Series solutions

1 Real analytic solutions

To start with let us recall the Taylor series of a function $y(x)$ around the point x_0:

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n.$$

So if we know the derivatives of the function at a point then we can formally write its Taylor series. Let us take an example of IVP:

$$y' = y, \ y(0) = 1.$$

Then we can see from the equation $y'(0) = y(0) = 1$. We can differentiate the equation to get second derivative

$$y'' = y' = y = 1 \text{ at } 0.$$

Similarly we can find all derivatives as

$$y^{(m)}(0) = 1, \text{ for all } m \in \mathbb{N}$$

Therefore its Taylor series is

$$y(x) = \sum_{k=0}^{\infty} \frac{x^n}{n!} = e^x.$$

This is the solution of the IVP.

Question: Can we do this always?

Consider another example

$$y' + y = h(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}, \ \ y(0) = 0.$$

In this case again we get

$$y^{(m)}(0) = 0, \ \forall m \in \mathbb{N}.$$

Therefore Taylor series of $y(x)$ is identically equal to zero. However by integrating the equation we can write the solution as

$$y(x) = e^{-x}\int_0^x h(t)dt$$

this is not equal to zero for any $x > 0$. Therefore the formal Taylor series expansion need not be a solution of IVP.
From our experince in Calculus, we know that $h(x)$ is not *real analytic*. So that could be the problem. To make our ideas clear let us recall some basics on power series.

We recall the following results from calculus about the power series. Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ is called *power series* with center x_0. It is easy to see that a power series converges for $x = x_0$. Power series is a function of x provided it converges for x. If a power series converges, then the domain of convergence is either a bounded interval or the whole of $\mathbb{R}$. So it is natural to study the largest interval where the power series converges.

Theorem 1. *If $\sum a_n x^n$ converges at $x = r$, then $\sum a_n x^n$ converges for $|x| < |r|$.*

Proof. We can find $C > 0$ such that $|a_n x^n| \leq C$ for all n. Then

$$|a_n x^n| \leq |a_n r^n||\frac{x}{r}|^n \leq C|\frac{x}{r}|^n.$$

Conclusion follows from comparison theorem.

Theorem 2. *Consider the power series $\sum\limits_{n=0}^{\infty} a_n x^n$. Suppose*

$$\beta = \limsup_n \sqrt[n]{|a_n|}$$

and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then

1. $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges for $|x| < R$*
2. $\sum\limits_{n=0}^{\infty} a_n x^n$ *diverges for $|x| > R$.*
3. *No conclusion if $|x| = R$.*

In case the limit exists in the definition of β, then

$$\beta = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Within the interval of convergence we can differentiate the series term-by-term and integrate term-by-term. Indeed we have

⟨dif⟩ **Theorem 3.** *Suppose that the power series* $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ *for* $|x-x_0| < R$ *has radius of convergence* $R > 0$ *and sum equal to* $f(x)$. *Then* f *is differentiable in* $|x-x_0| < R$ *and* $f'(x) = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}$ *for* $|x-x_0| < R$.

An important consequence of this is

⟨zerothm⟩ **Theorem 4.** *If* $\sum a_k x^k \equiv 0$ *then* $a_k = 0$ *for all* k

Proof. Taking $x = 0$ we get $a_0 = 0$. Then by above theorem we can differentiate the series to get

$$\sum k a_k x^{k-1} \equiv 0$$

Again taking $x = 0$ we get $a_1 = 0$. Proceeding this way we get $a_k = 0$ for all k. □

Definition 1. *A function* $g(x)$ *defined in an interval* I *containing the point* x_0 *is called* **real analytic** *at* x_0 *if it can be represented as power series around* x_0. *That is there exist constants* $c_k, k = 0, 1, 2...$ *such that the following series converges for* $|x - x_0| < r$, *for some* $r > 0$.

$$g(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k.$$

Remark 1. *If* g *is real analytic function at a point* x_0, *then* $c_k = \frac{g^{(k)}(x_0)}{k!}$. *That is* g *is equal to its Taylor series.*

Let us consider the example: $y'' = y$. Let us assume that the function $y(x)$ is equal to its power series $\sum c_k x^k$. Then by the above theorem 3, we get

$$y'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}, \quad y''(x) = \sum_{k=2}^{\infty} k(k-1) x^{k-2}.$$

