STAT 709: My notes

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Contents

Chapter 1

Intro

1.1 Textbook

As of Fall 2018, this class uses the book *Mathematical Statistics* by Jun Shao (2nd. edition). Unless otherwise noted, all definitions, lemmas, proposition, theorems, etc. can be found there. Numbering might not match.

1.2 Conventions re: ∞

We will use the following conventions:

- $\infty + x = \infty, x \in \mathbb{R}$
- $x \cdot \infty = \infty$ if $x > 0$
- $x \cdot \infty = -\infty$ if $x < 0$
- $0 \cdot \infty = 0$
- $\infty + \infty = \infty$
- $\infty^a = \infty, \forall a > 0$
- $\infty - \infty$ and $\frac{\infty}{\infty}$ are not defined

Chapter 2

Lecture Notes

2.1 Chapter 1: Probability Theory

2.1.1 Lecture 1: Measure space, measurable function, and integration

2.1.1.1 σ-fields

Definition 2.1 (σ-field (or σ-algebra))**.** A $\mathcal{F}$ collection of subsets of Ω is called a σ-field (or σ-algebra) if the following three conditions hold:

i) $\emptyset \in \mathcal{F}$
ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
iii) If $A_i \in \mathcal{F}$ for all $i = 1, 2, \ldots$, then $\bigcup_i A_i \in \mathcal{F}$.

Example 2.1 (A Few σ-fields)**.** There are some trivial examples. One is the example where $\mathcal{F} = \{\emptyset, \Omega\}$. It is easy to check that the three conditions are met for this collection of subsets. Another trivial example would be $\mathcal{F} = \mathbb{P}(\Omega)$[1].

The simplest non-trivial example is $\mathcal{F} = \left\{\emptyset, A, A^C, \Omega\right\}$ where $A \subset \Omega$. Since this collection of subsets is so small, it is easy to check the three conditions mentioned above.

Definition 2.2 (Measurable Space)**.** If $\mathcal{F}$ is a σ-field on Ω, then we call $(\Omega, \mathcal{F})$ a ***measurable space***.

2.1.1.2 σ-field generated by a collection of subsets

Sometimes we are interested in a specific collection of subsets, $\mathcal{C}$, that is NOT a σ-field. But since all the machinery we will develop works with σ-fields, we are interested in creating a σ-field that contains $\mathcal{C}$. So we introduce the notion of a *σ-field generated by a collection of subsets*.

Definition 2.3 (Generated σ-field)**.** The smallest σ-field containing a collection of subsets, $\mathcal{C}$, is called the σ-field generated by $\mathcal{C}$.

$\sigma(\mathcal{C})$ is used to denote the σ-field generated by $\mathcal{C}$, and is by definition the smallest σ-field that contains $\mathcal{C}$: if $\mathcal{F}$ is a σ-field with $\mathcal{C} \subset \mathcal{F}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

2.1.1.3 Borel σ-field

A particular important σ-field is the Borel σ-field. In general, this is defined as the σ-field generated by the collection of all open subsets of a specific topology. In particular, if we consider $\mathbb{R}^k$ is the k-dimensional

[1]P is used to denote the collection of all subsets.

Euclidean space, $\mathcal{O} = \left\{O \subseteq \mathbb{R}^k \,|O \text{ open set}\right\}$, then $\sigma(\mathcal{O}) = \mathcal{B}^k$ (the Borel σ-field on $\mathbb{R}^k$).

It can be shown that the σ-field generated by the collection of all closed sets is also the Borel σ-field.

Sometimes it is useful to be able to limit ourselves to a subspace of $\mathbb{R}^k$. In such cases, we can create the Borel σ-field on that subspace in the following way: if $C \in \mathcal{B}^k$, then $\mathcal{B}_C = \left\{C \cap B \,\middle|\, B \in \mathcal{B}^k\right\}$ is the Borel σ-field on C.

2.1.1.4 Measures

Definition 2.4 (Measure)**.** Let $(\Omega, \mathcal{F})$ is a measurable space

$\nu : \mathcal{F} \to \mathbb{R} \cup \{\infty\}$ is said to be a ***measure*** if

i) $0 \leq \nu(A) \leq \infty$ for all $A \in \mathcal{F}$
ii) $\nu(\emptyset) = 0$
iii) If $A_i \in \mathcal{F}$ for $i = 1, 2, ...,$ and $A_i \cap A_j = \emptyset, \forall i \neq j$ (i.e. A_i's are pairwise disjoint), then it must hold that

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$$

Example 2.2 (Counting Measure)**.** The *counting measure* is simply the measure that returns the number of elements in a set.

Example 2.3 (Lebesgue Measure)**.** The measure $m : \mathbb{R} \to \mathbb{R}$ satisfying that for all intervals $[a, b], a < b$,

$$m([a, b]) = b - a,$$

is called the *lebesgue measure*. This measure is unique.

Definition 2.5 (σ-finite measures)**.** A measure ν is called *σ-finite* if and only if there exists a sequence $\{A_1, A_2, \ldots\}$ such that $\cup A_i = \Omega$ and $\nu(A_i) < \infty, \forall i$.

Definition 2.6 (Measure Space)**.** If ν is a measure on $\mathcal{F}$, and $(\Omega, \mathcal{F})$ is a measurable space, then $(\Omega, \mathcal{F}, \nu)$ is a ***measure space***.

Example 2.4. Both the Lebesque measure is σ-finite.

The counting measure is σ-finite if and only if Ω is countable.

Definition 2.7 (Probability Space)**.** If $(\Omega, \mathcal{F}, \nu)$ is a measurable space with $\nu(\Omega) = 1$, then it is called a ***probability space***.

Proposition 2.1 (Properties of measures)**.** *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. Then the following holds:*

i) ***Monotonicity:*** *if $A \subseteq B$, then $\nu(A) \leq \nu(B)$*
ii) ***Subadditivity:*** *for any sequence, $A_1, A_2, \ldots,$*

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i)$$

iii) ***Continuity:*** *if $A_1 \subset A_2 \subset \ldots$ (or $A_1 \supset A_2 \supset \ldots$), then*

$$\nu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n),$$

where

$$\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i \left(or = \bigcap_{i=1}^{\infty} A_i\right)$$

Definition 2.8 (Cumulative Distribution Function)**.** The *cumulative distribution function* (c.d.f.) of a measure ν is defined as

$$F(x) = \nu((-\infty, x]), x \in \mathbb{R}.$$

There is a one-to-one correpsondence between probability measures on $(\mathbb{R}, \mathcal{B})$ and the set of c.d.f.'s.

Proposition 2.2 (Properties of c.d.f.'s)**.** *i) For a c.d.f,* $\mathcal{F}$*, on* $\mathbb{R}$*, it holds that: a)* $F(-\infty) = 0$ *b)* $F(\infty) = 1$ *c)* $x \leq y \Rightarrow F(x) \leq F(y)$ *(non-decreasing) d)* $\lim_{y \to x^+} F(y) = F(x)$ *(right continuous) ii) If a function* $F : \mathbb{R} \to \mathbb{R}$ *satisfies the four conditions above, it is a c.d.f. of a unique probability measure on* $(\mathbb{R}, \mathcal{B})$*.*

Proposition 2.3 (The Product Measure Theorem)**.** *If* $(\Omega_i, \mathcal{F}_i, \nu_i), i = 1, \ldots, k$ *are measure spaces with* σ*-finite measures. Then there exists a unique measure on the* σ*-field* $\sigma(\mathcal{F}_1 \times \ldots \mathcal{F}_k)$*:*

$$\nu_1 \times \cdots \times \nu_k(A_1 \times \cdots \times A_k) = \nu_1(A_1) \ldots \nu_k(A_k)$$

for all $A_i \in \mathcal{F}_i, i = 1, \ldots, k$*.*

This measure is called the product measure.

