Logic and Proof Hilary 2016

Examples of Ground Resolution Proofs

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In this lecture we show how to use the Ground Resolution Theorem, proved in the last lecture, to do some deduction in first-order logic.

1 Ground Resolution Theorem

Recall that the process of eliminating existential quantifiers by introducing extra function and constant symbols is called *Skolemisation*. The extra symbols introduced are called *Skolem functions*. We begin with a slight generalisation of a theorem that was stated in the previous lecture. In this generalisation we consider Skolemising a collection of formulas rather than a single formula.

Theorem 1. Let $F_1, \ldots, F_n$ be closed rectified formulas in prenex form with respective Skolem forms $G_1, \ldots, G_n$. Assume that each G_i is constructed using a different set of Skolem functions. Then $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is satisfiable if and only if $G_1 \wedge G_2 \wedge \ldots \wedge G_n$ is satisfiable.

Recall that a ground term is a term that does not contain any variables. Given a quantifier-free formula F, a *ground instance* of F is a formula obtained by replacing all the variables in F with ground terms.

The following is a slight generalisation of the version of the Ground Resolution Theorem proved in the last lecture. Before we considered only a single formula in Skolem form. Here we consider a conjunction of such formulas, which is more convenient for the applications below.

Theorem 2 (Ground Resolution Theorem)**.** Let $F_1, \ldots, F_n$ be closed formulas in Skolem form whose respective matrices $F_1^* \wedge \ldots \wedge F_n^*$ are in **CNF**. Then $F_1 \wedge \ldots \wedge F_n$ is unsatisfiable if and only if there is a propositional resolution proof of $\square$ from the set of ground instances of clauses from $F_1^*, \ldots, F_n^*$.

2 Examples

In this section we give two examples of the use of the Ground Resolution Theorem.

Example 3. We would like to formalise the following statements in first-order logic and to use ground resolution to show that (a), (b) and (c) together entail (d).

(a) Everyone at Oriel is either lazy, a rower or a drunk.

(b) All rowers are lazy.

(c) Someone at Oriel is not drunk.

(d) Someone at Oriel is lazy.

$$\cfrac{\cfrac{\cfrac{\cfrac{\{\neg R(a), L(a)\} \qquad \{\neg O(a), L(a), R(a), D(a)\}}{\{L(a), \neg O(a), D(a)\}} \qquad \{\neg O(a), \neg L(a)\}}{\{\neg O(a), D(a)\}} \qquad \{\neg D(a)\}}{\{\neg O(a)\}} \qquad \{O(a)\}}{\Box}$$

Figure 1: The nature of Oriel students

We translate (a), (b), (c) and the negation of (d) into closed formulas of first-order logic as follows.

$$\begin{aligned}
F_1 &= \forall x\,(O(x) \to (L(x) \vee R(x) \vee D(x)))\\
F_2 &= \forall x\,(R(x) \to L(x))\\
F_3 &= \exists x\,(O(x) \wedge \neg D(x))\\
F_4 &= \neg\exists x\,(O(x) \wedge L(x))\,.
\end{aligned}$$

Next we translate F_1, F_2, F_3 and F_4 to Skolem form. To do this we bring all quantifiers to the outside, eliminate existential quantifiers by introducing Skolem functions and finally bring the matrix of each formula into **CNF**. This yields

$$\begin{aligned}
G_1 &= \forall x\,(\neg O(x) \vee L(x) \vee R(x) \vee D(x))\\
G_2 &= \forall x\,(\neg R(x) \vee L(x))\\
G_3 &= O(a) \wedge \neg D(a)\\
G_4 &= \forall x\,(\neg O(x) \vee \neg L(x))\,.
\end{aligned}$$

where a is a fresh constant symbol.

Now we deduce the empty clause $\Box$ from ground instances of clauses in the respective matrices of the Skolem-form formulas $G_1, \ldots, G_4$. Note that these formulas are defined over a signature with a single constant symbol a, which is therefore the only ground term. The proof is shown in Figure 1.

Example 4. Using ground resolution we show that

$$\forall x\,\exists y\,(P(x) \to Q(y)) \to \exists y\,\forall x\,(P(x) \to Q(y))$$

is a valid sentence.

We can show this by showing that the negation is unsatisfiable. The negation can be written:

$$\forall x\,\exists y\,(P(x) \to Q(y)) \wedge \neg\exists y\,\forall x\,(P(x) \to Q(y))\,.$$

We bring each conjunction to Skolem form, yielding

$$\begin{aligned}
F_1 &= \forall x\,(\neg P(x) \vee Q(f(x)))\\
F_2 &= \forall y\,(P(g(y)) \wedge \neg Q(y))\,.
\end{aligned}$$

Note that F_1 and F_2 are defined over a signature with no constants and so there are no ground terms. We remedy this problem by introducing a single new constant symbol a. Now the set of ground terms is $\{a, f(a), g(a), f(f(a)), f(g(a), \ldots\}$. We can now derive $\Box$ by the propositional resolution proof in Figure 2 which every leaf is a ground instance of a clause from the respective matrices of F_1 and F_2.

$$\dfrac{\dfrac{\{P(g(a))\} \qquad \{\neg P(g(a)), Q(f(g(a)))\}}{\{Q(f(g(a)))\}} \qquad \{\neg Q(f(g(a)))\}}{\Box}$$

Figure 2: Ground Resolution proof for Example 4