Tutorial Sheet 4::: MAL 101::: Differential Equations

(1) Verify that f is a particular solution of the given (IVPs) Initial Value Problems:

(a) $\frac{dy}{dx} = y - x, \quad y(0) = 3, \quad f(x) = 2e^x + x + 1.$

(b) $\frac{dy}{dx} = y \, \tan x, \quad y(0) = \frac{1}{2}\pi, \quad f(x) = \frac{\pi}{2} \, \sec x.$

(2) Solve the following separable equations:

(a) $(x-4)y^4 \; dx - x^3(y^2-3) \; dy = 0$; (b) $x \sin y \; dx + (x^2+1)\cos y \; dy = 0.$

(c) $y' = \frac{x^2+y^2}{xy}$ (Substitute $\frac{y}{x} = u$); (d) $y' + \operatorname{cosec} y = 0.$

(3) Solve the following equation by reducing it to a separable equation:
$(x^2 - 3y^2) \; dx + 2xy \; dy = 0.$

(4) Determine whether or not each of the given equations is exact; solve those that are exact.

(a) $(6xy + 2y^2 - 5) \; dx + (3x^2 + 4xy - 6) \; dy = 0$;
(b) $(y^2+1)\cos x \; dx + 2y \sin x \; dy = 0$;
(c) $(2xy + 1) \; dx + (x^2 + 4y) \; dy = 0$;
(d) $(3x^2y + 2) \; dx - (x^3 + y) \; dy = 0.$
(e) $-2xy\sin(x^2)dx + \cos(x^2)dy = 0.$
(f) $(e^{(x+y)} - y)dx + (xe^{(x+y)} + 1)dy = 0$

(5) Solve the IVPs.

(a) $(2xy - 3) \; dx + (x^2 + 4y) \; dy = 0, \quad y(1) = 2$;
(b) $(3x^2y^2 - y^3 + 2x) \; dx + (2x^3y - 3xy^2 + 1) \; dy = 0, \quad y(-2) = 1.$
(c) $y\frac{dy}{dx} + 4x = 0, \quad y(0) = 2.$
(d) $\frac{dr}{d\theta} = b\left(\frac{dr}{d\theta}\cos\theta + r\sin\theta\right), \quad r(\pi/2) = \pi.$

(6) **Falling Body:** Consider a stone falling freely through the air. Assuming that the air resistance is negligible and the acceleration due to gravity $g = 9.8 \; m/s^2$, construct the resulting ODE and find its solution, if the initial position is h_0 and the initial velocity is v_0.

(7) **Subsonic Flight:** The efficiency of engines of subsonic airplanes depends on the air pressure and (usually) is maximum at 36000 ft. The rate of change of air pressure $y'(x)$ is proportional to air pressure $y(x)$ at height x. If y_0 is the pressure at sea level and the pressure decreases to half at 18000 ft, then find the air pressure at 36000 ft.

(8) **Dryer:** In a laundry dryer loss of moisture is directly proportional to moisture content of the laundry. If wet laundry loses one fourth of its moisture in the first 10 minutes, when will the laundry be 95% dry.

(9) Find all the curves in the xy-plane whose tangents pass through the point (a, b).

(10) Under what conditions on constants A, B, C, and D, is $(Ax + By)dx + (Cx + Dy)dy = 0$ is exact. Solve the equation when it is exact.

(11) Determine the constant A such that the equation is exact, and solve the resulting exact equation:

(a) $(x^2 + 3xy)dx + (Ax^2 + 4y)dy = 0$; (b) $(\frac{1}{x^2} + \frac{1}{y^2})dx + (\frac{Ax+1}{y^3})dy = 0.$

(12) Determine the most general function $N(x, y)$ such that the equation is exact:

(a) $(x^2 + xy^2) \; dx + N(x,y) \; dy = 0$; (b) $(x^{-2}y^{-2} + xy^{-3}) \; dx + N(x,y) \; dy = 0.$

(13) Consider the differential equation $(y^2 + 2xy) \; dx - x^2 \; dy = 0.$

(a) Observe that this equation is not exact. Multiply the given equation through by y^n, where n is an integer, and then determine n so that y^n is an integrating factor of the given equation. Solve the resulting exact equation.

(b) Show that $y = 0$ is a solution of the original nonexact equation but is not a solution of the essentially equivalent exact equation found in (a).

(14) Consider a differential equation of the form

$$[y + xf(x^2 + y^2)]\ dx + [yf(x^2 + y^2) - x]\ dy = 0.$$

(a) Show that an equation of this form is not exact.

(b) Show that $1/(x^2 + y^2)$ is an integrating factor of an equation of this form.

(15) Use the result of the above exercise to solve the equation

$$[y + x(x^2 + y^2)^2]\ dx + [y(x^2 + y^2)^2 - x]\ dy = 0.$$

(16) Find all solutions of the following equations:

(a) $y' - 2y = 1$; (b) $y' + y = e^x$; (c) $y' - 2y = x^2 + x$;

(d) $3y' + y = 2e^{-x}$; (e) $y' + 3y = e^x$.

(17) Consider the equation $y' + (\cos x)y = e^{-\sin x}$.

(a) Find the solution ϕ which satisfies $\phi(\pi) = \pi$.

(b) Show that any solution ϕ has the property that $\phi(\pi k) - \phi(0) = \pi k$, where k is any integer.

(18) Solve the Bernoulli's equations:

(a) $y' - 2xy = xy^2$; (b) $y' + y = xy^3$.

(19) Solve the following nonlinear ODEs.

(a) $y' \sin 2y + x \cos 2y = 2x$

(b) $2yy' + y^2 \sin x = \sin x \quad y(0) = \sqrt{5}$

(20) Solve the following IVP:

$$(x-1)y' = 2y, \quad y(1) = 1$$

Explain the results in view of the theory of existence and uniqueness of IVPs.

(21) Find all the initial conditions, such that corresponding IVP, with ODE,

$$(x^2 - 4x)y' = (2x - 4)y$$

has no solution, unique solution and more than one solutions.

(22) Show that the Lipschitz condition is satisfied by the function $|\sin y| + x$ at every point on the xy-plane though its partial derivative with respect to y does not exist on the line $y = 0$.

(23) Apply Picard's iteration method to the following problems. Do three steps of the iteration.

(a) $y' = y, y(0) = 1$; (b) $y' = x + y, y(0) = -1$.

(24) Consider the following IVP:

$$ydy = xdx, \ \ y(0) = \beta.$$

Find all possible $\beta \in \mathbb{R}$ for which the IVP has

(a) a unique solution,

(b) more than one solutions,

(c) no solutions.

(25) Consider the following IVP: $\frac{dy}{dx} = f(x, y)$. $y(0) = 0$ where

$$f(x, y) = \frac{\sin(x+y) + \cos(x+y)}{1 + x^2 + y^2}, \quad \text{for } x, y \in [-1, 1].$$

Using existence uniqueness theorem, find the largest interval in which it has a unique solution.

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