Assignment 4 (~~*Marks: 7*~~ ***Diwali Offer*** *Marks: 14, Due date: Nov. 01*) **COL703, Aug-Nov 2022**

Please note that there will be zero tolerance for dishonest means like copying solutions from others, and even letting others copy your solution, in any assignment/quiz/exam. If you are found indulging in such an activity, your answer-paper/code will not be evaluated and your participation/submission will not be counted. Second-time offenders will be summarily awarded an F grade. The onus will be on the supposed offender to prove his or her innocence.

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1. [~~**1 marks**~~ ***Diwali offer*: 2 marks**] Formalise the following as sentences of first order logic. Use $B(x)$ for "x is a barber", and $S(x,y)$ for "x shaves y".

   (a) Every barber shaves all persons who do not shave themselves.

   (b) No barber shaves any person who shaves himself.

   Convert your answers to Skolem form and use ground resolution to show that (c), given below, is a consequence of (a) and (b).

   (c) There are no barbers.

2. [~~**1.5 marks**~~ ***Diwali offer*: 3 marks**] Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties.

   - If $\mathcal{A} \sim \mathcal{B}$ then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
   - If $\mathcal{A} \sim \mathcal{B}$ then for each variable x we have (i) for each $a \in U_{\mathcal{A}}$ there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$, and (ii) for all $b \in U_{\mathcal{B}}$ there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

   Prove that if $\mathcal{A} \sim \mathcal{B}$ then for any formula F, $\mathcal{A} \models F$ iff $\mathcal{B} \models F$. You may assume that F is built from atomic formulas using the connectives $\wedge$ and $\neg$, and the quantifier $\exists$.

3. [~~**1 marks**~~ ***Diwali offer*: 2 marks**] Express the following by formulas of first-order logic, using predicates $H(x)$ for "x is happy", $R(x)$ for "x is rich", $G(x)$ for "x is a graduate", and $C(x,y)$ for "y is a child of x".

   (a) Any person is happy if all their children are rich.

   (b) All graduates are rich.

   (c) Someone is a graduate if they are a child of a graduate.

   (d) All graduates are happy.

   Use first-order resolution to show that (d) is entailed by (a), (b), and (c). Indicate the substitutions in each resolution step.

4. [~~**1 marks**~~ ***Diwali offer*: 2 marks**] Let $A(x_1, \ldots, x_n)$ be a formula with no quantifiers and no function symbols. Prove that $\forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n)$ is satisfiable if and only if it is satisfiable in an interpretation with there being just one element in the universe.

5. In this question, we work with first-order logic without equality.

   (a) [~~**1 marks**~~ ***Diwali offer*: 2 marks**] Consider a signature σ containing only a binary relation R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model $\mathcal{A}$ of F_n has at least n elements.

(b) [~~**1.5 marks**~~ ***Diwali offer*: 3 marks**] Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using the question 2 (above), or otherwise, show that any satisfiable σ-formula has a model where the universe has at most 2^k elements.