Assignment 4

① a. $\forall y \forall x \, (B(x) \land \neg S(y,y) \rightarrow S(x,y))$

b. $\forall x \forall y \, (B(x) \land S(y,y) \longrightarrow \neg B(x \quad \neg S(x,y))$

c. $\forall x \neg B(x)$

~~Skolem form~~

a. $\forall y \forall x \, (B(x) \land \neg S(y,y) \rightarrow S(x,y))$

$\forall y \forall x \, (\neg(B(x) \land \neg S(y,y)) \lor S(x,y))$

$\forall y \forall x \, (\neg B(x) \lor S(y,y) \lor S(x,y))$

b. $\forall x \forall y \, (\neg(B(x) \land S(y,y)) \lor \neg S(x,y))$

$\forall x \forall y \, ((\neg B(x)) \lor (\neg S(y,y)) \lor (\neg S(x,y)))$

c. $\forall x \neg B(x)$

$(\neg c)$ is $\exists x \, B(x)$

1. $\{\neg B(a) \; S(b,b) \; S(a,b)\}$

2. $\{\neg B(a), \neg S(b,b), \neg S(a,b)\}$

3. $\{B(a)\}$

4. $\{\neg S(b,b), \neg S(a,b)\}$ Resolving 2,3

5. $\{S(b,b), S(a,b)\}$ Resolving 1,3.

6. $\{\square\}$ Resolving 4,5.

hence proved.

2

Induction on structure of F.

Base case: $A \models F$ iff $B \models F$ when F is atomic formula. (1st definition of $A \sim B$)
(given)

Induction case:

a) $F = \neg F_1$

$A \models F$ iff $A \models \neg F_1$ iff $\underbrace{A \not\models F_1 \text{ iff } B \not\models F_1}_{\text{Induction hypothesis}}$ iff $B \models \neg F_1$.
iff $B \models F$

hence proved.

b) $F = F_1 \wedge F_2$

$A \models F$ iff $A \models F_1 \wedge F_2$ iff $A \models F_1$ and $A \models F_2$
iff $B \models F_1$ and $B \models F_2$ iff $B \models F_1 \wedge F_2$ iff $B \models F$

By induction hypothesis.

hence proved.

c) $F = \exists x F_1$

$A \models F$ iff $A \models \exists x F_1$ iff $A_{[x \mapsto a]} \models F_1$.

iff $B_{[x \mapsto b]} \models F_1$ iff $B \models \exists x F_1$ iff $B \models F$.

(by definition of $A \sim B$ and Induction hypothesis)

hence proved.

(3) a) $\forall x((\forall y(C(x,y) \rightarrow R(y))) \rightarrow H(x))$

$\forall x((\forall y(\neg C(x,y) \vee R(y)) \rightarrow H(x))$

$\forall x(\neg(\forall y(\neg C(x,y) \vee R(y))) \vee H(x))$

$\forall x(\exists y(C(x,y) \wedge \neg R(y)) \vee H(x))$

$\forall x((C(x,f(x)) \wedge \neg R(f(x))) \vee H(x))$

~~$\forall x((C(x,f(x)) \vee H(x)) \wedge (C(x,f(x)) \vee H(x)$~~

$\forall x((C(x,f(x)) \vee H(x)) \wedge (\neg R(f(x)) \vee H(x)))$

~~$\{C(a,f(a)), H(a)\}, \{\neg R(f(a)), H(a)\}$~~

$\{C(x,f(x)), H(x)\}$ $\{\neg R(f(x)), H(x)\}$.

