Estimating network-mediated causal effects via spectral embeddings

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NetSci 2024

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This is joint work

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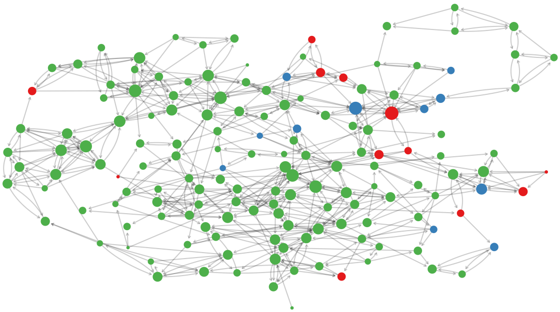
Motivation: understand causes of teenage smoking (Michell and West, 1996)



Tobacco use ● Never ● Occasional ● Regular

- 129 middle-schoolers in Glasgow
- **Social network** based on self-reported friendships
- **Auxiliary data:** spending money, leisure activities
- **Demographics:** sex, age
- **Behavior:** alcohol, cannabis and tobacco use

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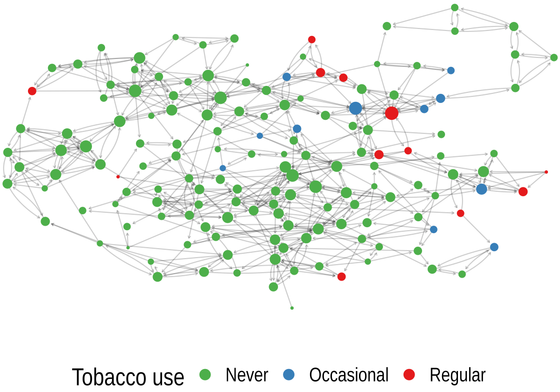

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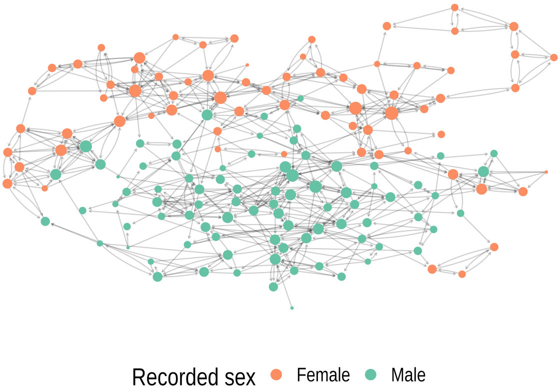
Question: how does sex influence tobacco use?

Smoking is a sexually differentiated behavior and also a social behavior



Direct effect of sex on smoking

- Social and cultural expectations could lead to more or less smoking

Indirect effect of sex on smoking

- Students to be friends with other students of same sex…
- …friends induce one another to smoke

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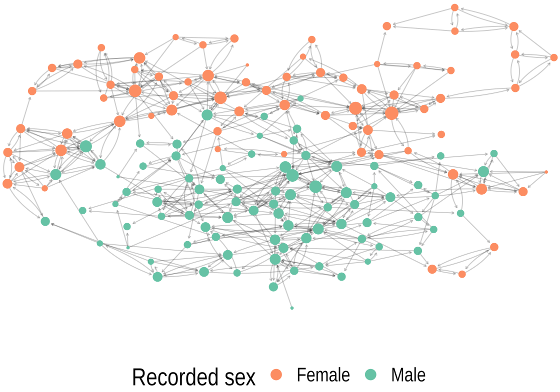

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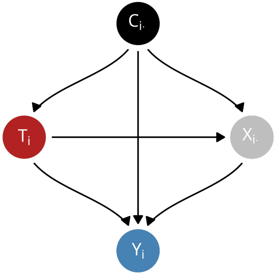
Should anti-smoking interventions target the direct or the social mechanism?

Defining causal effects is straightforward when there is no social network

Treatment	T_i	$\in \{0, 1\}$
Outcome	Y_i	$\in \mathbb{R}$
Mediators	$X_{i\cdot}$	$\in \mathbb{R}^{1\times d}$
Confounders	$C_{i\cdot}$	$\in \mathbb{R}^{1\times p}$



Definition (Average treatment effect)

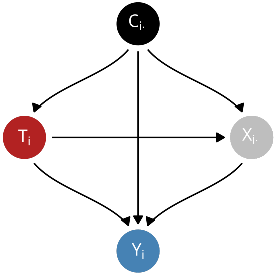
$$\Psi_{\mathrm{ate}} = \mathbb{E}[Y(1) - Y(0)]$$

$Y(t)$ is the counterfactual outcome under treatment value $t \in \{0, 1\}$.

