Estimating network-mediated causal effects via spectral embeddings

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12:54

AWARENESS

Stable awareness allows us to notice our thoughts, emotions, and impulses. It helps maintain a sense of inner balance, to be fo at work, and be fully present in relationship

PART 1
MINDFULNESS
14/16

PART 2
ATTENTION
0/16

SERIES 1: THE MINDFULNESS EXPERIMEN

In this series we'll learn how to be mindful and apply mindfulness practice in everyday life.

LEARN 6 mins • Aug 3, 2021

Good afternoon, Anisha

Next Up

LESSON 5 min
The Pull of Distraction

Foundations
Learn the foundational skills of training your mind.
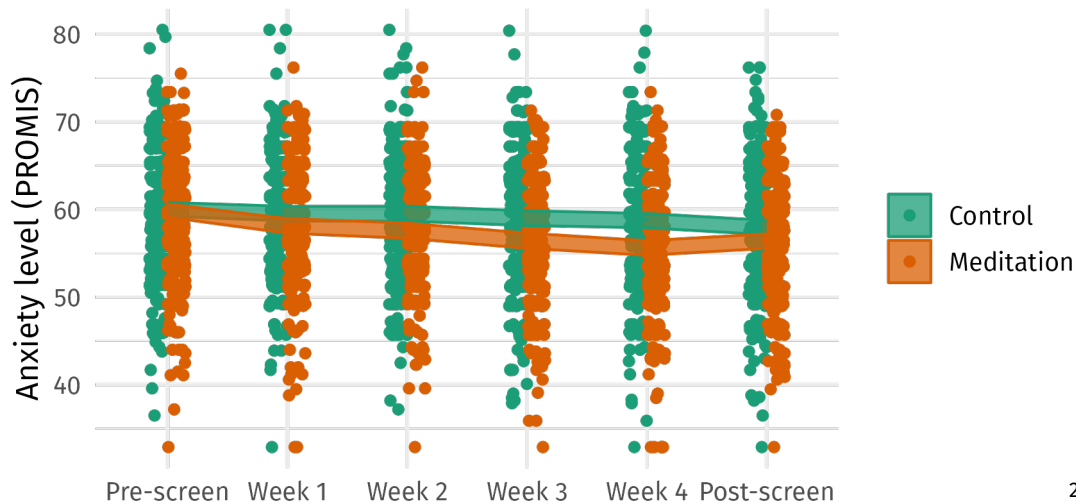
10/10 Lessons   13/13 Meditations

Looking Within

dfulness puts us in touch with a sense of ll-being that isn't so tied to the shifting circumstances of daily life.

03:01 / 04:44

We study the causal effect of four weeks of meditation

Counterfactual outcomes

anxiety, no meditation	$Y_i(0)$	$\in \mathbb{R}$
anxiety, meditation	$Y_i(1)$	$\in \mathbb{R}$

Observed data

Treatment	meditation	T_i	$\in \{0, 1\}$
Outcome	anxiety	Y_i	$\in \mathbb{R}$

$$\Psi_{\text{ate}} = \mathbb{E}[Y_i(1) - Y_i(0)]$$
$$= \mathbb{E}[Y_i \mid T_i = 1] - \mathbb{E}[Y_i \mid T_i = 0]$$
$$\widehat{\Psi}_{\text{ate}} = -2.7 \pm 1.3$$
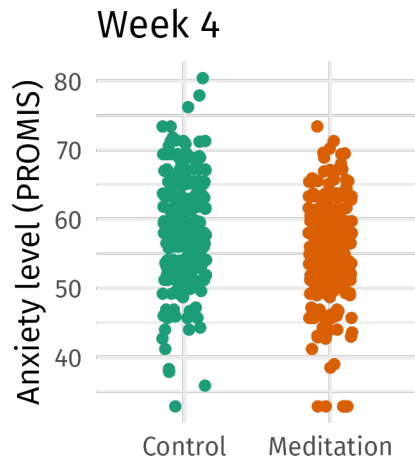
Week 4

