

STAT 709: My notes

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Chapter 1

Intro

1.1 Textbook

As of Fall 2018, this class uses the book *Mathematical Statistics* by Jun Shao (2nd. edition). Unless otherwise noted, all definitions, lemmas, proposition, theorems, etc. can be found there. Numbering might not match.

1.2 Conventions re: ∞

We will use the following conventions:

- $\infty + x = \infty, x \in \mathbb{R}$
- $x \cdot \infty = \infty$ if $x > 0$
- $x \cdot \infty = -\infty$ if $x < 0$
- $0 \cdot \infty = 0$
- $\infty + \infty = \infty$
- $\infty^a = \infty, \forall a > 0$
- $\infty - \infty$ and $\frac{\infty}{\infty}$ are not defined

Chapter 2

Lecture Notes

2.1 Chapter 1: Probability Theory

2.1.1 Lecture 1: Measure space, measurable function, and integration

2.1.1.1 σ -fields

Definition 2.1 (σ -field (or σ -algebra)). A \mathcal{F} collection of subsets of Ω is called a σ -field (or σ -algebra) if the following three conditions hold:

- i) $\emptyset \in \mathcal{F}$
- ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- iii) If $A_i \in \mathcal{F}$ for all $i = 1, 2, \dots$, then $\bigcup_i A_i \in \mathcal{F}$.

Example 2.1 (A Few σ -fields). There are some trivial examples. One is the example where $\mathcal{F} = \{\emptyset, \Omega\}$. It is easy to check that the three conditions are met for this collection of subsets. Another trivial example would be $\mathcal{F} = \mathbb{P}(\Omega)$ ¹.

The simplest non-trivial example is $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ where $A \subset \Omega$. Since this collection of subsets is so small, it is easy to check the three conditions mentioned above.

Definition 2.2 (Measurable Space). If \mathcal{F} is a σ -field on Ω , then we call (Ω, \mathcal{F}) a *measurable space*.

2.1.1.2 σ -field generated by a collection of subsets

Sometimes we are interested in a specific collection of subsets, \mathcal{C} , that is NOT a σ -field. But since all the machinery we will develop works with σ -fields, we are interested in creating a σ -field that contains \mathcal{C} . So we introduce the notion of a *σ -field generated by a collection of subsets*.

Definition 2.3 (Generated σ -field). The smallest σ -field containing a collection of subsets, \mathcal{C} , is called the σ -field generated by \mathcal{C} .

$\sigma(\mathcal{C})$ is used to denote the σ -field generated by \mathcal{C} , and is by definition the smallest σ -field that contains \mathcal{C} : if \mathcal{F} is a σ -field with $\mathcal{C} \subset \mathcal{F}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

2.1.1.3 Borel σ -field

A particular important σ -field is the Borel σ -field. In general, this is defined as the σ -field generated by the collection of all open subsets of a specific topology. In particular, if we consider \mathbb{R}^k is the k -dimensional

¹ \mathbb{P} is used to denote the collection of all subsets.

Euclidean space, $\mathcal{O} = \{O \subseteq \mathbb{R}^k \mid O \text{ open set}\}$, then $\sigma(\mathcal{O}) = \mathcal{B}^k$ (the Borel σ -field on \mathbb{R}^k).

It can be shown that the σ -field generated by the collection of all closed sets is also the Borel σ -field.

Sometimes it is useful to be able to limit ourselves to a subspace of \mathbb{R}^k . In such cases, we can create the Borel σ -field on that subspace in the following way: if $C \in \mathcal{B}^k$, then $\mathcal{B}_C = \{C \cap B \mid B \in \mathcal{B}^k\}$ is the Borel σ -field on C .

2.1.1.4 Measures

Definition 2.4 (Measure). Let (Ω, \mathcal{F}) is a measurable space

$\nu : \mathcal{F} \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be a *measure* if

- i) $0 \leq \nu(A) \leq \infty$ for all $A \in \mathcal{F}$
- ii) $\nu(\emptyset) = 0$
- iii) If $A_i \in \mathcal{F}$ for $i = 1, 2, \dots$, and $A_i \cap A_j = \emptyset, \forall i \neq j$ (i.e. A_i 's are pairwise disjoint), then it must hold that

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$

Example 2.2 (Counting Measure). The *counting measure* is simply the measure that returns the number of elements in a set.

Example 2.3 (Lebesgue Measure). The measure $m : \mathbb{R} \rightarrow \mathbb{R}$ satisfying that for all intervals $[a, b], a < b$,

$$m([a, b]) = b - a,$$

is called the *lebesgue measure*. This measure is unique.

Definition 2.5 (σ -finite measures). A measure ν is called *σ -finite* if and only if there exists a sequence $\{A_1, A_2, \dots\}$ such that $\cup A_i = \Omega$ and $\nu(A_i) < \infty, \forall i$.

