Logic and Proof Hilary 2016

Exercise Sheet 2

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1. Using resolution, show that $A \wedge B \wedge C$ is a consequence of

$$F = (\neg A \vee B) \wedge (\neg B \vee C) \wedge (A \vee \neg C) \wedge (A \vee B \vee C)\,.$$

2. Consider the function $\text{parity}_n : \{0,1\}^n \to \{0,1\}$ that takes value 1 precisely when an even number of its arguments have value 1.

   (a) Write down a CNF-formula G_4 on propositional variables $P_1, \ldots, P_4$ whose truth table represents parity_4.

   (b) Compute the reduced ordered binary decision diagram $\mathcal{D}_4$ for parity_4 under the orderings $(P_1, \ldots, P_4)$ and $(P_4, \ldots, P_1)$.

   (c) Compare the length of G_4 with the size of $\mathcal{D}_4$, and extrapolate to G_n and $\mathcal{D}_n$. (You need not justify your answer.)

3. Show that for any CNF formula F one can compute in polynomial time an equisatifiable formula $G_1 \wedge G_2$, with G_1 a Horn formula and G_2 a 2-CNF formula. Justify your answer.

   (**Hint:** Consider first the case that F consists of a single clause.)

4. A *renamable Horn formula* is a CNF formula that can be turned into a Horn formula by negating (all occurrences of) some of its variables. For example,

$$(P_1 \vee \neg P_2 \vee \neg P_3) \wedge (P_2 \vee P_3) \wedge (\neg P_1)$$

   can be turned into a Horn formula by negating P_1 and P_2.

   Given a CNF-formula F, show how to derive a 2-CNF formula G such that G is satisfiable if and only if F is a renamable Horn formula. Show moreover that one can derive a renaming that turns F into a Horn formula from a satisfying assignment for G.

5. Using resolution, or otherwise, show that there is a polynomial-time algorithm to decide satisfiability of those CNF formulas F in which each propositional variable occurs at most twice. Explain why your answer is correct and briefly explain why it meets the required time bound.

6. *Positive resolution* is a restriction of ordinary resolution, which is defined as follows: derive a resolvent from C_1 and C_2 only if C_1 is a positive clause, i.e., it consists only of positive literals. By adapting the completeness proof from lectures, show that if F is an unsatisfiable CNF formula then one can derive the empty clause from F using only positive resolution.

7. Suppose that $\boldsymbol{S} \models F$ for some formula F and set of formulas $\boldsymbol{S}$. Show that there is a finite set $\boldsymbol{S}_0 \subseteq \boldsymbol{S}$ such that $\boldsymbol{S}_0 \models F$.

8. Given an undirected graph $G = (V, E)$, a set of vertices $S \subseteq V$ is a *clique* if every pair of distinct vertices $u, v \in S$ are connected by an edge and S is an *independent set* if no pair of distinct vertices $u, v \in S$ is connected by an edge. Now consider the following two statements:

   (A) Every infinite graph either has an infinite clique or an infinite independent set.

   (B) For all k there exists n such that any graph with n vertices has a clique of size k or an independent set of size k.

   The goal of this question is to show that (A) implies (B).[1]

   (a) Carefully formulate the negation of (B).

   (b) Assuming the negation of (B), use the Compactness Theorem to prove the negation of (A), i.e., that there is an infinite graph with no infinite clique and no infinite independent set.

[1] As an optional exercise, beyond the scope of the course, you can try to prove (A). This result is Theorem 1 in `https://www.dpmms.cam.ac.uk/~par31/notes/ramsey.pdf`. Combining this with 8(b) we obtain a proof of (B).