Definition 2.9 (Joint and Marginal c.d.f.'s)**.** The *join c.d.f.* of a probability measure on $(\mathbb{R}^k, \mathcal{B}^k)$ is defined as

$$F(x_1, \ldots, x_k) = P((-\infty, x_1] \times \cdots \times (-\infty, x_k]), \quad x_i \in \mathbb{R}.$$

2.2 Lecture 2: 9/11

Definition 2.10 (Measurable Function)**.** Let $(\Omega, \mathcal{F})$ and $(\Lambda, \mathcal{G})$ be measurable spaces. Let $f : \Omega \to \Lambda$.

f is called a *measurable function* if and only if

$$f^{-1}(\mathcal{G}) \subset \mathcal{F}(\text{i.e.} f^{-1}(G) \in \mathcal{F} \forall G \in \mathcal{G}).$$

Note that if $\mathcal{F}$ is the collection of all subsets of Ω, then all functions are measurable.

Definition 2.11 (σ-field generated by a function)**.** Let f be as in @ref{measurable-function}. Then $f^{-1}(\mathcal{G})$ is a sub-σ-field of $\mathcal{F}$. We call it the σ-field generated by f, and denote it by $\sigma(f)$.

Definition 2.12 (Borel Functions)**.** A function from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$ is called a *Borel function$ if it is measurable.

Proposition 2.4 (Properties of Borel Functions)**.** *Let* $(\Omega, \mathcal{F})$ *be a measurable space.*

- *i) A function is Borel if and only if* $f^{-1}(a, \infty) \in \mathcal{F}$ *for all* $a \in \mathbb{R}$*.*
- *ii) If* f *and* g *are Borel, then so are* fg *and* $af + bg$*, where* $a, b \in \mathbb{R}$*. Also, if* $g(\omega) \neq 0$ *for all* $\omega \in \Omega$*, then* f/g *is also Borel.*
- *iii) If* $f_1, f_2, \ldots$ *are all Borel functions, then so are* $\sup_n f_n$*,* $\inf_n f_n$*,* $\limsup_n f_n$*, and* $\liminf_n f_n$*. Furthermore, the set*

$$A = \left\{\omega \in \Omega | \lim_{n \to \infty} f_n(\omega) \text{ exists}\right\}$$

is an event, and the function

$$h(\omega) = \begin{cases} \lim_{n \to \infty} f_n \omega & \omega \in A \\ f_1(\omega) & \omega \notin A \end{cases}$$

is borel.

- *iv) Suppose that* f *is measurable from* $(\Omega, \mathcal{F})$ *to* $(\Lambda, \mathcal{G})$ *and* g *is measurable from* $(\Lambda, \mathcal{G})$ *to* $(\Delta, \mathcal{H})$*. Then the composite function* $g \circ f$ *is measurable from* $(\Omega, \mathcal{F})$ *to* $(\Delta, \mathcal{H})$*.*
- *v) Let* Ω *be a Borel set in* $\mathbb{R}^p$*. If* f *is a continuous function from* Ω *to* $\mathbb{R}^q$*, then* f *is measurable.*

Proposition 2.5. *For any non-negative Borel function f there exists a sequence of non-negative simple functions $f_1, f_2, \dots$ such that*

$$f_n \to f \text{ for } n \to \infty$$

Definition 2.13 (Distribution)**.** Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and f a measurable function from this measure space into the measurable space $(\Lambda, \mathcal{G})$.

The measure defined as

$$\nu \circ f^{-1}(B) = \nu(f \in B) = \nu(f^{-1}(B)), \quad B \in \mathcal{G}$$

is called the *induced measure* by f.

If ν is a probability measure and f is a random variable (i.e. a Borel function), then $\nu \circ f^{-1}$ is called the *distribution* (or *law*) of f, and is denoted ν_f.

Notice that there are many notations for the same thing. If P is a probability measure and X a random variable, then

$$P_X(B) = P(X \in B) = P(X^{-1}(B)) = P \circ X^{-1}.$$

2.2.1 Integration

Definition 2.14 (Simple Function)**.** A function ϕ is called a *simple function* if it is of the form

$$\phi = \sum_{i=1}^{\infty} a_i 1_{A_i},$$

where $A_1, A_2, \dots$ are sets. If $a_i \geq 0$ for all $i \geq 1$, then ϕ is a non-negative simple function.
Definition 2.15 (Integral of a Non-negative Simple Function)**.** The integral of a non-negative simple function ϕ with respect to a measure ν is defined as

$$\int \phi d\nu = \sum_{i=1}^{k} a_i \nu(A_i).$$

Definition 2.16 (Integral of Non-negative Borel Function)**.** Let f be a non-negative Borel function. Let $\mathcal{S}_f$ be the collection of ALL non-negative simple function with $\phi(\omega) \leq f(\omega), \forall \omega \in \Omega$.

The integral of f with respect to ν is defined as

$$\int f d\nu = \sup \left\{ \int \phi d\nu \, | \phi \in \mathcal{S}_f \right\}.$$

Note: one consequence of this is that for any non-negative Borel function, there exists a sequence of simple functions $\phi_1, \phi_2, \dots$ such that $0 \leq \phi_i \leq f$ for all i and

$$\lim_{n\to\infty} \int \phi_n d\nu = \int f d\nu$$

Definition 2.17 (Integral of General Borel Function)**.** Let f be a Borel function, and let $f_+(\omega) = \max\{f(\omega), 0\}$ (i.e. the positive part) and $f_-(\omega) = \max\{-f(\omega), 0\}$ (i.e. the negative part). If at least one of $\int f_+ d\nu$ and $\int f_- d\nu$ is finite, we say that $\int f d\nu$ exists and

$$\int f d\nu = \int f_+ d\nu - \int f_- d\nu.$$

Definition 2.18 (Integrable Functions)**.** When $\int f d\nu < \infty$, i.e. the integral of both the positive and negative part of f is finite, we say that f is *integrable*.

Note: as a consequence of the definition of an integrable function we have that a Borel function is integrable if and only if $|f|$ is integrable. (This is true since $|f| = f_+ + f_-$.)

Notation: There are many different ways to write down an integral:

$$\int f d\nu = \int_\Omega f d\nu = \int f(\omega) d\nu = \int f(\omega) d\nu(\omega) = \int f(\omega)\nu(d\omega),$$

and if F is the c.d.f. (2.8) of a probability measure P on $(\mathbb{R}^k, \mathcal{B}^k)$,

$$\int f(x) dP = \int f(x) dF(x) = \int f dF$$

Proposition 2.6 (Linearity of Integrals)**.** *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and f and g be Borel functions.*

i) If $\int f d\nu$ exists, then for any $a \in \mathbb{R}$, $\int (af) d\nu$ exists, and

$$\int (af) d\nu = a \int f d\nu$$

ii) If $\int f d\nu$ and $\int g d\nu$ both are well defined, then $\int (f+g) d\nu$ exists and

$$\int (f+g) d\nu = \int f d\nu + \int g d\nu$$

Proof. Show that it holds for indicator functions, simple functions, non-negative functions, and then all functions. □

Definition 2.19 (Almost Everywhere or Almost Surely)**.** A statement is said to be true ν-a.e. (or ν-a.s.) if it is true for all $\omega \notin N$ and $\nu(N) = 0$.
Proposition 2.7 (a.e. for integrals)**.** *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, and f and g be Borel functions.*

i) If $f \le g$ ν-a.e., then $\int f d\nu \le \int g d\nu$, given that both integrals exist
ii) If $f \ge 0$ ν-a.e. and $\int f d\nu = 0$, then $f = 0$ ν-a.e.

Proof. ii) Let $A = \{f > 0\}$ and $A_n = \{f \ge n^{-1}\}$, $n = 1, 2, \ldots$. Then $A_n \subset A$ for any n and $\lim_{n\to\infty} A_n = \cup A_n = A$ (**show that this holds**).

