MTL101::Linear Algebra and Differential Equations

Tutorial 1

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Question 1

Suppose we have a system of three linear equations in real coefficients and in two unknowns:

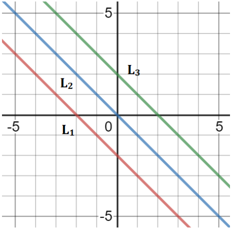
$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has a unique solution.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i \in \mathbb{R}$, $(i = 1, 2, 3)$, so that the above statements hold.

Solution: (a) The system has no solutions

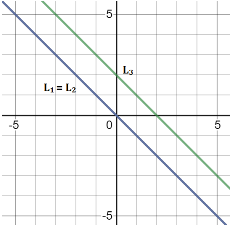
- All three lines are parallel.

$$L_1 : x + y = -2$$
$$L_2 : x + y = 0$$
$$L_3 : x + y = 2$$



- Two lines coincide and other line is parallel to them.

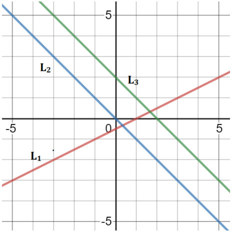
$$L_1 : x + y = 0$$
$$L_2 : x + y = 0$$
$$L_3 : x + y = 2$$



Question 1(a) Contd.

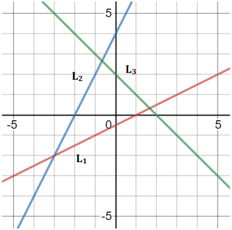
- Two lines are parallel and the third line intersects the other two lines.

$$L_1 : x - 2y = 1$$
$$L_2 : x + y = 0$$
$$L_3 : x + y = 2$$



- Three lines intersect at three different points.

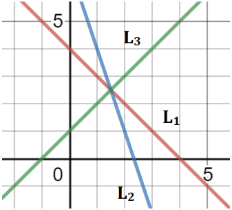
$$L_1 : x - 2y = 1$$
$$L_2 : 2x - y = -4$$
$$L_3 : x + y = 2$$



(b) The system has a unique solution

All the three lines have only one common point of intersection.

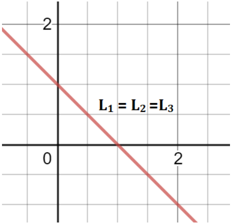
$$L_1 : x + y = 4$$
$$L_2 : 3x + y = 7$$
$$L_3 : x - y = -1$$



(c) The system has an infinitely many solutions.

All three lines coincide.

$$L_1 : x + y = 1$$
$$L_2 : x + y = 1$$
$$L_3 : x + y = 1$$



Question 2

Suppose we have a system of three linear equations in real coefficients and in three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

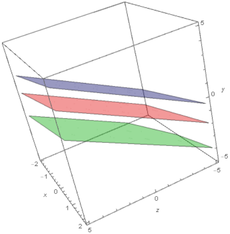
Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has a unique solution.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i, d_i \in \mathbb{R}, (i = 1, 2, 3)$, so that the above statements hold.

Question 2(a)

Solution: (a) The system has no solutions.

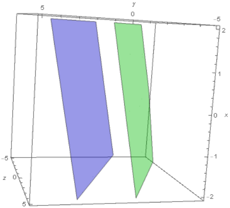
- All three planes are parallel.

$$P_1 : x - 2y + z = -2$$
$$P_2 : x - 2y + z = 1$$
$$P_3 : x - 2y + z = 5$$



- Two planes coincide and other plane is parallel.

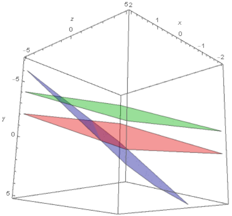
$$P_1 : x - 2y + z = -2$$
$$P_2 : x - 2y + z = 5$$
$$P_3 : x - 2y + z = 5$$



Question 2(a) Contd.

