# Assignment-2

1) Given $X \vdash \alpha$
$\Rightarrow$ To show $X \models \alpha$

By induction on the steps of derivation from $X$ to $\alpha$.
$\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ where $\alpha_n = \alpha$.

Base case: $n=1$
$X \vdash \alpha$
Then either $\alpha$ is a member of $X$ or it is an instance of axiom.
In either case $X \models \alpha$ $\because \alpha$ is ~~alwa~~ true.

Induction case $n>1$.
$\alpha_n$ = member of $X$ or instance of axiom,
both have been handled in base case

$\alpha_n$ = derived using MP.
from $\alpha_j, \alpha_j \Rightarrow \alpha_n$

So $X \vdash \alpha_j$ $X \models \alpha_j$
$X \vdash \alpha_j \Rightarrow \alpha_n$ $X \models \alpha_j \Rightarrow \alpha_n$

$X \models$ ~~$\alpha$~~ $\alpha_n$. hence proved.

$\Leftarrow$ Given $X \models \alpha$
to show $X \vdash \alpha$

Proof: Given $X \models \alpha$.
By compactness Thm, we know $\exists Y \subseteq_{fin} X$ s.t $Y \models \alpha$.
say $Y = \{\beta_1, \beta_2 \cdots \beta_n\}$

$\{\beta_1, \beta_2 \cdots \beta_n\} \models \alpha$.
Now ~~$\beta_n$~~ this means when $\beta_n$ is true $\alpha$ is true
$\vdash (\beta_1 \rightarrow \cdots (\beta_{n-1} \rightarrow \underbrace{(\beta_n \rightarrow \alpha)}_{\text{True}}))$ (by induction)
$\cdots$
is valid.
hence
~~using completeness Thm,~~

$\vdash$ ~~$B$~~ $(\beta_1 \rightarrow (\beta_2 \rightarrow \cdots \beta_{n-1} \rightarrow (\beta_n \rightarrow \alpha)) \ldots$
using Deduction Thm. $n$ times
$\{\beta_1, \beta_2 \cdots \beta_n\} \vdash_B \alpha$.
Thus use this subset in proving $X \vdash \alpha$
hence proved.

(2) (a) First we construct a maximal FSS.

$\bar{\Phi}$ is the enumeration of all formulae that can be made using $\neg$ and $\vee$ on atomic propositions and other formulae

Claim: $\bar{\Phi}$ is countably infinite set

$S = \{ \neg, \vee, p_1, p_2 \dots \}$

$S^2 = \cancel{app}\ S \times S$

$S^3 = S \times S \times S$.

Now we know $\{\neg, \vee\}$ is adequate (proved in Assignment 1)

hence $S' = S \cup S^2 \cup S^3 \cup \dots$

will be union of countably infinite sets, where each set is countably infinite, hence $S'$ is countably infinite

Also $S'$ lists all possible formulae.

$\bar{\Phi} = S'$

Actual proof: Say $X$ is an FSS

$X_0 = X \qquad \bar{\Phi} = \{ \alpha_1, \alpha_2 \dots \}$

$$X_n = \begin{cases} X_{n-1} \cup \{\alpha_n\} & \text{if } X_{n-1} \cup \{\alpha_n\} \text{ is FSS.} \\ X_{n-1} & \text{o.w.} \end{cases}$$

Thus, $Y = \bigcup_{n \geq 0} X_n$ is a maximal FSS.

Proof: $Y$ is FSS.

By contradiction.

say $\exists\ Z \subseteq_{fin} Y$. which is ~~not~~ ~~FSS~~ unsat.

$Z = \{ \beta_1, \beta_2 \dots \beta_n \}$

$\beta_i = \alpha_{i_1}$ [ based on the position of $\beta_t$ in ~~the~~ $\bar{\Phi}$.

let $j = \max(\alpha_{i_1}, \alpha_{i_2} \dots \alpha_{i_n})$

So when $X_{j+1}$ was being formed, if $Z$ had been unsat, then $X_{j+1}$ would not be FSS, which is a contradiction.