Then, by (iii) of 2.1, $\lim_{n\to\infty} \nu(A_n) = \nu(A)$. By part (i) and proposition 2.6, we get that, for any n,

$$n^{-1}\nu(A_n) = \int n^{-1} I_{A_n} d\nu \le \int f I_{A_n} d\nu \le \int f d\nu = 0.$$

□

2.2.2 Radon-Nikodym Derivatives

2.3 Lecture 3: 9/13

Proof of Ex 1.11: for any borel set A: if A = (-infty, x), it holds. If not, then...

π- and λ-systems.

Definition: π-system

If $\mathcal{C}$ is a collection of subsets, and it holds that $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$.

For a pi system, $\sigma(\mathcal{C}) = \mathcal{B}$

Proposition 2.8 (Calculus with Radon-Nikodym Derivatives)**.** *Let ν be a σ-finite measure on a measure space $(\Omega, \mathcal{F})$. All other measures discussed in the following are defined on $(\Omega, \mathcal{F})$.*

i) If λ is a meaure, $\lambda << \nu$, and $f \geq 0$, then

$$\int f d\lambda = \int f \frac{d\lambda}{d\nu} d\nu.$$

ii) If λ_i, $i = 1, 2$, are measures and $\lambda_i << \nu$, then $\lambda_1 + \lambda_2 << \nu$ and

$$\frac{d(\lambda_1 + \lambda_2)}{d\nu} = \frac{d\lambda_1}{d\nu} + \frac{d\lambda_2}{d\nu} \quad \nu\text{-}a.e.$$

iii) If τ is a measure, λ a σ-finite measure, and $\tau << \lambda << \nu$, then

$$\frac{d\tau}{d\nu} = \frac{d\tau}{d\lambda} \frac{d\lambda}{d\nu} \quad \nu\text{-}a.e..$$

iv) Let $(\Omega_i, \mathcal{F}_i, \nu_i)$ be a measure space and ν_i be σ-finite, $i = 1, 2$. Let λ_i be a σ-finite measure on $(\Omega_i, \mathcal{F}_i)$ and $\lambda_1 << \nu_i$, $i = 1, 2$. Then $\lambda_1 \times \lambda_2 << \nu_1 \times \nu_2$ and

$$\frac{d(\lambda_1 \times \lambda_2)}{d(\nu_1 \times \nu_2)}(\omega_1, \omega_2) = \frac{d\lambda_1}{d\nu_1}(\omega_1) \frac{d\lambda_2}{d\nu_2}(\omega_2) \quad \nu_1 \times \nu_2\text{-}a.e.$$

.

Chapter 3

Discussion Notes

3.1 Discussion 1: 5/14

3.1.1 σ-fields

Exercise 3.1 (Countable intersection/union of σ-fields)**.** Let $\mathcal{F}_n, n = 1, 2, \ldots$ be a sequance of σ-fields on Ω. Show the following:

a) $\cap_{i=1}^{\infty} \mathcal{F}_n$ is a σ-field.
b) If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$, then $\cup_{i=1}^{\infty} \mathcal{F}_n$ is not necessarily a σ-field.

Solution (ref(exr:ex11) a)). We need to show (i)-(iii) from definition 2.1.

i) Since $\mathcal{F}_n$ are all σ-fields, $\emptyset \in \mathcal{F}_n$ for all $n = 1, 2, \ldots$. Hence, $\emptyset \in \cap_{i=1}^{\infty} \mathcal{F}_n$.
ii) As (i).
iii) Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets from $\cap_{i=1}^{\infty} \mathcal{F}_n$. Then, for all i, $A_i \in \mathcal{F}_n$ for all n. Since F_n is a σ-field, $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_n$ for all n, and so $\cup_{i=1}^{\infty} A_i \in \cap_{i=1}^{\infty} \mathcal{F}_n$.

Hence, $\cap_{i=1}^{\infty} \mathcal{F}_n$ is a σ-field.

Solution (ref(exr:ex11) b)). Let $\Omega = [0, 1]$, and $\mathcal{F}_n = \sigma \left\{ [0, \frac{1}{2^n}), [\frac{1}{2^n}, \frac{2}{2^n}), \ldots, [\frac{2^{n-1}}{2^n}, 1) \right\}$.

Now, consider the set $B_n = [0, \frac{1}{2^n})$. Clearly $B_n \in \mathcal{F}_n$ for all n, hence $B_n \in \cup_{i=1}^{\infty} \mathcal{F}_n$. Hower, $\cap_{i=1}^{\infty} B_n = \{0\} \notin \cup_{i=1}^{\infty} \mathcal{F}_n$. So $\cup_{i=1}^{\infty} \mathcal{F}_n$ is not closed under countable intersection, i.e. it is not a σ-algebra.

3.1.2 $\pi - \lambda$ systems

Definition 3.1 (π-system)**.** Let $\mathcal{D}$ be a collection of subsets of Ω. $\mathcal{D}$ is said to be a ***π-system*** if it is closed under intersection, i.e. if

$$A, B \in \mathcal{D} \Rightarrow A \cap B \in \mathcal{D}.$$

Definition 3.2 (λ-system)**.** Let $\mathcal{L}$ be a collection of subsets of Ω. $\mathcal{L}$ is said to be a ***λ-system*** if it satisfies that

i) $\Omega \in \mathcal{L}$,
ii) If $A, B \in \Omega$ with $A \subset B$, then $B \setminus A \in \Omega$
iii) If $A_n \in \mathcal{L}$ and $A_n \subset A_{n+1}$ for all n, then

$$\cup_{i=1}^{\infty} A_n \in \mathcal{L}$$

Theorem 3.1 ($\pi - \lambda$ Theorem)**.** *If $\mathcal{D}$ is a π-system and $\mathcal{L}$ is a λ-system s.t. $\mathcal{D} \subset \mathcal{L}$, the $\sigma\{(D)\} \subset \mathcal{L}$.*
Exercise 3.2 (Proof of the $\pi - \lambda$ Theorem)**.**
Solution. Proof hints:

1) If $\mathcal{L}_t$ is λ-system for all $t \in I$, $\mathcal{D} \subset \mathcal{L}_t$, then $\cap_{t\in I}\mathcal{L}_t$ is a λ-system. Denote this $\mathcal{L}(\mathcal{D})$ (smallest λ-system containing $\mathcal{D}$).
2) If $\mathcal{L}$ is a π-system AND a λ-system, then $\mathcal{L}$ is a σ-field.
3) If $\mathcal{D}$ is π-system, then $\mathcal{L}(\mathcal{D})$ is π-system.

By 1)-3), $\mathcal{L} = \sigma(\mathcal{D})$, which implies ...

3.1.3 The "Good Sets" Principle

Exercise 3.3. Let $\mathcal{P}$ be a π-system, and ν_1 and ν_2 two measures that agree on $\mathcal{P}$, i.e.

$$\nu_1(A) = \nu_2(A) \text{ for all } A \in \mathcal{P}.$$

Assume there is a sequence of sets $A_n \in \mathcal{P}$ with $A_n \uparrow \Omega$ and $\nu_i(A_n) < \infty$ for all n.

Use the $\pi - \lambda$ theorem to prove that ν_1 and ν_2 agree on $\sigma(\mathcal{P})$.
Solution. Let $\mathcal{F}_n$ be given by

$$\mathcal{F}_n = \{A \in \sigma(\mathcal{P})\,|\nu_1(A \cap A_n) = \nu_2(A \cap A_n)\forall n\}$$

Let $A \in \mathcal{P}$. Since $A_n \in \mathcal{P}$ for all n and $\mathcal{P}$ is a π-system, $A \cap A_n \in \mathcal{P}$. So $\nu_1(A \cap A_n) = \nu_2(A \cap A_n)$, hence $\mathcal{P} \subset \mathcal{F}_n$.

By definition, $\mathcal{F}_n \subset \sigma(\mathcal{P})$, so $\mathcal{P} \subset \mathcal{F}_n \subset \sigma(\mathcal{P})$.