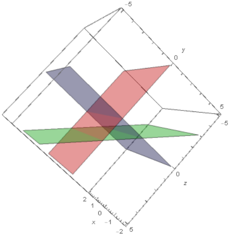
- Two planes are parallel and the third plane intersects the other two planes.

$$P_1 : x + y - z = 1$$
$$P_2 : x - 2y + z = 1$$
$$P_3 : x - 2y + z = 5$$



- Line of intersection of two planes is parallel to the third plane.

$$P_1 : z = 0$$
$$P_2 : y = 0$$
$$P_3 : y + z = 2$$

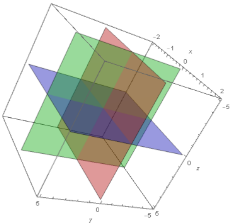


Question 2(b), 2(c)

(b) The system has a unique solution.

Line of intersection of two planes intersects the third plane.

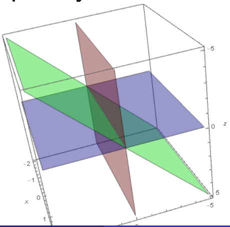
$$P_1 : x = 0$$
$$P_2 : y = 0$$
$$P_3 : z = 0$$



(c) The system has an infinitely many solutions.

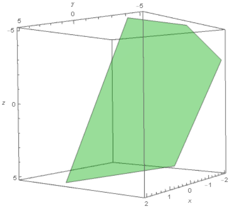
- Line of intersection of two planes completely lie on the third plane.

$$P_1 : y + z = 0$$
$$P_2 : y = 0$$
$$P_3 : z = 0$$



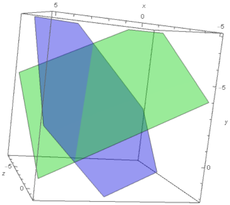
- All three planes coincide.

$$P_1 : x - 2y + z = 5$$
$$P_2 : x - 2y + z = 5$$
$$P_3 : x - 2y + z = 5$$



- Two planes coincide and the third plane intersects these planes.

$$P_1 : x + y - z = 1$$
$$P_2 : x - 2y + z = 5$$
$$P_3 : x - 2y + z = 5$$



Question 3

Suppose we have a system of two linear equations in real coefficients and three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

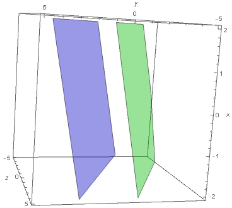
Interpret geometrically the following statements:
(a) the system has no solutions.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i, d_i \in \mathbb{R}, (i = 1, 2, 3)$, so that the above statements hold. Can you have unique solution ?

Question 3(a)

Solution: (a) The system has no solution

Two planes are parallel.

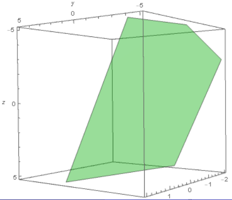
$$P_1 : x - 2y + z = -2$$
$$P_2 : x - 2y + z = 5$$



(c) The system has an infinitely many solutions

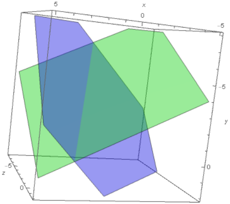
- Two planes coincide.

$$P_1 : x - 2y + z = 5$$
$$P_2 : x - 2y + z = 5$$



- Two planes intersects each other.

$$P_1 : x + y - z = 1$$
$$P_2 : x - 2y + z = 5$$



As the number of unknown variables are greater than the number of equations, the system cannot have a unique solution.

Question 4

Let $\vec{y} = A\vec{x}$ and $\vec{x} = B\vec{w}$ with A and B being 2×2 matrices and $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^2$. If $\vec{y} = C\vec{w}$, then find the relation between A, B and C.

Solution: Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

- The y_1y_2-coordinate system is related to the x_1x_2-coordinate system by the formula $\vec{y} = A\vec{x}$.
- The x_1x_2-coordinate system is related to the w_1w_2-coordinate system by the formula $\vec{x} = B\vec{w}$.
- The y_1y_2-coordinate system is related to the w_1w_2-coordinate system by the formula $\vec{y} = C\vec{w}$. Therefore for every $\vec{w} \in \mathbb{R}^2$,

$$C\vec{w} = \vec{y} = A\vec{x} = A(B\vec{w}) = (AB)\vec{w}.$$

- Consequently, we have $C = AB$.

Question 5

Let L_1, L_2 are lower triangular and U_1, U_2 are upper triangular, then which of the following matrices are lower triangular and upper triangular ?
(a) $L_1 + L_2$ (b) $U_1 + L_2$ (c) U_1^2 (d) $L_1 U_1$ (e) $U_1 L_2$

Recall: $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ real matrices.