The meditation program is designed to alter latent cognitive factors

1. Awareness (mindful action)
2. Connection (social connection, reducing loneliness)
3. Insight (cognitive defusion)
4. Purpose (presence of meaning)

The hope: improving these latent cognitive factors reduces anxiety

Multi-stage effects can be formalized as causal mediation

Decompose effect of T_i on Y_i:

1. Effect operating along $T_i \rightarrow Y_i$ path (direct)
   - i.e., guided breathing reduces anxiety

2. Effect operating along $T_i \rightarrow X_{i\cdot} \rightarrow Y_i$ path (indirect)
   - i.e., meditation program decreases loneliness, which in turn decreases anxiety

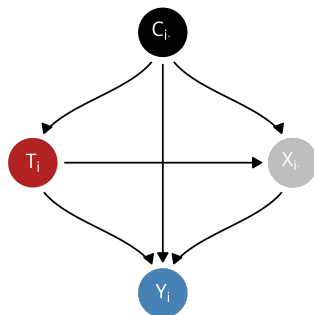$$\Psi_{\text{ate}} = \Psi_{\text{nde}} + \Psi_{\text{nie}}$$
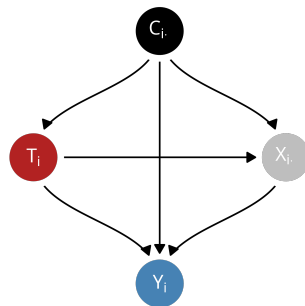
Natural direct and indirect effects are defined counterfactually

Treatment	meditation	T_i	$\in \{0, 1\}$
Outcome	anxiety	Y_i	$\in \mathbb{R}$
Mediators	cognitive state	$X_{i\cdot}$	$\in \mathbb{R}^{1 \times d}$
Confounders	age & sex	$C_{i\cdot}$	$\in \mathbb{R}^{1 \times p}$

$$\Psi_{\mathrm{nde}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t^*)) - Y_i(t^*, X_{i\cdot}(t^*))]$$
$$\Psi_{\mathrm{nie}} = \mathbb{E}[Y_i(t, X_{i\cdot}(t)) - Y_i(t, X_{i\cdot}(t^*))]$$

If we knew cognitive state, we could use standard
tools. But cognitive state is latent!

Psychologists measure latent cognitive state using surveys

For example, the NIH Toolbox loneliness survey

1. I feel alone and apart from others
2. I feel left out
3. I feel that I am no longer close to anyone
4. I feel alone
5. I feel lonely

Never	Rarely	Sometimes	Usually	Always
1 ○	2 ○	3 ○	4 ○	5 ○

The experimenters ran weekly surveys of the study participants

- NIH Toolbox Loneliness (5 questions)
- Five Facet Mindfulness Questionnaire Acting with Awareness subscale (8 questions)
- Drexel Defusion Scale (10 questions)
- Meaning in Life Questionnaire (10 questions)

Another survey: the Meaning in Life Questionnaire

1. I understand my life's meaning.
2. I am looking for something that makes my life feel meaningful.
3. I am always looking to find my life's purpose.
4. My life has a clear sense of purpose.
5. I have a good sense of what makes my life meaningful.
6. I have discovered a satisfying life purpose.
7. I am always searching for something that makes my life feel significant.
8. I am seeking a purpose or mission for my life
9. My life has no clear purpose.
10. I am searching for meaning in my life.

