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On the Gibbs phenomenon I: recovering exponential accuracy from the Fourier partial sum of a nonperiodic analytic function *

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Abstract

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It is well known that the Fourier series of an analytic and periodic function, truncated after $2N+1$ terms, converges *exponentially* with N, even in the maximum norm. It is also known that if the function is *not* periodic, the rate of convergence deteriorates; in particular, there is no convergence in the maximum norm, although the function is still analytic. This is known as the *Gibbs phenomenon*. In this paper we show that the first $2N+1$ Fourier coefficients contain enough information about the function, so that an exponentially convergent approximation (in the *maximum* norm) can be constructed. The proof is a constructive one and makes use of the Gegenbauer polynomials $C_n^\lambda(x)$. It consists of two steps. In the first step we show that the first m coefficients of the Gegenbauer expansion (based on $C_n^\lambda(x)$, for $0 \leqslant n \leqslant m$) of *any* L_2 function can be obtained, within exponential accuracy, provided that both λ and m are proportional to (but smaller than) N. In the second step we construct the Gegenbauer expansion based on C_n^λ, $0 \leqslant n \leqslant m$, from the coefficients found in the first step. We show that this series converges exponentially with N, provided that the original function is analytic (though nonperiodic). Thus we prove that *the Gibbs phenomenon can be completely overcome.*

Keywords: Gibbs phenomenon; Fourier series; Gegenbauer polynomials; exponential accuracy.

1. Introduction

We deal, in this paper, with a prototype of the Gibbs phenomenon, and show how to eliminate it. Consider an analytic but nonperiodic function $f(x)$ defined in $[-1, 1]$. Notice that

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$f(x)$ has a discontinuity at the boundary $x = \pm 1$ if it is extended periodically with period 2. The Fourier coefficients of $f(x)$ are defined by

$$\hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x)\, \mathrm{e}^{-ik\pi x}\, \mathrm{d}x. \tag{1.1}$$

Assume that the first $2N+1$ Fourier coefficients $\hat{f}(k)$, $|k| \leqslant N$, are known but the function $f(x)$ is not. Our objective is to recover the function $f(x)$ for $-1 \leqslant x \leqslant 1$ with exponential accuracy in the maximum norm. The traditional Fourier partial sum using the first $2N+1$ modes

$$f_N(x) = \sum_{k=-N}^{N} \hat{f}(k)\, \mathrm{e}^{ik\pi x} \tag{1.2}$$

does a poor job: it produces a first-order approximation to $f(x)$ with an error $O(1/N)$ away from the boundary $x = \pm 1$, and shows $O(1)$ spurious oscillations near the boundary $x = \pm 1$, known as the Gibbs phenomenon. Thus there is no convergence in the maximum norm. When one uses a filter in the Fourier space

$$f_N^{\sigma}(x) = \sum_{k=-N}^{N} \sigma_k^N \hat{f}(k)\, \mathrm{e}^{ik\pi x}, \tag{1.3}$$

where $\sigma_k^N = \overline{\sigma_{-k}^N}$ are suitably defined real or complex numbers which tend to zero when $|k|$ tends to N, the situation becomes better: one can get exponential accuracy away from the boundary $x = \pm 1$ if σ_k^N are chosen as suitable real numbers [7,9,10,12], or one can get exponential accuracy up to one boundary $x = -1$ or $x = 1$ if σ_k^N are chosen as suitable complex numbers [3]. In these cases the approximation is still in the space spanned by the first $2N+1$ trigonometric polynomials and is a convolution of the original Fourier partial sum with some filter kernel which is an approximate two-sided or one-sided δ-function, hence it cannot be exponentially accurate in the maximum norm for $-1 \leqslant x \leqslant 1$. For the one-sided filters introduced in [3] one can use two different approximations in $-1 \leqslant x \leqslant 0$ and in $0 < x \leqslant 1$, right-sided for the former and left-sided for the latter, to obtain exponential convergence globally.

In this paper we adopt a different point of view. The idea is the following: we realize that the problem with the Fourier approximation is the nonperiodicity of the function and the fact that the functions $\mathrm{e}^{ik\pi x}$ are the solutions of a *regular* Sturm–Liouville problem. In [6] it is shown that expanding an analytic, nonperiodic function $f(x)$ by the eigenfunctions of a *singular* Sturm–Liouville problem yields rapid convergence. For example, a Chebyshev or Legendre expansion of $f(x)$ converges exponentially. Thus, if the first $2N+1$ Fourier coefficients can provide enough information to reconstruct the coefficients of an expansion based on a singular Sturm–Liouville problem, we might recover the accuracy. Unfortunately, one can *not* recover the coefficients of the Chebyshev or the Legendre expansion within high enough accuracy.

In this paper we show that from the first $2N+1$ Fourier coefficients of an analytic but nonperiodic function, one can get the first $m \sim N$ coefficients in the *Gegenbauer* expansion based on the Gegenbauer polynomials $C_n^{\lambda}(x)$, provided that the parameter λ, appearing in the weight function $(1-x^2)^{\lambda-1/2}$, *grows with the number of Fourier modes* N. We prove that this yields exponential accuracy in the maximum norm.

Our proof consists of two separate and independent steps. The first step (Section 3) is to show that given the Fourier partial sum of the first $2N+1$ Fourier modes, of an arbitrary L_2-function $f(x)$, it is possible to recover the partial sum of the first m terms in the Gegenbauer expansion of the same function to exponential accuracy (in the maximum norm) by letting the parameter λ and number of terms m in the Gegenbauer expansion grow linearly with N. In this step $f(x)$ needs not be smooth. Any L_2-function will do. We denote the error between the exact Gegenbauer coefficients and the one obtained from the Fourier coefficients the *truncation error*. The results of this section are summarized in Theorem 3.4.

In the second step (called the *regularization error*), we prove the exponential convergence, in the maximum norm, of the Gegenbauer expansion of an analytic function when λ grows linearly with m. This is done in Section 4. The second step has its own interest: it is an exponential convergence proof in the maximum norm for such Gegenbauer expansions of analytic functions, where λ increases with the number of the terms used in the approximation. The results of this section are summarized in Theorem 4.5.

In Section 2 we bring some results concerning Gegenbauer polynomials that are relevant to the proof and the computations.

Finally, in Section 5, we bring the main theorem demonstrating that one can construct an exponentially convergent approximation to an analytic, nonperiodic function, from its first $2N+1$ Fourier coefficients.

In Section 6 we demonstrate the theory with some numerical examples. Of special interest is Example 6.1, concerning the function $f(x)=x$. This function was used originally (in 1898) to demonstrate the Gibbs phenomenon.

We will use A or $\tilde{A}$ for a generic constant independent of all the growing parameters throughout this paper. The actual value of A or $\tilde{A}$ may be different in different locations.