Decomposing causal effects is straightforward when there is no social network

Two distinct mechanisms:

1. Direct effect along $T_i \to Y_i$ path
2. Indirect effect along $T_i \to X_{i\cdot} \to Y_i$ path



Definition (Natural direct and indirect effects)

$$
\begin{aligned}
\Psi_{\text{ate}} &= \Psi_{\text{nde}} + \Psi_{\text{nie}} \\
&= \mathbb{E}[Y(1, M(0)) - Y(0, M(0))] + \mathbb{E}[Y(1, M(1)) - Y(1, M(0))]
\end{aligned}
$$

$Y(t, m)$ is the counterfactual outcome under treatment t and mediator m

We want to decompose node-level effects into direct and social components

Network $A \in \mathbb{R}^{n \times n}$

For each node i:

- Treatment $T_i \in \{0, 1\}$
- Outcome $Y_i \in \mathbb{R}$
- Confounders $C_{i\cdot} \in \mathbb{R}^p$

The network A is noisy and complex but often exhibits high levels of homophily

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The network A is noisy and complex but often exhibits high levels of homophily

Assume each node belongs to a social group $X_{i.}$ and behaviors vary with social group

Stochastic blockmodels are the canonical model for social groups

Nodes colored by social group

$X_{i\cdot} \in \{0,1\}^d$ one-hot indicator of social group

$B \in [0,1]^{d \times d}$ between-group friend probabilities

Friendships depend on social group and B

$\mathbb{P}\left(A_{ij} = 1 \,\middle|\, X\right) = X_{i\cdot} B X_{j\cdot}^T$

<u>Social groups X are latent (i.e., unobserved)</u>

We have very rich and general models for latent social groups

Nodes colored by social group

- Degree-correction
- Mixed-membership social groups
- Overlapping social groups
- Group-specific popularity parameters

Definition (Random dot product graph)

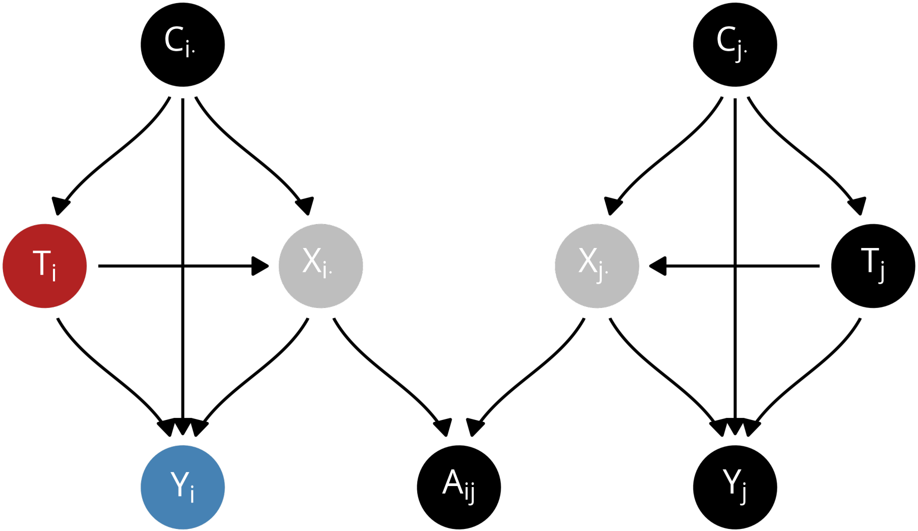
A symmetric, weighted adjacency matrix

$A = XX^T + E$

$X_{1\cdot}, \ldots, X_{n\cdot} \in \mathbb{R}^{1 \times d}$ i.i.d. latent positions

