COL703 Quiz 4

Sreemanti Dey

TOTAL POINTS

3 / 4

QUESTION 1

1 Q1 **1 / 1**

- ✓ **+ 1 pts** **Correct**
- **+ 0.5 pts** Partially correct
- **+ 0 pts** Incorrect

QUESTION 2

2 Q2 **0 / 1**

- **+ 1 pts** Correct
- **+ 0.5 pts** Partially Correct
- ✓ **+ 0 pts** **Incorrect**

QUESTION 3

3 Q3 **1 / 1**

- ✓ **+ 1 pts** **Correct**
- **+ 0.5 pts** Partially Correct
- **+ 0 pts** Incorrect

QUESTION 4

4 Q4 **1 / 1**

- ✓ **+ 1 pts** **Correct**
- **+ 0.5 pts** Partially correct
- **+ 0 pts** Incorrect

total sheets = 2 [Main sheet + 1 extra sheet]

1 + 0 + 1 + 1 = 3

Quiz 4 *(Sep 19, Marks: 4, Duration: 50 mins)* **COL703, Aug-Nov 2022**

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1. **[1 marks]** Prove the following lemma, which we had used for proving the Compactness theorem (recall that Φ was used to denote the set of propositional logic formulas):

For all $Z \subseteq \Phi$ and all $\beta \in \Phi$, $Z \models \beta$ iff $Z \cup \{\neg\beta\}$ is not satisfiable.

2. **[1 marks]** Show that if $\alpha \wedge \beta$ is consistent, then both α and β are consistent. Recall that α is said to be consistent if $\nvdash \neg\alpha$.

3. **[1 marks]** Consider the CNF formula ϕ shown below. Write a 3-CNF formula ψ such that ψ is satisfiable *iff* ϕ is satisfiable.

$\phi = (\neg x_1 \vee \neg x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1) \wedge (x_1)$

4. **[1 marks]** Let $X \subseteq \Phi$ and $\alpha \in \Phi$, where Φ denotes the set of propositional logic formulas. The use of strong completeness ($X \models \alpha$ *iff* $X \vdash \alpha$) can lead to an alternative, shorter, proof of compactness ($X \models \alpha$ *iff there exists* $Y \subseteq_{fin} X$, $Y \models \alpha$). Give that proof.

1) $\Rightarrow$ If $Z \cup \{\neg\beta\}$ is unsat, then $Z \models \beta$.

Proof: Consider all valuations under which Z is true.
Then under all these valuations $\neg\beta$ must be false. since $Z \cup \{\neg\beta\}$ is unsat. (otherwise, we would have v_Z makes $Z \cup \{\neg\beta\}$ true, which would mean $Z \cup \{\neg\beta\}$ is sat)

Thus under all valuations where Z is true,
β must be true (LEM).

Thus. $Z \models \beta$. (proved)

$\Leftarrow$ If $Z \models \beta$, then $Z \cup \{\neg\beta\}$ is unsat.

Proof: Under all valuations v_Z under which Z is true,
β must be true (def of $Z \models \beta$).
Hence $\neg\beta$ must be false under all these v_Z. (LEM)
Thus whenever Z is true, $\neg\beta$ is false $\Rightarrow Z \cup \{\neg\beta\}$ is false,
thus there is no valuation under which $\neg\beta$ is true when Z is true. On the other hand, when Z is false, then $Z \cup \{\neg\beta\}$ is also unsat. Hence proved.

2) Proving the contrapositive:

If either α or β is inconsistent, then $\alpha \wedge \beta$ is inconsistent.

WLG, consider α is inconsistent,

then $\vdash \neg\alpha$, (by def).

We need to prove $\vdash \neg(\alpha \wedge \beta)$

$\vdash \neg\alpha \vee \neg\beta$

Using natural deduction for proof.

1. $\neg\alpha$ premise
2. $\neg\alpha \vee \neg\beta$ $\vee_{i_1}$ 1.

