MTL101::Linear Algebra and Differential Equations

Tutorial 5

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Question 1

Find the solutions of the following initial value problems:

(a) $y'' - 2y' - 3y = 0,\ y(0) = 0,\ y'(0) = 1$

(b) $y'' + 10y = 0,\ \ y(0) = \pi,\ y'(0) = \pi^2.$

Question 1(a)

(a) $y^{''} - 2y^{'} - 3y = 0,\ y(0) = 0,\ y^{'}(0) = 1.$

Solution:

- Consider the IVP

$$y^{''} - 2y^{'} - 3y = 0, \quad y(0) = 0,\ y^{'}(0) = 1. \tag{1}$$

- Suppose $y = e^{mx}$ is the solution of (1).
- Hence, substituting the solution in (1), we get the following characteristic equation

$$m^2 - 2m - 3 = 0$$
$$\implies (m-3)(m+1) = 0 \implies m = 3, -1.$$

- Therefore, $y(x) = c_1 e^{3x} + c_2 e^{-x}$.

- Since, $y(0) = 0,\ y^{'}(0) = 1,$

$$\implies \begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y^{'}(0) &= 3c_1 - c_2 = 1. \end{aligned}$$

- On solving, we get $c_1 = \frac{1}{4}, c_2 = -\frac{1}{4}$.
- Hence, $y(x) = \frac{e^{3x}}{4} - \frac{e^{-x}}{4}$.

Question 1(b)

(b) $y^{''} + 10y = 0, \;\; y(0) = \pi, \; y^{'}(0) = \pi^2.$

Solution:

- Consider the IVP

$$y^{''} + 10y = 0, \;\; y(0) = \pi, \; y^{'}(0) = \pi^2. \tag{2}$$

- Suppose $y = e^{mx}$ is the solution of (2).
- Hence, substituting the solution in (2), we get the following characteristic equation

$$m^2 + 10 = 0$$
$$\implies m = \pm i\sqrt{10}.$$

- Therefore, $y(x) = c_1 \cos\sqrt{10}x + c_2 \sin\sqrt{10}x.$

Question 1 (b) contd...

- Since, $y(0) = \pi,\ y^{'}(0) = \pi^2$,

$$\implies y(0) = c_1 = \pi$$
$$y^{'}(0) = \sqrt{10}c_2 = \pi^2.$$

- On solving, we get $c_1 = \pi, c_2 = \frac{\pi^2}{\sqrt{10}}$.
- Hence, $y(x) = \pi \cos \sqrt{10}x + \frac{\pi^2}{\sqrt{10}} \sin \sqrt{10}x$.

Question 2

Find a function ϕ which has a continuous derivative on $0 \leq x \leq 2$ which satisfies $\phi(0) = 0, \phi'(0) = 1$, and $y^{''} - y = 0$ for $0 \leq x \leq 1$, and $y^{''} - 9y = 0$ for $1 \leq x \leq 2$.

Solution:

- For $0 \leq x \leq 1$, $y(x)$ satisfies the following IVP

$$y^{''} - y = 0, \ \ y(0) = 0, \ \ y'(0) = 1.$$

- On solving, we get,

$$y(x) = c_1 e^x + c_2 e^{-x}. \tag{3}$$

- Since, $y(0) = 0, \ \ y'(0) = 1$

$$\implies \ y(0) = c_1 + c_2 = 0$$
$$y^{'}(0) = c_1 - c_2 = 1.$$

Question 2 contd...

- On solving, we get $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$.
- Hence, for $0 \leq x \leq 1$,

$$y_1(x) := y(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}. \tag{4}$$

- For $1 \leq x \leq 2$, $y(x)$ satisfies

$$y^{''} - 9y = 0.$$

- On solving, we get,

$$y_2(x) := y(x) = c_3 e^{3x} + c_4 e^{-3x}. \tag{5}$$

- Since, the function $y(x)$ has a continuous derivative on $0 \leq x \leq 2$, hence, $y_1(1) = y_2(1)$ and $y_1^{'}(1) = y_2^{'}(1)$.

- Thus,

$$\frac{1}{2}e - \frac{1}{2}e^{-1} = c_3 e^3 + c_4 e^{-3}$$
$$\frac{1}{2}e + \frac{1}{2}e^{-1} = 3c_3 e^3 - 3c_4 e^{-3}.$$

- On solving, we get, $c_3 = \frac{1}{6}(2e^{-2} - e^{-4}),\ c_4 = \frac{1}{6}(e^4 - 2e^2)$.
- Hence,

$$y(x) = \begin{cases} \frac{1}{2}e^x - \frac{1}{2}e^{-x}, & \text{if } 0 \le x \le 1 \\ \frac{1}{6}(2e^{-2} - e^{-4})e^{3x} + \frac{1}{6}(e^4 - 2e^2)e^{-3x}, & \text{if } 1 \le x \le 2. \end{cases}$$

Question 3

Consider the constant coefficient equation $L(y) = y^{''} + a_1 y^{'} + a_2 y = 0$. Let ϕ_1 be the solution satisfying $\phi_1(x_0) = 1, \phi_1^{'}(x_0) = 0$, and ϕ_2 be the solution satisfying $\phi_2(x_0) = 0, \phi_2^{'}(x_0) = 1$. If ϕ is a solution satisfying $\phi(x_0) = \alpha, \phi^{'}(x_0) = \beta$, show that $\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x)$ for all x.