E independent sub-gamma noise

Goal: estimate direct and indirect effects as mediated by latent social groups



We want to use a regression estimator for direct and indirect effects

Assumption

$$\mathbb{E}[Y_i \mid T_i, C_{i\cdot}, X_{i\cdot}] = \beta_0 + T_i\beta_{\mathsf{t}} + C_{i\cdot}\beta_{\mathsf{c}} + X_{i\cdot}\beta_{\mathsf{x}} \quad \text{outcome model}$$
$$\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}] = \theta_0 + T_i\theta_{\mathsf{t}} + C_{i\cdot}\Theta_{\mathsf{c}} \quad \text{mediator model}$$

Fact: semi-parametric identification of natural mediated effects

Under previous assumption, when natural direct and indirect effect are non-parametrically identified, we have

$$\Psi_{\text{nde}} = \beta_{\mathsf{t}}$$
$$\Psi_{\text{nie}} = \theta_{\mathsf{t}}\,\beta_{\mathsf{x}}$$

Problem: We don't see the latent social groups X, can't fit regressions

Latent positions X can be estimated via principle components analysis

Definition (ASE)

Given a network A, the d-dimensional adjacency spectral embedding of A is

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2}$$

where $\widehat{U}\widehat{S}\widehat{U}^T$ is the rank-d truncated singular value decomposition of A.

Lemma

Under a suitable network model, there is a $d \times d$ orthogonal matrix Q such that

$$\max_{i \in [n]} \left\| \widehat{X}_{i\cdot} - X_{i\cdot} Q \right\| = o_p(1).$$

Estimated latent positions can plug directly into ordinary least squares

Let $\widehat{D} = \begin{bmatrix} 1 & T & C & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$ and $L = \begin{bmatrix} 1 & T & C \end{bmatrix} \in \mathbb{R}^{n \times (p+2)}$.

$$\begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_\mathrm{t} \\ \widehat{\beta}_\mathrm{c} \\ \widehat{\beta}_\mathrm{x} \end{bmatrix} = \left(\widehat{D}^T \widehat{D} \right)^{-1} \widehat{D}^T Y \quad \text{and} \quad \widehat{\Theta} = \left(L^T L \right)^{-1} L^T \widehat{X}.$$

$$\widehat{\Psi}_\mathrm{nde} = \widehat{\beta}_\mathrm{t} \quad \text{and} \quad \widehat{\Psi}_\mathrm{nie} = \widehat{\theta}_\mathrm{t} \, \widehat{\beta}_\mathrm{x}$$

Ordinary least squares regression estimates are asymptotically normal

Theorem

Under a suitably well-behaved network model and some moments conditions on regression errors, there is an unknown orthogonal matrix Q such that

$$\sqrt{n}\,\widehat{\Sigma}_{\beta}^{-1/2}\begin{pmatrix}\widehat{\beta}_{\mathrm{w}} - \beta_{\mathrm{w}} \\ Q\,\widehat{\beta}_{\mathrm{x}} - \beta_{\mathrm{x}}\end{pmatrix} \to \mathcal{N}(0, I_d), \textit{and}$$

$$\sqrt{n}\,\widehat{\Sigma}_{\mathrm{vec}(\Theta)}^{-1/2}\left(\mathrm{vec}\left(\widehat{\Theta}\,Q^T\right) - \mathrm{vec}(\Theta)\right) \to \mathcal{N}\left(0, I_{pd}\right).$$

where $\widehat{\Sigma}_{\mathrm{vec}(\Theta)}^{-1/2}$ and $\widehat{\Sigma}_{\beta}^{-1/2}$ are the typical heteroscedastic robust covariance estimators, with $\widehat{X}$ plugged in for X.

Note: must correctly specify dimension of latent positions d

Aside: our regression results are substantially more general than similar work

- Network can be weighted, rather than binary
- No parametric assumptions on edge noise
- No parametric assumptions on regression errors
- Regression errors can be heteroscedastic
- Can model latent positions as outcomes

Principle components + ordinary least squares is a very general, distributionally agnostic tool for network regression

<u>Folklore</u>: similar results hold for asymmetric A, rectangular A, bipartite A, graph Laplacian embeddings rather than adjacency matrix embeddings, general regression M-estimators other than ordinary least squares

Causal estimators are asymptotically normal

Theorem

Under the same statistical assumptions as before, plus counterfactual assumptions required for causal identification,

$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nde}}}\left(\widehat{\Psi}_{\mathrm{nde}} - \Psi_{\mathrm{nde}}\right) \to \mathcal{N}(0,1), \text{ and}$$
$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nie}}}\left(\widehat{\Psi}_{\mathrm{nie}} - \Psi_{\mathrm{nie}}\right) \to \mathcal{N}(0,1).$$

where $\widehat{\sigma}^2_{\mathrm{nde}}$ and $\widehat{\sigma}^2_{\mathrm{nie}}$ are rather unfriendly variance estimators derived via the delta method and the previous theorem.