- A is lower triangular matrix if $a_{ij} = 0$ for $i < j$.
- B is upper triangular matrix if $b_{ij} = 0$ for $i > j$.
- If $C = A + B$ and $C = [c_{ij}]$, then $c_{ij} = a_{ij} + bij$.
- If $D = AB$ and $D = [d_{ij}]$, then

$$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Question 5(a), 5(b)

Solution: (a) $L_1 + L_2$

- Let $L_1 = [p_{ij}]$, $L_2 = [q_{ij}]$ and $L_1 + L_2 = [l_{ij}]$.
- As L_1, L_2 are lower triangular matrices, $p_{ij} = 0$ and $q_{ij} = 0$ for $i < j$.
- $l_{ij} = p_{ij} + q_{ij} = 0$ for $i < j$.
- Hence, $L_1 + L_2$ is a lower triangular matrix.

Solution: (b) $U_1 + L_2$

- $U_1 + L_2$ may neither be a lower triangular nor upper triangular matrix.
- Consider

$$U_1 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Question 5(c)

Solution: (c) U_1^2

- Let $U_1 = [u_{ij}]$ and $U_1^2 = [v_{ij}]$.
- As U_1 is upper triangular, $u_{ij} = 0$ for $i > j$.
- For $i > j$

$$\begin{aligned} v_{ij} &= \sum_{k=1}^{n} u_{ik} u_{kj} \\ &= \sum_{k=1}^{j} u_{ik} u_{kj} + \sum_{k=j+1}^{n} u_{ik} u_{kj} \\ &= 0. \end{aligned}$$

- Hence, U_1^2 is an upper triangular matrix.

Question 5(d), 5(e)

Solution: (d) $L_1 U_1$, **(e)** $U_1 L_2$

- Both $L_1 U_1$ and $U_1 L_2$ may neither be an upper triangular matrix nor a lower triangular matrix.
- Consider

$$U_1 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \quad L_1 = L_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

- Then

$$L_1 U_1 = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix},$$
$$U_1 L_2 = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}.$$

Question 6

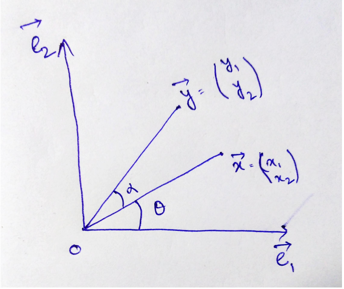
Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ and } B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

Show that $\vec{y} = A\vec{x}$ is the rotation of vector $\vec{x}$ counter clockwise by angle α. Also compute $\vec{y} = A^n\vec{x}$ and $\vec{y} = AB\vec{x}$. Interpret the results geometrically.

Solution:

- Let θ be the angle between $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- Then $x_1 = |\vec{x}| \cos\theta$ and $x_2 = |\vec{x}| \sin\theta$.



Question 6 Contd.

- $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is a new position of $\vec{x}$ after rotating counter clockwise by an angle α.
- Angle between $\vec{y}$ and $\vec{e_1}$ is $\theta + \alpha$.
- Therefore,

$$\begin{aligned} y_1 &= |\vec{y}|\cos(\theta+\alpha) & y_2 &= |\vec{y}|\sin(\theta+\alpha) \\ &= |\vec{x}|\cos\theta\cos\alpha - |\vec{x}|\sin\theta\sin\alpha & &= |\vec{x}|\sin\theta\cos\alpha + |\vec{x}|\cos\theta\sin\alpha \\ &= x_1\cos\alpha - x_2\sin\alpha & &= x_1\sin\alpha + x_2\cos\alpha \end{aligned}$$

- Hence,

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\vec{x}$$

- Therefore, A rotates vector $\vec{x}$ by angle α.

Question 6 Contd.

- To understand the behavior of $\vec{y} = AB\vec{x}$ on $\vec{x}$, note that
  - The product,

$$AB = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

  - Hence, AB rotates the vector $\vec{x}$ counter clockwise by angle $\alpha + \beta$.
- To understand the behavior of $\vec{y} = A^n\vec{x}$ on $\vec{x}$, note that
  - $A^2 = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}$ and using induction

    $A^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$.
  - Hence, A^n rotates the vector $\vec{x}$ counter clockwise by angle $n\alpha$.

