}_{\mathbb{R}^p} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1 \times d}} \underbrace{\beta_x}_{\mathbb{R}^d}$$

Ex: I'm on a frisbee team, and the frisbee team goes to the Great Dane together after each game

Semi-parametric causal identification

Recall the regression models:

$$\underbrace{\mathbb{E}[Y_i | T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_t}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\beta_c}_{\mathbb{R}^p} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1 \times d}} \underbrace{\beta_x}_{\mathbb{R}^d},$$

$$\underbrace{\mathbb{E}[X_{i\cdot} | T_i, C_{i\cdot}]}_{\mathbb{R}^{1 \times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1 \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_t}_{\mathbb{R}^{1 \times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_c}_{\mathbb{R}^{p \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_{tc}}_{\mathbb{R}^{p \times d}}.$$

Then:

$$\Psi_{nde}(t, t^*) = (t - t^*) \beta_t$$

$$\Psi_{nie}(t, t^*) = (t - t^*) \theta_t \beta_x + (t - t^*) \mu_c \Theta_{tc} \beta_x.$$

Estimation

Regression estimators

Challenge: regression models depend on X , but we never see X . Luckily we can estimate it!

Definition (ASE)

Given a network A , the \hat{d} -dimensional adjacency spectral embedding of A is

$$\hat{X} = \hat{U}\hat{S}^{1/2}$$

where $\hat{U}\hat{S}\hat{U}^T$ is the rank- \hat{d} truncated singular value decomposition of A .

Note that the analyst must specify \hat{d}

Uniform consistency of the adjacency spectral embedding

Well-known that \widehat{X} is a good estimate of X

Lemma

Under a suitably well-behaved network model, if \widehat{d} is correctly specified or consistently estimated, there is some $d \times d$ orthogonal matrix Q such that

$$\max_{i \in [n]} \left\| \widehat{X}_{i \cdot} - X_{i \cdot} Q \right\| = o_p(1).$$

\hat{X} can be plugged in for X just fine

Let $\hat{D} = \begin{bmatrix} 1 & T & C & \hat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$ and
 $L = \begin{bmatrix} 1 & T & C & T \cdot C \end{bmatrix} \in \mathbb{R}^{n \times (2p+2)}.$

We estimate β_w and β_x via ordinary least squares as follows

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_t \\ \hat{\beta}_c \\ \hat{\beta}_x \end{bmatrix} = (\hat{D}^T \hat{D})^{-1} \hat{D}^T Y.$$

Similarly, we estimate Θ via ordinary least squares as

$$\hat{\Theta} = (L^T L)^{-1} L^T \hat{X}.$$

Causal estimators

To estimate Ψ_{nde} and Ψ_{nie} in our semi-parametric setting, we combine regression coefficients from the network regression models:

$$\begin{aligned}\hat{\Psi}_{cde} &= \hat{\Psi}_{nde} = (t - t^*) \hat{\beta}_t && \text{and} \\ \hat{\Psi}_{nie} &= (t - t^*) \hat{\theta}_t \hat{\beta}_x + (t - t^*) \cdot \hat{\mu}_c \cdot \hat{\Theta}_{tc} \hat{\beta}_x.\end{aligned}$$

It's standard to fit two regressions and multiply coefficients to estimate an indirect effect like this ([VanderWeele and Vansteelandt, 2014](#)).

Main result

Theorem (Regression coefficients are asymptotically normal)

Under some mild assumptions, there is some unknown matrix Q such that

$$\sqrt{n} \widehat{\Sigma}_{\beta}^{-1/2} \begin{pmatrix} \widehat{\beta}_w - \beta_w \\ Q \widehat{\beta}_x - \beta_x \end{pmatrix} \rightarrow \mathcal{N}(0, I_d), \text{ and}$$

$$\sqrt{n} \widehat{\Sigma}_{\text{vec}(\Theta)}^{-1/2} \left(\text{vec}(\widehat{\Theta} Q^T) - \text{vec}(\Theta) \right) \rightarrow \mathcal{N}(0, I_{pd}).$$

where $\widehat{\Sigma}_{\text{vec}(\Theta)}^{-1/2}$ and $\widehat{\Sigma}_{\beta}^{-1/2}$ are the typical heteroscedasticity robust covariance estimators, with \widehat{X} plugged in for X .

Corollary

Theorem (Causal estimators are asymptotically normal)

Under the same statistical assumptions as before, plus mediating homophily,

$$\sqrt{n \hat{\sigma}_{\text{nde}}^2} (\hat{\Psi}_{\text{nde}} - \Psi_{\text{nde}}) \rightarrow \mathcal{N}(0, 1), \text{ and}$$

$$\sqrt{n \hat{\sigma}_{\text{nie}}^2} (\hat{\Psi}_{\text{nie}} - \Psi_{\text{nie}}) \rightarrow \mathcal{N}(0, 1).$$

where $\hat{\sigma}_{\text{nde}}^2$ and $\hat{\sigma}_{\text{nie}}^2$ are rather unfriendly variance estimators derived via the delta method and the previous theorem.