Now, if we can prove that $\mathcal{F}_n$ is a λ-system for all n, then by the $\pi - \lambda$ theorem (theorem 3.1), we have that $\sigma(\mathcal{P}) \subset \mathcal{F}_n$, which combined with the paragraph above gives us that $\sigma(\mathcal{P}) = \mathcal{F}_n$, hence ν_1 and ν_2 agree on $\sigma(\mathcal{P})$.

So let us show that $\mathcal{F}_n$ is indeed a λ-system:

i) $\Omega \in \mathcal{F}_n$. Since $A_n \uparrow \Omega$, we can use continuity of measure (proposition 2.1) to conclude that

$$\begin{aligned}\lim_{n\to\infty} \nu_i(A_n) &= \nu_i(\lim_{n\to\infty} A_n)\\ &= \nu_i(\Omega).\end{aligned}$$

Since $\nu_1(A_n) = \nu_2(A_n)$ $(A_n \in \mathcal{P})$, it holds that $\nu_1(\Omega) = \nu_2(\Omega)$, so $\Omega \in \mathbb{P} \subset \mathcal{F}_n$.

ii) Let $A, B \in \mathcal{F}_n$ with $A \subset B$. So,

$$\begin{aligned}\nu_1((A \setminus B) \cap A_n) &= \nu_1(A \cap A_n) - \nu_1(B \cap A_n)\\ &= \nu_2(A \cap A_n) - \nu_2(B \cap A_n)\\ &= \nu_2((A \setminus B) \cap A_n),\end{aligned}$$

which means that $A \setminus B \in \mathcal{F}_n$.

iii) Let $B_i \in \mathcal{F}_n$ s.t. $B_i \subset B_{i+1}$. Then, once again using continuity of measures to move limits around, we have

$$\begin{aligned}
\nu_1\left(\cup_{i=1}^{\infty} B_i \cap A_n\right) &= \nu_1\left(\cup_{i=1}^{\infty}(B_i \cap A_n)\right) && (3.1)\\
&= \lim_{i\to\infty} \nu_1(B_i \cap A_n) && (3.2)\\
&= \lim_{i\to\infty} \nu_2(B_i \cap A_n) && (3.3)\\
&= \nu_2\left(\cup_{i=1}^{\infty}(B_i \cap A_n)\right) && (3.4)\\
&= \nu_2\left(\cup_{i=1}^{\infty} B_i \cap A_n\right), && (3.5)
\end{aligned}$$

which gives us that $\cup_{i=1}^{\infty} B_i \in \mathcal{F}_n$, hence $\mathcal{F}_n$ is a λ-system.

3.1.4 From Indicator Function to General (Borel) Function

When we deine the Lebesgue integral, we define it in three steps.

1) First for indicator functions, which in turn is generalized to simple non-negative functions (i.e. linear combinations of indicator functions).
2) Second for any non-negative functions (which is done by utilizing that any such function can be described as the limit of a sequence of simple functions)
3) A general function (by separating the positive and negative parts)

Exercise 3.4. Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be a countable set, $\mathcal{F}$ all subsets of Ω, and ν the counting measure on Ω. Show that for any Borel function f, the integral of f with respect to ν is

$$\int f d\nu = \sum_{i=1}^{\infty} f(\omega_i) \tag{3.6}$$

Solution. Let $A \in \mathcal{F}$ and define $f = 1_A$. Then

$$\begin{aligned}
\int f d\nu &= \int_A d\nu\\
&= \nu(A)\\
&= \sum_{i=1}^{\infty} 1_A(\omega_i)
\end{aligned}$$

I.e. (3.6) holds for indicator functions, and hence also for simple functions.

Now, let f be a non-negative Borel function. Then we know that there exists a sequence $(f_n)_i^{\infty}$ of simple functions such that $f_n \uparrow f$. Then

$$\begin{aligned}
\int f d\nu &= \int \lim_{n\to\infty} f_n d\nu\\
&= \lim_{n\to\infty} \int f_n d\nu.
\end{aligned}$$

Since f_n is a simple function, we know that @ref{eq:ex14} holds. Hence,

$$\begin{aligned}\int f d\nu &= \lim_{n\to\infty} \sum_{i=1}^{\infty} f_n(\omega_i) \\ &= \sum_{i=1}^{\infty} \lim_{n\to\infty} f_n(\omega_i) \\ &= \sum_{i=1}^{\infty} f(\omega_i),\end{aligned}$$

and so @ref{eq:ex14} holds for non-negative Borel functions.

Finally, let f be any Borel function. Then we can write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Then both f_+ and f_- are non-negative Borel functions, hence @ref{eq:ex14} holds for both. So

$$\begin{aligned}\int f d\nu &= \int f_+ d\nu - \int f_- d\nu \\ &= \sum_{i=1}^{\infty} f_+(\omega_i) - \sum_{i=1}^{\infty} f_-(\omega_i) \\ &= \sum_{i=1}^{\infty} f_+(\omega_i) - f_-(\omega_i) \\ &= \sum_{i=1}^{\infty} f(\omega_i).\end{aligned}$$

So (3.6) holds for all Borel functions.

3.1.5 Switch the Order of Integration and Limit

Exercise 3.5 (Generalized Dominated Convergence Theorem)**.** If $\lim f_n = f$ and there exists a sequence of integrable functions $g_1, g_2, g_3, \ldots$ such that

- $|f_n| \leq g_n$ a.e.
- $g_n \to g$ a.e.
- $\lim_{n\to\infty} \int g_n d\nu = \int g d\nu$

then

$$\int \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int f_n d\nu \tag{3.7}$$

Chapter 4

Homework

4.1 First Exam Period

4.1.1 Assigned Problems

Exercise 4.1 (Ex 2)**.** Let $\mathcal{C}$ be a collection of subsets of Ω and let $\Gamma = \{\mathcal{F} | \mathcal{F}$ is a σ-field on Ω and $\mathcal{C} \subset \mathcal{F}\}$. Show that $\Gamma \neq \emptyset$ and $\sigma(\mathcal{C}) = \cap_{\mathcal{F}\in\Gamma}\mathcal{F}$.
Solution (Ex 2). Let $\mathbb{P}(\Omega)$ be the collection of all subsets of Ω. We know that this is a σ-field. It also contains $\mathcal{C}$. Hence, $\Gamma \neq \emptyset$.

By definition, $\sigma(\mathcal{C})$ is the smallest σ-field that contains $\mathcal{C}$, hence $\sigma(\mathcal{C}) \in \Gamma$ and $\sigma(\mathcal{C}) \subset \mathcal{F}$ for all $\mathcal{F} \in \Gamma$. Therefore, $\sigma(\mathcal{C}) \subset \cap_{\mathcal{F}\in\Gamma}\mathcal{F}$. But since $\sigma(\mathcal{C}) \in \Gamma$, $\sigma(\mathcal{C}) \in \cap_{\mathcal{F}\in\Gamma}\mathcal{F}$, which in turn ensures that $\cap_{\mathcal{F}\in\Gamma}\mathcal{F} \subset \sigma(\mathcal{C})$.

Hence $\sigma(\mathcal{C}) = \cap_{\mathcal{F}\in\Gamma}\mathcal{F}$.
Exercise 4.2 (Ex 5)**.** a) Let $\mathcal{C}$ be a π-system and $\mathcal{D}$ be a λ-system such that $\mathcal{C} \subset \mathcal{D}$. Show that $\sigma(\mathcal{C}) \subset \mathcal{D}$.
Solution (Ex 5).
Exercise 4.3 (Ex 12)**.** Let ν and λ be two measures on $(\Omega, \mathcal{F})$ such that $\nu(A) = \lambda(A)$ for any $A \in \mathcal{C} \subset \mathcal{F}$, where $\mathcal{C}$ is a π-system (3.1). Assume that ν is σ-finite (2.5).