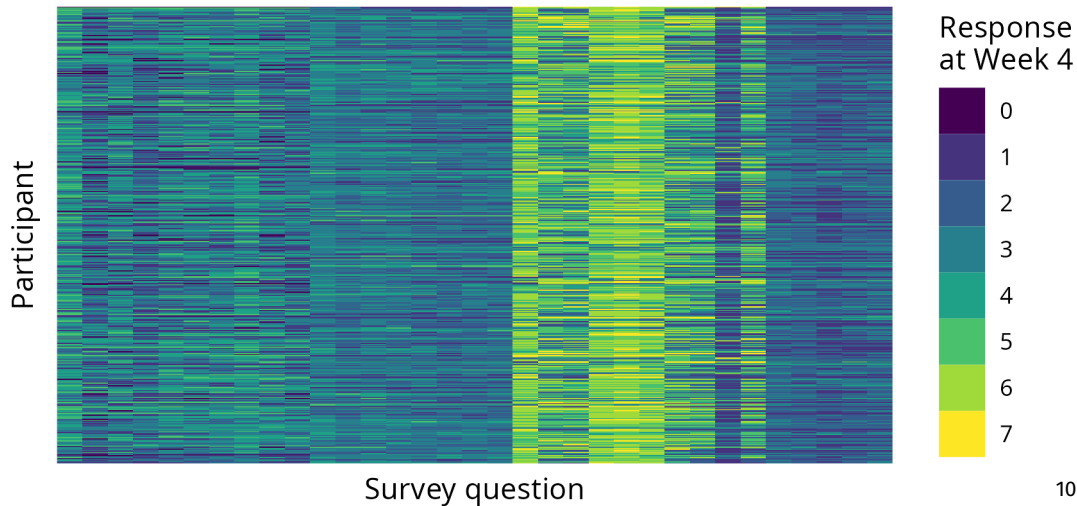| Absolutely untrue	Mostly untrue	Somewhat untrue	Can't say true or false	Somewhat true	Mostly true	Absolutely true
1 ○	2 ○	3 ○	4 ○	5 ○	6 ○	7 ○

The survey responses form a bipartite network

We model survey responses using a factor model

Suppose $A \in \mathbb{R}^{n \times m}$ is the matrix of survey responses. Then

$$\mathbb{E}[A \mid X, Z] = XZ^T$$

Participant embeddings $\quad X \quad \in \mathbb{R}^{n \times d}$

Question embeddings $\quad\quad Z \quad \in \mathbb{R}^{m \times d}$

We estimate latent cognitive state by embedding the network

Definition (ASE)

Given a network *A*, the *d*-dimensional <u>adjacency spectral embeddings</u> of *A* is

$$\widehat{X} = \widehat{U}\widehat{S}^{1/2} \text{ and } \widehat{Z} = \widehat{V}\widehat{S}^{1/2}$$
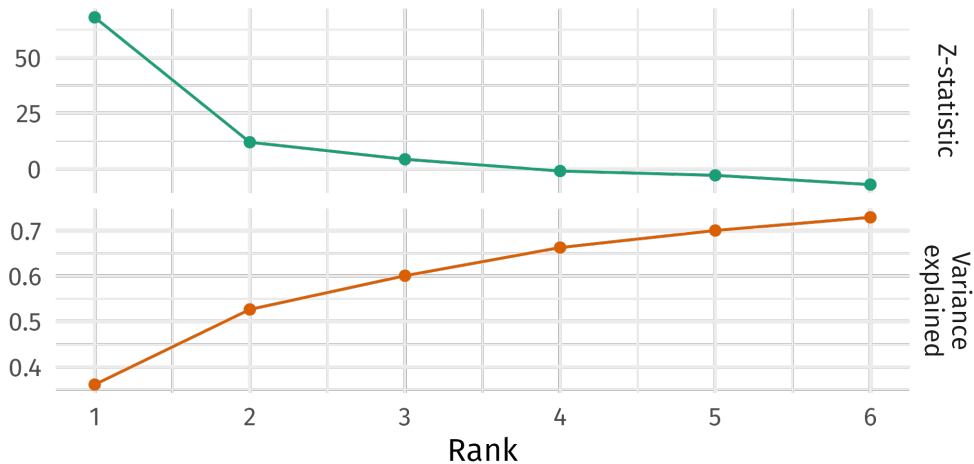
where $\widehat{U}\widehat{S}\widehat{V}^T$ is the rank-*d* truncated singular value decomposition of *A*.