2. Preliminaries

In this section we will introduce the Gegenbauer polynomials and discuss some of their asymptotic behavior. We rely heavily on the standardization of [2], although for our purpose a different scaling might have been more natural.

We start by defining the Gegenbauer polynomials $C_n^\lambda(x)$ in the following definition.

Definition 2.1. The *Gegenbauer polynomial* $C_n^\lambda(x)$ is the polynomial of order n that satisfies

$$\int_{-1}^{1} (1-x^2)^{\lambda-1/2} C_k^\lambda(x) C_n^\lambda(x)\,\mathrm{d}x = 0, \quad k \neq n, \tag{2.1}$$

and (for $\lambda \geqslant 0$)

$$C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n!\,\Gamma(2\lambda)}. \tag{2.2}$$

Note that the Gegenbauer polynomials thus defined are not orthonormal. In fact, the norm of $C_n^\lambda(x)$ is given by the following lemma.

Lemma 2.2. *The Gegenbauer polynomials defined above satisfy*

$$\int_{-1}^{1}(1-x^2)^{\lambda-1/2}C_n^\lambda(x)C_n^\lambda(x)\,\mathrm{d}x=h_n^\lambda, \tag{2.3}$$

where

$$h_n^\lambda=\pi^{1/2}C_n^\lambda(1)\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)(n+\lambda)}. \tag{2.4}$$

For the proof see [2, p.174].

We are ready now to deal with the asymptotics of the Gegenbauer polynomials for large n and λ. For this we need the next lemma.

Lemma 2.3 (Stirling). *For any number x such that $x \geqslant 1$ we have*

$$\Gamma(x+1)\leqslant(2\pi)^{1/2}x^{x+1/2}\,\mathrm{e}^{-x}\,\mathrm{e}^{1/12}, \tag{2.5}$$

$$\Gamma(x+1)\geqslant(2\pi)^{1/2}x^{x+1/2}\,\mathrm{e}^{-x}. \tag{2.6}$$

Lemma 2.4. *There exists a constant A independent of λ and n such that*

$$h_n^\lambda\leqslant A\frac{\lambda^{1/2}}{(n+\lambda)}C_n^\lambda(1), \tag{2.7}$$

$$h_n^\lambda\geqslant A^{-1}\frac{\lambda^{1/2}}{(n+\lambda)}C_n^\lambda(1). \tag{2.8}$$

The proof follows from (2.4) and the Stirling formula (2.5), (2.6).

Finally we would like to quote the *Rodrigues' formula* [2, p.175].

Lemma 2.5. *The Gegenbauer polynomials are explicitly given by*

$$(1-x^2)^{\lambda-1/2}C_n^\lambda(x)=\frac{(-1)^n}{2^n n!}G(\lambda,n)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^{n+\lambda-1/2}\right], \tag{2.9}$$

where $G(\lambda, n)$ is defined by

$$G(\lambda,n)=\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})}. \tag{2.10}$$

3. The truncation error

In this section we consider an arbitrary L_2-function $f(x)$ defined in $[-1, 1]$. We assume that the first $2N+1$ Fourier coefficients $\hat{f}(k)$, as defined in (1.1), are given. We are interested in recovering (within exponential accuracy in the maximum norm) the first m coefficients in the Gegenbauer expansion of $f(x)$:

$$f(x)=\sum_{l=0}^{\infty}\hat{f}^{\lambda}(l)C_l^{\lambda}(x), \tag{3.1}$$

where the Gegenbauer coefficients are defined by

$$\hat{f}^{\lambda}(l)=\frac{1}{h_l^{\lambda}}\int_{-1}^{1}(1-x^2)^{\lambda-1/2}f(x)C_l^{\lambda}(x)\,\mathrm{d}x, \tag{3.2}$$

with h_l^{λ} given by (2.4).

Since we do not know the function $f(x)$, but rather its truncated Fourier series $f_N(x)$ (defined in (1.2)), we have only an approximation to $\hat{f}^{\lambda}(l)$, which we denote by $\hat{g}^{\lambda}(l)$, given by

$$\hat{g}^{\lambda}(l)=\frac{1}{h_l^{\lambda}}\int_{-1}^{1}(1-x^2)^{\lambda-1/2}f_N(x)C_l^{\lambda}(x)\,\mathrm{d}x. \tag{3.3}$$

Notice that $\hat{g}^{\lambda}(l)$ depends on N. At this stage we would like to define the *truncation error*

$$\mathrm{TE}(\lambda,m,N)=\max_{-1\leqslant x\leqslant 1}\left|\sum_{l=0}^{m}(\hat{f}^{\lambda}(l)-\hat{g}^{\lambda}(l))C_l^{\lambda}(x)\right|. \tag{3.4}$$

The truncation error is the difference between the Gegenbauer expansion (with m terms) of the function $f(x)$ and that of the *truncated Fourier series* $f_N(x)$. It measures the error in the finite Gegenbauer expansion due to truncating the Fourier series.

In the next two theorems we bound the truncation error in terms of the number of given Fourier coefficients N, the number of Gegenbauer polynomials m, and λ.

Theorem 3.1. *If $f(x)$ is an L_2-function on* $[-1, 1]$, *then there exists a constant A which is independent of λ, m and N, such that the truncation error defined in* (3.4), *satisfies the following estimate*:

$$\mathrm{TE}(\lambda,m,N)\leqslant A\frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{(m-1)!\,\Gamma(2\lambda)}\left(\frac{2}{\pi N}\right)^{\lambda-1}. \tag{3.5}$$

Proof. As a first step we consider the special function $f(x)=\mathrm{e}^{\mathrm{i}n\pi x}$ with $|n|>N$. In this special case $f_N(x)=0$ and we obtain

$$(\hat{f}^{\lambda}(l)-\hat{g}^{\lambda}(l))C_l^{\lambda}(1)=\frac{C_l^{\lambda}(1)}{h_l^{\lambda}}\int_{-1}^{1}(1-x^2)^{\lambda-1/2}\,\mathrm{e}^{\mathrm{i}n\pi x}C_l^{\lambda}(x)\,\mathrm{d}x. \tag{3.6}$$