Definition 2.6 (Measure Space). If ν is a measure on \mathcal{F} , and (Ω, \mathcal{F}) is a measurable space, then $(\Omega, \mathcal{F}, \nu)$ is a *measure space*.

Example 2.4. Both the Lebesgue measure is σ -finite.

The counting measure is σ -finite if and only if Ω is countable.

Definition 2.7 (Probability Space). If $(\Omega, \mathcal{F}, \nu)$ is a measurable space with $\nu(\Omega) = 1$, then it is called a *probability space*.

Proposition 2.1 (Properties of measures). Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. Then the following holds:

- i) **Monotonicity:** if $A \subseteq B$, then $\nu(A) \leq \nu(B)$
- ii) **Subadditivity:** for any sequence, A_1, A_2, \dots ,

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i)$$

- iii) **Continuity:** if $A_1 \subset A_2 \subset \dots$ (or $A_1 \supset A_2 \supset \dots$), then

$$\nu\left(\lim_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} \nu(A_n),$$

where

$$\lim_{n \rightarrow \infty} A_n = \bigcup_{i=1}^{\infty} A_i \left(\text{or} = \bigcap_{i=1}^{\infty} A_i \right)$$

Definition 2.8 (Cumulative Distribution Function). The *cumulative distribution function* (c.d.f.) of a measure ν is defined as

$$F(x) = \nu((-\infty, x]), x \in \mathbb{R}.$$

There is a one-to-one correspondence between probability measures on $(\mathbb{R}, \mathcal{B})$ and the set of c.d.f.'s.

Proposition 2.2 (Properties of c.d.f.'s). *i) For a c.d.f. F , on \mathbb{R} , it holds that: a) $F(-\infty) = 0$ b) $F(\infty) = 1$ c) $x \leq y \Rightarrow F(x) \leq F(y)$ (non-decreasing) d) $\lim_{y \rightarrow x^+} F(y) = F(x)$ (right continuous) ii) If a function $F: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the four conditions above, it is a c.d.f. of a unique probability measure on $(\mathbb{R}, \mathcal{B})$.*

Proposition 2.3 (The Product Measure Theorem). *If $(\Omega_i, \mathcal{F}_i, \nu_i), i = 1, \dots, k$ are measure spaces with σ -finite measures. Then there exists a unique measure on the σ -field $\sigma(\mathcal{F}_1 \times \dots \times \mathcal{F}_k)$:*

$$\nu_1 \times \dots \times \nu_k(A_1 \times \dots \times A_k) = \nu_1(A_1) \dots \nu_k(A_k)$$

for all $A_i \in \mathcal{F}_i, i = 1, \dots, k$.

This measure is called the *product measure*.

Definition 2.9 (Joint and Marginal c.d.f.'s). The *join c.d.f.* of a probability measure on $(\mathbb{R}^k, \mathcal{B}^k)$ is defined as

$$F(x_1, \dots, x_k) = P((-\infty, x_1] \times \dots \times (-\infty, x_k]), \quad x_i \in \mathbb{R}.$$

2.2 Lecture 2: 9/11

Definition 2.10 (Measurable Function). Let (Ω, \mathcal{F}) and (Λ, \mathcal{G}) be measurable spaces. Let $f: \Omega \rightarrow \Lambda$.

f is called a *measurable function* if and only if

$$f^{-1}(\mathcal{G}) \subset \mathcal{F} \text{ (i.e. } f^{-1}(G) \in \mathcal{F} \forall G \in \mathcal{G}).$$

Note that if \mathcal{F} is the collection of all subsets of Ω , then all functions are measurable.

Definition 2.11 (σ -field generated by a function). Let f be as in @ref{measurable-function}. Then $f^{-1}(\mathcal{G})$ is a sub- σ -field of \mathcal{F} . We call it the σ -field generated by f , and denote it by $\sigma(f)$.

Definition 2.12 (Borel Functions). A function from (Ω, \mathcal{F}) to $(\mathbb{R}, \mathcal{B})$ is called a *Borel function* if it is measurable.

Proposition 2.4 (Properties of Borel Functions). *i) A function is Borel if and only if $f^{-1}(a, \infty) \in \mathcal{F}$ for all $a \in \mathbb{R}$.*

ii) If f and g are Borel, then so are fg and $af + bg$, where $a, b \in \mathbb{R}$. Also, if $g(\omega) \neq 0$ for all $\omega \in \Omega$, then f/g is also Borel.

iii) If f_1, f_2, \dots are all Borel functions, then so are $\sup_n f_n, \inf_n f_n, \limsup_n f_n$, and $\liminf_n f_n$. Furthermore, the set

$$A = \left\{ \omega \in \Omega \mid \lim_{n \rightarrow \infty} f_n(\omega) \text{ exists} \right\}$$

is an event, and the function

$$h(\omega) = \begin{cases} \lim_{n \rightarrow \infty} f_n(\omega) & \omega \in A \\ f_1(\omega) & \omega \notin A \end{cases}$$

iv) Suppose that f is measurable from (Ω, \mathcal{F}) to (Λ, \mathcal{G}) and g is measurable from (Λ, \mathcal{G}) to (Δ, \mathcal{H}) . Then the composite function $g \circ f$ is measurable from (Ω, \mathcal{F}) to (Δ, \mathcal{H}) .

v) Let Ω be a Borel set in \mathbb{R}^p . If f is a continuous function from Ω to \mathbb{R}^q , then f is measurable.

