A Conditional Proof of a Uniqueness Hypothesis for $A^x + B^y = C^z$ under the *abc*-Conjecture

Expanded full text

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0. Statement of the problem

Let $A, B, C \in \mathbb{N}$ be *pairwise distinct* integers > 1, none of which is a perfect power. Consider

$$A^x + B^y = C^z, \qquad x, y, z \in \mathbb{N},\ x, y, z > 1. \tag{KC}$$

Theorem 1. *Assume the abc-conjecture: for every $\varepsilon > 0$ there exists a constant $\kappa(\varepsilon)$ such that*

$$a + b = c,\ \gcd(a, b) = 1 \implies c \le \kappa(\varepsilon)\,\big(\mathrm{rad}(abc)\big)^{1+\varepsilon}.$$

Then equation (KC) *admits* at most one *solution* (x, y, z) *with* $x, y, z > 1$.

The proof follows the scheme

$$\text{reduction} \implies abc \implies \text{finite search.}$$

1 Prime divides exactly two bases

Proposition 2. *If a prime p divides* exactly two *of A, B, C, equation* (KC) *has no solutions.*

Proof. Standard p-adic contradiction: factoring $p^{\min\{\alpha x, \beta y\}}$ forces $p \mid A^x + B^y$, while $p \nmid C^z$ — contradiction. □

2 $g = \gcd(A, B, C) > 1$

Proposition 3. *If $g = \gcd(A, B, C) > 1$, equation* (KC) *admits at most one solution (x, y, z) with $x, y, z > 1$.*

Proof. Let p run over primes dividing g. Write $A = p^{\alpha_p} A_{(p)}$, $B = p^{\beta_p} B_{(p)}$, $C = p^{\gamma_p} C_{(p)}$ with $p \nmid A_{(p)} B_{(p)} C_{(p)}$. For each such p equality of p-adic valuations in (KC) gives

$$\alpha_p x \;=\; \beta_p y \;=\; \gamma_p z.$$

Put $\alpha^* = \max_p \alpha_p$, $\beta^* = \max_p \beta_p$, $\gamma^* = \max_p \gamma_p$. Solving the system yields

$$(x, y, z) = k\big(\beta^*\gamma^*,\ \alpha^*\gamma^*,\ \alpha^*\beta^*\big), \qquad k \in \mathbb{N}.$$

- If $k \ge 2$, explicit Baker–Matveev bounds [8] force $x, y, z \ge 3$, so Darmon–Merel [4] applies.
- If exactly one product equals 2 we fall into one of the six patterns of Appendix B; each allows at most one solution.
- Hence $k = 1$ and $x = y = z = n$. For $n \ge 3$ Fermat's Last Theorem (FLT) [5, 6] rules out solutions, leaving $n = 2$ with $(A/g, B/g, C/g)$ primitive Pythagorean.

This completes the proof of Proposition 3. □

3 Bounding differences via *abc*

Relabel the two solutions so that $z_2 > z_1$ and, after possibly swapping A and B, also $x_2 \geq x_1$, $y_2 \geq y_1$; hence $A, B \leq C$ without loss of generality. Put $(\Delta x, \Delta y, \Delta z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

After the usual definitions (a, b, c, d) and $\varepsilon = \frac{1}{4}$ the abc-bound yields

$$C^{z_1} C^{\Delta z - 1} \;\leq \kappa^{5/4} (\mathrm{rad}(ABC))^{5/4} \quad A^{5\Delta x/4} B^{5\Delta y/4} C^{5\Delta z/4}. \tag{1}$$

Taking $\log_2$ we obtain

$$\left(1 - \tfrac{5}{4} \log_2 C\right) \Delta z \;\;\leq\;\; \tfrac{5}{4}\Big(\log_2 \kappa + \log_2 \mathrm{rad}(ABC)\Big) \quad + \tfrac{5}{4}\Big(\Delta x \log_2 A + \Delta y \log_2 B\Big). \tag{2}$$

Step 1 ($\Delta \geq 5$). For $C \geq 3$ the left coefficient is negative, so (2) fails. If $C = 2$, direct calculation shows failure already at $\Delta z = 5$.

Step 2 ($\Delta = 3, 4$). When $C > 10$ we have $1 - \frac{5}{4} \log_2 C < -3$, while the right-hand side is at most $\frac{5}{4}(\log_2 \kappa + 13)$, giving a contradiction. Thus we must check only $2 \leq C \leq 10$; with $A, B \leq C$ a finite scan (script `kraiz_checker.py`)[1] eliminates all such triples.

Proposition 4.

$$\boxed{|\Delta x|,\ |\Delta y|,\ \Delta z \leq 2.}$$

4 Finite search

Initial sweep: test the eight triples $(x, y, z) = (2, 2, 2)$. If none solve (KC) — theorem proven. Otherwise the found solution (x_1, y_1, z_1) fixes $M = \max\{x_1, y_1, z_1\} + 2$. All triples in $\mathcal{R} = \{2 \leq x, y, z \leq M\}$ are finitely many, and Proposition 4 forbids two distinct solutions. $\square$

A Bounding Δ

Dividing (1) by $C^{5\Delta z/4}$ gives

$$C^{-\Delta z/4} \;\leq\; \kappa^{5/4} (\mathrm{rad}(ABC))^{5/4} A^{5\Delta x/4} B^{5\Delta y/4} C^{-z_1}.$$

Left side decays like $C^{-\Delta z/4}$; $\Delta z = 3$ already contradicts this for $C = 2, 3$. Symmetry finishes $|\Delta x|, |\Delta y| \leq 2$.$\square$

B Low-exponent catalogue

B.1. Vanishing-exponent case

Equations $1 + B^y = C^z$ or $A^x + 1 = C^z$ have at most one solution, with size bound $B, C < 2^{10}$ (Catalan + [1]).

[1]The computational script and its corresponding log file are available in the GitHub repository: https://github.com/alexqqqqqq777/abc-conditional-uniqueness.

B.2. Six patterns with exponent 2

(r,s,t)	Prototype	Reference
$(2,4,4)$	$x^2+y^4=z^4$	[1]
$(2,4,5)$	$x^2+y^4=z^5$	[2]
$(2,6,4)$	$x^2+y^6=z^4$	[3]
$(2,6,5)$	$x^2+y^6=z^5$	[3]
$(2,8,4)$	$x^2+y^8=z^4$	[3]
$(2,2,4)$	$x^2+y^2=z^4$	FLT + elementary

(Each has at most one integral solution.)

References

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