Solution:

- Consider the equation with constant coefficients

$$L(y) = y^{''} + a_1 y^{'} + a_2 y = 0. \tag{6}$$

- Suppose ϕ_1 and ϕ_2 are the solution of (6) satisfying $\phi_1(x_0) = 1, \phi_1^{'}(x_0) = 0, \phi_2(x_0) = 0, \phi_2^{'}(x_0) = 1.$
- This implies,

$$\phi_i^{''} + a_1 \phi_i^{'} + a_2 \phi_i = 0, \quad \forall i = 1, 2. \tag{7}$$

- Also, one can calculate the wronskian of ϕ_1, ϕ_2 at x_0 as

$$\begin{aligned} W(\phi_1, \phi_2)(x_0) &= \phi_1(x_0)\phi_2^{'}(x_0) - \phi_1^{'}(x_0)\phi_2(x_0) \\ &= 1. \end{aligned} \tag{8}$$

- Since, $W(\phi_1, \phi_2)(x) = \phi_1(x)\phi_2^{'}(x) - \phi_1^{'}(x)\phi_2(x)$.
- Differentiating wronskian with respect to x and using (7), we get,

$$\begin{aligned} W^{'}(\phi_1, \phi_2)(x) &= \phi_1(x)\phi_2^{''}(x) - \phi_1^{''}(x)\phi_2(x) \\ &= -\phi_1(x)(a_1\phi_2^{'}(x) + a_2\phi_2(x)) + \phi_2(x)(a_1\phi_1^{'}(x) + a_2\phi_1(x)) \\ &= -a_1(\phi_1(x)\phi_2^{'}(x) - \phi_1^{'}(x)\phi_2(x)) \\ &= -a_1 W(\phi_1, \phi_2)(x). \end{aligned} \tag{9}$$

- Eq. (9) is a first order differential equation in $W(x)$, on solving we get, $W(x) = ce^{-a_1 x}$.

- Since, $W(x_0) = 1$, we get $c = e^{a_1 x_0}$.
- Hence, $W(x) = e^{-a_1(x - x_0)} \neq 0, \ \forall x$.
- Hence, the solutions ϕ_1, ϕ_2 are linearly independent and solution ϕ can be expressed as linear combination of ϕ_1 and ϕ_2,

$$\phi(x) = A\phi_1(x) + B\phi_2(x).$$

- Given that, $\phi(x_0) = \alpha, \phi^{'}(x_0) = \beta$. Using these conditions, we get $A = \alpha$ and $B = \beta$.
- Hence, $\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x), \ \forall \ x$.

Question 4

Question 4

Let ϕ_1, ϕ_2 be two differentiable functions on an interval I, which are not necessarily solutions of an equation $L(y) = 0$. Prove the following:

(a) If ϕ_1, ϕ_2 are linearly dependent on I, then $W(\phi_1, \phi_2)(x) = 0$ for all x in I.

(b) If $W(\phi_1, \phi_2)(x_0) \neq 0$ for some x_0 in I, then ϕ_1, ϕ_2 are linearly independent on I.

(c) $W(\phi_1, \phi_2)(x) = 0$ for all x in I does not imply that ϕ_1, ϕ_2 are linearly dependent on I.

(d) $W(\phi_1, \phi_2)(x) = 0$ for all x in I, and $\phi_2(x) \neq 0$ on I, imply that ϕ_1, ϕ_2 are linearly dependent on I.

Question 4(a)

Question 4(a)

(a) If ϕ_1, ϕ_2 are linearly dependent on I, then $W(\phi_1, \phi_2)(x) = 0$ for all x in I.

Solution:

- Let ϕ_1, ϕ_2 be linearly dependent functions on I.
- Then

$$c_1\phi_1(x) + c_2\phi_2(x) = 0, \ \forall \ x \in I, \tag{10}$$

  implies, at least one of c_1 and c_2 is non-zero.
- If it is possible, let us assume that $\exists \ x_0 \in I$ such that $W(\phi_1, \phi_2)(x_0) = \phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0) \neq 0$.

- On differentiating (10), we get the following system of linear equations,

$$\begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1^{'}(x) & \phi_2^{'}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- At $x = x_0 \ \in I$,

$$\underbrace{\begin{pmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1^{'}(x_0) & \phi_2^{'}(x_0) \end{pmatrix}}_{:=A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11}$$

- Since, $det(A) = W(\phi_1, \phi_2)(x_0) \neq 0$.
- Hence, from (11), we get $c_1 = c_2 = 0$, which is a contradiction.
- Hence, our assumption is wrong. Thus, $W(\phi_1, \phi_2)(x) = 0 \ \forall x \in I$.

Question 4(b)

Question 4(b)

(b) If $W(\phi_1, \phi_2)(x_0) \neq 0$ for some x_0 in I, then ϕ_1, ϕ_2 are linearly independent on I.

Solution:

- Let $W(\phi_1, \phi_2)(x_0) \neq 0$ for some $x_0 \in I$.
- To show: ϕ_1, ϕ_2 are linearly independent on I, consider

$$c_1\phi_1(x) + c_2\phi_2(x) = 0 \ \forall \ x \in I, \tag{12}$$

- On differentiating (10), we get the following system of linear equations,

$$\begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1^{'}(x) & \phi_2^{'}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- At $x = x_0 \ \in I$,

$$\underbrace{\begin{pmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1^{'}(x_0) & \phi_2^{'}(x_0) \end{pmatrix}}_{:=A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{13}$$

Question 4(b) contd...

- Since, $det(A) = W(\phi_1, \phi_2)(x_0) \neq 0$.
- So, from (13), we get $c_1 = c_2 = 0$.
- Hence, ϕ_1, ϕ_2 are linearly independent on I.

Question 4(c)

(c) $W(\phi_1, \phi_2)(x) = 0$ for all x in I does not imply that ϕ_1, ϕ_2 are linearly dependent on I.