Proposition 2.5. *For any non-negative Borel function f there exists a sequence of non-negative simple functions f_1, f_2, \dots such that*

$$f_n \rightarrow f \text{ for } n \rightarrow \infty$$

Definition 2.13 (Distribution). Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and f a measurable function from this measure space into the measurable space (Λ, \mathcal{G}) .

The measure defined as

$$\nu \circ f^{-1}(B) = \nu(f \in B) = \nu(f^{-1}(B)), \quad B \in \mathcal{G}$$

is called the *induced measure* by f .

If ν is a probability measure and f is a random variable (i.e. a Borel function), then $\nu \circ f^{-1}$ is called the *distribution* (or *law*) of f , and is denoted ν_f .

Notice that there are many notations for the same thing. If P is a probability measure and X a random variable, then

$$P_X(B) = P(X \in B) = P(X^{-1}(B)) = P \circ X^{-1}.$$

2.2.1 Integration

Definition 2.14 (Simple Function). A function ϕ is called a *simple function* if it is of the form

$$\phi = \sum_{i=1}^{\infty} a_i 1_{A_i},$$

where A_1, A_2, \dots are sets. If $a_i \geq 0$ for all $i \geq 1$, then ϕ is a non-negative simple function.

Definition 2.15 (Integral of a Non-negative Simple Function). The integral of a non-negative simple function ϕ with respect to a measure ν is defined as

$$\int \phi d\nu = \sum_{i=1}^k a_i \nu(A_i).$$

Definition 2.16 (Integral of Non-negative Borel Function). Let f be a non-negative Borel function. Let \mathcal{S}_f be the collection of ALL non-negative simple function with $\phi(\omega) \leq f(\omega), \forall \omega \in \Omega$.

The integral of f with respect to ν is defined as

$$\int f d\nu = \sup \left\{ \int \phi d\nu \mid \phi \in \mathcal{S}_f \right\}.$$

Note: one consequence of this is that for any non-negative Borel function, there exists a sequence of simple functions ϕ_1, ϕ_2, \dots such that $0 \leq \phi_i \leq f$ for all i and

$$\lim_{n \rightarrow \infty} \int \phi_n d\nu = \int f d\nu$$

Definition 2.17 (Integral of General Borel Function). Let f be a Borel function, and let $f_+(\omega) = \max\{f(\omega), 0\}$ (i.e. the positive part) and $f_-(\omega) = \max\{-f(\omega), 0\}$ (i.e. the negative part). If at least one of $\int f_+ d\nu$ and $\int f_- d\nu$ is finite, we say that $\int f d\nu$ exists and

$$\int f d\nu = \int f_+ d\nu - \int f_- d\nu.$$

Definition 2.18 (Integrable Functions). When $\int f d\nu < \infty$, i.e. the integral of both the positive and negative part of f is finite, we say that f is *integrable*.

Note: as a consequence of the definition of an integrable function we have that a Borel function is integrable if and only if $|f|$ is integrable. (This is true since $|f| = f_+ + f_-$.)

Notation: There are many different ways to write down an integral:

$$\int f d\nu = \int_{\Omega} f d\nu = \int f(\omega) d\nu = \int f(\omega) d\nu(\omega) = \int f(\omega) \nu(d\omega),$$

and if F is the c.d.f. (2.8) of a probability measure P on $(\mathbb{R}^k, \mathcal{B}^k)$,

$$\int f(x) dP = \int f(x) dF(x) = \int f dF$$

Proposition 2.6 (Linearity of Integrals). Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and f and g be Borel functions.

i) If $\int f d\nu$ exists, then for any $a \in \mathbb{R}$, $\int (af) d\nu$ exists, and

$$\int (af) d\nu = a \int f d\nu$$

ii) If $\int f d\nu$ and $\int g d\nu$ both are well defined, then $\int (f + g) d\nu$ exists and

$$\int (f + g) d\nu = \int f d\nu + \int g d\nu$$

Proof. Show that it holds for indicator functions, simple functions, non-negative functions, and then all functions. \square

Definition 2.19 (Almost Everywhere or Almost Surely). A statement is said to be true ν -a.e. (or ν -a.s.) if it is true for all $\omega \notin N$ and $\nu(N) = 0$.

