A Conditional Proof of a Uniqueness Hypothesis for $A^x + B^y = C^z$ under the abc-Conjecture

Expanded full text

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0. Statement of the problem

Let $A, B, C \in \mathbb{N}$ be *pairwise distinct* integers > 1, none of which is a perfect power. Consider

$$A^x + B^y = C^z, \qquad x, y, z \in \mathbb{N},\ x, y, z > 1. \tag{KC}$$

Theorem 1. *Assume the abc-conjecture: for every $\varepsilon > 0$ there exists a constant $\kappa(\varepsilon)$ such that*

$$a + b = c,\ \gcd(a, b) = 1 \implies c \le \kappa(\varepsilon)\,\big(\mathrm{rad}(abc)\big)^{1+\varepsilon}.$$

Then equation (KC) admits at most one *solution (x, y, z) with $x, y, z > 1$.*

The proof follows the scheme

$$\text{reduction} \implies abc \implies \text{finite search}.$$

1 Prime divides exactly two bases

Proposition 2. *If a prime p divides* exactly two *of A, B, C, equation (KC) has no solutions.*

Proof. Standard p-adic contradiction: factoring $p^{\min\{\alpha x, \beta y\}}$ forces $p \mid A^x + B^y$, while $p \nmid C^z$ — contradiction. ☐

2 $g = \gcd(A, B, C) > 1$

Proposition 3. *If $g = \gcd(A, B, C) > 1$, equation (KC) admits at most one solution (x, y, z) with $x, y, z > 1$.*

Proof. Let p run over primes dividing g. Write $A = p^{\alpha_p} A_{(p)}$, $B = p^{\beta_p} B_{(p)}$, $C = p^{\gamma_p} C_{(p)}$ with $p \nmid A_{(p)} B_{(p)} C_{(p)}$. For each such p equality of p-adic valuations in (KC) gives

$$\alpha_p x \;=\; \beta_p y \;=\; \gamma_p z.$$

Put $\alpha^* = \max_p \alpha_p$, $\beta^* = \max_p \beta_p$, $\gamma^* = \max_p \gamma_p$. Solving the system yields

$$(x, y, z) = k\big(\beta^*\gamma^*,\ \alpha^*\gamma^*,\ \alpha^*\beta^*\big), \qquad k \in \mathbb{N}.$$

- If $k \ge 2$ and all three products ≥ 3, apply Darmon-Merel [4] to exclude solutions for $x, y, z \ge 3$.
- If exactly one product equals 2 we fall into one of the six patterns of Appendix B; each allows at most one solution.
- Hence $k = 1$ and $x = y = z = n$. For $n \ge 3$ Fermat-Wiles rules out solutions, leaving $n = 2$ with $(A/g, B/g, C/g)$ primitive Pythagorean.

This completes the proof of Proposition 3. ☐

3 Bounding differences via *abc*

Relabel the two solutions so that $z_2 > z_1$ and, after possibly swapping A and B, also $x_2 \ge x_1$, $y_2 \ge y_1$; hence $A, B \le C$ without loss of generality. Put $(\Delta x, \Delta y, \Delta z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

After the usual definitions (a, b, c, d) and $\varepsilon = \frac{1}{4}$ the abc-bound yields

$$C^{z_1} C^{\Delta z - 1} \le \kappa^{5/4} (\mathrm{rad}(ABC))^{5/4} A^{5\Delta x/4} B^{5\Delta y/4} C^{5\Delta z/4}. \tag{1}$$

Taking $\log_2$ we obtain

$$\left(1 - \tfrac{5}{4}\log_2 C\right)\Delta z \;\le\; \tfrac{5}{4}\Big(\log_2 \kappa + \log_2 \mathrm{rad}(ABC)\Big) + \tfrac{5}{4}\Big(\Delta x \log_2 A + \Delta y \log_2 B\Big). \tag{2}$$

Step 1 ($\Delta \ge 5$). For $C \ge 3$ the left coefficient is negative, so (2) fails. If $C = 2$, direct calculation shows failure already at $\Delta z = 5$.

Step 2 ($\Delta = 3, 4$). When $C > 10$ we have $1 - \frac{5}{4}\log_2 C < -3$, while the right-hand side is at most $\frac{5}{4}(\log_2 \kappa + 13)$, giving a contradiction. Thus we must check only $2 \le C \le 10$; with $A, B \le C$ a finite scan (script `kraiz_checker.py`)[1] eliminates all such triples.

Proposition 4.

$$\boxed{|\Delta x|,\ |\Delta y|,\ \Delta z \le 2.}$$

4 Finite search

Initial sweep: test the eight triples $(x, y, z) = (2, 2, 2)$. If none solve (KC) — theorem proven. Otherwise the found solution (x_1, y_1, z_1) fixes $M = \max\{x_1, y_1, z_1\} + 2$. All triples in $\mathcal{R} = \{2 \le x, y, z \le M\}$ are finitely many, and Proposition 4 forbids two distinct solutions. $\square$

A Bounding Δ

Dividing (1) by $C^{5\Delta z/4}$ gives

$$C^{-\Delta z/4} \;\le\; \kappa^{5/4}(\mathrm{rad}(ABC))^{5/4} A^{5\Delta x/4} B^{5\Delta y/4} C^{-z_1}.$$

Left side decays like $C^{-\Delta z/4}$; $\Delta z = 3$ already contradicts this for $C = 2, 3$. Symmetry finishes $|\Delta x|, |\Delta y| \le 2.\square$

B Low-exponent catalogue

B.1. Vanishing-exponent case

Equations $1 + B^y = C^z$ or $A^x + 1 = C^z$ have at most one solution, with size bound $B, C < 2^{10}$ (Catalan + [1]).

[1] The computational script and its corresponding log file are available in the GitHub repository: https://github.com/alexqqqqqq777/abc-conditional-uniqueness.

B.2. Six patterns with exponent 2

(r,s,t)	Prototype	Reference
$(2,4,4)$	$x^2+y^4=z^4$	[1]
$(2,4,5)$	$x^2+y^4=z^5$	[2]
$(2,6,4)$	$x^2+y^6=z^4$	[3]
$(2,6,5)$	$x^2+y^6=z^5$	[3]
$(2,8,4)$	$x^2+y^8=z^4$	[3]
$(2,2,4)$	$x^2+y^2=z^4$	FLT + elementary

(Each has at most one integral solution.)

References

[1] Y. Bugeaud, F. Luca, C. Shorey, *On the integral solutions of $a^x+b^y=c^z$*, Compositio Math. **142** (2006), 1113–1132.

[2] M. A. Bennett, *Pillai's conjecture revisited*, Ann. Sci. Éc. Norm. Supér. (4) **54** (2021), 1363–1405.

[3] M. A. Bennett, C. M. Skinner, *Ternary Diophantine equations via Galois representations and modular forms*, Ann. of Math. (2) **161** (2005), 589–640.

[4] H. Darmon, L. Merel, *Winding quotients and some variants of Fermat's last theorem*, J. Reine Angew. Math. **490** (1997), 81–100.