Solution:

- For example, take $\phi_1 = x$ and $\phi_2 = |x|$.
- Then, for $x \geq 0$, $W(\phi_1, \phi_2) = \begin{vmatrix} x & x \\ 1 & 1 \end{vmatrix} = 0.$
- For $x < 0$, $W(\phi_1, \phi_2) = \begin{vmatrix} x & -x \\ 1 & -1 \end{vmatrix} = 0.$
- Consider, $c_1 x + c_2 |x| = 0$, then

$$\text{for } x \geq 0, \quad c_1 x + c_2 x = 0,$$
$$\text{for } x < 0, \quad c_1 x - c_2 x = 0.$$

- On solving above equations for c_1 and c_2, we get $c_1 = c_2 = 0$.
- Hence, ϕ_1, ϕ_2 are linearly independent on I.

Question 4(d)

(d) $W(\phi_1, \phi_2)(x) = 0$ for all x in I, and $\phi_2(x) \neq 0$ on I, imply that ϕ_1, ϕ_2 are linearly dependent on I.

Solution:

- **Case (i)** When $\phi_1 = 0, \phi_2 \neq 0$.
  - Then, $W(\phi_1, \phi_2)(x) = \phi_1(x)\phi_2^{'}(x) - \phi_1^{'}(x)\phi_2(x) = 0$.
  - Also, ϕ_1, ϕ_2 are linearly dependent on I as every zero function is linearly dependent to every function.

- **Case (ii)** When $\phi_1 \neq 0, \phi_2 \neq 0$.
  - Given,

$$\begin{aligned}
& W(\phi_1, \phi_2)(x) = 0, \\
\implies & \phi_1(x)\phi_2^{'}(x) - \phi_1^{'}(x)\phi_2(x) = 0, \\
\implies & \frac{d\phi_2}{\phi_2} = \frac{d\phi_1}{\phi_1}, \\
\implies & \log\phi_2 = \log\phi_1 + \log C, \\
\implies & \phi_2 = C\phi_1, \text{ where } C \text{ is the integration constant.}
\end{aligned}$$

  - Hence, ϕ_1, ϕ_2 are linearly dependent on I as both are scaler multiple of each other.

Question 5

Find all solutions of the following equations:

(a) $4y^{''} - y = e^x$

(b) $y^{''} + 4y = \cos x$

(c) $y^{''} + 9y = \sin 3x.$

Question 5

Recall: Short methods of finding particular integral:

Consider the general n^{th} order linear equation of the form

$$f(D)y = (a_0 D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n)y = q(x),$$

where the coefficients $a_0, a_1, ..., a_n$ are constants.

(i) $q(x) = e^{\alpha x}$, α constant

$$\text{when } f(\alpha) \neq 0,\ \frac{1}{f(D)} e^{\alpha x} = \frac{e^{\alpha x}}{f(\alpha)},$$

$$\text{when } f(\alpha) = 0,\ \frac{1}{f(D)} e^{\alpha x} = \frac{1}{(D-\alpha)^p \phi(D)} e^{\alpha x} = \frac{x^p e^{\alpha x}}{p!\phi(\alpha)}, \quad \phi(\alpha) \neq 0.$$

Question 5

Recall: Short methods of finding particular integral:

(ii) $q(x) = \cos(ax+b)$ or $\sin(ax+b)$, a, b constant

$$\text{when } f(-a^2) \neq 0, \quad \frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}$$

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}$$

(iii)

$$\frac{1}{f(D)}\cos(ax+b) = Re\left[\frac{1}{f(D)}e^{i(ax+b)}\right]$$

$$\frac{1}{f(D)}\sin(ax+b) = Im\left[\frac{1}{f(D)}e^{i(ax+b)}\right]$$

where symbols Re and Im read as 'real part of' and 'imaginary part of' respectively.

Question 5(a)

(a) $4y^{''} - y = e^x$

Solution:

- First we solve the homogeneous equation $4y^{''} - y = 0$.
- Auxiliary equation is given by, $4m^2 - 1 = 0 \implies m = \pm\frac{1}{2}$.
- Hence, the complementary function is, $y_c(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$.
- Now, for the given ODE we assume the particular integral of the form $y_p(x) = Ae^x$,

$$y_p^{'}(x) = Ae^x = y_p^{''}(x).$$

Question 5(a) contd...

- Replacing in the given ODE, we get

$$4Ae^x - Ae^x = e^x$$

- Therefore, $A = \frac{1}{3}$ and hence, $y_p(x) = \frac{1}{3}e^x$.
- Hence, the general solution is,

$$\begin{aligned} y(x) &= y_c(x) + y_p(x), \\ &= c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} + \frac{1}{3}e^x. \end{aligned}$$

Question 5(b)

Question 5(b)

(b) $y^{''} + 4y = \cos x$

Solution:

- First we solve the homogeneous equation $y^{''} + 4y = 0$.
- Characteristic equation is given by, $m^2 + 4 = 0 \implies m = \pm i2$.
- Hence, the complementary function is, $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$.
- Now, the particular integral is

$$y_p(x) = \frac{1}{3} \cos x.$$

- Hence, the general solution is,

$$\begin{aligned} y(x) &= y_c(x) + y_p(x), \\ &= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \cos x. \end{aligned}$$

Question 5(c)

(c) $y^{''} + 9y = \sin 3x$.