Proposition 2.7 (a.e. for integrals). Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and f and g be Borel functions.

i) If $f \leq g$ ν -a.e., then $\int f d\nu \leq \int g d\nu$, given that both integrals exist

ii) If $f \geq 0$ ν -a.e. and $\int f d\nu = 0$, then $f = 0$ ν -a.e.

Proof. ii) Let $A = \{f > 0\}$ and $A_n = \{f \geq n^{-1}\}$, $n = 1, 2, \dots$. Then $A_n \subset A$ for any n and $\lim_{n \rightarrow \infty} A_n = \cup A_n = A$ (**show that this holds**).

Then, by (iii) of 2.1, $\lim_{n \rightarrow \infty} \nu(A_n) = \nu(A)$. By part (i) and proposition 2.6, we get that, for any n ,

$$n^{-1} \nu(A_n) = \int n^{-1} I_{A_n} d\nu \leq \int f I_{A_n} d\nu \leq \int f d\nu = 0.$$

\square

2.2.2 Radon-Nikodym Derivatives

2.3 Lecture 3: 9/13

Proof of Ex 1.11: for any borel set A: if $A = (-\infty, x)$, it holds. If not, then...

π - and λ -systems.

Definition: π -system

If \mathcal{C} is a collection of subsets, and it holds that $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$.

For a π system, $\sigma(\mathcal{C}) = \mathcal{B}$

Proposition 2.8 (Calculus with Radon-Nikodym Derivatives). *Let ν be a σ -finite measure on a measure space (Ω, \mathcal{F}) . All other measures discussed in the following are defined on (Ω, \mathcal{F}) .*

i) *If λ is a measure, $\lambda \ll \nu$, and $f \geq 0$, then*

$$\int f d\lambda = \int f \frac{d\lambda}{d\nu} d\nu.$$

ii) *If λ_i , $i = 1, 2$, are measures and $\lambda_i \ll \nu$, then $\lambda_1 + \lambda_2 \ll \nu$ and*

$$\frac{d(\lambda_1 + \lambda_2)}{d\nu} = \frac{d\lambda_1}{d\nu} + \frac{d\lambda_2}{d\nu} \quad \nu\text{-a.e.}$$

iii) *If τ is a measure, λ a σ -finite measure, and $\tau \ll \lambda \ll \nu$, then*

$$\frac{d\tau}{d\nu} = \frac{d\tau}{d\lambda} \frac{d\lambda}{d\nu} \quad \nu\text{-a.e.}$$

iv) *Let $(\Omega_i, \mathcal{F}_i, \nu_i)$ be a measure space and ν_i be σ -finite, $i = 1, 2$. Let λ_i be a σ -finite measure on $(\Omega_i, \mathcal{F}_i)$ and $\lambda \ll \nu_i$, $i = 1, 2$. Then $\lambda_1 \times \lambda_2 \ll \nu_1 \times \nu_2$ and*

$$\frac{d(\lambda_1 \times \lambda_2)}{d(\nu_1 \times \nu_2)}(\omega_1, \omega_2) = \frac{d\lambda_1}{d\nu_1}(\omega_1) \frac{d\lambda_2}{d\nu_2}(\omega_2) \quad \nu_1 \times \nu_2\text{-a.e.}$$

Chapter 3

Discussion Notes

3.1 Discussion 1: 5/14

3.1.1 σ -fields

Exercise 3.1 (Countable intersection/union of σ -fields). Let $\mathcal{F}_n, n = 1, 2, \dots$ be a sequence of σ -fields on Ω . Show the following:

- a) $\cap_{i=1}^{\infty} \mathcal{F}_n$ is a σ -field.
- b) If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$, then $\cup_{i=1}^{\infty} \mathcal{F}_n$ is not necessarily a σ -field.

Solution (ref(exr:ex11) a)). We need to show (i)-(iii) from definition 2.1.

- i) Since \mathcal{F}_n are all σ -fields, $\emptyset \in \mathcal{F}_n$ for all $n = 1, 2, \dots$. Hence, $\emptyset \in \cap_{i=1}^{\infty} \mathcal{F}_n$.
- ii) As (i).
- iii) Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets from $\cap_{i=1}^{\infty} \mathcal{F}_n$. Then, for all i , $A_i \in \mathcal{F}_n$ for all n . Since \mathcal{F}_n is a σ -field, $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_n$ for all n , and so $\cup_{i=1}^{\infty} A_i \in \cap_{i=1}^{\infty} \mathcal{F}_n$.

Hence, $\cap_{i=1}^{\infty} \mathcal{F}_n$ is a σ -field.

Solution (ref(exr:ex11) b)). Let $\Omega = [0, 1]$, and $\mathcal{F}_n = \sigma \left\{ [0, \frac{1}{2^n}), [\frac{1}{2^n}, \frac{2}{2^n}), \dots, [\frac{2^{n-1}}{2^n}, 1) \right\}$.

