# Solution of tutorial

**Solution1. Linear Operator:** An operator $\hat{\boldsymbol{L}}$ is linear if $\hat{\boldsymbol{L}}\left(a\psi(x) + b\varphi(x)\right) =$ a$\hat{\boldsymbol{L}}$ $\psi(x)$ + b$\hat{\boldsymbol{L}}$ $\varphi(x)$.

**i)**
$$\hat{A}\psi(x) = x\,\psi(x)$$
$$a\hat{A}\left(\psi(x)\right) = ax\psi(x)$$
$$b\hat{A}\left(\varphi(x)\right) = bx\varphi(x)$$
$$\hat{A}\left(a\psi(x) + b\varphi(x)\right) = x\,(a\psi(x) + b\varphi(x))$$

$\hat{A}$ is a linear operator

*iv)*
$$\hat{D}\psi(x) = \psi^2(x)$$
$$a\hat{D}\left(\psi(x)\right) = a\psi^2(x)$$
$$b\hat{D}\left(\varphi(x)\right) = b\varphi^2(x)$$
$$\hat{D}\left(a\psi(x) + b\varphi(x)\right) = \left(a\psi(x) + b\varphi(x)\right)^2 \neq a\psi^2(x) + b\varphi^2(x)$$

$\hat{D}$ is not a linear operator.

*vi)*
$$\hat{F} = \frac{\partial^2}{\partial x^2}$$
$$a\frac{\partial^2}{\partial x^2}\left(\psi(x)\right) = a\frac{\partial^2\psi(x)}{\partial x^2}$$

$$b\frac{\partial^2}{\partial x^2}\left(\varphi(x)\right) = b\frac{\partial^2\varphi(x)}{\partial x^2}$$

$$\frac{\partial^2}{\partial x^2}\left(a\psi(x) + \ b\varphi(x)\right) = a\frac{\partial^2\psi(x)}{\partial x^2} + b\frac{\partial^2\varphi(x)}{\partial x^2}$$

$\hat{F}$ is a linear operator

**Vii)** $$\hat{H}\psi(x) = \frac{1}{n}\psi(x)$$

$$a\hat{H}\left(\psi(x)\right) = \frac{a}{n}\psi(x)$$

$$b\hat{H}\left(\varphi(x)\right) = \frac{b}{n}\varphi(x)$$

$$\hat{H}\left(a\psi(x) + \ b\varphi(x)\right) = \frac{a\psi(x)+b\varphi(x)}{n}$$

$$\hat{H}\left(a\psi(x) + \ b\varphi(x)\right) = \frac{a}{n}\psi(x) + \frac{b}{n}\varphi(x)$$

$\hat{H}$ is a linear operator.

*(ix)* $$\hat{I}\ \psi(x) = \psi(x)$$

$$a\hat{I}\ \left(\psi(x)\right) = a\ \psi(x)$$

$$b\hat{I}\ \left(\varphi(x)\right) = b\ \varphi(x)$$

$$\hat{I}\ \left(a\psi(x) + \ b\varphi(x)\right) = \ a\ \psi(x) + b\ \varphi(x)$$

$\hat{I}$ is a linear operator

**Solution2.** Equivalent operator of $\left(\frac{\partial}{\partial x}\right)(x)$**,** let us consider a wave function $\boldsymbol{\psi}(\mathbf{x})$

$$\left\{\left(\frac{\partial}{\partial x}\right)(x)\right\}\boldsymbol{\psi}(\mathbf{x}) = \frac{\partial\left(x\boldsymbol{\psi}(\mathbf{x})\right)}{\partial x}$$

$$\frac{\partial\left(x\boldsymbol{\psi}(\mathbf{x})\right)}{\partial x} = x\frac{\partial\boldsymbol{\psi}(\mathbf{x})}{\partial x} + \boldsymbol{\psi}(\mathbf{x}) = \left\{x\frac{\partial}{\partial x} + 1\right\}\boldsymbol{\psi}(\mathbf{x})$$

$$\left(\frac{\partial}{\partial x}\right)(x) = \left(x\frac{\partial}{\partial x} + 1\right)$$

**Solution3.** Momentum operator ($\hat{p}_x = -i\hbar\, \partial / \partial x$) is Hermitian**.**

Proof. An operator is A is said to be Hermitian, if it satisfies the Hermitian condition,

$$\int_{-\infty}^{\infty} \boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{t})(\boldsymbol{A}\,\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t}))\, \boldsymbol{dx} = \int_{-\infty}^{\infty} (\boldsymbol{A}\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{t}))^*\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})\, \boldsymbol{dx} \qquad \textbf{(1)}$$

