MTL101: General Strategies to solve Second Order ODE

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1 Introduction

Hope all are doing fine in these testing times of the pandemic. This short document provides a summary to the different ways of solving the second order ODE. Why is this needed? Now, that we have seen so many methods of solving the differential equations, it is quite natural that you forget a method or two in the examinations and be like "Ahh! How did this get out of my mind?" Secondly, the order of applying these techniques is also important. You can toss you head for some minutes applying homogeneity concepts and end up finding that the equation was variable separable. Here, we will stick to second order equations.

2 Homogeneous Equation

The equation at hand is

$$y'' + py' + qy = 0$$

dropping the function notation for shorthand. Now the methods which follow are as per my order of preference and it can greatly simplify you task if you hold on to the given order and go to the next method down the list only if the previous one fails.

2.1 Constant coefficients

This has rightly been placed here. You know the solutions in and out and should be the first check if you encounter a differential equation.
Identification— p and q are constants.
Solution tactic— Replace y'' with m^2 and y' with m to get the auxiliary equation as

$$m^2 + pm + q = 0$$

Solve this equation.

- Case 1—Roots are real and distinct i.e. say m_1 and m_2. Then the general solution is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{1}$$

- Case 2—Roots are real but equal say m. Then the general solution is given as

$$y = (c_1 + c_2 x) e^{mx} \tag{2}$$

- Case 3—Roots are imaginary say of the form $\alpha \pm i\beta$. Then the general solution is of the form

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \tag{3}$$

2.2 Cauchy Euler equation

The method here is also pre-known.
Identification——Common looks make it easy to identify.

$$x^2y'' + axy' + by = 0$$

where a and b are constants
Method to solve——

- Set $x = e^z$ to change independent variable from x to z. Replace $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ with $\frac{dy}{dz}$ and $\frac{d^2y}{dz^2}$ by using chain rule. This reduces the equation to the form of constant coefficients in z. Solve this differential equation and then replace back $z = \ln x$.
- Another common method alternatively used is as follows. Substitute $y = x^m$ and then we get an auxiliary equation of the form

  $$m^2 + (a-1)m + b$$

  Now we calculate the roots of the equation and three cases arise

  - Case 1——Roots are real and distinct i.e. say m_1 and m_2. Then the general solution is given as

    $$y = c_1x^{m_1} + c_2x^{m_2} \tag{4}$$

  - Case 2——Roots are real but equal say m. Then the general solution is given as

    $$y = (c_1 + c_2 \ln x)x^m \tag{5}$$

  - Case 3——Roots are imaginary say of the form $\alpha \pm i\beta$. Then the general solution is of the form

    $$y = x^\alpha(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)) \tag{6}$$

2.3 Reduction of Order

2.3.1 Substitution

This method reduces the equation to first order and hence makes solutions easier to obtain. The plus point of this method is that it works even when the equation is not linear and might be the only approach in some cases.

- Case 1——$f(x, y', y'') = 0$.Put $z = y'$
- Case 2——$f(y, y', y'')$. Put $z = y'$ and change the independent variable to y by the use of chain rule. The process goes like this

  $$y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy}z$$

  . Substitution into the original equation makes it first order.

Though it looks simpler, I would recommend using the next method first, if one of the solutions is known.

2.3.2 Multiplicative function

In case one of the solutions is known, say y_1, then the other solution can be found out by assuming $y_2 = vy_1$ where v is some function of x. Substitute in the equation to get the function v as

$$v = \int \frac{e^{-\int p(x)dx}}{y_1^2}dx \tag{7}$$

3 Non-Homogeneous Equation

The equation looks like

$$y'' + p(x)y' + q(x)y = r(x)$$

The solution looks like

$$y = y_C + y_P$$

where y_C is the solution of the corresponding homogeneous equation and y_P is the particular solution free of arbitrary constants. The solution steps are:

1. Solve the homogeneous equation using the techniques mentioned in the section on Homogeneous equations.
2. Find the particular solution.

3.1 Method of Undetermined Coefficients

The method is easy one but has its own limitations. It generally works only when a and b are constants and $r(x)$ is one of the following (Substitutions shown alongside)

1. $ke^{\gamma x} \longrightarrow \quad y_P = Ce^{\gamma x}$
2. $kx^n \longrightarrow \quad y_P = k_n x^n + k_{n-1} x^{n-1} ..... + k_0$
3. $k \sin x \longrightarrow \quad y_P = c_1 \cos x + c_2 \sin x$
4. $k \cos x \longrightarrow \quad y_P = c_1 \cos x + c_2 \sin x$
5. $ke^{\gamma x} \sin x \longrightarrow \quad y_P = e^{\gamma x}(c_1 \cos x + c_2 \sin x)$
6. $ke^{\gamma x} \cos x \longrightarrow \quad y_P = e^{\gamma x}(c_1 \cos x + c_2 \sin x)$

Assuming the solutions shown above we substitute them in the the equation and find the values of the assumed constants.

3.2 Variation of Parameters

This solution method is a general one. We have calculated the solutions to the homogeneous equation say y_1 and y_2. Then the particular solution is given by

$$y_P(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

where W signifies the Wronskian of y_1 and y_2 taken in the same order.