b) $\forall x((G(x) \rightarrow R(x))$

$\forall x(\neg G(x) \vee R(x))$

$\{\neg G(x), R(x)\}$

~~$\{\neg G(a), R(a)\}$~~

~~also $\{\neg G(f(a)), R(f(a))\}$~~

c) $\forall x \forall y(G(x) \wedge C(x,y) \rightarrow G(y))$

$\forall y \forall x \quad (\neg(G(x) \wedge C(x,y)) \vee G(y))$

~~$\{\neg G(a), \neg C(a,f), G(y)\}$~~

~~$\{\neg G(a), \neg C(a,f(a)), G(f(a))\}$~~

$\{\neg G(x), \neg C(x,y), G(y)\}$

d) $\forall x(G(x) \rightarrow H(x))$

not d $\quad \neg\forall x(G(x) \rightarrow H(x))$

$\exists x \neg((G(x) \rightarrow H(x))$

$\exists x \quad \neg(\neg G(x) \vee H(x))$

$\exists x(G(x) \wedge \neg H(x))$

$\{G(a)\}$, $\{\neg H(a)\}$

~~a~~ ~~$\{G(a)\}$~~

$\{\lnot G(a),$ ~~$\lnot C(a, f(a)), G(f(a))$~~

Final set of clauses

1. $\{C(x_1, f(x_1)), H(x_1)\}$
2. $\{\lnot R(f(x_2)), H(x_2)\}$
3. $\{\lnot G(x_3), R(x_3)\}$
4. $\{\lnot G(x_4), \lnot C(x_4, y_5), G(y_5)\}$
5. $\{G(a)\}, \{\lnot H(a)\}$

$\theta = [a/x_4]$

in 4,5

6. $\{C(a, y_5), \lnot G(y_5)\}$

$\theta = [y_5/x_3]$

in 3,4.

7. $\{\lnot R(y_5), \lnot G(x_3), \lnot C(x_4, y_5)\}$

~~$G(a)$~~

$\theta = [a/x_3]$

in 5 & 7.

8. $\{\lnot R(y_5), \lnot C(a, y_5)\}$

~~$\theta = [y/x]\ [a/x_4]$~~

~~9.~~ $\theta =$ ~~add~~ $[y_5/y_5]$ Identity substitution

in 6 and 8.

9. $\{ Q(y_5), \neg R(y_5) \}$

$$\theta = [y_5 / x_3]$$

in 3 and 9

10. $\{ \square \}$

hence proved

(4) $\rightarrow$ If $\forall x_1 \ldots \forall x_n A(x_1, \ldots x_n)$ is sat in an interpretation with a single element in the universe, then we can ~~so~~ simply use this interpretation to make $\forall x_1 \ldots \forall x_n A(x_1 \ldots x_n)$ sat.

$\because$ by defn of satisfiability, we have to find an interpretation that satisfies the formula

hence this direction is trivial

proved.

$\leftarrow$

say $M \models \forall x_1 \forall x_2 \ldots \forall x_n A(x_1 \ldots x_n)$

~~say $M \models A(a, a, \ldots a)$~~

consider one single element in the universe of M ~~$\{a\}$~~ $\{a\}$.

and N be the structure which has this.

i.e. (N is substructure of M)

Now,

$M \models A(a, a, \ldots a)$ ($\because M \models \forall x_1 \forall x_2 \ldots \forall x_n A(x_1 \ldots x_n)$ and a lies in universe of M. hence true)

Thus. $N \models A(a, a, \ldots a)$ ($\because N$ has $\{a\}$ in its universe ? and A is quantifier-free)

$N \models \forall x_1 \forall x_2 \ldots \forall x_n A(x_1 \ldots x_n)$ ($\because \{a\}$ is only element in the universe of N).

Thus. ~~N satisfies~~ proved.

5a) We prove by construction.

$$F_n = \exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{i=1}^{n} R(x_i, x_{n-i+1}) \wedge \left(\bigwedge_{\substack{\forall i \forall j \\ j \neq n-i+1}} \neg R(x_i, x_j) \right)$$

and

$$F_1 = \exists x_1 R(x_1, x_1)$$

Claim-1: F_n is sat.

F_1 is sat under the model A

A: $U_A = \{a\}$

~~Rel~~ $R = \{(a,a)\}$.

F_n is sat under the model B.

B. $U_B = \{a_1, a_2 \ldots a_n\}$

$R = \{(a_i, a_j)\}$ ~~$\forall i \forall j$~~ $j = n-i+1$ for $i = 1$ to n.

Thus by def

$$\underbrace{\bigwedge_{i=1}^{n} R(a_i, a_{n-i+1})}_{\text{true}} \wedge \underbrace{\left(\bigwedge_{\substack{\forall i \forall j \\ j \neq n-i+1}} \underbrace{\neg R(a_i, a_j)}_{\text{false}} \right)}_{\text{true}}$$

hence B satisfies F_n.

Claim 2: For any model A' with less than n elements, we prove that F_n is unsat.

