Estimating network-mediated causal effects via spectral embeddings

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Recent Developments in Causal Inference

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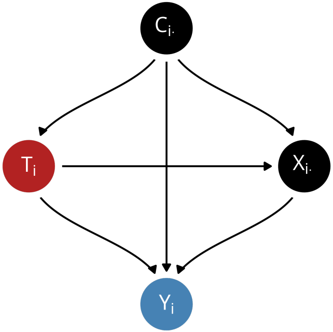
Causal mediation

Treatment	T_i	$\in \{0, 1\}$
Outcome	Y_i	$\in \mathbb{R}$
Mediators	$X_{i\cdot}$	$\in \mathbb{R}^{1 \times d}$
Confounders	$C_{i\cdot}$	$\in \mathbb{R}^{1 \times p}$



Decompose effect of T_i on Y_i:

1. Effect operating along $T_i \to Y_i$ path (direct)
2. Effect operating along $T_i \to X_{i\cdot} \to Y_i$ path (indirect)

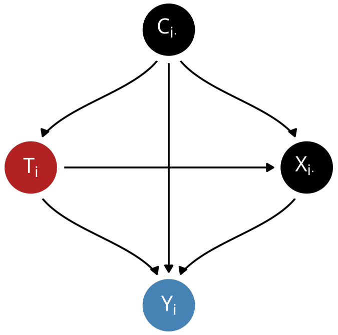
Decomposing the average treatment effect

$Y_i(t)$: counterfactual value of Y_i when T_i is set to t. Average treatment effect:

$$\Psi_{\text{ate}} = \mathbb{E}[Y_i(t) - Y_i(t^*)]$$

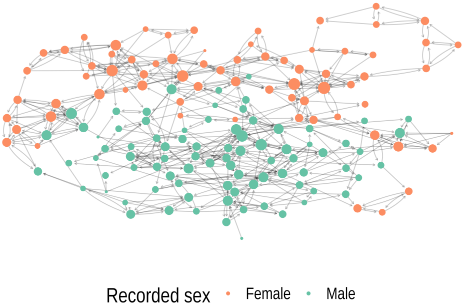
which decomposes into the natural direct effect and the natural indirect effect

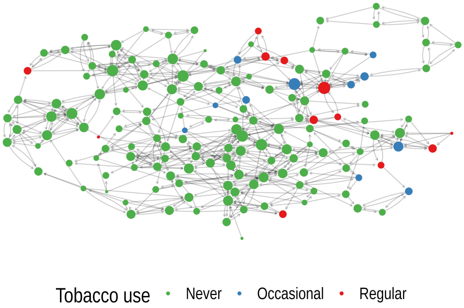
$$\Psi_{\text{ate}} = \Psi_{\text{nde}} + \Psi_{\text{nie}}$$
$$\Psi_{\text{nde}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t^*)) - Y_i(t^*, X_{i\cdot}(t^*))]$$
$$\Psi_{\text{nie}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t)) - Y_i(t, X_{i\cdot}(t^*))]$$



Social networks as mediators

Mediation in an adolescent social network

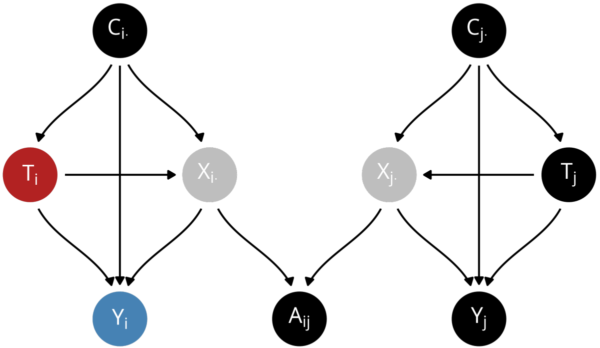




Teenage Friends and Lifestyle Study (wave 1), Glasgow, 1996

We propose a model where social groups in networks mediate causal effects

Adjacency matrix	A	$\in \mathbb{R}^{n \times n}$
Edge $i \sim j$	A_{ij}	$\in \mathbb{R}$
Treatment	T_i	$\in \{0, 1\}$
Outcome	Y_i	$\in \mathbb{R}$
Confounders	$C_{i\cdot}$	$\in \mathbb{R}^{1 \times p}$
Friend group (latent)	$X_{i\cdot}$	$\in \mathbb{R}^{1 \times d}$



Structural causal model for network mediation in a network with two nodes *i* and *j*

The challenge: the friend groups $X_{i\cdot}$ are not observed!