Roughly speaking, one can argue that this integral is rapidly decreasing when n increases; this is so because the integral is proportional to the nth Fourier coefficient of the function $(1-x^2)^{\lambda-1/2}C_l^{\lambda}(x)$ which is analytic and has λ periodic derivatives. It is nice to know that an explicit expression of this integral appears in the literature [2, p.213]:

$$\frac{1}{h_l^{\lambda}}\int_{-1}^{1}(1-x^2)^{\lambda-1/2}\,\mathrm{e}^{\mathrm{i}n\pi x}C_l^{\lambda}(x)\,\mathrm{d}x=\Gamma(\lambda)\left(\frac{2}{\pi n}\right)^{\lambda}\mathrm{i}^l(l+\lambda)J_{l+\lambda}(\pi n), \tag{3.7}$$

where $J_\nu(x)$ is the Bessel function. Since $|J_\nu(x)| \leqslant 1$ for all x and $\nu \geqslant 0$ [1, p.362], we have, for $0 \leqslant l \leqslant m$,

$$\begin{aligned} \left|(\hat{f}^\lambda(l) - \hat{g}^\lambda(l))C_l^\lambda(1)\right| &\leqslant C_l^\lambda(1)\Gamma(\lambda)\left(\frac{2}{\pi|n|}\right)^\lambda (l+\lambda) \\ &= \frac{(l+\lambda)\Gamma(l+2\lambda)\Gamma(\lambda)}{l!\,\Gamma(2\lambda)}\left(\frac{2}{\pi|n|}\right)^\lambda \\ &\leqslant \frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{m!\,\Gamma(2\lambda)}\left(\frac{2}{\pi|n|}\right)^\lambda, \end{aligned} \tag{3.8}$$

where in the second step we used the formula (2.2) for $C_l^\lambda(1)$, and in the last step we used the fact that $(l+\lambda)\Gamma(l+2\lambda)/l!$ is an increasing function of l.

We now return to the general function $f(x)$, which satisfies

$$f(x) - f_N(x) = \sum_{|n|>N} \hat{f}(n)\,\mathrm{e}^{in\pi x}. \tag{3.9}$$

Since $f(x)$ is an L_2-function, its Fourier coefficients $\hat{f}(n)$ are uniformly bounded:

$$|\hat{f}(n)| \leqslant A. \tag{3.10}$$

We thus have, using the result for the special case $\mathrm{e}^{in\pi x}$ in (3.8),

$$\begin{aligned} \left|(\hat{f}^\lambda(l) - \hat{g}^\lambda(l))C_l^\lambda(1)\right| &\leqslant A\frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{m!\,\Gamma(2\lambda)} \sum_{|n|>N}\left(\frac{2}{\pi|n|}\right)^\lambda \\ &\leqslant \tilde{A}\frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{m!\,\Gamma(2\lambda)}\left(\frac{2}{\pi N}\right)^{\lambda-1}, \end{aligned} \tag{3.11}$$

for all $0 \leqslant l \leqslant m$.

We can now estimate the truncation error (3.4) by

$$\begin{aligned} \mathrm{TE}(\lambda, m, N) &\leqslant m \max_{0\leqslant l\leqslant m}\ \max_{-1\leqslant x\leqslant 1}\left|(\hat{f}^\lambda(l) - \hat{g}^\lambda(l))C_l^\lambda(x)\right| \\ &\leqslant m \max_{0\leqslant l\leqslant m}\left|(\hat{f}^\lambda(l) - \hat{g}^\lambda(l))\right|C_l^\lambda(1) \\ &\leqslant mA\frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{m!\,\Gamma(2\lambda)}\left(\frac{2}{\pi N}\right)^{\lambda-1} \\ &= A\frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{(m-1)!\,\Gamma(2\lambda)}\left(\frac{2}{\pi N}\right)^{\lambda-1}, \end{aligned}$$

where in the second step we used the fact that $|C_l^\lambda(x)| \leqslant C_l^\lambda(1)$ for all $-1 \leqslant x \leqslant 1$ [2, p.206], and in the third step we used (3.11).

The theorem is now proven. □

Remark 3.2. The approximate Gegenbauer coefficients $\hat{g}^{\lambda}(l)$ in (3.3) can be explicitly expressed in terms of the Fourier coefficients $\hat{f}(k)$ as

$$\hat{g}^{\lambda}(l)=\delta_{0l}\hat{f}(0)+\Gamma(\lambda)\mathrm{i}^{l}(l+\gamma)\sum_{0<|k|\leqslant N} J_{l+\lambda}(\pi k)\left(\frac{2}{\pi k}\right)^{\lambda}\hat{f}(k). \tag{3.12}$$

Equation (3.12) follows immediately from the definition (3.3) of $\hat{g}^{\lambda}(l)$ and the integration formula (3.7).

For fixed λ, the truncation error (3.5) decays algebraically as $\mathrm{O}(1/N^{\lambda-1})$. However, if both λ and m grow linearly with N, the truncation error can be made exponentially small. In fact, we can state the following theorem.

Theorem 3.3. *If $\lambda=\alpha N$ and $m=\beta N$, where α and β are positive constants, then the truncation error defined in* (3.4) *satisfies*

$$\mathrm{TE}(\alpha N,\beta N,N)\leqslant AN^{2}q^{N}, \tag{3.13}$$

where

$$q=\frac{(\beta+2\alpha)^{\beta+2\alpha}}{(2\pi\,\mathrm{e})^{\alpha}\alpha^{\alpha}\beta^{\beta}}. \tag{3.14}$$

In particular, if $\alpha=\beta=\frac{2}{27}\pi\approx\frac{1}{4}$, then

$$q=\mathrm{e}^{-2\pi/27}\approx 0.8<1.$$

Proof. We use the Stirling formula (2.5), (2.6) to obtain, from the previous estimates on the truncation error in (3.5) and some simple algebra,

$$\begin{aligned}\mathrm{TE}(\alpha N,\beta N,N)&\leqslant A(\beta+\alpha)N\frac{\Gamma((\beta+2\alpha)N)\Gamma(\alpha N)}{(\beta N-1)!\,\Gamma(2\alpha N)}\left(\frac{2}{\pi N}\right)^{\alpha N-1}\\&\leqslant\tilde{A}N^{2}q^{N},\end{aligned}$$

with q defined by (3.14). If we take $\alpha=\beta$ in (3.14), we obtain

$$q=\left(\frac{27\beta}{2\pi\,\mathrm{e}}\right)^{\beta},$$

which attains its minimum value $q=\mathrm{e}^{-2\pi/27}$ at $\beta=\frac{2}{27}\pi$. $\square$

We would like to point out that we choose $\alpha=\beta$ in Theorem 3.3 simply to show that it is possible to obtain exponentially small truncation errors. This may not be the best choice in practice. We can easily verify that, for fixed α, (3.14) defines a q which is an increasing function of β. This is not surprising since the truncation error should be bigger if there are more terms in the Gegenbauer expansion to approximate. However, we will see in the next section that the regularization error will be smaller if m is bigger. In practice one might try to choose m to maintain some balance between these two errors. For a fixed β, (3.14) defines a q which attains its minimum at

$$\alpha = \tfrac{1}{4}\left\{\pi - 2\beta + \sqrt{\pi(\pi - 4\beta)}\right\}, \tag{3.15}$$

if $\beta \leqslant \frac{1}{4}\pi$. For example, if $\beta = \frac{2}{27}\pi$ and $\alpha \approx 1.33$ is chosen according to (3.15), then q given by (3.14) is approximately 0.49, much smaller than the minimum value 0.8 obtained with the restriction $\alpha = \beta$.

