COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 15 & 16 (Predicate Logic – Semantics, Normal Forms)

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October 6th and 10th, 2022

Semantics

$$\forall x \forall y\ (P(x, y) \to \exists z\ (\neg(z = x) \land \neg(z = y) \land P(x, z) \land P(z, y)))$$

- suppose variables take values from real numbers
- P(x, y) represents $x < y$

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- suppose variables take values from real numbers
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$$\forall x \forall y\ (P(x, y) \to \exists z\ (\neg(z = x) \wedge \neg(z = y) \wedge P(x, z) \wedge P(z, y)))$$

- suppose variables take values from natural numbers
- P(x, y) represents $x < y$

Semantics

Reference material:

https://www.cs.ox.ac.uk/people/james.worrell/lecture9-2015.pdf

σ-structure

Given a signature σ, a σ-*structure* (or *assignment*) $\mathcal{A}$ consists of:

- a non-empty set $U_{\mathcal{A}}$ called the *universe* of the structure;
- for each k-ary predicate symbol P in σ, a k-ary relation $P_{\mathcal{A}} \subseteq \underbrace{U_{\mathcal{A}} \times \cdots \times U_{\mathcal{A}}}_{k}$;
- for each k-ary function symbol symbol f in σ, a k-ary function, $f_{\mathcal{A}} : \underbrace{U_{\mathcal{A}} \times \cdots \times U_{\mathcal{A}}}_{k} \to U_{\mathcal{A}}$;
- for each constant symbol c, an element $c_{\mathcal{A}}$ of $U_{\mathcal{A}}$;
- for each variable x an element $x_{\mathcal{A}}$ of $U_{\mathcal{A}}$.

Evaluating terms

- For a constant symbol c we define $\mathcal{A}[\![c]\!] \stackrel{\text{def}}{=} c_{\mathcal{A}}$.
- For a variable x we define $\mathcal{A}[\![x]\!] \stackrel{\text{def}}{=} x_{\mathcal{A}}$.
- For a term $f(t_1, \ldots, t_k)$, where f is a k-ary function symbol and $t_1, \ldots, t_k$ are terms, we define $\mathcal{A}[\![f(t_1, \ldots, t_k)]\!] \stackrel{\text{def}}{=} f_{\mathcal{A}}(\mathcal{A}[\![t_1]\!], \ldots, \mathcal{A}[\![t_k]\!])$.

Satisfaction relation

1. $\mathcal{A} \models P(t_1, \ldots, t_k)$ if and only if $(\mathcal{A}[\![t_1]\!], \ldots, \mathcal{A}[\![t_k]\!]) \in P_{\mathcal{A}}$.
2. $\mathcal{A} \models (F \wedge G)$ if and only if $\mathcal{A} \models F$ and $\mathcal{A} \models G$.
3. $\mathcal{A} \models (F \vee G)$ if and only if $\mathcal{A} \models F$ or $\mathcal{A} \models G$.
4. $\mathcal{A} \models \neg F$ if and only if $\mathcal{A} \not\models F$.
5. $\mathcal{A} \models \exists x\, F$ if and only if there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \models F$.
6. $\mathcal{A} \models \forall x\, F$ if and only if $\mathcal{A}_{[x \mapsto a]} \models F$ for all $a \in U_{\mathcal{A}}$.

If we are working in first-order logic with equality then we additionally have

7. $\mathcal{A} \models t_1 = t_2$ if and only if $\mathcal{A}[\![t_1]\!] = \mathcal{A}[\![t_2]\!]$.

Some definitions

- quantifier-depth of a formula

  atomic formulas have 0 qd; $\text{qd}(\neg\phi) = \text{qd}(\phi)$;
  $\text{qd}(\phi \vee \psi) = \text{qd}(\phi \wedge \psi) = \mathit{max}(\text{qd}(\phi), \text{qd}(\psi))$
  $\text{qd}(\exists x\ \phi) = \text{qd}(\forall x\ \phi) = 1 + \text{qd}(\phi)$

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- ground terms – variable-free terms

- a closed formula or a sentence – formula with no free variables

  e.g. $\forall x \forall y \forall z\ (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ is closed
  $\forall x\ (x < (y + 1))$ is not closed

Examples

- undirected graph as a σ-structure (σ containing one binary relation E), with the interpretation of E as the edge relation

