

Examples of Ground Resolution Proofs

James Worrell

In this lecture we show how to use the Ground Resolution Theorem, proved in the last lecture, to do some deduction in first-order logic.

1 Ground Resolution Theorem

Recall that the process of eliminating existential quantifiers by introducing extra function and constant symbols is called *Skolemisation*. The extra symbols introduced are called *Skolem functions*. We begin with a slight generalisation of a theorem that was stated in the previous lecture. In this generalisation we consider Skolemising a collection of formulas rather than a single formula.

Theorem 1. Let F_1, \dots, F_n be closed rectified formulas in prenex form with respective Skolem forms G_1, \dots, G_n . Assume that each G_i is constructed using a different set of Skolem functions. Then $F_1 \wedge F_2 \wedge \dots \wedge F_n$ is satisfiable if and only if $G_1 \wedge G_2 \wedge \dots \wedge G_n$ is satisfiable.


Recall that a ground term is a term that does not contain any variables. Given a quantifier-free formula F , a *ground instance* of F is a formula obtained by replacing all the variables in F with ground terms.

The following is a slight generalisation of the version of the Ground Resolution Theorem proved in the last lecture. Before we considered only a single formula in Skolem form. Here we consider a conjunction of such formulas, which is more convenient for the applications below.

Theorem 2 (Ground Resolution Theorem). Let F_1, \dots, F_n be closed formulas in Skolem form whose respective matrices $F_1^* \wedge \dots \wedge F_n^*$ are in **CNF**. Then $F_1 \wedge \dots \wedge F_n$ is unsatisfiable if and only if there is a propositional resolution proof of \square from the set of ground instances of clauses from F_1^*, \dots, F_n^* .

2 Examples

In this section we give two examples of the use of the Ground Resolution Theorem.

 **Example 3.** We would like to formalise the following statements in first-order logic and to use ground resolution to show that (a), (b) and (c) together entail (d).

- (a) Everyone at Oriel is either lazy, a rower or a drunk.
- (b) All rowers are lazy.
- (c) Someone at Oriel is not drunk.
- (d) Someone at Oriel is lazy.

$$\begin{array}{c}
\frac{\{\neg R(a), L(a)\} \quad \{\neg O(a), L(a), R(a), D(a)\}}{\{L(a), \neg O(a), D(a)\}} \quad \frac{\{\neg O(a), \neg L(a)\}}{\{\neg O(a), D(a)\}} \quad \frac{\{\neg D(a)\}}{\{\neg O(a)\}} \\
\frac{\{\neg O(a), D(a)\} \quad \{\neg O(a)\}}{\square} \quad \{O(a)\}
\end{array}$$

Figure 1: The nature of Oriel students

We translate (a), (b), (c) and the negation of (d) into closed formulas of first-order logic as follows.

$$\begin{aligned}
F_1 &= \forall x (O(x) \rightarrow (L(x) \vee R(x) \vee D(x))) \\
F_2 &= \forall x (R(x) \rightarrow L(x)) \\
F_3 &= \exists x (O(x) \wedge \neg D(x)) \\
F_4 &= \neg \exists x (O(x) \wedge L(x)).
\end{aligned}$$

Next we translate F_1 , F_2 , F_3 and F_4 to Skolem form. To do this we bring all quantifiers to the outside, eliminate existential quantifiers by introducing Skolem functions and finally bring the matrix of each formula into **CNF**. This yields

$$\begin{aligned}
G_1 &= \forall x (\neg O(x) \vee L(x) \vee R(x) \vee D(x)) \\
G_2 &= \forall x (\neg R(x) \vee L(x)) \\
G_3 &= O(a) \wedge \neg D(a) \\
G_4 &= \forall x (\neg O(x) \vee \neg L(x)).
\end{aligned}$$

where a is a fresh constant symbol.

Now we deduce the empty clause \square from ground instances of clauses in the respective matrices of the Skolem-form formulas G_1, \dots, G_4 . Note that these formulas are defined over a signature with a single constant symbol a , which is therefore the only ground term. The proof is shown in Figure 1.

Example 4. Using ground resolution we show that

$$\forall x \exists y (P(x) \rightarrow Q(y)) \rightarrow \exists y \forall x (P(x) \rightarrow Q(y))$$

is a valid sentence.

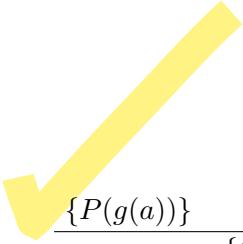
We can show this by showing that the negation is unsatisfiable. The negation can be written:

$$\forall x \exists y (P(x) \rightarrow Q(y)) \wedge \neg \exists y \forall x (P(x) \rightarrow Q(y)).$$

We bring each conjunction to Skolem form, yielding

$$\begin{aligned}
F_1 &= \forall x (\neg P(x) \vee Q(f(x))) \\
F_2 &= \forall y (P(g(y)) \wedge \neg Q(y)).
\end{aligned}$$

Note that F_1 and F_2 are defined over a signature with no constants and so there are no ground terms. We remedy this problem by introducing a single new constant symbol a . Now the set of ground terms is $\{a, f(a), g(a), f(f(a)), f(g(a)), \dots\}$. We can now derive \square by the propositional resolution proof in Figure 2 which every leaf is a ground instance of a clause from the respective matrices of F_1 and F_2 .



$$\begin{array}{c}
 \frac{\{P(g(a))\} \quad \{\neg P(g(a)), Q(f(g(a)))\}}{\{Q(f(g(a)))\}} \quad \frac{\quad}{\{ \neg Q(f(g(a))) \}} \\
 \hline
 \square
 \end{array}$$

Figure 2: Ground Resolution proof for Example 4