Substituting these into the equation $y'' - y = 0$ we get

$$\begin{aligned} 0 &= \sum_{k=2}^{\infty} k(k-1)x^{k-2} - \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_k\right] x^k \end{aligned}$$

Therefore we get the recurrence relation

$$c_{k+2} = \frac{c_k}{(k+2)(k+1)}, \ k = 1, 2, 3...$$

Therefore

$$\begin{aligned}
k = 0 : &\implies c_2 = \frac{c_0}{2} \\
k = 1 : &\implies c_3 = \frac{c_1}{3 \cdot 2} \\
k = 2 : &\implies c_4 = \frac{c_2}{4 \cdot 3} = \frac{c_0}{4!} \\
k = 3 : &\implies c_5 = \frac{c_1}{5!}
\end{aligned}$$

Iterating this we get

$$c_{2k} = \frac{c_0}{(2k)!}, \; c_{2k+1} = \frac{c_1}{(2k+1)!}$$

Therefore, if $c_0 = c_1$ we get

$$y(x) = c_0 \sum \frac{x^k}{k!} = c_0 e^x$$

and if $c_1 = -c_0$ then $y(x) = c_0 e^{-x}$. Since linear second order equation can have only two linearly independent solutions, e^x and e^{-x} are the only L.I. solutions. □

Let us consider another example with variable coefficients.

Example 2. $y'' - 2xy' + 2y = 0$.

Writing $y(x) = \sum a_k x^k$ and differentiating term by term we get

$$\begin{aligned}
xy' &= \sum_{k=1}^{\infty} k a_k x^k \\
y'' &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k
\end{aligned}$$

Substituting in the equation we get

$$y'' - 2xy' + 2y = \sum [(k+2)(k+1)a_{k+2} - 2k a_k + 2a_k] x^k = 0$$

Then by Theorem 4 above we get

$$(k+2)(k+1)a_{k+2} - 2k a_k + 2a_k = 0, \; \forall k$$

That is

$$2a_2 + 2a_0 = 0 \implies a_2 = -a_0$$

and

$$a_{k+2} = \frac{2(k-1)}{(k+2)(k+1)} a_k$$

Now $k = 1$ implies $a_3 = 0 \cdot a_1 = 0$ and hence

$$a_{2k+1} = 0 \text{ for all } k = 0, 1, 2..$$

Also $k = 2$ implies $a_4 = \frac{2a_2}{12} = -\frac{a_0}{6}$. All other even coefficients can be calculated as multiple of a_0. Therefore we get

$$y(x) = a_1 x + a_0(1 - \frac{x^4}{6} + ...)$$

The two L.I. solutions are $y_1(x) = x$ and $y_2(x) = 1 - \frac{x^4}{6} + ...$ To see the convergence of y_2 we use the recurrence relation. The coefficients of y_2 satisfy

$$\frac{a_{k+2}}{a_k} = \frac{2(k-1)}{(k+2)(k+1)} \to 0 \text{ as } k \to \infty.$$

Hence by ratio test, power series converges with radius of convergence equals to ∞. $\square$

Then it is natural to ask the following:

Q1: Can we apply this method for equations like $y'' + e^x y = 0$?
Q2: What is the most general class of coefficients for which the power series solution exists?

To answer the first question, we need the following Cauchy product

Definition 3. *Let $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ be two power series. The Cauchy product of these two series is $\sum_{k=0}^{\infty} c_n x^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.*

Theorem 5. *If $\sum a_k x^k$ converges for $|x| < R_1$ and $\sum b_k x^k$ converged for $|x| < R_2$ then their Cauchy product $\sum c_n x^n$ converges for $|x| < R$ where $R \geq \min\{R_1, R_2\}$.*

Remark 2. *It is possible to have series with finite radius of convergence and their product has infinite radius of convegence. For example take $f(x) = \frac{1+x}{1-x}$ and $g(x) = \frac{1-x}{1+x}$. Both of them has radius of convergence 1 but their product has ∞.*

Theorem 6. *Suppose $a(x)$ and $b(x)$ are real analytic in $|x| < R$. Then all solutions of the equation*

$$y'' + a(x)y' + b(x)y = 0$$

are real analytic in $|x| < R$.

