2.48

(a) using $\left(c|\varphi_i\rangle\langle\varphi_j|\right)^\dagger = c^*|\varphi_j\rangle\langle\varphi_i|$

$\Rightarrow \hat{A}^\dagger = |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| + |\varphi_3\rangle\langle\varphi_3| + i|\varphi_2\rangle\langle\varphi_1|$

$-|\varphi_3\rangle\langle\varphi_1| - i|\varphi_1\rangle\langle\varphi_2| - |\varphi_1\rangle\langle\varphi_3| = \hat{A} \Rightarrow$ Hermitian

using $|\varphi_i\rangle\langle\varphi_j| = \delta_{ij}$

$\hat{A}^2 = |\varphi_1\rangle\langle\varphi_1| - i|\varphi_1\rangle\langle\varphi_2| - |\varphi_1\rangle\langle\varphi_3| + |\varphi_2\rangle\langle\varphi_2|$

$+ i|\varphi_2\rangle\langle\varphi_1| - |\varphi_3\rangle\langle\varphi_1| + i|\varphi_2\rangle\langle\varphi_1| + |\varphi_3\rangle\langle\varphi_3|$

$+ |\varphi_2\rangle\langle\varphi_2| - i|\varphi_2\rangle\langle\varphi_3| - |\varphi_3\rangle\langle\varphi_1| + i|\varphi_3\rangle\langle\varphi_2|$

$+ |\varphi_3\rangle\langle\varphi_3| - i|\varphi_1\rangle\langle\varphi_2| + |\varphi_1\rangle\langle\varphi_1| - |\varphi_1\rangle\langle\varphi_3|$

$+ |\varphi_1\rangle\langle\varphi_1|$

$= 3|\varphi_1\rangle\langle\varphi_1| - 2i|\varphi_1\rangle\langle\varphi_2| - 2|\varphi_1\rangle\langle\varphi_3| + 2|\varphi_2\rangle\langle\varphi_2|$

$+ 2i|\varphi_2\rangle\langle\varphi_1| - 2|\varphi_3\rangle\langle\varphi_1| + 2|\varphi_3\rangle\langle\varphi_3| - i|\varphi_2\rangle\langle\varphi_3|$

$+ i|\varphi_3\rangle\langle\varphi_2| \neq \hat{A} \Rightarrow \hat{A}$ is not a projection operator

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(b) $A_{ij} = \langle \varphi_i | A | \varphi_j \rangle$

$$\Rightarrow \hat{A} = \begin{pmatrix} 1 & -i & -1 \\ i & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

(c) $$\det(A-a) = \begin{vmatrix} 1-a & -i & -1 \\ i & 1-a & 0 \\ -1 & 0 & 1-a \end{vmatrix} = 0$$

$$\Rightarrow a = (1 \,,\, 1+\sqrt{2} \,,\, 1-\sqrt{2})$$

for $a=1$

$$\begin{pmatrix} 1 & -i & -1 \\ i & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x = 0 \\ z = -iy \end{matrix}$$

normalize $\rightarrow |a_1 = 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

for $a = 1+\sqrt{2} \rightarrow |a_2 = 1+\sqrt{2}\rangle = \frac{1}{2} \begin{pmatrix} -\sqrt{2} \\ -i \\ 1 \end{pmatrix}$

" $a = 1-\sqrt{2} \rightarrow |a_3 = 1-\sqrt{2}\rangle = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ -i \\ 1 \end{pmatrix}$