Question 7

Which of the following matrices are row echelon and row reduced echelon matrix. Give a reason when the matrix is not row reduced echelon.

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Recall: Row echelon matrix

- All zero-rows are at the bottom.
- The leading coefficient of a non-zero row is always strictly left to the leading coefficient of the next row.

Row reduced echelon matrix

- It is a row echelon matrix.
- The leading entry in each non-zero row is equal to 1.
- Each column containing a leading 1 has zeros in all its other entries.

Question 7

Solution:

- $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ is a row echelon matrix. It is not a row reduced echelon matrix as the leading coefficient of the second row is not equal to 1.
- $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ is a row reduced echelon matrix.
- $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not a row echelon matrix as a zero row is lying above a non-zero row.
- $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a row reduced echelon matrix.

Question 8

Find row reduced echelon matrix which is row equivalent to the matrices in the previous question and their transposes.

Recall: A is row equivalent to the matrix B if A can be obtained from B by finitely many row operations.

Solution:

- $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Question 8

Row reduced echelon matrix of transpose matrix

- $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 5 & 3 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 3 & 0 \end{pmatrix}$
  $\xrightarrow{R_3 \to R_3 - 5R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 3 & 0 \end{pmatrix}$, then proceed as above.

Question 8 Contd.

- $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
  $$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
- $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
  $$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 9

Show that every elementary matrix is invertible and inverse is an elementary matrix

Recall:

1. A square matrix E of size n is called an elementary matrix if there is an elementary row operation q such that $E = q(I_n)$.
2. Let q be an elementary row operation, then, for $A \in M_{m\times n}(\mathbb{R})$
$$q(A) = q(I_m)A$$
3. If matrix A is obtained from identity matrix I by a certain row operation q then A^{-1} is obtained from I by its inverse operation q^{-1}, i.e.,
   - If q is swapping operation then $q^{-1} = q$.
   - If q is multiplication of a row by x (where $x \neq 0$) then q^{-1} is multiplication of a row by x^{-1}.
   - If q is an adding operation i.e $R_l \to R_l + xR_j$, then q^{-1} will be $R_l \to R_l - xR_j$.

Question 9

Solution:

- Let E be an elementary matrix. Therefore, $E = q(I)$ for some elemenatry operation q.
- Define $F = q^{-1}(I)$, where q^{-1} is the reverse operation of q.
- Then,

$$\begin{aligned} I = (q^{-1} \circ q)(I) &= q^{-1} \circ (q(I)) \\ &= q^{-1}(E) \\ &= q^{-1}(I)E \quad \text{(see 'Recall, point-2')} \\ &= FE \end{aligned}$$

- Similarly,

$$\begin{aligned} I = (q \circ q^{-1})(I) &= q \circ (q^{-1}(I)) \\ &= q(F) \\ &= q(I)F \quad \text{(see 'Recall, point-2')} \\ &= EF \end{aligned}$$

Question 9 Contd.

- Thus $I = FE = EF$, i.e., E is invertible and the inverse is given by F.
- As, by definition, F is obtained by a single row operation applied on I, therefore F is an elementary matrix.

Question 10

Compute the rank of the following matrices. Using the rank determine which of these matrices are invertible.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}.$$

Recall:

- The rank of a matrix A is the number of non-zero rows in the Row Echelon form of A.
- A matrix $A_{n\times n}$ is invertible $\iff$ Rank$(A) =$ n.
- Row-equivalent matrices have the same rank.

Question 10

Solution: Using elementary row operations we have,

- $$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

  As the rank $A = 2$, the above matrix is not invertible.

Question 10 Contd.

- $$\begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

  As the rank $A = 3$, the above matrix is invertible.

- $$\begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 3 & -4 \\ 0 & 2 & 3 \\ 0 & 4 & 6 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 3 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

  As the rank $A = 2$, the above matrix is not invertible.

Question 11

Find the inverse of the invertible matrices in the previous question by reducing the matrix to row reduced echelon form (identity matrix).