We summarize the results of this section in the next theorem.

Theorem 3.4 (The exponential decay of the truncation error). *Let $f(x)$ be an $L_2[-1, 1]$-function, and $\hat{f}(k)$, $-N \leqslant k \leqslant N$, its Fourier coefficients defined in* (1.1). *Let $\hat{f}^{\lambda}(l)$ be the Gegenbauer expansion coefficients of $f(x)$ defined in* (3.2), *and let $\hat{g}^{\lambda}(l)$ be the Gegenbauer coefficients of the truncated Fourier series $f_N(x)$ defined in* (1.2), *$\hat{g}^{\lambda}(l)$ are given explicitly in* (3.12).

Then, if $\lambda = m = \beta N$, where $\beta < \frac{2}{27}\pi e$, the truncation error decays exponentially *with the number of Fourier modes N, i.e.,*

$$\mathrm{TE}(\beta N, \beta N, N) = \max_{-1 \leqslant x \leqslant 1}\left|\sum_{l=0}^{m}\left(\hat{f}^{\lambda}(l) - \hat{g}^{\lambda}(l)\right)C_l^{\lambda}(x)\right| \leqslant A N^2 q^N, \tag{3.16}$$

with

$$q = \left(\frac{27\beta}{2\pi\,e}\right)^{\beta} < 1.$$

4. The regularization error

In this section we would like to establish error estimates for approximating an analytic function $f(x)$ on $[-1, 1]$ by its Gegenbauer expansion based on the Gegenbauer polynomials $C_n^{\lambda}(x)$. Since our goal is to remove the Gibbs phenomenon, we will use the maximum norm. In the last section we have shown that we can get the Gegenbauer partial sum of the first m terms of any L_2-function from its Fourier partial sum of the first $2N + 1$ modes with exponential accuracy in the maximum norm, if λ and m are both growing linearly with N. Thus in this section we will consider the case of large λ and m.

We will assume that $f(x)$ is an analytic function on $[-1, 1]$, satisfying the following assumptions.

Assumption 4.1. There exist constants $\rho \geqslant 1$ and $C(\rho)$ such that, for every $k \geqslant 0$,

$$\max_{-1 \leqslant x \leqslant 1}\left|\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x)\right| \leqslant C(\rho)\frac{k!}{\rho^k}. \tag{4.1}$$

This is a standard assumption for analytic functions. ρ is actually the distance from $[-1, 1]$ to the nearest singularity of $f(x)$ in the complex plane (see, for example, [8]). The assumption can be modified using the techniques in [5].

Let us consider the Gegenbauer partial sum of the first m terms for the function $f(x)$ given by (3.1), with the Gegenbauer coefficients $\hat{f}^{\lambda}(l)$ defined by (3.2). We want to estimate the regularization error in the maximum norm:

$$\mathrm{RE}(\lambda, m) = \max_{-1 \leqslant x \leqslant 1} \left| f(x) - \sum_{l=0}^{m} \hat{f}^{\lambda}(l) C_l^{\lambda}(x) \right|. \tag{4.2}$$

We start by estimating the Gegenbauer coefficients $\hat{f}^{\lambda}(l)$.

Lemma 4.2. *The Gegenbauer coefficient* $\hat{f}^{\lambda}(l)$, *as defined in* (3.2), *of an analytic function satisfying Assumption* 4.1, *can be bounded by*

$$|\hat{f}^{\lambda}(l)| \leqslant A \frac{C(\rho)\Gamma(\lambda + \frac{1}{2})\Gamma(l + 2\lambda)}{h_l^{\lambda}(2\rho)^l \Gamma(2\lambda)\Gamma(l + \lambda + 1)}. \tag{4.3}$$

Proof. We start by using the definition (3.2) for $\hat{f}^{\lambda}(l)$. We replace the term $(1-x^2)^{\lambda-1/2}C_l^{\lambda}(x)$ by the Rodrigues' formula (2.9), (2.10) to get

$$\hat{f}^{\lambda}(l) = \frac{(-1)^l G(\lambda, l)}{h_l^{\lambda} 2^l l!} \int_{-1}^{1} f(x) \frac{\mathrm{d}^l}{\mathrm{d}x^l}\left[(1-x^2)^{l+\lambda-1/2}\right] \mathrm{d}x,$$

where $G(\lambda, l)$ is defined in (2.10). Integrating by parts l times we get

$$\hat{f}^{\lambda}(l) = \frac{G(\lambda, l)}{h_l^{\lambda} 2^l l!} \int_{-1}^{1} \frac{\mathrm{d}^l f}{\mathrm{d}x^l}(x)(1-x^2)^{l+\lambda-1/2} \, \mathrm{d}x.$$

We now use Assumption 4.1 to estimate the derivative $\mathrm{d}^l f(x)/\mathrm{d}x^l$, thus obtaining

$$|\hat{f}^{\lambda}(l)| \leqslant \frac{G(\lambda, l)C(\rho)}{h_l^{\lambda} 2^l \rho^l} \int_{-1}^{1} (1-x^2)^{l+\lambda-1/2} \, \mathrm{d}x.$$

Since $C_0^{\lambda}(x) = 1$, the remaining integral is simply $h_0^{l+\lambda}$ and can be obtained from (2.3), (2.4):

$$|\hat{f}^{\lambda}(l)| \leqslant \frac{G(\lambda, l)C(\rho)\sqrt{\pi}\,\Gamma(l + \lambda + \frac{1}{2})}{h_l^{\lambda} 2^l \rho^l (l + \lambda)\Gamma(l + \lambda)},$$

and finally using the definition of $G(\lambda, l)$ from (2.10) we get (4.3). □

The estimate (4.3) can be used naturally to get an estimate in the weighted L_2-norm. This would have been more transparent if we had adopted an orthogonal Gegenbauer basis rather than (2.2). However, the aim of this paper is to establish estimates in the maximum norm. We thus state the following theorem.