Lemma

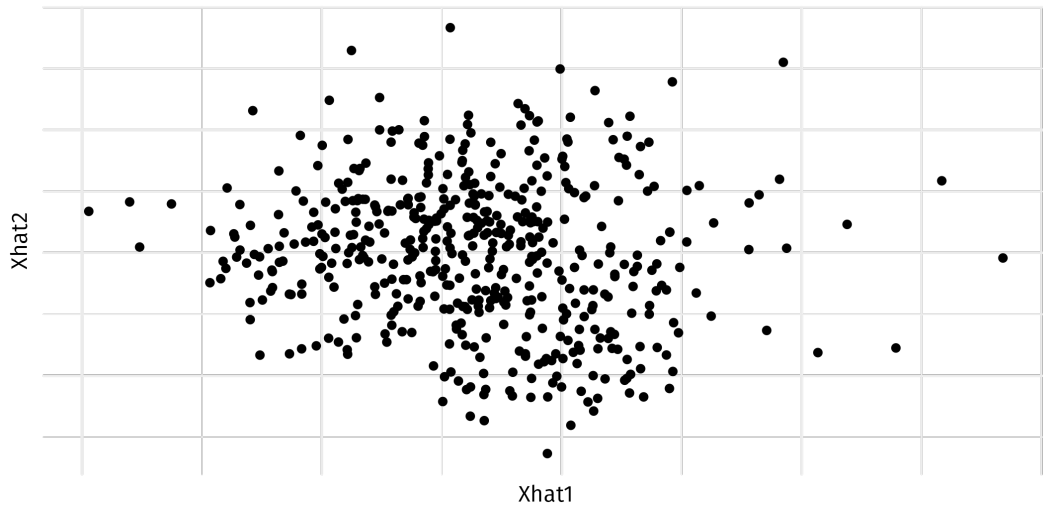*Under a suitable low-rank model, there are $d \times d$ orthogonal matrices Q_1, Q_2 such that*

$$\max_{i \in [n]} \left\| \widehat{X}_{i\cdot} - X_{i\cdot}Q_1 \right\| = o_p(1) \text{ and } \max_{i \in [n]} \left\| \widehat{V}_{i\cdot} - V_{i\cdot}Q_2 \right\| = o_p(1).$$
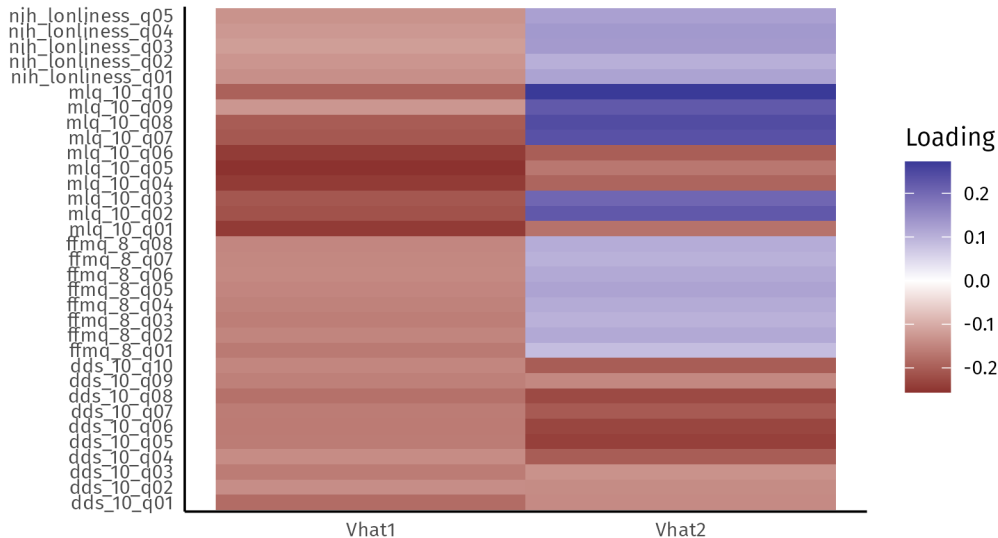
We must estimate the number of latent factors d

