Peer effects in the linear-in-means model may be inestimable even when identified

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Summary

Linear-in-means models are widely used to investigate peer effects. Identifying peer effects in these models is challenging, but conditions for identification are well-known. However, even when peer effects are identified, they may not be estimable, due to an asymptotic colinearity issue: as sample size increases, peer effects become more and more linearly dependent. We show that asymptotic colinearity occurs whenever nodal covariates are independent of the network and the minimum degree of the network is growing. Asymptotic colinearity can cause estimators to be inconsistent or to converge at slower than expected rates. We also demonstrate that dependence between nodal covariates and network structure can alleviate colinearity issues in random dot product graphs. These results suggest that linear-in-means models are less reliable for studying peer influence than previously believed.

1. Introduction

The linear-in-means model is a canonical approach to estimating social influence in social networks. Suppose there is a network with n nodes, encoded by a symmetric adjacency matrix $A \in \mathbb{R}^{n \times n}$. In binary networks, $A_{ij} = 1$ if nodes i and j form an edge, and $A_{ij} = 0$ otherwise, though we note that our results do not require that A be binary. Each node i is associated with an outcome $Y_i \in \mathbb{R}$ and a covariate $T_i \in \mathbb{R}$. Letting $\mathcal{N}(i) = \{j \in [n] : A_{ij} = 1\}$ denote the neighbors of node i in the network, the treatment and outcome of the neighbors are allowed to influence the outcome of node i as follows:

$$Y_i = \alpha + \frac{\beta}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} Y_j + \gamma T_i + \frac{\delta}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} T_j + \varepsilon_i. \tag{1}$$

The coefficient β , typically called the "contagion term", measures how peer outcomes Y_j influence the outcome Y_i at vertex i. This is variously referred to elsewhere in the literature as an "exogeneous spatial lag" (Florax & Folmer, 1992), a "spatial autoregression" or an "endogeneous peer effect" (Manski, 1993). Similarly, the coefficient δ , typically called the "interference term", measures how peer treatments T_j influence i's outcome Y_i . Elsewhere in the literature, δ is variously referred

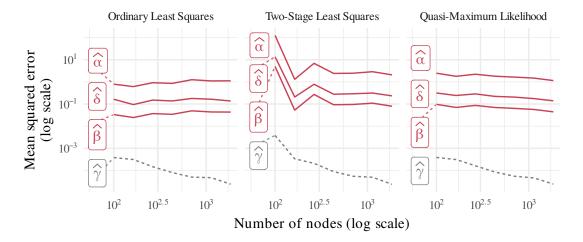


Fig. 1: Mean squared error of estimates. Each panel corresponds to a different estimator. Within a panel, the x-axis represents the sample size on a log scale, and the y-axis represents the Monte Carlo estimate of mean squared error, also on a log scale. Each line corresponds to a single coefficient; solid red lines are asymptotically colinear, dashed gray lines are not.

to as a "contextual peer effect" (Manski, 1993), an "exogeneous peer effect" (Manski, 1993), a "spatial Durbin term," or a "spatially lagged X" term (Anselin, 1988).

The linear-in-means model has been the subject of considerable attention due to challenges identifying the peer effects (see Bramoullé et al., 2020, for a recent review). Manski (1993) famously showed that peer effects are not identified in highly structured social networks. Much of the difficulty comes from the presence of *Y* on both sides of equation (1). Manski's result was subsequently generalized by Proposition 1 of Bramoullé et al. (2009), which showed that identification of the linear-in-means model is possible, provided that there are some open triangles ("intransivity") or some variation in group sizes in the network. Since most social networks feature open triangles or variation in group sizes, the linear-in-means model is generally understood to be identified.

Unfortunately, identification in the linear-in-means model does not imply estimability of the parameters. To briefly highlight the problem, consider Fig. 1, which shows the estimation error of three estimators applied to an identified model in a simulation study. All networks in the simulation exhibit high levels of intransitivity, yet none of the estimators are able to recover the regression coefficients α , β and δ . The mean-squared error of the estimates does not decrease with growing sample size. These simulation results are contrary to expectations that these estimators are consistent at \sqrt{n} -rates.

The issue is a degeneracy in the design matrix of the linear-in-means model. While the columns of the design matrix are linearly independent for every finite sample size n, the contagion and the interference columns become colinear with the intercept in the large sample limit, a setting sometimes referred to as "nearly singular design" (Phillips, 2016; Knight, 2008). Our main result states that the peer effect terms of the design matrix converge uniformly to constant multiples of the intercept column. The consequence of this colinearity is that peer effects can be inestimable even when they are identified.

To develop some intuition for the problem, consider a randomized experiment on a social network where each person in the network is treated independently with probability $\pi \in (0,1)$.

For every node i = 1, 2, ..., n, the interference term corresponds to the portion of treated peers, which is an average over $|\mathcal{N}(i)|$ peers. As the network grows, the portion of treated peers converges to π , the probability of treatment, and under mild conditions, the convergence is uniform over all nodes. Thus, the interference term in the design matrix becomes colinear with the intercept term in large samples, regardless of whether the network is intransitive or not. Perhaps surprisingly, a similar intuition holds for the contagion term.

Asymptotic colinearity is in general poorly-understood, and may have different effects on different estimators, especially those making different bias-variance tradeoffs. Asymptotic colinearity can lead to inconsistency, or consistency at slower than usual rates. In some regimes, asymptotic colinearity does not affect estimation (Lee, 2004). As a consequence, whenever asymptotic colinearity is possible, the behavior of most estimators is ambiguous. In a simulation study, our simulations suggest that ordinary least squares, two-stage least squares, and quasi-maximum likelihood estimators are inconsistent for regression coefficients under asymptotic colinearity, or that at least that that fail to achieve expected parametric rates. Further, we develop a lower bound on the estimation error for the ordinary least squares estimator and prove that ordinary least squares is often an inconsistent estimator when there is asymptotic colinearity. These results suggest that analyzing randomized experiments on networks with the linear-in-means model may lead to inconsistent estimates of peer effects, or confidence intervals that do not achieve their nominal coverage rates. They may also explain the poor predictive performance of linear-in-means models in comparison to other network regression models (Li et al., 2019; Le & Li, 2022).

We also show a limited positive result for network covariates that are dependent on the network. In the special case of a network generated according to the random dot product graph (Athreya et al., 2018), a low-rank network model that generalizes the stochastic blockmodel, we show that certain covariates are sufficiently dependent on network structure to prevent most asymptotic colinearity problems.

Notation

For a matrix A, let ||A||, $||A||_F$ and $||A||_{2,\infty}$ denote the spectral, Frobenius, and two-to-infinity norms, respectively. For a matrix A, we write A_i for its i-th row and $A_{\cdot j}$ for its j-th column. We use standard Landau notation, e.g., $O(a_n)$ and $o(a_n)$ to denote growth rates, as well as the probabilistic variants $O_P(a_n)$ and $O_P(a_n)$. For example, g(n) = O(f(n)) means that for some constant C > 0, |g(n)| < Cf(n) for all suitably large n. In proofs, C denotes a constant not depending on the number of vertices n, whose precise value may change from line to line, and occasionally within the same line.

2. The linear-in-means model

In the linear-in-means model, each outcome Y_i is a function of all the other outcomes $Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n$. This simultaneity (sometimes known as the "reflection problem", see Manski, 1993) makes it challenging to understand the data generating process of the linear-in-means model. In this section, we introduce the reduced form of the linear-in-means model, explain the generative process, and describe conditions for identification. Understanding identification in the linear-in-means provides insight into asymptotic colinearity and why it might pose a problem for estimation. In essence, asymptotic colinearity corresponds to identification getting weaker and weaker with increasing sample size.

We begin by expressing the linear-in-means model in matrix-vector form. Define the degree matrix $D = \text{diag}(d_1, d_2, \dots, d_n)$, where $d_i = \sum_j A_{ij}$. Let $G = D^{-1}A$ be the row-normalized adjacency matrix. Multiplication by G thus maps nodal values to the neighborhood averages of these

values, in the sense that $[GY]_i = d_i^{-1} \sum_j A_{ij} Y_j$ (see Fig. 2 for an illustration). Incorporating this notation, we have

$$Y = \alpha 1_n + \beta GY + T\gamma + GT\delta + \varepsilon, \tag{2}$$

where $GY \in \mathbb{R}^n$ encodes the vector of neighborhood averages of Y and $GT \in \mathbb{R}^n$ encodes the neighborhood averages of T. Thus, the design matrix of the linear-in-means model is

$$W_n = \begin{bmatrix} 1_n GY T GT \end{bmatrix}. \tag{3}$$

We assume, as is typical, that the errors ε are mean-zero and independent of the network A and nodal covariates T. We don't make parametric assumptions on the noise terms ε . Later on, we will consider multiple nodal covariates, allowing $T_i \in \mathbb{R}^p$ to be vector-valued. Thus, $\gamma, \delta \in \mathbb{R}^p$ are allowed to be vector-valued.

To work with equation (2), we solve for Y and consider the reduced form specification of the model, which is possible provided that $I - \beta G$ is invertible (i.e., when $|\beta| < 1$). Employing the Neumann expansion $(I - \beta G)^{-1} = \sum_{k=0}^{\infty} \beta^k G^k$, we can write

$$Y = (I - \beta G)^{-1} (1_n \alpha + T\gamma + GT\delta + \varepsilon) = \sum_{k=0}^{\infty} \beta^k G^k (1_n \alpha + T\gamma + GT\delta + \varepsilon). \tag{4}$$

This reduced form describes how outcomes Y can be generated given a network A, nodal covariates T and errors ε . The expression further suggests that Y can be interpreted as the equilibrium state reached after repeated neighborhood averaging on the network. That is, one way to sample an outcome vector Y is to construct an initial outcome vector $Y^{(0)} = \alpha 1_n + \gamma T + \delta GT + \varepsilon$, and then to repeatedly diffuse this outcome over the network according to G, weighting by G each time. To be very concrete: once G has been sampled, compute the scaled neighborhood average value G and G and G and G and G and G and add it to G to construct a new outcome vector G and so on. Repeating this process infinitely many times produces the equilibrium value G in equation G, which is guaranteed to be finite and unique by the fact that G is

Interpreting the regression coefficients $(\alpha, \beta, \gamma, \delta)$ in light of the repeated diffusion process for Y can be challenging, as the typical interpretation of linear regression coefficients no longer applies. Vazquez-Bare (2023) and LeSage & Pace (2009, Chapter 2) discuss these considerations in detail. Similarly, estimating the regression coefficients requires specialized techniques, as the contagion and interference terms introduce dependence between entries of the outcome vector Y. Some approaches to estimation are given by Ord (1975); Kelejian & Prucha (2001); Lee (2002, 2003, 2004); Kelejian & Prucha (2007); Lee et al. (2010); Su (2012); Drukker et al. (2013); Lin & Lee (2010) and surveyed in Bivand et al. (2021).

In order for inference based on the linear-in-means model to be meaningful, the regression coefficients must be uniquely determined given an infinite amount of data. A key identification result was given in Bramoullé et al. (2009). It expresses, algebraically, that peer effects are identified when there are nodes in a network that share a mutual neighbor, but are not neighbors themselves (for example, nodes *B* and *D* in Fig. 2).

Proposition 1 (Bramoullé et al. 2009). Fix n. Suppose $\mathbb{E}[\varepsilon \mid T] = 0$ and let

$$Y = 1_n \alpha + GY\beta + T\gamma + GT\delta + \varepsilon.$$

Suppose that $|\beta| < 1$ and $\gamma\beta + \delta \neq 0$. If I, G and G^2 are linearly independent in the sense that $aI + bG + cG^2 = 0$ only if a = b = c = 0, then α, β, γ and δ are identified. If I, G and G^2 are linearly dependent and no node is isolated, then $(\alpha, \beta, \gamma, \delta)$ are not identified.

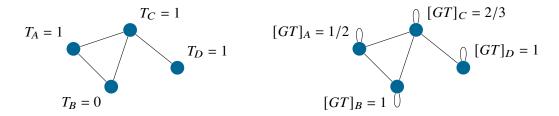


Fig. 2: Neighborhood averaging. (Left) A binary covariate T on small network. (Right) The average values of T in each node's neighborhood. For example, node A is connected to nodes B and C, the average value of T in neighborhood centered on A is 1/2 (the value of T at node A is excluded from this calculation.). Similarly, the average value of T in the neighborhood centered on T is 1.

The condition $\gamma\beta + \delta \neq 0$ means that there is either some interference effect δ , or some direct effect γ and some contagion effect β , and if both are present, they don't cancel each other out. Proposition 1 implies that one can examine a network and read off whether the linear-in-means model is identified by performing an eigendecomposition.

Remark 1. We briefly clarify a statement about Proposition 1 in Bramoullé et al. (2009, p. 44), which reads: "Our identification results are asymptotic in nature [...]. They characterize when social effects can, or cannot, be disentangled if we are not limited in the number of observations we can obtain." This use of "asymptotic" is slightly non-standard, as "asymptotic" typically refers to the limiting setting where sample sizes tend to infinity. In contrast, Proposition 1 characterizes when there is a one-to-one correspondence between the conditional expectation of Y and the regression coefficients $(\alpha, \beta, \gamma, \delta)$ given a network A with a fixed number of nodes n. Our own results (Theorems 1 and 2 below), are asymptotic in the more traditional sense of the sample size n (i.e., the number of vertices) growing to infinity.

Proposition 1 implies the following empirical test for identifiability.

Proposition 2. If G has three or more distinct eigenvalues, then I, G and G^2 are linearly independent.

See the Appendix C for a proof. For many random network models, such as random dot product graphs and stochastic blockmodels, G has three or more distinct eigenvalues with high probability, implying that the linear-in-means is identified. Similarly, many empirical networks satisfy this condition.

The conditions of Proposition 1 can be related to the identifying conditions for a typical linear model, where the regression coefficients are identified if and only if the columns of the design matrix are linearly independent. In the linear-in-means model, if there are no isolated nodes, the coefficients $(\alpha, \beta, \gamma, \delta)$ are identified if and only if $1_n, T, GT$ and $\mathbb{E}[GY \mid T]$ are linearly independent (Bramoullé et al., 2009, Appendix B). Later, we show that $GY = \mathbb{E}[GY \mid T] + o_P(1)$ in randomized experiments, such that linear independence of $1_n, T, GT$ and GY is nearly the same as linear independence of $1_n, T, GT$ and $\mathbb{E}[GY \mid T]$ in large samples.

3. Peer effects are asymptotically colinear in random experiments

The coefficients in the linear-in-means model are identified in finite samples, provided that the network obeys the conditions of Proposition 1. However, the design matrix can become degenerate as the sample size increases.

Definition 1. Suppose $W_n^T W_n/n$ converges to a limit Σ . Two or more columns of the design matrix given in equation (3) are asymptotically colinear when the corresponding columns of Σ are linearly dependent.

Asymptotic colinearity occurs under very mild conditions on the network. Crucially, asymptotic colinearity occurs even when the identifying conditions of Proposition 1 hold.

LEMMA 1. Suppose that (1) the nodal covariates $T_1, T_2, ..., T_n$ are independent with shared $mean \tau \in \mathbb{R}$, and T is independent of A; (2) the centered nodal covariates $\{T_i - \tau : i \in 1, 2, ..., n\}$, are independent (v, b)-subgamma random variables; (3) the regression errors $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are independent subgamma random variables with parameters not depending on n, and further are independent of $T_1, ..., T_n$; and (4) the adjacency matrix A contains only non-negative entries and does not contain any self-loops, such that $A_{ii} = 0$ for all i = 1, 2, ..., n.

If the degrees of the network grow such that

$$\max_{i \in [n]} \frac{1}{d_i^2} \sum_{j=1}^n A_{ij}^2 = o\left(\frac{1}{v \log^2 n}\right) \text{ and } \max_{i,j \in [n]} \frac{A_{ij}}{d_i} = o\left(\frac{1}{b \log n}\right). \tag{5}$$

then

$$\max_{i \in [n]} \left\| [GT]_i - \tau \right\| = o(1) \ almost \ surely$$

and

$$\max_{i \in [n]} \left\| [GY]_i - \eta \right\| = o(1) \ almost surely,$$

where

$$\eta = \frac{\alpha + (\gamma + \delta)\tau}{1 - \beta}.$$
 (6)

A proof can be found in the Appendix. The first condition requires that nodal covariates T are independent of the network, and independent across nodes. The second condition is effectively a technical condition, and requires that the tail behavior of the nodal covariates decays at, or faster than, that of Gamma random variables. This ensures that averages of the T_i are well-behaved. The class of subgamma distributions is broad, and includes as special cases the Bernoulli, Poisson, Exponential, Gamma, and Gaussian distributions, as well as any sub-Gaussian or squared sub-Gaussian distribution and all bounded distributions (Boucheron et al., 2013; Vershynin, 2020). Similarly, the third condition requires that the regression errors ε are not too heavy-tailed.

The fourth condition requires that the network has no self-loops. Weighted networks are allowed so long as all edges have non-negative edges weights. The condition in equation (5) requires that the size of each neighborhood is growing. Notably, it does not require that the network is binary (i.e., $A_{ij} \in \{0, 1\}$), but rather only requires that the edge weights be non-negative. That is, $A_{ij} \geq 0$ denotes the strength of the connection between nodes i and j. When the network is binary, such that $A_{ij} \in \{0, 1\}$, the condition on the degrees reduces to $\min_{i \in [n]} d_i = \omega(\log n)$. That is, the size of the smallest neighborhood must grow faster than $\log n$. This implies that no nodes in the network are isolated. In the more general case of a weighted, non-negative network, the condition

in equation (5) essentially requires that no individual edge accounts for too much of the total "weight" incident on any vertex. Although we present Lemma 1 in the context of scalar nodal covariates, we note that the lemma can be extended to multiple nodal covariates provided that the subgamma assumption holds for each covariate.

When all of the conditions hold, the interference term GT and the contagion term GY become colinear with the intercept in the large-network limit (i.e., as the number of vertices n grows).

Remark 2. Lemma 1 implies that the columns of the design matrix corresponding to the intercept α , the contagion parameter β and the interference parameter δ are asymptotically colinear under the simulations in Bramoullé et al. (2009).

As discussed in the introduction, the intuition for the colinearity between the intercept and interference term GT is that neighborhood averages of T converge to the expected value of T when T is independent of the network. What is perhaps surprising is that the contagion term GY behaves similarly to the interference term GT. This can be better seen by multiplying by G in the reduced form of equation (4). When no node is isolated,

$$\begin{split} GY &= \sum_{k=0}^{\infty} \beta^k G^{k+1} (1_n \alpha + T \gamma + G T \delta + \varepsilon) \\ &= \frac{\alpha}{1-\beta} G 1_n + \gamma G T + (\gamma \beta + \delta) \sum_{k=0}^{\infty} \beta^k G^{k+2} T + \sum_{k=0}^{\infty} \beta^k G^{k+1} \varepsilon. \end{split}$$

Since $G1_n = 1_n$, the first right-hand term is a constant vector. We have already argued that the second term converges to a constant. The third right-hand term expands to $GT + \beta G^2T + \beta^2 G^3T + \cdots$. When GT is near-constant, we have $G^2T = G(GT) \approx GT$, and higher-order summands of this term behave similarly. The final right-hand term of GY is zero in expectation and thus irrelevant for identification pursues. Nonetheless, one can see that, by a similar argument as for the third term, the fourth term should converge to a column vector of zeroes.

Altogether, the implication is that the contagion term GY converges to a constant when the interference term GT converges to a constant. This is particularly concerning, because GT is simply a collection of averages with shared expectation, and we thus anticipate that GT will converge to τ , the expectation of T, under a wide variety of circumstances, leading to colinearity issues. Note that the conditions of Lemma 1 describe one set of sufficient conditions for GT and GY to converge uniformly to constants, but these conditions are by no means necessary. We expect that GT and GY will converge to constants under other conditions not listed here.

The next result shows that asymptotic colinearity degrades the performance of ordinary least squares estimators, and in sufficiently dense networks causes inconsistency.

Theorem 1. Let $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ be the vector of ordinary least squares estimates of $(\alpha, \beta, \gamma, \delta)$, based on an n-by-n network, and suppose that as n grows, the sequence of networks is such that

$$||G||_F^2 = o(n). (7)$$

Suppose that the adjacency matrix A contains only non-negative entries and does not contain any self-loops, such that $A_{ii} = 0$ for all i = 1, 2, ..., n; the nodal covariates $T_1, T_2, ..., T_n$ are independent with shared mean $\tau \in \mathbb{R}$; the regression errors $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ and the centered nodal covariates $\{T_i - \tau : i = 1, 2, ..., n\}$ are independent sub-Gaussian random variables with parameters not depending on n; and the vectors T and ε are independent given A. Then if $\beta = 0$,

$$\min\{|\hat{\alpha} - \alpha|, |\widehat{\beta} - \beta|\} = \Omega_P(1)$$

and

$$|\hat{\delta} - \delta| = \Omega_P \left(\frac{1}{\|G\|_F} \right). \tag{8}$$

If $\beta \neq 0$,

$$\min\{|\hat{\alpha} - \alpha|, |\widehat{\beta} - \beta|\} = \Omega_P \left(\frac{1}{\|G\|_F}\right).$$

Finally, under the stronger growth assumption $||G||_F^2 = o(\sqrt{n})$, equation (8) continues to hold for all values of β .

Theorem 1 lower bounds the estimation error of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\delta}$. The precise lower bounds depend on the density of the network (as enforced by rates on the Frobenius norm of G), and whether or not there is contagion (i.e., whether $\beta=0$ or not). In binary networks, letting d_{\min} denote the minimum degree of the network, the condition $\|G\|_F^2=o(n)$ is satisfied when d_{\min} grows at any rate. In this setting, when there is no contagion, $\hat{\alpha}$ and $\hat{\beta}$ are inconsistent, and $\hat{\delta}$ is at best consistent at the slower rate $\sqrt{n/d_{\min}}$. Note that, when the minimum degree grows linearly in the number of nodes n, $\hat{\delta}$ will also be inconsistent.

When contagion is present (i.e., $\beta \neq 0$), $\hat{\alpha}$ and $\hat{\beta}$ are at best consistent at rate $\sqrt{n/d_{\min}}$. In contagious settings, our results only lower bound the estimation error in $\hat{\delta}$ when the network is moderately dense. In particular, in a binary network, the condition $||G||_F^2 = o(\sqrt{n})$ is satisfied when d_{\min} grows at $\omega(\sqrt{n})$ rates. When this holds, $\hat{\delta}$ again is at best consistent at the slower rate $\sqrt{n/d_{\min}}$.

Remark 3 (Non-random covariates). Since we assume that T_1, T_2, \ldots, T_n are independent covariates with shared expectation, Theorem 1 does not apply if the nodal covariates T are fixed and non-random.

Two previous results provide valuable context for Theorem 1. Under the assumption that nodal covariates T are fixed, do not exhibit asymptotic colinearity, and there is no contextual peer effects (i.e., $\delta = 0$), Lee (2002) showed that ordinary least estimates of α , β and γ are inconsistent when d_{\min} is bounded by a constant, consistent at d_{\min} -rates when d_{\min} is $O(\sqrt{n})$ and consistent at $1/\sqrt{n}$ -rates when $d_{\min} = \omega(\sqrt{n})$. The deviation from our results in the random nodal variate setting, which state the estimation error increases rather than decreases with d_{\min} , is intriguing. From an applied perspective, it is perhaps unsettling that the behavior of ordinary least squares changes so dramatically under fixed and random models of the data, and that in the fixed setting, density is good for the estimates, whereas in the random setting, density is bad for the estimates.

As an additional point of comparison, Lee (2004) carefully studied the behavior of a quasi-maximum likelihood estimator under asymptotic colinearity, again under the assumption that the nodal covariates T_i are fixed, and again without an interference term (i.e., $\delta = 0$). Again, the precise behavior of the quasi-maximum likelihood estimator depended on the growth rate of d_{\min} . For bounded d_{\min} , as in sparse networks, standard \sqrt{n} -rates and asymptotic normal distributions obtain. When d_{\min} diverges, some coefficients can only by estimated at $\sqrt{n/d_{\min}}$ -rates. When the minimum degree d_{\min} grows at linearly in n, the estimator can be inconsistent.

4. PEER EFFECTS ARE PARTIALLY COLINEAR IN RANDOM DOT PRODUCT GRAPHS

We next consider a model where there is sufficient dependence between some nodal covariates and the network structure so that that the neighborhood averages converge to non-constant values

(see also Case, 1991; Martellosio, 2022, on the importance of the relationship between the network and nodal covariates). As our above results illustrate, when the network is independent of nodal covariates, interference terms and contagion terms both become colinear with the intercept. When the network is dependent on nodal covariates, the story becomes more complicated. We first introduce a general probabilistic model for random networks. This model includes as special cases a number of widely-used network models that may be more familiar to practitioners, and we discuss these special-case models in more detail below.

DEFINITION 2 (RANDOM DOT PRODUCT GRAPH, YOUNG & SCHEINERMAN 2007). Let F be a distribution on \mathbb{R}^d such that $0 \le x^T y$ for all $x, y \in \text{supp } F$ and the convex cone of supp F is d-dimensional. Draw X_1, X_2, \ldots, X_n independently and identically from F, and collect these in the rows of $X \in \mathbb{R}^{n \times d}$ for ease of notation. Conditional on these n vectors, which we call latent positions, generate edges by drawing $\{A_{ij}: 1 \le i < j \le n\}$ as independent (v,b)-subgamma random variables with $\mathbb{E}[A_{ij} \mid X] = \rho X_i^T X_j$, where $\rho \in [0,1]$. Then we say that A is distributed according to an n-vertex random dot product graph with latent position distribution F, (v,b)-subgamma edges and sparsity factor ρ . We write $(A,X) \sim \text{RDPG}(F,n)$, with the subgamma and sparsity parameters made clear from the context.

Definition 2 is a slight generalization of the "classical" random dot product graph (Young & Scheinerman, 2007; Athreya et al., 2018), in that it permits weighted edges, rather than binary edges. The random dot product graph as defined in Young & Scheinerman (2007) assumes that $0 \le x^T y \le 1$ for all $x, y \in \text{supp } F$, in order to ensure that these inner products yield edge probabilities. We do not require this restriction for our results, but the above definition recovers the binary setting as a special case.

Under the random dot product graph, each node in a network is associated with a latent vector, and these latent vectors characterize the propensities for pairs of vertices to form edges with one other. Specifically, nodes close to one another in latent space are more likely to form edges, and nodes far apart are unlikely to form edges. When nodes cluster in the latent space, the result is that edges are more likely to form between nodes with similar latent characteristics. This manifests as homophily in the resulting network.

More concretely for practitioners, degree-corrected stochastic blockmodels are submodels of the random dot product graph (Definition 2).