Takeaways

- We developed a method to decompose causal effects into effects operating along direct and indirect pathways in a low-rank latent space
- ”Latent causal effects” sounds scary, but can often just look at your data to interpret them

Thank you! Questions?

Hayes, Alex, Mark M. Fredrickson, and Keith Levin. "Estimating Network-Mediated Causal Effects via Spectral Embeddings." arXiv, April 14, 2023. *https://arxiv.org/abs/2212.12041*.

Read the manuscript at

R package latentnetmediate

Stay in touch

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I'm looking for a post-doc starting Fall 2024, say hi if this work interests you!

These slides are available at

https://github.com/alexpghayes/2024-06-17-netsci-slides

Peer effects

- Identifying peer effects in these models is very nuanced
  - Not one but two forthcoming manuscripts about this
- When peer effects are identified, ordinary least squares sometimes works
- When peer effects are unidentified, typically due to aliasing with β_{x} or β_0
  - Aliasing might occur only in the asymptotic limit

$$\mathbb{E}[Y_i \mid T, C_{i\cdot}, X_{i\cdot}, Y_{-i}, A] = \beta_0 + T_i\beta_{\mathsf{t}} + C_{i\cdot}\beta_{\mathsf{c}} + X_{i\cdot}\beta_{\mathsf{x}} + \underbrace{\beta_{\mathsf{gt}} \sum_{j \neq i} \frac{A_{ij}T_j}{\sum_i A_{ij}}}_{\text{interference}} + \underbrace{\beta_{\mathsf{gy}} \sum_{j \neq i} \frac{A_{ij}Y_j}{\sum_i A_{ij}}}_{\text{“contagion”}}$$

Lots of research attention focused on using this kind of model for causal inference

Identifying assumptions

T

he random variables $(Y_i, Y_i(t,x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:
$$\text{if } T_i = t\text{, then } X_{i\cdot}(t) = X_{i\cdot} \text{ with probability 1, and}$$
$$\text{if } T_i = t \text{ and } X_{i\cdot} = x\text{, then } Y_i(t,x) = Y_i \text{ with probability 1}$$

2. Sequential ignorability:
$$\{Y_i(t^*,x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot} \quad \text{and} \quad \{Y_i(t^*,x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$$

3. Positivity:
$$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0 \text{ for each } x \in \operatorname{supp}(X_{i\cdot})$$
$$\mathbb{P}(t \mid C_{i\cdot}) > 0 \text{ for each } t \in \operatorname{supp}(T_i)$$

Semi-parametric network model

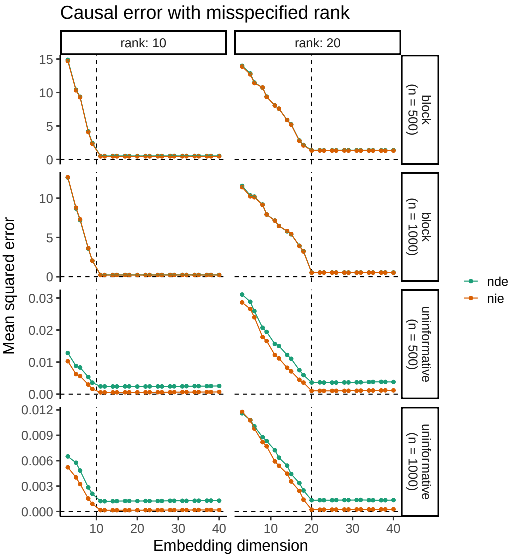
Definition

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \mid X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\operatorname{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.

Remark

$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q, the latent positions X are only identifiable up to an orthogonal transformation.

Choosing $\widehat{d}$: overestimating the embedding dimension is fine



References

Michell, L. and P. West (1996). Peer pressure to smoke: The meaning depends on the method. Health Education Research 11(1), 39–49.