Logic and Proof Hilary 2016

Exercise Sheet 1

Dr James Worrell

1. Let F, G and H be formulas and let $\boldsymbol{S}$ be a set of formulas. Which of the following statements are true? Justify your answer.
   (a) If F is unsatisfiable, then $\neg F$ is valid.
   (b) If $F \to G$ is satisfiable and F is satisfiable, then G is satisfiable.
   (c) $P_1 \to (P_2 \to (P_3 \to \ldots (P_n \to P_1) \ldots))$ is valid.
   (d) $\boldsymbol{S} \models F$ and $\boldsymbol{S} \models \neg F$ cannot both hold.
   (e) If $\boldsymbol{S} \models F \vee G$, $\boldsymbol{S} \cup \{F\} \models H$ and $\boldsymbol{S} \cup \{G\} \models H$, then $\boldsymbol{S} \models H$.

2. Let F and G be two formulas.
   (a) Explain the difference between F and G being **equisatisfiable** and them being **logically equivalent**.
   (b) Explain very briefly the difference between $F \leftrightarrow G$ and $F \equiv G$.

3. Give an equational proof of the following equivalence, justifying each step with reference to the Boolean algebra axioms and the Substitution Rule as appropriate.

$$\neg((\neg P \vee Q) \wedge P) \vee Q \equiv \textbf{true}$$

4. Suppose that F and G are formulas such that $F \models G$.
   (a) Show that if F and G have no variable in common then either F is unsatisfiable or G is valid.
   (b) Now let F and G be arbitrary formulas. Show that there is a formula H, mentioning only propositional variables common to F and G, such that $F \models H$ and $H \models G$.

   Hint. Recall that every truth table is realised by some propositional formula and consider what the truth table of H ought to look like: under which assignments must H be true and under which assignments must H be false?

5. A **perfect matching** in an undirected graph $G = (V, E)$ is a subset of the edges $M \subseteq E$ such that every vertex $v \in V$ is an endpoint of exactly one edge in M. Given a finite graph G, describe how to obtain a propositional formula φ_G such that φ_G is satisfiable if and only if G has a perfect matching. The formula φ_G should be computable from G in time polynomial in $|V|$.

6. Fix a non-empty set U. A U**-assignment** is a function from the collection of propositional variables to the power set of U, that is, $\mathcal{A}$ maps each propositional variable to a subset of U. Such an assignment is extended to all formulas as follows:

- $\mathcal{A}[\![\mathbf{false}]\!] = \emptyset$ and $\mathcal{A}[\![\mathbf{true}]\!] = U$;
- $\mathcal{A}[\![F \wedge G]\!] = \mathcal{A}[\![F]\!] \cap \mathcal{A}[\![G]\!]$;
- $\mathcal{A}[\![F \vee G]\!] = \mathcal{A}[\![F]\!] \cup \mathcal{A}[\![G]\!]$;
- $\mathcal{A}[\![\neg F]\!] = U \setminus \mathcal{A}[\![F]\!]$.

Say that a formula F is U-**valid** if $\mathcal{A}[\![F]\!] = U$ for all U-assignments $\mathcal{A}$.

(a) Show that if F is U-valid then F is valid with respect to the standard semantics defined in the lecture notes.

Hint: Show that each standard assignment $\mathcal{A}$ can be "simulated" by a certain U-assignment $\mathcal{A}'$.

(b) Show that if F is valid then F is U-valid.

Hint: Fix an arbitrary $u \in U$ and argue that $u \in \mathcal{A}[\![F]\!]$.

7. (a) Write down a **DNF**-formula equivalent to $(P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge \cdots \wedge (P_n \vee Q_n)$.

   (b) Prove that any **DNF**-formula equivalent to the above formula must have at least 2^n clauses.