Solution: estimate friend groups $X_{i\cdot}$ using community models!

We use a semi-parametric estimator that accommodates:

- Stochastic blockmodels
- Degree-corrected stochastic blockmodels
- Mixed-membership stochastic blockmodels
- Overlapping stochastic blockmodels
- Random dot product graphs
- Etc

Intuition: stochastic blockmodels

d communities or "blocks"

$X_{i\cdot} \in \{0,1\}^d$ one-hot indicator of node i's block

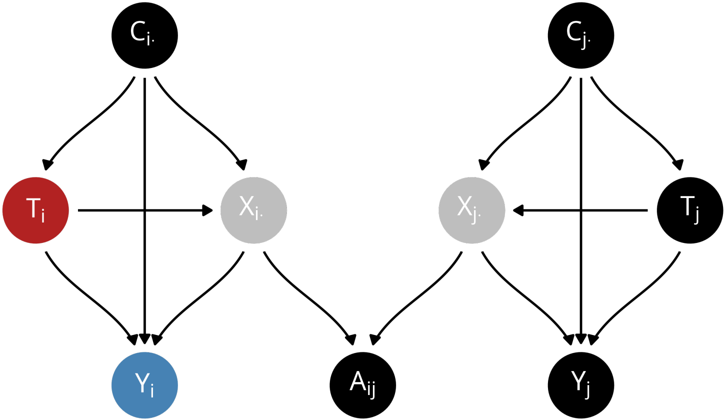
X is latent (i.e. unobserved)

$B \in [0,1]^{d \times d}$ inter-block edge probabilities

Friendships depend on group memberships and B

$\mathbb{P}\left(A_{ij} = 1 \,\middle|\, X\right) = X_{i\cdot} B X_{j\cdot}^T$

Returning to the structural causal model for a moment



Don't know X but can estimate it!

Definition (ASE)

Given a network A, the $\widehat{d}$-dimensional adjacency spectral embedding of A is

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2}$$

where $\widehat{U}\widehat{S}\widehat{U}^T$ is the rank-$\widehat{d}$ truncated singular value decomposition of A.

Lemma

Under a suitably well-behaved network model, if $\widehat{d}$ is correctly specified or consistently estimated, there is some $d \times d$ orthogonal matrix Q such that

$$\max_{i \in [n]} \left\| \widehat{X}_{i\cdot} - X_{i\cdot} Q \right\| = o_p(1).$$

Semi-parametric regression estimators

If the previous DAG is correct, under the additional assumption that:

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_{\mathsf{t}}}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\beta_{\mathsf{c}}}_{\mathbb{R}^{p}} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1\times d}} \underbrace{\beta_{\mathsf{x}}}_{\mathbb{R}^{d}},$$

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1\times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_{\mathsf{t}}}_{\mathbb{R}^{1\times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{\mathsf{c}}}_{\mathbb{R}^{p\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_{\mathsf{tc}}}_{\mathbb{R}^{p\times d}}.$$

Then:

$$\Psi_{\mathrm{nde}}(t, t^*) = (t - t^*)\,\beta_{\mathsf{t}}$$
$$\Psi_{\mathrm{nie}}(t, t^*) = (t - t^*)\,\theta_{\mathsf{t}}\,\beta_{\mathsf{x}} + (t - t^*)\,\mu_c\,\Theta_{\mathsf{tc}}\,\beta_{\mathsf{x}}.$$

Plug $\widehat{X}$ into regression estimator

Let $\widehat{D} = \begin{bmatrix} 1 & T & C & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n\times(2+p+d)}$ and $L = \begin{bmatrix} 1 & T & C & T\cdot C \end{bmatrix} \in \mathbb{R}^{n\times(2p+2)}$.

$$\begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_\mathsf{t} \\ \widehat{\beta}_\mathsf{c} \\ \widehat{\beta}_\mathsf{x} \end{bmatrix} = \left(\widehat{D}^T\widehat{D}\right)^{-1}\widehat{D}^T Y \quad \text{and} \quad \widehat{\Theta} = \left(L^T L\right)^{-1} L^T \widehat{X}.$$

$$\widehat{\Psi}_{\mathrm{nde}} = (t - t^*)\,\widehat{\beta}_\mathsf{t} \qquad \text{and}$$

$$\widehat{\Psi}_{\mathrm{nie}} = (t - t^*)\,\widehat{\theta}_\mathsf{t}\,\widehat{\beta}_\mathsf{x} + (t - t^*)\cdot\widehat{\mu}_c\cdot\widehat{\Theta}_\mathsf{tc}\,\widehat{\beta}_\mathsf{x}.$$