On replacing the operator A by momentum operator in the above condition,

$$\int_{-\infty}^{\infty} \boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{t})\, \widehat{\boldsymbol{P}}_{\boldsymbol{x}}\varphi(\boldsymbol{x},\boldsymbol{t})\, \boldsymbol{dx} = \int_{-\infty}^{\infty} (\widehat{\boldsymbol{P}}_{\boldsymbol{x}}\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{t}))^*\varphi(\boldsymbol{x},\boldsymbol{t})\, \boldsymbol{dx}$$

Now, put the value of momentum operator**,** $\widehat{\boldsymbol{P}}_{\boldsymbol{x}} = -\boldsymbol{i}\hbar\frac{\partial}{\partial x}$

$$\int_{-\infty}^{\infty} \boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{t})(-\boldsymbol{i}\hbar\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}\ )\ \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})\ \boldsymbol{dx} = \int_{-\infty}^{\infty} \left\{-\boldsymbol{i}\hbar\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{t})\right\}^* \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})\ \boldsymbol{dx}$$

On taking LHS,

$$-\boldsymbol{i}\hbar\int_{-\infty}^{\infty} \boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{t})\frac{\boldsymbol{\partial}\varphi(\boldsymbol{x},\boldsymbol{t})}{\boldsymbol{\partial x}}\ \boldsymbol{dx} =$$

$$= -\boldsymbol{i}\hbar\ \{[\boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\boldsymbol{\partial\psi}^*(\boldsymbol{x},\boldsymbol{t})}{\boldsymbol{\partial x}}\ \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})\ \boldsymbol{dx}\}$$

In the above equation 1st term equals to zero because at infinite position for a confined particle wave function should be zero.

$$= \boldsymbol{i}\hbar\int_{-\infty}^{\infty}\frac{\boldsymbol{\partial\psi}^*(\boldsymbol{x},\boldsymbol{t})}{\boldsymbol{\partial x}}\ \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})\ \boldsymbol{dx} = \int_{-\infty}^{\infty}\left\{-\boldsymbol{i}\hbar\frac{\boldsymbol{\partial\psi}(\boldsymbol{x},\boldsymbol{t})}{\boldsymbol{\partial x}}\right\}^* \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})\ \boldsymbol{dx}$$

LHS and RHS are equal, momentum operator follow the condition of Hermitian operator. Hence momentum operator is a Hermitian operator, $(\widehat{\boldsymbol{P}}_{\boldsymbol{x}})^{\dagger} = (\widehat{\boldsymbol{P}}_{\boldsymbol{x}})$

Similarly calculate for further power of momentum operator.

**Solution 4.** The normalization constant and expectation values of $\widehat{\langle x\rangle}$ $and$ $\langle\hat{p}_x\rangle$ for the following wave function:

$$\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{C}\ \boldsymbol{e}^{\boldsymbol{i}\boldsymbol{k}_0\boldsymbol{x}}\boldsymbol{e}^{\left[\frac{-(\boldsymbol{x}-\boldsymbol{x}_0)^2}{4\boldsymbol{a}^2}\right]}\boldsymbol{e}^{\frac{-\boldsymbol{i}\boldsymbol{E}_0\boldsymbol{t}}{\hbar}} \qquad \textbf{\textit{(1)}}$$

Here $C$, $x_0$, $k_0$, $a$, $E_0$ are constants. Consider the $x_0$, $k_0$, $a$, $E_0$ are real.

To calculate normalization constant C, apply normalization condition

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \; dx = \int_{-\infty}^{\infty} \psi^*(x,t)\, \psi(x,t)\, dx = 1 \qquad (2)$$

$$\psi^*(x,t) = \; C^* e^{-ik_0 x} e^{\left[\frac{-(x-x_0)^2}{4a^2}\right]} e^{\frac{iE_0 t}{\hbar}}$$

$$\int_{-\infty}^{\infty} \psi(x,t)\psi^*(x,t) \; dx$$

$$= \int_{-\infty}^{\infty} C^* e^{-ik_0 x} e^{\left[\frac{-(x-x_0)^2}{4a^2}\right]} e^{\frac{iE_0 t}{\hbar}} \, C \, e^{ik_0 x} e^{\left[\frac{-(x-x_0)^2}{4a^2}\right]} e^{\frac{-iE_0 t}{\hbar}} \; dx = 1$$

$$\int_{-\infty}^{\infty} |C|^2 e^{\left[\frac{-(x-x_0)^2}{4a^2}\right]} \; e^{\left[\frac{-(x-x_0)^2}{4a^2}\right]} dx = 1$$

$$\int_{-\infty}^{\infty} |C|^2 \;\; e^{\left[\frac{-(x-x_0)^2}{2a^2}\right]} dx = 1$$

$$|C|^2 \; \int_{-\infty}^{\infty} e^{\left[\frac{-(x-x_0)^2}{2a^2}\right]} \; dx = 1 \qquad (3)$$

On applying the integration on Gaussian function, $\int_{-\infty}^{\infty} e^{[-\beta z^2]} \; dz = \sqrt{\frac{\pi}{\beta}}$, where $\beta$ is a constant.

