Assignment-2

$\Leftarrow$

1. Given $X \vdash \alpha$

$\Rightarrow$ To show $X \vDash \alpha$.

By induction on the steps of derivation from X to α.

$\{\alpha_1, \alpha_2 \dots \alpha_n\}$ where $\alpha_n = \alpha$.

Base case: $n=1$

$X \vdash \alpha$

Then either α is a member of X or it is an instance of axiom.

In either case $X \vDash \alpha$ $\because \alpha$ is ~~alwa~~ true under the given premises.

Induction case $n>1$.

α_n = member of X or instance of axiom,

both have been handled in base case

α_n = derived using MP.

from $\alpha_j, \alpha_j \Rightarrow \alpha_n$

So $X \vdash \alpha_j$ $X \vDash \alpha_j$

~~$X \vdash \alpha_j \Rightarrow \alpha_n$ $X \vdash \alpha_j \Rightarrow \alpha_n$~~

$X \vdash \alpha_j \Rightarrow \alpha_n$ $X \vDash \alpha_j \Rightarrow \alpha_n$ using I.H.

Thus $X \vDash$ ~~α_j~~ α_n. hence proved.

using MP.

$\Rightarrow$

Given $X \vDash \alpha$

to show $X \vdash \alpha$

Proof: Given $X \vDash \alpha$,

By compactness Thm, we know $\exists\, Y \subseteq_{fin} X$ s.t $Y \vDash \alpha$.

say $Y = \{\beta_1, \beta_2 \dots \beta_n\}$

$\{\beta_1, \beta_2 \dots \beta_n\} \vDash \alpha$.

Now ~~β_n~~ this means when $\beta_1, \beta_2 \dots \beta_n$ is true α is true

$\vDash (\overset{T}{\beta_1} \rightarrow \dots (\overset{T}{\beta_{n-1}} \rightarrow \underbrace{(\overset{T}{\beta_n} \rightarrow \overset{T}{\alpha})}_{\text{True}}))$ (by induction)

$\underbrace{\dots}_{T}$

is valid.

hence using completeness Theorem,

~~using completeness Thm,~~

$\vdash (\beta_1 \rightarrow (\beta_2 \rightarrow \dots \beta_{n-1} \rightarrow (\beta_n \rightarrow \alpha)))$.

using Deduction Thm. n times

$\{\beta_1, \beta_2 \dots \beta_n\} \vdash \alpha$.

Thus use this subset in proving $X \vdash \alpha$

hence proved.

2) a) Let X be an FSS. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of ϕ.
We define an infinite sequence of sets $X_0, X_1, \ldots$ as follows:

- $X_0 = X$
- For $i \geq 0$, $X_{i+1} = \begin{cases} X_i \cup \{\alpha_i\} & \text{if } X_i \cup \{\alpha_i\} \text{ is FSS} \\ X_i & \text{otherwise} \end{cases}$

Each set in this sequence is FSS by construction and
$X_0 \subseteq X_1 \subseteq X_2 \ldots$ Let $Y = \bigcup_{i \geq 0} X_i$.

We claim that Y is maximal FSS.

Claim: ~~Proof.~~ Y is FSS.

By contradiction.
say $\exists\ Z \subseteq_{fin} Y$. which is ~~not~~ ~~FSS~~ unsat.

$Z = \{ \beta_1, \beta_2 \cdots \beta_n \}$.

$\beta_i = \alpha_{i_1}$ [based on the position of β_t in ~~the~~ $\bar{\phi}$.

let $j = \max(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_n})$ Thus $Z \subseteq_{fin} X_{j+1}$ in the sequence $X_0 \subseteq X_1 \cdots \subseteq Y$.

So when X_{j+1} was being formed, if Z had been unsat, then X_{j+1} would not be FSS, which is a contradiction.

Hence Y is FSS

Claim: Y is maximal

Proof: by contradiction.

Suppose $Y \cup \{\alpha\}$ is FSS for some formula $\alpha \notin Y$.

Let $\alpha = \beta_j$ in our enumeration of $\bar{\Phi}$.

$\because$ $\beta_j \notin Y$, β_j was not added at step $j+1$ in our construction. In other words, which means $X_j \cup \{\beta_j\}$ is not FSS.

$\exists$ $Z \subseteq_{fin} X_j$ s.t. $Z \cup \{\beta_j\}$ is unsat. Since $X_j \subseteq Y$, we must have $Z \subseteq_{fin} Y$ which is a contradiction that $Y \cup \{\alpha\}$ is FSS.