Main result

Theorem (Regression coefficients are asymptotically normal)

Under a suitably well-behaved network model and some moments conditions on regression errors, there is an unknown orthogonal matrix Q such that

$$\sqrt{n}\,\widehat{\Sigma}_{\beta}^{-1/2}\begin{pmatrix}\widehat{\beta}_w - \beta_w \\ Q\,\widehat{\beta}_x - \beta_x\end{pmatrix} \to \mathcal{N}(0, I_d), \textit{and}$$

$$\sqrt{n}\,\widehat{\Sigma}_{\mathrm{vec}(\Theta)}^{-1/2}\left(\mathrm{vec}\left(\widehat{\Theta}\,Q^T\right) - \mathrm{vec}(\Theta)\right) \to \mathcal{N}\left(0, I_{pd}\right).$$

where $\widehat{\Sigma}_{\mathrm{vec}(\Theta)}^{-1/2}$ *and* $\widehat{\Sigma}_{\beta}^{-1/2}$ *are the typical heteroscedasticity robust covariance estimators, with* $\widehat{X}$ *plugged in for* X*.*

Corollary

Theorem (Causal estimators are asymptotically normal)

Under the same statistical assumptions as before, plus mediating homophily,

$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nde}}}\left(\widehat{\Psi}_{\mathrm{nde}} - \Psi_{\mathrm{nde}}\right) \to \mathcal{N}(0,1), \text{ and}$$
$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nie}}}\left(\widehat{\Psi}_{\mathrm{nie}} - \Psi_{\mathrm{nie}}\right) \to \mathcal{N}(0,1).$$

where $\widehat{\sigma}^2_{\mathrm{nde}}$ and $\widehat{\sigma}^2_{\mathrm{nie}}$ are rather unfriendly variance estimators derived via the delta method and the previous theorem.

Application to Glasgow data

- Estimated effects are adjusted for possible confounding by age and church attendance.
- Estimated effects vary with the chosen dimension *d* of the latent space
- Over-specifying *d* is typically okay, but under-specifying *d* leads to a failure to capture social structure in *X*
- Once we capture enough social structure in *X*, we see a significant indirect social effect that leads adolescent girls to smoke more

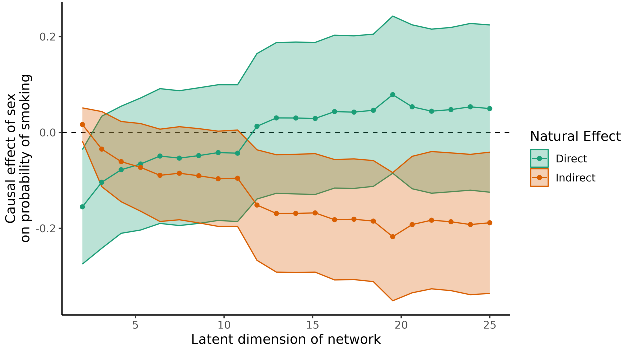


Figure 1: Estimated direct and indirect effects of sex on tobacco usage in the Glasgow social network. Positive values indicate a greater propensity for adolescent boys to smoke, negative effects a greater propensity for adolescent girls to smoke.

Thank you! Questions?