Example 1 (Poisson Degree-Corrected Stochastic Blockmodel). The Poisson degree-corrected stochastic blockmodel (Rohe et al., 2018; Karrer & Newman, 2011) is an undirected model of community membership, with d communities. Each node, index by i, is assigned a block $z(i) \in \{1, 2, ..., d\}$ with probability $pr(z(i) = k) = \pi_k$, and a degree-correction parameter θ_i , which describes its propensity to connect with other nodes. Conditional on block memberships and degree-correction parameters, edges are generated independently between every pair of vertices in the network according to a Poisson distribution. The expected number of edges between two vertices depends on their community memberships, their degree correction parameters, a positive semi-definite matrix $B \in [0,1]^{d \times d}$ of inter-block edge formation probabilities, and a scaling factor $\rho \in [0,1]$. That is,

$$\mathbb{E}[A_{ij} = 1 \mid z(i), z(j), \theta_i, \theta_j] = \rho \,\theta_i B_{z(i), z(j)} \theta_j.$$

Mixed-membership stochastic blockmodels and overlapping stochastic blockmodels, with and without degree-correction, and with Poisson or Bernoulli edges are also special cases of our model in Definition 2 (Latouche et al., 2011; Airoldi et al., 2008; Jin et al., 2024; Zhang et al., 2020; Rohe & Zeng, 2023).

The key feature of random dot product graphs is that the latent positions X and the network A are highly dependent on one another. Thus, if the latent positions X are incorporated into a linear-in-means models as nodal covariates, the neighborhood averages GX and GY will not converge to constants, and there is thus the potential for $(\alpha, \beta, \gamma, \delta)$ to avoid the asymptotic colinearity issue highlighted in Lemma 1 and Theorem 1. Theorem 2 shows, however, that some regression terms are still colinear in the asymptotic limit.

Theorem 2. Suppose that (A, X) are sampled from a random dot product model where X is rank d with probability 1. Let ε be a vector of mean zero, i.i.d. $(v_{\varepsilon}, b_{\varepsilon})$ -subgamma random variables, with $(v_{\varepsilon}, b_{\varepsilon})$ not depending on n, and let

$$Y = \alpha 1_n + \beta GY + X\gamma + GX\delta + \varepsilon$$

for $\alpha, \beta \in \mathbb{R}$ and $\gamma, \delta \in \mathbb{R}^d$, and the conditions of Proposition 1 hold. Suppose that X has $k \geq 2d$ distinct rows. Then, under suitable technical conditions, the columns of design matrix corresponding to $(\alpha, \beta, \delta_1, \delta_2, \ldots, \delta_d)$ are asymptotically colinear. If any two elements of $(\alpha, \beta, \delta_1, \delta_2, \ldots, \delta_d)$ are equal to zero, there is no asymptotic colinearity.

Theorem 2 states that the linear-in-means model with latent positions as nodal covariates is typically asymptotically well-behaved provided that there a mild amount of variation in X, which will induce degree heterogeneity, and at most d-2 columns of GX are present in the generative model. See Appendix A for a proof and details about the technical conditions. Two entries of $(\alpha, \beta, \delta_1, \delta_2, \ldots, \delta_d)$ need to be set to zero in the data generating process to achieve a non-singular design matrix in the limit because GX is asymptotically colinear with both the intercept column 1_n and the contagion column GY. There is additionally a knife-edge failure case that we discuss in Appendix A.

Since the latent positions X are unobserved, one might wonder if Theorem 2 has practical applications. To begin with, we note that the model considered in Theorem 2 is estimable. The typical approach is to estimate X via the adjacency spectral embedding (Sussman et al., 2014; Athreya et al., 2018) and use the estimated values \hat{X} as plug-in replacements for X (Hayes et al., 2023; McFowland & Shalizi, 2021; Le & Li, 2022; Nath et al., 2023). For instance, Nath et al. (2023) considers a linear-in-means model under the restriction that $\delta = 0$, and extends the quasimaximum likelihood estimator of Lee (2004) to a bias-corrected variant accounting for estimation error of \hat{X} about X. Theorem 2 clarifies when asymptotic colinearity occurs, explaining some intriguing features of the simulations presented in Fig. 2 of Nath et al. (2023), such as the use of a rank-two mixing matrix B in a four block stochastic blockmodel. The use of a rank-deficit B matrix is equivalent to forcing the latent positions X to have four distinct rows, or k = 2d in our notation, which is exactly the minimum amount of degree heterogeneity required for β to be distinguishable from γ . Our theory further clarifies that Nath et al. (2023) could include an intercept column in their regression specification, if they so desired.

5. Simulation study on finite-sample consequences of asymptotic colinearity

We illustrate the consequences of Theorems 1 and 2 via simulation. Our simulations demonstrate that, even though our results are asymptotic, estimators are negatively affected by the near-perfect multicolinearity in finite samples. This is the case even though all the conditions of Proposition 1 are satisfied, such that α, β, δ and γ are identified for finite n.

All networks in our simulations below are generated from a Poisson degree-corrected stochastic blockmodel with n nodes and four equally probably blocks (see Example 1 above). The edge

formation matrix $B \in [0, 1]^{4 \times 4}$ is set to

$$B = \begin{bmatrix} 0.5 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.5 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.5 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.5 \end{bmatrix},$$

and the sparsity parameter ρ is set so that the expected mean degree of the network is $2n^{0.7}$. Degree-correction parameters $\theta_1, \theta_2, \dots, \theta_n$ are sampled independently from a continuous uniform distribution supported on the interval [1, 2]. Networks sampled from this model will have three distinct eigenvalues with high probability, and thus satisfy the conditions of Proposition 1.

We consider three distinct generative models for nodal outcomes $Y_1, Y_2, ..., Y_n$. In the *Bernoulli* model, there is a single nodal covariate $T_i \sim \text{Bern}(0.5)$ sampled independently for all nodes, and independently of the network. The regression model is then

$$Y = \alpha 1_n + \beta GY + \gamma T + \delta GT + \varepsilon,$$

and we fix $\alpha = 3$, $\beta = 0.2$, $\gamma = 4$, $\delta = 2$ and sample $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ independently and identically with $\sigma = 0.1$. α , β and δ are asymptotically aliased in the Bernoulli model.

In the *Unrestricted* model, the nodal covariates are the latent positions of the stochastic blockmodel, such that is $T_i = X_i \in \mathbb{R}^4$. Recall that $X_i \in R^4$ where $X = US^{1/2}$ and USU^T is the eigendecomposition of $\mathbb{E}[A \mid z(1), z(2), \dots, z(n), \theta]$ (recall that z_i is the block membership of node i; see Example 1). The nodal regression model is thus

$$Y = \alpha 1_n + \beta GY + X\gamma + GX\delta + \varepsilon,$$

where we again set $\alpha = 3$, $\beta = 0.2$ and sample $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ independently and identically with $\sigma = 0.1$. Since $X_i \in \mathbb{R}^4$, $\gamma, \delta \in \mathbb{R}^4$ and we fix $\delta = (2, 2, 2, 2)$ and $\gamma = (1.5, 2.5, 3.5, 4.5)$. α, β and δ are asymptotically aliased in this model.

Finally, the *Restricted* model is the same as the *Unrestricted* model, but with the additional constraint that $\delta = (0, 0, 2, 2)$, such that the design matrix is full rank in the limit and there is no asymptotic colinearity, as per Theorem 2.

We vary the sample size (i.e., the number of vertices n) on a logarithmic scale, considering $n \in \{100, 163, 264, 430, 698, 1135, 1845, 3000\}$, and replicate our experiments 100 times for each simulation setting. In each setting, we estimate the parameters of the linear-in-means model using ordinary least squares (Lee, 2002; Trane, 2023), two-stage least squares (Kelejian & Prucha, 1998; Lee, 2003; Piras & Postiglione, 2022), and a quasi-maximum likelihood estimator (Lee, 2004; Nath et al., 2023).

In Fig. 3, we plot the mean squared error of the estimated coefficients. All estimators that we consider fail to recover the regression coefficients that correspond to asymptotically colinear columns of the design matrix. First, consider the ordinary least squares estimates, and observe that mean squared error decreases for all asymptotically unaliased coefficients (gray), as expected. In contrast, mean squared error for asymptotically colinear coefficients (colored) is either increasing or constant as sample size grows. This agrees with our theoretical results indicating that ordinary least squares cannot be a consistent estimator for the asymptotically colinear coefficients. Note, however, that this holds across all three models considered in the simulation, not just ordinary least squares: the performance of the two-stage least squares and quasi-maximum likelihood estimators is largely similar. These simulations suggest that ordinary least squares, two-stage least squares, and quasi-maximum likelihood estimators are inconsistent for regression coefficients under asymptotic colinearity, or that at least that they fail to achieve expected parametric rates.

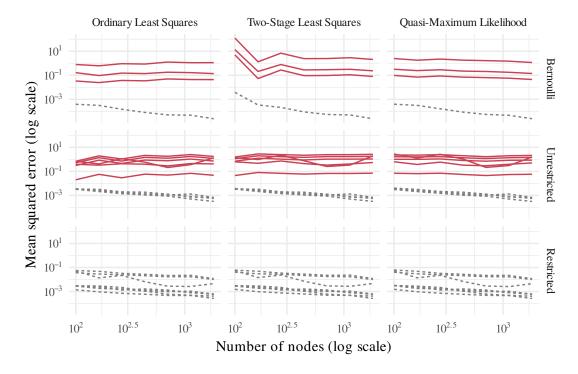


Fig. 3: Mean squared error of estimates. Each row of panels denotes a different simulation setting, and each column of panels corresponds to a different estimator. Within a panel, the x-axis represents the sample size on a log scale, and the y-axis represents the Monte Carlo estimate of mean squared error, also on log scale. Each line corresponds to a single coefficient. Solid red lines are asymptotically colinear, dashed gray lines are not.

In Fig. 4, we explicitly show that the columns of the design matrix become more and more colinear as sample size increases. To measure colinearity, we use variance inflation factors. The variance inflation factor for the *i*-th estimated coefficient is defined as $1/(1-R_i^2)$ where R_i is the coefficient of determination in the regression where the *i*-th covariate is predicted based on all other covariates, using ordinary least squares (typically, it is considered inappropriate to compute variance inflation factors for the intercept, but here we are not interested in the variance multiplier interpretation of variance inflation factors, but rather a simple metric for colinearity; Fox & Monette, 1992).

In the *Bernoulli* model, the contagion and interference terms are converging to constant multiples of the intercept, and in the *Unrestricted* model, the intercept and contagion term are becoming closer and closer to the span of the interference terms. In the *Restricted* model, where all coefficients are asymptotically identified, there is still some colinearity of the intercept, interference and contagion terms, but not as much as in the *Unrestricted* model. The variance inflation factors in the *Restricted* model are either constant or growing slowly as a function of n. In the *Bernoulli* model, the variance inflation factors for the intercept α , interference effect δ , and contagion effect β are orders of magnitude larger than in the *Restricted* model and increasing multiplicatively with sample size. In the *Unrestricted* model, the variance inflation factor for the unidentified coefficients are also growing rapidly as a function of sample size. These results are exactly as expected based on Theorems 1 and 2.

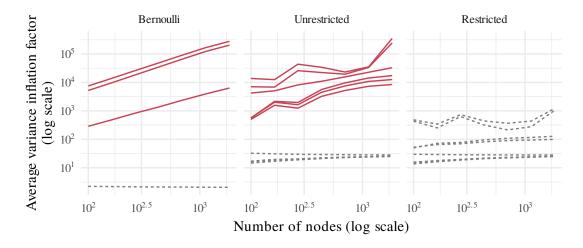


Fig. 4: Average variance inflation factors. Each panel denotes a different simulation setting. The x-axis represents the sample size on a log scale, and the y-axis represents the Monte Carlo estimate of mean variance inflation factor, also on log scale. Each line corresponds to a single coefficient. Solid red lines are asymptotically colinear, dashed gray lines are not.

6. Discussion

Our results tell a cautionary tale about estimating peer effects using the linear-in-means model. We show that peer effects can be inestimable even when they are identified. Whenever nodal covariates are independent of network structure, and the minimum degree of the network grows at $\omega(\log n)$ -rates, peer effect columns of the design matrix are asymptotically colinear. Our negative results do not rely on any particular network model or structure in the network beyond a lower bound on the minimum degree. Further, we prove that ordinary least squares is inconsistent in some settings. We confirm via simulation that ordinary least squares, two-stage least squares, and quasi-maximum likelihood estimators all exhibit concerning behavior. Previous theory has largely assumed that these colinearity problems do not exist.

We further showed in the limited setting of random dot product graphs that dependence between nodal covariates and network structure can prevent colinearity issues. It is an open question whether longitudinal models of peer influence suffer from this same issue (Zhu et al., 2017; McFowland & Shalizi, 2021; Katsouris, 2024), or if estimability challenges can be resolved via additional assumptions on the error covariance (Rose, 2017).

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A. Theorem 2: details and proof sketch

In this section we present some preliminaries and then sketch a proof of Theorem 2. Let $\mu \in \mathbb{R}^d$ denote the mean of the latent position distribution F. We take F to be fixed with respect to n, but we allow both the subgamma parameters $(\nu, b) = (\nu_n, b_n)$ and the sparsity parameter $\rho = \rho_n$ to vary with n, to capture the widely observed phenomenon of sparse networks (i.e., the expected number of edges grows at a rate slower than n^2). While these parameters are allowed to depend on n, we suppress the subscripts on ν, b and ρ in the sequel for the sake of readability. Our results require a handful of assumptions on the behavior of the latent position distribution F and on the growth of the parameters ν, b and ρ .

Assumption 1. The edge-level (v, b)-subgamma parameters and the sparsity ρ are such that

$$\rho = \omega \left(\frac{\log^2 n}{n^{1/2}} \right) \tag{1}$$

and

$$\frac{v+b^2}{\rho} = \Theta(1). \tag{2}$$

Assumption 1 primarily ensures that the network A is dense. In the context of a binary network, the restriction on ρ is equivalent to requiring all degrees in the network to grow at rates faster than $\log^2 n\sqrt{n}$.

Remark 4. The $\omega(n^{-1/2}\log^2 n)$ lower bound on ρ required by Assumption 1 is larger than the more typical $\omega(n^{-1}\log^c n)$ often encountered in results related to the random dot product graph (see, e.g., Rubin-Delanchy et al., 2022; Levin et al., 2022). While a more careful analysis might yield a less strict lower bound, we observe that this lower bound matches, up to the logarithmic factor, the lower bound required for convergence of ordinary least squares estimates considered in Lee (2002). It may be of interest to pursue the asymptotics of GX and GY in sparser regimes using concentration inequalities specialized to the binary case (Tropp, 2015; Lei & Rinaldo, 2015), but we do not pursue that here.

Assumption 2. The latent position distribution F and the sparsity parameter ρ are such that

$$\min_{i \in 1, \dots, n} |X_i^T \mu| = \omega \left(\frac{\log^2 n}{\sqrt{n} \rho} \right) \text{ almost surely.}$$
 (3)

Assumption 2 requires that the latent positions X cannot have small inner products with each other. Roughly, this prevents the minimum expected degree of the network from getting too small.

Assumption 3. The latent position distribution F is such that

$$\max_{i \in 1, \dots, n} ||X_i|| = o(n^{1/2}) \text{ almost surely.}$$
 (4)

Assumption 3 gives uniform control of the norm of the X_i .

Assumption 4. The latent position distribution F is such that

$$\mathbb{E}||X_1||^2 < \infty. \tag{5}$$

Lastly, Assumption 4 requires that F has finite second moment. With these assumptions in hand, we are ready to state our main result, with the help of some additional notation. Denote by Λ the second moment matrix of F,

$$\Lambda = \mathbb{E}\left[X_1 X_1^T\right]. \tag{6}$$

Additionally, define the random diagonal matrix

$$H = \operatorname{diag}(X_1^T \mu, X_2^T \mu, \dots, X_n^T \mu) \in \mathbb{R}^{n \times n}.$$
 (7)

and the population-level matrix

$$\Gamma = \left(I - \beta \mathbb{E} \left[\frac{X_1 X_1^T}{X_1^T \mu} \right] \right)^{-1} - I. \tag{8}$$

Under these assumptions, the key result is the following lemma, which describes the asymptotic behavior of the GX and GY columns of the design matrix.

Lemma 2. Suppose that (A, X) are sampled from a random dot product model where X is rank d with probability 1. Let ε be a vector of mean zero, i.i.d. $(v_{\varepsilon}, b_{\varepsilon})$ -subgamma random variables, with $(v_{\varepsilon}, b_{\varepsilon})$ not depending on n, and let

$$Y = \alpha 1_n + \beta G Y + X \gamma + G X \delta + \varepsilon \tag{9}$$

for $\alpha, \beta \in \mathbb{R}$ and $\gamma, \delta \in \mathbb{R}^d$. Suppose that Assumptions 1, 2, 3, and 4 all hold. Let $\tilde{\gamma} = \Lambda \gamma + \Gamma \Lambda(\beta \gamma + \delta)$. Then

$$\left\|GX - \left(H^{-1}X\Lambda\right)\right\|_{2,\infty} = o(1)$$
 almost surely.

and

$$\left\|GY - \left(\frac{\alpha}{1-\beta}1_n + H^{-1}X\tilde{\gamma}\right)\right\|_{2,\infty} = o(1)$$
 almost surely.

See Appendix H for a proof. Given these results, we next seek to understand the asymptotic design matrix. Let

$$W_n = \begin{bmatrix} 1_n \ GY \ X \ GX \end{bmatrix}$$
 and $W = \begin{bmatrix} 1_n \ \left(\frac{\alpha}{1-\beta} 1_n + H^{-1} X \tilde{\gamma}\right) \ X \ H^{-1} X \Lambda \end{bmatrix}$.

The asymptotic design matrix W will always be rank deficient, but the precise rank depends on X.

First, we observe that $W \in \mathbb{R}^{n \times (2d+2)}$ has maximum rank 2d. To see this, note that $H^{-1}X\Lambda$ is colinear with the intercept, by definition of H and Λ . Similarly, $H^{-1}X\Lambda$ is colinear with $\frac{\alpha}{1-\beta}1_n + H^{-1}X\tilde{\gamma}$. That is, in the asymptotic limit GX is colinear with both the intercept 1_n and GY. Provided that we avoid the knife-edge failure case $\tilde{\gamma} = \mu$, $\frac{\alpha}{1-\beta}1_n + H^{-1}X\tilde{\gamma}$ is not colinear with the intercept. Thus, the maximum rank of W is 2d + 2 - 2 = 2d.

Next we develop a condition such that the maximum rank of 2d is attained. Note that ehe population matrix W obtains rank 2d whenever $\begin{bmatrix} X & H^{-1}X \\ X \\ X \end{bmatrix}$ is full-rank, or, equivalently, when $\begin{bmatrix} X & H^{-1}X \\ X \end{bmatrix}$ is full-rank, since Λ is invertible by definition. Intuitively, distinguishing the direct effect γ from the interference term δ depends heavily on the presence of degree heterogeneity within the network. In particular, the (scaled) degree normalization matrix $n\rho D^{-1}$ converges to H^{-1} in the large-n limit, which encodes the expected degree of each node, conditional on the latent positions X. Thus, W obtains rank 2d when information contained in the latent positions X and expected degree scaled latent positions $H^{-1}X$ is linearly independent. This should generally be the case in, for example, degree-corrected stochastic blockmodels. We present a sufficient condition for $\begin{bmatrix} X & H^{-1}X \end{bmatrix}$ to be full-rank below, although it is fairly hard to interpret.

PROPOSITION 3. Let $\mu \in \mathbb{R}^d$ and suppose that $Y_1, Y_2, \ldots, Y_d, Z_1, Z_2, \ldots, Z_d \in \mathbb{R}^d$ are rows of $X \in \mathbb{R}^{n \times d}$ such that Y_1, Y_2, \ldots, Y_d are linearly independent and Z_1, Z_2, \ldots, Z_d are linearly independent. Collecting these vectors in the rows of $Y \in \mathbb{R}^{d \times d}$ and $Z \in \mathbb{R}^{d \times d}$, respectively, define

$$H_Y = \operatorname{diag}\left(Y_1^T \mu, Y_2^T \mu, \dots, Y_d^T \mu\right)$$
 and $H_Z = \operatorname{diag}\left(Z_1^T \mu, Z_2^T \mu, \dots, Z_d^T \mu\right)$.

Provided that

$$Z^{-1}H_Z^{-1}Z - Y^{-1}H_Y^{-1}Y \in \mathbb{R}^{d \times d}$$

is invertible, then the matrix

$$M = \left[X \ H^{-1} X \right] \in \mathbb{R}^{n \times 2d} \tag{10}$$

has rank 2d.

Note that Proposition 3 further implies that X is linearly independent of the intercept 1_n by our previous observation that $H^{-1}X$ is always colinear with the intercept.

Proof. We begin by observing that it will suffice to show that

$$\tilde{M} = \begin{bmatrix} Y & H_Y^{-1}Y \\ Z & H_Z^{-1}Z \end{bmatrix} \in \mathbb{R}^{2d \times 2d}$$

has full rank, since $\tilde{M} \in \mathbb{R}^{2d \times 2d}$ comprises 2d rows of the matrix M defined in equation (10). Since Y is invertible by linear independence of $Y_1, Y_2, \dots, Y_d, \tilde{M}$ is invertible if and only if the Schur complement

$$H_Z^{-1}Z-ZY^{-1}H_Y^{-1}Y\in\mathbb{R}^{d\times d}$$

is invertible. Multiplying by appropriate matrices, invertibility of the above matrix is equivalent to invertibility of

$$Z^{-1}H_Z^{-1}Z - Y^{-1}H_Y^{-1}Y,$$

completing the proof.

With Lemma 2 and Proposition 3 in hand, we are ready to sketch a proof of Theorem 2.

Proof sketch for Theorem 2. By Lemma 2, $\|W - W_n\|_{2,\infty} = o(1)$ almost surely. The asymptotic design matrix $W \in \mathbb{R}^{2d+2}$ has maximum rank 2d. This follows as $H^{-1}X\Lambda$ is colinear with the intercept, by definition of H and Λ . Similarly, $H^{-1}X\Lambda$ is colinear with $\frac{\alpha}{1-\beta}1_n + H^{-1}X\tilde{\gamma}$. Provided that we avoid the knife-edge failure case $\tilde{\gamma} = \mu$, $\frac{\alpha}{1-\beta}1_n + H^{-1}X\tilde{\gamma}$ is not colinear with the intercept. Thus, the maximum rank of W is 2d + 2 - 2 = 2d. If condition of Proposition 3 is satisfied, $[XH^{-1}X]$ has rank 2d, which implies that W has rank 2d, completing the proof.

Remark 5 (Identification failure in stochastic blockmodels). The condition of Proposition 3 requires that X has at least 2d distinct rows. We consider one common scenario where this is not the case. Suppose F is a mixture distribution over d points $\delta_1, \delta_2, ..., \delta_d \in \mathbb{R}^d$ and $\delta_1, ..., \delta_d$ are linearly independent. F could thus present represent the distribution of the latent positions of X for any stochastic blockmodel with full-rank mixing matrix B.

Let $\Delta = (\delta_1^T, \delta_2^T, ..., \delta_d^T)^T \in \mathbb{R}^{d \times d}$, such that Δ is full-rank by hypothesis. Suppose that $n_1, ..., n_d$ points are sampled from each of the atoms $\delta_1, ..., \delta_d$. Then, without loss of generality, we can reorder the rows of X and write

$$X = \begin{bmatrix} 1_{n_1} & 0 & \dots & 0 \\ 0 & 1_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1_{n_d} \end{bmatrix} \Delta \text{ and } H^{-1}X = \begin{bmatrix} (\delta_1^T \mu)^{-1} 1_{n_1} & 0 & \dots & 0 \\ 0 & (\delta_2^T \mu)^{-1} 1_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\delta_d^T \mu)^{-1} 1_{n_d} \end{bmatrix} \Delta$$

such that X and $H^{-1}X$ are colinear and $[X \ H^{-1}X] \in \mathbb{R}^{n \times 2d}$ only has rank d. Thus, the columns corresponding to direct effect γ and the interference effect δ are asymptotically colinear in stochastic blockmodels without degree correction.

B. Concentration inequalities and moment bounds

Here we collect a few technical results regarding moment bounds and concentration inequalities to be used in our main results.

DEFINITION 3 (BOUCHERON ET AL. 2013). Let Z be a mean-zero random variable with cumulant generating function $\psi_Z(t) = \log \mathbb{E}[e^{tZ}]$. Z is subgamma with parameters $v \ge 0$ and $b \ge 0$ if

$$\psi_{Z}(t) \leq \frac{t^{2} \nu}{2(1-bt)} \ \ and \ \ \psi_{-Z}(t) \leq \frac{t^{2} \nu}{2(1-bt)} \ \ for \ all \ \ t < 1/b.$$

If this is the case, we will often simply write that Z is (v, b)-subgamma.

Lemma 3 (Boucheron et al. (2013) Chapter 2). Suppose that Z is a (v, b)-subgamma random variable. Then for all t > 0,

$$\Pr\left[|X| > \sqrt{2\nu t} + bt\right] \le \exp\{-t\}$$

The following are basic results concerning subgamma random variables, which we prove for the sake of completeness.

Lemma 4. Let $Z_1, Z_2, ..., Z_n$ be a collection of independent (v, b)-subgamma random variables and let c > 0 be a constant. Then it holds with probability at least $1 - Cn^{-c}$ that

$$\max_{i \in [n]} |Z_i| \le C\sqrt{\nu + b^2} \log n,$$

Proof. For $t \ge 0$, applying a union bound followed by Lemma 3,

$$\Pr[\max_{i} |Z_{i}| > \sqrt{2\nu t} + bt] \le \sum_{i=1}^{n} \Pr[|Z_{i}| > \sqrt{2\nu t} + bt] \le n \exp\{-t\}.$$

Taking $t = C \log n$ for C > 0 chosen suitably large, it holds that with probability at least $1 - n^{-c}$,

$$\max_{i} |Z_{i}| \leq \sqrt{2C\nu \log n} + Cb \log n \leq C\sqrt{\nu + b^{2}} \log n,$$

as we set out to show. \Box

Lemma 5. Let $Z_1, Z_2, ..., Z_n$ be a collection of independent (v, b)-subgamma random variables and let $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ be nonnegative. Then, defining $S_n = \sum_i \alpha_i Z_i$, for any t > 0, for any constant c > 0, it holds with probability at least $1 - 2n^{-c}$ that

$$|S_n| \le C(v^{1/2} + b) \left(\sum_{i=1}^n \alpha_i^2\right)^{1/2} \log n$$
 (1)

and

$$|S_n| = O\left((v^{1/2} + b)\left(\sum_{i=1}^n \alpha_i^2\right)^{1/2} \log n\right)$$
 almost surely.