Thank you! Questions?

Read the manuscript at

<https://arxiv.org/abs/2212.12041>

R package [netmediate](#)

Stay in touch

 [@alexphayes](#)

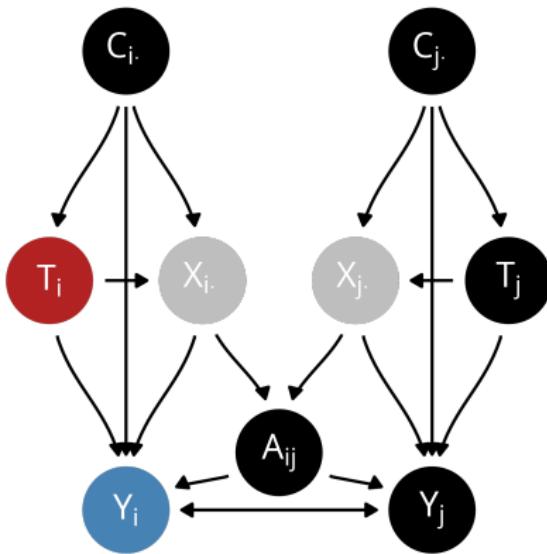
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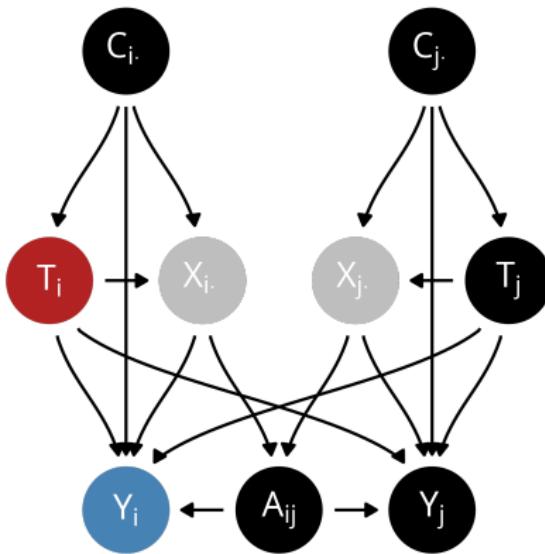
I'm looking for a post-doc, say hi if this work interests you!

Disambiguation: contagion is not allowed



Contagion ($Y_j \rightarrow Y_i$) is not allowed

Disambiguation: interference is not allowed



Interference ($T_j \rightarrow Y_i$) is not allowed

More on interference and contagion

Interference and contagion effects are allowed so long as they happen in the latent space. Suppose

$$\mathbb{E}[Y_i | W_{i\cdot}, X_{i\cdot}] = W_{i\cdot}\beta_w + X_{i\cdot}\beta'_x + \delta_y \sum_j X_{i\cdot}^T X_{j\cdot} Y_j$$

This latent space contagion model is a special parametric case of the regression outcome model (take $\beta_x = \beta'_x + X^T Y \delta_y$).

Semi-parametric network model

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let

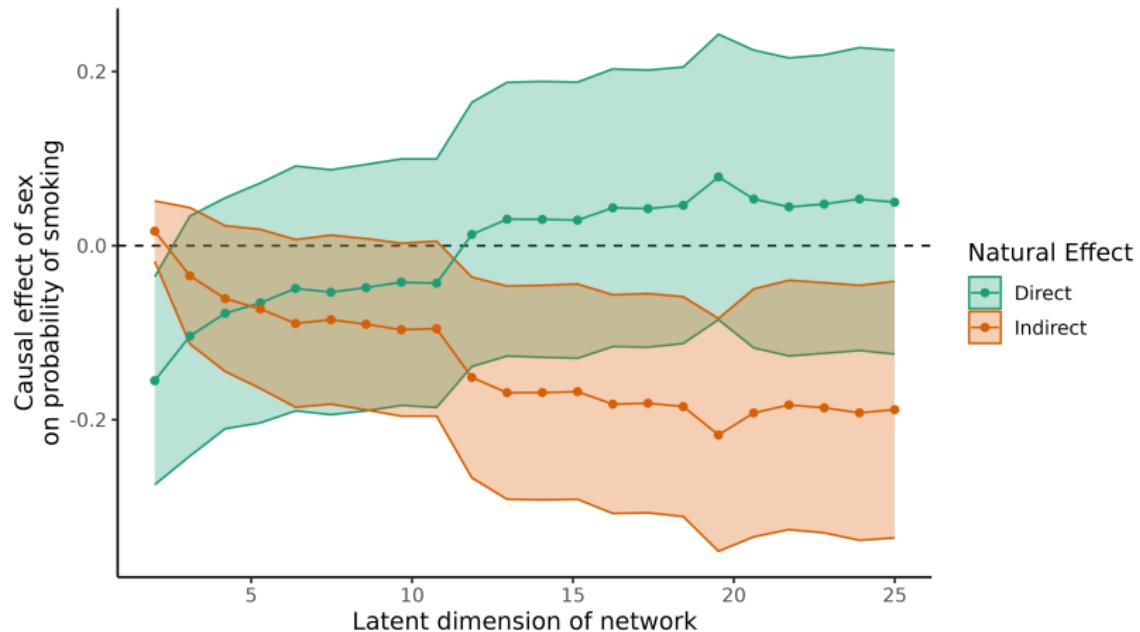
$P = \mathbb{E}[A | X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows X_1, \dots, X_n . That is, P has $\text{rank}(P) = d$ and is positive semi-definite with eigenvalues