Proof. See the text book.

Legendre Equation: One of the important differential equation that appears in Mathematical physics is the Legendre equation:

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0, \ p \in \mathbb{R}.$$

If we write this equation as

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{p(p+1)}{(1-x^2)}y = 0$$

then we see that the functions a_1 and a_2 given by

$$a_1(x) = \frac{2x}{(1-x^2)}, \text{ and } a_2(x) = \frac{p(p+1)}{(1-x^2)},$$

are real analytic at $x = 0$ with the series converges in $(-1, 1)$. Therefore by the above theorem the problem will have two linearly independent real analytic solutions. We can write the serieses of a_1 and a_2 as

$$a_1(x) = (-2x)\sum_{k=0}^{\infty} x^{2k}, \ \ a_2(x) = \sum_{k=0}^{\infty} p(p+1)x^{2k}, \ \text{ for } |x| < 1.$$

By assuming $y(x) = \sum c_k x^k$ we see that

$$\begin{aligned} (-2x)y'(x) &= \sum_{k=0}^{\infty} -2kc_k x^k \\ y''(x) &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k \\ -x^2y'' &= \sum_{k=0}^{\infty} -k(k-1)c_k x^k \end{aligned}$$

Therefore

$$\begin{aligned} (1-x^2)y'' - 2xy' + p(p+1)y &= \sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} - k(k-1)c_k - 2kc_k + p(p+1)c_k\right] x^k \\ &= \sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} + (p+k+1)(p-k)c_k\right] x^k \end{aligned}$$

Therefore by Theorem 4 we get the recurrence relation

$$(k+2)(k+1)c_{k+2} + (p+k+1)(p-k)c_k = 0, \ k = 0, 1, 2, ... \tag{1.1}$$ recrel

Hence we get

$$c_2 = -\frac{p(p+1)}{2}c_0, \qquad c_3 = -\frac{(p+2)(p-1)}{3\cdot 2}c_1$$
$$c_4 = \frac{(p+3)(p+1)p(p-2)}{4\cdot 3\cdot 2}c_0, \quad c_5 = \frac{(p+4)(p+2)(p-1)(p-3)}{5\cdot 4\cdot 3\cdot 2}c_1.$$

Therefore we can write the solution as

$$\begin{aligned} y(x) &= c_0\left(1 - \frac{p(p+1)}{2!}x^2 + ...\right) + c_1\left(x - \frac{(p+2)(p-1)}{3!}x^3 + ...\right) \\ &= c_0\phi_1(x) + c_1\phi_2(x). \end{aligned}$$

The adjacent coefficients of the series in ϕ_1 and ϕ_2 are related by (1.1) and hence

$$\left|\frac{c_{k+2}}{c_k}\right| = \left|\frac{(p+k+1)(p-k)}{(k+2)(k+1)}\right| \to 0, \text{ as } k \to \infty.$$

By ratio test the series converges. Also,

$$W(\phi_1, \phi_2)(0) = 1.$$

Therefore ϕ and ϕ_2 are two linearly independent solutions.

Legendre Polynomials: We note that when p is a non-negative even integer $p = 2m, m = 0, 1, 2..$ then ϕ_1 has only a finite number of non-zero terms viz a polynomial of order $2m$. for example

$$\begin{aligned} p = 0 : &\implies \phi_1(x) = 1 \\ p = 2 : &\implies \phi_1(x) = 1 - 3x^2 \end{aligned}$$

Similarly when p is a positive odd integer $p = 2m+1$ then ϕ_2 is a polynomial of order $2m+1$. For example,

$$\begin{aligned} p = 1 : &\implies \phi_1(x) = x \\ p = 3 : &\implies \phi_1(x) = x - \frac{5}{3}x^3 \end{aligned}$$

Definition 4. *The polynomial solution P_n of degree n of Legendre equation satisfying $P_n(1) = 1$ is called $n-$th Legendre polynomial. These are explicitly given by*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

2 Regular singular points: Frobenius solutions

Let $a(x) : (-l, l)\backslash\{0\} \to \mathbb{R}$ be a continuous function. The point $x = 0$ is called a singular point if $|a(x)| \to \infty$ as $x \to 0$. Now we define

Definition 5. Regular singular point: *A point $x = x_0$ is a a regular singular point for the second order equation*

$$y'' + a(x)y' + b(x)y = 0$$

if $(x - x_0)a(x)$ and $(x - x_0)^2 b(x)$ are real analytic at $x = x_0$.