Hence $Y$ is FSS

Claim: $Y$ is maximal

Proof: ~~Say~~ by contradiction.

Say $\exists\, \alpha$ which is not included in $Y$ but make $Y \cup \{\alpha\}$ FSS.

Say its position in $\bar{\phi}$ is $k$.

Then when constructing $X_K$, we should have ~~constructed~~ add $\alpha$ by definition of $X_K$, hence a contradiction.

Claim 1: If $\beta$ is consistent, ~~it~~ $\beta$ is sat

Proved in class

Claim 2: If $\beta$ is inconsistent, $\beta$ is unsat.

Proof: $\vdash \neg\beta$

$\models \neg\beta$ ~~B~~ using completeness & soundness.

~~B~~ $\neg\beta$ is valid
$\beta$ is unsat. $\Big\}$ proved in class.

(b) $\to$ First we show
$\{\alpha, \neg\alpha\}$ both cannot be in $X$.

Contradiction: If both were in $X$, then $\{\alpha, \neg\alpha\}$ is a finite subset of $X$ that is unsat, which is a contradiction.

$\to$ Say neither $\alpha$ nor $\neg\alpha$ is in $X$.

Since $X$ is max FSS, there must be subsets $B \subseteq_{fin} X$ and $C \subseteq_{fin} X$ st $B \cup \{\alpha\}$ is unsat and $C \cup \{\neg\alpha\}$ is unsat.

$B = \{\beta_1, \beta_2 \dots \beta_m\}$

$C = \{\gamma_1, \gamma_2 \dots \gamma_k\}$

Thus we have $\beta_1 \wedge \beta_2 \wedge \dots \beta_m \wedge \alpha$ unsat and
$\gamma_1 \wedge \gamma_2 \dots \wedge \gamma_k \wedge \neg\alpha$ unsat

$\hat{\beta} \wedge \alpha$ unsat

$\hat{\gamma} \wedge \neg\alpha$ unsat.

Using claim 1. : if $\beta$ is unsat, $\beta$ is inconsistent.

$\vdash \neg(\hat{\beta} \wedge \alpha) \qquad \vdash \neg\hat{\beta} \vee \neg\alpha \qquad \vdash \alpha \Rightarrow \neg\hat{\beta}$

$\vdash \neg(\hat{\gamma} \wedge \neg\alpha) \qquad \vdash \neg\hat{\gamma} \vee \alpha \qquad \vdash \neg\alpha \Rightarrow \neg\hat{\gamma}$

Using the thesis proved in quiz.

$\vdash (\alpha \Rightarrow \beta) \Rightarrow ((\delta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \delta) \Rightarrow (\beta \vee \gamma))$

$\vdash (\alpha \Rightarrow \neg\hat{\beta}) \Rightarrow ((\neg\alpha \Rightarrow \hat{\gamma}) \Rightarrow ((\alpha \vee \neg\alpha) \Rightarrow (\neg\hat{\beta} \vee \neg\hat{\gamma}))$

Using Deduction Thm,

$\vdash (\neg\hat{\beta} \vee \neg\hat{\gamma}) \qquad \vdash \neg(\hat{\beta} \wedge \hat{\gamma})$ hence $\hat{\beta} \wedge \hat{\gamma}$ is inconsistent.

using claim 2., $\hat{\beta} \wedge \hat{\gamma}$ is unsat. which is a contradiction.

Thus $\alpha \in X$ iff $\neg\alpha \notin X$.
proved.

(c)

Claim 1: If $\alpha \vee \beta$ is consistent, then either $\alpha$ is consistent or $\beta$ is consistent.

Contrapositive: If both $\alpha$ and $\beta$ are inconsistent, $\alpha \vee \beta$ is ~~so~~ inconsistent.

1. $\vdash \neg\beta$ ($\beta$ is inconsistent) premise
2. $\neg\alpha \vdash \neg\beta$ (adding premise ~~is~~ does not matter)
3. $\vdash \neg\alpha \Rightarrow \neg\beta$
4. $\vdash (\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ A3 instance

~~5.~~ Replacing $\alpha$ with $\beta$.