Cross-validated eigenvalue method selects d = 2

Participants embed into an inocuous latent space

- Psychologists hoped to find four latent factors corresponding to loneliness, meaning in life, etc, etc
- We only found two "do you feel good", and "do you feel bad"
- People basically answer all the questions exactly the same
- Still curious how much the latent well-being factors account for decrease in anxiety

We use survey responses to estimate latent participant embeedings

The identifying assumptions can be expressed counterfactually

The random variables $(Y_i, Y_i(t,x), X_{i\cdot}, X_{i\cdot}(t), C_{i\cdot}, T_i)$ are independent over $i \in [n]$ and obey the following three properties.

1. Consistency:

   if $T_i = t$, then $X_{i\cdot}(t) = X_{i\cdot}$ with probability 1, and

   if $T_i = t$ and $X_{i\cdot} = x$, then $Y_i(t,x) = Y_i$ with probability 1

2. Sequential ignorability:

   $$\{Y_i(t^*,x), X_{i\cdot}(t)\} \perp\!\!\!\perp T_i \mid C_{i\cdot} \quad \text{and} \quad \{Y_i(t^*,x)\} \perp\!\!\!\perp X_{i\cdot} \mid T_i = t, C_{i\cdot}$$

3. Positivity:

   $$\mathbb{P}(x \mid T_i, C_{i\cdot}) > 0 \text{ for each } x \in \text{supp}(X_{i\cdot})$$
   $$\mathbb{P}(t \mid C_{i\cdot}) > 0 \text{ for each } t \in \text{supp}(T_i)$$

If the non-parametric identification conditions hold and also

$$\underbrace{\mathbb{E}[Y_i \mid T_i, C_{i\cdot}, X_{i\cdot}]}_{\mathbb{R}} = \underbrace{\beta_0}_{\mathbb{R}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\beta_t}_{\mathbb{R}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\beta_c}_{\mathbb{R}^p} + \underbrace{X_{i\cdot}}_{\mathbb{R}^{1\times d}} \underbrace{\beta_x}_{\mathbb{R}^d},$$

$$\underbrace{\mathbb{E}[X_{i\cdot} \mid T_i, C_{i\cdot}]}_{\mathbb{R}^{1\times d}} = \underbrace{\theta_0}_{\mathbb{R}^{1\times d}} + \underbrace{T_i}_{\{0,1\}} \underbrace{\theta_t}_{\mathbb{R}^{1\times d}} + \underbrace{C_{i\cdot}}_{\mathbb{R}^{1\times p}} \underbrace{\Theta_c}_{\mathbb{R}^{p\times d}}$$

Then

$$\Psi_{\mathrm{nde}}(t, t^*) = (t - t^*)\,\beta_t$$
$$\Psi_{\mathrm{nie}}(t, t^*) = (t - t^*)\,\theta_t\,\beta_x$$

We using estimated $\widehat{X}$ in place of unobserved X in regression estimators

Let $\widehat{D} = \begin{bmatrix} 1 & T & C & \widehat{X} \end{bmatrix} \in \mathbb{R}^{n \times (2+p+d)}$ and $L = \begin{bmatrix} 1 & T & C \end{bmatrix} \in \mathbb{R}^{n \times (p+2)}$.

$$\begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_t \\ \widehat{\beta}_c \\ \widehat{\beta}_x \end{bmatrix} = \left(\widehat{D}^T \widehat{D} \right)^{-1} \widehat{D}^T Y \quad \text{and} \quad \widehat{\Theta} = \left(L^T L \right)^{-1} L^T \widehat{X}.$$

$$\widehat{\Psi}_{\mathrm{nde}} = (t - t^*) \, \widehat{\beta}_t$$
$$\widehat{\Psi}_{\mathrm{nie}} = (t - t^*) \, \widehat{\theta}_t \, \widehat{\beta}_x$$