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0 = \lambda_{d+1} = \dots = \lambda_n$. Conditional on X , the upper-triangular elements of $A - P$ are independent (ν_n, b_n) -sub-gamma random variables.

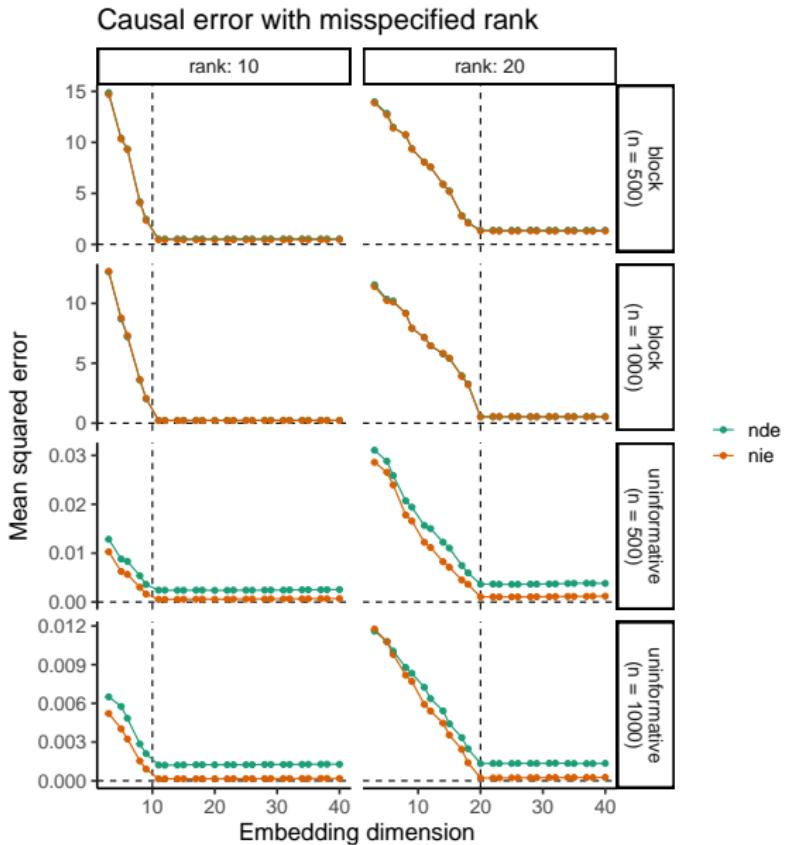
Semi-parametric network model: identification of X

$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q , the latent positions X are only identifiable up to an orthogonal transformation.

Choosing \hat{d} : do a multiverse analysis



Choosing \hat{d} : overestimating the embedding dimension is fine



Identifying assumptions

The random variables $(Y_i, Y_i(t, x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:

if $T_i = t$, then $X_{i\cdot}(t) = X_{i\cdot}$ with probability 1, and

if $T_i = t$ and $X_{i\cdot} = x$, then $Y_i(t, x) = Y_i$ with probability 1

2. Sequential ignorability:

$\{Y_i(t^*, x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot}$ and $\{Y_i(t^*, x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$

3. Positivity:

$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0$ for each $x \in \text{supp}(X_{i\cdot})$

$\mathbb{P}(t \mid C_{i\cdot}) > 0$ for each $t \in \text{supp}(T_i)$

Interventions allowed

Provided that controls C_i are sufficiently informative about group membership $X_{i.}$, treatment T_i is allowed to cause:

- Changes in popularity within a group
- Movement to a new friend group
- Becoming a member of a new friend group while remaining in current friend group
- Friendships becoming more or less likely between distinct friend groups
- Combinations of the above

See Appendix of manuscript for details.

References

VanderWeele, T. and S. Vansteelandt (2014, January).
Mediation Analysis with Multiple Mediators. Epidemiologic
methods 2(1), 95–115.