For simplicity if we take $x_0 = 0$ then a second order equation with a regular singular point at $x = 0$ has the form

$$x^2y'' + xa(x)y' + b(x)y = 0$$

where a, b are real analytic at 0.

Important Idea: Assume $y(x) = x^r \sum_{k=0}^{\infty} c_k x^k$ and find $c_k's$

Rationale: Coefficients are singular. So expect solutions also to be singular

Let us illustrate the method for

Example 6. *Consider the equation $L(y) = x^2y'' + \frac{3}{2}xy' + xy = 0$.*

Solution: Let us assume $y(x) = x^r \sum c_k x^k$. Then

$$\begin{aligned} y'(x) &= \sum c_k (k + r)x^{k+r-1} \\ y''(x) &= \sum c_k (k + r)(k + r - 1)x^{k+r-2}. \end{aligned}$$

Therefore substituting in the equation

$$\begin{aligned} 0 = L(y) &= \sum_{k=0}^{\infty} \left((k + r)(k + r - 1)c_k + \frac{3}{2}(k + r)c_k \right) x^{k+r} + \sum_{k=1}^{\infty} c_{k-1}x^{k+r} \\ &= (r(r - 1) + \frac{3}{2}r)c_0x^r + \sum_{k=1}^{\infty} \left[\left((k + r)(k + r - 1) + \frac{3}{2}(k + r) \right) c_k + c_{k-1} \right] x^{k+r} \\ &= p(r)c_0x^r + \sum_{k=1}^{\infty} [p(r + k)c_k + c_{k-1}]\, x^k = 0. \end{aligned}$$

here $p(r) = r(r - 1) + \frac{3}{2}r$ is called **indicial polynomial**. Now by Theorem 4, we get

$$p(r) = 0 \ (c_0 \neq 0, why?) \ \text{ and } \ p(r + k)c_k = -c_{k-1}.$$

This implies $r_1 = 0, r_2 = -\frac{1}{2}$.

$$p(r+k)c_k = -c_{k-1} \implies c_k = \frac{(-1)^k c_0}{p(r+k)p(r+k-1)...p(r+1)},\ k=1,2,...$$

1^{st} **solution:** Take $r = r_1 = 0$. Then $p(r+k) = p(k) \neq 0$ for all $k = 1,2,..$ (since the only other root is $-\frac{1}{2}$). Hence all other coefficients c_k can be computed using

$$c_k = \frac{(-1)^k c_0}{p(k)p(k-1)...p(1)},\ k=1,2,...$$

Hence the first solution is

$$\phi_1(x) = c_0 + c_0 \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{p(k)p(k-1)...p(1)}$$

2^{nd} **Solution:** Taking $r = r_2 = -\frac{1}{2}$, we see that $p(r+k) = p(k-\frac{1}{2}) \neq 0$ for any $k = 1,2,...$ since the other root is 0. Hence all other coefficients c_k can be computed using

$$c_k = \frac{(-1)^k c_1}{p(k-\frac{1}{2})p(k-\frac{3}{2})...p(\frac{1}{2})}, k=1,2,3,$$

and the solution will be

$$\phi_2(x) = c_1 x^{-\frac{1}{2}} + c_1 x^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{p(k-\frac{1}{2})p(k-\frac{3}{2})...p(\frac{1}{2})}$$

Convergence: The recurrence relation implies

$$\left|\frac{c_{k+1}}{c_k}\right| \leq \left|\frac{1}{p(k+r+1)}\right| \to 0 \text{ as } k \to \infty.$$

Hence by ratio test the series converges for all x. However the second solution has singularity at $x = 0$. □

