

Exercise Sheet 5

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1. Consider the theory \mathbf{T} of the structure $\mathcal{A} = (\mathbb{N}, 0, s, <)$, where s is the unary function given by $s(n) = n + 1$.
 - (a) Suppose that F is a conjunction of atomic formulas, all of which mention the variable x . Show that there is a quantifier-free formula G such that $\mathbf{T} \models \exists x F \leftrightarrow G$.
 - (b) By following the reasoning in the lecture notes, conclude from (i) that \mathbf{T} has quantifier elimination.
 - (c) Say that $S \subseteq \mathbb{N}$ is *definable* if there is a formula F with one free variable x such that $\mathcal{A}_{[x \mapsto a]} \models F$ if and only if $a \in S$. Given that \mathbf{T} has quantifier elimination, show that the definable subsets of \mathbb{N} are the finite and cofinite subsets of \mathbb{N} .
2. This question concerns the theory \mathbf{T} of the structure $\mathcal{A} = (\mathbb{N}, 0, 1, +, <, \{P_k\}_k)$, where for each integer $k > 1$, the unary predicate $P_k(n)$ holds if and only if n is divisible by k . You are given that \mathbf{T} has quantifier elimination.
 - (a) Say that a set $S \subseteq \mathbb{N}$ is *ultimately periodic* if there exist positive integers n_0 and p such that for all $n \geq n_0$, $n \in S$ iff $n + p \in S$. Show that any quantifier-free formula that mentions a single variable x defines an ultimately periodic subset of \mathbb{N} .
 - (b) Using your answer to part (a), or otherwise, show that there is no formula on free variables x, y and z that defines the multiplication relation $M = \{(a, b, c) \in \mathbb{N}^3 : ab = c\}$ on the structure \mathcal{A} .