Solution:

- Consider $[A|I]$, where A is the matrix considered in Question 10(b) and I is the 3×3 identity matrix.
- $\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ -1 & -1 & 5 & 0 & 1 & 0 \\ 2 & 7 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 5 & -2 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 3R_2]{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -5 & -3 & 1 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{2} R_3]{R_1 \to R_1 + 3R_3}$

Question 11

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -11 & 3 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -11 & 3 \\ 0 & 1 & 0 & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

Therefore, $A^{-1} = \begin{pmatrix} -16 & -11 & 3 \\ \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$

Question 12

Write down the following matrices as the product of elementary matrices (whenever possible).

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}.$$

Recall:

- A is row equivalent to I if and only if I is row equivalent to A. If $I = (q_s \circ ... \circ q_2 \circ q_1)(A)$, then $A = (q_1^{-1} \circ ... \circ q_{s-1}^{-1} \circ q_s^{-1})(I)$.
- That is, $A = E_1^{-1} E_2^{-1} \dots E_s^{-1}$, where $E_k = q_k(I)$.

Question 12

Solution:

- $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{q_1: R_2 \to R_2 - 4R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{q_2: R_1 \to R_1 - 2R_2}$$
  $$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{q_3: R_1 \to R_1 - 3R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  Hence, A is row equivalent to I, and $A = E_1^{-1} E_2^{-1} E_3^{-1}$, where

  $$E_1 = q_1(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}$ is not invertible since rank is 2, see Question 10.

Question 13

Solve the following systems of equations by reducing the augmented matrix to the row reduced echelon form:

a) $x_1 - x_2 + 2x_3 = 1, 2x_1 + 2x_3 = 1, x_1 - 3x_2 + 4x_3 = 2,$

b) $x_1 + 7x_2 + x_3 = 4, x_1 - 2x_2 + x_3 = 0, -4x_1 + 5x_2 + 9x_3 = -9,$

c) $x_2 + 5x_3 = -4, x_1 + 4x_2 + 3x_3 = -2, 2x_1 + 7x_2 + x_3 = -1,$

d) $-2x_1 - 3x_2 + 4x_3 = 5, x_2 - x_3 = 4, x_1 + 3x_2 - x_3 = 2.$

Solution: a) $x_1 - x_2 + 2x_3 = 1, 2x_1 + 2x_3 = 1, x_1 - 3x_2 + 4x_3 = 2$

- Above system of equations can be written as $Ax = b$, i.e.,

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Question 13(a)

- Using elementary row transformations on augmented matrix $(A|B)$ we have,

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2/2]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- Since, rank $A = 2$ and rank $(A|b) = 2$. Also, rank $A =$ rank $(A|b) < 3$, above system of equations have infinite solutions.
- Here x_3 is the free variable. Let $x_3 = a$. We have,

$$x_1 - x_2 + 2x_3 = 1, \quad x_2 - x_3 = -\frac{1}{2}$$

Therefore, $(x_1\ x_2\ x_3)^T = \left(\frac{1}{2} - a \quad a - \frac{1}{2} \quad a\right)^T$

Question 13(b)

(b) $x_1 + 7x_2 + x_3 = 4, x_1 - 2x_2 + x_3 = 0, -4x_1 + 5x_2 + 9x_3 = -9$

- Above system of equations can be written as $Ax = b$, i.e.,

$$\begin{pmatrix} 1 & 7 & 1 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -9 \end{pmatrix}$$

- Using elementary row operations on augmented matrix $(A|B)$ we have,

$$\left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right) \xrightarrow[R_3 \to R_3 + 4R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & -9 & 0 & -4 \\ 0 & 33 & 13 & 7 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2/9} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & -1 & 0 & -\frac{4}{9} \\ 0 & 33 & 13 & 7 \end{array}\right) \xrightarrow{R_3 \to R_3 + 33R_2}$$

Question 13(b) Contd.

$$\left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & -1 & 0 & -\frac{4}{9} \\ 0 & 0 & 13 & -23/39 \end{array}\right) \xrightarrow{R_3 \to R_3/13} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & -1 & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & -23/39 \end{array}\right)$$

- As rank $A =$ rank $(A|b) = 3$, above system of equations has a unique solution. Hence, we have

$$x_1 + 7x_2 + x_3 = 4, \quad -x_2 = -\frac{4}{9}, \quad x_3 = -\frac{23}{39}.$$

- So, by backward substitution we get,

$$(x_1 \ x_2 \ x_3)^T = \left(\frac{173}{117} \ \ \frac{4}{9} \ \ -\frac{23}{39}\right)^T$$