Now let us look at the most general case where $a(x)$ and $b(x)$ are real analytic. So let us assume $a(x) = \sum \alpha_k x^k,\ \ b(x) = \sum \beta_k x^k$ and $y(x) = \sum c_k x^{k+r}$ then

$$\begin{aligned}
y'(x) &= \sum (k+r)c_k x^{k+r-1} \\
y''(x) &= \sum (k+r-1)(k+r)c_k x^{k+r-2} \\
b(x) &= x^r \sum c_k x^k \sum \beta_k x^k = x^r \sum \tilde{\beta}_k x^k, \quad \tilde{\beta}_k = \sum_{j=0}^{k} c_j \beta_{k-j} \\
xa(x)y'(x) &= x^r \sum (k+r)c_k x^k \sum \alpha_k x^k = x^r \sum \tilde{\alpha}_k x^k, \quad \tilde{\alpha}_k = \sum_{j=0}^{k} (j+r)c_j \alpha_{k-j}.
\end{aligned}$$

Substituting this in $L(y) = 0$ we get

$$\begin{aligned} 0 = L(y) = x^2y'' + xa(x)y' + b(x)y &= x^r \sum \left[(k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k\right] x^k \\ &= \sum_{k=0}^{\infty} \left[(k+r)(k+r-1)c_k + \sum_{j=0}^{k}(j+r)c_j\alpha_{k-j} + \sum_{j=0}^{k} c_j\beta_{k-j}\right] x^{k+r} \end{aligned}$$

Therefore by Theorem 4, we get

$$(k+r)(k+r-1)c_k + (k+r)c_k\alpha_0 + c_k\beta_0 + \sum_{j=0}^{k-1}(j+r)c_j\alpha_{k-j} + \sum_{j=0}^{k-1} c_j\beta_{k-j} = 0,\ k = 0, 1, 2...$$

That is writing the coefficient of c_0 and $c_k, k = 1, 2, ..$ we see

$$p(r)c_0 + p(r+k)c_k + \sum_{j=0}^{k-1}(j+r)c_j\alpha_{k-j} + \sum_{j=0}^{k-1} c_j\beta_{k-j} = 0$$

where $p(r) = r(r-1) + r\alpha_0 + \beta_0$ is called the **indicial polynomial.**

By assuming $c_0 \neq 0$, we get

$$p(r) = 0 \implies r = r_1 \text{ and } r = r_2$$

For each of the values of r_1 and r_2 we see that $c_k, k = 1, 2, 3..$ satisfies

$$p(r+k)c_k = -\sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] c_j$$

So if $p(r+k)$ is not zero for any k, then the above equation determines the c_k uniquely and hence the solutions. But this may not be the case as r_2 can be equal to $r_1 + k$ for some k. Therefore we have the following several cases:

Case I: $r_1 \neq r_2,\ r_1 < r_2,\ r_2 \neq r_1 + k$ for any k.

In this case all coefficients $c_k, k = 1, 2, ...$ can be computed using

$$c_k(r) = \frac{-1}{p(r+k)} \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] c_j \tag{2.1}$$

coef

The linearly independent solutions are

$$\phi_1(x) = c_0 x^{r_1} + x^{r_1} \sum_{k=1}^{\infty} c_k(r_1) x^k, \; c_0 \neq 0$$
$$\phi_2(x) = c_0 x^{r_2} + x^{r_2} \sum_{k=1}^{\infty} c_k(r_2) x^k, \; c_0 \neq 0$$

An example of this type is computed above.

Case II: $r_1 = r_2$

In this case $p(r_1) = 0$ and $p'(r_1) = 0$. Writing the solution $y(x)$ as

$$y(x) = c_0 + \sum_{k=1}^{\infty} c_k(r) x^{k+r}$$

we may see $L(y)$ as a function of x and r. By assuming that $c_k(r)$ satisfy (2.1), we get

$$\begin{aligned} L(y) &= c_0 q(r) + \sum_{k=1}^{\infty} c_k(r) x^{k+r} \\ &= c_0 q(r) \end{aligned}$$

Therefore

$$L\left(\frac{\partial}{\partial r} y\right) = \frac{\partial}{\partial r} L(y)(x,r) = c_0 (p'(r) + (\log r) p(r)) x^r = 0, \text{ at } r = r_1.$$

