## Tutorial Sheet 5::: MTL101::: Second order ODE

(1) Find the solutions of the following initial value problems:
(a) $y'' - 2y' - 3y = 0, y(0) = 0, y'(0) = 1$ (b) $y'' + 10y = 0, y(0) = \pi, y'(0) = \pi^2$.

(2) Find a function $\phi$ which has a continuous derivative on $0 \leq x \leq 2$ which satisfies $\phi(0) = 0$, $\phi'(0) = 1$, and $y'' - y = 0$ for $0 \leq x \leq 1$, and $y'' - 9y = 0$ for $1 \leq x \leq 2$.

(3) Consider the constant coefficient equation $L(y) = y'' + a_1 y' + a_2 y = 0$. Let $\phi_1$ be the solution satisfying $\phi_1(x_0) = 1$, $\phi_1'(x_0) = 0$, and $\phi_2$ be the solution satisfying $\phi_2(x_0) = 0$, $\phi_2'(x_0) = 1$. If $\phi$ is a solution satisfying $\phi(x_0) = \alpha$, $\phi'(x_0) = \beta$, show that $\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x)$ for all $x$.

(4) Let $\phi_1$, $\phi_2$ be two differentiable functions on an interval $I$, which are not necessarily solutions of an equation $L(y) = 0$. Prove the following:

(a) If $\phi_1$, $\phi_2$ are linearly dependent on $I$, then $W(\phi_1, \phi_2)(x) = 0$ for all $x$ in $I$.

(b) If $W(\phi_1, \phi_2)(x_0) \neq 0$ for some $x_0$ in $I$, then $\phi_1$, $\phi_2$ are linearly independent on $I$.

(c) $W(\phi_1, \phi_2)(x) = 0$ for all $x$ in $I$ does not imply that $\phi_1$, $\phi_2$ are linearly dependent on $I$.

(d) $W(\phi_1, \phi_2)(x) = 0$ for all $x$ in $I$, and $\phi_2(x) \neq 0$ on $I$, imply that $\phi_1$, $\phi_2$ are linearly dependent on $I$.

(5) Find all solutions of the following equations:
(a) $4y'' - y = e^x$ (b) $y'' + 4y = \cos x$ (c) $y'' + 9y = \sin 3x$.

(6) Let $L(y) = y'' + a_1 y' + a_2 y = 0$, where $a_1, a_2$ are constants, and let $p$ be the characteristic equation $p(r) = r^2 + a_1 r + a_2$.

(a) If $A, \alpha$ are constants and $p(\alpha) \neq 0$, show that there is a solution $\phi$ of $L(y) = Ae^{\alpha x}$ of the form $\phi(x) = Be^{\alpha x}$, where $B$ is a constant.
(b) Compute a particular solution of $L(y) = Ae^{\alpha x}$ in case $p(\alpha) = 0$.

(7) Are the following set of functions defined on $-\infty < x < \infty$ linearly dependent or independent there? Why?

(a) $\phi_1(x) = 1, \phi_2(x) = x, \phi_3(x) = x^3$
(c) $\phi_1(x) = x, \phi_2(x) = e^{2x}, \phi_3(x) = |x|$.

(8) Use the method of undetermined coefficients to find a particular solution of each of the following equations:

(a) $y'' + 4y = \cos x$ (b) $y'' + 4y = \sin 2x$
(c) $y'' - y' - 2y = x^2 + \cos x$ (d) $y'' + 9y = x^2 e^{3x}$.

(9) Find a real solution.
(a) $x^2y'' - 4xy' + 6y = 0$, (b) $4x^2y'' + 12xy' + 3y = 0$, (c) $x^2y'' + 7xy' + 9y = 0$,
(d) $x^2y'' - 2.5xy' - 2y = 0$, (e) $x^2y'' + 7xy' + 13y = 0$.

(10) Solve the initial value problems.
(a) $x^2y'' - 2xy' + 2y = 0$, $y(1) = 1.5$, $y'(1) = 1$.
(b) $x^2y'' + 3xy' + y = 0$, $y(1) = 3$, $y'(1) = -4$.
(c) $x^2y'' - 3xy' + 4y = 0$, $y(1) = 0$, $y'(1) = 3$.

(11) Find all solutions of the following equations:

(a) $y''' - 8y = 0$ (b) $y^{(4)} + 16y = 0$
(c) $y^{(100)} + 100y = 0$ (e) $y^{(4)} - 16y = 0$

(12) Use the variation of parameters method to solve the following equations:

(a) $y''' - y' = x$, (b) $y^{(4)} + 16y = \cos x$, (c) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x$.