7.18

(a) we are going to use these relations

$$Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\, e^{\pm i\varphi} \sin\theta\cos\theta$$

$$\boxed{Y_{l,m} = Y_{l,m}(\theta,\varphi)}$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\, e^{\pm 2i\varphi} \sin^2\theta$$

now

$$\frac{xz}{r^2} = \frac{r\sin\theta\cos\varphi\, r\cos\theta}{r^2} = \sin\theta\cos\theta\cos\varphi = \frac{1}{2}\sin\theta\cos\theta\,(e^{i\varphi}+e^{-i\varphi})$$

$$= \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(Y_{2,-1} - Y_{2,1}\right)$$

$$\frac{x^2-y^2}{r^2} = \frac{r^2\sin^2\theta\cos^2\varphi - r^2\sin^2\varphi\sin^2\theta}{r^2}$$

$$= \sin^2\theta\,(\cos^2\varphi - \sin^2\varphi) = \sin^2\theta\cos 2\varphi$$

$$= \frac{1}{2}\sin^2\theta\,(e^{2i\varphi} + e^{-2i\varphi}) = \frac{1}{2}\sqrt{\frac{32\pi}{15}}\left(Y_{2,2} + Y_{2,-2}\right)$$

(b)

$$\langle 1,0|\frac{xz}{r^2}|1,1\rangle = \frac{1}{2}\sqrt{\frac{8\pi}{15}}\,\langle 1,0|Y_{2,-1} - Y_{2,1}|1,1\rangle$$

$$\underline{\underline{\overset{(7.401)}{=}}}\ \frac{1}{2}\sqrt{\frac{8\pi}{15}}\Big(\underbrace{\langle 1,2;1,-1|1,0\rangle}_{\sqrt{\frac{3}{10}}} - \underbrace{\langle 1,2;1,1|1,0\rangle}_{0}\Big)$$

$$\underbrace{\langle 1||Y^{(2)}||1\rangle}_{(7.404)}$$

$$= \frac{1}{\cancel{2}}\sqrt{\frac{\cancel{8\pi}}{15}}\sqrt{\frac{3}{10}}\left(\sqrt{\frac{\cancel{5}}{\cancel{4\pi}}}\left(-\sqrt{\frac{\cancel{2}}{\cancel{5}}}\right)\right) = -\sqrt{\frac{1}{50}} = -\frac{\sqrt{2}}{10}$$

$$\langle 1,1 \left| \frac{x^2-y^2}{r^2} \right| 1,-1\rangle$$

$$= \frac{1}{2}\sqrt{\frac{32\pi}{15}}\left(\langle 1,1 | Y_{2,2} + Y_{2,-2} | 1,-1\rangle \right)$$

$$= \frac{1}{2}\sqrt{\frac{32\pi}{15}}\left(\overbrace{\langle 1,2;-1,2 | 1,1\rangle}^{\sqrt{\frac{3}{5}}} \langle 1 \| Y^{(2)} \| 1\rangle + \underbrace{\langle 1,2;-1,-2 | 1,1\rangle}_{0} \langle 1 \| Y^{(2)} \| 1\rangle \right)$$

$$= \frac{1}{2}\sqrt{\frac{32\cancel{\pi}}{\cancel{15}_{5}}}\sqrt{\frac{\cancel{3}}{5}}\left(-\sqrt{\frac{2}{\cancel{5}}}\sqrt{\frac{\cancel{5}}{4\cancel{\pi}}}\right) = -\frac{8}{\sqrt{25}} = -\frac{8}{5}$$