- σ with one binary relation $<$, interpreted in the usual way over integers, rationals, reals

More definitions

- a first-order formula F over σ is satisfiable if there is a σ-structure $\mathcal{A}$ such that $\mathcal{A} \models F$
- if F is not satisfiable, it is called unsatisfiable
- F is called valid if (and only if) $\neg F$ is unsatisfiable
- for a set of formulas $\mathcal{S}$, we say $\mathcal{S} \models F$ to mean that every σ-structure $\mathcal{A}$ that satisfies $\mathcal{S}$ also satisfies F

Exercise

Consider a signature σ with a constant symbol 0, a unary function symbol s, and a unary predicate symbol P.

Is $P(0) \wedge \forall x\ (P(x) \rightarrow P(s(x))) \wedge \exists x\ \neg P(x)$ satisfiable?

Relevance Lemma

Suppose $\mathcal{A}$ and $\mathcal{A}'$ are σ-assignments with the same universe, and identical interpretation of the predicate, function, and constant symbols in σ.

If $\mathcal{A}$ and $\mathcal{A}'$ give the same interpretation to each variable occurring free in some σ-formula F,

then $\mathcal{A} \models F$ iff $\mathcal{A}' \models F$.

Proof: reading exercise (page 5 of `https://www.cs.ox.ac.uk/people/james.worrell/lecture9-2015.pdf`)

Special case

If F is a closed formula (or a sentence), and $\mathcal{A}$ and $\mathcal{A}'$ are assignments that only differ in interpretation of variables,

then $\mathcal{A} \models F$ iff $\mathcal{A}' \models F$.

Logical equivalence

First-order formulas F and G are logically equivalent, denoted $F \equiv G$, if for all σ-assignments $\mathcal{A}$, we have $\mathcal{A} \models F$ iff $\mathcal{A} \models G$.

Example

$$\neg\forall x\ F \;\equiv\; \exists x \neg F$$

Example

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$$(\forall x\ F \wedge G) \ \equiv\ \forall x\ (F \wedge G) \quad \text{(if } x \text{ does not occur free in } G\text{)}$$

Prenex Normal Form

A predicate logic formula is in prenex normal form if it written as a string of quantifiers and bound variables, called the prefix, followed by a quantifier-free part, called the matrix.

$$Q_1 y_1 Q_2 y_2 \ldots Q_n y_n \; F$$

where $Q_i \in \{\forall, \exists\}$, $n \geq 0$, and F contains no quantifiers.

Every formula is equivalent to a formula in prenex normal form.

Example

$$\neg(\exists x \; P(x, y) \vee \forall z \; Q(z)) \wedge \exists w \; Q(w)$$

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F,

then $\mathcal{A} \models F[t/x] \quad \text{iff} \quad \mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Renaming bound variables

Let F denote the formula $Qx\ G$ where Q is a quantifier. Let y be a variable that does not occur in G.

Then $F \equiv Qy\ (G[y/x])$.

Rectified formulas

A formula is rectified if no variable occurs both bound and free and if all quantifiers in the formula refer to different variables.

You can rectify a formula by renaming bound variables.

Example: $\forall x \, \exists y \; P(x, f(y)) \;\wedge\; \forall y \; (Q(x, y) \vee R(x))$

Every formula is equivalent to a rectified formula in prenex form.

Skolem Form

We say that a rectified prenex formula is in Skolem form if it does not contain any occurrence of the existential quantifier.

A rectified prenex formula can be transformed to an equisatisfiable formula in Skolem form by using extra function symbols.

$\forall x\ \exists y\ P(x, y)$ and $\forall x\ P(x, f(x))$ are equisatisfiable.

Skolem Form

Let $F = \forall y_1 \, \forall y_2 \, \ldots \, \forall y_n \, \exists z \; G$ be a rectified formula. Given a function symbol f of arity n that does not appear in F, write

$$F' = \forall y_1 \, \forall y_2 \, \ldots \, \forall y_n \; G[f(y_1, y_2, \ldots, y_n)/z].$$

Then F and F' are equisatisfiable.

Example

$$\forall x\ \exists y\ \forall z\ \exists w\ (\neg P(a, w) \vee Q(f(x), y))$$

Next week

- Herbrand theorem, Ground resolution
- Undecidability result

Thank you!