Show that $\nu(A) = \lambda(A)$ for all $A \in \sigma(\mathcal{C})$.
Solution (Ex 12). Let $\mathcal{F} = \{A \in \sigma(\mathcal{C}) | \nu(A) = \lambda(A)\}$. Then $\mathcal{C} \subset \mathcal{F}$. If we can show that $\mathcal{F}$ is a σ-field, then $\sigma(\mathcal{C}) \subset \mathcal{F}$ (since $\sigma(\mathcal{C})$ is the smallest σ-field that contains $\mathcal{C}$), which proves that $\nu(A) = \lambda(A)$ for all $A \in \sigma(\mathcal{C})$.
Exercise 4.4 (Ex 14)**.** Prove proposition 1.4 (proposition 2.4)
Solution (Ex 14 (i)). Assume f is Borel. Then $f^{-1}(A) \in \mathcal{F}$ for all open sets $A \in \mathcal{B}$, hence $f^{-1}(a,\infty) \in \mathcal{F}$.

Now assume $f^{-1}(a,\infty) \in \mathcal{F}$ for all $a \in \mathbb{R}$, and let $\mathcal{G} = \{A \in \mathcal{B} | f^{-1}(A) \in \mathcal{F}\}$. So, $(a,\infty) \in \mathcal{G}$ for all $a \in \mathbb{R}$. If we can show that $\mathcal{G}$ is a σ-field, then we will have that $\sigma((a,\infty)) = \mathcal{B} \subset \mathcal{G}$, hence $f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$, meaning that f is measurable.

So let us prove that $\mathcal{G}$ is a σ-field.

a) First of all, $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$.
b) Second, let $A \in \mathcal{G}$. Since $f^{-1}(A^C) = (f^{-1}(A))^C \in \mathcal{F}$ ($\mathcal{F}$ is a σ-field and $f^{-1}(A) \in \mathcal{F}$, so $(f^{-1}(A))(^C) \in \mathcal{F}$).
c) Finally, let $A_1, A_2, \ldots$ be a sequence of sets such that $A_i \in \mathcal{G}$ for all i. Then $f^{-1}\left(\cup_{i=1}^{\infty} A_i\right) = \cup_{i=1}^{\infty} f^{-1}(A_i)$. Since $f^{-1}(A_i) \in \mathcal{F}$ for all i and $\mathcal{F}$ is a σ-field, $\cup_{i=1}^{\infty} f^{-1}(A_i) \in \mathcal{F}$, so $\cup_{i=1}^{\infty} A_i \in \mathcal{G}$.

So $\mathcal{G}$ is a σ-field, which concludes the proof.
Solution (Ex 14 (ii)). Assume f and g are Borel functions. Let $a, b \in \mathbb{R}$. af is Borel, since

$$(af)^{-1}((c,\infty)) = \{\omega \in \Omega : a \cdot f(\omega) \in (c,\infty)\}.$$

If $a \neq 0$,

$$\begin{aligned}(af)^{-1}((c,\infty)) &= \left\{\omega \in \Omega : f(\omega) \in (\tfrac{c}{a},\infty)\right\} \\ &= f^{-1}(\tfrac{c}{a},\infty).\end{aligned}$$

Since f is Borel, this is a measurable set (by (i)). If $a = 0$, then

$$(af)^{-1}((c,\infty)) = \begin{cases}\Omega & \text{if } c \leq 0 \\ \emptyset & \text{if } c < 0\end{cases}$$

In either case, $(af)^{-1}((c,\infty)) \in \mathcal{F}$. Since it holds that for all $a, c \in \mathbb{R}$ that $(af)^{-1}((c,\infty)) \in \mathcal{F}$, af is measurable by (i).

Let $c \in \mathbb{R}$. Now consider the sum of f and g:

$$\begin{aligned}(f+g)^{-1}((c,\infty)) &= \{\omega \in \Omega : f(\omega) + g(\omega) > c\} \\ . &= \cup_{t\in\mathbb{Q}}\{\omega \in \Omega : f(\omega) > c - t\} \cap \{\omega \in \Omega : g(\omega) > t\} \\ &= \cup_{t\in\mathbb{Q}} f^{-1}((c-t,\infty)) \cap g^{-1}((t,\infty)).\end{aligned}$$

Since f and g are both measurable, $f^{-1}((c-t,\infty)) \in \mathcal{F}$ and $g^{-1}((t,\infty)) \in \mathcal{F}$ for all $t \in \mathbb{R}$. Hence, the intersection of the two is measurable for any $t \in \mathbb{R}$, which in turn implies that the union over all rational numbers is measurable (countable union of measurable sets). Hence, $f + g$ is measurable.

Combine the two results to get the final result.[1]
Solution (Ex 14 (iii)).
Solution (Ex 14 (iv)). Assume f is measurable from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{G})$, and g measureable from $(\Lambda, \mathcal{G})$ to $(\Delta, \mathcal{H})$. Let $H \in \mathcal{H}$. We want to show that $(g \circ f)^{-1}(H) \in \mathcal{F}$, since this would mean $g \circ f$ is measurable. So,

$$\begin{aligned}(g \circ f)^{-1}(H) &= \{\omega \in \Omega | g(f(\omega)) \in H\} \\ &= \left\{\omega \in \Omega | f(\omega) \in g^{-1}(H)\right\} \\ &= f^{-1}(g^{-1}(H)).\end{aligned}$$

Since g is measurable, $g^{-1}(H) \in \mathcal{G}$, and since f is measurable, $f^{-1}(g^{-1}(H)) \in \mathcal{F}$. So, $g \circ f$ is measurable.
Solution (Ex 14 (v)). Let $f : \Omega \to \mathbb{R}^p$, where Ω is a borel set. Assume f is continuous. Then, if A is an open set, $f^{-1}(A)$ is an open set, and therefore borel. Hence, $f^{-1}((a,\infty))$ is a borel set for all a, and by (i) we have that f is a borel function.
Exercise 4.5 (Ex 19)**.** Let $\{f_n\}$ be a sequence of Borel functions on a measurable space. Show that

a) $\sigma(f_1, f_2, \ldots) = \sigma(\cup_{j=1}^{\infty}\sigma(f_j)) = \sigma(\cup_{j=1}^{\infty}\sigma(f_1, \ldots, f_j))$.
b) $\sigma(\limsup_n f_n) \subset \cap_{n=1}^{\infty}\sigma(f_n, f_{n+1}, \ldots)$.

Solution (Ex 19).
Exercise 4.6 (Ex 24)**.** Let f be an integrable function on $(\Omega, \mathcal{F}, \nu)$. Show that for each $\epsilon > 0$, there exists a δ_ϵ such that for $A \in \mathcal{F}$:

$$\nu(A) < \delta_\epsilon \Rightarrow \int_A |f| d\nu < \epsilon.$$

[1]Note: This could be done in one step, but I found it easier to split up into two.

Solution. Let $\epsilon > 0$, $A \in \mathcal{F}$ with $\nu(A) < \delta_\epsilon = \frac{\epsilon}{\sup_{\omega \in A} |f(\omega)|}$. Then

$$\begin{aligned}\int_A |f| d\nu &\leq \int_A \sup_{\omega \in A} |f(\omega)| d\nu \\ &= \sup_{\omega \in A} |f(\omega)| \nu(A) \\ &< \epsilon.\end{aligned}$$

Exercise 4.7 (Ex 30)**.** For any c.d.f. F and any $a \geq 0$, show that $\int [F(x+a) - F(x)] dx = a$
Solution. ALEX!
Exercise 4.8 (Ex 34)**.** Prove proposition 2.8
Solution (Ex 34 i))*.* Let g be the unique function denoted by $\frac{d\lambda}{d\nu}$. Assume $f = 1_A$ for some $A \in \mathcal{F}$. Since $\lambda << \nu$, we know that $\lambda(A) = \int_A g d\nu$. So,

$$\begin{aligned}\int f d\lambda &= \int 1_A d\lambda \\ &= \lambda(A) \\ &= \int_A g d\nu \\ &= \int 1_A g d\nu = \int f g d\nu.\end{aligned}$$

Hence, (i) is true for all indicator functions, and so by linearity of integrals (2.6) for all non-negative simple functions.