Regression coefficients are asymptotically normal

Theorem

Under a suitably well-behaved network model and some moment conditions on regression errors, there is an unknown orthogonal matrix Q such that

$$\sqrt{n}\,\widehat{\Sigma}_{\beta}^{-1/2} \begin{pmatrix} \widehat{\beta}_{\mathsf{w}} - \beta_{\mathsf{w}} \\ Q\,\widehat{\beta}_{\mathsf{x}} - \beta_{\mathsf{x}} \end{pmatrix} \to \mathcal{N}(0, I_d), \text{ and}$$

$$\sqrt{n}\,\widehat{\Sigma}_{\mathsf{vec}(\Theta)}^{-1/2} \left(\mathsf{vec}\left(\widehat{\Theta}\,Q^T\right) - \mathsf{vec}(\Theta) \right) \to \mathcal{N}(0, I_{pd}).$$

where $\widehat{\Sigma}_{\mathsf{vec}(\Theta)}^{-1/2}$ and $\widehat{\Sigma}_{\beta}^{-1/2}$ are the typical heteroscedasticity robust covariance estimators, with $\widehat{X}$ plugged in for X.

Causal estimators are asymptotically normal

Theorem

*Under the same statistical assumptions as before, plus mediating homophily,*

$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nde}}}\left(\widehat{\Psi}_{\mathrm{nde}} - \Psi_{\mathrm{nde}}\right) \to \mathcal{N}(0, 1),\ and$$

$$\sqrt{n\,\widehat{\sigma}^2_{\mathrm{nie}}}\left(\widehat{\Psi}_{\mathrm{nie}} - \Psi_{\mathrm{nie}}\right) \to \mathcal{N}(0, 1).$$

where $\widehat{\sigma}^2_{\mathrm{nde}}$ and $\widehat{\sigma}^2_{\mathrm{nie}}$ are variance estimators derived via the delta method and the previous theorem.