No, Sreemanti. As we had discussed, $\vee$-intro makes this too simple. The exercise was to use Hilbert's style proof. This was unstated, for which I apologize, but it should have been clear as we did a similar exercise in the class (and had discussed how natural deduction is not to be used for this).

hence $\neg\alpha \vee \neg\beta$ is also derivable if $\neg\alpha$ is derivable

hence $(\alpha \wedge \beta)$ is inconsistent, hence proved.

Similarly for β.

proved.

3) $\cancel{\phi_1 = (\neg x_1 \wedge \neg x_4)}$

$\cancel{\psi_1 = (\neg x_1 \wedge \neg x_4 \wedge u_1)}$

$\phi_1 = (\neg x_1 \vee \neg x_4)$

$\psi_1 = (\neg x_1 \vee \neg x_4 \vee u_1) \wedge (\neg x_1 \vee \neg x_4 \vee \neg u_1)$

where u_1, q_1, p_1, p_2 are literals. (atomic prop.).

$\phi_2 = (x_1 \vee \neg x_2 \vee \neg x_3)$

$\psi_2 = (x_1 \vee \neg x_2 \vee \neg x_3)$

$\phi_3 = (\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1)$

$\psi_3 = \cancel{(\neg x_2 \vee \neg x_3 \vee v_1) \wedge (x_4 \vee x_1 \vee \neg v_1)}\ (\neg x_2 \vee \neg x_3 \vee q_1) \wedge (x_4 \vee x_1 \vee \neg q_1)$

$\phi_4 = x_1$

$\psi_4 = \cancel{x_1 \vee u_1 \vee p_2}\ (x_1 \vee p_1 \vee p_2) \wedge (x_1 \vee \neg p_1 \vee p_2) \wedge (x_1 \vee \neg p_1 \vee \neg p_2) \wedge (x_1 \vee p_1 \vee \neg p_2)$

Thus

$\cancel{\psi = (\neg x_1 \vee \neg x_4 \vee u_1) \wedge (\neg x_1 \vee \neg x}$

$\psi = \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$.

$= (\neg x_1 \vee \neg x_4 \vee u_1) \wedge (\neg x_1 \vee \neg x_4 \vee \neg u_1) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3 \vee q_1)$
$\wedge (x_4 \vee x_1 \vee \neg q_1) \wedge (x_1 \vee p_1 \vee p_2) \wedge (x_1 \vee \neg p_1 \vee p_2) \wedge$
$(x_1 \vee \neg p_1 \vee \neg p_2) \wedge (x_1 \vee p_1 \vee \neg p_2)$

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4) $\Leftarrow$ ~~$\Rightarrow$~~ If $\exists Y \subseteq_{fin} X, Y \models \alpha$, then $X \models \alpha$.

Proof: using strong completeness,

$Y \models \alpha$ means $Y \vdash \alpha$.

Thus there is a derivation for α using Y, say the steps are $\{\beta_1, \beta_2 \ldots \beta_n = \alpha\}$ thus, using this derivation. (since ~~prem~~ $Y \subseteq X$, thus all formulae in Y are in X hence using that derivation we have $X \vdash \alpha$)

Thus $X \models \alpha$ (strong completeness)

Sure. You could have also argued this using valuations (see page 13, bottom, logic notes).

$\Rightarrow$ ~~$\Leftarrow$~~ If $X \models \alpha$, then $\exists Y \subseteq_{fin} X, Y \models \alpha$.

this is the forward direction

Proof: By strong completeness,

$X \vdash \alpha$.

Now since $X \vdash \alpha$, thus we must have a sequence of derivation such that the last derivation (formula) gives α.

If the seq had been ∞, we would not have been able to find/determine the last element of the sequence but here we know, hence the derivation must be finite

$\Rightarrow$ ~~X must b~~ The derivation must be using a finite subset of formulae in X, call it Y.

hence $Y \vdash \alpha$.

again by strong completeness,

$Y \models \alpha$.