Put $(x - x_0) = y$ in the equation (3), $dx = dy$

$$|C|^2 \int_{-\infty}^{\infty} e^{\left[\frac{-(y)^2}{2a^2}\right]} \; dy = |C|^2 \sqrt{2\pi}\, a = 1$$

$$|C|^2 \sqrt{2\pi}\, a = 1$$

$$|C|^2 = \frac{1}{\sqrt{2\pi}\, a}$$

$$C = \sqrt{\frac{1}{\sqrt{2\pi}\, a}}$$

Expectation value of quantity, $\langle \widehat{\mathrm{A}} \rangle = \dfrac{\int_{-\infty}^{\infty} \psi^*(x,t) A \psi(x,t)\, dx}{\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\, dx}$

Expectation value of x, $\langle \widehat{x} \rangle = \dfrac{\int_{-\infty}^{\infty} \psi^*(x,t) X \psi(x,t)\, dx}{\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\, dx}$

On putting the value of wave function,

$$\langle \widehat{x} \rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, a}\, x\, e^{\left[\frac{-(x-x_0)^2}{2a^2}\right]} dx$$

$\langle \widehat{x} \rangle = \frac{1}{\sqrt{2\pi}\, a} \int_{-\infty}^{\infty} x \;\; e^{\left[\frac{-(x-x_0)^2}{2a^2}\right]} dx$

Let $(x - x_0) = y,$

$\langle \widehat{x} \rangle = \frac{1}{\sqrt{2\pi}\, a} \int_{-\infty}^{\infty} (y + x_0)\, e^{\left[\frac{-(y)^2}{2a^2}\right]} \; dy$

On further solving, $\langle \widehat{x} \rangle = \frac{1}{\sqrt{2\pi}\, a}\, x_0 \sqrt{2\pi}\, a = x_0$

$\langle \widehat{x} \rangle = x_0$

Similarly, calculate $\langle \widehat{p}_x \rangle$ by putting the value of $p_x$ operator $\widehat{p}_x = -i\hbar \frac{\partial}{\partial x}$

**Solution5** (i) $\left[\widehat{X}, \widehat{P}_x\right] = ?$

Let ψ(x) be the position wave function,

$$[\hat{X}, \hat{P}_x]\psi(x) = (\hat{X}\hat{P}_x - \hat{P}_x\hat{X})\psi(x)$$

Put the value of momentum operator, $\hat{P}_x = -i\hbar\frac{\partial}{\partial x}$

$$x\left(-i\hbar\frac{\partial\psi(x)}{\partial x}\right) + i\hbar\frac{\partial}{\partial x}(x\psi(x)) = x\left(-i\hbar\frac{\partial\psi(x)}{\partial x}\right) + i\hbar\psi(x) + i\hbar\, x\frac{\partial\psi(x)}{\partial x}$$

$$[\hat{X}, \hat{P}_x]\psi(x) = i\hbar\psi(x)$$

$$[\hat{X}, \hat{P}_x] = i\hbar$$

(iv) $[\hat{L}_x, \hat{L}_y] = ?$

$\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ are the orbital angular momentum operator such that,

$\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y = -i\hbar\left(\hat{Y}\frac{\partial}{\partial z} - \hat{Z}\frac{\partial}{\partial y}\right),$

$\hat{L}_y = \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z = -i\hbar\left(\hat{Z}\frac{\partial}{\partial x} - \hat{X}\frac{\partial}{\partial z}\right),$ and

$$\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x = -i\hbar\left(\hat{X}\frac{\partial}{\partial y} - \hat{Y}\frac{\partial}{\partial x}\right)$$

On applying wave function

$$[\hat{L}_x, \hat{L}_y]\psi = [\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z]\psi$$

$$[\hat{L}_x, \hat{L}_y]\psi = \{[\hat{Y}\hat{P}_z, \hat{Z}\hat{P}_x] - [\hat{Y}\hat{P}_z, \hat{X}\hat{P}_z] - [\hat{Z}\hat{P}_y, \hat{Z}\hat{P}_x] + [\hat{Z}\hat{P}_y, \hat{X}\hat{P}_z]\}\psi$$

$$[\hat{L}_x, \hat{L}_y]\psi = \{\hat{Y}[\hat{P}_z, \hat{Z}]\hat{P}_x + \hat{X}[\hat{Z}, \hat{P}_z]\hat{P}_y\}\psi = i\hbar\,(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x)\psi = i\hbar\hat{L}_z\,\psi$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$