Tutorial Sheet 6::: MTL 101::: Systems of ODEs and Laplace Transforms

(1) Find the real general solution of the following systems.
 (a) $y_1' = -8y_1 - 2y_2,\ y_2' = 2y_1 - 4y_2,$
 (b) $y_1' = -3y_1 - y_2 + 2y_3,\ y_2' = -4y_2 + 2y_3,\ y_3' = y_2 - 5y_3,$
 (c) $y_1' = -y_1 - 4y_2 + 2y_3,\ y_2' = 2y_1 + 5y_2 - y_3,\ y_3' = 2y_1 + 2y_2 + 2y_3.$

(2) Solve the following IVPs.
 (a) $y_1' = 2y_1 + 2y_2,\ y_2' = 5y_1 - y_2,\ y_1(0) = 0,\ y_2(0) = -7.$
 (b) $y_1' = -14y_1 + 10y_2,\ y_2' = -5y_1 + y_2,\ y_1(0) = -1,\ y_2(0) = 1.$

(3) Solve the following system of equations.
 (a) $y_1' = y_2 + e^{3t},\ y_2' = y_1 - 3e^{3t}$
 (b) $y_1' = 3y_1 + y_2 - 3\sin 3t,\ y_2' = 7y_1 - 3y_2 + 9\cos 3t - 16\sin 3t$
 (c) $y_1' = -2y_1 + y_2,\ y_2' = -y_1 + e^t.$

(4) Solve the following IVP.
 (a) $y_1' = y_2 - 5\sin t,\ y_2' = -4y_1 + 17\cos t,\ y_1(0) = 5,\ y_2(0) = 2.$
 (b) $y_1' = y_1 + 4y_2 - t^2 + 6t,\ y_2' = y_1 + y_2 - t^2 + t - 1,\ y_1(0) = 2,\ y_2(0) = -1.$
 (c) $y_1' = 5y_1 + 4y_2 - 5t^2 + 6t + 25,\ y_2' = y_1 + 2y_2 - t^2 + 2t + 4,\ y_1(0) = 0,\ y_2(0) = 0.$

(5) Find the Laplace transform of the following functions.
 $\cos^2 wt,\ e^t\cosh 3t,\ \sin 2t\cos 2t,\ e^{-\alpha t}\cos\beta t,\ \sinh t\cos t,\ 2e^{-t}\cos^2\frac{1}{2}t.$

(6) Find the inverse Laplace transform of the following functions.
 $\frac{5s}{s^2-25},\ \frac{1-7s}{(s-3)(s-1)(s+2)},\ \frac{2s^3}{s^4-1},\ \frac{2}{s^2+s+\frac{1}{2}}$

(7) Solve the following IVP using Laplace transform.
 (a) $y'' - y' - 2y = 0,\ y(0) = 8,\ y'(0) = 7.$
 (b) $y'' + 2y' - 3y = 6e^{-2t},\ y(0) = 2,\ y'(0) = -14.$

(8) Find the Laplace transform of the following functions (where u is the unit step function):
 $tu(t-1),\ e^{-2t}u(t-3),\ 4u(t-\pi)\cos t.$

(9) Find the inverse Laplace transform of the following functions:
 $e^{-3s}/s^3,\ 3(1-e^{-\pi s})/(s^2+9),\ se^{-2s}/(s^2+\pi^2)$

(10) Solve the following IVP.
 (a) $y'' + 6y' + 8y = e^{-3t} - e^{-5t},\ y(0) = 0,\ y'(0) = 0.$
 (b) $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0,\ y'(0) = 0.$
 (c) $y'' + 4y' + 5y = \delta(t-1)$, ($\delta$ is the Dirac's Delta) $y(0) = 0,\ y'(0) = 3.$
 (d) $y'' + 5y' + 6y = u(t-1) + \delta(t-2)$ (where u, δ are the step function and the dirac's Delta function), $y(0) = 0$ and $y'(0) = 1.$

(11) Find the Laplace transform (by differentiation) of the following functions:
 $t^2\cosh\pi t,\ te^{-t}\sin t,\ t^2\cos wt$

(12) Find inverse Laplace transform of the following functions by differentiation or integration:
 $\frac{1}{(s-3)^3},\ \frac{2s+6}{(s^2+6s+10)^2},\ \ln(\frac{s+a}{s+b}),\ \cot^{-1}\frac{s}{\pi}.$

(13) Compute convolution of the following:
 $1 * \sin wt,\ e^t * e^{-1},\ \cos wt * \sin wt,\ u(t-1) * t^2,\ u(t-3) * e^{2t}.$

(14) Use convolution theorem to compute the inverse transform:

$$\frac{6}{s(s+3)},\ \frac{s^2}{(s^2+w^2)^2},\ \frac{e^{-as}}{s(s+s-2)},\ \frac{w}{s^2(s^2+w^2)},\ \frac{1}{(s+3)(s-2)}$$

(15) Solve IVP by using convolution.
 (a) $y'' + y = 3\cos 2t;\ y(0) = 0,\ y'(0) = 0.$
 (b) $y'' + 2y' + 2y = 5u(t-2\pi)\sin t;\ y(0) = 1,\ y'(0) = 0.$
 (c) $y'' + y = r(t), r(t) = 4t$ if $1 < t < 2$ and 0 otherwise; $y(0) = 0,\ ,y'(0) = 0.$
 (d) $y'' + 3y' + 2y = r(t),\ r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0,\ y'(0) = 0.$

(16) Solve the integral equations using Laplace transform.

$$y(t) = 1 + \int_0^t y(r)\,dr,\ y = 2t - 4\int_0^t y(r)(t-r)\,dr,\ y(t) = 1 - \sinh t + \int_0^t (1+r)y(t-r)\,dr.$$

(17) Use partial fraction method to find the Laplace transform of the following:

$$\frac{6}{(s+2)(s-4)}, \quad \frac{s^2+9s-9}{s^3-9s}, \quad \frac{s^3+6s^2+14s}{(s+2)^4}.$$

(18) Derive the following formulae.

(a) $\mathcal{L}^{-1}\{\frac{1}{s^4+4a^4}\} = \frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$,

(b) $\mathcal{L}^{-1}\{\frac{s}{s^4+4a^4}\} = \frac{1}{2a^2} \sinh at \sin at$,

(19) Solve the following IVPs (using Laplace transform).

(a) $y_1' = -y_1 + y_2,\ y_2' = -y_1 - y_2,\ y_1(0) = 1,\ y_2(0) = 0$,

(b) $y_1'' + y_2 = -5\cos 2t,\ y_2'' + y_1 = 5\cos 2t,\ y_1(0) = 1,\ y'(0) = 1,\ y_2(0) = -1, y_2'(0) = 1$.

(c) $y_1' = 2y_1 + 4y_2 + 64tu(t-1),\ y_2' = y_1 + 2y_2,\ y_1(0) = -4,\ y_2(0) = -4$.

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