Claim 1: If β is consistent, then β is sat
Proved in class

Claim 2: If β is inconsistent, β is unsat.

Proof: $\vdash \neg\beta$

$\models \neg\beta$ using completeness & soundness.

$\neg\beta$ is valid ~~proved in class.~~

~~β is unsat.~~

Thus $\neg\beta$ is always true hence β is always false.

Thus β is unsat.

Claim 3.

$\vdash (\alpha \Rightarrow \beta) \Rightarrow ((\delta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \delta) \Rightarrow (\beta \vee \gamma))$

to prove $\alpha \Rightarrow \beta,\ \delta \Rightarrow \gamma,\ \alpha \vee \delta \vdash \beta \vee \gamma$

to prove $\alpha \Rightarrow \beta,\ \delta \Rightarrow \gamma,\ \neg\alpha \Rightarrow \delta \vdash \neg\beta \Rightarrow \gamma$.

using $a \Rightarrow b,\ b \Rightarrow c \vdash a \models c$ (in class)

to prove $\underbrace{\alpha \Rightarrow \beta}$, $\underbrace{\neg\alpha \Rightarrow \gamma} \vdash \neg\beta \Rightarrow \gamma$

$\underbrace{\neg\alpha \vee \beta}$

$\neg\beta \Rightarrow \neg\alpha$.

to prove $\neg\beta \Rightarrow \gamma$ hence proved.

(b) $\rightarrow$ First we show
$\{\alpha, \neg\alpha\}$ both cannot be in X

contradiction: If both were in X, then $\{\alpha, \neg\alpha\}$ is a finite subset of X that is unsat, which is a contradiction.

$\rightarrow$ Say neither α nor $\neg\alpha$ is in X.

Since X is max FSS, there must be sub sets $B \subseteq_{fin} X$ and $C \subseteq_{fin} X$

st $B \cup \{\alpha\}$ is unsat and $C \cup \{\neg\alpha\}$ is unsat.

$B = \{\beta_1, \beta_2 \cdots \beta_m\}$

$C = \{\gamma_1, \gamma_2 \cdots \gamma_k\}$

Thus we have $\beta_1 \wedge \beta_2 \wedge \cdots \beta_m \wedge \alpha$ unsat and
$\gamma_1 \wedge \gamma_2 \cdots \wedge \gamma_k \wedge \neg\alpha$ unsat

$\hat{\beta} \wedge \alpha$ unsat

$\hat{\gamma} \wedge \neg\alpha$ unsat.

Using claim 1: if β is unsat, β is inconsistent.

$\vdash \neg(\hat{\beta} \wedge \alpha)$ $\quad \vdash \neg\hat{\beta} \vee \neg\alpha$ $\quad \vdash \alpha \Rightarrow \neg\hat{\beta}$

$\vdash \neg(\hat{\gamma} \wedge \neg\alpha)$ $\quad \vdash \neg\hat{\gamma} \vee \alpha$ $\quad \vdash \neg\alpha \Rightarrow \neg\hat{\gamma}$

Using the following a claim proved in claim 3. ~~thesis proved in quiz~~

$\vdash (\alpha \Rightarrow \beta) \Rightarrow ((\delta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \delta) \Rightarrow (\beta \vee \gamma))$

$\vdash (\alpha \Rightarrow \neg\hat{\beta}) \Rightarrow ((\neg\alpha \Rightarrow \neg\hat{\gamma}) \Rightarrow ((\alpha \vee \neg\alpha) \Rightarrow (\neg\hat{\beta} \vee \neg\hat{\gamma}))$

using Deduction Thm,

$\vdash (\neg\hat{\beta} \vee \neg\hat{\gamma})$ $\quad \vdash \neg(\hat{\beta} \wedge \hat{\gamma})$ $\quad$ hence $\hat{\beta} \wedge \hat{\gamma}$ is inconsistent.

using claim 2, $\hat{\beta} \wedge \hat{\gamma}$ is unsat. which is a contradiction.

Thus $\alpha \in X$ iff $\neg\alpha \notin X$.
proved.

(c)

Claim 1: If $\alpha \vee \beta$ is consistent, then either α is consistent or β is consistent.

<u>Contrapositive</u>: If both α and β are inconsistent, $\alpha \vee \beta$ is ~~so~~ inconsistent.

