2-6

(a)

$$\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{15}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{15} + \frac{1}{3} + \frac{1}{5} = \frac{9}{15}$$

(b)

To finding expectation value of $\hat{B}$ with state $|\psi\rangle$ first we find $\hat{B}|\psi\rangle$

$$\begin{aligned}\hat{B}|\psi\rangle = \hat{B}\left(\frac{1}{\sqrt{15}}|\phi_1\rangle + \frac{1}{\sqrt{3}}|\phi_2\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle\right) &= \frac{1}{\sqrt{15}}\hat{B}|\phi_1\rangle + \frac{1}{\sqrt{3}}\hat{B}|\phi_2\rangle + \frac{1}{\sqrt{5}}\hat{B}|\phi_3\rangle \\ &= \frac{1}{\sqrt{15}}(3\,(1)^2 - 1)|\phi_1\rangle + \frac{1}{\sqrt{3}}(3\,(2)^2 - 1)|\phi_2\rangle + \frac{1}{\sqrt{5}}(3\,(3)^2 - 1)|\phi_3\rangle \\ &= \frac{2}{\sqrt{15}}|\phi_1\rangle + \frac{11}{\sqrt{3}}|\phi_2\rangle + \frac{26}{\sqrt{5}}|\phi_3\rangle\end{aligned}$$

Then we conclude

$$\langle\psi|\hat{B}|\psi\rangle = \frac{1}{\sqrt{15}}\frac{2}{\sqrt{15}} + \frac{1}{\sqrt{3}}\frac{11}{\sqrt{3}} + \frac{1}{\sqrt{5}}\frac{26}{\sqrt{5}} = \frac{2}{15} + \frac{11}{3} + \frac{26}{5} = \frac{135}{15} = 9$$

For this reason, $|\psi\rangle$ is not normalize so

$$\langle\hat{B}\rangle_\psi = \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{9}{\frac{9}{15}} = 15$$

(c)

$$\langle\psi|\hat{B}^2|\psi\rangle = \langle\psi|\hat{B}\hat{B}|\psi\rangle$$

By assuming that $|\phi\rangle = \hat{B}|\psi\rangle$ we have

$$\langle\phi|\phi\rangle = \left(\frac{2}{\sqrt{15}}\right)^2 + \left(\frac{11}{\sqrt{3}}\right)^2 + \left(\frac{26}{\sqrt{5}}\right)^2 = \frac{4}{15} + \frac{121}{3} + \frac{676}{5} = \frac{2637}{15}$$

$$\langle\hat{B}^2\rangle_\psi = \frac{\langle\psi|\hat{B}^2|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\frac{2637}{15}}{\frac{9}{15}} = 293$$