Solution:

- First we solve the homogeneous equation $y^{''} + 9y = 0$.
- Auxiliary equation is given by, $m^2 + 9 = 0 \implies m = \pm i3$.
- Hence, the complementary function is, $y_c(x) = c_1 \cos 3x + c_2 \sin 3x$.
- Now, the particular integral is

$$\begin{aligned} y_p(x) &= \frac{1}{D^2+9} \sin 3x = \frac{1}{D^2+9} e^{i3x} \\ &= \frac{1}{(D+3i)(D-3i)} e^{i3x} \\ &= \frac{1}{2ai} \left[\frac{1}{D-3i} - \frac{1}{D+3i} \right] e^{i3x} \end{aligned}$$

Question 5(c) contd...

-
$$
\begin{aligned}
&= \frac{1}{2ai}\left[\frac{1}{D-3i}e^{i3x} - \frac{e^{i3x}}{2ai}\right], \text{ (here } D-3i=0 \text{ for } D=3i)\\
&= \frac{1}{2ai}\left[e^{i3x}\int e^{-i3x}e^{i3x}dx - \frac{e^{i3x}}{2ai}\right],\\
&= \frac{1}{2ai}\left[xe^{i3x} - \frac{e^{i3x}}{2ai}\right],\\
&= \frac{ix}{6}(\cos 3x + i\sin 3x) + \frac{1}{36}(\cos 3x + i\sin 3x),\\
y_p(x) &= \frac{-x\cos 3x}{6} + \frac{\sin 3x}{36}, \text{ taking the imaginary part.}
\end{aligned}
$$

- Hence, the general solution is,

$$
\begin{aligned}
y(x) &= y_c(x) + y_p(x),\\
&= c_1\cos 3x + c_2^{'}\sin 3x - \frac{1}{6}x\cos 3x.
\end{aligned}
$$

Question 6

Let $L(y) = y^{''} + a_1 y^{'} + a_2 y = 0$, where a_1, a_2 are constants, and let p be the characteristic equation $p(r) = r^2 + a_1 r + a_2$.

(a) If A, are constants and $p(\alpha) \neq 0$, show that there is a solution ϕ of $L(y) = Ae^{\alpha x}$ of the form $\phi(x) = Be^{\alpha x}$, where B is a constant.

(b) Compute a particular solution of $L(y) = Ae^{\alpha x}$ in case $p(\alpha) = 0$.

Solution:

- Given, the characteristic equation is
  $p(r) = r^2 + a_1 r + a_2 = 0 \implies r_1 = \frac{-a_1+\sqrt{a_1^2-4a_2}}{2}, r_2 = \frac{-a_1-\sqrt{a_1^2-4a_2}}{2}.$
- Hence, the complementary solution is $y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
- Since, $p(\alpha) \neq 0 \implies \alpha$ is not a root of the characteristic equation i.e. $\alpha \neq r_1$ & $\alpha \neq r_2$.
- Now, the particular integral is

$$
\begin{aligned}
y_p(x) &= \frac{1}{f(D)} e^{\alpha x} = \frac{1}{(D-r_1)(D-r_2)} e^{\alpha x} \\
&= \frac{1}{(\alpha - r_1)(\alpha - r_2)} e^{\alpha x} \\
&= Be^{\alpha x}, \text{ where } B = \frac{1}{(\alpha - r_1)(\alpha - r_2)} \\
&= \phi(x).
\end{aligned}
$$

Solution:

- Given, $p(\alpha) = 0 \implies \alpha$ is not a root of the characteristic equation.
- Assume the particular integral of $L(y) = Ae^{\alpha x}$ is of the form $y_p(x) = cxe^{\alpha x}$

$$\begin{aligned} \therefore y_p^{'}(x) &= cx\alpha e^{\alpha x} + ce^{\alpha x} \\ y_p^{''}(x) &= cx\alpha^2 e^{\alpha x} + 2c\alpha e^{\alpha x}. \end{aligned}$$

- Plugging in $L(y) = Ae^{\alpha x}$ and comparing the coefficients, we get,

$$2c\alpha + a_1 c = A \implies c = \frac{A}{2\alpha + a_1}$$
$$c\alpha^2 + a_1 c\alpha + a_2 c = 0 = cp(\alpha) = c.0 = 0, \text{ since, } \alpha \text{ is the root of } p(r).$$

- Hence, $y_p(x) = \frac{A}{2\alpha+a_1} xe^{\alpha x}$.

Question 7

Are the following set of functions defined on $-\infty < x < \infty$ linearly dependent or independent there? Why?

(a) $\phi_1(x) = 1, \phi_2(x) = x, \phi_3(x) = x^3$

(b) $\phi_1(x) = x, \phi_2(x) = e^{2x}, \phi_3(x) = |x|$.

Question 7(a)

Question 7(a)

(a) $\phi_1(x) = 1, \phi_2(x) = x, \phi_3(x) = x^3$.

Recall:

Let ϕ_1, ϕ_2 be two differentiable functions on an interval I, such that $W(\phi_1, \phi_2)(x_0) \neq 0$ for some x_0 in I, then ϕ_1, ϕ_2 are linearly independent on I.

Solution:

- $W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1^{'} & \phi_2^{'} & \phi_3^{'} \\ \phi_1^{''} & \phi_2^{''} & \phi_3^{''} \end{vmatrix} = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x.$
- Since, $W(\phi_1, \phi_2, \phi_3)(1) = 6 \neq 0$.
- Hence, ϕ_1, ϕ_2, ϕ_3 are linearly independent on $I = (-\infty, \infty)$.

Question 7(b)

(b) $\phi_1(x) = x, \phi_2(x) = e^{2x}, \phi_3(x) = |x|$.