Theorem 4.3. *If* $f(x)$ *is an analytic function on* $[-1, 1]$ *satisfying Assumption* 4.1, *then the regularization error defined in* (4.2) *can be bounded by*

$$\mathrm{RE}(\lambda, m) \leqslant A \frac{C(\rho)\Gamma(\lambda + \frac{1}{2})\Gamma(m + 2\lambda + 1)}{m\sqrt{\lambda}\,(2\rho)^m \Gamma(2\lambda)\Gamma(m + \lambda)}. \tag{4.4}$$

Proof. By (4.3) and (2.8) we obtain

$$|\hat{f}^\lambda(l)|C_l^\lambda(1) \leqslant A\frac{C(\rho)\Gamma(\lambda+\frac{1}{2})\Gamma(l+2\lambda)}{\sqrt{\lambda}\,(2\rho)^l\Gamma(2\lambda)\Gamma(l+\lambda)}. \tag{4.5}$$

If we define

$$B(l) = A\frac{C(\rho)\Gamma(\lambda+\frac{1}{2})\Gamma(l+2\lambda)}{\sqrt{\lambda}\,(2\rho)^l\Gamma(2\lambda)\Gamma(l+\lambda)},$$

then clearly for $m \leqslant l$,

$$\frac{B(l+1)}{B(l)} = \frac{l+2\lambda}{2\rho(1+\lambda)} \leqslant \frac{1+2\lambda/m}{\rho(2+2\lambda/m)} \leqslant \frac{1+2\lambda/m}{2+2\lambda/m}.$$

We can thus sum (4.5) for $m+1 \leqslant l < \infty$ to obtain

$$\begin{aligned}\mathrm{RE}(\lambda, m) &= \left|\max_{-1\leqslant x\leqslant 1}\sum_{l=m+1}^{\infty}\hat{f}^\lambda(l)C_l^\lambda(x)\right| \leqslant \sum_{l=m+1}^{\infty}|\hat{f}^\lambda(l)|C_l^\lambda(1)\\ &\leqslant \sum_{l=m+1}^{\infty}B(l) \leqslant B(m+1)\frac{2(m+\lambda)}{m}\\ &\leqslant A\frac{C(\rho)\Gamma(\lambda+\frac{1}{2})\Gamma(m+2\lambda+1)}{m\sqrt{\lambda}\,(2\rho)^m\Gamma(2\lambda)\Gamma(m+\lambda)},\end{aligned}$$

where we have used again the fact that $|C_l^\lambda(x)| \leqslant C_l^\lambda(1)$ for all $-1 \leqslant x \leqslant 1$ in the second step. This finishes the proof. □

About the size of the regularization error (4.4) when λ depends linearly on m, as is the case in Theorem 3.3 for truncation errors, we can state the following theorem.

Theorem 4.4. *If $\lambda = \gamma m$, where γ is a positive constant, then the regularization error defined in (4.2) satisfies*

$$\mathrm{RE}(\gamma m, m) \leqslant Aq^m, \tag{4.6}$$

where q is given by

$$q = \frac{(1+2\gamma)^{1+2\gamma}}{\rho 2^{1+2\gamma}\gamma^\gamma(1+\gamma)^{1+\gamma}}, \tag{4.7}$$

which is always less than 1. In particular, if $\gamma = 1$ and $m = \beta N$ where β is a positive constant, then

$$\mathrm{RE}(\beta N, \beta N) \leqslant Aq^N, \tag{4.8}$$

with

$$q = \left(\frac{27}{32\rho}\right)^\beta. \tag{4.9}$$

Proof. We use the Stirling formula (2.5), (2.6) to replace the Gamma functions in (4.4). A little bit of algebra brings us to (4.6), (4.7) if $\lambda = \gamma m$. Notice that the constant A in (4.6) contains the contribution of ρ related terms. It is easy to verify that q defined by (4.7) is a strictly increasing function of γ and tends to $1/\rho \leqslant 1$ when γ tends to infinity. Hence we have $q < 1/\rho \leqslant 1$ for all $\gamma > 0$. As for the proof of (4.8), (4.9), we simply plug in $\gamma = 1$ and $m = \beta N$ into (4.6), (4.7). □

Finally we summarize the results of this section in the following theorem.

Theorem 4.5 (The exponential decay of regularization error). *Let $f(x)$ be an analytic function on $[-1, 1]$ satisfying Assumption* 4.1. *Let $\hat{f}^{\lambda}(l)$, $0 \leqslant l \leqslant m$, be its Gegenbauer coefficients defined in* (3.2). *Furthermore, assume that $\lambda = m = \beta N$. Then*

$$\max_{-1 \leqslant x \leqslant 1} \left| f(x) - \sum_{l=0}^{m} \hat{f}^{\lambda}(l) C_l^{\lambda}(x) \right| \leqslant A q^N, \tag{4.10}$$

where

$$q = \left(\frac{27}{32\rho} \right)^{\beta}.$$

5. The main theorem

In this section we bring the main theorem demonstrating that one can construct an exponentially convergent (in the *maximum norm*) approximation to an analytic, nonperiodic function, from its first $2N+1$ Fourier coefficients. The method is indicated in the last two sections. Namely, first we get, from the Fourier coefficients, an approximation to the first $m = \beta N$ Gegenbauer coefficients. This approximation, by virtue of the discussion in Section 3, converges exponentially fast to the true coefficients, provided that λ grows with N. We then construct the partial Gegenbauer expansion which converges exponentially to $f(x)$ by virtue of the discussion in Section 4.

We are ready to state the following theorem.

Theorem 5.1 (Removal of the Gibbs phenomenon). *Consider an analytic and nonperiodic function $f(x)$ on $[-1, 1]$, satisfying*

$$\max_{1 \leqslant x \leqslant 1} \left| \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x) \right| \leqslant C(\rho) \frac{k!}{\rho^k}, \quad \rho \geqslant 1. \tag{5.1}$$

Assume that the Fourier coefficients

$$\hat{f}(k) = \frac{1}{2} \int_{-1}^{1} f(x)\, \mathrm{e}^{-ik\pi x}\, \mathrm{d}x$$

are known for $-N \leqslant k \leqslant N$.

Let $\hat{g}^\lambda(l)$, $0 \leqslant l \leqslant m$, be the Gegenbauer expansion coefficients of $f_N(x) = \sum_{k=-N}^{N} \hat{f}(k)\, \mathrm{e}^{ik\pi x}$ explicitly given by

$$\hat{g}^\lambda(l) = \delta_{0l}\hat{f}(0) + \Gamma(\lambda)\mathrm{i}^l(l+\lambda) \sum_{0<|k|\leqslant N} J_{l+\lambda}(\pi k)\left(\frac{2}{\pi k}\right)^\lambda \hat{f}(k). \tag{5.2}$$

Then *for $\lambda = m = \beta N$, where $\beta < \frac{2}{27}\pi\mathrm{e}$,*

$$\max_{-1\leqslant x\leqslant 1}\left| f(x) - \sum_{l=0}^{m} \hat{g}^\lambda(l) C_l^\lambda(x)\right| \leqslant AN^2 q_{\mathrm{T}}^N + \bar{A} q_{\mathrm{R}}^N, \tag{5.3}$$

where

$$q_{\mathrm{T}} = \left(\frac{27\beta}{2\pi\,\mathrm{e}}\right)^\beta < 1, \qquad q_{\mathrm{R}} = \left(\frac{27}{32\rho}\right)^\beta < 1.$$

Proof. We start by noting that (5.2), for the approximate Gegenbauer coefficients $\hat{g}^\lambda(l)$, follows from Remark 3.2.