Proof. By a basic property of subgamma random variables (see Boucheron et al., 2013, Chapter 2), $\alpha_i Z_i$ is (ν_i, b_i) -subgamma, where $\nu_i = \alpha_i^2 \nu$ and $b_i = \alpha_i b_i$, and the random sum $S_n = \sum_i \alpha_i Z_i$ is a subgamma random variable with parameters

$$\bar{v} = \sum_{i=1}^{n} v_i = v \sum_{i=1}^{n} \alpha_i^2$$

$$\bar{b} = \max_{i \in [n]} b_i = b \max_{i \in [n]} \alpha_i \le b \sqrt{\sum_{i=1}^{n} \alpha_i^2}.$$

Thus, applying Lemma 3, for any t > 0,

$$\Pr\left[|S_n| > \sqrt{2\bar{\nu}t} + \bar{b}t\right] \le \exp\{-t\}.$$

Taking $t = C \log n$ for suitably large C > 0 and noting that $\bar{b} \le C \bar{v}^{1/2}$ for suitably-chosen constant C > 0, it follows that

$$\Pr\left[|S_n| > C(\bar{v}^{1/2}\log^{1/2}n + \bar{b}\log n\right] \le 2n^{-c}.$$

Observing that

$$\bar{v}^{1/2} \log^{1/2} n + \bar{b} \log n \le (v^{1/2} + b) \left(\sum_{i=1}^{n} \alpha_i^2 \right)^{1/2} \log n$$

establishes Equation (1). Taking $c > 1 + \epsilon$ and applying the Borel-Cantelli lemma then implies that

$$|S_n| = O\left(\bar{v}^{1/2} \log^{1/2} n + \bar{b} \log n\right)$$
 almost surely,

which completes the proof.

Lemma 6. Let $X_1, X_2, ..., X_n \in \mathbb{R}^d$ and let $\xi_1, \xi_2, ..., \xi_n$ be conditionally independent (v, b)-subgamma random variables given $X_1, X_2, ..., X_n$. Then with probability at least $1 - 2n^{-3}$,

$$\left\| \sum_{j=1}^{n} \xi_j X_j \right\| \le C \sqrt{\sum_{j=1}^{n} \|X_j\|^2} \sqrt{\nu + b^2} \log n.$$

Proof. Define the random vector $\widetilde{\xi} \in \mathbb{R}^d$ according to

$$\widetilde{\xi}_k = \sum_{i=1}^n \xi_j X_{jk}$$

for $k \in [d]$. We observe that conditional on $X_1, X_2, \ldots, X_n, \widetilde{\xi}_k$ is a sum of subgamma random variables. Applying Bernstein's inequality (Boucheron et al., 2013, Corollary 2.11), for t > 0,

$$\Pr\left[|\widetilde{\xi}_k| > t\right] \le 2 \exp\left\{\frac{-t^2}{2(\gamma_{ik} + b_{ik}t)}\right\},$$

where

$$v_{ik} = v \sum_{j=1}^{n} X_{jk}^{2} \quad \text{and}$$

$$b_{ik} = b \max_{i \in [n]} |X_{jk}|.$$

Taking $t = C\sqrt{v_{ik} + b_{ik}^2} \log n$ for C > 0 chosen suitably large, it holds with probability at least $1 - 2n^{-3}$ that

$$|\widetilde{\xi}_k| \le C \left(\nu \sum_{j=1}^n X_{jk}^2 + b^2 \max_{j \in [n]} |X_{jk}|^2 \right)^{1/2} \log n$$

$$\le C ||X_{\cdot k}||^2 \sqrt{\nu + b^2} \log n.$$

A union bound over all $j \in [d]$ followed by taking square roots implies that with probability at least $1 - 2n^{-3}$,

$$\|\widetilde{\xi}\| \le C \sqrt{\sum_{j=1}^n \|X_j\|^2} \sqrt{\nu + b^2} \log n,$$

completing the proof.

Lemma 7. Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a network with degree matrix $D \in \mathbb{R}^{n \times n}$, and let $\varepsilon \in \mathbb{R}^n$ be a vector of independent mean zero (v, b)-subgamma random variables with ε independent of A. Then, letting $G = D^{-1}A$, it holds with high probability that

$$\max_{i \in [n]} |[G\varepsilon]_i| \leq C \max_{i \in [n]} \sqrt{\nu_i \log^2 n} + C \max_{i \in [n]} b_i \log n,$$

where

$$v_i = v \sum_{j=1}^n \frac{A_{ij}^2}{D_i^2}$$
 and $b_i = b \max_{j \in [n]} \frac{A_{ij}}{D_i}$. (2)

Proof. Unrolling the definition, for $i \in [n]$,

$$[G\varepsilon]_i = \frac{1}{D_i} \sum_{j=1}^n A_{ij}\varepsilon_j,$$

which is a weighted sum of subgamma random variables. Lemma 5, conditional on A, implies that for t > 0,

$$\Pr\left[|[G\varepsilon]_i| > t \mid A\right] \le 2\exp\left\{\frac{-t^2}{2(\nu_i + b_i t)}\right\},\,$$

where v_i and b_i are as defined in Equation (2). Taking $t = C\sqrt{v_i + b_i^2} \log n$ for C > 0 suitably large, it holds with probability at least $1 - 2n^{-3}$ that

$$|[G\varepsilon]_i| \le C\sqrt{\nu_i + b_i^2} \log n.$$

Noting that C can be chosen independently of the index i, a union bound over all $i \in [n]$ implies that with probability at least $1 - 2n^{-2}$,

$$\max_{i \in [n]} |[G\varepsilon]_i| \le C \max_{i \in [n]} \sqrt{\nu_i + b_i^2} \log n \le C \max_{i \in [n]} \sqrt{\nu_i \log^2 n} + C \max_{i \in [n]} b_i \log n,$$

as we set out to show. \Box

Lemma 8. Suppose that for all $n \ge 1$, $Z \in \mathbb{R}^n$ is a vector of independent v_Z -subgaussian random variables with v_Z constant with respect to n and $M = M_n \in \mathbb{R}^{n \times n}$ is a (possibly random) matrix such that Z is independent of M. Then

$$\left| Z^{\top} M Z - \mathbb{E} Z^{\top} M Z \right| = O_P(\|M\|_F).$$

Proof. By the Hanson-Wright inequality (Rudelson & Vershynin, 2013; Vershynin, 2020),

$$\Pr\left[\left|Z^{\top}MZ - \mathbb{E}Z^{\top}MZ\right| > t\right] \le 2\exp\left\{-c\min\left\{\frac{t^2}{\nu_Z^2\|M\|_F^2}, \frac{t}{\nu_Z\|M\|}\right\}\right\}.$$

Setting $t = C\nu_Z ||M||_F$, we have

$$\Pr\left[\left|\frac{1}{n}Z^{\top}MZ - \frac{1}{n}\mathbb{E}Z^{\top}MZ\right| > C\nu_Z \|M\|_F\right] \le 2\exp\left\{-C\min\left\{1, \frac{\|M\|_F}{\|M\|}\right\}\right\} = 2\exp\{-C\}.$$

Choosing C > 0 suitably large makes this right-hand probability arbitrarily small, and it follows that

$$\left| Z^{\top} M Z - \mathbb{E} Z^{\top} M Z \right| = O_P(\|M\|_F),$$

as we set out to show. \Box

Lemma 9. Suppose that $Z \in \mathbb{R}^n$ is a vector of independent mean-zero random variables with shared variance σ_Z^2 and shared fourth moment ζ_4 and let $M \in \mathbb{R}^{n \times n}$ be a fixed matrix with Z not depending on M. Then

$$\mathbb{E}Z^{\mathsf{T}}MZ = \sigma_Z^2 \operatorname{tr} M \tag{3}$$

and

$$\mathbb{E}(Z^{\top}MZ)^2 \leq C\zeta_4 \left[\|M\|_F^2 + (\operatorname{tr} M)^2 \right].$$

Proof. Since Z is mean zero,

$$\mathbb{E}Z^{\mathsf{T}}MZ = \sigma_Z^2 \operatorname{tr} M$$

and Equation (3) is immediate.

Expanding the quadratic,

$$\mathbb{E}(Z^{\top}MZ)^{2} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} M_{ij} M_{k\ell} \mathbb{E} Z_{i} Z_{j} Z_{k} Z_{\ell}.$$

Noting that the expectations disappear unless $|\{i, j, k, \ell\}| \in \{2, 4\}$, we have

$$\mathbb{E}(Z^{\top}MZ)^{2} = \zeta_{4} \sum_{i=1}^{n} M_{ii}^{2} + \sigma_{Z}^{4} \sum_{i \neq j} (M_{ii}M_{jj} + M_{ij}^{2} + M_{ij}M_{ji}).$$

Applying Jensen's inequality and using the fact that $2ab \le (a^2 + b^2)$,

$$\mathbb{E}(Z^{\top}MZ)^{2} \leq C\zeta_{4} \left(\|M\|_{F}^{2} + \sum_{i \neq j} M_{ii} M_{jj} \right) \leq C\zeta_{4} \left[\|M\|_{F}^{2} + (\operatorname{tr} M)^{2} \right],$$

completing the proof.

Lemma 10. Suppose that $Z, \tilde{Z} \in \mathbb{R}^n$ are independent mean-zero random vectors. Let the entries of Z have independent entries with shared variance σ_Z^2 and shared fourth moment ζ_4 . Similarly, let the entries of \tilde{Z} have shared variance $\sigma_{\tilde{Z}}^2$ and shared fourth moment $\tilde{\zeta}_4$. Let $M \in \mathbb{R}^{n \times n}$ be a fixed matrix with Z and \tilde{Z} not depending on M. Then

$$\mathbb{E}\left(Z^{\top}M\tilde{Z}\right)^{2} = \sigma_{Z}^{2}\sigma_{\tilde{Z}}^{2}\|M\|_{F}^{2}.$$

Proof. Expanding the quadratic,

$$\mathbb{E}\left(Z^{\top}M\tilde{Z}\right)^{2} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \mathbb{E}M_{ij}M_{k\ell}Z_{i}Z_{k}\tilde{Z}_{j}\tilde{Z}_{\ell}.$$

Using the independence structure of Z and \tilde{Z} ,

$$\mathbb{E}\left(Z^{\top}M\tilde{Z}\right)^{2} = \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij}^{2} \sigma_{Z}^{2} \sigma_{\tilde{Z}}^{2},$$

completing the proof.

C. Proof of Proposition 2

Proof. By Proposition 1, it is sufficient to show that I, G and G^2 are linearly independent as vectors in $\mathbb{R}^{n\times n}$. Toward this end, let $c_1, c_2, c_3 \in \mathbb{R}$ be such that $c_1I + c_2G + c_3G^2 = 0$. The characteristic polynomial of this matrix must have n roots all equal to zero. It follows that for any eigenvalue λ of G, we must have $c_1 + c_2\lambda + c_3\lambda^2 = 0$. This quadratic has at most two distinct solutions for c_1, c_2, c_3 not all zero. Since G has more than two distinct eigenvalues, it follows that we must have $c_1 = c_2 = c_3 = 0$.

C.1. Degree concentration

The following result relate to control of the degree and associated quantities (e.g., their conditional expectations) to be used in the proof of our main results.

Lemma 11. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ . Denote the (conditional) expected degree by

$$\delta_i = \mathbb{E}[D_i \mid X] = \rho \sum_{i: j \neq i} X_i^T X_j. \tag{1}$$

Then with probability $1 - O(n^{-2})$,

$$\max_{i \in [n]} |D_i - \delta_i| \le C\sqrt{\nu + b^2} \sqrt{n} \log n.$$

Proof. Fix $i \in [n]$. We observe that

$$D_i - \delta_i = \sum_{j \in [n] \setminus \{i\}} (A_{ij} - \rho X_i^T X_j)$$

is, conditional on X, a sum of (v, b)-subgamma random variables. Applying Lemma 5 with suitably chosen constants, it holds with probability at least $1 - 2n^{-3}$ that

$$|D_i - \delta_i| \le C\sqrt{\nu + b^2} \sqrt{n} \log n.$$

A union bound over $i \in [n]$ completes the result.

Lemma 12. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ with F and the sparsity parameter ρ obeying the growth assumption in Equation (3).

Define the minimum expected degree

$$\delta_{\min} = \min_{i \in [n]} \delta_i,\tag{2}$$

where δ_i is as defined in Equation (1). Then

$$\min_{i\in[n]}D_i=\Omega(\delta_{\min}).$$

Proof. Applying Lemma 11, it holds with high probability that

$$\min_{i \in [n]} D_i \ge \delta_{\min} - C\sqrt{\nu + b^2} n^{1/2} \log n. \tag{3}$$

By Lemma 16,

$$\delta_{\min} = \Omega(n\rho \min_{i \in [n]} X_i^T \mu).$$

By our growth assumption in Equation (3) and the fact that $\rho \leq 1$,

$$n\rho \min_{i\in[n]} X_i^T \mu = \omega(\sqrt{n\rho}\log n) = \omega(\sqrt{\nu + b^2}\sqrt{n}\log n).$$

It follows that $\delta_{\min} = \omega(\sqrt{v + b^2}n^{1/2}\log n)$. Applying this to Equation (3) completes the proof.

Lemma 13. Suppose that $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ , obeying Assumption 1 and the growth assumption in Equation (3).

With δ_i as defined in Equation (1), it holds with probability at least $1 - O(n^{-2})$ that for all $i \in [n]$,

$$\left| \frac{1}{D_i} - \frac{1}{\delta_i} \right| \le \frac{C\sqrt{\nu + b^2}\sqrt{n}\log n}{\delta_i^2}.$$

Proof. By Lemma 16,

$$\delta_{\min} = \Omega(n\rho \min_{i \in [n]} X_i^T \mu).$$

Combining this with our growth assumption in Equation (3) and the fact that $\rho \leq 1$, then applying our growth assumption in Equation (2),

$$\delta_{\min} = \omega \left(\sqrt{\nu + b^2} \sqrt{n} \log n \right).$$

It follows that, applying Lemma 11 and trivially bounding $\delta_i \geq \delta_{\min}$, it holds for all $i \in [n]$ that

$$D_i \ge \delta_i - |D_i - \delta_i| \ge \delta_i - C\sqrt{\nu + b^2} n^{1/2} \log n \ge \delta_i (1 - o(1)).$$
 (4)

Applying Lemma 11 once more, it holds for all $i \in [n]$ that

$$\left|\frac{1}{D_i} - \frac{1}{\delta_i}\right| = \frac{|D_i - \delta_i|}{D_i \delta_i} \le \frac{C\sqrt{\nu + b^2} n^{1/2} \log n}{D_i \delta_i}.$$

Lower-bounding D_i using Equation (4) completes the proof.

LEMMA 14. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ , with F obeying Assumption 3. Let $r, q \in [0, \infty)$ be such that the expectation

$$\Xi_{r,q} = \mathbb{E}\frac{\|X_1\|^r}{(X_1^T \mu)^q}$$

exists and is finite. Then

$$\sum_{i=1}^{n} \frac{n^{q-1} \rho^{q} \|X_{i}\|^{r}}{\delta_{i}^{q}} \le (1 + o(1)) \Xi_{r,q}.$$

Proof. By Lemma 15, it holds uniformly over $i \in [n]$ that

$$\min_{i \in [n]} \frac{\delta_i}{n \rho X_i^T \mu} \ge 1 - o(1),$$

from which it follows that uniformly over $i \in [n]$,

$$\frac{n\rho}{\delta_i} \le \frac{1}{X_i^T \mu} \left(1 + o(1) \right).$$

Therefore,

$$\sum_{i=1}^{n} \frac{n^{q-1} \rho^{q} \|X_{i}\|^{r}}{\delta_{i}^{q}} = \frac{1}{n} \sum_{i=1}^{n} \|X_{i}\|^{r} \left(\frac{n\rho}{\delta_{i}}\right)^{q} \leq \frac{(1+o(1))^{q}}{n} \sum_{i=1}^{n} \frac{\|X_{i}\|^{r}}{\left(X_{i}^{T} \mu\right)^{q}}.$$

Applying the law of large numbers completes the proof.

LEMMA 15. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ , with F obeying Assumption 3. Then with high probability it holds that

$$\max_{i \in [n]} \frac{\left| \delta_i - n\rho X_i^T \mu \right|}{\delta_i} = o(1) \ almost surely. \tag{5}$$

and

$$\max_{i \in [n]} \frac{n\rho X_i^T \mu}{\delta_i} = \Theta(1) \ almost \ surely. \tag{6}$$

Proof. Recalling the definition of δ_i from Equation (1), for any $i \in [n]$,

$$\frac{\delta_i}{n\rho} - X_i^T \mu = \frac{1}{n} \sum_{j:j \neq i} X_i^T X_j - X_i^T \mu = X_i^T (\bar{X} - \mu) - \frac{\|X_i\|^2}{n}.$$

Applying the triangle inequality, Cauchy-Schwarz and standard concentration inequalities (Boucheron et al., 2013; Vershynin, 2020),

$$\left| \frac{\delta_i}{n\rho} - X_i^T \mu \right| \le \|X_i\| \|\bar{X} - \mu\| + \frac{\|X_i\|^2}{n} \le \frac{C\|X_i\| \log n}{\sqrt{n}} + \frac{\|X_i\|^2}{n}. \tag{7}$$

It follows that

$$\max_{i \in [n]} \frac{\left| \delta - n \rho X_i^T \mu \right|}{n \rho X_i^T \mu} \leq \frac{C \log n}{\sqrt{n}} \max_{i \in [n]} \frac{\left\| X_i \right\|}{X_i^T \mu} + \frac{C}{n} \max_{i \in [n]} \left\| X_i \right\|^2.$$

Applying Lemma 46 and our growth assumption in Equation (4),

$$\max_{i \in [n]} \frac{\left| \delta - n\rho X_i^T \mu \right|}{n\rho X_i^T \mu} = o(1) \text{ almost surely.}$$
 (8)

Noting that for $i \in [n]$,

$$\frac{n\rho X_i^T \mu}{\delta_i} = \left(1 + \frac{\delta_i - n\rho X_i^T \mu}{n\rho X_i^T \mu}\right)^{-1},$$

Equation (8) implies that

$$\max_{i \in [n]} \frac{n\rho X_i^T \mu}{\delta_i} = 1 + o(1) \text{ almost surely,}$$

establishing Equation (6).

Multiplying through by appropriate quantities, for any $i \in [n]$

$$\frac{\left|\delta_i - n\rho X_i^T \mu\right|}{\delta_i} = \frac{\left|\delta_i - n\rho X_i^T \mu\right|}{n\rho X_i^T \mu} \frac{n\rho X_i^T \mu}{\delta_i}.$$

Taking the maximum over $i \in [n]$ followed by an application of Equations (6) and (8) yields Equation (5), completing the proof.

LEMMA 16. Let $(A, X) \sim \text{RDPG}(F, n)$ with sparsity parameter ρ and recall the definition of δ_{\min} from Equation (2). Suppose that F is such that the growth assumption in Equation (3) holds. Then

$$\delta_{\min} = \Omega\left(n\rho \min_{i \in [n]} X_i^T \mu\right).$$

Proof. We recall that for $i \in [n]$,

$$\delta_i = \rho \sum_{j: j \neq i} X_i^T X_j = (n\rho) X_i^T \mu + (n\rho) X_i^T (\bar{X} - \mu) - \rho \|X_i\|^2.$$

Taking the minimum over all $i \in [n]$,

$$\delta_{\min} \ge (n\rho) \left(\min_{i \in [n]} X_i^T \mu \right) + (n\rho) \left(\min_{i \in [n]} X_i^T (\bar{X} - \mu) \right) + \rho \min_{i \in [n]} \|X_i\|^2. \tag{9}$$

By standard concentration inequalities (Boucheron et al., 2013), $\|\bar{X} - \mu\| = O(n^{-1/2} \log n)$, and thus

$$\left| (n\rho) \min_{i \in [n]} X_i^T (\bar{X} - \mu) \right| \le (n\rho) \left\| \bar{X} - \mu \right\| \min_{i \in [n]} \|X_i\| \le C\rho \left(\sqrt{n} \log n \right) \min_{i \in [n]} \|X_i\|. \tag{10}$$

Bounding the minimum by the average and appealing to the law of large numbers,

$$\left(\sqrt{n}\log n\right)\min_{i\in[n]}\|X_i\|\leq \left(\sqrt{n}\log n\right)\frac{1}{n}\sum_{i\in[n]}\|X_i\|=O(\sqrt{n}\log n).$$

Applying our growth assumption in Equation (3) and using the fact that $\rho \leq 1$ by assumption,

$$\left(\sqrt{n}\log n\right)\min_{i\in[n]}\|X_i\|=o(n\min_{i\in[n]}X_i^T\mu).$$

Applying this to Equation (10),

$$(n\rho) \min_{i \in [n]} X_i^T (\bar{X} - \mu) = o(n\rho \min_{i \in [n]} X_i^T \mu).$$
 (11)

Again applying the law of large numbers,

$$\min_{i \in [n]} \|X_i\|^2 \le \frac{1}{n} \sum_{i=1}^n \|X_i\|^2 = O(1).$$

Since Equation (11) trivially implies $n \min_i X_i^T \mu = \omega(n^{-1})$, it follows that

$$\min_{i \in [n]} ||X_i||^2 = o(n \min_{i \in [n]} X_i^T \mu).$$

Applying the above display and Equation (11) to Equation (9),

$$\delta_{\min} \ge (1 - o(1)) (n\rho) \min_{i \in [n]} X_i^T \mu,$$

completing the proof.

C.2. Spectral results

Here we collect results related to the spectral properties of the adjacency matrix A and its expected value $P = \rho XX^T$ under the RPDG.

LEMMA 17. Let $(A, X) \sim \text{RDPG}(F, n)$ with sparsity parameter ρ . Then, defining

$$\mathcal{D} = \operatorname{diag}(\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}^{n \times n},\tag{12}$$

we have

$$\|\rho X^T \mathcal{D}^{-1} X\| = 1.$$

Proof. We note first that since for a matrix B, the matrices B^TB and BB^T have the same non-zero eigenvalues. As such, $||B^TB|| = ||BB^T||$. In particular, taking $B = \mathcal{D}^{-1/2}X$,

$$\|\rho X^T \mathcal{D}^{-1} X\| = \|\rho \mathcal{D}^{-1/2} X X^T \mathcal{D}^{-1/2}\|.$$
 (13)

Now, suppose that $u \in \mathbb{R}^n$ is an eigenvector of $\rho \mathcal{D}^{-1/2} X X^T \mathcal{D}^{-1/2}$ with eigenvalue λ , so that

$$\rho \mathcal{D}^{-1/2} X X^T \mathcal{D}^{-1/2} u = \lambda u.$$

Then we have

$$\rho \mathcal{D}^{-1} X X^T (\mathcal{D}^{-1/2} u) = \mathcal{D}^{-1/2} \left(\rho \mathcal{D}^{-1/2} X X^T \mathcal{D}^{-1/2} u \right) = \lambda \mathcal{D}^{-1/2} u.$$

In particular, λ is also an eigenvalue of $\rho \mathcal{D}^{-1}XX^T$, albeit with a different eigenvector. It follows that, using the fact that $\mathcal{D}^{-1}XX^T$ is row-stochastic and thus has largest eigenvalue 1,

$$\left\|\rho \mathcal{D}^{-1/2} X X^T \mathcal{D}^{-1/2}\right\| = \left\|\rho \mathcal{D}^{-1} X X^T\right\| = 1.$$

Combining this with Equation (13) above completes the proof.

The following lemmas collect a few basic facts about the random walk Laplacian G.

Lemma 18. Let $G = D^{-1}A$ be the transition matrix of a network with adjacency matrix A and degree matrix D. The eigenvalues of G are all real, and $||G|| \le 1$.

Proof. To see that all eigenvalues of G are real, let $\lambda \in \mathbb{R}$ be such that $D^{-1/2}AD^{-1/2}u = \lambda u$ for some eigenvector $u \in \mathbb{R}^n$. Then, multiplying by $D^{-1/2}$ on both sides, $D^{-1}AD^{-1/2}u = \lambda D^{-1/2}u$, so that $D^{-1/2}u$ is an eigenvector of G with eigenvalue λ . Thus, G and $D^{-1/2}AD^{-1/2}$ have the same spectrum. Since $D^{-1/2}AD^{-1/2}$ is symmetric, all its eigenvalues are real.

Since $G = D^{-1}A$ is row stochastic, $||G|| \le 1$ follows from the Perron-Frobenius theorem.

LEMMA 19. Let $G = D^{-1}A$ be the transition matrix of a network with adjacency matrix A and degree matrix D. If $|\beta| < 1$ then $I - \beta G$ is invertible and has all eigenvalues in the interval $1 \pm \beta$. Further,

$$\|(I - \beta G)^{-1}\|_{\infty} \le \frac{1}{1 - |\beta|}.$$
 (14)

Proof. By Lemma 18, all eigenvalues of βG have absolute value at most $|\beta|$, from which all eigenvalues of $I - \beta G$ have absolute value in $1 \pm \beta$. In particular, the eigenvalues of $I - \beta G$ are bounded away from zero, ensuring that the matrix is invertible.

To prove Equation (14), note that by the Neumann expansion and the triangle inequality,

$$\|(I - \beta G)^{-1}\|_{\infty} \le \sum_{q=0}^{\infty} |\beta|^q \|G^q\|_{\infty}.$$

Since G^q is row-stochastic for any q = 0, 1, 2, ..., we have $||G^q||_{\infty} \le 1$ trivially, and thus

$$\|(I - \beta G)^{-1}\|_{\infty} \le \sum_{q=0}^{\infty} |\beta|^q = \frac{1}{1 - |\beta|},$$

completing the proof.

LEMMA 20. Let $(A, X) \sim \text{RDPG}(F, n)$. Then

$$||X|| = O(\sqrt{n})$$
 almost surely.

Proof. By the law of large numbers,

$$\left\| \frac{X^T X}{n} - \Lambda \right\| = o(1).$$

Multiplying through by n,

$$\left\| X^T X - n\Lambda \right| = o(n).$$

Applying the triangle inequality and using this bound,

$$\left\|X\right\|^2 = \left\|X^TX\right\| \leq \left\|n\Lambda\right\| + \left\|X^TX - n\Lambda\right\| \leq n\left\|\Lambda\right\| + o(n).$$

Taking square roots and recalling that Λ is assumed constant completes the proof.

Lemma 21. Let $(A, X) \sim \text{RDPG}(F, n)$ with sparsity parameter ρ . Denoting $P = \rho X X^T$, with probability at least $1 - O(n^{-2})$,

$$||A - P|| \le C\sqrt{\nu + b^2}\sqrt{n}\log n.$$

Proof. This is a special case of Lemma 5 in Levin et al. (2022), obtained by setting N = 1 and taking all the v_{ij} and b_{ij} parameters to be identically v and b, respectively.

D. Proof of Lemma 1

Here, we provide proof details for Lemma 1 and Theorem 1. The proof of Lemma 1 relies on two basic lemmas.