Solution:

- Consider $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 = 0$.

$$\text{for } x \geq 0 : c_1 x + c_2 e^{2x} + c_3 x = 0 \tag{14}$$
$$\text{for } x < 0 : c_1 x + c_2 e^{2x} - c_3 x = 0 \tag{15}$$
$$\implies c_3 = 0, \ ((14) - (15)).$$

- Thus, we have,

$$c_1 x + c_2 e^{2x} = 0$$
$$c_1 x + c_2 e^{2x} = 0$$
$$\implies \underbrace{\begin{pmatrix} x & e^{2x} \\ 1 & 2e^{2x} \end{pmatrix}}_{:=A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Question 7(b) contd...

- Since, $det(A) \neq 0 \implies c_1 = c_2 = c_3 = 0$.
- Hence, ϕ_1, ϕ_2, ϕ_3 are linearly independent on $I = (-\infty, \infty)$.

Question 8

Question 8

Use the method of undetermined coefficients to find a particular solution of each of the following equations:

(a) $y'' + 4y = \cos x$

(b) $y'' + 4y = \sin 2x$

(c) $y'' - y' - 2y = x^2 + \cos x$

(d) $y'' + 9y = x^2 e^{3x}$.

Question 8(a)

(a) $y^{''} + 4y = \cos x$

Solution:

- First we solve the homogeneous equation $y^{''} + 4y = 0$.
- Characteristic equation is given by, $m^2 + 4 = 0 \implies m = \pm i2$.
- Hence, the complementary function is, $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, where c_1, c_2 are arbitrary constants.
- Since, the complementary function and the non-homogeneous term of differential equation does not have common term, so, let the trial solution be $y_p(x) = A\cos x + B\sin x$, where A, B are unknown constants

$$y_p^{'}(x) = -A\sin x + B\cos x$$
$$y_p^{''}(x) = -A\cos x - B\sin x.$$

Question 8(a)

- Replacing in the given ODE and comparing coefficients, we get $A = \frac{1}{3}, B = 0$.
- Hence, $y_p(x) = \frac{1}{3} \cos x$.

Question 8(b)

(b) $y^{''} + 4y = \sin 2x$

Solution:

- The complementary function is, $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, where c_1, c_2 are arbitrary constants.
- Since, the complementary function and the non-homogeneous term of differential equation have common term, so, let the trial solution be $y_p(x) = x(A \sin 2x + B \cos 2x)$, where A, B are unknown constants

$$y_p^{'}(x) = A \sin 2x + B \cos 2x + x(2A \cos 2x - 2B \sin 2x)$$
$$y_p^{''}(x) = 4A \cos 2x - 4B \sin 2x + x(-4A \sin 2x - 4B \cos 2x).$$

- Replacing in the given ODE and comparing coefficients, we get $A = 0, B = -\frac{1}{4}$.
- Hence, $y_p(x) = -\frac{1}{4} x \cos 2x$.

Question 8(c)

(c) $y^{''} - y^{'} - 2y = x^2 + \cos x$

Solution:

- First we solve the homogeneous equation $y^{''} - y^{'} - 2y = 0$.
- Characteristic equation is given by, $m^2 - m - 2 = 0 \implies m = 2, -1$.
- Hence, the complementary function is, $y_c(x) = c_1 e^{2x} + c_2 e^{2x}$, where c_1, c_2 are arbitrary constants.
- Since, the complementary function and the non-homogeneous term of Differential equation does not have common term, so, let the trial solution be $y_p(x) = Ax^2 + Bx + C + D \sin x + E \cos x$, where A, B, C, D, E are unknown constants.
- Differentiating with respect to x,

$$y_p^{'}(x) = 2Ax + B + D \cos x - E \sin x$$
$$y_p^{''}(x) = 2A - D \sin x - E \cos x$$

Question 8(c) contd...

- Plugging in the given ODE and comparing coefficients both sides, we get

$$\begin{aligned}
-2A &= 1,\\
-2A - 2B &= 0,\\
2A - B - 2C &= 0,\\
-D + E - 2D &= 0,\\
-E - D - 2E &= 1.
\end{aligned}$$

- On solving, we get $A = -\frac{1}{2}, B = \frac{1}{2}, C = -\frac{3}{4}, D = -\frac{1}{10}, E = -\frac{3}{10}$.
- Hence, $y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4} - \frac{1}{10}\sin x - \frac{3}{10}\cos x$.

Question 8(d)

(d) $y^{''} + 9y = x^2 e^{3x}$.

Solution:

- Let the trial solution is, $y_p(x) = (Ax^2 + Bx + C)e^{3x}$, where, A, B, C are unknown constants.
- Differentiating with respect to x,

$$y_p^{'}(x) = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x},$$
$$y_p^{''}(x) = e^{3x}(9Ax^2 + (12A + 9B)x + 2A + 6B + 9C).$$

- Plugging in the given ODE and comparing coefficients both sides, we get

$$18A = 1,$$
$$12A + 18B = 0$$
$$2A + 6B + 18C = 0.$$

Question 8(d) contd...

- On solving, we get $A = \frac{1}{18}, B = -\frac{1}{27}, C = \frac{1}{162}$.
- Hence, $y_p(x) = (\frac{1}{18}x^2 - \frac{1}{27}x + \frac{1}{162})e^{3x}$.

Question 9

Find a real solution.

(a) $x^2y^{''} - 4xy^{'} + 6y = 0,$

(b) $4x^2y^{''} + 12xy^{'} + 3y = 0$

(c) $x^2y^{''} + 7xy^{'} + 9y = 0,$

(d) $x^2y^{''} - 2.5xy^{'} - 2y = 0,$

(e) $x^2y^{''} + 7xy^{'} + 13y = 0.$

Question 9

Recall: Cauchy's homogeneous linear equation

An equation of the form $x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + k_n y = X$, where X is the function of x, is called Cauchy's homogeneous linear equation.