In order to establish (5.3), we introduce the Gegenbauer coefficients of the function $f(x)$ and denote them by $\hat{f}^\lambda(l)$; they are defined in (3.2).

We have

$$\begin{aligned} &\max_{-1\leqslant x\leqslant 1}\left| f(x) - \sum_{l=0}^{m} \hat{g}^\lambda(l) C_l^\lambda(x)\right| \\ &\quad\leqslant \max_{-1\leqslant x\leqslant 1}\left| f(x) - \sum_{l=0}^{m} \hat{f}^\lambda(l) C_l^\lambda(x)\right| \\ &\quad\quad + \max_{-1\leqslant x\leqslant 1}\left| \sum_{l=0}^{m} \hat{f}^\lambda(l) C_l^\lambda(x) - \sum_{l=0}^{m} \hat{g}^\lambda(l) C_l^\lambda(x)\right|. \end{aligned}$$

The first term is the regularization error and has been estimated in Theorem 4.5. The second term is the truncation error and has been estimated in Theorem 3.4. The theorem is thus proved. □

A few remarks are in order.

Remark 5.2. The proof is constructive: given $2N+1$ Fourier coefficients $\hat{f}(k)$, $-N \leqslant k \leqslant N$, one gets explicitly the approximate Gegenbauer coefficients $\hat{g}^\lambda(l)$, $0 \leqslant l \leqslant m$, and the Gegenbauer series can be explicitly constructed.

Remark 5.3. The reconstruction is not optimal, since no effort has been spent to optimize the parameters.

6. Numerical results

In this section we demonstrate the theory using numerical examples. We implement the method in the following way. Assuming that the first $2N+1$ Fourier coefficients of $f(x)$, $\hat{f}(k)$

for $-N \leqslant k \leqslant N$, as defined in (1.1), are given. We compute the approximate Gegenbauer coefficients $\hat{g}^\lambda(l)$, for $0 \leqslant l \leqslant m$, defined in (3.3), using the following formula given in Remark 3.2:

$$\hat{g}^\lambda(l) = \delta_{0l}\hat{f}(0) + \Gamma(\lambda)\mathrm{i}^l(l+\lambda) \sum_{0<|k|\leqslant N} J_{l+\lambda}(\pi k)\left(\frac{2}{\pi k}\right)^\lambda \hat{f}(k). \tag{6.1}$$

We compute the Bessel function $J_\nu(x)$ using an IMSL routine. Once the approximate Gegenbauer coefficients $\hat{g}^\lambda(l)$ are obtained, we can compute the approximation to $f(x)$ by directly summing

$$g_m^\lambda(x) = \sum_{l=0}^{m} \hat{g}^\lambda(l) C_l^\lambda(x), \tag{6.2}$$

as long as we can compute the Gegenbauer polynomial $C_l^\lambda(x)$ accurately. The formula we used to compute $C_l^\lambda(x)$ is

$$C_l^\lambda(\cos\theta) = \sum_{k=0}^{l} \frac{\Gamma(k+\lambda)}{k!\,\Gamma(\lambda)} \frac{\Gamma(l-k+\lambda)}{(l-k)!\,\Gamma(\lambda)} \cos(l-2k)\theta, \tag{6.3}$$

which can be found in [1, p.175].

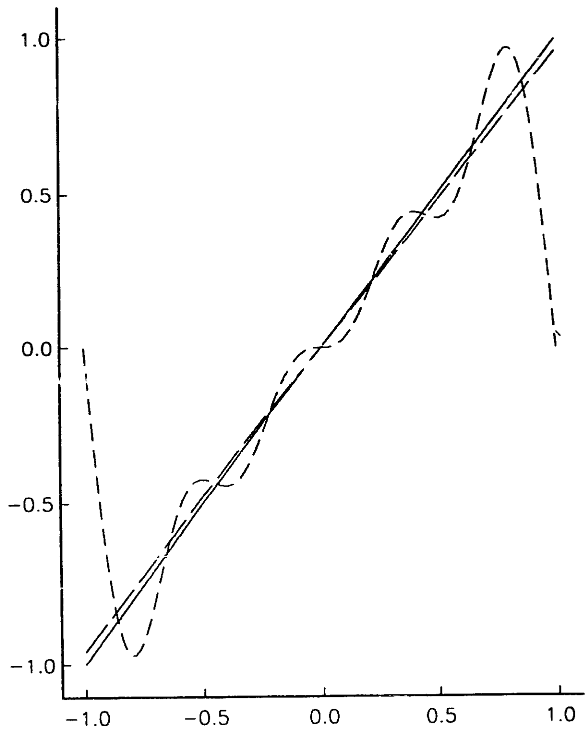

Fig. 1. The function $f(x)=x$ (background solid line); the Fourier partial sum $f_N(x)$ defined in (1.2) with $N=4$ (short dashed line), and the approximation $g_m^\lambda(x)$ defined in (6.2) through the Gegenbauer polynomials with $N=4$ and $m=\lambda=\frac{1}{4}N$ in the long dashed line.

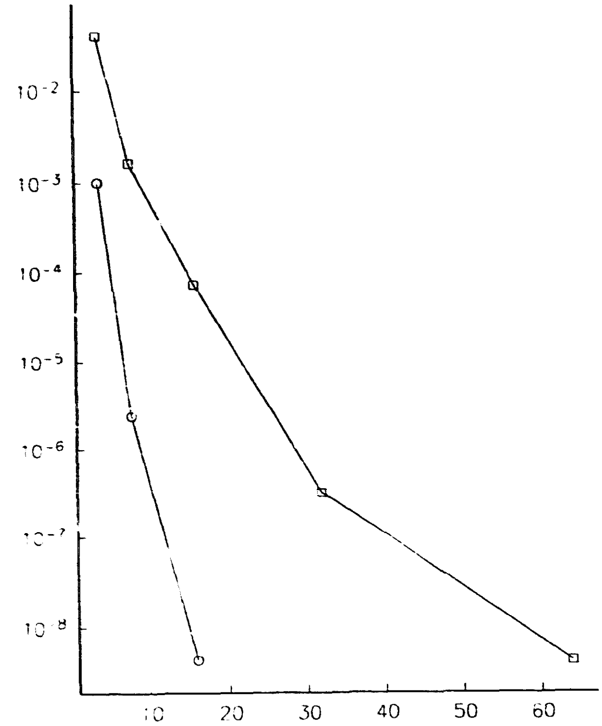

Fig. 2. For the function $f(x)=x$, the errors in the maximum norm, $\max_{-1\leqslant x\leqslant 1}|f(x)-g_m^\lambda(x)|$, where $g_m^\lambda(x)$ is defined in (6.2), in a logarithmic scale, versus N ($2N+1$ is the number of Fourier modes given). The squares are for the case with $m=\lambda=\frac{1}{4}N$; the circles are for the case with $m=\frac{1}{4}N$ and λ determined by (3.15).