Lemma 22. Under the setting of Lemma 1,

$$\max_{i \in [n]} |[GT]_i - \tau| \to 0 \quad almost surely.$$

Proof. For each $i \in [n]$, define

$$\tilde{T}_i = [GT]_i = [D^{-1}AT]_i = \frac{1}{d_i} \sum_{i=1}^n A_{ij}T_j.$$

For ease of notation, define $W_{ij} = A_{ij}/d_i$ for each $i, j \in [n]$, so that

$$\tilde{T}_i = \sum_{i=1}^n W_{ij} T_j.$$

Recalling that $\mathbb{E}[T_i \mid A] = \tau$, Lemma 7 implies that

$$\max_{i \in [n]} \left| \tilde{T}_i - \tau \right| \le C \left(\sqrt{\nu} \log n \right) \max_{i \in [n]} \sqrt{\sum_{j=1}^n \frac{A_{ij}^2}{d_i^2}} + C \left(b \log n \right) \max_{i \in [n]} \max_{j \in [n]} \frac{A_{ij}}{d_i}.$$

Applying the Borel-Cantelli lemma conditional on any sequence of networks obeying our growth assumptions in Equation (5),

$$\max_{i \in [n]} |\tilde{T}_i - \tau| = o(1) \text{ almost surely,}$$

as we set out to show.

Lemma 23. Under the same setting as Lemma 22, defining

$$\xi = (I - \beta G)^{-1} G^2 T = \sum_{k=0}^{\infty} \beta^k G^{k+2} T \in \mathbb{R}^n,$$

we have

$$\max_{i \in [n]} |\xi_i - \tau| \to 0 \quad almost surely.$$

Proof. We observe that trivially, since G is row-stochastic,

$$(I - \beta G)^{-1}G(\tau 1_n) = \tau 1_n.$$

Thus, for $i \in [n]$,

$$\begin{split} \xi_i - \tau &= \left[(I - \beta G)^{-1} G^2 T \right]_i - \tau = \left[(I - \beta G)^{-1} \, G \, (GT - \tau \mathbb{1}_n) \right]_i \\ &= \sum_{j=1}^n \left[(I - \beta G)^{-1} \, G \right]_{ij} \left[(GT)_j - \tau \mathbb{1}_n \right]_j. \end{split}$$

Applying the triangle inequality and bounding the entries of $GT - \tau 1_n$ by their maximum,

$$|(I - \beta G)^{-1} G (GT - \tau 1_n)|_i \le \left(\max_{i \in [n]} |[GT]_i - \tau| \right) \sum_{j=1}^n |(I - \beta G)^{-1} G|_{ij}.$$

By Lemma 22, it will suffice for us to establish that

$$\max_{i \in [n]} \sum_{j=1}^{n} \left| (I - \beta G)^{-1} G \right|_{ij} = O(1). \tag{1}$$

We observe that if $\beta = 0$, then for any $i \in [n]$,

$$\sum_{j=1}^{n} \left| (I - \beta G)^{-1} G \right|_{ij} = \sum_{j=1}^{n} |G|_{ij} = \frac{1}{d_i} \sum_{j=1}^{n} A_{ij} = 1,$$

and Equation (1) holds.

If $\beta \neq 0$, then by the Neumann expansion,

$$\sum_{j=1}^{n} \left| (I - \beta G)^{-1} G \right|_{ij} = \sum_{j=1}^{n} \left| \sum_{q=0}^{\infty} \beta^{q} G^{q+1} \right|_{ij} \le \frac{1}{|\beta|} \sum_{j=1}^{n} \sum_{q=1}^{\infty} |\beta|^{q} |G^{q}|_{ij}. \tag{2}$$

Since each G^q is itself a transition matrix, we have for any $i \in [n]$ and any $q = 1, 2, 3, \ldots$

$$\sum_{j=1}^{n} [G^q]_{ij} = 1.$$

It follows that

$$\sum_{j=1}^{n} \left| (I - \beta G)^{-1} G \right|_{ij} \le \frac{1}{|\beta|} \sum_{q=1}^{\infty} |\beta|^{q} = \frac{1}{|\beta|(1 - |\beta|)}.$$

Since this right-hand side is constant with respect to n and does not depend on $i \in [n]$, we have established Equation (1), completing the proof.

With the above two lemmas in hand, Lemma 1 follows straightforwardly.

Proof of Lemma 1. By Lemma 22,

$$\max_{i \in [n]} |[GT]_i - \tau| \to 0 \quad \text{almost surely,}$$
 (3)

where we remind the reader that $\tau = \mathbb{E}T_1$ is the shared expectation of the node-level covariates. Similarly, by Lemma 23,

$$\max_{i \in [n]} \left| \left[(I - \beta G)^{-1} G^2 T \right]_i - \tau \right| \to 0 \quad \text{almost surely.}$$
 (4)

By an argument that mirrors that in Lemma 22,

$$\max_{i \in [n]} \left| \left[(I - \beta G)^{-1} G \epsilon \right]_i \right| \to 0 \quad \text{almost surely.}$$
 (5)

Multiplying both sides of Equation (4) by G,

$$GY = \frac{\alpha}{1-\beta} \mathbf{1}_n + (I-\beta G)^{-1} GT\gamma + (I-\beta G)^{-1} G^2 T\delta + (I-\beta G)^{-1} G\epsilon.$$

Applying the triangle inequality and Cauchy-Schwarz,

$$\begin{aligned} \max_{i \in [n]} \left| [GY]_i - \frac{\alpha}{1 - \beta} - \tau(\gamma + \delta) \right| &\leq \|\gamma\| \max_{i \in [n]} \left\| \left[(I - \beta G)^{-1} GT \right]_i - \tau \right\| \\ &+ \|\delta\| \max_{i \in [n]} \left\| (I - \beta G)^{-1} G^2 T - \tau \right\| \\ &+ \max_{i \in [n]} \left| \left[(I - \beta G)^{-1} G \epsilon \right]_i \right|. \end{aligned}$$

Taking $\eta = \alpha/(1-\beta) + \tau(\gamma + \delta)$ and applying Equations (3), (4) and (5), we have shown that

$$\max_{i \in [n]} |[GY]_i - \eta| \to 0 \quad \text{almost surely,}$$

completing the proof.

E. Proof of Theorem 1

Here, we prove Theorem 1. The proof consists of two basic steps. The first step shows that $\hat{\theta}_n - \theta_n \in \mathbb{R}^4$ concentrates, asymptotically, on a subspace of dimension two. This step relies, essentially, on controlling the variation of the design matrix W_n degined in Equation (3), about a simpler design matrix. These technical lemmas are proved in Section F below. Our second step consists in showing that the portion of $\hat{\theta}_n - \theta_n$ that concentrates on this subspace lower bounded in magnitude by $\|G\|_F$. To show this, we use the Payley-Zygmund inequality to lower bound several inner products between ε and the columns of W_n . These lower-bounds are established in Section G.

Proof of Theorem 1. Recall that we have defined $\hat{\theta}_n = (\hat{\alpha}, \widehat{\beta}, \hat{\gamma}, \hat{\delta})$ to be the ordinary least squares estimate of $\theta = (\alpha, \beta, \gamma, \delta)$. Note that we subscript $\hat{\theta}_n$ to remind the reader that its behavior changes as the network grows. Since $Y = W_n \theta + \varepsilon$, it follows that

$$\hat{\theta}_n - \theta = \left(W_n^{\mathsf{T}} W_n\right)^{-1} W_n^{\mathsf{T}} \varepsilon. \tag{1}$$

Write the singular value decomposition of W_n as

$$W_n = \widehat{V}\widehat{S}\widehat{U}^{\top} \in \mathbb{R}^{n \times 4}$$
.

where, letting $\hat{s}_1 \ge \hat{s}_2 \ge \hat{s}_3 \ge \hat{s}_4$ denote the singular values of W_n ,

$$\widehat{S} = \operatorname{diag}(\widehat{s}_1, \widehat{s}_2, \widehat{s}_3, \widehat{s}_4) = \begin{bmatrix} \widehat{S}_{>} & 0 \\ 0 & \widehat{S}_0 \end{bmatrix} \in \mathbb{R}^{4 \times 4},$$

where $\widehat{S}_{>} = \operatorname{diag}(\widehat{s}_1, \widehat{s}_2)$ and $\widehat{S}_0 = \operatorname{diag}(\widehat{s}_3, \widehat{s}_4)$. We partition the columns of \widehat{U} and \widehat{V} correspondingly as

$$W_n = \widehat{V}_{>} \widehat{S}_{>} \widehat{U}_{>}^{\top} + \widehat{V}_0 \widehat{S}_0 \widehat{U}_0^{\top}. \tag{2}$$

That is, letting $\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4 \in \mathbb{R}^4$ denote the right singular vectors of W_n corresponding to $\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4$, respectively,

$$\widehat{U}_{>} = \left[\hat{u}_1 \ \hat{u}_2 \right] \in \mathbb{R}^{4 \times 2} \text{ and } \widehat{U}_0 = \left[\hat{u}_3 \ \hat{u}_4 \right] \in \mathbb{R}^{4 \times 2}, \tag{3}$$

with $\widehat{V}_{>}$, $\widehat{V}_{0} \in \mathbb{R}^{n}$ defined analogously.

Rewriting Equation (1) accordingly, we have

$$\hat{\theta}_n - \theta = \widehat{U}_0 \widehat{S}_0^{-2} \widehat{U}_0^\top W_n^\top \varepsilon + \widehat{U}_{>} \widehat{S}_{>}^{-2} \widehat{U}_{>}^\top W_n^\top \varepsilon. \tag{4}$$

Applying submultiplicativity followed by Lemmas 27 and 28 (both proved below in Section F),

$$\left\|\widehat{U}_{>}\widehat{S}_{>}^{-2}\widehat{U}_{>}^{\top}W_{n}^{\top}\varepsilon\right\| \leq \left\|\widehat{S}_{>}^{-2}\right\|\left\|W_{n}^{\top}\varepsilon\right\| = O_{P}\left(\frac{1}{\sqrt{n}}\right).$$

Applying this fact to Equation (4),

$$\hat{\theta}_n - \theta = \widehat{U}_0 \widehat{S}_0^{-2} \widehat{U}_0^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon + o_P \left(\frac{1}{\sqrt{n}} \right). \tag{5}$$

As population analogues to \widehat{U}_0 and $\widehat{U}_{>}$ defined above, define $U_0, U_{>} \in \mathbb{R}^{4 \times 2}$ according to

$$U_0 = \begin{bmatrix} u_{\eta} & u_{\tau} \end{bmatrix}$$
 and $U_> = \begin{bmatrix} e_3 & u_{\circ} \end{bmatrix}$, (6)

where $u_{\eta}, u_{\tau}, u_{\circ}, e_3 \in \mathbb{R}^4$ are given by

$$u_{\tau} = \frac{1}{\sqrt{1+\tau^2}} \begin{bmatrix} -\tau \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad u_{\eta} = \frac{1}{\sqrt{1+\eta^2}} \begin{bmatrix} -\eta \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_{\circ} = \frac{1}{\sqrt{1+\eta^2+\tau^2}} \begin{bmatrix} 1 \\ \eta \\ 0 \\ \tau \end{bmatrix} \quad \text{and} e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{7}$$

where $\tau = \mathbb{E}T_1$ and η is as defined in Equation (6).

Let $Q \in \mathbb{R}^{2 \times 2}$ be the orthogonal matrix guaranteed by Lemma 31. Adding and subtracting appropriate quantities,

$$\widehat{U}_0 \widehat{S}_0^{-2} \widehat{U}_0^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon = U_0 Q \widehat{S}_0^{-2} \widehat{U}_0^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon + (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} \widehat{U}_0^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon. \tag{8}$$

Again adding and subtracting appropriate quantities and applying the triangle inequality,

$$\left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} \widehat{U}_0^\top W_n^\top \varepsilon \right\| \le \left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top W_n^\top \varepsilon \right\| + \left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} Q^\top U_0^\top W^\top \varepsilon \right\|. \tag{9}$$

By basic properties of the norm,

$$\left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top W_n^\top \varepsilon \right\| \leq \left\| \widehat{U}_0 - U_0 Q \right\|^2 \left\| \widehat{S}_0^{-2} \right\| \left\| W_n^\top \varepsilon \right\|.$$

Applying Lemmas 31, 27 and 28 (all proved below in Section F),

$$\left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top W_n^\top \varepsilon \right\| = O_P \left(\frac{1}{\sqrt{n}} \right). \tag{10}$$

Again applying basic properties of the norm followed by Lemmas 31 and 27,

$$\left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} Q^\top U_0^\top W^\top \varepsilon \right\| \le \left\| U_0^\top W^\top \varepsilon \right\| O_P \left(\frac{1}{\|G\|_F \sqrt{n}} \right)$$

Recalling that U_0 has u_{τ} and u_{η} as its columns, Lemma 28 implies that

$$||U_0^\top W^\top \varepsilon|| = O_P (||G||_F + ||G||_F^2 \mathbf{1}_{\{\beta \neq 0\}}).$$

Thus, we have

$$\left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} Q^\top U_0^\top W^\top \varepsilon \right\| = O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right). \tag{11}$$

Applying Equations (10) and (11) to Equation (9),

$$\left\| (\widehat{U}_0 - U_0 Q) \widehat{S}_0^{-2} \widehat{U}_0^\top W_n^\top \varepsilon \right\| = O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

Applying this to Equation (8) in turn,

$$\widehat{U}_0\widehat{S}_0^{-2}\widehat{U}_0^\top W_n^\top \varepsilon = U_0 Q \widehat{S}_0^{-2} \widehat{U}_0^\top W_n^\top \varepsilon + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

Finally, plugging this into Equation (5),

$$\hat{\theta}_n - \theta = U_0 Q \widehat{S}_0^{-2} \widehat{U}_0^\top W_n^\top \varepsilon + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right). \tag{12}$$

Define a population design matrix

$$\overline{W}_n = \left[1_n \ \eta 1_n \ T \ \tau 1_n \right]. \tag{13}$$

and note that u_{τ} and u_{η} as defined in Equation (7) are in the null space of \overline{W}_n , so that

$$\overline{W}U_0=0.$$

Adding and subtracting appropriate quantities,

$$U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon = U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^{\mathsf{T}} (W_n - \overline{W}_n)^{\mathsf{T}} \varepsilon + U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^{\mathsf{T}} \overline{W}_n^{\mathsf{T}} \varepsilon.$$
(14)

Applying basic properties of the norm along with Lemmas 27, 31 and 28,

$$\left\| U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top (W_n - \overline{W}_n)^\top \varepsilon \right\| \le \|\widehat{S}_0^{-2}\| \|\widehat{U}_0 - U_0 Q\| \|(W_n - \overline{W}_n)^\top \varepsilon\| = O_P \left(\frac{1}{\sqrt{n}} \right). \tag{15}$$

Applying basic properties of the norm and recalling that U_0 and Q have spectral norms at most 1,

$$\left\| U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top \overline{W}_n^\top \varepsilon \right\| \le \left\| \widehat{S}_0^{-2} \right\| \left\| Q - U_0^\top \widehat{U} \right\| \left\| \overline{W}^\top \varepsilon \right\|$$

Applying Lemma 27 and using the law of large numbers to write $\|\overline{W}_n^{\mathsf{T}}\varepsilon\| = O_P(\sqrt{n})$,

$$\left\| U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top \overline{W}_n^\top \varepsilon \right\| \le \left\| Q - U_0^\top \widehat{U} \right\| O_P \left(\frac{\sqrt{n}}{|G|_F^2} \right). \tag{16}$$

By standard arguments (see, e.g., Cape et al., 2019, Lemma 6.7),

$$\left\| Q - U_0^{\mathsf{T}} \widehat{U} \right\| \le \left\| \sin \Theta(U_0, \widehat{U}_0) \right\|^2, \tag{17}$$

where $\Theta(U_0, \widehat{U}_0)$ denotes the diagonal matrix of principal angles between the subspaces spanned by U_0 and \widehat{U}_0 (see, e.g., Bhatia, 1997, Chapter VII)

Applying the Davis-Kahan theorem (Davis & Kahan, 1970; Yu et al., 2015),

$$\left\| \sin \Theta(U_0, \widehat{U}_0) \right\| \le \frac{\left\| W^{\mathsf{T}} W - \overline{W}^{\mathsf{T}} \overline{W} \right\|_F}{\Omega_P(n)},$$

where we have used Lemma 30 to lower-bound the eigengap of $\overline{W}^{\mathsf{T}}\overline{W}$. Applying Lemma 29,

$$\left\|\sin\Theta(U_0,\widehat{U}_0)\right\| = O_P\left(\frac{\|G\|_F}{\sqrt{n}}\right).$$

Plugging this bound into Equation (17),

$$\left\| Q - U_0^{\mathsf{T}} \widehat{U} \right\| = O_P \left(\frac{\|G\|_F^2}{n} \right). \tag{18}$$

Plugging Equation (18) into Equation (16),

$$\left\| U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^{\top} \overline{W}_n^{\top} \varepsilon \right\| = O_P \left(\frac{1}{\sqrt{n}} \right).$$

Applying this and Equation (15) to Equation (14),

$$\left\| U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0 Q)^\top W_n^\top \varepsilon \right\| = O_P \left(\frac{1}{\sqrt{n}} \right). \tag{19}$$

Adding and subtracting appropriate quantities in Equation (12).

$$\hat{\theta}_n - \theta = U_0 Q \widehat{S}_0^{-2} U_0^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon + U_0 Q \widehat{S}_0^{-2} (\widehat{U}_0 - U_0)^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right),$$

and Equation (19) yields

$$\hat{\theta}_n - \theta = U_0 Q \hat{S}_0^{-2} Q^\top U_0^\top W_n^\top \varepsilon + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right). \tag{20}$$

It remains for us to remove the orthogonal transformation on \widehat{S}_0^{-2} , which we do following an argument similar to Lemma 17 in Lyzinski et al. (2017). Adding and subtracting appropriate quantities,

$$Q\widehat{S}_0^{-2} = (Q - \widehat{U}_0^\top U_0)\widehat{S}_0^{-2} + \widehat{U}_0^\top U_0\widehat{S}_0^{-2}.$$

Applying Equation (18) and Lemma 27,

$$Q\widehat{S}_{0}^{-2} = \widehat{U}_{0}^{\top} U_{0} \widehat{S}_{0}^{-2} + O_{P} \left(\frac{1}{n}\right).$$

Again adding and subtracting appropriate quantities,

$$Q\widehat{S}_{0}^{-2} = \widehat{U}_{0}^{\top} \left(U_{0} \widehat{S}_{0}^{-2} U_{0}^{\top} - \widehat{U}_{0} \widehat{S}^{-2} \widehat{U}_{0}^{\top} \right) U_{0} + \widehat{S}^{-2} \widehat{U}_{0}^{\top} U_{0} + O_{P} \left(\frac{1}{n} \right). \tag{21}$$

Adding and subtracting appropriate quantities and using submultiplicativity of the norm,

$$\left\| U_0 \widehat{S}_0^{-2} U_0^{\mathsf{T}} - \widehat{U}_0 \widehat{S}^{-2} \widehat{U}_0^{\mathsf{T}} \right\| \le \|\widehat{S}_0^{-2}\| \|\widehat{U}_0 - U_0 Q\| \left(\|\widehat{U}_0\| + \|U_0\| \right),$$

so that Lemma 27 and Lemma 31 imply

$$\left\| U_0 \widehat{S}_0^{-2} U_0^{\top} - \widehat{U}_0 \widehat{S}^{-2} \widehat{U}_0^{\top} \right\| = O_P \left(\frac{1}{\sqrt{n} \|G\|_F} \right).$$

Applying this bound to Equation (21) and recalling that $||G||_F \ge 1$,

$$Q\widehat{S}_0^{-2} = \widehat{S}^{-2}\widehat{U}_0^{\top}U_0 + O_P\left(\frac{1}{\sqrt{n}}\right).$$

It follows that

$$Q\widehat{S}_0^{-2} - \widehat{S}_0^{-2}Q = \widehat{S}^{-2}\left(\widehat{U}_0^\top U_0 - Q\right),$$

and another application of Equation (18) and unitary invariance of the norm yields

$$\left\| Q \widehat{S}_0^{-2} Q^\top - \widehat{S}_0^{-2} \right\| = O_P \left(\frac{1}{\sqrt{n}} \right).$$

Adding and subtracting appropriate quantities in Equation (20) and applying this fact,

$$\hat{\theta}_n - \theta = U_0 \widehat{S}_0^{-2} U_0^\top W_n^\top \varepsilon + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

Now, applying a unit-norm contrast $z \in \mathbb{R}^4$,

$$z^{\top} \left(\hat{\theta}_n - \theta \right) = z^{\top} U_0 \widehat{S}_0^{-2} U_0^{\top} W_n^{\top} \varepsilon + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

For ease of notation, define the random vector $h = (h_1, h_2)^{\top} \in \mathbb{R}^2$ as

$$h = U_0^\top W_n^\top \varepsilon = \begin{bmatrix} u_\tau^\top W_n^\top \varepsilon \\ u_\eta^\top W_n^\top \varepsilon \end{bmatrix}.$$

Then we have, writing $\widehat{S}_0 = \operatorname{diag}(\widehat{s}_{\tau}, \widehat{s}_{\eta})$,

$$\boldsymbol{z}^{\top} \boldsymbol{U}_{0} \widehat{\boldsymbol{S}}_{0}^{-2} \boldsymbol{U}_{0}^{\top} \boldsymbol{W}_{n}^{\top} \boldsymbol{\varepsilon} = \left[\boldsymbol{z}^{\top} \boldsymbol{u}_{\tau} \ \boldsymbol{z}^{\top} \boldsymbol{u}_{\eta} \right] \widehat{\boldsymbol{S}}_{0}^{-2} \boldsymbol{h},$$

so that

$$z^{\top} \left(\hat{\theta}_n - \theta \right) = \left(\frac{z^{\top} u_{\tau} h_1}{\hat{s}_{\tau}^2} + \frac{z^{\top} u_{\eta} h_2}{\hat{s}_{\eta}^2} \right) + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right). \tag{22}$$

Applying Lemma 33 with M = G,

$$|h_1| = \left| u_{\tau}^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon \right| = \left| (T - \tau 1_n)^{\mathsf{T}} G \varepsilon \right| = \Omega_P(\|G\|_F). \tag{23}$$

If $\beta \neq 0$, Lemma 37 yields

$$|h_2| = \Omega ||G||_F^2,$$

while if $\beta = 0$, Lemma 39 yields

$$|h_2| = \Omega ||G||_F$$
.

Combining the above two displays,

$$|h_2| = \Omega_P \Big(||G||_F + ||G||_F^2 \mathbf{1}_{\{\beta \neq 0\}} \Big). \tag{24}$$

We observe that if z is any of the 4-dimensional basis vectors e_1 , e_2 or e_4 , then at least one of $z^T u_\tau$ and $z^T u_\eta$ are nonzero. Taking $z = e_4$ in Equation (22), we have $e_4^T u_\tau = 1$ while $e_2^T u_\eta = 0$, so that

$$\hat{\delta} - \delta = e_4^{\top} (\hat{\theta}_n - \theta) = \frac{h_1}{\hat{s}_{\tau}^2} + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

Applying Equation (23) and using Lemma 27 to lower-bound the terms in the denominators arising from \widehat{S}_0 ,

$$\left|\hat{\delta} - \delta\right| = \Omega_P \left(\frac{1}{\|G\|_F}\right) + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}}\right).$$

When $\beta = 0$, our growth bound in Equation (7) implies

$$\left|\hat{\delta} - \delta\right| = \Omega_P \left(\frac{1}{\|G\|_F}\right),\tag{25}$$

while when $\beta = 0$, the stronger assumption $||G||_F^2 = o(n)$ is sufficient to ensure

$$\left|\hat{\delta} - \delta\right| = \Omega_P \left(\frac{1}{\|G\|_F}\right). \tag{26}$$

Taking $z = e_2$ in Equation (22), we have $e_2^{\mathsf{T}} u_{\tau} = 0$ while $e_2^{\mathsf{T}} u_{\eta} = 1$, so that

$$\widehat{\beta} - \beta = \frac{h_2}{\widehat{s}_\eta^2} + O_P \bigg(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \bigg).$$

Applying Equation (24) and using Lemma 27 to lower-bound the terms in the denominators arising from \widehat{S}_0 ,

$$\left|\widehat{\beta} - \beta\right| = \Omega_P \left(\frac{1}{\|G\|_F} + \mathbf{1}_{\{\beta \neq 0\}}\right) + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}}\right),$$

and our growth assumption in Equation (7) is sufficient to ensure that

$$\left|\widehat{\beta} - \beta\right| = \Omega_P \left(\frac{1}{\|G\|_F} + \mathbf{1}_{\{\beta \neq 0\}}\right). \tag{27}$$

Finally, taking $z = e_1$, we have $e_1^{\mathsf{T}} u_{\tau} = -\tau$ and $e_1^{\mathsf{T}} u_{\eta} = -\eta$, so that

$$\hat{\alpha} - \alpha = e_1^{\top} \left(\hat{\theta}_n - \theta \right) = \frac{-\tau h_1}{\hat{s}_{\tau}^2} + \frac{-\eta h_2}{\hat{s}_n^2} + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

Observing that h_1 and h_2 both contain factors of the form $\varepsilon^T M(T - \tau 1_n)$ and that h_2 contains a quadratic form in ε , we can ensure that h_1 and h_2 do not additively cancel. Equations (23) and (24) along with using

Lemma 27 to lower-bound the terms in the denominators arising from \widehat{S}_0 yield

$$|\hat{\alpha} - \alpha| = \Omega \frac{1}{\|G\|_F} + \mathbf{1}_{\{\beta \neq 0\}} + O_P \left(\frac{1 + \|G\|_F \mathbf{1}_{\{\beta \neq 0\}}}{\sqrt{n}} \right).$$

Once again, our growth assumption in Equation (7) is sufficient to ensure that

$$|\hat{\alpha} - \alpha| = \Omega \frac{1}{\|G\|_E} + \mathbf{1}_{\{\beta \neq 0\}}.$$
 (28)

Equations (26), (25), (27) and (28) are precisely what we set out to show.

F. Controlling the Design Matrix in Theorem 1

Here we collect results related to the design matrix W_n , defined in Equation (3). Our first four technical lemmas are used to control the behavior of W_n about \overline{W}_n as defined in Equation (13). W_n and \overline{W}_n differ in the columns corresponding to the interference and contagion terms. These two columns are controlled by Lemmas 24 and 25, respectively.