Therefore if ϕ_1 is the solution corresponding to $r = r_1$ then $\frac{\partial \phi_1}{\partial r}$ at $r = r_1$ is the second solution. Therefore,

$$\phi_1(x) = \sum_{k=0}^{\infty} c_k(r_1) x^{k+r_1} = c_0 x^r + \sum_{k=1}^{\infty} c_k(r_1) x^{k+r_1}$$

$$\begin{aligned} \phi_2(x) = \left.\frac{\partial \phi_1}{\partial r}\right|_{r=r_1} &= x^{r_1} \sum_{k=1}^{\infty} c_k'(r_1) x^k + (\log x) x^{r_1} \sum_{k=0}^{\infty} c_k(r_1) x^k \\ &= x^{r_1} \sum_{k=1}^{\infty} c_k'(r_1) x^k + (\log x) \phi_1(x) \end{aligned}$$

Practically, one assumes the following for calculating the second solution

$$\phi_2(x) = x^{r_1} \sum_{k=1}^{\infty} b_k x^k + (\log x) \phi_1(x)$$

and obtain the coefficients b_k by substituting this in the given equation.

Example 7. *Consider the Bessel equation:* $x^2y'' + xy' + x^2y = 0$

In this case $a(x) = 1, b(x) = x^2$. Therefore $\alpha_0 = 1, \beta_0 = 0$.
Indicial polynomial: $p(r) = r(r-1)\alpha_0 r + \beta_0 = r^2$. The roots are $r_1 = 0$ and $r_2 = 0$. Then the possible solutions are of the form

$$\phi_1(x) = \sum c_k x^k, \text{ and } \phi_2(x) = (\log x)\phi_1 + \sum_{k=1}^{\infty} b_k x^k$$

Then

$$\begin{aligned} 0 = x^2y'' + xy' + x^2y &= \sum_{k=2}^{\infty} c_k k(k-1)x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^{k+2} \\ &= c_1 x + \sum_{k=2}^{\infty} \left[(k(k-1)+k)c_k + c_{k-2}\right] x^k \end{aligned}$$

Therefore

$$c_1 = 0, \text{ and } c_k = -\frac{c_{k-2}}{k^2}, k = 2, 3, ...$$

This implies

$$c_{2k+1} = 0, \; c_{2k} = \frac{(-1)^k}{2^2 \cdot 4^2 \cdots (2m)^2} = \frac{(-1)^k}{2^{2k}(k!)^2}, \; k = 1, 2, 3...$$

Therefore the first solution is

$$\phi_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}$$

To get the second solution we substitute ϕ_2 into the equation

$$\begin{aligned} \phi_2'(x) &= \sum_{k=1}^{\infty} kc_k x^{k-1} + \frac{\phi_1}{x} + \log x \phi_1' \\ \phi_2''(x) &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \frac{\phi_1}{x^2} + \frac{2}{x}\phi_1' + (\log x)\phi_1'' \end{aligned}$$

Therefore

$$\begin{aligned} L(\phi_2) &= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=1}^{\infty} c_k x^{k+2} + 2x\phi_1' + \log x L(\phi_1) \\ &= c_1 x + c_2 2^2 x^2 + \sum_{k=3}^{\infty} \left[(k(k-1)+k)c_k + c_{k-1}\right] x^k + 2x\phi_1' = 0 \end{aligned}$$

Again by theorem 4 we get

$$c_1 = 0$$
$$c_1 x + c_2 2^2 x^2 + \sum_{k=3}^{\infty} \left[(k(k-1)+k)c_k + c_{k-1}\right] x^k = -2\sum_{k=1}^{\infty} \frac{(-1)^k 2k x^{2k}}{2^{2k}(m!)^2}$$

Since the right hand side series has only even powers of x, we get all odd coefficients c_{2k+1} as

$$c_1 = 0,$$
$$k^2 c_k = c_{k-2},\ k = 3,5,7,\ldots \implies c_{2k+1} = 0\ \forall k \in \mathbb{N}$$

Comparing the even powers, we can compute all even coefficients c_{2k} as

$$4c_2 = 1$$
$$(2k)^2 c_{2k} + c_{2k-2} = \frac{(-1)^{k+1} k}{2^{2k-2}(k!)^2},\ k = 2,3,4\ldots$$