Now, consider the set $B_n = [0, \frac{1}{2^n})$. Clearly $B_n \in \mathcal{F}_n$ for all n , hence $B_n \in \cup_{i=1}^{\infty} \mathcal{F}_n$. However, $\cap_{i=1}^{\infty} B_n = \{0\} \notin \cup_{i=1}^{\infty} \mathcal{F}_n$. So $\cup_{i=1}^{\infty} \mathcal{F}_n$ is not closed under countable intersection, i.e. it is not a σ -algebra.

3.1.2 $\pi - \lambda$ systems

Definition 3.1 (π -system). Let \mathcal{D} be a collection of subsets of Ω . \mathcal{D} is said to be a π -**system** if it is closed under intersection, i.e. if

$$A, B \in \mathcal{D} \Rightarrow A \cap B \in \mathcal{D}.$$

Definition 3.2 (λ -system). Let \mathcal{L} be a collection of subsets of Ω . \mathcal{L} is said to be a λ -**system** if it satisfies that

- i) $\Omega \in \mathcal{L}$,
- ii) If $A, B \in \mathcal{L}$ with $A \subset B$, then $B \setminus A \in \mathcal{L}$
- iii) If $A_n \in \mathcal{L}$ and $A_n \subset A_{n+1}$ for all n , then

$$\cup_{i=1}^{\infty} A_n \in \mathcal{L}$$

Theorem 3.1 ($\pi - \lambda$ Theorem). *If \mathcal{D} is a π -system and \mathcal{L} is a λ -system s.t. $\mathcal{D} \subset \mathcal{L}$, then $\sigma\{(D)\} \subset \mathcal{L}$.*

Exercise 3.2 (Proof of the $\pi - \lambda$ Theorem).

Solution. Proof hints:

- 1) If \mathcal{L}_t is λ -system for all $t \in I$, $\mathcal{D} \subset \mathcal{L}_t$, then $\cap_{t \in I} \mathcal{L}_t$ is a λ -system. Denote this $\mathcal{L}(\mathcal{D})$ (smallest λ -system containing \mathcal{D}).
- 2) If \mathcal{L} is a π -system AND a λ -system, then \mathcal{L} is a σ -field.
- 3) If \mathcal{D} is π -system, then $\mathcal{L}(\mathcal{D})$ is π -system.

By 1)-3), $\mathcal{L} = \sigma(\mathcal{D})$, which implies ...

3.1.3 The “Good Sets” Principle

Exercise 3.3. Let \mathcal{P} be a π -system, and ν_1 and ν_2 two measures that agree on \mathcal{P} , i.e.

$$\nu_1(A) = \nu_2(A) \text{ for all } A \in \mathcal{P}.$$

Assume there is a sequence of sets $A_n \in \mathcal{P}$ with $A_n \uparrow \Omega$ and $\nu_i(A_n) < \infty$ for all n .

Use the $\pi - \lambda$ theorem to prove that ν_1 and ν_2 agree on $\sigma(\mathcal{P})$.

Solution. Let \mathcal{F}_n be given by

$$\mathcal{F}_n = \{A \in \sigma(\mathcal{P}) \mid \nu_1(A \cap A_n) = \nu_2(A \cap A_n) \forall n\}$$

Let $A \in \mathcal{P}$. Since $A_n \in \mathcal{P}$ for all n and \mathcal{P} is a π -system, $A \cap A_n \in \mathcal{P}$. So $\nu_1(A \cap A_n) = \nu_2(A \cap A_n)$, hence $\mathcal{P} \subset \mathcal{F}_n$.

By definition, $\mathcal{F}_n \subset \sigma(\mathcal{P})$, so $\mathcal{P} \subset \mathcal{F}_n \subset \sigma(\mathcal{P})$.

Now, if we can prove that \mathcal{F}_n is a λ -system for all n , then by the $\pi - \lambda$ theorem (theorem 3.1), we have that $\sigma(\mathcal{P}) \subset \mathcal{F}_n$, which combined with the paragraph above gives us that $\sigma(\mathcal{P}) = \mathcal{F}_n$, hence ν_1 and ν_2 agree on $\sigma(\mathcal{P})$.

So let us show that \mathcal{F}_n is indeed a λ -system:

- i) $\Omega \in \mathcal{F}_n$. Since $A_n \uparrow \Omega$, we can use continuity of measure (proposition 2.1) to conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu_i(A_n) &= \nu_i(\lim_{n \rightarrow \infty} A_n) \\ &= \nu_i(\Omega). \end{aligned}$$

Since $\nu_1(A_n) = \nu_2(A_n)$ ($A_n \in \mathcal{P}$), it holds that $\nu_1(\Omega) = \nu_2(\Omega)$, so $\Omega \in \mathcal{F}_n$.