Lemma 24. Under the assumptions of Theorem 1,

$$||GT - \tau 1_n|| = \Theta_P(||G||_F).$$

Proof. Writing $\dot{T} = T - \tau 1_n$ for ease of notation, we first observe that since $G1_n = 1_n$,

$$GT - \tau \mathbf{1}_n = G(T - \tau \mathbf{1}_n) = G\dot{T}.$$

Expanding the Euclidean norm,

$$||GT - \tau \mathbf{1}_n||^2 = \dot{T}^\top G^\top G \dot{T},$$

and thus

$$\mathbb{E} \|GT - \tau \mathbf{1}_n\|^2 = \sigma_T^2 \operatorname{tr} G^{\mathsf{T}} G = \sigma_T^2 \|G\|_F^2. \tag{1}$$

Applying Lemma 8 with $M = G^{T}G$,

$$\|GT - \tau \mathbf{1}_n\|^2 \le \mathbb{E}\dot{T}^{\top} G^{\top} G \dot{T} + O_P \Big(\sigma_T^2 \|G^{\top} G\|_F \Big). = \sigma_T^2 \|G\|_F^2 + O_P \Big(\|G^{\top} G\|_F \Big).$$

Using the trivial upper bound $||G^{\top}G||_F \le ||G||_F^2$ and taking square roots,

$$||GT - \tau 1_n|| = O_P(||G||_F). \tag{2}$$

Expanding the quartic and writing $M = G^{T}G$ for ease of notation,

$$\mathbb{E} \|G(T - \tau \mathbf{1}_n)\|^4 = \mathbb{E} \left(\dot{T}^{\top} M \dot{T}\right)^2 = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{\ell=1}^n M_{ij} M_{k\ell} \mathbb{E} \dot{T}_i \dot{T}_j \dot{T}_k \dot{T}_{\ell}.$$

Observing that the summands are zero except when $|\{i, j, k, \ell\}| \in \{2, 4\}$ and using the fact that $M = G^{T}G$ is symmetric,

$$\mathbb{E}\left\|G(T-\tau \mathbf{1}_n)\right\|^4 = \left(\mathbb{E}\dot{T}_1^4\right)\sum_{i=1}^n M_{ii}^2 + \sigma_T^4\sum_{i\neq j}\left(M_{ii}M_{jj} + M_{ij}^2\right).$$

Applying Jensen's inequality and collecting terms,

$$\mathbb{E} \|G(T - \tau 1_n)\|^4 \le \left(\mathbb{E}\dot{T}_1^4\right) \left(\sum_{i=1}^n \sum_{j=1}^n M_{ij}^2 + \sum_{i \neq j} M_{ii} M_{jj}\right). \tag{3}$$

Again recalling that $M = G^{T}G$ and using basic properties of the Frobenius norm,

$$\sum_{i=1}^{n} \sum_{i=1}^{n} M_{ij}^{2} = \|G^{\top}G\|_{F}^{2} \le \|G\|_{F}^{4}$$

and

$$\sum_{i \neq j} M_{ii} M_{jj} = \sum_{i \neq j} \sum_{k=1}^{n} \sum_{\ell=1}^{n} G_{ki}^{2} G_{\ell j}^{2} \le ||G||_{F}^{4}.$$

Plugging the above two displays into Equation (3),

$$\mathbb{E} \|G(T - \tau 1_n)\|^4 \le \left(\mathbb{E}\dot{T}_1^4\right) \|G\|_F^4. \tag{4}$$

Applying the Paley-Zygmund inequality, for any $c \in [0, 1]$,

$$\Pr\left[\|GT - \tau \mathbf{1}_n\|^2 \ge c\mathbb{E}\|GT - \tau \mathbf{1}_n\|^2\right] \ge (1 - c)^2 \frac{\mathbb{E}^2 \|GT - \tau \mathbf{1}_n\|^2}{\mathbb{E}\|GT - \tau \mathbf{1}_n\|^4}.$$

Applying Equations (1) and (4),

$$\Pr\left[\|GT - \tau 1_n\|^2 \ge c \mathbb{E} \|GT - \tau 1_n\|^2\right] \ge \frac{(1 - c)^2 \sigma_T^4}{\mathbb{E} \dot{T}_1^4}.$$

Choosing $c \in [0, 1]$ constant with respect to n, we conclude that

$$||GT - \tau \mathbf{1}_n||^2 = \Omega_P \Big(\mathbb{E} ||GT - \tau \mathbf{1}_n||^2 \Big).$$

Applying Equation (1) a second time and taking square roots,

$$||GT - \tau \mathbf{1}_n|| = \Omega_P(||G||_F).$$

Combining this with Equation (2) completes the proof.

Lemma 25. Under the assumptions of Theorem 1,

$$||GY - \eta 1_n|| = \Theta_P(||G||_F).$$

Proof. Multiplying by G in Equation (2),

$$\begin{split} GY &= \frac{\alpha}{1-\beta} \mathbf{1}_n + \gamma (I-\beta G)^{-1} GT + \delta (I-\beta G)^{-1} G^2 T + (I-\beta G)^{-1} \varepsilon \\ &= \frac{\alpha}{1-\beta} \mathbf{1}_n + (I-\beta G)^{-1} \left(\gamma G + \delta G^2 \right) T + (I-\beta G)^{-1} \varepsilon, \end{split}$$

so that, writing $\dot{T} = T - \tau 1_n \in \mathbb{R}^n$ for ease of notation,

$$GY - \eta \mathbf{1}_n = (I - \beta G)^{-1} \left(\gamma G + \delta G^2 \right) \dot{T} + (I - \beta G)^{-1} G \varepsilon.$$

Applying the triangle inequality,

$$||GY - \eta \mathbf{1}_n|| \le ||(I - \beta G)^{-1} \left(\gamma G + \delta G^2 \right) \dot{T}|| + ||(I - \beta G)^{-1} G \varepsilon||.$$
 (5)

We first note that, writing

$$\widetilde{G} = (I - \beta G)^{-1} \left(\gamma G + \delta G^2 \right) \tag{6}$$

for ease of notation,

$$\left\| (I - \beta G)^{-1} \left(\gamma G + \delta G^2 \right) \dot{T} \right\|^2 = \dot{T}^\top \widetilde{G}^\top \widetilde{G} \dot{T}. \tag{7}$$

Applying Lemma 8 with $M = \widetilde{G}^{\top} \widetilde{G}$, we have

$$\left|\dot{T}^{\top}\widetilde{G}^{\top}\widetilde{G}\dot{T}\right| \leq \mathbb{E}\dot{T}^{\top}\widetilde{G}^{\top}\widetilde{G}\dot{T} + O_{P}\left(\sigma_{T}^{2}\|\widetilde{G}^{\top}\widetilde{G}\|_{F}\right) = \sigma_{T}^{2}\operatorname{tr}\widetilde{G}^{\top}\widetilde{G} + O_{P}\left(\sigma_{T}^{2}\|\widetilde{G}^{\top}\widetilde{G}\|_{F}\right).$$

Noting that $\|\widetilde{G}^{\top}\widetilde{G}\|_F \leq \|\widetilde{G}\|_F^2 = \operatorname{tr} \widetilde{G}^{\top}\widetilde{G}$, it follows that

$$\dot{T}^{\top} \widetilde{G}^{\top} \widetilde{G} \dot{T} = O_P \Big(\| \widetilde{G} \|_F^2 \Big). \tag{8}$$

Recalling the definition of \widetilde{G} from Equation (6) and using submultiplicativity,

$$\left\|\widetilde{G}\right\|_F^2 \leq \left\|(I - \beta G)^{-1}\right\|^2 \left\|G\right\|_F^2 \leq \frac{(\gamma + \delta)^2}{(1 - \beta)^2} \operatorname{tr} G^{\top} G.$$

Taking square roots in Equation (8) and applying the above bound, we conclude that

$$\left\| \widetilde{G}\dot{T} \right\| = O_P(\|G\|_F). \tag{9}$$

Again applying Lemma 8,

$$\varepsilon^{\top} G^{\top} (I - \beta G)^{-\top} (I - \beta G)^{-1} G \varepsilon = \sigma_{\varepsilon}^{2} \operatorname{tr} G^{\top} (I - \beta G)^{-\top} (I - \beta G)^{-1} G + O_{P} \Big(\big\| G^{\top} (I - \beta G)^{-\top} (I - \beta G)^{-1} G \big\|_{F} \Big).$$

An argument similar to that given above to bound Equation (7) yields

$$||(I - \beta G)^{-1} G \varepsilon|| = O_P(||G||_F).$$

Applying this and Equation (9) to Equation (5),

$$||GY - \eta \mathbf{1}_n|| = O_P\left(\sqrt{\operatorname{tr} G^\top G}\right). \tag{10}$$

Expanding Y according to Equation (2) and writing $\dot{T} = T - \tau 1_n$ for ease of notation,

$$GY - \eta \mathbf{1}_n = (I - \beta G)^{-1} (\gamma G + \delta G^2) \dot{T} + (I - \beta G)^{-1} G \varepsilon.$$

It follows that, writing $K = (I - \beta G)^{-T} (I - \beta G)^{-1}$ for ease of notation,

$$||GY - \eta \mathbf{1}_n||^2 = \dot{T}^{\mathsf{T}} (\gamma G + \delta G^2)^{\mathsf{T}} K (\gamma G + \delta G^2) \dot{T} + \varepsilon^{\mathsf{T}} G^{\mathsf{T}} K G \varepsilon + 2\varepsilon^{\mathsf{T}} G^{\mathsf{T}} K (\gamma G + \delta G^2) \dot{T}.$$

$$(11)$$

Taking expectations, the cross-term vanishes since ε and T are independent by assumption, and thus

$$\mathbb{E} \|GY - \eta \mathbf{1}_n\|^2 = \sigma_T^2 \|(I - \beta G)^{-1} (\gamma G + \delta G^2)\|_E^2 + \sigma_E^2 \|(I - \beta G)^{-1} G\|_E^2.$$

Using Lemma 19 to lower-bound the singular values of $(I - \beta G)^{-1}$ and applying basic properties of the Frobenius norm,

$$\mathbb{E} \|GY - \eta \mathbf{1}_n\|^2 \ge \frac{\sigma_T^2}{(1 - \beta)^2} \|\gamma G + \delta G^2\|_F^2 + \frac{\sigma_{\varepsilon}^2}{(1 - \beta)^2} \|G\|_F^2,$$

Again using basic properties of the Frobenius norm,

$$\|\gamma G + \delta G^2\|_F^2 = \|(\gamma I + \delta G)G\|_F^2 \ge (\gamma - \delta)^2 \|G\|_F,$$

so that

$$\mathbb{E} \|GY - \eta \mathbf{1}_n\|^2 \ge C \|G\|_F^2 \tag{12}$$

for suitably-chosen constant C > 0.

Again expanding the Euclidean norm as in Equation (11) and using $(a + b)^2 \le 2(a^2 + b^2)$,

$$\begin{split} \|GY - \eta \mathbf{1}_n\|^4 &= \left[\varepsilon^\top (I - \beta G)^{-1} G \varepsilon + \dot{T}^\top (\gamma G + \delta G^2)^\top K (\gamma G + \delta G^2) \dot{T} + 2 \varepsilon^\top G^\top K (\gamma G + \delta G^2) \dot{T} \right]^2 \\ &\leq 4 \left(\varepsilon^\top G^\top K (\gamma G + \delta G^2) \dot{T} \right)^2 + 4 \left(\varepsilon^\top (I - \beta G)^{-1} G \varepsilon \right)^2 \\ &\quad + 4 \left(\dot{T}^\top (\gamma G + \delta G^2)^\top K (\gamma G + \delta G^2) \dot{T} \right)^2. \end{split}$$

Taking expectations and applying Lemma 9,

$$\mathbb{E} \|GY - \eta \mathbf{1}_{n}\|^{4} \leq 4\mathbb{E} \left(\varepsilon^{\top} G^{\top} K (\gamma G + \delta G^{2}) \dot{T} \right)^{2} + 4 (\mathbb{E} \varepsilon_{1}^{4}) \left[\|(I - \beta G)^{-1} G\|_{F}^{2} \left(\operatorname{tr} (I - \beta G)^{-1} G \right)^{2} \right]$$

$$+ 4 (\mathbb{E} \dot{T}_{1}^{4}) \left[\|(\gamma G + \delta G^{2})^{\top} K (\gamma G + \delta G^{2})\|_{F}^{2} + \left(\operatorname{tr} (\gamma G + \delta G^{2})^{\top} K (\gamma G + \delta G^{2}) \right)^{2} \right].$$

Using the fact that for $M \in \mathbb{R}^{n \times n}$ with real eigenvalues, we have

$$\operatorname{tr} M^2 = \sum_{i=1}^n \lambda_i^2(M) = \operatorname{tr} M^\top M = ||M||_F^2,$$

basic properties of the Frobenius norm and Lemma 19 imply

$$\mathbb{E} \|GY - \eta \mathbf{1}_n\|^4 \le 4\mathbb{E} \left(\varepsilon^\top G^\top K (\gamma G + \delta G^2) \dot{T} \right)^2 + \frac{C(\mathbb{E}\varepsilon_1^4 + \mathbb{E}\dot{T}_1^4)}{(1 - \beta)^2} \|G\|_F^2.$$

Applying Lemma 10,

$$\mathbb{E} \|GY - \eta \mathbf{1}_n\|^4 \le 4\sigma_{\varepsilon}^2 \sigma_T^2 \|G^{\top} K(\gamma G + \delta G^2)\|_F^2 + \frac{C(\mathbb{E}\varepsilon_1^4 + \mathbb{E}\dot{T}_1^4)}{(1 - \beta)^2} \|G\|_F^2.$$

Again using basic properties of the Frobenius norm and Lemma 19,

$$\mathbb{E} \|GY - \eta 1_n\|^4 \le C \|G\|_F^2.$$

Since $||G||_F \ge ||G|| = 1$, it holds trivially that

$$\mathbb{E} \|GY - \eta 1_n\|^4 \le C \|G\|_F^4. \tag{13}$$

Applying the Paley-Zygmund inequality, for any $\theta \in [0, 1]$,

$$\Pr\left[\|GY - \eta \mathbf{1}_n\|^2 \ge \theta \mathbb{E} \|GY - \eta \mathbf{1}_n\|^2\right] \ge (1 - \theta)^2 \frac{\mathbb{E}^2 \|GY - \eta \mathbf{1}_n\|^2}{\mathbb{E} \|GY - \eta \mathbf{1}_n\|^4}.$$

Taking $\theta \in (0, 1)$ to be a constant with respect to n and applying Equations (12) and (13),

$$||GY - \eta 1_n||^2 = \Omega_P (\mathbb{E}||GY - \eta 1_n||^2).$$

Applying Equation (12) again and taking square roots,

$$||GY - \eta 1_n|| = \Omega_P(||G||_F).$$

Combining this with Equation (10) completes the proof.

With Lemmas 24 and 25 in hand, we are prepared to show that W_n concentrates about \overline{W}_n .

LEMMA 26. Under the assumptions of Theorem 1,

$$\left\|W_n - \overline{W}_n\right\| = O_P(\|G\|_F).$$

Proof. Recalling the definitions of W_n and \overline{W}_n from Equations (3) and (13),

$$||W_n - \overline{W}_n||^2 \le ||W_n - \overline{W}_n||_F^2 = ||GY - \eta \mathbf{1}_n||^2 + ||GT - \tau \mathbf{1}_n||^2.$$

Applying Lemmas 24 and 25 and taking square roots completes the proof.

It will be helpful below to have control over the singular values of W_n and its analogue \overline{W}_n . Indeed, the behavior of the two smallest singular values of W_n lies at the heart of Theorem 1. Toward this end, we state the following result.

Lemma 27. Let $\hat{s}_1 \geq \hat{s}_2 \geq \hat{s}_3 \geq \hat{s}_4 \geq 0$ be the singular values of W_n sorted in non-increasing order. Then

$$\hat{s}_1 \ge \hat{s}_2 = \Omega_P(\sqrt{n}) \tag{14}$$

and

$$\hat{s}_4 \le \hat{s}_3 = O_P(\|G\|_F). \tag{15}$$

Further, we have the lower-bound

$$\hat{s}_3 \ge \hat{s}_4 = \Omega_P \left(\sqrt{\operatorname{tr} G^\top G} \right). \tag{16}$$

Proof. Recall the vectors u_{τ} , u_{η} , e_3 , $u_{\circ} \in \mathbb{R}^4$ from Equation (7), and note that these form an orthonormal basis for \mathbb{R}^4 . By the min-max characterization of singular values,

$$\hat{s}_1 \ge \hat{s}_2 \ge \min \left\{ \| W^\top e_3 \|, \| W^\top u_\circ \| \right\}. \tag{17}$$

By the law of large numbers,

$$||W^{\top}e_3|| = \sqrt{\sum_{i=1}^n T_i} = \Omega_P(\sqrt{n}).$$
 (18)

Recalling the definition of u_{\circ} from Equation (7),

$$||W^{\top}u_{\circ}||^{2} = n + \eta||GY||^{2} + \tau||GT||^{2} \ge n.$$

Taking square roots,

$$||W^{\mathsf{T}}u_{\circ}|| = \Omega_{P}(\sqrt{n}). \tag{19}$$

Applying Equations (18) and (19) to Equation (17) yields Equation (14). Again appealing to the min-max characterization of singular values and unrolling the definition of W_n ,

$$\hat{s}_4 \leq \hat{s}_3 \leq \max\left\{\|W_n^\top u_\tau\|, \|W_n^\top u_n\|\right\} = \max\left\{\|GT - \tau 1_n\|, \|GY - \eta 1_n\|\right\}.$$

Applying Lemmas 24 and 25,

$$\hat{s}_4 \leq \hat{s}_3 = O_P(\|G\|_F),$$

yielding Equation (15).

Once more applying the min-max characterization of singular values, noting that $u_{\eta}, u_{\tau}, u_{\circ}, e_3 \in \mathbb{R}^4$ constructed above form an orthonormal basis,

$$\hat{s}_3 \ge \hat{s}_4 \ge \min \left\{ \|We_3\|, \|Wu_\circ\|, \|W_nu_\tau\|, \|W_nu_\eta\| \right\}.$$

Applying Equations (18) and (19),

$$\hat{s}_3 \ge \hat{s}_4 \ge \min \left\{ C\sqrt{n}, \|W_n u_\tau\|, \|W_n u_n\| \right\}.$$

Applying Lemmas 24 and 25 along with the fact that $n \ge ||G||_F^2$ yields Equation (16), completing the proof.

Lemma 28. Under the assumptions of Theorem 1, with $u_{\tau}, u_{\eta} \in \mathbb{R}^4$ as defined in Equation (7),

$$\|u_{\tau}^{\mathsf{T}}W_{n}^{\mathsf{T}}\varepsilon\| = O_{P}(\|G\|_{F}).$$
 (20)

and

$$\|u_{\eta}^{\top}W_{n}^{\top}\varepsilon\| = O_{P}\left(\mathbf{1}_{\{\beta\neq0\}}\|G\|_{F}^{2} + \|G\|_{F}\right). \tag{21}$$

Further,

$$\left\| (W_n - \overline{W})^\top \varepsilon \right\| = O_P(\sqrt{n}). \tag{22}$$

and

$$||W_n^{\top} \varepsilon|| = O_P(\sqrt{n}). \tag{23}$$

Proof. Recalling the definition of $u_{\tau} \in \mathbb{R}^4$,

$$u_{\tau}^{\top}W_{n}^{\top}\varepsilon = (GT - \tau 1_{n})^{\top}\varepsilon.$$

Define $v = (GT - \tau 1_n)/\|GT - \tau 1_n\| \in \mathbb{R}^n$, taking v = 0 if $GT - \tau 1_n = 0$. Then standard concentration inequalities (Boucheron et al., 2013; Vershynin, 2020) yield that

$$\left|u_{\tau}^{\mathsf{T}}W_{n}^{\mathsf{T}}\varepsilon\right| = \|GT - \tau \mathbf{1}_{n}\|v^{\mathsf{T}}\varepsilon = O_{P}(\|GT - \tau \mathbf{1}_{n}\|).$$

pplying Lemma 24 yields Equation (20).

Unrolling the definition of u_{η} and W_n ,

$$u_{\eta}^{\top}W_{n}^{\top}\varepsilon=(GY-\eta 1_{n})^{\top}\varepsilon.$$

Expanding Y according to Equation (2),

$$u_n^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon = \varepsilon^{\mathsf{T}} G (I - \beta G)^{-1} \varepsilon + (T - \tau 1_n)^{\mathsf{T}} (I - \beta G)^{-1} (\gamma G + \delta G^2) \varepsilon. \tag{24}$$

By an argument similar to that given above.

$$\left| (T - \tau 1_n)^\top (I - \beta G)^{-1} (\gamma G + \delta G^2) \varepsilon \right| \le O_P(\|G\|_F).$$

Applying this and Lemma 8 to Equation (24),

$$\left|u_{\eta}^{\top}W_{n}^{\top}\varepsilon\right| \leq \sigma_{\varepsilon}^{2}\operatorname{tr}G(I-\beta G)^{-1} + O_{P}\left(\left\|G(I-\beta G)^{-1}\right\|_{F}\right) + O_{P}(\left\|G\right\|_{F}).$$

Applying basic properties of the Frobenius norm and Lemma 19,

$$\left| u_{\eta}^{\mathsf{T}} W_{n}^{\mathsf{T}} \varepsilon \right| \leq \sigma_{\varepsilon}^{2} \operatorname{tr} G (I - \beta G)^{-1} + O_{P} (\|G\|_{F}).$$

If $\beta = 0$, the trace disappears, since our assumption of no self-loops in our network forces tr G = 0. On the other hand, if $\beta \neq 0$, by basic properties of the norm and using the fact that $||G||_F \geq ||G|| \geq 1$, we have

$$\operatorname{tr} G(I - \beta G)^{-1} \le ||G||_F ||(I - \beta G)^{-1}||_F \le \frac{1}{1 - |\beta|} ||G||_F^2.$$

Thus, we have shown that

$$\left|u_{\eta}^{\top}W_{n}^{\top}\varepsilon\right|=O_{P}\left(\|G\|_{F}+\|G\|_{F}^{2}\mathbf{1}_{\{\beta\neq0\}}\right),$$

which yields Equation (21).

To see Equation (22), note that

$$(W_n - \overline{W}_n)^{\top} \varepsilon = \begin{bmatrix} 0 \\ (GT - \tau \mathbf{1}_n)^{\top} \varepsilon \\ 0 \\ (GY - \eta \mathbf{1}_n)^{\top} \varepsilon \end{bmatrix},$$

and apply Lemmas 24 and 25.

Finally, to see Equation (23), first note that since u_{τ} , u_n , u_o , $e_3 \in \mathbb{R}^4$ form an orthonormal basis,

$$\left\| W_n^\top \varepsilon \right\|^2 = \left| u_\tau^\top W_n^\top \varepsilon \right|^2 + \left| u_n^\top W_n^\top \varepsilon \right|^2 + \left| u_\circ^\top W_n^\top \varepsilon \right|^2 + \left| e_3^\top W_n^\top \varepsilon \right|^2.$$

Applying Equations (20) and (21)

$$\left\|W_n^{\top}\varepsilon\right\|^2 = \left\|u_{\circ}^{\top}W_n^{\top}\varepsilon\right\|^2 + \left\|e_{3}^{\top}W_n^{\top}\varepsilon\right\|^2 \cdot + O_P(\operatorname{tr} G^{\top}G).$$

Applying standard concentration inequalities,

$$|e_3^\top W_n^\top \varepsilon| = |T^\top \varepsilon| = O_P(\sqrt{n}),$$

so that

$$\left\|W_n^{\top}\varepsilon\right\|^2 = \left|u_{\circ}^{\top}W_n^{\top}\varepsilon\right|^2 + O_P(n) + O_P(\operatorname{tr} G^{\top}G).$$

Recalling the definition of $u_{\circ} \in \mathbb{R}^4$ from Equation (7) and applying standard concentration inequalities,

$$u_{\circ}^{\top}W_{n}^{\top}\varepsilon = (1+\tau+\eta)1_{n}^{\top}\varepsilon = O_{P}(\sqrt{n}),$$

whence

$$\left\|W_n^{\top} \varepsilon\right\|^2 = O_P(n) + O_P(\operatorname{tr} G^{\top} G).$$

Noting that tr $G^{\top}G \le n\|G\|^2 = n$ and taking square roots yields Equation (23), completing the proof. \Box LEMMA 29. *Under the assumptions of Theorem 1*,

$$\left\|W_n^{\top}W_n - \overline{W}_n^{\top}\overline{W}_n\right\| = O_P(\sqrt{n}\|G\|_F).$$

Proof. Adding and subtracting appropriate quantities and applying the triangle inequality,

$$\left\| W_n^\top W_n - \overline{W}_n^\top \overline{W}_n \right\| \le 2 \| (W_n - \overline{W}_n)^\top \overline{W} \| + \| W_n - \overline{W} \|^2.$$

Applying Lemma 26,

$$\left\| W_n^\top W_n - \overline{W}_n^\top \overline{W}_n \right\| \le 2 \| (W_n - \overline{W}_n)^\top \overline{W} \| + O_P \left(\|G\|_F^2 \right). \tag{25}$$

Recalling the definitions of W_n and \overline{W}_n from Equations (3) and (13), respectively,

$$(W_n - \overline{W}_n)^\top \overline{W} = \left[0 \ GY - \eta \mathbf{1}_n \ 0 \ GT - \tau \mathbf{1}_n \right]^\top \left[\mathbf{1}_n \ \tau \mathbf{1}_n \ T \ \eta \mathbf{1}_n \right],$$

so that, upper bounding the spectral norm by the Frobenius norm,

$$\left\| (W_n - \overline{W}_n)^\top \overline{W} \right\|^2 \le \left(1 + \tau^2 + \eta^2 \right) \left[1_n^T \left(GY - \eta 1_n \right) \right]^2 + \left(1 + \tau^2 + \eta^2 \right) \left[1_n^T \left(GT - \tau 1_n \right) \right]^2 + \left[T \top \left(GY - \eta 1_n \right) \right]^2.$$
(26)

Applying Cauchy-Schwarz,

$$\left\| (W_n - \overline{W}_n)^\top \overline{W} \right\|^2 \leq \left[(1 + \tau^2 + \eta^2) n + \|T\|^2 \right] \left[\|GY - \eta \mathbf{1}_n\|^2 + \|GT - \tau \mathbf{1}_n\|^2 \right].$$

Applying Lemmas 24 and 25,

$$\left\| (W_n - \overline{W}_n)^{\top} \overline{W} \right\|^2 \le \left[(1 + \tau^2 + \eta^2) n + \|T\|^2 \right] O_P (\|G\|_F^2).$$

Applying the law of large numbers, $||T||^2 = O_P(n)$, and thus, taking square roots,

$$\|(W_n - \overline{W}_n)^\top \overline{W}\| = O_P(\sqrt{n} \|G\|_F).$$

Plugging this into Equation (25),

$$\left\| W_n^\top W_n - \overline{W}_n^\top \overline{W}_n \right\| = O_P \left(\sqrt{n} \|G\|_F \right) + O_P \left(\|G\|_F^2 \right).$$

Noting that $||G||_F^2 \le n||G||^2 = n$ completes the proof.

With the above lemmas in hand, we can control the error between the right singular vectors of W and their analogues in \overline{W} . As a preliminary, we note that by construction, \overline{W} has two non-zero singular values and two singular values that are identically zero.

Lemma 30. Under the assumptions of Theorem 1, with \overline{W}_n as defined in Equation (13) and $U_0 \in \mathbb{R}^{4 \times 2}$ as defined in Equation (6), we have

$$\overline{W}_n^{\mathsf{T}} U_0 = 0 \in \mathbb{R}^{4 \times 2}. \tag{27}$$

Further, with u_{\circ} , $e_3 \in \mathbb{R}^4$ as defined in Equation (7),

$$\min\left\{\|\overline{W}^{\top}\overline{W}_{n}u_{\circ}\|,\|\overline{W}^{\top}\overline{W}_{n}e_{3}\|\right\} = \Omega_{P}(n). \tag{28}$$

Proof. Recalling that $u_{\tau}, u_{\eta} \in \mathbb{R}^4$ are the columns of U_0 , it is simple to verify that by construction,

$$\overline{W}_n u_\tau = \overline{W}_n u_\eta = 0 \in \mathbb{R}^4,$$

and Equation (27) follows.