5. $\vdash (\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\alpha \Rightarrow \beta) \Rightarrow \alpha)$
6. $\vdash (\neg\alpha \Rightarrow \beta) \Rightarrow \alpha$ MP 3,5
7. $\vdash \neg\alpha \Rightarrow \neg(\neg\alpha \Rightarrow \beta)$ using $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$ proved in (class + quiz)
8. $\vdash \neg\alpha$ premise
9. $\vdash \neg(\neg\alpha \Rightarrow \beta)$ MP 8,7
10. $\vdash \neg(\alpha \vee \beta)$

hence $\alpha \vee \beta$ is inconsistent, ~~which is a contradiction~~.

~~Thus~~ hence proved.

Claim 2: If either $\alpha$ is consistent or $\beta$ is consistent, then $\alpha \vee \beta$ is consistent.

Contrapositive: If $\alpha \vee \beta$ is inconsistent, then both $\alpha$ & $\beta$ are ~~so~~ inconsistent.

1. $\vdash \neg(\alpha \vee \beta)$ premise
2. $\vdash \neg(\neg\alpha \Rightarrow \beta)$

~~3. $\beta \vdash \neg(\neg\alpha \Rightarrow \beta)$~~

~~4. $\vdash \beta \Rightarrow \neg(\neg\alpha \Rightarrow \beta)$~~

~~5.~~

3. $\vdash \beta \Rightarrow (\neg\alpha \Rightarrow \beta)$ A1 instance
4. $\vdash \neg(\neg\alpha \Rightarrow \beta) \Rightarrow \neg\beta$
5. $\vdash \neg\beta$ MP 2,4.

1. $\neg(\alpha \vee \beta)$

2. $\neg(\neg\alpha \Rightarrow \beta)$

3. $\alpha, \neg\alpha \vdash (\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ A3 instance

4. $\alpha, \neg\alpha \vdash \alpha$

5. $\alpha, \neg\alpha \vdash \neg\alpha$

6. $\alpha, \neg\alpha \vdash \neg\alpha \Rightarrow (\neg\beta \Rightarrow \neg\alpha)$ A1 instance

7. $\alpha, \neg\alpha \vdash (\neg\beta \Rightarrow \neg\alpha)$ MP 5,6.

8. $\alpha, \neg\alpha \vdash ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ MP 7,3.

9. $\alpha, \neg\alpha \vdash \alpha \Rightarrow (\neg\beta \Rightarrow \alpha)$ A1 instance

10. $\alpha, \neg\alpha \vdash (\neg\beta \Rightarrow \alpha)$ MP 4,9

11. $\alpha, \neg\alpha \vdash \beta$ MP 10,8

12. $\alpha \vdash (\neg\alpha \Rightarrow \beta)$ ~~MP~~

13. $\vdash \alpha \Rightarrow (\neg\alpha \Rightarrow \beta)$

14. $\vdash \neg(\neg\alpha \Rightarrow \beta) \Rightarrow \neg\alpha$ using $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$

15. $\vdash \neg\alpha$ MP 2,14.

hence $\alpha$ is inconsistent and $\beta$ is inconsistent.

Proof:

$\alpha \vee \beta \in X \iff \alpha \vee \beta$ is sat $\iff \alpha \vee \beta$ is consistent $\iff$ either $\alpha$ is
($\because \alpha \vee \beta$ is finite subset) ~~case~~

consistent or $\beta$ is consistent $\iff$ ~~either $\alpha \in X$ or $\beta \in X$~~ either $\alpha$ is sat or $\beta$ is sat $\iff$ either $\alpha \in X$ or $\beta \in X$.
($\because \alpha, \beta$ are finite subsets of $X$)

hence proved.

(d) Proof: By induction on the structure of $\alpha$.

Base case: $\alpha = p$, $p \in P$ [P is list of propositions]

Then $v_x \models p$ iff (by definition of $v_x$) $p \in X$.