Read the manuscript at https://arxiv.org/abs/2212.12041

R package netmediate

Stay in touch

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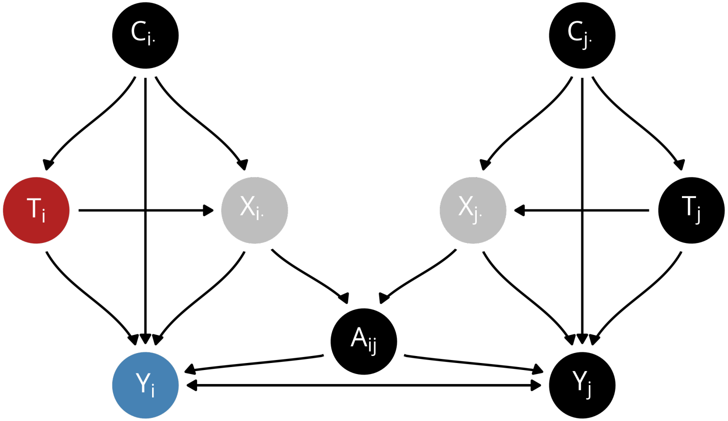
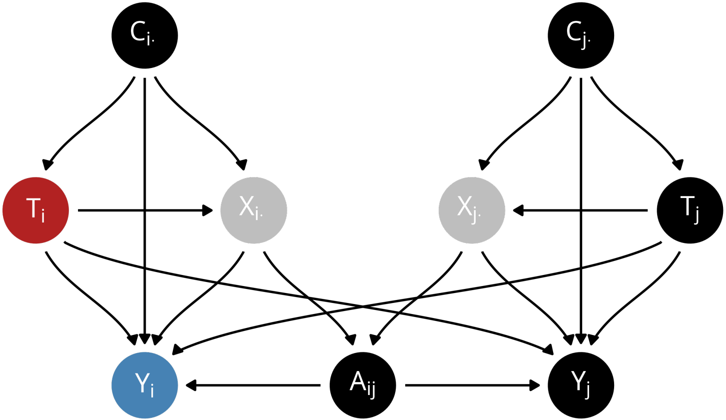
I'm looking for a post-doc starting Fall 2024, say hi if this work interests you!

Disambiguation: contagion ($Y_j \to Y_i$) is not allowed



Disambiguation: interference ($T_j \to Y_i$) is not allowed



More on interference and contagion

Interference and contagion effects are allowed so long as they happen in the latent space. Suppose

$$\mathbb{E}[Y_i \mid W_{i\cdot}, X_{i\cdot}] = W_{i\cdot}\beta_{\mathsf{w}} + X_{i\cdot}\beta'_{\mathsf{x}} + \delta_{\mathsf{y}} \sum_j X_{i\cdot}^T X_{j\cdot} Y_j$$

This latent space contagion model is a special parametric case of the regression outcome model (take $\beta_{\mathsf{x}} = \beta'_{\mathsf{x}} + X^T Y \delta_{\mathsf{y}}$).

Semi-parametric network model

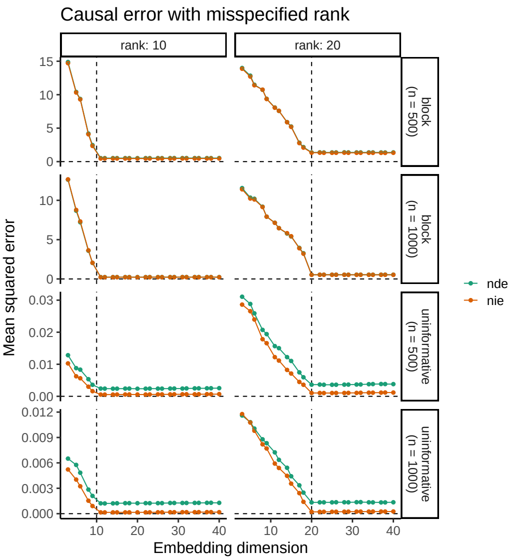
Definition

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \,|\, X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\text{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.

Remark

$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q, the latent positions X are only identifiable up to an orthogonal transformation.

Choosing $\widehat{d}$: overestimating the embedding dimension is fine



Identifying assumptions

Definition

The random variables $(Y_i, Y_i(t,x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:

   if $T_i = t$, then $X_{i\cdot}(t) = X_{i\cdot}$ with probability 1, and

   if $T_i = t$ and $X_{i\cdot} = x$, then $Y_i(t,x) = Y_i$ with probability 1

2. Sequential ignorability:

$$\{Y_i(t^*, x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot} \quad \text{and} \quad \{Y_i(t^*, x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$$

3. Positivity:

$$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0 \text{ for each } x \in \mathrm{supp}(X_{i\cdot})$$

$$\mathbb{P}(t \mid C_{i\cdot}) > 0 \text{ for each } t \in \mathrm{supp}(T_i)$$

Interventions allowed

Provided that controls $C_{i\cdot}$ are sufficiently informative about group membership $X_{i\cdot}$, treatment T_i is allowed to cause:

- Changes in popularity within a group
- Movement to a new friend group
- Becoming a member of a new friend group while remaining in current friend group
- Friendships becoming more or less likely between distinct friend groups
- Combinations of the above

See Appendix of manuscript for details.