Recalling u_{\circ} and e_3 from Equation (7), note that

$$\left\| \overline{W}^{\top} \overline{W} e_3 \right\| = \left\| \overline{W}^{\top} T \right\| \ge T^{\top} T = \Omega_P(n), \tag{29}$$

where the last equality follows from the law of large numbers. Similarly, one can verify that

$$\left\| \overline{W}^{\mathsf{T}} \overline{W} u_{\circ} \right\| = \Omega_{P}(n). \tag{30}$$

Combining the above two displays,

$$\min\{\|\overline{W}^{\top}\overline{W}e_3\|,\|\overline{W}^{\top}\overline{W}u_\circ\|\} = \Omega_P(n),$$

establishing Equation (28).

We can now use Lemma 30 to control the singular vectors of W_n .

Lemma 31. Under the assumptions of Theorem 1, let $W_n \in \mathbb{R}^{n \times 4}$ and $\overline{W}_n \in \mathbb{R}^{n \times 4}$ be as defined in Equations (3) and (13), respectively, and let $\widehat{U}_0, U_0 \in \mathbb{R}^4$ be as defined in Equations (3) and (6), respectively. Then there exists an orthogonal $Q \in \mathbb{R}^{2 \times 2}$ such that

$$\left\|\widehat{U}_0 - U_0 Q\right\|_F = O_P \left(\frac{\|G\|_F}{\sqrt{n}}\right).$$

Proof. By Lemma 30,

$$\overline{W}U_0 = 0u_{\tau} = \overline{W}u_n = 0.$$

Thus, we may without loss of generality take the columns of U_0 to be the right singular vectors of \overline{W} corresponding to singular value 0.

By the Davis-Kahan theorem (see, e.g., Yu et al., 2015, Theorem 2), there exists a sequence of orthogonal matrices $Q = Q_n \in \mathbb{R}^{2 \times 2}$ such that

$$\left\|\widehat{U}_0 - U_0 Q\right\|_F \le \frac{\left\|W^\top W - \overline{W}^\top \overline{W}\right\|_F}{\Omega_P(n)},\tag{31}$$

where we have used Lemmas 30 and 27 to lower-bound the eigengap as

$$\min\{\|\overline{W}^{\top}\overline{W}e_3\|,\|\overline{W}^{\top}\overline{W}u_\circ\|\} - \hat{s}_3^2 = \Omega_P(n) - O_P(\|G\|_F^2) = \Omega_P(n),$$

noting that the second equality follows from our growth assumption in Equation (7). Applying Lemma 29 to Equation (31),

$$\left\|\widehat{U}_0-U_0Q\right\|_F=O_P(\|G\|_F)\sqrt{n}.$$

Noting that $||G||_F^2 \le n||G|| = n$ completes the proof.

G. Paley-Zygmund Bounds for Theorem 1

In this section, we establish the lower-bounds on entries of $U_0^\top W_n^\top \varepsilon$ needed in the proof of Theorem 1. We make frequent use of the Payley-Zygmund inequality, which states that for any $c \in [0, 1]$ and non-negative random variable Z with finite second moment,

$$\Pr[Z \ge c \mathbb{E}Z] \ge (1-c)^2 \frac{\mathbb{E}^2 Z}{\mathbb{E}Z^2}.$$

Lemma 32. Under the assumptions of Theorem 1, suppose that $M \in \mathbb{R}^{n \times n}$ is independent of T and ε and define Then

$$\mathbb{E}\left[(T - \tau \mathbf{1}_n)^{\mathsf{T}} M \varepsilon \right]^2 = \sigma_{\varepsilon}^2 \sigma_T^2 \|M\|_F^2 \tag{1}$$

and, writing τ_4 for the fourth central moment of T_1

$$\mathbb{E}\left[(T - \tau \mathbf{1}_n)^{\mathsf{T}} M \varepsilon \right]^4 \le C \mathbb{E} \varepsilon_1^4 \tau_4 \|M\|_F^4. \tag{2}$$

Proof. For ease of notation, define $\dot{T} = T - \tau 1_n$ and

$$Z = \left[(T - \tau \mathbf{1}_n)^{\mathsf{T}} M \varepsilon \right]^2 = \left(\dot{T}^{\mathsf{T}} M \varepsilon \right)^2. \tag{3}$$

Since ε and \dot{T} are independent of one another and have independent mean-zero subgaussian entries, Lemma 10 yields Equation (1).

Expanding the quadratic,

$$\begin{split} Z^2 &= \left(\sum_{i=1}^n \sum_{j=1}^n M_{ij} \dot{T}_i \varepsilon_j\right)^4 \\ &= \sum_{i_1, j_1} \sum_{i_2, j_2} \sum_{i_3, j_3} \sum_{i_4, j_4} M_{i_1 j_1} M_{i_2 j_2} M_{i_3 j_3} M_{i_4 j_4} \dot{T}_{i_1} \dot{T}_{i_2} \dot{T}_{i_3} \dot{T}_{i_4} \varepsilon_{j_1} \varepsilon_{j_2} \varepsilon_{j_3} \varepsilon_{j_4}. \end{split}$$

Taking expectations, all summands with a singleton i_k or j_k have mean zero, whence

$$\begin{split} \mathbb{E}Z^2 &= \tau_4 \left(\mathbb{E}\varepsilon_1^4 \right) \sum_{i,j} M_{ij}^4 + \tau_4 \sigma_{\varepsilon}^4 \sum_{i=1}^n \sum_{j \neq \ell} M_{ij}^2 M_{i\ell}^2 \\ &+ \left(\mathbb{E}\varepsilon_1^4 \right) \sigma_T^4 \sum_{i \neq k} \sum_{j=1}^n M_{ij}^2 M_{kj}^2 + \sigma_{\varepsilon}^4 \sigma_T^4 \sum_{\substack{i,j,k,\ell \\ \text{distinct}}} \left[M_{ij}^2 M_{k\ell}^2 + M_{ij} M_{i\ell} M_{kj} M_{k\ell} \right]. \end{split}$$

Applying Jensen's inequality and the fact that

$$M_{ij}M_{i\ell}M_{kj}M_{k\ell} \leq \frac{1}{2}(M_{ij}M_{k\ell} + M_{i\ell}M_{kj})^2,$$

it follows that

$$\mathbb{E}Z^2 \leq C\tau_4(\mathbb{E}\varepsilon_1^4) \left(\sum_{i=1}^n \sum_{j=1}^n M_{ij}^2 \right)^2 = C\tau_4(\mathbb{E}\varepsilon_1^4) \|M\|_F^4,$$

establishing Equation (2) and completing the proof.

Lemma 33. Under the assumptions of Theorem 1 and suppose that $M \in \mathbb{R}^{n \times n}$ is independent of T and ε . Then

$$\left| (T - \tau 1_n)^{\top} M \varepsilon \right| = \Omega_P(\|M\|_F).$$

Proof. Define

$$Z = \left[(T - \tau 1_n)^{\top} M \varepsilon \right]^2.$$

By the Paley-Zygmund inequality, for any $c_0 \in [0, 1]$,

$$\Pr\left[Z \ge c_0 \mathbb{E}Z\right] \ge (1 - c_0)^2 \frac{\mathbb{E}^2 Z}{\mathbb{E}Z^2}.$$

Applying Lemma 32,

$$\Pr\left[Z \ge c_0 \mathbb{E}Z\right] \ge \frac{C(1-c_0)^2 \sigma_{\varepsilon}^4 \sigma_T^4}{\mathbb{E}\varepsilon_1^4 \tau_4},$$

where we remind the reader that τ_4 denotes the fourth central moment of T_1 . Taking c_0 suitably small, it follows that

$$|(T-\tau 1_n)^{\top} M \varepsilon| = \Omega_P (\sqrt{\mathbb{E}Z}).$$

A second application of Lemma 32 yields

$$\left| (T - \tau 1_n)^\top M \varepsilon \right| = \Omega_P(\|M\|_F),$$

as we set out to show.

Lemma 34. Under the assumptions of Theorem 1, let $M \in \mathbb{R}^{n \times n}$ be independent of ε . Then

$$\mathbb{E}\left(\varepsilon^{\top} M \varepsilon\right)^{2} \leq \left(\mathbb{E}\varepsilon_{1}^{4}\right) \left[2\|M\|_{F}^{2} + (\operatorname{tr} M)^{2}\right]. \tag{4}$$

Further, if the entries of M are non-negative,

$$\mathbb{E}\left(\varepsilon^{\mathsf{T}} M \varepsilon\right)^{2} \ge \sigma_{\varepsilon}^{2} \|M\|_{F}^{2}. \tag{5}$$

Proof. Expanding the square and using the fact that the entries of ε are independent and mean zero,

$$\mathbb{E}\left(\varepsilon^{T}M\varepsilon\right)^{2} = \left(\mathbb{E}\varepsilon_{1}^{4}\right)\sum_{i=1}^{n}M_{ii}^{2} + \sigma_{\varepsilon}^{4}\sum_{i\neq j}\left(M_{ij}^{2} + M_{ii}M_{jj} + M_{ij}M_{ji}\right).$$

The lower bound in Equation (4) now follows immediately. On the other hand, applying Jensen's inequality and rearranging the sums,

$$\mathbb{E}\left(\varepsilon^T M \varepsilon\right)^2 \leq \left(\mathbb{E}\varepsilon_1^4\right) \left[\left\|M\right\|_F^2 + \sum_{i \neq j} \left(M_{ii} M_{jj} + M_{ij} M_{ji}\right) \right].$$

Upper bounding $2M_{ij}M_{ji} \le M_{ij}^2 + M_{ji}^2$ and using the fact that $\sum_i M_{ii}^2 \ge$,

$$\mathbb{E}\left(\varepsilon^{T} M \varepsilon\right)^{2} \leq \left(\mathbb{E}\varepsilon_{1}^{4}\right) \left(2\|M\|_{F}^{2} + \sum_{i,j} M_{ii} M_{jj}\right),\,$$

which establishes Equation (4), completing the proof.

Lemma 35. Under the assumptions of Theorem 1, let $M \in \mathbb{R}^{n \times n}$ be independent of ε and T. Then

$$\mathbb{E}\left[\varepsilon^{\top}M(T-\tau \mathbf{1}_n)\right]^4 \leq 30\mathbb{E}\varepsilon_1^4\tau_4\|M\|_F^4$$

Proof. Expanding,

$$\begin{split} \left(\varepsilon^{\top} M \dot{T}\right)^{4} &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} N_{ij} \varepsilon_{i} \dot{T}_{j}\right)^{4} \\ &= \sum_{i_{1}=1}^{n} \sum_{j_{1}=1}^{n} \sum_{i_{2}=1}^{n} \sum_{j_{2}=1}^{n} \sum_{j_{3}=1}^{n} \sum_{j_{4}=1}^{n} \sum_{j_{4}=1}^{n} M_{i_{1}j_{1}} M_{i_{2}j_{2}} M_{i_{3}j_{3}} M_{i_{4}j_{4}} \varepsilon_{i_{1}} \varepsilon_{i_{2}} \varepsilon_{i_{3}} \varepsilon_{i_{4}} \dot{T}_{j_{1}} \dot{T}_{j_{2}} \dot{T}_{j_{3}} \dot{T}_{j_{4}} \end{split}$$

Taking expectations, notice that all summands disappear except those in which each each ε_i and \dot{T}_j appears with powers greater than 1, so that

$$\mathbb{E}\left(\varepsilon^{\top}M\dot{T}\right)^{4} = \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij}^{4} \mathbb{E}\varepsilon_{1}^{4} \tau_{4}$$

$$+ \sum_{i_{1} \neq i_{2}} \sum_{j_{1} \neq j_{2}} \left(M_{i_{1}j_{1}}^{2} M_{i_{2}j_{2}}^{2} + M_{i_{1}j_{2}}^{2} M_{i_{2}j_{1}}^{2} + 2M_{i_{1}j_{1}} M_{i_{1}j_{2}} M_{i_{2}j_{1}} M_{i_{2}j_{2}}\right) \sigma_{\varepsilon}^{4} \sigma_{T}^{4}$$

$$+ \sum_{i_{1} \neq i_{2}} \sum_{j=1}^{n} 6M_{i_{1}j}^{2} M_{i_{2}j}^{2} \sigma_{\varepsilon}^{4} \sigma_{T}^{4} + \sum_{i=1}^{n} \sum_{j_{1} \neq j_{2}} 6M_{ij_{1}}^{2} M_{ij_{2}}^{2} \sigma_{\varepsilon}^{4} \tau_{4}.$$

Applying Jensen's inequality,

$$\mathbb{E}\left(\varepsilon^{\top}M\dot{T}\right)^{4} \leq \mathbb{E}\varepsilon_{1}^{4}\tau_{4}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}M_{ij}^{4} + 6\sum_{i_{1}\neq i_{2}}\sum_{j=1}^{n}M_{i_{1}j}^{2}M_{i_{2}j}^{2} + 6\sum_{i=1}^{n}\sum_{j_{1}\neq j_{2}}M_{ij_{1}}^{2}M_{ij_{2}}^{2}\right.$$

$$\left. + 6\sum_{i_{1}\neq i_{2}}\sum_{j_{1}\neq j_{2}}\left(M_{i_{1}j_{1}}^{2}M_{i_{2}j_{2}}^{2} + M_{i_{1}j_{2}}^{2}M_{i_{2}j_{1}}^{2} + 2M_{i_{1}j_{1}}M_{i_{1}j_{2}}M_{i_{2}j_{1}}M_{i_{2}j_{2}}\right)\right].$$

Collecting terms and using the fact that $M_{ij}^2 \ge 0$,

$$\mathbb{E}\left(\varepsilon^{\top}M\dot{T}\right)^{4} \leq 6\mathbb{E}\varepsilon_{1}^{4}\tau_{4}\left[\left(\sum_{i,j}M_{ij}^{2}\right)^{2} + \sum_{i_{1}\neq i_{2}}\sum_{j_{1}\neq j_{2}}\left(M_{i_{1}j_{1}}M_{i_{2}j_{2}} + M_{i_{2}j_{1}}M_{i_{1}j_{2}}\right)^{2}\right].$$
 (6)

Since $(a+b)^2 \le 2a^2 + 2b^2$,

$$\begin{split} \sum_{i_1 \neq i_2} \sum_{j_1 \neq j_2} \left(M_{i_1 j_1} M_{i_2 j_2} + M_{i_2 j_1} M_{i_1 j_2} \right)^2 &\leq 2 \sum_{i_1 \neq i_2} \sum_{j_1 \neq j_2} \left(M_{i_1 j_1}^2 M_{i_2 j_2}^2 + M_{i_2 j_1}^2 M_{i_1 j_2}^2 \right) \\ &\leq 2 \sum_{i_1, j_2, k, \ell} \left(M_{ij}^2 M_{k\ell}^2 + M_{kj}^2 M_{i\ell}^2 \right), \end{split}$$

where we have again used the fact that $M_{ij}^2 \ge 0$ trivially. Rearranging sums slightly,

$$\sum_{i_1 \neq i_2} \sum_{j_1 \neq j_2} \left(M_{i_1 j_1} M_{i_2 j_2} + M_{i_2 j_1} M_{i_1 j_2} \right)^2 \le 4 \left(\sum_{i,j} M_{ij}^2 \right)^2.$$

Plugging this bound into Equation (6),

$$\mathbb{E}\left(\varepsilon^{\top}M\dot{T}\right)^{4} \leq 30\mathbb{E}\varepsilon_{1}^{4}\tau_{4}\left(\operatorname{tr}M^{\top}M\right)^{2},$$

as we set out to show.

LEMMA 36. Under the model in Equation (2), suppose that the network A is hollow. Then we have

$$\left(\operatorname{tr}(I - \beta G)^{-1} G \right)^2 \ge \frac{\beta^2}{(1+\beta)^2} \|G\|_F^4.$$

Proof. For ease of notation, let $1 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge -1$ denote the ordered eigenvalues of G. Applying the Neumann expansion and using the fact that tr G = 0 since A is hollow,

$$\operatorname{tr}(I - \beta G)^{-1}G = \sum_{q=0}^{\infty} \beta^q \operatorname{tr} G^{1+q} = \beta \sum_{q=0}^{\infty} \beta^q \operatorname{tr} G^{2+q}.$$

Noting that tr $G^{\ell} = \sum_{i} \lambda_{i}^{\ell}$ for any integer ℓ ,

$$\operatorname{tr}(I - \beta G)^{-1}G = \beta \sum_{q=0}^{\infty} \beta^q \sum_{i=1}^n \lambda_i^{2+q}.$$

Interchanging the sums,

$$\operatorname{tr}(I - \beta G)^{-1}G = \beta \sum_{i=1}^{n} \lambda_i^2 \sum_{q=0}^{\infty} (\beta \lambda_i)^q = \sum_{i=1}^{n} \frac{\beta \lambda_i^2}{1 - \beta \lambda_i}.$$

Squaring both sides and recalling that $|\beta \lambda_i| < 1$ for all $i \in [n]$,

$$\left(\operatorname{tr}(I - \beta G)^{-1}G\right)^{2} = \beta^{2}\left(\sum_{i=1}^{n} \frac{\lambda_{i}^{2}}{1 - \beta \lambda_{i}}\right)^{2} \ge \frac{\beta^{2}}{\max_{i}(1 - \beta \lambda_{i})^{2}}\left(\sum_{i=1}^{n} \lambda_{i}^{2}\right)^{2} . = \frac{\beta^{2}}{\max_{i}(1 - \beta \lambda_{i})^{2}}\|G\|_{F}^{4}.$$

Noting that $(1 - \beta \lambda_i)^2 \le (1 + \beta)^2$ (take $|\lambda_i| = 1$ with sign matching β),

$$\left(\operatorname{tr}(I - \beta G)^{-1}G\right)^2 \ge \frac{\beta^2}{(1+\beta)^2} \|G\|_F^4,$$

completing the proof.

Lemma 37. Under the assumptions of Theorem 1, suppose that $\beta \neq 0$. Then

$$\left|u_{\eta}^{\top}W_{n}^{\top}\varepsilon\right|=\Omega_{P}\left(\|G\|_{F}^{2}\right).$$

Proof. Expanding the matrix-vector products,

$$u_n^{\mathsf{T}} W_n^{\mathsf{T}} \varepsilon = (GY - \eta 1_n)^{\mathsf{T}} \varepsilon.$$

Expanding GY according to Equation (2) and recalling the definition of η from Equation (6),

$$\boldsymbol{u}_{\eta}^{\top}\boldsymbol{W}_{n}^{\top}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\top}\left(\boldsymbol{I} - \boldsymbol{\beta}\boldsymbol{G}\right)^{-1}\boldsymbol{G}\left[\left(\gamma\boldsymbol{I} + \delta\boldsymbol{G}\right)\left(\boldsymbol{T} - \tau\boldsymbol{1}_{n}\right) + \boldsymbol{\varepsilon}\right].$$

Define

$$\tilde{Z} = \left| \varepsilon^{\top} \left(I - \beta G \right)^{-1} G \left[(\gamma I + \delta G) (T - \tau 1_n) + \varepsilon \right] \right|. \tag{7}$$

Since $\tilde{Z} \ge 0$ by construction, the Paley-Zygmund inequality implies that for any $c_0 \in [0, 1]$,

$$\Pr\left[\tilde{Z} > c_0 \mathbb{E}\tilde{Z}\right] \ge (1 - c_0)^2 \frac{\mathbb{E}^2 \tilde{Z}}{\mathbb{E}\tilde{Z}^2}.$$
 (8)

By Jensen's inequality,

$$\mathbb{E}\tilde{Z} \geq \left| \mathbb{E}\varepsilon^{\top} \left(I - \beta G \right)^{-1} G \left[(\gamma I + \delta G) (T - \tau \mathbf{1}_n) + \varepsilon \right] \right| = \left| \mathbb{E}\varepsilon^{\top} \left(I - \beta G \right)^{-1} G \varepsilon \right|,$$

where we have used the fact that ε and $T - \tau 1_n$ are are independent and mean zero. It follows that

$$\mathbb{E}^2 \tilde{Z} \ge \sigma_{\varepsilon}^4 \left| \operatorname{tr} \left(I - \beta G \right)^{-1} G \right|^2. \tag{9}$$

We note that if we have $\beta = 0$, then this lower-bound would be vacuous, since tr G = 0, whence our assumption that $\beta \neq 0$. Applying Lemma 36 to lower-bound this trace,

$$\mathbb{E}^2 \tilde{Z} \ge \sigma_{\varepsilon}^4 \frac{|\beta|}{1 + |\beta|} \|G\|_F^4. \tag{10}$$

Expanding the square, we have

$$\tilde{Z}^{2} = \left[\varepsilon^{\top} \left(I - \beta G \right)^{-1} G \varepsilon \right]^{2} + \left[\varepsilon^{\top} \left(I - \beta G \right)^{-1} G (\gamma I + \delta G) (T - \tau 1_{n}) \right]^{2} + 2 \left[\varepsilon^{\top} \left(I - \beta G \right)^{-1} G \varepsilon \right] \left[\varepsilon^{\top} \left(I - \beta G \right)^{-1} G (\gamma I + \delta G) (T - \tau 1_{n}) \right].$$

Taking expectations, the cross-term disappears, since $T - \tau 1_n$ has independent mean zero entries, and we have

$$\mathbb{E}\tilde{Z}^{2} = \mathbb{E}\left[\varepsilon^{\top} (I - \beta G)^{-1} G \varepsilon\right]^{2} + \mathbb{E}\left[\varepsilon^{\top} (I - \beta G)^{-1} G (\gamma I + \delta G) (T - \tau 1_{n})\right]^{2}.$$
 (11)

Applying Lemma 34 with $M = (I - \beta G)^{-1} G$

$$\mathbb{E}\left[\varepsilon^{\top} \left(I - \beta G\right)^{-1} G \varepsilon\right]^{2} \leq \left(\mathbb{E}\varepsilon_{1}^{4}\right) \left[\left\|\left(I - \beta G\right)^{-1} G\right\|_{F}^{2} + \left(\operatorname{tr}\left(I - \beta G\right)^{-1} G\right)^{2}\right]. \tag{12}$$

Applying Lemma 32 with $M = (I - \beta G)^{-1} G(\gamma I + \delta G)$,

$$\mathbb{E}\left[\varepsilon^{\top}\left(I-\beta G\right)^{-1}G(\gamma I+\delta G)(T-\tau 1_{n})\right]^{2}=\sigma_{\varepsilon}^{2}\tau_{2}\left\|\left(I-\beta G\right)^{-1}G(\gamma I+\delta G)\right\|_{F}^{2}.$$

By submultiplicativity, it follows that

$$\begin{split} \mathbb{E}\left[\varepsilon^{\top}\left(I-\beta G\right)^{-1}G(\gamma I+\delta G)(T-\tau \mathbf{1}_{n})\right]^{2} &\leq \sigma_{\varepsilon}^{2}\tau_{2}\left\|\left(\gamma I+\delta G\right)\right\|^{2}\left\|\left(I-\beta G\right)^{-1}G\right\|_{F}^{2} \\ &\leq \sigma_{\varepsilon}^{2}\left(\gamma^{2}+\delta^{2}\right)\left\|\left(I-\beta G\right)^{-1}G\right\|_{F}^{2}. \end{split}$$

Applying this and Equation (12) to Equation (11),

$$\mathbb{E}\tilde{Z}^{2} \le C \left[\left\| (I - \beta G)^{-1} G \right\|_{F}^{2} + \left(\text{tr} \left(I - \beta G \right)^{-1} G \right)^{2} \right], \tag{13}$$

By basic properties of the Frobenius norm followed by Lemma 19,

$$\|(I - \beta G)^{-1} G\|_F^2 \le \|(I - \beta G)^{-1}\| \|G\|_F^2 \le \frac{1}{1 - |\beta|} \|G\|_F^2$$

Since $||G||_F^2 \ge ||G|| = 1$, we have $||G||_F^2 \le ||G||_F^4$ trivially, and thus

$$\left\| (I - \beta G)^{-1} G \right\|_F^2 \le \frac{\|G\|_F^4}{1 - |\beta|}.$$

Applying Lemma 36 and subsuming β into the constant term,

$$\|(I - \beta G)^{-1} G\|_F^2 \le C \left(\operatorname{tr}(I - \beta G)^{-1} G \right)^2$$

Applying this bound to Equation (13),

$$\mathbb{E}\tilde{Z}^2 \le C \left(\operatorname{tr} \left(I - \beta G \right)^{-1} G \right)^2. \tag{14}$$

Applying Equations (10) and (14) to Equation (8) and noting that σ_{ε}^2 and β are fixed with respect to n, we have shown that

$$\Pr\left[\tilde{Z} > c_0 \mathbb{E}\tilde{Z}\right] \ge C(1 - c_0)^2 > 0$$

for suitably-chosen constant C > 0. It follows that

$$\left| \varepsilon^{\top} \left(I - \beta G \right)^{-1} G \left[(\gamma I + \delta G) (T - \tau 1_n) + \varepsilon \right] \right| = \Omega_P (\mathbb{E} \tilde{Z}).$$

Applying Equation (10) once again,

$$\left| \varepsilon^{\top} \left(I - \beta G \right)^{-1} G \left[(\gamma I + \delta G) (T - \tau 1_n) + \varepsilon \right] \right| = \Omega_P \left(\|G\|_F^2 \right).$$

Thus,

$$\left| \varepsilon^{\top} \left(I - \beta G \right)^{-1} G \left[(\gamma I + \delta G) (T - \tau 1_n) + \varepsilon \right] \right| = \Omega_P (\operatorname{tr} G^{\top} G), \tag{15}$$

completing the proof.

We note that if $\beta = 0$, the lower-bound argument used above fails, as $tr(1 - \beta G)^{-1}G = tr G = 0$. For this setting, a more delicate argument is needed.

