

Exercise Sheet 3

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1. Write down formulas in first-order logic with equality expressing the following requirements.

- (a) Unary function f is an onto function.
- (b) Binary relation R is an equivalence relation.
- (c) Unary predicate P holds for exactly three different elements.

2. Consider the following structures over a signature with a single binary relation symbol R :

$$U_{\mathcal{A}} = \mathbb{N} \text{ and } R_{\mathcal{A}} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n < m\}$$

$$U_{\mathcal{B}} = \mathbb{Z} \text{ and } R_{\mathcal{B}} = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n < m\}$$

$$U_{\mathcal{C}} = \mathbb{Q} \text{ and } R_{\mathcal{C}} = \{(n, m) \in \mathbb{Q} \times \mathbb{Q} : n < m\}$$

Give a formula that is satisfied by \mathcal{B} but not by \mathcal{A} , and a formula that is satisfied by \mathcal{C} but not by \mathcal{B} .

3. Translate the following formula to rectified form, then to prenex form, and finally to Skolem form:

$$\forall z \exists y (Q(x, g(y), z) \vee \neg \forall x P(x)) \wedge \neg \forall z \exists x \neg R(f(x, z), z).$$

4. Are the following claims correct? Justify your answers.

- (a) For any formula F and term t , if F is valid then $F[t/x]$ is valid.
- (b) $\exists x (P(x) \rightarrow \forall y P(y))$ is valid.
- (c) For any formula F and constant symbol c , if $F[c/x]$ is valid and c does not appear in F then $\forall x F$ is valid.

5. Let σ be a signature with finitely many relation and constant symbols, but no function symbols.

- (a) Given σ -formulas G_1, \dots, G_n and a propositional formula F that mentions variables P_1, \dots, P_n , let $F[G_1/P_1, \dots, G_n/P_n]$ denote the σ -formula obtained by substituting G_i for all occurrences of P_i in F . Give a formal definition of $F[G_1/P_1, \dots, G_n/P_n]$.
- (b) Given propositional formulas $F \equiv F'$, both over variables P_1, \dots, P_n , and σ -formulas $G_1 \equiv G'_1, \dots, G_n \equiv G'_n$, show that $F[G_1/P_1, \dots, G_n/P_n] \equiv F'[G'_1/P_1, \dots, G'_n/P_n]$.
- (c) Fix $n \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many quantifier-free σ -formulas that use first-order variables x_1, \dots, x_n .
- (d) Fix $n, k \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many σ -formulas of quantifier depth at most k that use first-order variables x_1, \dots, x_n .

6. Fix a signature σ . Consider a relation \sim on σ -assignments that satisfies the following two properties:

- (P1) If $\mathcal{A} \sim \mathcal{B}$ then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
- (P2) If $\mathcal{A} \sim \mathcal{B}$ then for each variable x we have (i) for each $a \in U_{\mathcal{A}}$ there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$, and (ii) for all $b \in U_{\mathcal{B}}$ there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

Prove that if $\mathcal{A} \sim \mathcal{B}$ then for any formula F , $\mathcal{A} \models F$ if and only if $\mathcal{B} \models F$. You may assume that F is built from atomic formulas using the connectives \wedge and \neg and the quantifier \exists .

7. In this question we work with first-order logic without equality.

- (a) Consider a signature σ containing only a binary relation symbol R . For each positive integer n show that there is a satisfiable σ -formula F_n such that every model \mathcal{A} of F_n has at least n elements.
- (b) Consider a signature σ containing only unary predicate symbols P_1, \dots, P_k . Using Question 6, or otherwise, show that any satisfiable σ -formula has a model with at most 2^k elements.