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By definition

$$w = -\frac{qB}{mc}$$

So, we have

$$\left[\hat{S}_y, \hat{H}\right] = \left[\hat{S}_y, w\hat{S}_y\right] = 0$$

$$\left[\hat{S}_x, \hat{H}\right] = \left[\hat{S}_x, w\hat{S}_y\right] = w\left[\hat{S}_x, \hat{S}_y\right] = i\hbar w\hat{S}_z$$

$$\left[\hat{S}_z, \hat{H}\right] = \left[\hat{S}_z, w\hat{S}_y\right] = w\left[\hat{S}_z, \hat{S}_y\right] = -i\hbar w\hat{S}_x$$

From this relation

$$\frac{d\hat{A}_H(t)}{dt} = \frac{1}{i\hbar}\left[\hat{A}_H, \hat{H}\right] = \frac{1}{i\hbar}e^{\frac{i\hat{H}t}{\hbar}}\left[\hat{A}(0), \hat{H}\right]e^{-i\frac{\hat{H}t}{\hbar}}$$

It results

$$\frac{d\hat{S}_x(t)}{dt} = \frac{1}{i\hbar}\left[\hat{S}_x(t), \hat{H}\right] = \frac{1}{i\hbar}e^{\frac{i\hat{H}t}{\hbar}}\left[\hat{S}_x(0), \hat{H}\right]e^{-i\frac{\hat{H}t}{\hbar}} = \frac{i\hbar w}{i\hbar}e^{\frac{i\hat{H}t}{\hbar}}\hat{S}_z(0)e^{-i\frac{\hat{H}t}{\hbar}} = w\hat{S}_z(t)$$

$$\frac{d\hat{S}_y(t)}{dt} = \frac{1}{i\hbar}\left[\hat{S}_y(t), \hat{H}\right] = \frac{1}{i\hbar}e^{\frac{i\hat{H}t}{\hbar}}\left[\hat{S}_y(0), \hat{H}\right]e^{-i\frac{\hat{H}t}{\hbar}} = 0$$

$$\frac{d\hat{S}_z(t)}{dt} = \frac{1}{i\hbar}\left[\hat{S}_z(t), \hat{H}\right] = \frac{1}{i\hbar}e^{\frac{i\hat{H}t}{\hbar}}\left[\hat{S}_z(0), \hat{H}\right]e^{-i\frac{\hat{H}t}{\hbar}} = \frac{-i\hbar w}{i\hbar}e^{\frac{i\hat{H}t}{\hbar}}\hat{S}_x(0)e^{\frac{-i\hat{H}t}{\hbar}} = -w\hat{S}_x(t)$$

therefore

$$\hat{S}_y(t) = \hat{S}_y(0)$$

$$\frac{d^2\hat{S}_x(t)}{dt^2} = w\frac{d\hat{S}_z(t)}{dt} = w\left(-w\hat{S}_x(t)\right) = -w^2\hat{S}_x(t)$$

$$\frac{d^2\hat{S}_z(t)}{dt^2} = w\frac{d\hat{S}_x(t)}{dt} = -w\left(+w\hat{S}_z(t)\right) = -w^2\hat{S}_z(t)$$

Solving these two equations yields

$$\hat{S}_x(t) = \hat{A}\sin wt + \hat{B}\cos wt$$

$$\hat{S}_z(t) = \hat{C}\sin wt + \hat{D}\cos wt$$

In these relations $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\widehat{D}$ are time independent operators. Imposing initial conditions at $t = 0$ we can find these constants.

$$\hat{S}_x(0) = \hat{B}$$

$$\hat{S}_z(0) = \hat{D}$$

$$w\hat{S}_z(0) = \left.\frac{d\hat{S}_x(t)}{dt}\right|_{t=0} = w\hat{A}$$

$$-w\hat{S}_x(0) = \left.\frac{d\hat{S}_z(t)}{dt}\right|_{t=0} = w\hat{C}$$

Thus

$$\hat{S}_x(t) = \hat{S}_z(0)\sin wt + \hat{S}_x(0)\cos wt$$

$$\hat{S}_z(t) = -\hat{S}_x(0)\sin wt + \hat{S}_z(0)\cos wt$$

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Translated by: @PhysicsDirectory