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The total angular momentum of the system is obtained by coupling $S_1 = \frac{1}{2}$ and $S_2 = \frac{1}{2}$: $\vec{\hat{S}} = \vec{\hat{S}}_1 + \vec{\hat{S}}_2$

$$\hat{S}^2 = \hat{S}_1{}^2 + \hat{S}_2{}^2 + 2\vec{\hat{S}}_1.\vec{\hat{S}}_2$$

This leads to

$$\vec{\hat{S}}_1.\vec{\hat{S}}_2 = \frac{1}{2}\left(\hat{S}^2 - \hat{S}_1{}^2 - \hat{S}_2{}^2\right)$$

and when this is inserted into the system's Hamiltonian it yields

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}\left(\hat{S}_1{}^2 + \vec{\hat{S}}_1.\vec{\hat{S}}_2\right) - \frac{\varepsilon_2}{\hbar}S_z = \frac{\varepsilon_1}{\hbar^2}\left(\hat{S}_1{}^2 + \frac{1}{2}\left(\hat{S}^2 - \hat{S}_1{}^2 - \hat{S}_2{}^2\right)\right) - \frac{\varepsilon_2}{\hbar}\hat{S}_z$$
$$= \frac{\varepsilon_1}{2\hbar^2}\left(\hat{S}^2 + \hat{S}_1{}^2 - \hat{S}_2{}^2\right) - \frac{\varepsilon_2}{\hbar}\hat{S}_z$$

Notice that the operators $\hat{H}, \hat{S}_1{}^2, \hat{S}_2{}^2, \hat{S}^2$, and $\hat{S}_z$ mutually commute; we denote their joint eigenstates by$|s, m\rangle$ The energy levels of $\hat{H}$ are thus given by

$$\hat{H}|s,m\rangle = \left[\frac{\varepsilon_1}{2\hbar^2}\left(\hat{S}^2 + \hat{S}_1{}^2 - \hat{S}_2{}^2\right) - \frac{\varepsilon_2}{\hbar}\hat{S}_z\right]|s,m\rangle$$
$$= \left[\frac{\varepsilon_1\hbar^2}{2\hbar^2}\left(s(s+1) + s_1(s_1+1) - s_2(s_2+1)\right) - \frac{\varepsilon_2 m\hbar}{\hbar}\right]|s,m\rangle$$
$$= \frac{\varepsilon_1}{2}s(s+1) - m\varepsilon_2|s,m\rangle$$

since $s_1 = s_2 = \frac{1}{2}$.

$$|s_1 - s_2| \le s \le |s_1 + s_2|\,, -s \le m \le s$$

So s can have zero or 1 as its values, when it is zero, m must be zero too, and if it is 1, m can be ± 1 , 0 thus if $E_{s,m}$ are eigen values of Hamiltonian we have:

$$E_{s,m} = \frac{\varepsilon_1}{2}s(s+1) - m\varepsilon_2$$

$E_{0,0} = 0$ not acceptable

$E_{1,0} = \varepsilon_1$ without degeneracy

$E_{1,1} = \varepsilon_1 - \varepsilon_2$ without degeneracy

$E_{1,-1} = \varepsilon_1 + \varepsilon_2$ without degeneracy