- ii) Let $A, B \in \mathcal{F}_n$ with $A \subset B$. So,

$$\begin{aligned} \nu_1((A \setminus B) \cap A_n) &= \nu_1(A \cap A_n) - \nu_1(B \cap A_n) \\ &= \nu_2(A \cap A_n) - \nu_2(B \cap A_n) \\ &= \nu_2((A \setminus B) \cap A_n), \end{aligned}$$

which means that $A \setminus B \in \mathcal{F}_n$.

- iii) Let $B_i \in \mathcal{F}_n$ s.t. $B_i \subset B_{i+1}$. Then, once again using continuity of measures to move limits around, we have

$$\nu_1(\cup_{i=1}^{\infty} B_i \cap A_n) = \nu_1(\cup_{i=1}^{\infty} (B_i \cap A_n)) \quad (3.1)$$

$$= \lim_{i \rightarrow \infty} \nu_1(B_i \cap A_n) \quad (3.2)$$

$$= \lim_{i \rightarrow \infty} \nu_2(B_i \cap A_n) \quad (3.3)$$

$$= \nu_2(\cup_{i=1}^{\infty} (B_i \cap A_n)) \quad (3.4)$$

$$= \nu_2(\cup_{i=1}^{\infty} B_i \cap A_n), \quad (3.5)$$

which gives us that $\cup_{i=1}^{\infty} B_i \in \mathcal{F}_n$, hence \mathcal{F}_n is a λ -system.

3.1.4 From Indicator Function to General (Borel) Function

When we define the Lebesgue integral, we define it in three steps.

- 1) First for indicator functions, which in turn is generalized to simple non-negative functions (i.e. linear combinations of indicator functions).
- 2) Second for any non-negative functions (which is done by utilizing that any such function can be described as the limit of a sequence of simple functions)
- 3) A general function (by separating the positive and negative parts)

Exercise 3.4. Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be a countable set, \mathcal{F} all subsets of Ω , and ν the counting measure on Ω . Show that for any Borel function f , the integral of f with respect to ν is

$$\int f d\nu = \sum_{i=1}^{\infty} f(\omega_i) \quad (3.6)$$

Solution. Let $A \in \mathcal{F}$ and define $f = 1_A$. Then

$$\begin{aligned} \int f d\nu &= \int_A d\nu \\ &= \nu(A) \\ &= \sum_{i=1}^{\infty} 1_A(\omega_i) \end{aligned}$$

I.e. (3.6) holds for indicator functions, and hence also for simple functions.

Now, let f be a non-negative Borel function. Then we know that there exists a sequence $(f_n)_i^{\infty}$ of simple functions such that $f_n \uparrow f$. Then

$$\begin{aligned} \int f d\nu &= \int \lim_{n \rightarrow \infty} f_n d\nu \\ &= \lim_{n \rightarrow \infty} \int f_n d\nu. \end{aligned}$$

Since f_n is a simple function, we know that [@ref{eq:ex14}](#) holds. Hence,

$$\begin{aligned}
\int f d\nu &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} f_n(\omega_i) \\
&= \sum_{i=1}^{\infty} \lim_{n \rightarrow \infty} f_n(\omega_i) \\
&= \sum_{i=1}^{\infty} f(\omega_i),
\end{aligned}$$

and so (3.6) holds for non-negative Borel functions.

Finally, let f be any Borel function. Then we can write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Then both f_+ and f_- are non-negative Borel functions, hence (3.6) holds for both. So

$$\begin{aligned}
\int f d\nu &= \int f_+ d\nu - \int f_- d\nu \\
&= \sum_{i=1}^{\infty} f_+(\omega_i) - \sum_{i=1}^{\infty} f_-(\omega_i) \\
&= \sum_{i=1}^{\infty} f_+(\omega_i) - f_-(\omega_i) \\
&= \sum_{i=1}^{\infty} f(\omega_i).
\end{aligned}$$

So (3.6) holds for all Borel functions.

3.1.5 Switch the Order of Integration and Limit

Exercise 3.5 (Generalized Dominated Convergence Theorem). If $\lim_{n \rightarrow \infty} f_n = f$ and there exists a sequence of integrable functions g_1, g_2, g_3, \dots such that

- $|f_n| \leq g_n$ a.e.
- $g_n \rightarrow g$ a.e.
- $\lim_{n \rightarrow \infty} \int g_n d\nu = \int g d\nu$

then

$$\int \lim_{n \rightarrow \infty} f_n d\nu = \lim_{n \rightarrow \infty} \int f_n d\nu \tag{3.7}$$

Chapter 4

Homework

4.1 Chapter 1

4.1.1 First Exam Period

Exercise 4.1 (Ex 2). Let \mathcal{C} be a collection of subsets of Ω and let $\Gamma = \{\mathcal{F} | \mathcal{F} \text{ is a } \sigma\text{-field on } \Omega \text{ and } \mathcal{C} \subset \mathcal{F}\}$. Show that $\Gamma \neq \emptyset$ and $\sigma(\mathcal{C}) = \cap_{\mathcal{F} \in \Gamma} \mathcal{F}$.

