

Estimating network-mediated causal effects via spectral embeddings

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2023-08-09 @ JSM 2023

Recent Developments in Causal Inference

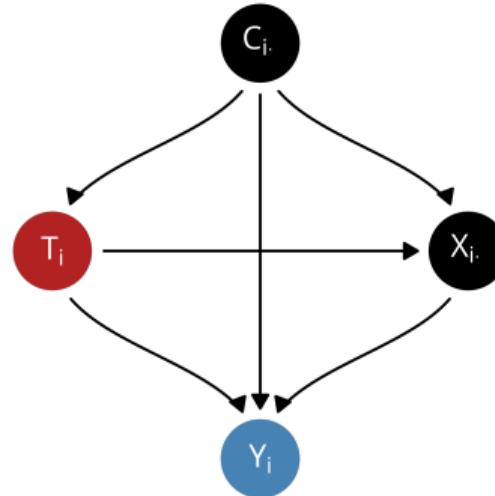
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Causal mediation

Treatment	$T_i \in \{0, 1\}$
Outcome	$Y_i \in \mathbb{R}$
Mediators	$X_{i \cdot} \in \mathbb{R}^{1 \times d}$
Confounders	$C_{i \cdot} \in \mathbb{R}^{1 \times p}$

Decompose effect of T_i on Y_i :

1. Effect operating along $T_i \rightarrow Y_i$ path
(direct)
2. Effect operating along $T_i \rightarrow X_{i \cdot} \rightarrow Y_i$ path (indirect)



$$\Psi_{\text{ate}} = \Psi_{\text{nde}} + \Psi_{\text{nie}}$$

$$\Psi_{\text{nde}} = \mathbb{E}[Y_i(t, X_{i \cdot}(t^*)) - Y_i(t^*, X_{i \cdot}(t^*))]$$

$$\Psi_{\text{nie}} = \mathbb{E}[Y_i(t, X_{i \cdot}(t)) - Y_i(t, X_{i \cdot}(t^*))]$$

Social networks as mediators

Mediation in an adolescent social network



Recorded sex ● Female ● Male



Tobacco use ● Never ● Occasional ● Regular

Teenage Friends and Lifestyle Study (wave 1), Glasgow, 1996

We propose a model where social groups in networks mediate causal effects

Adjacency matrix

$$A \in \mathbb{R}^{n \times n}$$

Edge $i \sim j$

$$A_{ij} \in \mathbb{R}$$

Treatment

$$T_i \in \{0, 1\}$$

Outcome

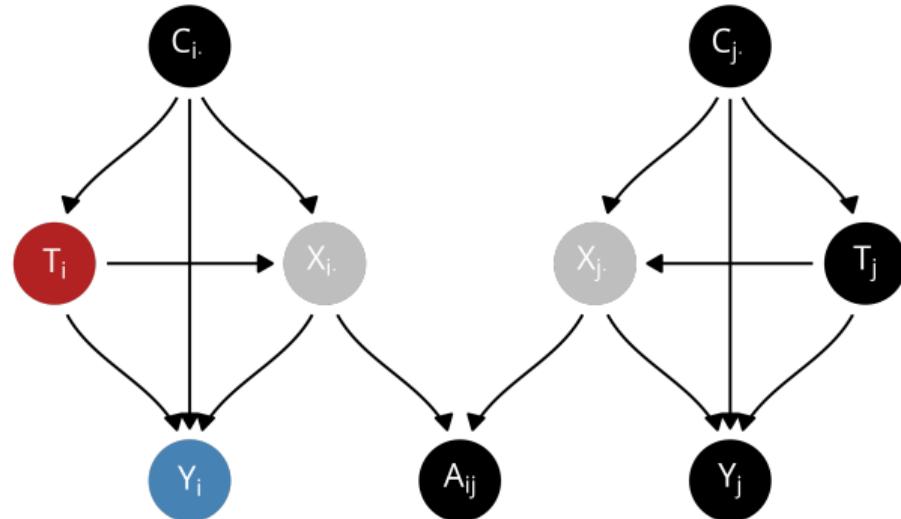
$$Y_i \in \mathbb{R}$$

Confounding

$$C_{i \cdot} \in \mathbb{R}^{1 \times p}$$

Friend group (latent)

$$X_{i \cdot} \in \mathbb{R}^{1 \times d}$$



Structural causal model for network mediation in a network with two nodes i and j

The challenge: the friend groups X_i are not observed!