Question 13 (c)

(c) $x_2 + 5x_3 = -4,\ x_1 + 4x_2 + 3x_3 = -2,\ 2x_1 + 7x_2 + x_3 = -1$

- Above system of equations can be written as $Ax = b$, i.e.,

$$\begin{pmatrix} 0 & 1 & 5 \\ 1 & 4 & 3 \\ 2 & 7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -1 \end{pmatrix}.$$

- Using elementary row operations on augmented matrix $(A|b)$ we have,

$$\left(\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{array}\right)$$

Question 13(c) Contd.

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

- Hence, rank $A = 2$ and rank $(A|b) = 3$. As, rank $A \neq$ rank $(A|b)$, above system of equation has no solution.

Question 13(d)

(d) $-2x_1 - 3x_2 + 4x_3 = 5,\ x_2 - x_3 = 4,\ x_1 + 3x_2 - x_3 = 2$

- Above system of equations can be written as $Ax = b$, i.e.,

$$\begin{pmatrix} -2 & -3 & 4 \\ 0 & 1 & -1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

- Using elementary row operations on augmented matrix $(A|B)$ we have,

$$\left(\begin{array}{ccc|c} -2 & -3 & 4 & 5 \\ 0 & 1 & -1 & 4 \\ 1 & 3 & -1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 4 \\ -2 & -3 & 4 & 5 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 2R_1} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 3 & 2 & 9 \end{array}\right) \xrightarrow{R_3 \to R_3 - 3R_2}$$

Question 13(d) Contd.

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 5 & -3 \end{array}\right) \xrightarrow{R_3 \to R_3/5} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -\frac{3}{5} \end{array}\right)$$

- As rank $A =$ rank $(A|B) = 3$, above system of equations have a unique solution. Hence, we have

$$x_3 = -3/5, \quad x_2 - x_3 = 4, \quad x_1 + 3x_2 - x_3 = 2.$$

- So, by backward substitution we get,

$$(x_1 \ x_2 \ x_3)^T = \left(-\frac{44}{5} \quad \frac{17}{5} \quad -\frac{3}{5}\right)^T$$

Question 14

Consider the following system of equations:

$$x + 2y + z = 3, ay + 5z = 10, 2x + 7y + az = b.$$

a) Find all values of a for which the following system of equations has a unique solution. b) Find all pairs (a, b) for which the system has more than one solution.

Solution:

- Above system of equations can be written as $Ax = b$, i.e.,

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & a & 5 \\ 2 & 7 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ b \end{pmatrix}$$

- The system of equations will have a unique solution iff $|A| \neq 0$.
- Since, $|A| = a^2 - 2a - 15 = (a - 5)(a + 3)$. Therefore, the above system of equation has a unique solution iff $a \neq 5, -3$.

Case 1 : When a=-3

- Using elementary row operations on augmented matrix we have,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 2 & 7 & -3 & b \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 0 & 3 & -5 & b-6 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 0 & 0 & 0 & b+4 \end{array}\right)$$

- rank $A = 2$. Above system of equation will have infinite solutions iff rank $A =$rank $(A|B) < 3$. This implies that rank $(A|B) = 2$.
- This gives, $b + 4 = 0$, i.e., $b = -4$.
- Hence, for $(a, b) = (-3, -4)$ the system has more than one solution.

Case 2 : When a = 5

- Using elementary row transformations on augmented matrix $(A|B)$ we have,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 2 & 7 & 5 & b \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & 3 & 3 & b-6 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3/3]{R_2 \to R_2/5} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & \frac{b-6}{3} \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & \frac{b-12}{3} \end{array}\right)$$

- rank $A = 2$. Above system of equation will have infinite solutions ifF rank $A =$rank $(A|B) < 3$. This implies that rank $(A|B) = 2$.
- Therefore, $\frac{b-12}{3} = 0$, i.e., $b = 12$.
- Hence, for $(a, b) = (5, 12)$ the system has more than one solution.

Question 15

Find $a, b, c, p, q \in \mathbb{R}$ such that the following system has a solution:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & p \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Solution: The system of equation written in the form of $Ax = b$ has a solution if

- rank $A =$rank $(A|b) = 3$. In this case it will have a unique solution.
- rank $A =$ rank $(A|b) < 3$. In this case it will have infinite solutions.