Now, let f be a general non-negative Borel function. Then we know that there exists a sequence of simple functions $\phi_1, \phi_2, \ldots$ such that $\phi_n \uparrow f$. Hence, utilizing the monotone convergence theorem and the fact that we know (i) holds for simple functions,

$$\begin{aligned}\int f d\lambda &= \int \lim_{n\to\infty} \phi_n d\lambda \\ &= \lim_{n\to\infty} \int \phi_n d\lambda \\ &= \lim_{n\to\infty} \int \phi_n g d\nu \\ &= \int \lim_{n\to\infty} \phi_n g d\nu \\ &= \int f g d\nu,\end{aligned}$$

and so we have shown that (i) holds for any non-negative Borel function.
Solution (Ex 34 ii))*.* Assume $\lambda_1 << \nu$ and $\lambda_2 << \nu$. Then

$$\begin{aligned}(\lambda_1 + \lambda_2)(A) &= \lambda_1(A) + \lambda_2(A) \\ &= \int_A g_1 d\nu + \int_A g_2 d\nu \\ &= \int_A (g_1 + g_2) d\nu,\end{aligned}$$

so $\lambda_1 + \lambda_2 << \nu$, and

$$\frac{d(\lambda_1 + \lambda_2)}{d\nu} = g_1 + g_2 = \frac{d\lambda_1}{d\nu} + \frac{d\lambda_2}{d\nu}.$$

Solution (Ex 34 iii)). Since $\tau << \lambda$,

$$\tau(A) = \int_A \frac{d\tau}{d\lambda} d\lambda.$$

Since $\lambda << \nu$, we can use (i) with $f = \frac{d\tau}{d\lambda}$, to get

$$\tau(A) = \int_A \frac{d\tau}{d\lambda}\frac{d\lambda}{d\tau} d\tau,$$

which tells us that $\tau << \nu$ and

$$\frac{d\tau}{d\nu} = \frac{d\tau}{d\lambda}\frac{d\lambda}{d\tau}.$$

Solution (Ex 34 iv)). By definition, $(\lambda_1 \times \lambda_2)(A) = \int_A d(\lambda_1 \times \lambda_2) = \int 1_A d(\lambda_1 \times \lambda_2)$. Since $1_A \geq 0$, we can use Fubini to get

$$(\lambda_1 \times \lambda_2)(A) = \int \int 1_A d\lambda_1 d\lambda_2.$$

Since $\lambda_1 << \nu_1$, we can use (i) with $f = 1_A$ (1_A is non-negative) to obtain that

$$(\lambda_1 \times \lambda_2)(A) = \int \int 1_A \frac{d\lambda_1}{d\nu_1} d\nu_1 d\lambda_2,$$

and then, since $\lambda_2 << \nu_2$, using (i) again with $f = \int 1_A \frac{d\lambda_1}{d\nu_1} d\nu_1$ (which is non-negative) to get

$$(\lambda_1 \times \lambda_2)(A) = \int \int 1_A \frac{d\lambda_1}{d\nu_1} d\nu_1 \frac{d\lambda_2}{d\nu_2} d\nu_2.$$

Finally, using Fubini again we get

$$\begin{aligned}(\lambda_1 \times \lambda_2)(A) &= \int \int 1_A \frac{d\lambda_1}{d\nu_1}\frac{d\lambda_2}{d\nu_2} d\nu_1 d\nu_2 \\ &= \int_A \frac{d\lambda_1}{d\nu_1}\frac{d\lambda_2}{d\nu_2} d(\nu_1 \times \nu_2).\end{aligned}$$

I.e. $\lambda_1 \times \lambda_2 << \nu_1 \times \nu_2$ and $\frac{d(\lambda_1 \times \lambda_2)}{d(\nu_1 \times \nu_2)} = \frac{d\lambda_1}{d\nu_1}\frac{d\lambda_2}{d\nu_2}$.

Exercise 4.9 (Ex 35)**.** Let $\{a_n\}$ be a sequence of positive numbser with $\sum_{i=1}^{\infty} a_n = 1$, and $\{P_n\}$ a sequence of probability measure on a common measurable space, $(\Omega, \mathcal{F})$. Define $P = \sum_{n=1}^{\infty} P_n$.

a) Show that P is a probability measure.
b) Let ν be a σ-finite measure. Show that

$$P_n << \nu \text{ for all } n \in \mathbb{N} \quad \iff \quad P << \nu.$$

c) Derive the Lebesgue p.d.f. of P when P_n is the gamma distribution $\Gamma(\alpha, n^{-1})$ with $\alpha > 1$ and $a_n \propto n^{-\alpha}$.

Solution (Ex 35 a)). Need to show that $P = \sum_{n=1}^{\infty} a_n P_n$ is a probability measure. So we check the three properties for a probability measure (2.4), with the extra property that $P(\Omega) = 1$:

i) $P(A) = \sum_{n=1}^{\infty} a_n P_n(A) \geq 0$ for all A since $a_n > 0$ for all n by assumption, and $P_n(A) \geq 0$ for all n, since P_n is a probability measure. Furthermore, $P(\Omega) = \sum_{n=1}^{\infty} a_n P_n(\Omega) = \sum_{n=1}^{\infty} a_n = 1$, where the second equality holds since P_n is a probability measure (by assumption), and the last equality is exactly the assumption we made about the a_ns. I.e. $0 \leq P(A) \leq 1$.

ii) Since $P_n(\emptyset) = 0$, $P(\emptyset) = \sum_{n=1}^{\infty} a_n P_n(\emptyset) = 0$.

iii) Let $A_1, A_2, \ldots$ be a countable sequence of pairwise disjoint sets. Then using that P_n is a measure for all n,

$$\begin{aligned}
P(\cup_{i=1}^{\infty} A_i) &= \sum_{n=1}^{\infty} a_n P_n\left(\cup_{i=1}^{\infty} A_i\right) \\
&= \sum_{n=1}^{\infty} a_n \sum_{i=1}^{\infty} P_n(A_i) \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} a_n P_n(A_i) \\
&= \sum_{i=1}^{\infty} P(A_i).
\end{aligned}$$

Solution (Ex 35 b)). Assume $P << \nu$. Let $A \in \mathcal{F}$ with $\nu(A) = 0$. Assume there exists $n \in \mathbb{N}$ such that $P_n(A) > 0$. Then $P(A) > a_n P_n(A) > 0$. But $P << \nu$ implies that $P(A) = 0$, so by contradiction, $P_n(A) = 0$ for all $n \in \mathbb{N}$.

Now assume $P_n << \nu$. Let $A \in \mathcal{F}$ with $\nu(A) = 0$. Then $P_n(A) = 0$ for all $n \in \mathbb{N}$. Hence, $P(A) = \sum_{n=1}^{\infty} a_n P_n(A) = 0$, which means that $P << \nu$.

Exercise 4.10 (Ex 50)**.**
Exercise 4.11 (Ex 55)**.** ALEX
Exercise 4.12 (Ex 56)**.** ALEX
Exercise 4.13 (Ex 65)**.** ALEX
Exercise 4.14 (Ex 74)**.** ALEX
Exercise 4.15 (Ex 83)**.** ALEX
Exercise 4.16 (Ex 85)**.**
Exercise 4.17 (Ex 93)**.** ALEX
Exercise 4.18 (Ex 99)**.** ALEX
Exercise 4.19 (Ex 101)**.**
Exercise 4.20 (Ex 106)**.** ALEX
Exercise 4.21 (Ex 115)**.**
Exercise 4.22 (Ex 117)**.**
Exercise 4.23 (Ex 126)**.** Prove (vii) of Theorem 1.8
Solution. Let $X_n \to dX$, $P(X = c) = 1$.