Latent factors increase anxiety, meditation decreases anxiety

Outcome regression

Control coefficients not displayed
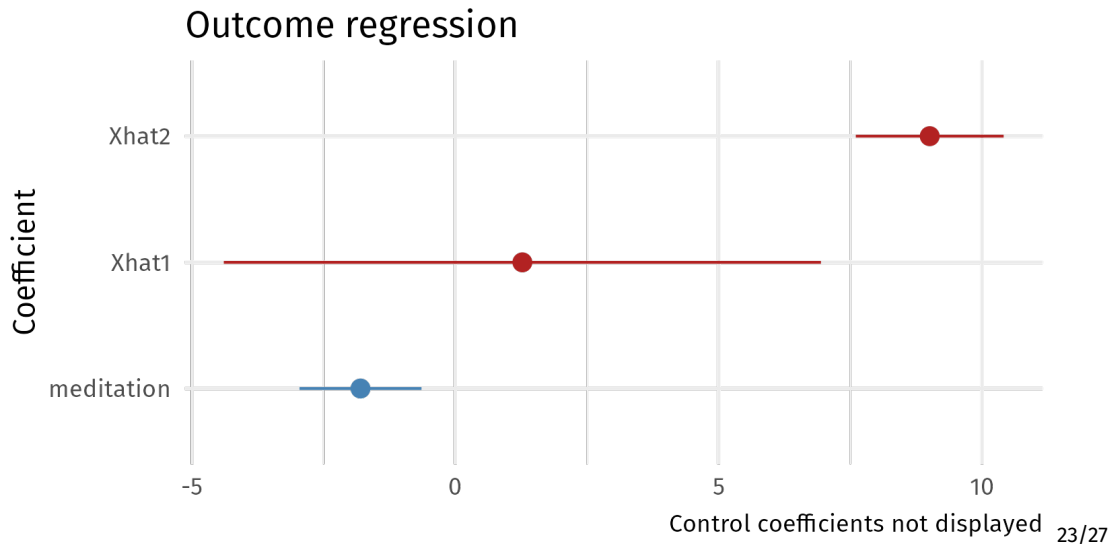
Meditation causes a small but significant shift in latent space
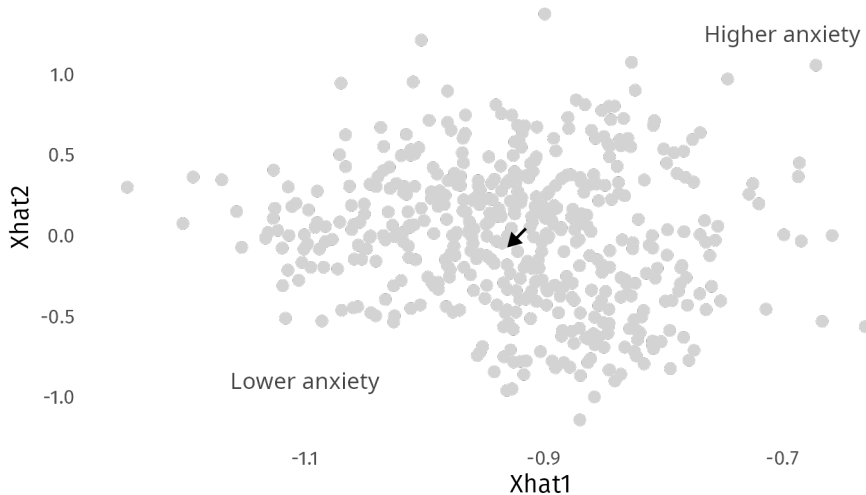
$$\widehat{\Psi}_{\mathrm{ate}} = -2.7 \pm 1.3$$
$$\widehat{\Psi}_{\mathrm{nde}} = -1.8 \pm 1.1$$
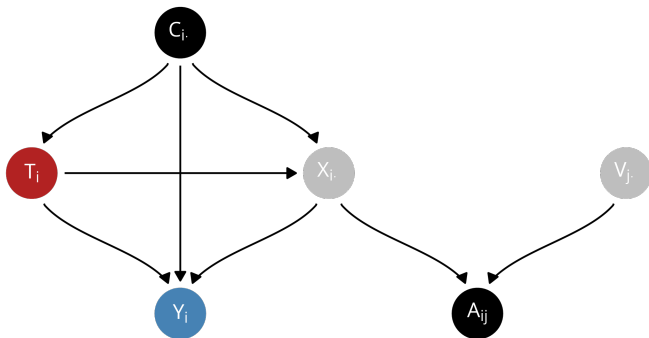$$\widehat{\Psi}_{\mathrm{nie}} = -1 \pm 0.7$$
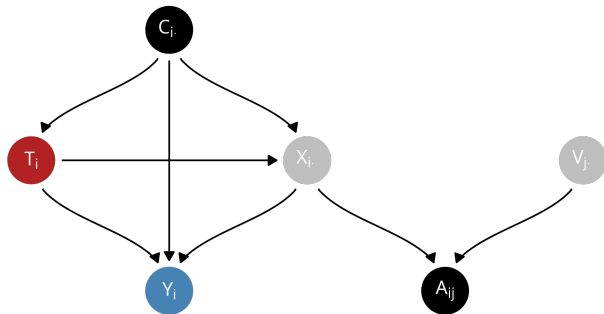
- We developed a method to decompose causal effects into effects operating along direct and indirect pathways in a low-rank latent space
- "Latent causal effects" sounds scary, but can often just look at your data to interpret them

$$\Psi_{\mathrm{ate}} = \Psi_{\mathrm{nde}} + \Psi_{\mathrm{nie}}$$
$$\widehat{\Psi}_{\mathrm{ate}} = -2.7 \pm 1.3$$
$$\widehat{\Psi}_{\mathrm{nde}} = -1.8 \pm 1.1$$
$$\widehat{\Psi}_{\mathrm{nie}} = -1 \pm 0.7$$