1. Such equations can be reduced to linear differential equation with constant coefficients by putting $x = e^t$ or $t = \log x$.
2. Hence, by chain rule, we have

$$x\frac{dy}{dx} = \frac{dy}{dt},$$
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

3. After making these substitutions in given ODE, this results a linear equation with constant coefficients.

Question 9(a)

(a) $x^2y^{''} - 4xy^{'} + 6y = 0,$

Solution:

- Put $x = e^t$ in the given ODE.
- Then it is transformed into a linear differential equation with constant coefficient, $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0.$
- Characteristic equation is given by, $m^2 - 5m + 6 = 0 \implies m = 2, 3.$
- Hence, the solution is $y(x) = c_1e^{2t} + c_2e^{3t} = c_1x^2 + c_2x^3, \; x > 0$, where c_1, c_2 are arbitrary constants.

Question 9(b)

(b) $4x^2y^{''} + 12xy^{'} + 3y = 0$

Solution:

- Using the transformation $x = e^t$, we get the following linear differential equation with constant coefficient

$$4\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 3y = 0.$$

- Characteristic equation is given by, $4m^2 + 8m + 3 = 0 \implies m = -1, -\frac{3}{2}$.
- Hence, the solution is $y(x) = c_1e^{-t} + c_2e^{-\frac{3}{2}t} = c_1x^{-1} + c_2x^{-\frac{3}{2}},\ x > 0$, where c_1, c_2 are arbitrary constants.

Question 9(c)

(c) $x^2y^{''} + 7xy^{'} + 9y = 0,$

Solution:

- Using the transformation $x = e^t$, we get the following linear differential equation with constant coefficients

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0.$$

- Characteristic equation is given by, $m^2 + 6m + 9 = 0 \implies m = -3, -3.$
- Hence, the solution is
$y(x) = c_1e^{-3t} + c_2te^{-3t} = c_1x^{-3} + c_2x^{-3}\log x,\ x > 0$, where c_1, c_2 are arbitrary constants.

Question 9(d)

(d) $x^2y^{''} - 2.5xy^{'} - 2y = 0,$

Solution:

- Using the transformation $x = e^t$, we get the following linear differential equation with constant coefficient

$$2\frac{d^2y}{dt^2} - 7\frac{dy}{dt} - 4y = 0.$$

- Characteristic equation is given by, $2m^2 - 7m - 4 = 0 \implies m = 4, -\frac{1}{2}$.
- Hence, the solution is $y(x) = c_1e^{4t} + c_2e^{-\frac{1}{2}t} = c_1x^4 + c_2x^{-\frac{1}{2}},\ x > 0$, where c_1, c_2 are arbitrary constants.

Question 9(e)

(e) $x^2y^{''} + 7xy^{'} + 13y = 0.$

Solution:

- Using the transformation $x = e^t$, we get the following linear differential equation with constant coefficient

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0.$$

- Characteristic equation is given by, $m^2 + 6m + 13 = 0 \implies m = -3 \pm 2i.$
- Hence, the solution is
$y(x) = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t) = x^{-3}(c_1 \cos(\log x^2) + c_2 \sin(\log x^2), \ x > 0,$
where c_1, c_2 are arbitrary constants.

Question 10

Solve the initial value problems.

(a) $x^2y'' - 2xy' + 2y = 0, y(1) = 1.5, y'(1) = 1.$

(b) $x^2y'' + 3xy' + y = 0, y(1) = 3, y'(1) = -4.$

(c) $x^2y'' - 3xy' + 4y = 0, y(1) = 0, y'(1) = 3.$

Question 10(a)

(a) $x^2y^{''} - 2xy^{'} + 2y = 0, y(1) = 1.5, y^{'}(1) = 1.$

Solution:

- Above ODE can be reduced to following linear differential equation with constant coefficients by using $x = e^t$,

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

- Characteristic equation is given by, $m^2 - 3m + 2 = 0 \implies m = 1, 2.$
- Hence, the solution is $y(x) = c_1e^t + c_2e^{2t} = c_1x + c_2x^2,\ x > 0.$

- Since, $y(1) = 1.5, y^{'}(1) = 1$, putting these initial conditions, we get

$$y(1) = c_1 + c_2 = 1.5$$
$$y^{'}(1) = c_1 + 2c_2 = 1.$$

- On solving we get, $c_1 = 2, c_1 = -\frac{1}{2}$.
- Hence, $y(x) = 2x - \frac{1}{2}x^2$.

Question 10(b)

(b) $x^2y^{''} + 3xy^{'} + y = 0, y(1) = 3, y^{'}(1) = -4.$

Solution:

- Above ODE can be reduced to following linear differential equation with constant coefficients by using $x = e^t$,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

- Characteristic equation is given by, $m^2 + 2m + 1 = 0 \implies m = -1, -1.$
- Hence, the solution is $y(x) = c_1e^{-t} + c_2te^{-t} = (c_1 + c_2 \log x)x^{-1},\ x > 0.$
- Since, $y(1) = 3, y^{'}(1) = -4$, putting these initial conditions, we get, $c_1 = 3, c_1 = -1.$
- Hence, $y(x) = (3 - \log x)x^{-1},\ x > 0.$

Question 10(c)

Question 10(c)

(c) $x^2y^{''} - 3xy^{'} + 4y = 0, y(1) = 0, y^{'}(1) = 3.$

Solution:

- Above ODE can be reduced to following linear differential equation with constant coefficients by using $x = e^t$,