Apply triangle inequality and assumption:

$$\lim_{n\to\infty} P(||X_n - c|| > \epsilon) \leq \lim_{n\to\infty} P(||X_n - X|| > \epsilon) + \lim_{n\to\infty} P(||X - c|| > \epsilon) = 0.$$

Exercise 4.24 (Ex 127)**.**
Exercise 4.25 (Ex 128)**.**
Exercise 4.26 (Ex 137)**.** Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of R.Vs. Assume $X_n \to_d X$ and $P_{Y_n|X_n=x_n} \to_w P_Y$ almost surely for every sequence of numbers $\{x_n\}$, where X and Y are independent random variables. Show that $X_n + Y_n \to_d X + Y$.
Solution.

$$P_{Y_n,X_n} = P_{Y_n|X_n} \cdot P_{X_n} \to P_Y P_X = P_{Y,X},$$

where the last equality holds due to independence of X and Y.
Exercise 4.27 (Ex 138)**.**
Exercise 4.28 (Ex 140)**.** $X_n \sim N(\mu_n, \sigma_n^2, n \in \mathbb{N}$ and $X \sim N(\mu, \sigma^2)$. Show that $X_n \to_d X \iff \mu_n \to \mu$ and $\sigma_n \to \sigma$.

Solution. Assume $\mu_n \to \mu$ and $\sigma_n \to \sigma$. Then $f_n(x) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{\frac{-(x-\mu_n)^2}{\sigma_n}} \to f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{\sigma}}$ for all x. Hence, $X_n \to_d X$.

Assume $X_n \to_d X$, and assume for contradiciton that $(\mu_n, \sigma_n^2) \to (a, b^2) \neq (\mu, \sigma^2)$. This implies that $f_n = \frac{1}{\sqrt{2\pi}\sigma_n} e^{\frac{-(x-\mu_n)^2}{\sigma_n}} \to f = \frac{1}{\sqrt{2\pi}b} e^{\frac{-(x-a)^2}{b}}$, hence $X_n \to_d Y$ where $Y \sim N(a, b^2)$. Since $(a, b) \neq (\mu, \sigma^2)$, and we know the limiting distribution is unique, this contradicts our assumption.
Exercise 4.29 (Ex 142)**.** f_n is the Lebesgue p.d.f. of the t-distribution t_n. Show that $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$, where f is the Lebesgue p.d.f. for standard normal.

Solution. By definition, $f_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{\frac{-(n+1)}{2}}$.

Note that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. So, since $\sqrt{1 + \frac{x^2}{n}} \to 0$,

$$\left(1 + \frac{x^2}{n}\right)^{\frac{-(n+1)}{2}} = \frac{1}{\left(1 + \frac{x^2/2}{n/2}\right)^{-n/2} \sqrt{1 + \frac{x^2}{n}}} \to e^{-x^2/2}.$$

Since $\lim_{n\to\infty} \frac{\Gamma(n+c)}{\Gamma(n)n^c} = 1$,

$$\lim_{n\to\infty} \frac{\Gamma(\frac{n}{2} + \frac{1}{2})}{\Gamma(\frac{n}{2})\sqrt{n/2}} = 1.$$

So

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{\frac{-(n+1)}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \lim_{n\to\infty} \frac{\Gamma(\frac{n}{2} + \frac{1}{2})}{\Gamma(\frac{n}{2})\sqrt{n/2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\end{aligned}$$

Exercise 4.30 (Ex 146)**.** Let $U_1, U_2, \ldots$ be i.i.d. random variables, $U_i \sim U[0, 1]$. Let $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that $\sqrt{n}(Y_n - e) \to_d N(0, e^2)$.
Solution. Let $X_i = -log(U_i)$. Then $EX_1 = \text{Var}(X_1) = 1$, and implies $U_i = e^{-X_i}$. So

$$\begin{aligned}
Y_n &= \left(\prod_{i=1}^n e^{-X_u}\right)^{-1/n} \\
&= e^{\frac{1}{n}\sum_{i=1}^n X_i}.
\end{aligned}$$

By the CLT (corollary 1.2; page 69), $\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - EX_1) = \frac{n}{\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^n X_i - 1\right) \to_d N(0, 1)$.

Let $g = e^x$. Then $g'(x) = g(x)$, and $U_i = g(X_i)$. Using the delta method (specifically corollary 1.1; page 61), we get that

$$\sqrt{n}\,(Y_n - e) = \tfrac{n}{\sqrt{n}}\left(g\left(\tfrac{1}{n}\sum_{i=1}^{n} X_i\right) - g(1)\right) \to_d N(0, g(1)^2) = N(0, e^2).$$

Exercise 4.31 (Ex 149)**.** Let $X_1, \ldots X_n$ be i.i.d. random variables such that for $x = 3, 4, \ldots$, $P(X_1 = \pm x) = (2cx^2\log(x))^{-1}$, where $c = \sum_{x=3}^{\infty} \frac{x^{-2}}{\log(x)}$.

Show that $E|X_1| = \infty$, but $\frac{1}{n}\sum_{i=1}^{n} X_i \to_p 0$, using Theorem 1.13(i).
Solution. Notice that

$$E|X_1| = \sum_{x=3}^{\infty} 2\frac{x}{2cx^2\log(x)} = \frac{1}{c}\sum_{x=3}^{\infty}\frac{1}{x\log(x)} \geq \tfrac{1}{c}\int_3^{\infty}\frac{1}{x\log(x)}dx = \infty,$$

and that $EX_1 = 0$. (To see that the inequality above holds, draw it!)

Consider $nP(|X_1| > n)$:

$$\begin{aligned} nP(|X_1| > n) &= n\sum(x = n)^{\infty}\frac{1}{2cx^2\log(x)} \\ &\leq \frac{n}{c}\int_n^{\infty}\frac{1}{x^2\log(x)}dx \\ &\leq \frac{n}{c\log(n)}\int_n^{\infty}\frac{1}{x^2}dx \\ &= \frac{n}{c\log(n)}\frac{1}{n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

So by theorem 1.13(i), we $\frac{1}{n}\sum_{i=1}^{n} X_i - a_n \to_p 0$, where $a_n = E(X_1 I_{\{|X_1|\leq n\}}) \to 0$ (can be seen using an argument as above).
Exercise 4.32 (Ex 152)**.** Let $T_n = \sum_{i=1}^{n} X_i$, where $X_1, X_2, \ldots$ are independent and $P(X_n = \pm n^{\theta}) = 0.5$ for some $\theta > 0$.

a) Show that when $\theta < 0.5$, then $T_n/n \to_{a.s.} 0$
b) Show that when $\theta \geq 1$, then $T_n/n \to_p 0$ does NOT hold

Solution. (a)

Since $\theta < 0.5$, $2(\theta - 1) < -1$. So

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{E|X_n|^2}{n^2} &= \sum_{n=1}^{\infty}\frac{(n^{\theta})^2}{n^2} \\ &= \sum_{n=1}^{\infty} n^{2(\theta-1)} < \infty. \end{aligned}$$

By SLLN, $T_n/n = \frac{1}{n}\sum_{i=1}^{\infty} X_i \to_{a.s.} EX_1 = 0$.

(b)

Note that the p.d.f. of X_n is $F_n(x) = 0 1_{X_n < -n^{\theta}} + \frac{1}{2}1_{-n^{\theta}\leq X_n < n^{\theta}} + 1_{X_n \geq n^{\theta}}$. By definition, $X_n \to_d 0$ if and only if $F_n(x) \to F(x) = 1_{(X\geq 0)}$ for all x continuity points of F. For any $n > 1$, $F_n(\frac{-n^{\theta}}{2}) = 0.5 \neq F(\frac{-n^{\theta}}{2}) = 0$. So X_n does not converge to 0 in distribution. By theorem 1.8(iii), this means that X_n does not converge to 0 in probability (since convergence in probability implies convergence in distribution).
Exercise 4.33 (Ex 153)**.** Let $X_2, X_3, \ldots$ be independent random variables with $P(X_n = \pm\sqrt{n/\log(n)}) = 0.5$. Show that $\sum_{k=1}^{\infty}\frac{E|X_k|^p}{k^p} = \infty$ for all $p \in [1, 2]$, but $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{\infty} E|X_i|^2 = 0$.