Read the manuscript at *https://arxiv.org/abs/2212.12041*

R package latentnetmediate

Stay in touch

🐦 @alexpghayes

✉ alex.hayes@wisc.edu

Ⓦ *https://www.alexpghayes.com*

○ *https://github.com/alexpghayes*

I'm looking for a post-doc starting Fall 2024, say hi if this work interests you!
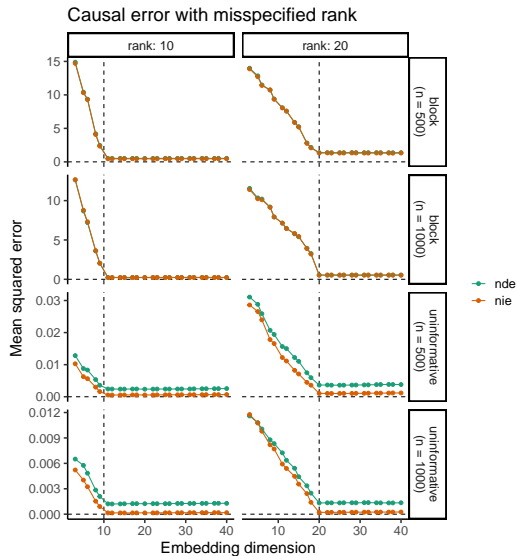
Semi-parametric network model

Definition

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \mid X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\mathrm{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.

Remark

$P = XX^T = (XQ)(XQ)^T$ for any $d \times d$ orthogonal matrix Q, the latent positions X are only identifiable up to an orthogonal transformation.

Causal error with misspecified rank

mindful action subscale

1. When I do things, my mind wanders off and I'm easily distracted.
2. I don't pay attention to what I'm doing because I'm daydreaming, worrying, or otherwise distracted.
3. I am easily distracted.
4. I find it difficult to stay focused on what's happening in the present.
5. It seems I am 'running on automatic' without much awareness of what I'm doing.
6. I rush through activities without being really attentive to them.
7. I do jobs or tasks automatically without being aware of what I'm doing.
8. I find myself doing things without paying attention.

Never or very rarely true	Rarely true	Sometimes true	Often true	Very often or always true
1 ○	2 ○	3 ○	4 ○	5 ○

Drexel Defusion Scale

1. Feelings of anger. You become angry when someone takes your place in a long line. To what extent would you normally be able to defuse from feelings of anger?

2. Cravings for food. You see your favorite food and have the urge to eat it. To what extent would you normally be able to defuse from cravings for food?

3. Physical pain. Imagine that you bang your knee on a table leg. To what extent would you normally be able to defuse from physical pain?

4. Anxious thoughts. Things have not been going well at school or your job, and work just keeps piling up. To what extent would you normally be able to defuse from anxious thoughts like "I'll never get this done."?

5. Thoughts of self. Imagine you are having a thought such as "no one likes me." To what extent would you normally be able to defuse from negative thoughts about yourself?

6. Thoughts of hopelessness. You are feeling sad and stuck in a difficult

Network model

Definition

Let $A \in \mathbb{R}^{n \times n}$ be a random symmetric matrix, such as the adjacency matrix of an undirected graph. Let $P = \mathbb{E}[A \mid X] = XX^T$ be the expectation of A conditional on $X \in \mathbb{R}^{n \times d}$, which has independent and identically distributed rows $X_{1\cdot}, \ldots, X_{n\cdot}$. That is, P has $\mathrm{rank}(P) = d$ and is positive semi-definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 = \lambda_{d+1} = \cdots = \lambda_n$. Conditional on X, the upper-triangular elements of $A - P$ are independent (ν_n, b_n)-sub-gamma random variables.

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