Solution (Ex 2). Let $\mathbb{P}(\Omega)$ be the collection of all subsets of Ω . We know that this is a σ -field. It also contains \mathcal{C} . Hence, $\Gamma \neq \emptyset$.

By definition, $\sigma(\mathcal{C})$ is the smallest σ -field that contains \mathcal{C} , hence $\sigma(\mathcal{C}) \in \Gamma$ and $\sigma(\mathcal{C}) \subset \mathcal{F}$ for all $\mathcal{F} \in \Gamma$. Therefore, $\sigma(\mathcal{C}) \subset \cap_{\mathcal{F} \in \Gamma} \mathcal{F}$. But since $\sigma(\mathcal{C}) \in \Gamma$, $\sigma(\mathcal{C}) \in \cap_{\mathcal{F} \in \Gamma} \mathcal{F}$, which in turn ensures that $\cap_{\mathcal{F} \in \Gamma} \mathcal{F} \subset \sigma(\mathcal{C})$.

Hence $\sigma(\mathcal{C}) = \cap_{\mathcal{F} \in \Gamma} \mathcal{F}$.

Exercise 4.2 (Ex 3). Let Ω, \mathcal{F}_j , $j = 1, 2, \dots$, be measurable spaces such that $\mathcal{F}_j \subset \mathcal{F}_{j+1}$. Is $\cup_j \mathcal{F}_j$ a σ -field?

Solution (Ex 3). No.

Let $\Omega = [0, 1]$, and $\mathcal{F}_n = \sigma\left\{[0, \frac{1}{2^n}), [\frac{1}{2^n}, \frac{2}{2^n}), \dots, [\frac{2^{n-1}}{2^n}, 1)\right\}$.

Now, consider the set $B_n = [0, \frac{1}{2^n})$. Clearly $B_n \in \mathcal{F}_n$ for all n , hence $B_n \in \cup_{i=1}^{\infty} \mathcal{F}_i$. However, $\cap_{i=1}^{\infty} B_i = \{0\} \notin \cup_{i=1}^{\infty} \mathcal{F}_i$. So $\cup_{i=1}^{\infty} \mathcal{F}_i$ is not closed under countable intersection, i.e. it is not a σ -algebra.

Exercise 4.3 (Ex 5). a) Let \mathcal{C} be a π -system and \mathcal{D} be a λ -system such that $\mathcal{C} \subset \mathcal{D}$. Show that $\sigma(\mathcal{C}) \subset \mathcal{D}$.

Solution (Ex 6).

Exercise 4.4 (Ex 6). Prove part (ii) and (iii) of proposition 2.1.

Solution (Ex 6). i) Let $A \subset B$. Then $B \setminus A \cap A = \emptyset$, hence $\nu(B) = \nu((B \setminus A) \cup A) = \nu(B \setminus A) + \nu(A) \geq \nu(A)$.

ii) Let A_1, A_2, \dots be a sequence of sets. Define $B_i = A_i \setminus (\cup_{k=1}^{i-1} A_k)$. Then the B_i s are pairwise disjoint. Hence,

$$\begin{aligned} \nu(\cup_{i=1}^{\infty} A_i) &= \nu(\cup_{i=1}^{\infty} B_i) \\ &= \sum_{i=1}^{\infty} \nu(B_i) \\ &= \sum_{i=1}^{\infty} \nu(A_i \setminus (\cup_{k=1}^{i-1} A_k)). \end{aligned}$$

Since $A_i \setminus (\cup_{k=1}^{i-1} A_k) \subset A_i$, we use (i) to get the result:

$$\sum_{i=1}^{\infty} \nu(A_i \setminus (\cup_{k=1}^{i-1} A_k)) \leq \sum_{i=1}^{\infty} \nu(A_i)$$

iii) Let $A_1 \subset A_2 \subset A_3 \subset \dots$. Define $B_i = A_i \setminus A_{i-1}$. Then B_1, B_2, \dots is a sequence of pairwise disjoint sets, and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$. Hence,

$$\begin{aligned} \nu(\cup_{i=1}^k A_i) &= \nu(\cup_{i=1}^k B_i) \\ &= \sum_{i=1}^k \nu(B_i) \\ &= \sum_{i=1}^k \nu(A_i \setminus A_{i-1}) \\ &= \sum_{i=1}^k \nu(A_i) - \nu(A_{i-1}) \\ &= \nu(A_k). \end{aligned}$$

Taking the limit on both sides gives us

$$\nu(\cup_{i=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} \nu(A_n).$$

Exercise 4.5 (Ex 12). Let ν and λ be two measures on (Ω, \mathcal{F}) such that $\nu(A) = \lambda(A)$ for any $A \in \mathcal{C} \subset \mathcal{F}$, where \mathcal{C} is a π -system (3.1). Assume that ν is σ -finite (2.5).