Case III: Roots differ by integer., i.e., $r_2 = r_1 + m$ for some $m \in \mathbb{N}$
Recall that the relation satisfied by c_k:

$$p(r+k)c_k = -\sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] c_j := D_k \tag{2.2}$$ sprelation

Since $r_1 > r_2$ and p is second order polynomial $p(r_1 + k) \neq 0$ for all $k \in \mathbb{N}$. Therefore all coefficients can be computed using (2.2) to write the series solution of first solution ϕ_1. Now to find second solution we note that

$$p(r_2) = p(r_1 + m) = 0$$

Therefore it is not clear how to compute c_m from (2.2). However in **some special cases** D_m on the Right hand side of (2.2) also may become zero. In that case we can take c_m to be arbitrary and other coefficients can be calculated using (2.2). For example

Example 8. $x^2y'' + 2x^2y' - 2y' = 0$

In this case by taking $y(x) = \sum c_k x^{k+r}$ and substituting in the equation, we get

$$\begin{aligned} 0 =& x^2y'' + 2x^2y' - 2y' = \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} 2(k+r)c_k x^{k+r+1} - 2\sum_{k=0}^{\infty} c_k x^{k+r} \\ =& r(r-1)c_0 x^r + (-2)c_0 x^r + \sum_{k=1}^{\infty}\left[(k+r)(k+r-1)c_k + 2(k+r-1)c_{k-1} - 2c_k\right] x^{k+r} \end{aligned}$$

Therefore indicial polynomial is $p(r) = r^2 - r - 2$. The roots are $r_1 = -1$, and $r_2 = 2$. That is $r_2 = r_1 + 3$ implying $m = 3$. So c_3 needs to be seen carefully.
From the recurrence relation

$$((k+r)(k+r-1) - 2)\, c_k = -2(k+r-1)c_{k-1}$$

Taking $r = 2$ we can find all coefficients to get the first solution in the form

$$\phi_1(x) = x^2 \sum c_k x^k$$

Now taking $r = -1$ we get the relation

$$[(k-1)(k-2) - 2]\, c_k = -2(k-2)c_{k-1},\ k = 1, 2, 3...$$

Therefore

$$\begin{aligned}
k = 1 :&\implies -2c_1 = (-2)(-1)c_0 = 2c_0 = 2 \implies c_1 = -1\\
k = 2 :&\implies (-2)c_2 = (-2)(0)c_1 \implies c_2 = 0\\
k = 3 :&\implies (2-2)c_3 = (-2)(1)c_2 = 0
\end{aligned}$$

implying c_3 is arbitrary. So we may choose $c_3 = 0$. From the recurrence relation we will get $c_k = 0$ for all $k \geq 4$. hence the second solution is

$$\phi_2(x) = x^{-1}(1 - x)$$

On the other hand one may choose $c_3 \neq 0$. If $c_3 = 1$ then $c_4, c_5, ...$ can be found from the recurrence relation and the resultant solution will be

$$\begin{aligned}
y(x) &= x^{-1}(1-x) + x^{-1}(x^3 + c_4 x^4 + \cdots)\\
&= x^{-1}(1-x) + x^2(1 + c_4 x + \cdots) = \phi_2(x) + C\phi_1(x)
\end{aligned}$$

Hence for any choice of c_3 we will get only ϕ_1 and ϕ_2 above are the only L.I. solutions. $\square$

In general to find second solution we use $\phi_2 = v\phi_1$ and reducing the order we get

$$\begin{aligned}
v' &= \frac{1}{\phi_1^2} e^{-\int a_1(x)dx} = \frac{1}{x^{2r_1}(c_0 + c_1 x + \cdots)^2} e^{-\int \left(\alpha_0 x^{-1} + \alpha_1 + \alpha_2 x + \cdots\right)}\\
&= \frac{1}{x^{2r_1}(c_0 + c_1 x + \cdots)^2} e^{-\alpha_0 \log x - \alpha_1 x + \cdots}\\
&= \frac{x^{-\alpha_0}}{x^{2r_1}(c_0 + c_1 x + \cdots)^2} e^{-\alpha_1 x + \cdots} = \frac{1}{x^{2r_1 + \alpha_0}} g(x)
\end{aligned}$$

