4-27

In the region of $|x| \geq a$ it is obvious that $\psi(x)$ must be zero duo to infinite potential. In the region of $-a \leq x \leq a$ we have:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\big(E - V_0\delta(x)\big)\psi(x) = 0$$

Supposing $x \neq 0$

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0 \quad , \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$\psi(x)$ must be zero at $x = \pm a$ thus

$$\psi(x) = \begin{cases} 0 & ; \ |x| \geq a \\ A\sin k(x+a) & ; \ -a \leq x \leq a \\ B\sin k(x-a) & ; \ 0 \leq x \leq a \end{cases}$$

We want even solutions so $A = -B$

On the other hand, by discontinuity of wave function at $x = 0$ and integrating from Schrödinger equation we have:

$$\left.\frac{d\psi(x)}{dx}\right|_{x=0^+} - \left.\frac{d\psi(x)}{dx}\right|_{x=0^-} = \frac{2mV_0}{\hbar^2}\psi(0)$$

$$-Ak\cos ka - Ak\cos ka = \frac{2mV_0}{\hbar^2}A\sin ka$$

So

$$\tan ka = -\frac{k\hbar^2}{mV_0}$$

By solving above equation, we can find energy, meanwhile A can be found by normalization condition:

$$1 = \int_{-\infty}^{+\infty}|\psi(x)|^2 dx = \int_{-a}^{0} A^2\sin^2 k(x+a)\,dx + \int_{0}^{a} A^2\sin^2 k(x-a)\,dx = \frac{A^2}{2}\left(2a - \frac{\sin 2ka}{k}\right)$$

$$\Rightarrow A = \left(a - \frac{\sin 2ka}{2k}\right)^{\frac{-1}{2}}$$

Mohammad Behtaj & Adel Sepehri

Translate by: @PhysicsDirectory Telegram Channel