Solution: estimate friend groups X_i using community models!

We use a semi-parametric estimator that accommodates:

- Stochastic blockmodels
- Degree-corrected stochastic blockmodels
- Mixed-membership stochastic blockmodels
- Overlapping stochastic blockmodels
- Random dot product graphs
- Etc

Intuition: stochastic blockmodels



d communities or “blocks”

$X_{i \cdot} \in \{0, 1\}^d$ one-hot indicator of node i 's block

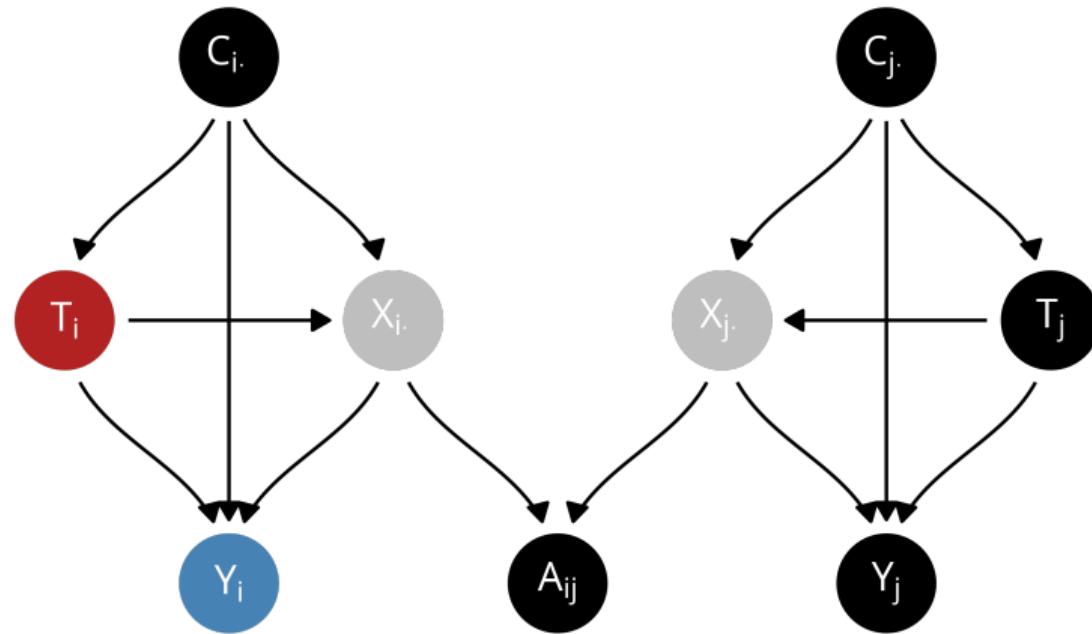
X is latent (i.e., unobserved)

$B \in [0, 1]^{d \times d}$ inter-block edge probabilities

Friendships depend on group memberships and B

$$\mathbb{P}(A_{ij} = 1 \mid X) = X_{i \cdot} B X_{j \cdot}^T$$

Returning to the structural causal model for a moment



Don't know X but can estimate it!

Definition (ASE)

Given a network A , the d -dimensional adjacency spectral embedding of A is

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2}$$

where $\widehat{U}\widehat{S}\widehat{U}^T$ is the rank- d truncated singular value decomposition of A .