Case 1: Unique Solution

- The system will have a unique solution iff A is invertible, equivalently $|A| \neq 0$.
- $|A| = q$, therefore the system of equation has a unique solution iff $q \neq 0$ and $p, a, b, c \in \mathbb{R}$.

Question 15

Case 2 : Infinite Solutions

- For this case, we must have rank $A =$ rank $(A|b) < 3$.
- Now, rank $A < 3$ iff $q = 0$, and in that case rank $A = 2$ because A has two non zero pivot columns.
- Consider $(A|b)$ when $q = 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & p & b \\ 0 & 0 & 0 & c \end{array}\right)$$

- Now, rank $(A|b) = 2$ when $c = 0$.
- Hence, the system of equations have infinite solutions if $c = q = 0$ and $p, a, b \in \mathbb{R}$.

Question 16

Assume A, B are square matrices os same size, then,
(a) $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$ and $\text{trace}(AB) = \text{trace}(BA)$.
(b) Let A, B are 2×2 matrices, then $\det(AB) = \det(A)\det(B)$.
(c) Find A, B such that $\det(A+B) \neq \det(A) + \det(B)$

Solution: (a) trace$(A+B) =$ **trace**$(A) +$ **trace**(B)

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrix.
- Then

$$\begin{aligned} trace(A) + trace(B) &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) \\ &= trace(A+B) \end{aligned}$$

Question 16 (a)

trace$(AB) =$ **trace**(BA)

- $(i,j)^{th}$ entry of matrix AB is given by $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

- Consider

$$\begin{aligned} trace(AB) &= \sum_{i=1}^{n}(AB)_{ii} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik} \\ &= trace(BA) \end{aligned}$$

Question 16 (b)

(b) Let A, B **are** 2×2 **matrices, then** $\mathbf{det}(AB) = \mathbf{det}(A)\mathbf{det}(B)$.
Solution:

- Let
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } \; B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
- As $det(A) = a_{11}a_{22} - a_{21}a_{12}$ and $det(B) = b_{11}b_{22} - b_{21}b_{12}$.
- Now,
$$\begin{aligned} det(AB) &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) \\ &\quad - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22}) \\ &= a_{11}a_{22}(b_{11}b_{22} - b_{21}b_{12}) - a_{12}a_{21}(b_{11}b_{22} - b_{21}b_{12}) \\ &= (a_{11}a_{22} - a_{21}a_{12})(b_{11}b_{22} - b_{21}b_{12}) \\ &= det(A)det(B). \end{aligned}$$

Question 16 (c)

(c) Find A, B **such that** $\mathbf{det}(A+B) \neq \mathbf{det}(A) + \mathbf{det}(B)$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then $det(A) = 1$, $det(B) = 1$ and $det(A + B) = 0$.

Question 17

Show that computation of nth order determinant using expansion needs atleast $n!$ multiplication. If a multiplication takes 10^{-9} sec on a computer, compute the time needed for computing determinant of a 25×25 matrix.

Solution:

- Let $f(n)$ be the numbers of multiplications required to find the determinant of a $n \times n$ matrix using cofactor expansion.
- We claim that $f(n) = \sum_{k=1}^{n-1} \frac{n!}{k!}$, for $n > 1$
- Cofactor expansion of determinant of $n \times n$ matrix A along j^{th} row is

$$det(A) = \sum_{k=1}^{n} a_{jk}(-1)^{j+k} det A_{jk}$$

where A_{jk} is the matrix formed by removing j^{th} row and k^{th} column.

Question 17

- Since each minor A_{jk} is of size $n-1 \times n-1$, it requires $f(n-1)$ multiplications.
- There are total of n minors.
- Also there are n more multiplications of each element of j^{th} row with the corresponding minor.
- By the above reasoning and absorbing multiplications of $(-1)^{i+j}$ in the addition, we have the following recursive formulae,

$$f(n) = nf(n-1) + n$$

- Now, using mathematical induction on $f(n)$.
- Hence, we obtain that $f(n) \geq n!$.
- As $25! \approx 10^{25}$. Therefore, time needed to compute the determinant of a 25×25 matrix will be 10^{16} seconds $\approx 3 \times 10^{9}$ years.