Lemma 38. Under the assumptions of Theorem 1, suppose that $\beta = 0$. Then there exist positive constants C_0 , C_1 such that

$$\mathbb{E}^{2} \left[\varepsilon^{\mathsf{T}} G(\gamma I + \delta G) (T - \tau 1_{n}) + \varepsilon^{\mathsf{T}} G \varepsilon \right]^{2} \ge C_{0} \|G\|_{F}^{4}$$
(16)

and

$$\mathbb{E}\left[\varepsilon^{\top}G(\gamma I + \delta G)(T - \tau 1_n) + \varepsilon^{\top}G\varepsilon\right]^4 \le C_1 \|G\|_F^4 \tag{17}$$

Proof. For ease of notation, define

$$N = G(\gamma I + \delta G) \text{ and } \dot{T} = T - \tau 1_n \tag{18}$$

along with the random variable

$$\tilde{Z}_0 = \left[\varepsilon^\top G(\gamma I + \delta G)(T - \tau 1_n) + \varepsilon^\top G \varepsilon \right]^2. \tag{19}$$

Expanding the square,

$$\tilde{Z}_0 = \left(\varepsilon^\top N \dot{T}\right)^2 + \left(\varepsilon^\top G \varepsilon\right)^2 + 2 \left(\varepsilon^\top N \dot{T}\right) \left(\varepsilon^\top G \varepsilon\right).$$

Taking expectations, the cross-term disappears because ε and \dot{T} are mean zero and independent of one another, and we have

$$\mathbb{E}\tilde{Z}_0 = \mathbb{E}\left(\varepsilon^{\top}G\varepsilon\right)^2 + \mathbb{E}\left(\varepsilon^{\top}N\dot{T}\right)^2.$$

Applying Lemma 34 with M = G, noting that G is entrywise non-negative by definition,

$$\mathbb{E}\left(\varepsilon^{\top}G\varepsilon\right)^{2} \geq \sigma_{\varepsilon}^{2} \|G\|_{F}^{2}.$$

It follows that

$$\mathbb{E}\tilde{Z}_0 \geq \mathbb{E}\left(\varepsilon^{\top}N\dot{T}\right)^2 + \sigma_{\varepsilon}^2 \|G\|_F^2.$$

Trivially lower-bounding $\mathbb{E}\left(\varepsilon^{\top}N\dot{T}\right)^{2}\geq0$ yields

$$\mathbb{E}\tilde{Z}_0 \geq \sigma_{\varepsilon}^4 \|G\|_F^2,$$

Squaring both sides yields Equation (16).

Recalling the definition of \tilde{Z}_0 from Equation (19),

$$\begin{split} \tilde{Z}_0^2 &= \left[\varepsilon^\top N \dot{T} + \varepsilon^\top G \varepsilon \right]^4 \\ &\leq 8 \left[\left(\varepsilon^\top N \dot{T} \right)^4 + \left(\varepsilon^\top G \varepsilon \right)^4 \right]. \end{split}$$

Taking expectations,

$$\mathbb{E} \tilde{Z}_0^2 \leq 8 \mathbb{E} \left(\varepsilon^\top N \dot{T} \right)^4 + 8 \mathbb{E} \left(\varepsilon^\top G \varepsilon \right)^4.$$

Applying Lemma 35 with M = N,

$$\mathbb{E}\tilde{Z}_0^2 \le 240\mathbb{E}\varepsilon_1^4 \tau_4 \|N\|_F^4 + 8\mathbb{E}\left(\varepsilon^\top G \varepsilon\right)^4. \tag{20}$$

Recalling that $G_{ii} = 0$ for all $i \in [n]$ and defining $\bar{G} = G + G^{T}$,

$$(\varepsilon^{\top} G \varepsilon)^{4} = \left(\sum_{i \neq j} G_{ij} \varepsilon_{i} \varepsilon_{j} \right)^{4} = \left[\sum_{i < j} (G_{ij} + G_{ji}) \varepsilon_{i} \varepsilon_{j} \right]^{4}$$

$$= \sum_{(i_{1}, j_{1}) \in \binom{n}{2}} \sum_{(i_{2}, j_{2}) \in \binom{n}{2}} \sum_{(i_{3}, j_{3}) \in \binom{n}{2}} \sum_{(i_{4}, j_{4}) \in \binom{n}{2}} \prod_{k=1}^{4} \bar{G}_{i_{k} j_{k}} \varepsilon_{i_{k}} \varepsilon_{j_{k}}.$$

We note that since $i_k \neq j_k$ in the above expression, no summand contains a power of ε_i higher than 4. Further, note that we have

$$\mathbb{E}\prod_{k=1}^{4}\bar{G}_{i_{k}j_{k}}\varepsilon_{i_{k}}\varepsilon_{j_{k}}=0$$

unless every element of $(i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4)$ is repeated at least once. It follows that, taking expectations and writing C_2^n to denote the set of all unordered pairs of elements from [n],

$$\begin{split} \mathbb{E} \left(\varepsilon^{\top} G \varepsilon \right)^{4} &= \sum_{i < j} \bar{G}_{ij}^{4} (\mathbb{E} \varepsilon_{1}^{4})^{2} + \sum_{\substack{(i_{1}, j_{1}), (i_{2}, j_{2}) \in C_{2}^{n} \\ |\{i_{1}, j_{1}\} \cap \{i_{2}, j_{2}\}| = 1}} \bar{G}_{i_{1} j_{1}}^{2} \bar{G}_{i_{2}, j_{2}}^{2} \sigma_{\varepsilon}^{2} \mathbb{E}^{2} \varepsilon_{1}^{3} \\ &+ \sum_{\substack{(i_{1}, j_{1}), (i_{2}, j_{2}) \in C_{2}^{n} \\ |\{i_{1}, j_{1}, i_{2}, j_{2}\}| = 3}} \bar{G}_{i_{1} j_{1}}^{2} \bar{G}_{i_{2}, j_{2}}^{2} \sigma_{\varepsilon}^{4} \mathbb{E} \varepsilon_{1}^{4} + \sum_{\substack{(i_{1}, j_{1}), (i_{2}, j_{2}), (i_{3}, j_{3}), (i_{4}, j_{4}) \in C_{2}^{n} \\ |\{i_{1}, j_{1}, i_{2}, j_{2}, i_{3}, j_{3}, i_{4}, j_{4}\}| = 4}} \bar{G}_{i_{1} j_{1}} \bar{G}_{i_{2} j_{2}} \bar{G}_{i_{3} j_{3}} \bar{G}_{i_{4} j_{4}} \sigma_{\varepsilon}^{8}. \end{split}$$

Applying Jensen's inequality to the moments of (entries of) ε ,

$$\mathbb{E}\left(\varepsilon^{\top}G\varepsilon\right)^{4} \leq \left(\mathbb{E}\varepsilon_{1}^{4}\right)^{2} \left[\sum_{i < j} \bar{G}_{ij}^{4} + \sum_{\substack{(i_{1},j_{1}),(i_{2},j_{2}) \in C_{2}^{n} \\ |\{i_{1},j_{1}\} \cap \{i_{2},j_{2}\}| = 1}} \bar{G}_{i_{1}j_{1}}^{2} \bar{G}_{i_{2},j_{2}}^{2} + \sum_{\substack{(i_{1},j_{1}),(i_{2},j_{2}) \in C_{2}^{n} \\ |\{i_{1},j_{1},(i_{2},j_{2}) \in C_{2}^{n} \\ |\{i_{1},j_{1},i_{2},j_{2}\}| = 3}} \bar{G}_{i_{1}j_{1}}^{2} \bar{G}_{i_{2},j_{2}}^{2} + \sum_{\substack{(i_{1},j_{1}),(i_{2},j_{2}),(i_{3},j_{3}),(i_{4},j_{4}) \in C_{2}^{n} \\ |\{i_{1},j_{1},i_{2},j_{2}\}| = 3}} \bar{G}_{i_{1}j_{1}} \bar{G}_{i_{2}j_{2}} \bar{G}_{i_{3}j_{3}} \bar{G}_{i_{4}j_{4}} \right].$$

Using the fact that G is entrywise non-negative,

$$\mathbb{E}\left(\varepsilon^{\top}G\varepsilon\right)^{4} \leq (\mathbb{E}\varepsilon_{1}^{4})^{2} \left[4\left(\operatorname{tr}G^{\top}G\right)^{2} + \sum_{\substack{(i_{1},j_{1}),(i_{2},j_{2}),(i_{3},j_{3}),(i_{4},j_{4}) \in C_{2}^{n} \\ |\{i_{1},j_{1},i_{2},j_{2},i_{3},j_{3},i_{4},j_{4}\}| = 4}} \bar{G}_{i_{1}j_{1}}\bar{G}_{i_{2}j_{2}}\bar{G}_{i_{3}j_{3}}\bar{G}_{i_{4}j_{4}} \right]. \tag{21}$$

Let us consider the sum

$$\sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3),(i_4,j_4)\in C_2^n\\|\{i_1,j_1,i_2,j_2,i_3,j_3,i_4,j_4\}|=4}} \bar{G}_{i_1j_1}\bar{G}_{i_2j_2}\bar{G}_{i_3j_3}\bar{G}_{i_4j_4}.$$
(22)

We note that since \bar{G} is symmetric by construction, each of the summands in this expression is of the form either $\bar{G}_{ii}^2 \bar{G}_{k\ell}^2$ or $\bar{G}_{ij} \bar{G}_{jk} \bar{G}_{k\ell} \bar{G}_{\ell i}$ for $\{i, j, k, \ell\}$ chosen distinct. Since

$$\bar{G}_{ij}\bar{G}_{jk}\bar{G}_{k\ell}\bar{G}_{\ell i} \leq \frac{1}{2} \left(\bar{G}_{ij}^2 \bar{G}_{k\ell}^2 + \bar{G}_{jk}^2 \bar{G}_{\ell i}^2 \right),$$

it follows that

$$\sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3),(i_4,j_4)\in C_2^n\\|\{i_1,j_1,i_2,j_2,i_3,j_3,i_4,j_4\}|=4}} \bar{G}_{i_1j_1}\bar{G}_{i_2j_2}\bar{G}_{i_3j_3}\bar{G}_{i_4j_4} \leq 2\sum_{\substack{i,j,k,\ell\\ \text{distinct}}} \bar{G}_{ij}^2\bar{G}_{k\ell}^2.$$

Recalling the definition of $\bar{G} = G + G^{T}$ and using the fact that G is entrywise non-negative and hollow, it follows that

$$\begin{split} \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3),(i_4,j_4) \in C_2^n \\ |\{i_1,j_1,i_2,j_2,i_3,j_3,i_4,j_4\}| = 4}} \bar{G}_{i_1j_1}\bar{G}_{i_2j_2}\bar{G}_{i_3j_3}\bar{G}_{i_4j_4} &\leq 8 \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{\ell=1}^n G_{ij}^2 G_{k\ell}^2 \\ &= 8 \left(\sum_{i=1}^n \sum_{j=1}^n G_{ij}^2 \right)^2 \\ &= 8 \left(\operatorname{tr} G^\top G \right)^2. \end{split}$$

Plugging this into Equation (21),

$$\mathbb{E}\left(\varepsilon^{\top}G\varepsilon\right)^{4} \le 12(\mathbb{E}\varepsilon_{1}^{4})^{2}\left(\operatorname{tr}G^{\top}G\right)^{2}.$$
(23)

Applying Equation (23) to Equation (20),

$$\mathbb{E} \tilde{Z}_0^2 \leq \left[240\mathbb{E}\varepsilon_1^4\tau_4\|N\|_F^4 + 96(\mathbb{E}\varepsilon_1^4)^2\right]\|G\|_F^4.$$

Bounding

$$||N||_F^4 \le ||\gamma I + \delta G|| ||G||_F^4 \le C||G||_F^2$$

and recalling that $\mathbb{E}\varepsilon_1^4$, τ_4 , γ and δ are fixed with respect to n yields Equation (16), completes the proof.

Lemma 39. Under the assumptions of Theorem 1, if $\beta = 0$, then

$$\left|u_{\eta}^{\top}W_{n}^{\top}\varepsilon\right|=\Omega_{P}(\|G\|_{F}).$$

Proof. Following the same expansion as in Lemma 37,

$$\boldsymbol{u}_{\eta}^{\top}\boldsymbol{W}_{n}^{\top}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\top}\left(\boldsymbol{I} - \boldsymbol{\beta}\boldsymbol{G}\right)^{-1}\boldsymbol{G}\left[\left(\gamma\boldsymbol{I} + \delta\boldsymbol{G}\right)\left(\boldsymbol{T} - \tau\boldsymbol{1}_{n}\right) + \boldsymbol{\varepsilon}\right].$$

When $\beta = 0$, the lower-bound in Equation (9) used in Lemma 37 becomes vacuous, and we must instead consider

$$\tilde{Z}_0 = \left(u_\eta^\top W_n^\top \varepsilon\right)^2 = \left[\varepsilon^\top G\left((\gamma I + \delta G)(T - \tau 1_n) + \varepsilon\right)\right]^2,\tag{24}$$

where we have used our assumption that $\beta = 0$ to simplify $(I - \beta G) = I$. The Paley-Zygmund inequality implies that for any $c_0 \in [0, 1)$,

$$\Pr\left[\tilde{Z}_0 > c_0 \mathbb{E} \tilde{Z}_0\right] \ge (1 - c_0)^2 \frac{\mathbb{E}^2 \tilde{Z}_0}{\mathbb{E} \tilde{Z}_0^2}.$$

Applying Lemma 38 to lower bound the numerator and upper bound the denominator,

$$\Pr\left[\tilde{Z}_0 > c_0 \mathbb{E} \tilde{Z}_0\right] \ge (1 - c_0)^2 \frac{C_0}{C_1},$$

where C_1 , C_0 are the constants guaranteed by Lemma 38. It follows that

$$\tilde{Z}_0 = \Omega_P(\mathbb{E}\tilde{Z}_0).$$

Applying Lemma 38 a second time to lower-bound the expectation, it follows that when $\beta = 0$,

$$\tilde{Z}_0 = \Omega_P \Big(\|G\|_F^2 \Big).$$

Recalling the definition of \tilde{Z}_0 from Equation (19), taking square roots completes the proof.

H. Proof of Lemma 2

Proof. By Lemma 41, established in Appendix I,

$$||GX - H^{-1}X\Lambda||_{2,\infty} = o(1) \text{ almost surely.}$$
 (1)

Multiplying by G in Equation (4),

$$GY = \frac{\alpha}{1-\beta} \mathbf{1}_n + (I-\beta G)^{-1} GX\gamma + (I-\beta G)^{-1} G^2 X\delta + (I-\beta G)^{-1} G\varepsilon, \tag{2}$$

where we have used the fact that G commutes with $(I - \beta G)^{-1}$, and the fact that

$$(I - \beta G)^{-1} 1_n = \frac{1}{1 - \beta} 1_n.$$

By Lemma 50, established in Appendix J,

$$\left\| (I - \beta G)^{-1} G^2 X - H^{-1} X \Gamma \Lambda \right\|_{2,\infty} = o(1) \text{ almost surely.}$$
 (3)

By Lemma 51, also established in Appendix J,

$$\left\| (I - \beta G)^{-1} GX - H^{-1}X (I + \beta \Gamma) \Lambda \right\|_{2,\infty} = o(1) \text{ almost surely.}$$
 (4)

Applying basic properties of the $(2, \infty)$ -norm and Lemmas 7 and 19,

$$\max_{i \in [n]} \left| \left[(I - \beta G)^{-1} G \varepsilon \right]_i \right| \leq \left\| (I - \beta G)^{-1} \right\|_{\infty} \max_{i \in [n]} \left| \left[G \varepsilon \right]_i \right| \\
\leq \frac{C}{1 - |\beta|} \left[v_{\varepsilon} \max_{i \in [n]} \sqrt{\sum_{j=1}^n \frac{A_{ij}^2}{D_i^2} \log^2 n} + b_{\varepsilon} \max_{i \in [n]} \max_{j \in [n]} \frac{A_{ij}}{D_i} \log n \right].$$
(5)

Applying Lemma 4, it holds with high probability that for all $i \in [n]$,

$$\sum_{j=1}^{n} A_{ij}^{2} \le 2 \sum_{j=1}^{n} (\rho X_{i}^{T} X_{j})^{2} + 2n(\nu + b^{2}) \log^{2} n.$$

Using this fact and applying Lemma 12,

$$\max_{i \in [n]} \sum_{j=1}^{n} \frac{A_{ij}^{2}}{D_{i}^{2}} \leq \frac{2n}{\delta_{\min}^{2}} \left[\frac{\rho^{2}}{n} \sum_{j=1}^{n} (X_{i}^{T} X_{j})^{2} + (\nu + b^{2}) \log^{2} n \right].$$

Applying the law of large numbers, our growth assumption in Equation (2) and the fact that $\rho \le 1$ by assumption, it holds with high probability that

$$\max_{i \in [n]} \sum_{i=1}^n \frac{A_{ij}^2}{D_i^2} \le \frac{Cn}{\delta_{\min}} \left(\rho^2 + (\nu + b^2) \log^2 n \right) \le \frac{C\rho n \log^2 n}{\delta_{\min}^2}.$$

Applying Lemma 16, it holds with high probability that

$$\max_{i \in [n]} \sqrt{\sum_{j=1}^{n} \frac{A_{ij}^2}{D_i^2}} \le \frac{C \log^2 n}{n\rho \left(\min_{i \in [n]} X_i^T \mu\right)^2}.$$

Invoking our growth assumption in Equation (3) and the fact that $\rho \leq 1$,

$$\max_{i \in [n]} \sqrt{\sum_{i=1}^{n} \frac{A_{ij}^2}{D_i^2} \log^2 n} = o(1) \text{ almost surely.}$$
 (6)

By Lemmas 4 and 12, recalling the definition of δ_{min} from Equation (2),

$$\max_{i \in [n]} \max_{j \in [n]} \frac{A_{ij} \log n}{D_i} \le \frac{C\sqrt{\nu + b} \log^2 n}{\delta_{\min}}.$$

Further applying Lemma 16 followed by our assumptions in Equations (2) and (3) and the fact that $\rho \leq 1$,

$$\max_{i \in [n]} \max_{j \in [n]} \frac{A_{ij} \log n}{D_i} \le \frac{C \log^2 n}{n \sqrt{\rho} \min_{i \in [n]} X_i^T \mu} = o(1) \text{ almost surely.}$$
 (7)

Applying Equations (6) and (7) to Equation (5),

$$\max_{i \in [n]} \left| \left[(I - \beta G)^{-1} G \varepsilon \right]_i \right| = o(1) \text{ almost surely.}$$
 (8)

Recalling Equation (2), applying the triangle inequality followed by Equations (1), (3), (4) and (8),

$$\begin{aligned} \left\| GY - \frac{\alpha}{1-\beta} \mathbf{1}_n - H^{-1}X \left(I + \beta \Gamma \right) \Lambda \gamma - H^{-1}X \Gamma \Lambda \delta \right\|_{2,\infty} \\ &\leq \left\| \gamma \right\| \left\| \left(I - \beta G \right)^{-1} GX - H^{-1}X \left(I + \beta \Gamma \right) \Lambda \right\|_{2,\infty} \\ &+ \left\| \delta \right\| \left\| \left(I - \beta G \right)^{-1} G^2 X - H^{-1}X \Gamma \Lambda \right\|_{2,\infty} + \left\| \left(I - \beta G \right)^{-1} G \varepsilon \right\|_{2,\infty} \\ &= o(1) \text{ almost surely,} \end{aligned}$$

as we set out to show.

I. Convergence of GX term

Here we establish the uniform entrywise convergence of GX to an appropriate limit object $H^{-1}X\Lambda$, as used in our proof of Theorem 2.

We require the following technical lemma.

Lemma 40. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ and suppose that Assumptions 1, 2 and 3 hold. Then

$$||n\rho D^{-1}X - H^{-1}X||_{2,\infty} = o(1)$$
 almost surely.

Proof. By Lemma 13, with high probability,

$$\|n\rho D^{-1} X - n\rho \mathcal{D}^{-1} X\|_{2,\infty} = \max_{i \in [n]} n\rho \left| \frac{1}{D_i} - \frac{1}{\delta_i} \right| \|X_i\|$$

$$\leq C\rho \sqrt{\nu + b^2} \left(n^{3/2} \log n \right) \max_{i \in [n]} \frac{\|X_i\|}{\delta_i^2}.$$

Multiplying through by appropriate quantities, applying Lemmas 15 and 46, and applying our growth assumption in Equation (2),

$$\left\|n\rho D^{-1}X - n\rho \mathcal{D}^{-1}X\right\|_{2,\infty} \leq \frac{C\sqrt{\nu + b^2}\log n}{\rho\sqrt{n}} \left(\max_{i\in[n]} \frac{\|X_i\|}{(X_i^T\mu)^2}\right) \left(\max_{i\in[n]} \frac{n\rho X_i^T\mu}{\delta_i}\right)^2 \leq \frac{C\log n}{\sqrt{n\rho}} \left(\max_{i\in[n]} \frac{1}{X_i^T\mu}\right).$$

Applying our growth assumption in Equation (3) and using the fact that $\rho \leq 1$,

$$||n\rho D^{-1}X - n\rho \mathcal{D}^{-1}X||_{2,\infty} = o(1)$$
 almost surely,

so that by the triangle inequality,

$$\|n\rho D^{-1}X - H^{-1}X\|_{2,\infty} \le \|n\rho \mathcal{D}^{-1}X - H^{-1}\|_{2,\infty} + o(1). \tag{1}$$

By definition and basic properties of the norm,

$$\left\|n\rho\mathcal{D}^{-1}X - H^{-1}X\right\|_{2,\infty} \leq \max_{i\in[n]} \left|\frac{n\rho}{\delta_i} - \frac{1}{X_i^T\mu}\right| \|X_i\| = n\rho \max_{i\in[n]} \left|\frac{1}{\delta_i} - \frac{1}{n\rho X_i^T\mu}\right| \|X_i\| \leq \max_{i\in[n]} \frac{\left|\delta_i - n\rho X_i^T\mu\right| \|X_i\|}{X_i^T\mu\delta_i}.$$

Applying Lemma 46,

$$\left\| n\rho \mathcal{D}^{-1} X - H^{-1} X \right\|_{2,\infty} \le \left(\max_{i \in [n]} \frac{\|X_i\|}{X_i^T \mu} \right) \left(\max_{i \in [n]} \frac{\left| \delta_i - n\rho X_i^T \mu \right|}{\delta_i} \right) \le C \left(\max_{i \in [n]} \frac{\left| \delta_i - n\rho X_i^T \mu \right|}{\delta_i} \right).$$

Applying Lemma 15,

$$||n\rho \mathcal{D}^{-1}X - H^{-1}X||_{2,\infty} = o(1)$$
 almost surely.

Applying this to Equation (1) completes the proof.

With Lemma 40 in hand, we are ready to prove our convergence result.

LEMMA 41. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ , and suppose that Assumptions 1, 2 and 3 hold. Then

$$||GX - H^{-1}X\Lambda||_{2,\infty} = o(1)$$
 almost surely.

Proof. Recalling $G = D^{-1}A$, by basic properties of the $(2, \infty)$ -norm,

$$||GX - D^{-1}PX||_{2,\infty} \le ||D^{-1}||_{\infty} ||(A - P)X||_{2,\infty}.$$

Recalling the definition of δ_{\min} from Equation (2) and applying Lemmas 6 and 12,

$$\|GX - D^{-1}PX\|_{2,\infty} \le \frac{C\sqrt{\nu + b^2} \log n}{\delta_{\min}} \left(\sum_{j=1}^n \|X_j\|^2\right)^{1/2}.$$

Multiplying through by appropriate quantities and applying the law of large numbers,

$$\|GX - D^{-1}PX\|_{2,\infty} \le \frac{C\sqrt{\nu + b^2}\sqrt{n}\log n}{\delta_{\min}}.$$

Further applying Lemma 16,

$$\left\|GX - D^{-1}PX\right\|_{2,\infty} \le \frac{C\sqrt{\nu + b^2}\log n}{\sqrt{n}\rho\min_{i\in[n]}X_i^T\mu}.$$

Our growth assumptions in Equations (2) and (3) and the fact that $\rho \leq 1$ then imply

$$||GX - D^{-1}PX||_{2,m} \to 0$$
 almost surely. (2)

Recalling that $P = \rho XX^T$ and using basic properties of the $(2, \infty)$ -norm,

$$\|D^{-1}PX - H^{-1}XX^{T}X\|_{2,\infty} = \left\| (n\rho D^{-1}X) \frac{X^{T}X}{n} - H^{-1}X \frac{X^{T}X}{n} \right\|_{2,\infty}$$

$$\leq \|\rho D^{-1}X - H^{-1}X\|_{2,\infty} \left\| \frac{X^{T}X}{n} \right\|.$$

By the law of large numbers, $n^{-1}X^TX \to \Lambda$ almost surely, and thus, since Λ is constant with respect to n,

$$||D^{-1}PX - H^{-1}XX^TX||_{2,\infty} \le C ||n\rho D^{-1}X - H^{-1}X||_{2,\infty}.$$

Applying Lemma 40,

$$\left\| D^{-1}PX - H^{-1}X \frac{X^T X}{n} \right\|_{2,\infty} = o(1).$$
 (3)

Once more applying basic properties of the $(2, \infty)$ -norm and standard (multivariate) concentration inequalities (Vershynin, 2020; Boucheron et al., 2013),

$$\left\| H^{-1} X \frac{X^T X}{n} - H^{-1} X \Lambda \right\|_{2,\infty} \le \left\| H^{-1} X \right\|_{2,\infty} \left\| \frac{X^T X}{n} - \Lambda \right\|$$

$$\le \frac{C \log n}{\sqrt{n}} \left\| H^{-1} X \right\|_{2,\infty}$$

$$= o(1) \text{ almost surely,}$$

$$(4)$$

where the final equality follows from Lemma 46. Applying the triangle inequality followed by Equations (2), (3) and (4) completes the proof.

I.1. Convergence of
$$X^TD^{-1}X$$
 term

Our results, in particular Lemma 50, rely upon the convergence of $X^T(I - \beta G)^{-1}D^{-1}X$ to a population analogue. This convergence in turn relies on convergence of $\rho X^TD^{-1}X$, which we establish below.

Lemma 42. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity factor ρ , and suppose that Assumptions 3 and 4 hold. Then

$$\left\| \rho X^T \mathcal{D}^{-1} X - \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right\| = o(1) \quad almost \ surely.$$

Proof. By the triangle inequality,

$$\left\| \rho X^T \mathcal{D}^{-1} X - \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right\| \leq \left\| \rho X^T \mathcal{D}^{-1} X - X^T (nH)^{-1} X \right\| + \left\| X^T (nH)^{-1} - \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right\|.$$

Applying Lemmas 43 and 44,

$$\left\| \rho X^T \mathcal{D}^{-1} X - \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right\| = o(1) \text{ almost surely,}$$

completing the proof.

LEMMA 43. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity factor ρ , and suppose that Assumptions 3 and 4 hold. Then

$$\|\rho X^T \mathcal{D}^{-1} X - X^T (nH)^{-1} X\| = O\left(\frac{\log n}{\sqrt{n}}\right)$$
 almost surely.

where H is as defined in Equation (7).

Proof. Expanding the matrix products and applying the triangle inequality,

$$\begin{split} \left\| \rho X^{T} \mathcal{D}^{-1} X - X^{T} (nH)^{-1} X \right\| &\leq \sum_{i=1}^{n} \|X_{i}\|^{2} \left| \frac{\rho}{\delta_{i}} - \frac{1}{n X_{i}^{T} \mu} \right| = \rho \sum_{i=1}^{n} \|X_{i}\|^{2} \left| \frac{1}{\delta_{i}} - \frac{1}{n \rho X_{i}^{T} \mu} \right| = \sum_{i=1}^{n} \frac{\|X_{i}\|^{2}}{n X_{i}^{T} \mu} \frac{|\delta_{i} - n \rho X_{i}^{T} \mu|}{\delta_{i}} \\ &\leq \left(\max_{i \in [n]} \frac{|\delta_{i} - n \rho X_{i}^{T} \mu|}{\delta_{i}} \right) \frac{1}{n} \sum_{i=1}^{n} \frac{\|X_{i}\|^{2}}{X_{i}^{T} \mu}. \end{split}$$

Applying Lemma 15 to control the maximum and applying the law of large numbers to control the sample mean,

$$\|\rho X^T \mathcal{D}^{-1} X - X^T (nH)^{-1} X\| = o(1) \text{ almost surely,}$$
 (5)

completing the proof.