Lemma

Under a suitable network model, there is a $d \times d$ orthogonal matrix Q such that

$$\max_{i \in [n]} \left\| \widehat{X}_{i \cdot} - X_{i \cdot} Q \right\| = o_p(1).$$

Note that we must correctly specify d

Semi-parametric regression models for mediation

If the previous DAG is correct, under the additional assumption that:

$$\underbrace{\mathbb{E}[Y_i | T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_t}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\beta_c}_{\mathbb{R}^p} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1 \times d}} \underbrace{\beta_x}_{\mathbb{R}^d},$$

$$\underbrace{\mathbb{E}[X_{i\cdot} | T_i, C_{i\cdot}]}_{\mathbb{R}^{1 \times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1 \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_t}_{\mathbb{R}^{1 \times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_c}_{\mathbb{R}^{p \times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1 \times p}} \underbrace{\Theta_{tc}}_{\mathbb{R}^{p \times d}}.$$

Then:

$$\Psi_{\text{nde}}(t, t^*) = (t - t^*) \beta_t$$

$$\Psi_{\text{nie}}(t, t^*) = (t - t^*) \theta_t \beta_x + (t - t^*) \mu_c \Theta_{tc} \beta_x.$$

Estimation: plug \widehat{X} into least squares

Let $\widehat{D} = \begin{bmatrix} 1 & T & C & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$ and $L = \begin{bmatrix} 1 & T & C & T \cdot C \end{bmatrix} \in \mathbb{R}^{n \times (2p+2)}$.

$$\begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_t \\ \widehat{\beta}_c \\ \widehat{\beta}_x \end{bmatrix} = (\widehat{D}^T \widehat{D})^{-1} \widehat{D}^T Y \quad \text{and} \quad \widehat{\Theta} = (L^T L)^{-1} L^T \widehat{X}.$$

$$\widehat{\Psi}_{\text{nde}} = (t - t^*) \widehat{\beta}_t \quad \text{and}$$

$$\widehat{\Psi}_{\text{nie}} = (t - t^*) \widehat{\theta}_t \widehat{\beta}_x + (t - t^*) \cdot \widehat{\mu}_c \cdot \widehat{\Theta}_{tc} \widehat{\beta}_x.$$

Main result

Theorem (Regression coefficients are asymptotically normal)

Under a suitably well-behaved network model and some moment conditions on regression errors, there is an unknown orthogonal matrix Q such that

$$\sqrt{n} \widehat{\Sigma}_{\beta}^{-1/2} \begin{pmatrix} \widehat{\beta}_w - \beta_w \\ Q \widehat{\beta}_x - \beta_x \end{pmatrix} \rightarrow \mathcal{N}(0, I_d), \text{ and}$$

$$\sqrt{n} \widehat{\Sigma}_{\text{vec}(\Theta)}^{-1/2} \left(\text{vec}(\widehat{\Theta} Q^T) - \text{vec}(\Theta) \right) \rightarrow \mathcal{N}(0, I_{pd}).$$

where $\widehat{\Sigma}_{\text{vec}(\Theta)}^{-1/2}$ and $\widehat{\Sigma}_{\beta}^{-1/2}$ are the typical heteroscedasticity robust covariance estimators, with \widehat{X} plugged in for X .

Corollary

Theorem (Causal estimators are asymptotically normal)

Under the same statistical assumptions as before, plus mediating homophily,

$$\sqrt{n \hat{\sigma}_{\text{nde}}^2} (\hat{\Psi}_{\text{nde}} - \Psi_{\text{nde}}) \rightarrow \mathcal{N}(0, 1), \text{ and}$$
$$\sqrt{n \hat{\sigma}_{\text{nie}}^2} (\hat{\Psi}_{\text{nie}} - \Psi_{\text{nie}}) \rightarrow \mathcal{N}(0, 1).$$

where $\hat{\sigma}_{\text{nde}}^2$ and $\hat{\sigma}_{\text{nie}}^2$ are rather unfriendly variance estimators derived via the delta method and the previous theorem.