We remark that the implementation described above is subject to roundoff effects for large λ and m. We use a CRAY Y-MP to carry out all the computations. Our implementation can give accurate result only when the error is no smaller than 10^{-9}. And we will show results only in those cases. A better way to implement this method might be through Chebyshev polynomials.

For the purpose of testing we take two functions.

Example 6.1. $f(x)=x$.

This is the original example used in 1898 to show the famous Gibbs phenomenon. For this particular function there is no regularization error as long as $m\geqslant 1$, hence all the errors observed result from the truncation error.

In Fig. 1, we show the exact function $f(x)=x$ in the solid line, the Fourier partial sum $f_N(x)$ with $N=4$ in the short dashed line, and the approximation through the Gegenbauer polynomials $g_m^\lambda(x)$, as defined in (6.2), with $N=4$ and $m=\lambda=\frac{1}{4}N$ in the long dashed line. We can clearly see that the Fourier partial sum $f_N(x)$ shows the Gibbs oscillations, while the approximation $g_m^\lambda(x)$ through the Gegenbauer polynomials is uniformly accurate. In Fig. 2 we draw the errors in the maximum norm, with a logarithmic scale, versus N (recall that $2N+1$ is the number of Fourier modes given), both with $m=\lambda=\frac{1}{4}N$ (squares) and with $m=\frac{1}{4}N$ and λ determined by (3.15) (circles). This picture confirms our estimates in (3.13)–(3.15) for the exponential convergence of the truncation errors.

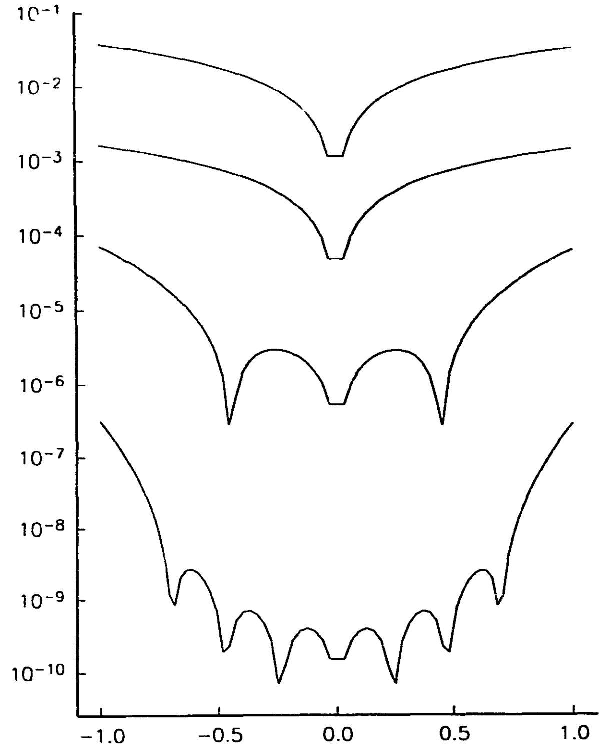

Fig. 3. For the function $f(x)=x$, in a logarithmic scale, the pointwise errors $|f(x)-g_m^\lambda(x)|$, where $g_m^\lambda(x)$ is defined in (6.2), of the case $m=\lambda=\frac{1}{4}N$ for $N=4, 8, 16, 32$.

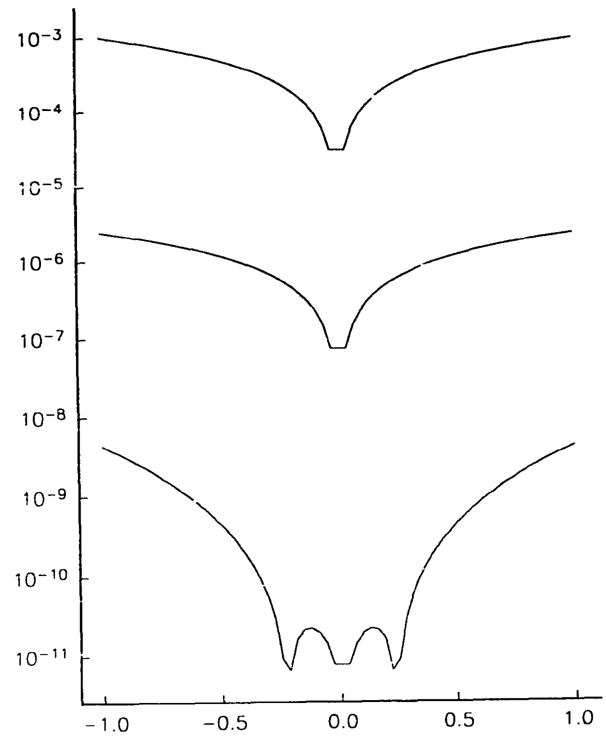

Fig. 4. For the function $f(x)=x$, in a logarithmic scale, the pointwise errors $|f(x)-g_m^\lambda(x)|$, where $g_m^\lambda(x)$ is defined in (6.2), of the case $m=\frac{1}{4}N$ and λ determined by (3.15), for $N=4, 8, 16$.

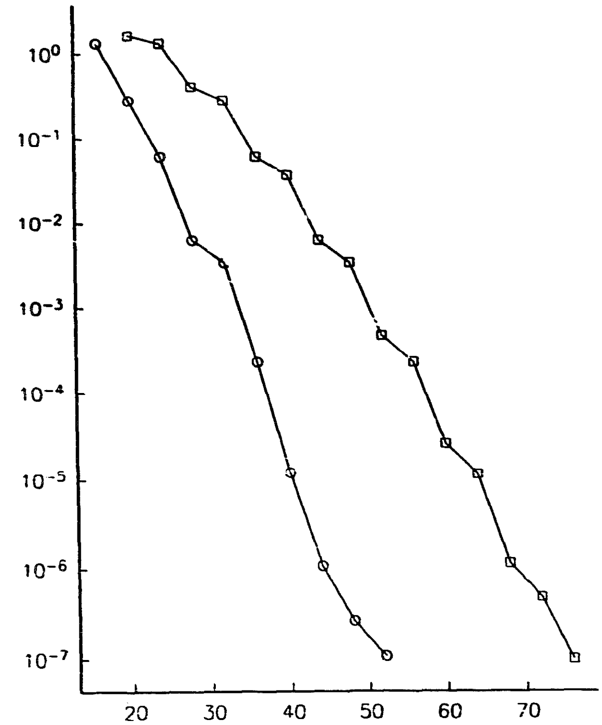

Fig. 5. For the function $f(x) = \cos[1.4\,\pi(x+1)]$, the errors in the maximum norm, $\max_{-1 \leqslant x \leqslant 1} |f(x) - g_m^\lambda(x)|$, where $g_m^\lambda(x)$ is defined in (6.2), in a logarithmic scale versus N ($2N+1$ is the number of Fourier modes given). The squares are for the case with $m = \lambda = \frac{1}{4}N$; the circles are for the case with $m = \lambda = \frac{2}{5}N$.