where $g(t) = \frac{e^{-\alpha_1 x + \cdots}}{(c_0 + c_1 x + \cdots)^2}$. Since $g(0) = \frac{1}{c_0^2} \neq 0$, $g(x)$ is real analytic at 0. This implies

$$g(x) = \sum g_k x^k$$

Therefore by taking $n = 2r_1 + \alpha_0$

$$v'(x) = b_0 x^{-n} + b_1 t^{-n+1} + \cdots + b_{n-1} t^{-1} + b_n + \cdots$$

This implies

$$v(x) = b_0 \frac{x^{-n+1}}{-n+1} + \cdots + b_{n-1} \log x + b_n x + \cdots$$

Hence the second solution ϕ_2 is

$$\begin{aligned}
\phi_2(x) = v\phi_1 &= b_{n-1}(\log x)\phi_1 + x^{r_1}(c_0 + c_1 x + \cdots)(b_0 \frac{x^{-n+1}}{-n+1} + \cdots) \\
&= b_{n-1}(\log x)\phi_1 + x^{r_1 - n + 1}(c_0 + c_1 x + \cdots)(\frac{b_0}{-n+1} + \cdots) \\
&= b_{n-1}(\log x)\phi_1 + x^{r_2} \sum d_k x^k.
\end{aligned}$$

To show $r_2 = 1 - n + r_1$ we notice that

$$p(r) = r(r-1) + r\alpha_0 + \beta_0, \quad \text{also } p(r) = (r - r_1)(r - r_2)$$

Sum of the roots $= \alpha_0 - 1 = -(r_1 + r_2) \implies r_2 = 1 - \alpha_0 - r - 1$. Now substituting $n = 2r_1 + \alpha_0$ we get $r_2 = 1 - n + r_1$. There in this case we substitute

$$\phi_2(x) = b_{n-1}(\log x)\phi_1 + x^{r_2} \sum_{k=0}^{\infty} d_k x^k$$

in the equation and find the unknowns $b_{n-1} \& d_k, k = 1, 2, 3...$ Note that in this case b_{n-1} can be equal to zero which is the special case discussed above.

As a result of the above discussed is summarized as

Theorem 7. *Suppose $a(x)$ and $b(x)$ are real analytic at $x = 0$ having radius of convergence R. Then the equation*

$$x^2 y'' + x a(x) y' + b(x) y = 0$$

has a solution in the form $x^r \sum c_k x^k$ (for some $r \in \mathbb{R}$) and the radius of convergence R. Other Linearly independent solution can be obtained by reduction of order.

Proof. Refer to the text book. □

Reference text book: E.A. Coddington, An introduction to ODE, PHI, 2003.

2.1 Problems

1. Find two linearly independent series solutions of

   (a) $y'' - xy' + 2y = 0$

   (b) $y'' + 3x^2y' - 2xy = 0$

   (c) $y'' + x^2y' + x^2y = 0$

   (d) $(1 + x^2)y'' + y = 0$

2. Consider the Chebyshev equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0, \ \alpha \in \mathbb{R}$$

   (a) Compute two linearly independent series solutions for $|x| < 1$.

   (b) Show that for each non-negative $\alpha = n$ there is a polynomial solution of degree n

3. Consider the Hermite equation

$$y'' - 2xy' + 2\alpha y = 0, \ \alpha \in \mathbb{R}$$

   (a) Compute two linearly independent series solutions.

   (b) Show that for each non-negative $\alpha = n$ there is a polynomial solution of degree n.

4. Show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0, & (n \neq m) \\ \frac{2}{2n+1} & n = m. \end{cases}$$

5. Show that there are constants $c_0, c_1, c_2, ...c_n$ such that

$$x^n = c_0P_0(x) + c_1P_1(x) + ... + c_nP_n(x)$$

6. Find all solutions of the following equations for $x > 0$:

   (a) $x^2y'' + 2xy' - 6y = 0$

   (b) $2x^2y'' + xy' - y = 0$

   (c) $x^2y'' - 5xy' + 9y = 0$

7. Find all solutions of the following equations for $x > 0$:

   (a) $3x^2y'' + 5xy' + 3xy = 0$

   (b) $x^2y'' + 3xy' + (1 + x)y = 0$

   (c) $x^2y'' - 2x(x + 1)y' + 2(x + 1)y = 0$