Application to Glasgow data

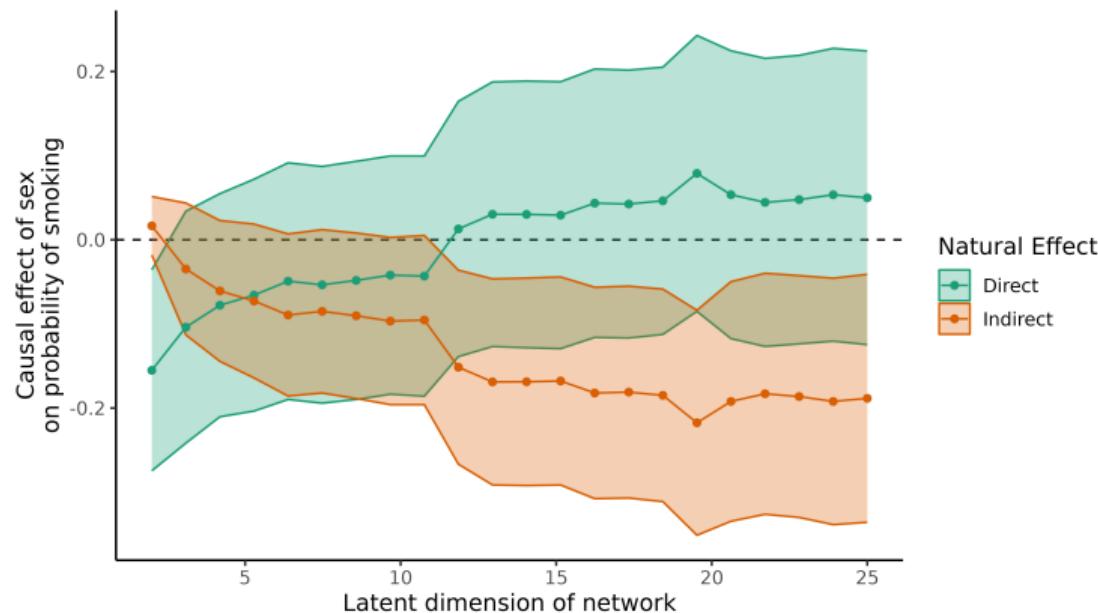


Figure 1: Estimated direct and indirect effects of sex on tobacco usage in the Glasgow social network, adjusted for age and church attendance. Positive values indicate a greater propensity for adolescent boys to smoke, negative effects a greater propensity for adolescent girls to smoke.

Thank you! Questions?

Read the manuscript at <https://arxiv.org/abs/2212.12041>

R package [netmediate](#)

Stay in touch

 [@alexphayes](#)

