COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 22 & 23 (Modal Logic)

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October 31st and November 7th

Why?

- In propositional logic, a valuation is a static assignment of truth values to atomic propositions.
- We may use atoms to describe properties of the current state of a program.
- In that case, the truth of an atom varies as the state changes.
- Modal logic is a framework to describe such a situation.

How?

- the idea is to look at a collection of possible valuations simultaneously
- each valuation represents a possible state of the world
- seperately, we specify how the possible worlds are connected to each other
- and enrich our logical language with a way of referring to truths across possible worlds

Syntax

Semantics

- a frame is a structure $F = (W, R)$
- W is a set of possible worlds; $R \subseteq W \times W$ is the accessibility relation
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- a model is a pair (F, V) where $F = (W, R)$ is a frame and $V : W \to 2^{\mathcal{P}}$ be a valuation
- the notion of truth is localised to each world in a model
- $M, w \models \alpha$ denotes that α is true at the world w in the model M

Satisfaction

Exercise

Verify that $M, w \vDash \Diamond\alpha$ iff there exists w' such that wRw', and $M, w' \vDash \alpha$.

Satisfiability and Validity

Examples of valid formulas

- every tautology of propositional logic is valid
- $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ is valid
- if α is valid, $\Box\alpha$ is also valid

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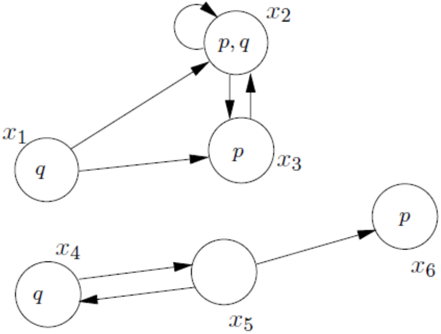
- $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ is valid
- if α is valid, $\Box\alpha$ is also valid
- what about substitution instances of propositional tautologies (e.g. $\Box\alpha \vee \neg\Box\alpha$)?

Example



Formula schemes

- it is sometimes useful to talk about a whole family of formulas which have the same 'shape'
- these are called formula schemes; e.g. $\phi \to \Box\Diamond\phi$
- $p \to \Box\Diamond p$, $q \to \Box\Diamond q$, and $(p \wedge \Diamond q) \to \Box\Diamond(p \wedge \Diamond q)$ are all instances of the above formula
- may think of them as under-specified parse-trees

Formula schemes

- semantically, a scheme can be thought of as a conjunction of all its instances
- we say that a world/model satisfies a scheme if it satisfies all its instances
- an instance being satisfied in a model does not imply that the entire scheme is satisfied in the model
- in contrast, if a frame satisfies an instance, it satisfies the entire scheme
- why? because frames have no information about truth or falsity of atomic propositions

Correspondence theory

- the modalities $\Box$ and $\Diamond$ can be used to describe interesting propoerties of the accessibility relation R of a frame (W, R)

- for a modal logic formula α, we identify a class of frames $\mathcal{C}_\alpha$ as follows:
  $F = (W, R) \in \mathcal{C}_\alpha$ iff for every valuation V over W, for every $w \in W$, and for every substitution instance β of α,
  $((W, R), V), w \models \beta$

- a class of frames $\mathcal{C}$ is characterized by a formula α if $\mathcal{C} = \mathcal{C}_\alpha$

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- the class of reflexive frames is characterised by the formula $\Box\alpha \rightarrow \alpha$

Exercise

Prove that the class of transitive frames is characterized by the formula $\Box\alpha \to \Box\Box\alpha$.

Proof:

Dual formulas

- transitivity $\Diamond\Diamond\alpha \to \Diamond\alpha$
- reflexivity ?

Exercise

The class of symmetric frames is characterized by the formula $\alpha \to \Box\Diamond\alpha$.

Axiomatizing Valid Formulas

- a formula α is valid if for every frame $F = (W, R)$, every model $M = (F, V)$ and every world $w \in W$, $M, w \vDash \alpha$
- we can refine this notion by restricting to a class of frames
- α is $\mathcal{C}$-valid if for every frame $F = (W, R)$ from the class $\mathcal{C}$, for every model ...
- dually, we can talk about $\mathcal{C}$-satisfiability

Axiomatizing Valid Formulas

Using Hilbert's system to prove soundness and completeness.

Consider the following axiom system K.

Hilbert's axioms for propositional logic (A1–A3) $+ \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ (K)

Inference rules:

Modus ponens (MP) $+$ from α, derive $\Box\alpha$ (G)

Soundness and Completeness for system K: For all formulas α, $\vdash_K \alpha$ *iff* $\models \alpha$.

Completeness for other classes of frames

Let system T be the set of axioms obtained by adding the following axiom to system K.

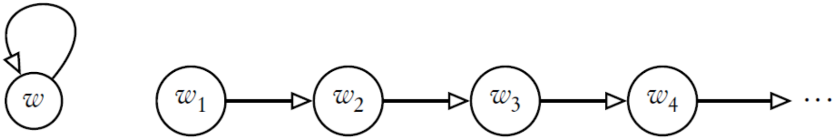
$\Box\alpha \rightarrow \alpha$ (T)

System T is sound and complete with respect to the class of reflexive frames.

Expressiveness

The class of irreflexive frames cannot be characterised in modal logic.

Intuition:
Consider the two frames shown here:



Suppose we choose the same valuation for all the worlds in these two frames.

Now, the claim is that the two models thus obtained (frames + valuation) cannot be distinguished by any modal logic formula. (We aren't going to formalize this, but the idea behind this proof is to induct on the structure of the formula.)

Expressiveness

The class of irreflexive frames cannot be characterised in modal logic.

Intuition: Let α be a formula that characterises irreflexive frames.

Since the first frame is not irreflexive, there must be a β, instance of α, such that β does not hold in w (for some model, i.e. under some valuation).

Let us choose the same valuation for every w_i in the second frame. Clearly, β is not satisfied at any w_i in the resulting model.

But this is a contradiction – because β is an instance of α, which characterises irreflexive frames, and the second frame is irreflexive!

Next class

- Binary Decision Diagrams

Next week

- FOL: Soundness and Completeness or Decidable Theories

Thank you!