Induction case:

There are 2 cases to consider:

i) $\alpha$ is of the form $\neg\beta$

$v_x \models \neg\beta$ iff (by def.) $v_x \not\models \beta$ iff (by I.H.) $\beta \notin X$ iff $\beta \in X$ (using part b)

ii) $\alpha$ is of the form $\beta \vee \gamma$.

$v_x \models \beta \vee \gamma$ iff (by def.) $v_x \models \beta$ or $v_x \models \gamma$ iff (by I.H.).

$\beta \in X$ or $\gamma \in X$ iff $(\beta \vee \gamma) \in X$.
(by part c)

Thus, in ~~all~~ all cases, this holds, thus. proved.

(e) Take any FSS X.

by part a) X can be extended to maximal FSS. ~~say~~ Y.

By part d) Y has a valuation $v_Y$ s.t. for every formula $\alpha$, $v_Y \models \alpha$ iff ~~$\alpha \in X$~~ $\alpha \in Y$.

Thus ~~Y~~ Y contains all formulae that were present in X, hence $v_Y$ makes all formulae in X true.

Thus $v_X \models X$.

hence proved.

(b) $\Leftarrow$ If $\exists Y \subseteq_{fin} X$ s.t $Y \models \alpha$, then $X \models \alpha$.

Consider $v_X \models X$, then $v_X \models Y$ $(\because Y \subseteq_{fin} X)$.

Hence $v_X \models \alpha$. using given statement

$\therefore v_X \models X$, ~~means~~ implies $v_X \models \alpha$.

Thus $X \models \alpha$. proved.

$\Rightarrow$ If $X \models \alpha$, then $\exists Y \subseteq_{fin} X$ s.t $Y \models \alpha$.

$X \models \alpha$ implies $X \cup \{\neg\alpha\}$ is unsat (proved in claim below)

Claim: $X \models \alpha$ implies $X \cup \{\neg\alpha\}$ is unsat.

Proof: If $Z \cup \{\neg\beta\}$ is unsat, then $Z \models \beta$.

Consider all valuations under which $Z$ is true. Then under all these valuation, $\neg\beta$ must be false since $Z \cup \{\neg\beta\}$ is unsat.

(Otherwise, we would have $v_z$ makes $Z \cup \{\neg\beta\}$ true which would mean $Z \cup \{\neg\beta\}$ is sat)

Thus under all valuation where $Z$ is true, $\beta$ must be true. (LEM)

Thus $Z \models \beta$ (proved)

$\Leftarrow$ If $Z \models \beta$, then $Z \cup \{\neg\beta\}$ is unsat.

Proof: Under all valuations $v_z$ under which $Z$ is true, $\beta$ must be true (def of $Z \models \beta$)

Hence $\neg\beta$ must be false under all these $v_z$ (LEM)

Thus whenever $Z$ is true, $\neg\beta$ is false $\Rightarrow$ $Z \cup \{\neg\beta\}$ is false.

thus there is no valuation under which $\neg\beta$ is true when $Z$ is true. On the other hand, $Z$ is false then $Z \cup \{\neg\beta\}$ is also unsat. Hence proved.

Theorem:

Proof: Given $X \vDash \alpha$.

$X \cup \{\neg\alpha\}$ is unsat.

Now ~~we~~ from part (e), we can say that if $X$ is FSS, then $\exists V_X$ s.t $V_X \vDash X$, hence $X$ is sat.

Thus if ~~X is FSS, then X is~~

if every finite subset of $X$ is sat, then $X$ is FSS, hence $X$ is sat.

Thus if $X$ is unsat, then $\exists Y \subseteq_{fin} X$ s.t $Y$ is unsat.

(contrapositive)

$\because X \cup \{\neg\alpha\}$ is unsat.

$\exists Y \subseteq_{fin} X \cup \{\neg\alpha\}$ s.t. $Y$ is unsat. Thus $(Y \setminus \{\neg\alpha\}) \cup \{\neg\alpha\}$ is unsat, where $(Y \setminus \{\neg\alpha\}) \subseteq_{fin} X$. This implies $Y \setminus \{\neg\alpha\} \vDash \alpha$.

hence proved.