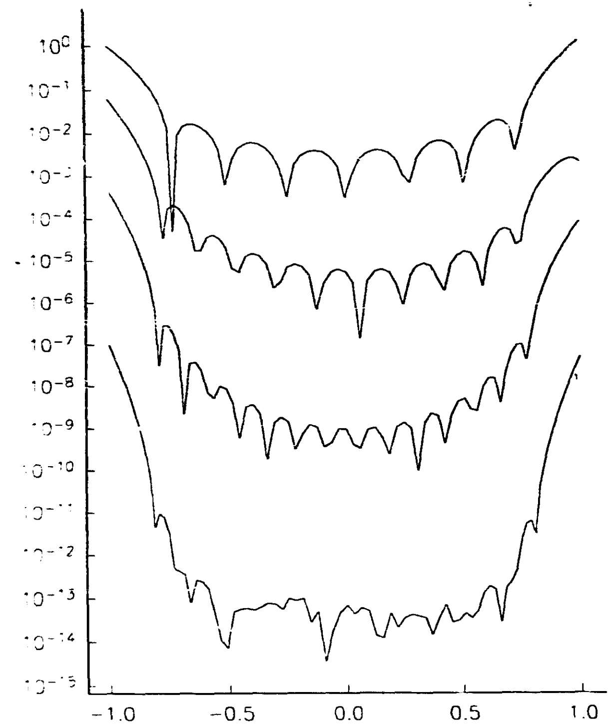

Fig. 6. For the function $f(x) = \cos[1.4\,\pi(x+1)]$, in a logarithmic scale, the pointwise errors $|f(x) - g_m^\lambda(x)|$, where $g_m^\lambda(x)$ is defined in (6.2), of the case $m = \lambda = \frac{1}{4}N$ for $N = 24, 36, 52, 76$.

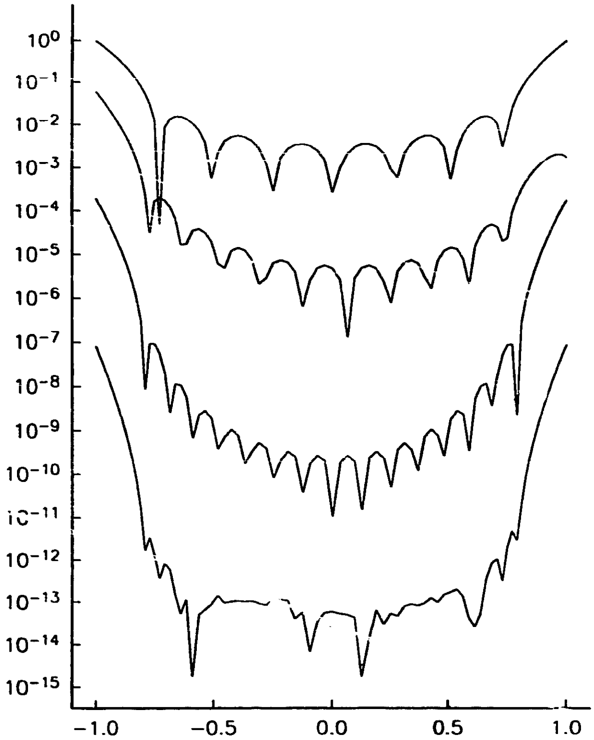

Fig. 7. For the function $f(x)=\cos[1.4\ \pi(x+1)]$, in a logarithmic scale, the pointwise errors $|f(x)-g_m^\lambda(x)|$, where $g_m^\lambda(x)$ is defined in (6.2), of the case $m=\lambda=\frac{2}{5}N$ for $N=16, 24, 36, 52$.

Next, we show the pointwise errors of the approximation with $m=\lambda=\frac{1}{4}N$ for $N=4, 8, 16, 32$ in Fig. 3, and the pointwise errors with $n=\frac{1}{4}N$ and λ determined by (3.15), for $N=4, 8, 16$, in Fig. 4, again both in logarithmic scales. We can observe exponential convergence both inside the interval and near the boundary, although the absolute error is several magnitudes smaller inside the interval than at the boundary.

We have also run the test for $f(x)=x^n$ with $n=2,\ldots,9$, obtaining similar results (not shown here).

Example 6.2. $f(x)=\cos[1.4\ \pi(x+1)]$.

This function satisfies Assumption 4.1 with arbitrary $\rho>1$. Both truncation and regularization error will be present. The choice $m=\lambda=\frac{1}{4}N$ gives the exponential convergence as expected. The choice $m=\lambda=\frac{2}{5}N$ gives better results in this case.

In Fig. 5 we draw the errors in the maximum norm versus N, with a logarithmic scale, both with $m=\lambda=\frac{1}{4}N$ (squares) and with $m=\lambda=\frac{2}{5}N$ (circles). The choice of $m=\frac{1}{4}N$ and λ determined by (3.15) does not converge in this case, indicating that (3.15) is probably a bad choice for the regularization errors.

Finally, again in logarithmic scales, we show the pointwise errors in the approximations with $m=\lambda=\frac{1}{4}N$ for $N=24, 36, 52, 76$ in Fig. 6, and the pointwise errors with $m=\lambda=\frac{2}{5}N$ for $N=16, 24, 36, 52$ in Fig. 7. We again observe exponential convergence both inside the interval and at the boundary, and the several magnitudes difference in the absolute errors inside the interval and at the boundary.

7. Further remarks

There are several topics which are not addressed in this paper and will be discussed in a future paper.

(1) Other choices of polynomials such as the Laguerre polynomials and Hermite polynomials.

(2) Algebraic convergence for C^k but not analytic functions.

(3) The case with exponential recovery in a subinterval $[a, b]$ of $[-1, 1]$. That is, given the first $2N+1$ Fourier modes for a function defined on $[-1, 1]$, find its Gegenbauer partial sum of the first m terms based on the scaled Gegenbauer polynomials on $[a, b]$, in which the function $f(x)$ is analytic. This would allow one to handle multiple discontinuities and discontinuities of unknown location.

(4) The procedure realized in the Fourier space as a convolution, similar to the approach used in [3].

(5) Efficient and stable numerical implementation of those methods.

(6) Extension to collocation techniques and other than the Fourier basis functions.

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