1. $\vdash \neg\beta$ (β is inconsistent) premise
2. $\neg\alpha \vdash \neg\beta$ (adding premise ~~is~~ does not matter)
3. $\vdash \neg\alpha \Rightarrow \neg\beta$ Deduction Thm
4. $\vdash (\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ A3 instance

~~6.~~ Replacing α with β.

5. $\vdash (\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\alpha \Rightarrow \beta) \Rightarrow \alpha)$
6. $\vdash (\neg\alpha \Rightarrow \beta) \Rightarrow \alpha$ MP 3,5
7. $\vdash \neg\alpha \Rightarrow \neg(\neg\alpha \Rightarrow \beta)$ using $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$ proved in (class + quiz)
8. $\vdash \neg\alpha$ premise
9. $\vdash \neg(\neg\alpha \Rightarrow \beta)$ MP 8,7
10. $\vdash \neg(\alpha \vee \beta)$

hence $\alpha \vee \beta$ is inconsistent, ~~which is a contradiction~~.

~~Thus~~ hence proved.

<u>Claim 2</u>: If either α is consistent or β is consistent, then $\alpha \vee \beta$ is consistent.

<u>Contrapositive</u>: If $\alpha \vee \beta$ is inconsistent, then both α & β are ~~so~~ inconsistent.

1. $\vdash \neg(\alpha \vee \beta)$ premise
2. $\vdash \neg(\neg\alpha \Rightarrow \beta)$

~~3. $\beta \vdash \neg(\neg\alpha \Rightarrow \beta)$~~

~~4. $\vdash \beta \Rightarrow \neg(\neg\alpha \Rightarrow \beta)$~~

~~5.~~

3. $\vdash \beta \Rightarrow (\neg\alpha \Rightarrow \beta)$ A1 instance
4. $\vdash \neg(\neg\alpha \Rightarrow \beta) \Rightarrow \neg\beta$ using $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$
5. $\vdash \neg\beta$ MP 2,4.

1. $\neg(\alpha \vee \beta)$

2. $\neg(\neg\alpha \Rightarrow \beta)$

3. $\alpha, \neg\alpha \vdash (\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ A3 instance

4. $\alpha, \neg\alpha \vdash \alpha$

5. $\alpha, \neg\alpha \vdash \neg\alpha$

6. $\alpha, \neg\alpha \vdash \neg\alpha \Rightarrow (\neg\beta \Rightarrow \neg\alpha)$ A1 instance

7. $\alpha, \neg\alpha \vdash (\neg\beta \Rightarrow \neg\alpha)$ MP 5,6.

8. $\alpha, \neg\alpha \vdash ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ MP 7,3

9. $\alpha, \neg\alpha \vdash \alpha \Rightarrow (\neg\beta \Rightarrow \alpha)$ A1 instance

10. $\alpha, \neg\alpha \vdash (\neg\beta \Rightarrow \alpha)$ MP 4,9

11. $\alpha, \neg\alpha \vdash \beta$ MP 10,8

12. $\alpha \vdash (\neg\alpha \Rightarrow \beta)$ ~~MP~~ Deduction Thm

13. $\vdash \alpha \Rightarrow (\neg\alpha \Rightarrow \beta)$ Deduction Thm

14. $\vdash \neg(\neg\alpha \Rightarrow \beta) \Rightarrow \neg\alpha$ using $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

15. $\vdash \neg\alpha$ MP 2,14.

hence α is inconsistent and β is inconsistent.

Proof:

$\alpha \vee \beta \in X \Leftrightarrow \alpha \vee \beta$ is sat $\Leftrightarrow \alpha \vee \beta$ is consistent (using claim 1 & 2 in part b) $\Leftrightarrow$ either α is

($\because \alpha \vee \beta$ is finite subset and X is maximal FSS)

~~case~~

~~either $\alpha \in X$ or $\beta \in X$~~ either α is sat or

consistent or β is consistent (proved above) $\Leftrightarrow$

β is sat (using claim 1 & 2 in part b) $\Leftrightarrow$ either $\alpha \in X$ or $\beta \in X$.

($\because \alpha, \beta$ are finite subsets of X) and X is maximal FSS.

hence proved.

(d) Proof: By induction on the structure of α.

Base case: $\alpha = p$, $p \in P$ [P is list of propositions]

Then $v_X \models p$ iff (by definition of v_X) $p \in X$.