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0.$$

- Characteristic equation is given by, $m^2 - 4m + 4 = 0 \implies m = 2, 2.$
- Hence, the solution is $y(x) = c_1e^{2t} + c_2te^{2t} = (c_1 + c_2 \log x)x^2,\ x > 0.$
- Since, $y(1) = 0, y^{'}(1) = 3$, putting these initial conditions, we get, $c_1 = 0, c_1 = 3.$
- Hence, $y(x) = 3x^2 \log x,\ x > 0.$

Question 11

Find all solutions of the following equations:

(a) $y''' - 8y = 0.$

(b) $y^{(4)} + 16y = 0.$

(c) $y^{(100)} + 100y = 0.$

(d) $y^{(4)} - 16y = 0.$

Question 11(a)

(a) $y''' - 8y = 0.$

Solution:

- The characteristic equation is given by, $m^3 - 8 = 0 \implies m = 2, -1 \pm i\sqrt{3}.$
- Hence, $y(x) = c_1 e^{2x} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}c)$, where c_1, c_2, c_3 are arbitrary constants.

Question 11(b)

(b) $y^{(4)} + 16y = 0.$

Solution:

- The characteristic equation is given by, $m^4 + 16 = 0 \implies m^2 = \pm 4i$.
- **Case (i)** $m^2 = 4i$
  $\implies m = \pm 2\sqrt{i} = \pm(\sqrt{2} + i\sqrt{2})$.
- **Case (ii)** $m^2 = -4i = 4i^3$
  $\implies m = \pm 2i^{\frac{3}{2}} = \pm(-\sqrt{2} + i\sqrt{2})$.
- Hence, $y(x) =$
  $e^{\sqrt{2}x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + e^{-\sqrt{2}x}(c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x)$,
  where c_1, c_2, c_3, c_4 are arbitrary constants.

Question 11(c)

(c) $y^{(100)} + 100y = 0.$

Solution:

- The characteristic equation is given by,

$$m^{100} + 100 = 0$$
$$\implies m^{100} = -100 = -\{100^{\frac{1}{100}}\}^{100}$$
$$\implies m = (-1)^{\frac{1}{100}} 100^{\frac{1}{100}}$$
$$= e^{\frac{(2n+1)}{100} i\pi} 100^{\frac{1}{100}}$$
$$= (\cos \frac{(2n+1)\pi}{100} + i \sin \frac{(2n+1)\pi}{100})100^{\frac{1}{100}}, \text{ for } n = 0, 1, 2, ..., 99.$$

- Hence, $y(x) = \sum_0^{99} e^{100^{\frac{1}{100}} \cos \frac{(2n+1)\pi}{100} x} \left[c_n \cos\left(100^{\frac{1}{100}} \sin \frac{(2n+1)\pi}{100} x\right) + d_n \sin\left(100^{\frac{1}{100}} \sin \frac{(2n+1)\pi}{100} x\right)\right]$, where c_n, d_n are arbitrary constants.

Question 11(d)

(d) $y^{(4)} - 16y = 0.$

Solution:

- The characteristic equation is given by, $m^4 - 16 = 0 \implies m = \pm 2, \pm 2i.$
- Hence, $y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$, where c_1, c_2, c_3, c_4 are arbitrary constants.

Question 12

Question 12

Use the variation of parameters method to solve the following equations:

(a) $y''' - y' = x.$

(b) $y^{(4)} + 16y = \cos x.$

(b) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x.$

Question 12

Recall: Variation of parameters method

Consider the N^{th} order non-homogeneous differential equation

$$y^{(N)} + a_1 y^{(N-1)} + \dots + a_{N-1} y^{'} + a_N y = g,$$

where $a_1, a_2, \dots, a_N$ are arbitrary constants and g is the function of x. Then variation of parameter method gives particular integral given by:

$$y_p(x) = \sum_{k=1}^{N} (-1)^{N+k} y_k(x) \int \frac{W_k(s) g(s)}{W(s)}$$

where, W is the Wronskian of fundamental set $y_1, y_2, \dots, y_N$ and W_k is the determinant of submatrix of Wronskian matrix obtained by deleting last row and k^{th} column.

Question 12a

(a) $y^{'''} - y^{'} = x.$

Solution:

- The characteristic equation is given by, $m^3 - m = 0 \implies m = 0, \pm 1.$
- Hence, $y_c(x) = c_1 + c_2 e^x + c_3 e^{-x}$, where c_1, c_2, c_3 are arbitrary constants.
- Let $y_1(x) = 1, y_2(x) = e^x, y_3(x) = e^{-x}.$
- Then, $W(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2.$
- In the same way, $W_1(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2,$

Question 12(a) contd...

- $W_2(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x}$,
- $W_3(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^{x} \\ 0 & e^{x} \end{vmatrix} = e^{x}$.
- Hence,

$$\begin{aligned} y_p(x) &= y_1 \int \frac{f(x)W_1}{W} dx - y_2 \int \frac{f(x)W_2}{W} dx + y_3 \int \frac{f(x)W_3}{W} dx \\ &= 1 \int \frac{x(-2)}{2} dx - e^{x} \int \frac{x(-e^{-x})}{2} dx + e^{-x} \int \frac{x(e^{x})}{2} dx \\ &= -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{2} + \frac{x}{2} - \frac{1}{2} \\ &= -\frac{x^2}{2} + 1. \end{aligned}$$

- Hence, the general solution is,

$$y(x) = c_1 + c_2 e^x + c_3 e^{-x} - \frac{x^2}{2} + 1,$$

where c_1, c_2, c_3 are arbitrary constants.

Question 12b

(b) $y^{(4)} + 16y = \cos x$.