Show that $\nu(A) = \lambda(A)$ for all $A \in \sigma(\mathcal{C})$.

Solution (Ex 12). Let $\mathcal{F} = \{A \in \sigma(\mathcal{C}) | \nu(A) = \lambda(A)\}$. Then $\mathcal{C} \subset \mathcal{F}$. If we can show that \mathcal{F} is a σ -field, then $\sigma(\mathcal{C}) \subset \mathcal{F}$ (since $\sigma(\mathcal{C})$ is the smallest σ -field that contains \mathcal{C}), which proves that $\nu(A) = \lambda(A)$ for all $A \in \sigma(\mathcal{C})$.

Exercise 4.6 (Ex 14). Prove proposition 1.4

Exercise 4.7 (Ex 15). Show that a monotone function from \mathbb{R} to \mathbb{R} is Borel, and a c.d.f. on \mathbb{R}^k is Borel.

Exercise 4.8 (Ex 17). Let f be a non-negative Borel function on (Ω, \mathcal{F}) . Show that f is the limit of a sequence of simple functions $\{\phi_n\}$ on (Ω, \mathcal{F}) with $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f$.

Exercise 4.9 (Ex 19). Let $\{f_n\}$ be a sequence of Borel functions on a measurable space. Show that

$$\text{a) } \sigma(f_1, f_2, \dots) = \sigma(\cup_{j=1}^{\infty} \sigma(f_j)) = \sigma(\cup_{j=1}^{\infty} \sigma(f_1, \dots, f_j)).$$

$$\text{b) } \sigma(\limsup_n f_n) \subset \cap_{n=1}^{\infty} \sigma(f_n, f_{n+1}, \dots).$$

Exercise 4.10 (Ex 24). Let f be an integrable function on $(\Omega, \mathcal{F}, \nu)$. Show that for each $\epsilon > 0$, there exists a δ_ϵ such that for $A \in \mathcal{F}$:

$$\nu(A) < \delta_\epsilon \Rightarrow \int_A |f| d\nu < \epsilon.$$

Solution. Let $\epsilon > 0$, $A \in \mathcal{F}$ with $\nu(A) < \delta_\epsilon = \frac{\epsilon}{\max_{\omega \in A} |f(\omega)|}$. Then

$$\begin{aligned} \int_A |f| d\nu &\leq \int_A \max_{\omega \in A} |f(\omega)| d\nu \\ &= \max_{\omega \in A} |f(\omega)| \nu(A) \\ &< \epsilon. \end{aligned}$$

Exercise 4.11 (Ex 30). For any c.d.f. F and any $a \geq 0$, show that $\int [F(x+a) - F(x)]dx = a$
Solution.

Exercise 4.12 (Ex 34). Prove proposition 2.8}

Solution (Ex 34 i)). Let g be the unique function denoted by $\frac{d\lambda}{d\nu}$. Assume $f = 1_A$ for some $A \in \mathcal{F}$. Since $\lambda \ll \nu$, we know that $\lambda(A) = \int_A g d\nu$. So,

$$\begin{aligned} \int f d\lambda &= \int 1_A d\lambda \\ &= \lambda(A) \\ &= \int_A g d\nu \\ &= \int 1_A g d\nu = \int f g d\nu. \end{aligned}$$

Hence, (i) is true for all indicator functions, and so by linearity of integrals (2.6) for all non-negative simple functions.

Now, let f be a general non-negative Borel function. Then we know that there exists a sequence of simple functions ϕ_1, ϕ_2, \dots such that $\phi_n \uparrow f$. Hence, utilizing the monotone convergence theorem and the fact that we know (i) holds for simple functions,

$$\begin{aligned} \int f d\lambda &= \int \lim_{n \rightarrow \infty} \phi_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int \phi_n g d\nu \\ &= \int \lim_{n \rightarrow \infty} \phi_n g d\nu \\ &= \int f g d\nu, \end{aligned}$$

and so we have shown that (i) holds for any non-negative Borel function.

Solution (Ex 34 ii)). Assume $\lambda_1 \ll \nu$ and $\lambda_2 \ll \nu$. Then

$$\begin{aligned} (\lambda_1 + \lambda_2)(A) &= \lambda_1(A) + \lambda_2(A) \\ &= \int_A g_1 d\nu + \int_A g_2 d\nu \\ &= \int_A (g_1 + g_2) d\nu, \end{aligned}$$

so $\lambda_1 + \lambda_2 \ll \nu$, and

$$\frac{d(\lambda_1 + \lambda_2)}{d\nu} = g_1 + g_2 = \frac{d\lambda_1}{d\nu} + \frac{d\lambda_2}{d\nu}.$$