LEMMA 44. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity factor ρ and suppose that Assumption 4 holds. Letting $H \in \mathbb{R}^{n \times n}$ be as defined in Equation (7),

$$\left\| X^T (nH)^{-1} X - \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right\| = o(1) \text{ almost surely.}$$

Proof. For ease of notation, write

$$\Xi = \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu}.$$

Expanding the matrix products,

$$X^{T}(nH)^{-1}X - \Xi = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{X_{i}X_{i}^{T}}{X_{i}^{T}\mu} - \Xi\right).$$

By the strong law of large numbers,

$$||X^T(nH)^{-1}X - \Xi|| = \left\| \frac{1}{n} \sum_{i=1}^n \left(\frac{X_i X_i^T}{X_i^T \mu} - \Xi \right) \right\| = o(1)$$
 almost surely,

as we set out to show.

Lemma 45. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity factor ρ and suppose that Assumptions 4, 1 and 2 hold. Then

$$\|\rho X^T D^{-1} X - \rho X^T \mathcal{D}^{-1} X\| = O\left(\frac{\log n}{\sqrt{n\rho}}\right)$$
 almost surely.

Proof. Expanding the matrix-vector products and applying the triangle inequality,

$$\|\rho X^T D^{-1} X - \rho X^T \mathcal{D}^{-1} X\| \le \rho \sum_{i=1}^n \left| \frac{1}{D_i} - \frac{1}{\delta_i} \right| \|X_i\|^2.$$

Applying Lemma 13, it holds with high probability that

$$\|\rho X^T D^{-1} X - \rho X^T \mathcal{D}^{-1} X\| \le C \rho \left(\sum_{i=1}^n \frac{\|X_i\|^2}{\delta_i^2} \right) \left(\sqrt{\nu + b^2} n^{1/2} \log n \right)$$

Multiplying through by appropriate quantities and applying Lemma 14 with r = q = 2 (noting that Lemma 14 applies, thanks to Corollary 1),

$$\|\rho X^T D^{-1} X - \rho X^T \mathcal{D}^{-1} X\| \le \frac{C\sqrt{\nu + b^2} \log n}{\sqrt{n}\rho} \left(\mathbb{E} \frac{\|X_1\|^2}{(X_1^T \mu)^2} + o(1) \right)$$

with high probability. Applying our growth assumptions in Equation (2) and using the fact that F is assumed fixed with respect to n completes the proof.

J. Convergence of Contagion Term

Lemma 46. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ . Then there exists a constant $c_F > 0$ depending on F but not on n such that

$$||H^{-1}X||_{2,\infty} \le c_F$$
 almost surely.

Proof. Recalling the definition of $H \in \mathbb{R}^{n \times n}$ from Equation (7),

$$||H^{-1}X||_{2,\infty} = \max_{i \in [n]} \frac{||X_i||}{||X_i^T\mu||} = \left(\min_{i \in [n]} \frac{||X_i^T\mu||}{||X_i||}\right)^{-1}.$$
 (1)

Since $X_1^T X_2 \ge 0$ with probability 1, we may assume without loss of generality that the support of F is contained in the positive orthant and that μ has all entries strictly positive. Defining $S_{\ge 0}$ to be the intersection of the unit sphere with the non-negative orthant, i.e.,

$$S_{\geq 0} = \{ u \in \mathbb{R}^d : ||u|| = 1, u_k \geq 0 \text{ for all } k \in [d] \},$$

we have

$$\min_{i \in [n]} \frac{X_i^T \mu}{\|X_i\|} = \min_{i \in [n]} \frac{X_i^T}{\|X_i\|} \mu \ge \inf_{u \in \mathcal{S}_{>0}} u^T \mu \ge \min_{k \in [d]} \mu_k.$$

Applying this to Equation (1) and noting that the right-hand side does not depend on n,

$$\|H^{-1}X\|_{2,\infty} \le \frac{1}{\min_{k \in [d]} \mu_k}$$
 almost surely.

Defining c_F to be this right-hand side completes the proof.

COROLLARY 1. Suppose that F is a distribution on \mathbb{R}^d obeying the assumptions in Definition 2 and suppose that F obeys the growth assumption in Equation (5). Then

$$\mathbb{E} \frac{\|X_1\|^2}{(X_i^T \mu)^2}$$
 and $\mathbb{E} \frac{\|X_1\|^4}{(X_i^T \mu)^2}$

exist and are finite.

Proof. By an argument largely identical to that in Lemma 46,

$$\sup_{x \in \text{supp } F} \frac{\|x\|}{x^T \mu} < \infty,$$

from which

$$\mathbb{E}\frac{\|X_1\|^2}{(X_i^T\mu)^2} \le \left(\sup_{x \in \text{supp } F} \frac{\|x\|}{x^T\mu}\right)^2 < \infty.$$

Similarly, applying our assumption in Equation (5),

$$\mathbb{E}\frac{\|X_1\|^4}{(X_i^T\mu)^2} \le \left(\sup_{x \in \operatorname{supp} F} \frac{\|x\|}{x^T\mu}\right)^2 \mathbb{E}\|X_1\|^2 < \infty,$$

completing the proof.

Lemma 47. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ , and suppose that Assumptions 1, 2, 3 and 4 hold. Then

$$\|(A-P)D^{-1}X\|_{2,\infty} = O\left(\frac{\log^2 n}{\sqrt{n\rho}}\right)$$
 almost surely.

Proof. Applying the triangle inequality,

$$\|(A-P)D^{-1}X\|_{2,m} \le \|(A-P)\mathcal{D}^{-1}X\|_{2,m} + \|(A-P)(D^{-1}-\mathcal{D}^{-1})X\|_{2,m}. \tag{2}$$

Applying Lemma 6 and a union bound over all $i \in [n]$, it holds with high probability that

$$\begin{aligned} \left\| (A - P) \mathcal{D}^{-1} X \right\|_{2,\infty} &= \max_{i \in [n]} \left\| \sum_{j=1}^{n} \frac{(A - P)_{ij} X_{j}}{\delta_{j}} \right\| \\ &\leq \left(\sum_{j=1}^{n} \frac{\|X_{j}\|^{2}}{\delta_{j}^{2}} \right)^{1/2} \sqrt{\nu + b^{2}} \log n \\ &\leq \frac{C \sqrt{\nu + b^{2}} \log n}{\sqrt{n} \rho} \left(\sum_{j=1}^{n} \frac{n \rho^{2} \|X_{j}\|^{2}}{\delta_{j}^{2}} \right)^{1/2}. \end{aligned}$$

Note that by Corollary 1, Lemma 14 applies with q = r = 2. Combining this with our assumption that $(v + b^2) = \Theta(\rho)$,

$$\left\| (A - P)\mathcal{D}^{-1} X \right\|_{2,\infty} \le \frac{C \log n}{\sqrt{n\rho}}.$$
 (3)

Applying the definition of the $(2, \infty)$ -norm followed by Lemmas 4 and 13,

$$\|(A - P)(D^{-1} - \mathcal{D}^{-1})X\|_{2,\infty} = \max_{i \in [n]} \left\| \sum_{j=1}^{n} (A - P)_{ij} \left(\frac{1}{D_j} - \frac{1}{\delta_j} \right) X_j \right\|$$

$$\leq C \left\| \sum_{j=1}^{n} \frac{X_j}{\delta_j^2} \right\| (\nu + b^2) \sqrt{n} \log^2 n.$$

Applying Lemma 14 with r = 1, q = 2,

$$\left\|(A-P)(D^{-1}-\mathcal{D}^{-1})X\right\|_{2,\infty} \leq \frac{C(\nu+b^2)\sqrt{n}\log^2 n}{n\rho^2} \leq \frac{C\log^2 n}{\sqrt{n}\rho},$$

where we have again used our growth assumption in Equation (2). Applying this and Equation (3) to Equation (2),

$$\left\| (A - P) D^{-1} X \right\|_{2,\infty} \le \frac{C \log n}{\sqrt{n\rho}} + \frac{C \log^2 n}{\sqrt{n\rho}} = O\left(\frac{\log^2 n}{\sqrt{n\rho}}\right),$$

completing the proof.

Lemma 48. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ and suppose that Assumptions 1, 2, 3 and 4 hold.

Let $q \ge 0$ be an integer. Then for any constant $\tau > 0$,

$$\left\| G^{q+2} X - \rho D^{-1} X \left(\rho X^T D^{-1} X \right)^q X^T X \right\|_{2,\infty} \le \frac{C(q+1)(1+\tau)^q \sqrt{n} \log^2 n}{\delta_{\min}}.$$

Proof. Recalling that $G = D^{-1}A$ and $P = \rho XX^{T}$,

$$G^{q+2}X = G^{q+1}(\rho D^{-1}X)(X^TX) + G^{q+1}D^{-1}(A-P)X.$$
(4)

Applying basic properties of the $(2, \infty)$ -norm followed by Lemma 12 and using the fact that G^{q+1} is row-stochastic,

$$\|G^{q+1}D^{-1}(A-P)X\|_{2,\infty} \le \|G^{q+1}\|_{\infty} \|D^{-1}\|_{\infty} \|(A-P)X\|_{2,\infty} \le \frac{C}{\delta_{\min}} \|(A-P)X\|_{2,\infty}.$$

Further applying Lemma 6 to control $||(A-P)X||_{2,\infty}$,

$$\|G^{q+1}D^{-1}(A-P)X\|_{2,\infty} \le \frac{C\sqrt{\nu+b^2}\log n}{\delta_{\min}} \left(\sum_{i=1}^n \|X_i\|^2\right)^{1/2}.$$

Multiplying through by appropriate quantities and applying the law of large numbers,

$$\|G^{q+1}D^{-1}(A-P)X\|_{2,\infty} \le \frac{C\sqrt{\nu+b^2\sqrt{n}\log n}}{\delta_{\min}}.$$

Rearranging Equation (4) and applying this bound,

$$\left\| G^{q+2} X - G^{q+1}(\rho D^{-1} X)(X^T X) \right\|_{2,\infty} \le \frac{C\sqrt{\nu + b^2} \sqrt{n} \log n}{\delta_{\min}}.$$
 (5)

Again recalling $G = D^{-1}A$ and adding and subtracting appropriate quantities,

$$G^{q+1}(\rho D^{-1}X)(X^TX) = G^q \rho D^{-1}X(\rho X^T D^{-1}X)(X^TX) + G^q D^{-1}(A - P)(\rho D^{-1}X)(X^TX).$$
(6)

Again applying basic properties of the $(2, \infty)$ -norm,

$$\left\|G^q D^{-1}(A-P)(\rho D^{-1}X)(X^TX)\right\|_{2,\infty} \leq \rho \left\|G^q\right\|_{\infty} \left\|D^{-1}\right\|_{\infty} \left\|(A-P) D^{-1}X\right\|_{2,\infty} \left\|X\right\|^2 d^{\frac{1}{2}} d^{\frac{1$$

Using the fact that G^q is row-stochastic and applying Lemmas 12, 20 and 47,

$$\|G^q D^{-1}(A - P)(\rho D^{-1}X)(X^T X)\|_{2,\infty} \le \frac{C\sqrt{n}\log n}{\delta_{\min}}.$$

Rearranging Equation (6), applying the triangle inequality and using this bound,

$$\left\| G^{q+1}(\rho D^{-1}X)(X^TX) - G^q(\rho D^{-1}X) \left(\rho X^T D^{-1}X \right) (X^TX) \right\|_{2,\infty} \le \frac{C\sqrt{n} \log n}{\delta_{\min}}.$$

Applying the triangle inequality and combining this bound with Equation (5),

$$\|G^{q+2}X - G^{q}(\rho D^{-1}X)(\rho X^{T}D^{-1}X)(X^{T}X)\|_{2,\infty} \le \frac{C\left(1 + \sqrt{\nu + b^{2}}\right)\sqrt{n}\log^{2}n}{\delta_{\min}}.$$

Applying our growth assumption in Equation (2) and using the fact that $\rho = O(1)$ trivially, it holds with high probability that

$$\left\| G^{q+2} X - G^{q} (\rho D^{-1} X) (\rho X^{T} D^{-1} X) (X^{T} X) \right\|_{2,\infty} \le \frac{C \sqrt{n} \log^{2} n}{\delta_{\min}}.$$
 (7)

If q = 0, our proof is complete, so suppose that $q \ge 1$. An argument essentially identical to that following Equation (6) yields

$$\begin{split} & \left\| G^q(\rho D^{-1}X)(\rho X^T D^{-1}X)(X^T X) - G^{q-1}(\rho D^{-1}X)(\rho X^T D^{-1}X)^2(X^T X) \right\|_{2,\infty} \\ & \leq \left\| G^{q-1} D^{-1}(A-P)(\rho D^{-1}X)(\rho X^T D^{-1}X)(X^T X) \right\|_{2,\infty} \\ & \leq \rho \left\| D^{-1} \right\|_{\infty} \left\| (A-P) D^{-1} \right\|_{2,\infty} \left\| \rho X^T D^{-1}X \right\| \left\| X^T X \right\| \\ & \leq \frac{C\sqrt{n} \log^2 n}{\delta_{\min}} \left\| \rho X^T D^{-1}X \right\|. \end{split}$$

Applying the triangle inequality and recursively repeating this argument,

$$\begin{split} \left\| G^q(\rho D^{-1}X)(\rho X^T D^{-1}X)(X^T X) - (\rho D^{-1}X)(\rho X^T D^{-1}X)^{q+1}(X^T X) \right\|_{2,\infty} \\ & \leq \frac{C\sqrt{n}\log^2 n}{\delta_{\min}} \sum_{m=1}^q \left\| \rho X^T D^{-1}X \right\|^m. \end{split}$$

Applying the triangle inequality and using Equation (7),

$$\left\| G^{q+2}X - (\rho D^{-1}X)(\rho X^T D^{-1}X)^{q+1}(X^T X) \right\|_{2,\infty} \le \frac{C\sqrt{n}\log^2 n}{\delta_{\min}} \sum_{m=0}^q \left\| \rho X^T D^{-1}X \right\|^m. \tag{8}$$

Applying Lemma 45 and using our growth assumption in Equation (1), since $\tau > 0$ is constant by assumptio is constant by assumption, it holds with high probability that for all suitably large n that

$$\|\rho X^T D^{-1} X\|^m \le (\|\rho X^T \mathcal{D}^{-1} X\| + \tau)^m \le (1 + \tau)^m,$$

where the second inequality follows from Lemma 17. Applying this bound to Equation (8) and trivially upper-bounding the sum,

$$\begin{split} \left\| G^{q+2} X - (\rho D^{-1} X) (\rho X^T D^{-1} X)^{q+1} (X^T X) \right\|_{2,\infty} & \leq \frac{C \sqrt{n} \log^2 n}{\delta_{\min}} \sum_{m=0}^q (1+\tau)^m \\ & \leq \frac{C (q+1) (1+\tau)^q \sqrt{n} \log^2 n}{\delta_{\min}}, \end{split}$$

as we set out to show. \Box

LEMMA 49. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ , and suppose that Assumptions 4, 1, 2 and 3 hold as well as the growth assumption in Equation (3). Then

$$\left\| \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X - H^{-1} X \Gamma \Lambda \right\|_{2, \infty} = o(1).$$

Proof. Recursively applying Lemma 45, for any $q \ge 0$,

$$\left\| \left(\beta \rho X^T D^{-1} X \right)^q - \left(\beta \rho X^T \mathcal{D}^{-1} X \right)^q \right\| \le \frac{C |\beta|^q \log n}{\sqrt{n\rho}} \sum_{m=0}^{q-1} \left(\frac{\log n}{\sqrt{n\rho}} \right)^m \\ \le \frac{C q |\beta|^q \log n}{\sqrt{n\rho}},$$

where the second inequality follows from our assumption in Equation (1).

Applying the Neumann expansion followed by the triangle inequality, the above display implies that

$$\left\| \left(I - \beta \rho X^T D^{-1} X \right)^{-1} - \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} \right\| \le \frac{C \log n}{\sqrt{n\rho}} \sum_{q=0}^{\infty} q |\beta|^q = O\left(\frac{\log n}{\sqrt{n\rho}} \right), \tag{9}$$

where we have used the fact that $|\beta| < 1$.

Applying the basic properties of the $(2, \infty)$ -norm and multiplying through by appropriate quantities,

$$\begin{split} & \left\| \rho D^{-1} X \left(I - \beta \rho X^{T} D^{-1} X \right)^{-1} X^{T} X - \rho D^{-1} X \left(I - \beta \rho X^{T} \mathcal{D}^{-1} X \right)^{-1} X^{T} X \right\|_{2,\infty} \\ & \leq \left\| n \rho D^{-1} X \right\|_{2,\infty} \left\| \left(I - \beta \rho X^{T} D^{-1} X \right)^{-1} - \left(I - \beta \rho X^{T} \mathcal{D}^{-1} X \right)^{-1} \right\| \left\| \frac{X^{T} X}{n} \right\| \\ & \leq \frac{C \sqrt{n} \log^{2} n}{\delta_{\min}} \left\| n \rho D^{-1} X \right\|_{2,\infty} \left\| \frac{X^{T} X}{n} \right\|. \end{split}$$

Applying the law of large numbers to the sample covariance $n^{-1}X^{T}X$, it follows that

$$\begin{split} & \left\| \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X - \rho D^{-1} X \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} X^T X \right\|_{2,\infty} \\ & \leq \frac{C \sqrt{n} \log^2 n}{\delta_{\min}} \left\| n \rho D^{-1} X \right\|_{2,\infty}. \end{split}$$

Applying Lemma 40 followed by Lemma 46,

$$\begin{split} & \left\| \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X - \rho D^{-1} X \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} X^T X \right\|_{2,\infty} \\ & \leq \frac{C \sqrt{n} \log^2 n}{\delta_{\min}} \left[\left\| H^{-1} X \right\|_{2,\infty} + o(1) \right] \leq \frac{C \sqrt{n} \log^2 n}{\delta_{\min}}. \end{split}$$

Applying Lemma 16 followed by our growth assumption in Equation (3)

$$\left\| \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X - \rho D^{-1} X \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} X^T X \right\|_{2,\infty}$$

$$\leq \frac{C \log^2 n}{\sqrt{n} \rho \min_{i \in [n]} X_i^T \mu} = o(1) \text{ almost surely.}$$
(10)

We note that Lemma 17 along with the fact that $|\beta \rho| < 1$ implies that

$$\left\| \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} \right\| = O(1).$$

It follows that, by basic properties of the $(2, \infty)$ -norm,

$$\left\| \left(n\rho D^{-1} X - H^{-1} X \right) \left(I - \beta \rho X^{T} \mathcal{D}^{-1} X \right)^{-1} \frac{X^{T} X}{n} \right\|_{2,\infty}$$

$$\leq \left\| n\rho D^{-1} X - H^{-1} X \right\|_{2,\infty} \left\| \frac{X^{T} X}{n} \right\|.$$

Applying Lemma 40 and the law of large numbers,

$$\left\| n\rho D^{-1} X \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} \frac{X^T X}{n} - H^{-1} X \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} \frac{X^T X}{n} \right\|_{2,\infty} = o(1). \tag{11}$$

By a similar argument, this time applying the law of large numbers and using Lemma 46 to control $H^{-1}X$.

$$\left\| H^{-1}X \left(I - \beta \rho X^{T} \mathcal{D}^{-1} X \right)^{-1} \frac{X^{T} X}{n} - H^{-1}X \left(I - \beta \rho X^{T} \mathcal{D}^{-1} X \right)^{-1} \Lambda \right\|_{2,\infty}$$

$$\leq \left\| H^{-1}X \right\|_{2,\infty} \left\| \frac{X^{T} X}{n} - \Lambda \right\| = o(1) \text{ almost surely.}$$
(12)

Applying basic properties of the $(2, \infty)$ -norm,

$$\left\| H^{-1}X \left(I - \beta \rho X^{T} \mathcal{D}^{-1}X \right)^{-1} \Lambda - H^{-1}X \left(I - \beta \mathbb{E} \frac{X_{1}X_{1}^{T}}{X_{1}^{T}\mu} \right)^{-1} \Lambda \right\|_{2,\infty}$$

$$\leq \left\| H^{-1}X \right\|_{2,\infty} \left\| \left(I - \beta \rho X^{T} \mathcal{D}^{-1}X \right)^{-1} - \left(I - \beta \mathbb{E} \frac{X_{1}X_{1}^{T}}{X_{1}^{T}\mu} \right)^{-1} \right\| \|\Lambda\|.$$

Lemma 46 and Lemma 42, along with the continuous mapping theorem imply that almost surely,

$$\left\| H^{-1} X \left(I - \beta \rho X^T \mathcal{D}^{-1} X \right)^{-1} \Lambda - H^{-1} X \left(I - \beta \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right)^{-1} \Lambda \right\|_{2, \infty} = o(1).$$
 (13)

Applying the triangle inequality and combining Equations (10), (11), (12) and (13),

$$\left\| \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X - H^{-1} X \left(I - \beta \mathbb{E} \frac{X_1 X_1^T}{X_1^T \mu} \right)^{-1} \Lambda \right\|_{2,\infty} = o(1)$$

almost surely, completing the proof.

LEMMA 50. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ and suppose that Assumptions 1, 2, 3 and 4 hold. Then

$$\|(I - \beta G)^{-1} G^2 X - H^{-1} X \Gamma \Lambda\|_{2,\infty} = o(1)$$
 almost surely.

Proof. Applying Lemma 48, and letting $\tau > 0$ be a constant of our choosing to be specified below, it holds with high probability that for all $q \ge 0$,

$$\left\| \beta^{q} G^{q+2} X - \beta^{q} \rho D^{-1} X \left(\rho X^{T} D^{-1} X \right)^{q} X^{T} X \right\|_{2,\infty} \le \frac{C |\beta|^{q} (q+1) (1+\tau)^{q} \sqrt{n} \log^{2} n}{\delta_{\min}}.$$
 (14)

Recalling that by the Neumann expansion we have

$$(I - \beta G)^{-1} G^2 X = \sum_{q=0}^{\infty} \beta^q G^{q+2} X,$$

Applying the triangle inequality and using the bound in Equation (14) yields that

$$\left\| (I - \beta G)^{-1} G^{2} X - \sum_{q=0}^{\infty} \beta^{q} \rho D^{-1} X \left(\rho X^{T} D^{-1} X \right)^{q} X^{T} X \right\|_{2,\infty}$$

$$\leq \sum_{q=0}^{\infty} |\beta|^{q} \left\| G^{q+2} X - \rho D^{-1} X \left(\rho X^{T} D^{-1} X \right)^{q} X^{T} X \right\|_{2,\infty}$$

$$\leq \frac{C \sqrt{n} \log^{2} n}{\delta_{\min}} \sum_{q=0}^{\infty} (q+1) (1+\tau)^{q} |\beta|^{q}.$$

Choosing τ small enough that $(1 + \tau)|\beta| < 1$, the infinite sum converges to a quantity depending only on constants τ and β , and it follows that

$$\left\| (I - \beta G)^{-1} G^2 X - \sum_{q=0}^{\infty} \beta^q \rho D^{-1} X \left(\rho X^T D^{-1} X \right)^q X^T X \right\|_{2,\infty} \le \frac{C \sqrt{n} \log^2 n}{\delta_{\min}}.$$

Applying Lemma 16 followed by our growth assumption in Equation (3),

$$\left\| (I - \beta G)^{-1} G^2 X - \sum_{q=0}^{\infty} \beta^q \rho D^{-1} X \left(\rho X^T D^{-1} X \right)^q X^T X \right\|_{2,\infty} = o(1) \text{ almost surely.}$$
 (15)

We observe that by Lemma 17 and Lemma 45, for any constant $\tau > 0$, it holds for all suitably large n that

$$\left\|\rho X^T D^{-1} X\right\|^m \leq \left(\left\|\rho X^T \mathcal{D}^{-1} X\right\| + \tau\right)^m \leq (1+\tau)^m\,,$$

Again choosing τ small enough that $(1 + \tau)|\beta| < 1$, we have

$$\|\rho \beta X^T D^{-1} X\| \le |\beta| \|\rho X^T D^{-1} X\| \le |\beta| (1+\tau) < 1,$$

so that the Neumann expansion converges, and

$$\sum_{q=0}^{\infty} \beta^{q} \rho D^{-1} X \left(\rho X^{T} D^{-1} X \right)^{q} X^{T} X = \rho D^{-1} X \left[\sum_{q=0}^{\infty} \left(\beta \rho X^{T} D^{-1} X \right)^{q} \right] X^{T} X$$
$$= \rho D^{-1} X \left(I - \beta \rho X^{T} D^{-1} X \right)^{-1} X^{T} X.$$

Plugging this into Equation (15), it holds with high probability that

$$\left\| (I - \beta G)^{-1} G^2 X - \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X \right\|_{2,\infty} \le \frac{C \sqrt{n} \log^2 n}{\delta_{\min}}.$$

Applying Lemma 16 followed by our growth assumption in Equation (3),

$$\left\| (I - \beta G)^{-1} G^2 X - \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X \right\|_{2,\infty} = o(1) \text{ almost surely.}$$

Lemma 49 implies that

$$\left\| \rho D^{-1} X \left(I - \beta \rho X^T D^{-1} X \right)^{-1} X^T X - H^{-1} X \Gamma \Lambda \right\|_{2,\infty} = o(1) \text{ almost surely.}$$

The triangle inequality and combining the above two displays completes the proof.

Lemma 51. Let $(A, X) \sim \text{RDPG}(F, n)$ with (v, b)-subgamma edges and sparsity parameter ρ and suppose that Assumptions 1, 2, 3 and 4 hold. Then

$$\left\| \left(I - \beta G \right)^{-1} G X - H^{-1} X \left(I + \beta \Gamma \right) \Lambda \right\|_{2,\infty} = o(1) \ \ almost \ surely.$$

Proof. Applying the Neumann expansion,

$$(I - \beta G)^{-1} GX = GX + \sum_{q=1}^{\infty} \beta^q G^{q+1} X = GX + \beta (I - \beta G)^{-1} G^2 X.$$

Applying the triangle inequality followed by Lemmas 41 and 50,

$$\begin{split} \left\| (I - \beta G)^{-1} G X - H^{-1} X \left(I + \beta \Gamma \right) \Lambda \right\|_{2,\infty} \\ & \leq \left\| G X - H^{-1} X \Lambda \right\|_{2,\infty} + |\beta| \left\| (I - \beta G)^{-1} G^2 X - H^{-1} X \Gamma \Lambda \right\|_{2,\infty} \\ & = o(1) \ \text{ almost surely,} \end{split}$$

as we set out to show.

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