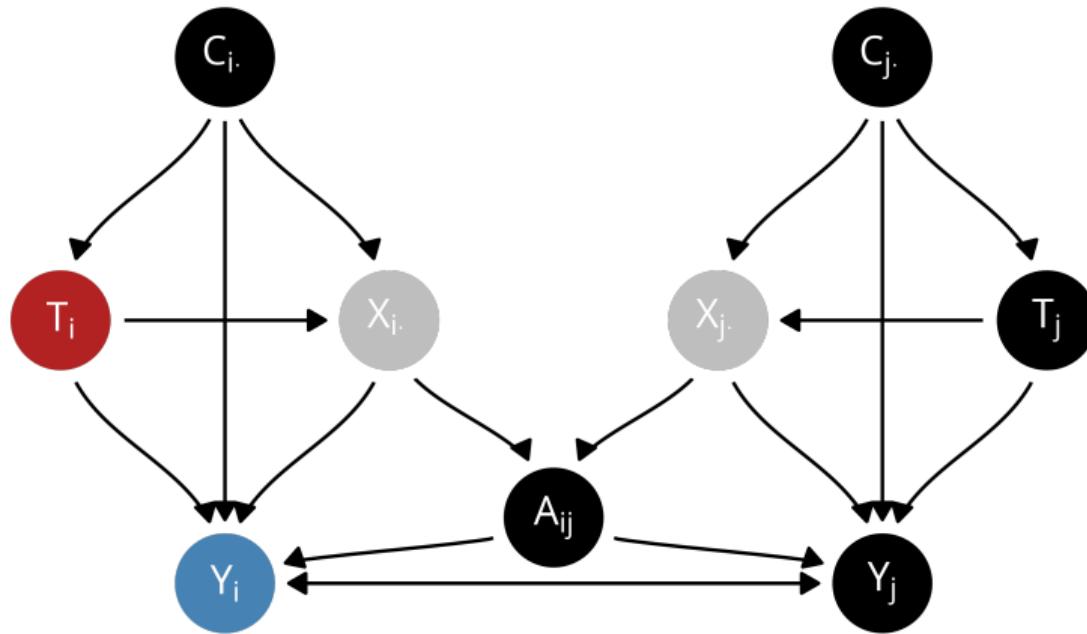
 alex.hayes@wisc.edu

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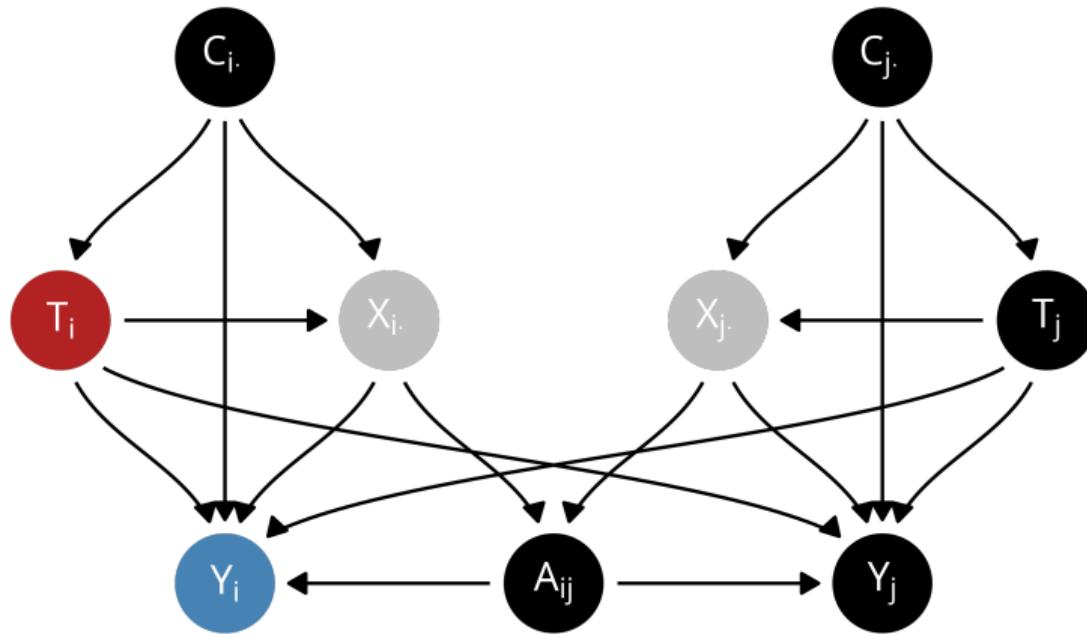
 <https://github.com/alexphayes>

I'm looking for a post-doc starting Fall 2024, say hi if this work interests you!

Disambiguation: contagion ($Y_j \rightarrow Y_i$) is not allowed



Disambiguation: interference ($T_j \rightarrow Y_i$) is not allowed



More on interference and contagion

Interference and contagion effects are allowed so long as they happen in the latent space. Suppose

$$\mathbb{E}[Y_i | W_{i\cdot}, X_{i\cdot}] = W_{i\cdot}\beta_w + X_{i\cdot}\beta'_x + \delta_y \sum_j X_{i\cdot}^T X_{j\cdot} Y_j$$

This latent space contagion model is a special parametric case of the regression outcome model (take $\beta_x = \beta'_x + X^T Y \delta_y$).