Induction case:

There are 2 cases to consider:

i) α is of the form $\neg\beta$

$v_X \models \neg\beta$ iff (by def.) $v_X \not\models \beta$ iff (by I.H.) $\beta \notin X$ iff $\beta \in X$ (using part b)

ii) α is of the form $\beta \vee \gamma$.

$v_X \models \beta \vee \gamma$ iff (by def.) $v_X \models \beta$ or $v_X \models \gamma$ iff (by I.H.).

$\beta \in X$ or $\gamma \in X$ iff $(\beta \vee \gamma) \in X$.
(by part c)

Thus, in ~~all~~ all case, this holds, thus proved.
($\because$ $\{\neg, \vee\}$ are adequate (proved earlier) in class.
and all ϕ_i's in Φ are made of $\neg, \vee$)

(e) Take any FSS X.

by part a) X can be extended to maximal FSS ~~say~~ Y.

By part d) Y has a valuation v_Y s.t. for every formula α, $v_Y \models \alpha$ iff ~~$\alpha \in X$~~ $\alpha \in Y$.

Thus ~~Y~~ Y contains all formulae that were present in X, hence v_Y makes all formulae in X true.

Thus ~~$v_Y \models X$~~. $v_Y \models X$.

hence proved.

(b) $\Leftarrow$ If $\exists Y \subseteq_{fin} X$ s.t $Y \models \alpha$, then $X \models \alpha$.

Consider $v_X \models X$, then $v_X \models Y$ $(\because Y \subseteq_{fin} X)$.

Hence $v_X \models \alpha$. using given statement

$\therefore v_X \models X$, ~~means~~ implies $v_X \models \alpha$.

Thus $X \models \alpha$. proved.

$\Rightarrow$ If $X \models \alpha$, then $\exists Y \subseteq_{fin} X$ s.t $Y \models \alpha$.

$X \models \alpha$ implies $X \cup \{\neg\alpha\}$ is unsat (proved in claim below)

Claim: $X \models \alpha$ $\Longleftrightarrow$ $X \cup \{\neg\alpha\}$ is unsat.

Proof: $\Leftarrow$ If $Z \cup \{\neg\beta\}$ is unsat, then $Z \models \beta$.

Consider all valuations under which Z is true. Then under all

these valuations, $\neg\beta$ must be false since $Z \cup \{\neg\beta\}$ is unsat.
(Otherwise, we would have v_Z makes $Z \cup \{\neg\beta\}$ true which would mean $Z \cup \{\neg\beta\}$ is sat)

Thus under all valuation where Z is true, β must be true. (LEM)

Thus $Z \models \beta$ (proved)

$\Rightarrow$ If $Z \models \beta$, then $Z \cup \{\neg\beta\}$ is unsat.

Proof: Under all valuations v_Z under which Z is true, β must be true (def of $Z \models \beta$)

Hence $\neg\beta$ must be false under all these v_Z (LEM)

Thus whenever Z is true, $\neg\beta$ is false $\Rightarrow$ $Z \cup \{\neg\beta\}$ is false.

thus there is no valuation under which $\neg\beta$ is true when Z is true. On the other hand, Z is false then $Z \cup \{\neg\beta\}$ is also unsat. Hence proved.

Theorem: If $X \models \alpha$, $\exists\, Y \subseteq_{fin} X$ s.t. $Y \models \alpha$.

Proof: Given $X \models \alpha$.

$X \cup \{\neg\alpha\}$ is unsat.

Now ~~we~~ from part (e), we can say that if X is FSS, then $\exists\, V_X$ s.t $V_X \models X$., hence X is sat.

Thus ~~if X is FSS, then X is~~ if every finite subset of X is sat, then X is FSS, hence X is sat.

Thus if X is unsat, then $\exists\, Y \subseteq_{fin} X$ s.t Y is unsat.

(contrapositive).

$\because X \cup \{\neg\alpha\}$ is unsat.

$\exists\, Y \subseteq_{fin} X \cup \{\neg\alpha\}$ s.t. Y is unsat. Thus $(Y \setminus \{\neg\alpha\}) \cup \{\neg\alpha\}$ is unsat, where $(Y \setminus \{\neg\alpha\}) \subseteq_{fin} X$. This implies $Y \setminus \{\neg\alpha\} \models \alpha$.

hence proved.