COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 17 & 18 (Herbrand's Theorem, Ground Resolution)

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Proofs for recap: Logical equivalence

$(\forall x\; F \wedge G) \;\equiv\; \forall x\;(F \wedge G)$ (if x does not occur free in G)

Proof:

Proofs for recap: Renaming bound variables

Let F denote the formula $Qx\ G$ where Q is a quantifier. Let y be a variable that does not occur in G.

Then $F \equiv Qy\ (G[y/x])$.

Proof:

Proofs for recap: Skolem Form

Let $F = \forall y_1 \, \forall y_2 \, \ldots \, \forall y_n \, \exists z \, G$ be a rectified formula. Given a function symbol f of arity n that does not appear in F, write

$$F' = \forall y_1 \, \forall y_2 \, \ldots \, \forall y_n \, G[f(y_1, y_2, \ldots, y_n)/z].$$

Then F and F' are equisatisfiable.

Proof:

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F,

then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Proof: reading exercise

Herbrand structure

universe is the set of ground terms

terms and function symbols being interpreted "as themselves"

built from syntax

Herbrand structure

herbrand structure means set of ground terms, ch=c, and fh(t1,..tk) = f(t1,..tk)

Definition 1. Let σ be a signature with at least one constant symbol. A σ-structure $\mathcal{H}$ is called a *Herbrand structure* if the following hold:

1. The universe $U_{\mathcal{H}}$ is the set of ground terms over σ.
2. For every constant symbol c in σ we have $c_{\mathcal{H}} = c$.
3. For every k-ary function symbol f in σ and for all ground terms $t_1, t_2 \ldots, t_n \in U_{\mathcal{H}}$ we have $f_{\mathcal{H}}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$.

we only have to give meaning to predicates in herbrand structure.

Interpretation of a ground term

Let $\mathcal{H}$ be a Herbrand structure, and t be a ground term.

Then, $\mathcal{H}[\![t]\!] = t$.

Translation Lemma for Herbrand structures

Let $\mathcal{H}$ be a Herbrand structure, F be a formula, and t be a ground term.

Then $\mathcal{H} \models F[t/x] \quad$ if and only if $\quad \mathcal{H}_{[x \mapsto t]} \models F$.

Herbrand's Theorem and Proof

To prove satisfiability of a formula, you have to show that it has a herbrand model when written as a closed formula in skolem form.

Let $F := \forall x_1 \dots \forall x_n \; F^*$ be a closed formula in Skolem form.

Then F is satisfiable iff it has a Herbrand model.

Proof:

Example

Is the following formula satisfiable?

$$F := \exists x_1 \exists x_2 \exists x_3 \ (\neg(\neg P(x_1) \rightarrow P(x_2)) \wedge \neg(\neg P(x_1) \rightarrow \neg P(x_3)))$$

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol
- does not work for $\forall x_1 \exists x_2 \; F^*$

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \ F^*$, where the matrix F^* does not contain any function symbol
- does not work for $\forall x_1 \exists x_2 \ F^*$
- the presence of a function symbols in its Skolem form makes each Herbrand structure infinite

Herbrand expansion

Let $F := \forall x_1 \ldots \forall x_n\ F^*$ be a closed formula in Skolem form with matrix F^*.

$E(F) := \{F^*[t_1/x_1] \ldots [t_n/x_n] \mid t_1 \ldots t_n \text{ are ground terms } \}$

Herbrand expansion

Let $F := \forall x_1 \dots \forall x_n \ F^*$ be a closed formula in Skolem form with matrix F^*.

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A closed formula F in Skolem form is satisfiable iff $E(F)$ is satisfiable when considered as a set of propositional formulas.

Proof:

Ground resolution

A closed formula F in Skolem form is unsatisfiable iff there is a propositional resolution proof of $\Box$ from $E(F)$.

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Ground resolution

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Proof:

$E(F)$ is unsat iff some finite subset of $E(F)$ is unsat. (Compactness theorem)

Ground resolution

A closed formula F in Skolem form is unsatisfiable iff there is a propositional resolution proof of $\square$ from $E(F)$.

Proof:
$E(F)$ is unsat iff some finite subset of $E(F)$ is unsat. (Compactness theorem)

Soundness and completeness of propositional resolution says that we can derive $\square$ from $E(F)$ using resolution.

Generalized version of Ground Resolution Theorem

Let $F_1, F_2, \ldots, F_n$ be closed formulas in Skolem form

whose respective matrices $F_1^*, F_2^*, \ldots, F_n^*$ are in CNF.

$F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is unsatisfiable iff there is a propositional resolution proof of $\Box$ from the ground instances[1] of clauses from $F_1^*, F_2^*, \ldots, F_n^*$.

[1] a ground instance of F is a formula obtained by replacing all variables in F with ground terms

Example

Let us use ground resolution to show that (a), (b), and (c) together entail (d).

(a) Everyone in the class is either sleepy, bored, or day-dreaming.

(b) All those who are bored are sleepy.

(c) Someone in the class is not day-dreaming.

(d) Someone in the class is sleepy.

Example

Show that $\forall x \; \exists y \; (P(x) \to Q(y)) \to \exists y \; \forall x \; (P(x) \to Q(y))$.

Semi-decidability of validity

Validity of first-order formulas is semi-decidable.

Next week

- Undecidability of satisfiability
- Resolution for Predicate Logic
- Soundness and Completeness

Thank you!