Semi-parametric network model

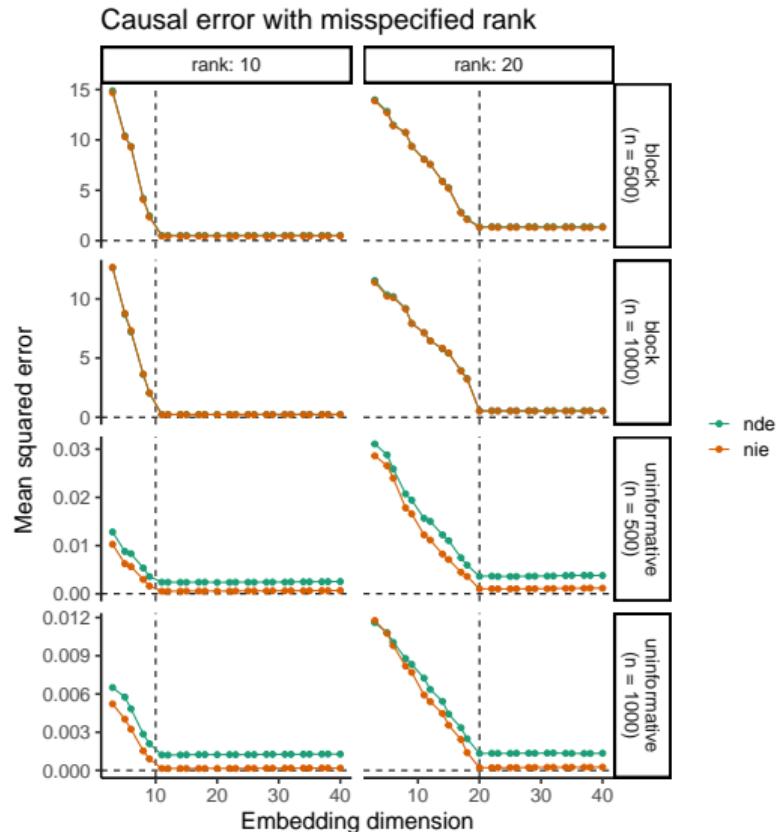
Definition

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A | X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows X_1, \dots, X_n . That is, P has $\text{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0 = \lambda_{d+1} = \dots = \lambda_n$. Conditional on X , the upper-triangular elements of $A - P$ are independent (ν_n, b_n) -sub-gamma random variables.

Remark

$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q , the latent positions X are only identifiable up to an orthogonal transformation.

Choosing \hat{d} : overestimating the embedding dimension is fine



Identifying assumptions

Definition

The random variables $(Y_i, Y_i(t, x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:

if $T_i = t$, then $X_{i\cdot}(t) = X_{i\cdot}$ with probability 1, and

if $T_i = t$ and $X_{i\cdot} = x$, then $Y_i(t, x) = Y_i$ with probability 1

2. Sequential ignorability:

$$\{Y_i(t^*, x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot} \quad \text{and} \quad \{Y_i(t^*, x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$$

3. Positivity:

$$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0 \text{ for each } x \in \text{supp}(X_{i\cdot})$$

$$\mathbb{P}(t \mid C_{i\cdot}) > 0 \text{ for each } t \in \text{supp}(T_i)$$

Interventions allowed

Provided that controls $C_i.$ are sufficiently informative about group membership $X_i.$, treatment T_i is allowed to cause:

- Changes in popularity within a group
- Movement to a new friend group
- Becoming a member of a new friend group while remaining in current friend group
- Friendships becoming more or less likely between distinct friend groups
- Combinations of the above

See Appendix of manuscript for details.