Solution. For any $p \in [1,2]$:

$$\sum_{n=2}^{\infty} \frac{E|X_n|^p}{n^p} = \sum_{n=2}^{\infty} \frac{(n/\log(n))^{p/2}}{n^p} = \sum_{n=2}^{/} infty \frac{1}{(n\log(n))^{p/2}} = \infty.$$

Use that $\log(x)$ is an increasing function:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=2}^{n} E|X_k|^2 &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=2}^{\infty} \frac{k}{\log(k)} \\
&\leq \lim_{n\to\infty} \frac{(n-1)\frac{n}{\log(n)}}{n^2} \\
&\leq \lim_{n\to\infty} \frac{n\frac{n}{\log(n)}}{n^2} \\
&\leq \lim_{n\to\infty} \frac{1}{\log(n)} = 0.
\end{aligned}$$

Exercise 4.34 (Ex 163)**.**
Exercise 4.35 (Ex 164)**.** Let $X_1, X_2, \ldots$ be independent variables with $P(X_j = \pm j^a) = P(X_j = 0) = \frac{1}{3}$, where $a > 0$, $j = 1, 2, \ldots$. Does Liapounov's condition hold?
Solution. $\sigma_n^= \sqrt{\sum_{j=1}^{n} \frac{2j^{2a}}{3}}$. So, want to see if

$$\frac{\sum_{j=1}^{n} E|X_j - EX_j|^{2+\delta}}{\sigma_n^{2+\delta}} \to 0.$$

So,

$$\frac{\sum_{j=1}^{n} \left(\frac{2}{3} j^{2a}\right)^{2+\delta}}{\left(\frac{2}{3}\sum_{j=1}^{n} j^{2a}\right)^{(2+\delta)/2}} = \frac{(\frac{2}{3})^{2+\delta}}{(\frac{2}{3})^{(2+\delta)/2}} \frac{\sum_{j=1}^{\infty} j^{(2+\delta)a}}{\left(\sum_{j=1}^{\infty} j^{2a}\right)^{(2+\delta)/2}}.$$

Choose $\delta = 2$. Using Jensens inequality with $\phi(x) = x^2$ and the ratio test, we see that

$$\frac{(\frac{2}{3})^4}{(\frac{2}{3})^2} \frac{\sum_{j=1}^{\infty}((j^a)^2)^2}{\left(\sum_{j=1}^{\infty} j^{2a}\right)^2} \to 0$$

4.1.2 Suggested Problems

Exercise 4.36 (Ex 3)**.** Let $\Omega, \mathcal{F}_j)$, $j = 1, 2, \ldots$, be measurable spaces such that $\mathcal{F}_j \subset \mathcal{F}_{j+1}$. Is $\cup_j \mathcal{F}_j$ a σ-field?
Solution (Ex 3). No.

Let $\Omega = [0,1]$, and $\mathcal{F}_n = \sigma\left\{[0, \frac{1}{2^n}), [\frac{1}{2^n}, \frac{2}{2^n}), \ldots, [\frac{2^{n-1}}{2^n}, 1)\right\}$.

Now, consider the set $B_n = [0, \frac{1}{2^n})$. Clearly $B_n \in \mathcal{F}_n$ for all n, hence $B_n \in \cup_{i=1}^{\infty} \mathcal{F}_i$. Hower, $\cap_{i=1}^{\infty} B_i = \{0\} \notin \cup_{i=1}^{\infty} \mathcal{F}_i$. So $\cup_{i=1}^{\infty} \mathcal{F}_i$ is not closed under countable intersection, i.e. it is not a σ-algebra.
Exercise 4.37 (Ex 6)**.** Prove part (ii) and (iii) of proposition 2.1.
Solution (Ex 6). i) Let $A \subset B$. Then $B \setminus A \cap A = \emptyset$, hence $\nu(B) = \nu((B \setminus A) \cup A) = \nu(B \setminus A) + \nu(A) \geq \nu(A)$.

ii) Let $A_1, A_2, \ldots$ be a sequence of sets. Define $B_i = A_i \setminus \left(\cup_{k=1}^{i-1} A_k\right)$. Then the B_is are pairwise disjoint. Hence,

$$\begin{aligned}\nu\left(\cup_{i=1}^{\infty} A_i\right) &= \nu\left(\cup_{i=1}^{\infty} B_i\right)\\ &= \sum_{i=1}^{\infty} \nu(B_i)\\ &= \sum_{i=1}^{\infty} \nu\left(A_i \setminus \left(\cup_{k=1}^{i-1} A_k\right)\right).\end{aligned}$$

Since $A_i \setminus \left(\cup_{k=1}^{i-1} A_k\right) \subset A_i$, we use (i) to get the result:

$$\sum_{i=1}^{\infty} \nu\left(A_i \setminus \left(\cup_{k=1}^{i-1} A_k\right)\right) \leq \sum_{i=1}^{\infty} \nu\left(A_i\right)$$

iii) Let $A_1 \subset A_2 \subset A_3 \subset \ldots$. Define $B_i = A_i \setminus A_{i-1}$. Then $B_1, B_2, \ldots$ is a sequence of pairwise disjoint sets, and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$. Hence,

$$\begin{aligned}\nu\left(\cup_{i=1}^{k} A_i\right) &= \nu\left(\cup_{i=1}^{k} B_i\right)\\ &= \sum_{i=1}^{k} \nu(B_i)\\ &= \sum_{i=1}^{k} \nu(A_i \setminus A_{i-1})\\ &= \sum_{i=1}^{k} \nu(A_i) - \nu(A_{i-1})\\ &= \nu(A_k).\end{aligned}$$

Taking the limit on both sides gives us

$$\nu\left(\cup_{i=1}^{k} A_i\right) = \lim_{n\to\infty} \nu(A_n).$$

Exercise 4.38 (Ex 15)**.** Show that a monotone function from $\mathbb{R}$ to $\mathbb{R}$ is Borel, and a c.d.f. on $\mathbb{R}^k$ is Borel.
Exercise 4.39 (Ex 17)**.** Let f be a non-negative Borel function on $(\Omega, \mathcal{F})$. Show that f is the limit of a sequence of simple functions $\{\phi_n\}$ on $(\Omega, \mathcal{F})$ with $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$.
Exercise 4.40 (Ex 23)**.**
Exercise 4.41 (Ex 25)**.**
Exercise 4.42 (Ex 31)**.**
Exercise 4.43 (Ex 36)**.**
Exercise 4.44 (Ex 46)**.**
Exercise 4.45 (Ex 53)**.**
Exercise 4.46 (Ex 57)**.**
Exercise 4.47 (Ex 61)**.**
Exercise 4.48 (Ex 66)**.**
Exercise 4.49 (Ex 70)**.**
Exercise 4.50 (Ex 73)**.**
Exercise 4.51 (Ex 78)**.**
Exercise 4.52 (Ex 79)**.**
Exercise 4.53 (Ex 81)**.**

Exercise 4.54 (Ex 82).
Exercise 4.55 (Ex 86).
Exercise 4.56 (Ex 88).
Exercise 4.57 (Ex 91).
Exercise 4.58 (Ex 97).
Exercise 4.59 (Ex 98).
Exercise 4.60 (Ex 102).
Exercise 4.61 (Ex 104).
Exercise 4.62 (Ex 114).
Exercise 4.63 (Ex 116).
Exercise 4.64 (Ex 116).
Exercise 4.65 (Ex 118).
Exercise 4.66 (Ex 119).
Exercise 4.67 (Ex 121).
Exercise 4.68 (Ex 122).
Exercise 4.69 (Ex 125).
Exercise 4.70 (Ex 136).
Exercise 4.71 (Ex 141).
Exercise 4.72 (Ex 144).
Exercise 4.73 (Ex 145).
Exercise 4.74 (Ex 150).
Exercise 4.75 (Ex 154).
Exercise 4.76 (Ex 156).
Exercise 4.77 (Ex 161).
Exercise 4.78 (Ex 166).