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(a)

From normalization condition we have:

$$1 = \langle R|R\rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} A^* e^{-\alpha r^2} A e^{-\alpha r^2} r^2 dr \sin\theta \, d\theta d\varphi = 4\pi|A|^2 \int_0^\infty e^{-2\alpha r^2} r^2 dr = 4\pi|A|^2 \frac{1}{4}\sqrt{\frac{\pi}{8\alpha^3}}$$
$$= |A|^2 \left(\frac{\pi}{2\alpha}\right)^{\frac{3}{2}}$$

$$\Rightarrow |A|^2 = \left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}}$$

so:

$$\langle R|\hat{H}|R\rangle = \left\langle R\left|-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2\right|R\right\rangle = \left\langle R\left|-\frac{\hbar^2}{2m}\nabla^2\right|R\right\rangle + \frac{1}{2}m\omega^2 4\pi|A|^2 \int_0^\infty e^{-2\alpha r^2} r^4 dr$$

But

$$\left\langle R\left|-\frac{\hbar^2}{2m}\nabla^2\right|R\right\rangle = \frac{\hbar^2}{2m}\int \vec{\nabla}(R^*(r)).\vec{\nabla}(R(r))d^3r = \frac{\hbar^2}{2m}|A|^2 \int_0^\infty \left(-2\alpha r e^{-\alpha r^2}\right)^2 4\pi r^2 dr$$
$$= \frac{8\hbar^2|A|^2\pi\alpha^2}{m}\int_0^\infty e^{-2\alpha r^2} r^4 dr$$

Thus

$$\langle \hat{H}\rangle_R = \langle R|\hat{H}|R\rangle = \left(\frac{8\hbar^2|A|^2\pi\alpha^2}{m} + 2\pi m\omega^2|A|^2\right)\int_0^\infty e^{-2\alpha r^2} r^4 dr = 2\pi|A|^2\left(\frac{4\hbar^2\alpha^2}{m} + m\omega^2\right)\frac{3}{8}\sqrt{\frac{\pi}{(2\alpha)^5}}$$
$$= \frac{3}{4}\pi^{\frac{3}{2}}\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}}\frac{1}{(2\alpha)^{\frac{5}{2}}}\left(\frac{4\hbar^2\alpha^2}{m} + m\omega^2\right) = \frac{3}{8}\left(\frac{4\hbar^2\alpha}{m} + \frac{m\omega^2}{\alpha}\right)$$

But the lowest value of $\langle \hat{H}\rangle_R$ happens at $\alpha_0 = \frac{m\omega}{2\hbar}$ which gives

$$\langle \hat{H}\rangle_R\Big|_{\alpha=\alpha_0} = \frac{3}{8}\left(\frac{4\hbar^2\alpha}{m} + m\omega^2\left(\frac{2\hbar}{m\omega}\right)\right) = \frac{3}{2}\hbar\omega$$

This value is exactly equal to ground state energy of spherical harmonic oscillator.

(b)

From normalization condition we have:

$$1 = \langle R|R\rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} |A|^2 e^{-2\alpha r} r^2 dr \sin\theta \, d\theta d\varphi = 4\pi|A|^2 \int_0^\infty e^{-2\alpha r} r^2 dr = \frac{\pi|A|^2}{\alpha^3}$$

$$\Rightarrow |A|^2 = \frac{\alpha^3}{\pi}$$

so:

$$
\begin{aligned}
\langle \hat{H} \rangle_R = \langle R | \hat{H} | R \rangle = \left\langle R \middle| -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 \middle| R \right\rangle = \left\langle R \middle| -\frac{\hbar^2}{2m}\nabla^2 \middle| R \right\rangle + \frac{1}{2}m\omega^2 \langle R | r^2 | R \rangle \\
= \frac{\hbar^2}{2m}\int \vec{\nabla}\big(R^*(r)\big).\vec{\nabla}\big(R(r)\big) d^3r + \frac{1}{2}m\omega^2 4\pi |A|^2 \int_0^\infty e^{-2\alpha r} r^4 dr \\
= \frac{\hbar^2 |A|^2}{2m}\int_0^\infty (-\alpha e^{-\alpha r})^2 4\pi r^2 dr + 2\pi m\omega^2 |A|^2 \int_0^\infty e^{-2\alpha r} r^4 dr \\
= \frac{2\pi\hbar^2 |A|^2 \alpha^2}{m}\frac{1}{4\alpha^3} + 2\pi m\omega^2 |A|^2 \frac{3}{4\alpha^5} = \frac{\alpha^3}{\pi}\frac{\pi\hbar^2 |A|^2 \alpha^2}{m}\frac{1}{2\alpha^3} + \pi m\omega^2 \frac{\alpha^3}{\pi}\frac{3}{2\alpha^5} \\
= \frac{\hbar^2\alpha^2}{2m} + \frac{3m\omega^2}{2\alpha^2}
\end{aligned}
$$

The lowest value of $\langle \hat{H} \rangle_R$ happens at $\alpha_0 = \left(\frac{3m^2\omega^2}{\hbar^2}\right)^{1/4}$ which gives

$$
\langle \hat{H} \rangle_R \Big|_{\alpha=\alpha_0} = \frac{\hbar^2}{2m}\left(\frac{3m^2\omega^2}{\hbar^2}\right)^{1/2} + \frac{3m\omega^2}{2}\left(\frac{3m^2\omega^2}{\hbar^2}\right)^{-1/2} = \frac{\sqrt{3}}{2}\hbar\omega + \frac{\sqrt{3}}{2}\hbar\omega = \sqrt{3}\hbar\omega
$$

Thus

$$
E_0{}^{Vm} = \sqrt{3}\,\hbar\omega
$$

(c)

In (a) we have:

$$
Relative\ errors = \left| \frac{E_0^{exact} - E_0^{VM}}{E_0^{exact}} \right| = \frac{\frac{3}{2}\hbar\omega - \frac{3}{2}\hbar\omega}{\frac{3}{2}\hbar\omega} = 0
$$

In (b) we have:

$$
Relative\ errors = \left| \frac{E_0^{exact} - E_0^{VM}}{E_0^{exact}} \right| = \frac{\frac{3}{2}\hbar\omega - \sqrt{3}\,\hbar\omega}{\frac{3}{2}\hbar\omega} = 0.13
$$

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