We already have a model (shown above) with n elements for F_n.

For F_1, this is trivial since ~~an~~ empty universe will not satisfy

$F_1 = \exists x_1 R(x_1, x_1)$

Now for F_n,

~~Say~~ We try replacing $\underbrace{x_j \text{ with } x_i}_{\Downarrow}$ and prove that it is not possible.

to reduce the requirement of n elements (of universe)
[our formula has $x_1 \cdots x_n$].

Consider some arbitrary x_i $i \in [1,n]$
and some x_j $j \in [1,n]$ $j \neq i$.
and we replace x_j with x_i everywhere in F_n.

~~Case-1: n is odd and j is middle element, thus j does not~~

Notation: $R(x_i, x_j)$ is a +ve predicate
$\neg R(x_i, x_j)$ is a -ve predicate.

Case-1: j does not occur among the predicates. Possible only when n is odd and j is middle element.

Now $\because j \neq i$ and j is not present among the predicates
i has to be present among the predicates ($\because$ only mid is missing according to the construction of the formula).

So say it is $R(x_i, x_k)$ and $R(x_k, x_i)$

Now, we also have $\neg R(x_j, x_k)$
$\because x_j$ is not among the predicates

Thus we have $R(x_i, x_k) \wedge \neg R(x_i, x_k)$
which makes the formula unsat.

Case-2: j occurs among the the predicates
So we have by construction of F,
$R(x_j, x_k) \wedge R(x_k, x_j)$
$k = n-j+1$

So if we replace x_j with x_i, we get $R(x_i, x_k)$
But, we also have $\neg R(x_i, x_{n-j+1})$ in F.
$\because i \neq j$ $\quad n-i+1 \neq n-j+1$

and $v = n-j+1$

hence $R(x_i, x_k) \wedge \neg R(x_i, x_k)$
make F unsat.

Thus proved.

5) b)

Given:

The signature σ consists of only unary predicate symbols $P_1 \ldots P_K$.
Thus any σ-formula, F we can form will ~~as~~ necessarily have the following properties:

(i) F will not have any function symbols ~~or co~~ in it.

~~Thus~~ It may have quantifiers.

We know ~~Thus~~ we can write any σ-formula F in prenex form. Thus we can write F in prenex form as well.

So structure of F will be of the form

~~$\exists \ldots \forall \ldots \exists x_1 \exists x_2 \ldots \forall \ldots$~~

$Q_1 Q_2 Q_3 \ldots$ $P_1(x_1)$ and/or $P_2(x_2)$ . . .

in prenex form.

(ii) Since we are in FOL without equality and we have unary predicates, hence we shall not be able to specify that 2 variables x_i & x_j have to be necessarily unique ~~&~~ ~~for~~ ~~(despite having~~ ~~similar prop~~ ~~(unless its nece~~
($\because$ we do not have = function symbol here).

Thus, if x_i and x_j, ~~being equal~~ ~~have similar properties necessary~~ ~~for being making~~ a ~~model~~ ~~formula~~ ~~satisfiable~~, ~~we can replace~~ are different in a model but making them same does not affect satisfiability of the ~~...~~ formula, we can replace x_j with x_i everywhere still maintain satisfiability.

Thus, consider any model A that satisfies ~~any~~ σ-~~for~~ formula F.

Now A could have ∞ universe.

~~Basis~~ Consider any element / constant a. $a \in U_A$

Now a ~~...~~ ~~can~~ has the following property

$\Rightarrow$ a is present in the domain of a subset of P_i where i ~~...~~ $\in [1, k]$.

Thus total subsets of $P_1 .. P_K = 2^k$
and hence we have 2^k elements having different properties.

Now consider a model ~~that~~ $A \models F$ that uses an extra element (not among 2^k discussed above).
Call this b.

So b shall match with exactly one element among these 2^k elements in terms of which ~~does~~ $P_1 .. P_K$ it belongs to, call that matched element c.

~~Thus~~ And since ~~there~~ we cannot enforce inequality of 2 variables, and both share same property, thus b can be replaced with c.
everywhere. hence we can reduce the distinct ~~vo~~ constants we used in ~~one~~ U_A one by one so that we end up with 2^k elements that are present in U_B. and still maintain satisfiability

Thus $B \models F$ hence proved.