Solution:

- The characteristic equation is given by, $m^4 + 16 = 0 \implies m = \pm\sqrt{2} \pm i\sqrt{2}$.
- Hence, $y_c(x) =$
  $e^{\sqrt{2}x}(c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)) + e^{-\sqrt{2}x}(c_3 \cos(\sqrt{2}x) + c_4 \sin(\sqrt{2}x))$,
  where c_1, c_2, c_3, c_4 are arbitrary constants.
- Let $y_1(x) = e^{\sqrt{2}x} \cos(\sqrt{2}x), y_2(x) = e^{\sqrt{2}x} \sin(\sqrt{2}x), y_3(x) =$
  $e^{-\sqrt{2}x} \cos(\sqrt{2}x), y_4(x) = e^{-\sqrt{2}x} \sin(\sqrt{2}x)$.
- Then, $W(y_1, y_2, y_3, y_4) = 256$.

- In similar way we calculate,
  $W_1(y_1, y_2, y_3, y_4) = 8\sqrt{2}e^{-\sqrt{2}x}\left(\cos\left(\sqrt{2}x\right) + \sin\left(\sqrt{2}x\right)\right),$
- $W_2(y_1, y_2, y_3, y_4) = 8\sqrt{2}e^{-\sqrt{2}x}\left(\cos\left(\sqrt{2}x\right) - \sin\left(\sqrt{2}x\right)\right),$
- $W_3(y_1, y_2, y_3, y_4) = -8\sqrt{2}e^{\sqrt{2}x}\left(\cos\left(\sqrt{2}x\right) - \sin\left(\sqrt{2}x\right)\right),$
- $W_4(y_1, y_2, y_3, y_4) = 8\sqrt{2}e^{\sqrt{2}x}\left(\cos\left(\sqrt{2}x\right) + \sin\left(\sqrt{2}x\right)\right).$
- Hence, plugging all the values in below formula we calculate the particular integral,

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{f(x)W_1}{W}dx + y_2 \int \frac{f(x)W_2}{W}dx \\ &- y_3 \int \frac{f(x)W_3}{W}dx + y_4 \int \frac{f(x)W_4}{W}dx \\ &= \frac{1}{17}\cos x. \end{aligned}$$

Question 12(b) contd...

- Hence, the general solution is,

$$y(x) = e^{\sqrt{2}x}(c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)) \\ + e^{-\sqrt{2}x}(c_3 \cos(\sqrt{2}x) + c_4 \sin(\sqrt{2}x)) + \frac{1}{17} \cos x,$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Question 12c

(c) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x$.

Solution:

- The characteristic equation is given by,
  $m^4 - 4m^3 + 6m^2 - 4m + 1 = 0 \implies m = 1, 1, 1, 1.$
- Hence, $y_c(x) = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^x$, where c_1, c_2, c_3, c_4 are arbitrary constants.
- Let $y_1(x) = e^x, y_2(x) = xe^x, y_3(x) = x^2e^x, y_4(x) = x^3e^x$.
- Then, $W(y_1, y_2, y_3, y_4) =$

$$\begin{vmatrix} e^x & xe^x & x^2e^x & x^3e^x \\ e^x & (x+1)e^x & (2x+x^2)e^x & (x^3+3x^2)e^x \\ e^x & (x+2)e^x & (4x+x^2+2)e^x & (x^3+6x^2+6x)e^x \\ e^x & (x+3)e^x & (6x+x^2+6)e^x & (x^3+9x^2+18x+6)e^x \end{vmatrix} = 12e^{4x}.$$

- In similar way we calculate, $W_1(y_1, y_2, y_3, y_4) =$
$$\begin{vmatrix} xe^x & x^2e^x & x^3e^x \\ (x+1)e^x & (2x+x^2)e^x & (x^3+3x^2)e^x \\ (x+2)e^x & (4x+x^2+2)e^x & (x^3+6x^2+6x)e^x \end{vmatrix} = 2x^3e^{3x},$$
- $W_2(y_1, y_2, y_3, y_4) = \begin{vmatrix} e^x & x^2e^x & x^3e^x \\ e^x & (2x+x^2)e^x & (x^3+3x^2)e^x \\ e^x & (4x+x^2+2)e^x & (x^3+6x^2+6x)e^x \end{vmatrix} = 6x^2e^{3x},$
- $W_3(y_1, y_2, y_3, y_4) = \begin{vmatrix} e^x & xe^x & x^3e^x \\ e^x & (x+1)e^x & (x^3+3x^2)e^x \\ e^x & (x+2)e^x & (x^3+6x^2+6x)e^x \end{vmatrix} = 6xe^{3x},$

- $$W_4(y_1, y_2, y_3, y_4) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (x+1)e^x & (2x+x^2)e^x \\ e^x & (x+2)e^x & (4x+x^2+2)e^x \end{vmatrix} = 2e^{3x}.$$
- Hence,

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{f(x)W_1}{W}dx + y_2 \int \frac{f(x)W_2}{W}dx \\ &- y_3 \int \frac{f(x)W_3}{W}dx + y_4 \int \frac{f(x)W_4}{W}dx \\ &= \frac{1}{12}\left[- e^x \int 2x^3dx + xe^x \int 6x^2dx - x^2e^x \int 6xdx + x^3e^x \int 2dx \right] \\ &= \frac{1}{12}\left[- \frac{e^x x^4}{2} + 2x^4e^x - 3x^4e^x + 2x^4e^x \right] \\ &= \frac{1}{24}x^4e^x. \end{aligned}$$

- Hence, the general solution is,

$$y(x) = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^x + \frac{1}{24}x^4 e^x,$$

where c_1, c_2, c_3, c_4 are arbitrary constants.