MTL101 :: Linear Algebra and Differential Equations

Tutorial 2

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Question 1

Question 1

Suppose $v_1 = (1, 2), v_2 = (0, 1) \in \mathbb{R}^2$.

(a) Describe geometrically the subsets $W_1 := \{tv_1 : t \in \mathbb{R}\}$, $W_2 := \{tv_2 : t \in \mathbb{R}\}$, $W_3 := \{sv_1 + tv_2 : s, t \in \mathbb{R}\}$ and $W_4 := \{sv_1 + tv_2 : 0 \leq s, t \leq 1\}$.

(b) Which of W_1, W_2, W_3, W_4 are subspaces of $\mathbb{R}^2$? Justify your answer in each case.

(c) Show that $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^2$.

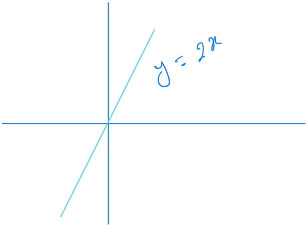
(d) Suppose $v_3 = (2, 3)$. Is $\{v_1, v_2, v_3\}$ linearly independent?

Question 1(a)

Suppose $v_1 = (1,2), v_2 = (0,1) \in \mathbb{R}^2$.
Describe geometrically the subsets $W_1 := \{tv_1 : t \in \mathbb{R}\}$, $W_2 := \{tv_2 : t \in \mathbb{R}\}$, $W_3 := \{sv_1 + tv_2 : s, t \in \mathbb{R}\}$ and $W_4 := \{sv_1 + tv_2 : 0 \leq s, t \leq 1\}$.

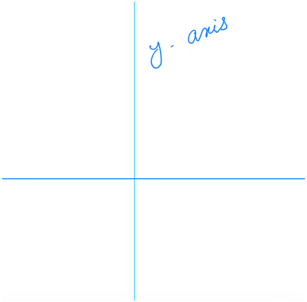
Solution:

- $W_1 := \{tv_1 : t \in \mathbb{R}\}$
  $= \{(t, 2t) : t \in \mathbb{R}\}$



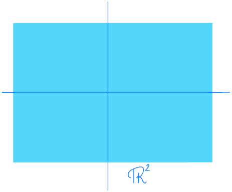
Question 1(a) Contd.

- $W_2 := \{tv_2 : t \in \mathbb{R}\}$
  $= \{(0, t) : t \in \mathbb{R}\}$

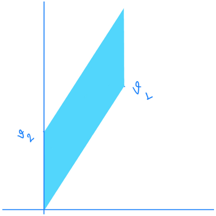


Question 1(a) Contd.

- $$W_3 := \{sv_1 + tv_2 : s, t \in \mathbb{R}\}$$
  $$= \{(s, 2s + t) : s, t \in \mathbb{R}\}$$



- $$W_4 := \{sv_1 + tv_2 : 0 \leq s, t \leq 1\}$$
  $$= \{(s, 2s + t) : 0 \leq s, t \leq 1\}$$



Question 1(b)

Question 1(b)

Which of W_1, W_2, W_3, W_4 are subspaces of $\mathbb{R}^2$? Justify your answer in each case.

Question 1(b)

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Which of W_1, W_2, W_3, W_4 are subspaces of $\mathbb{R}^2$? Justify your answer in each case.

Recall:

A nonempty subset W is said to be the subspace of V over field F if

(a) $u + v \in W$ for all $u, v \in W$,

(b) $au \in W$ for $a \in F, u \in W$.

Question 1(b) cont.

Solution:

- $W_1 := \{tv_1 : t \in \mathbb{R}\}$.
  - Let $u = t_1 v_1, v = t_2 v_1 \in W_1$, then
$$\begin{aligned} u + v &= t_1 v_1 + t_2 v_1 = (t_1 + t_2) v_1 \\ &= t_3 v_1 \in W_1 \text{ where } t_1 + t_2 = t_3 \in \mathbb{R}. \end{aligned}$$
  - Let $a \in \mathbb{R}$ and $u = tv_1 \in W$, then
$$au = atv_1 = t^{'} v_1 \in W_1, \text{where } at = t^{'} \in \mathbb{R}.$$
  - Hence, W_1 is a subspace of $\mathbb{R}^2$.

Question 1(b) cont.

Solution:

- $W_2 := \{tv_2 : t \in \mathbb{R}\}$.
  - Let $u = t_1 v_2, v = t_2 v_2 \in W_2$, then

$$\begin{aligned} u + v &= t_1 v_2 + t_2 v_2 = (t_1 + t_2)v_2 \\ &= t_3 v_2 \in W_2 \text{ where } t_1 + t_2 = t_3 \in \mathbb{R}. \end{aligned}$$

  - Let $a \in \mathbb{R}$ and $u = tv_2 \in W_2$, then

$$au = atv_2 = t^{'} v_2 \in W_2, \text{where } at = t^{'} \in \mathbb{R}.$$

  - Hence, W_2 is a subspace of $\mathbb{R}^2$.

Solution contd.

- $W_3 := \{sv_1 + tv_2 : s, t \in \mathbb{R}\}$.
  - Let $u = s_1 v_1 + t_1 v_2, v = s_2 v_1 + t_2 v_2 \in W_3$, then

$$\begin{aligned} u + v &= s_1 v_1 + t_1 v_2 + s_2 v_1 + t_2 v_2 \\ &= s_3 v_1 + t_3 v_2 \in W_3 \text{ where } s_1 + s_2 = s_3, t_1 + t_2 = t_3 \in \mathbb{R}. \end{aligned}$$

  - Let $a \in \mathbb{R}$ and $u = sv_1 + tv_2 \in W_2$, then

$$au = a(sv_1 + tv_2) = s^{'} v_1 + t^{'} v_2 \in W_3, \text{where } as = s^{'}, at = t^{'} \in \mathbb{R}.$$

  - Hence, W_3 is a subspace of $\mathbb{R}^2$.
- $W_4 := \{sv_1 + tv_2 : 0 \leq s, t \leq 1\}$.
  Let $v_1 \in W_4$ and $2 \in \mathbb{R}$. Then $2v_1 = (2, 4) \notin W_4$. Hence, W_4 is **NOT** a subspace of $\mathbb{R}^2$.

Question 1(c)

Question 1(c)

Show that $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^2$.

Question 1(c)

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Show that $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^2$.

Recall:

- A subset $X = \{v_1, v_2, ..., v_m\}$ of a vector space V over F is said to be linearly dependent if there exist scalars $a_1, a_2, ..., a_m \in F$ such that at least one of these scalars is nonzero and $a_1v_1 + a_2v_2 + \;\; + a_mv_m = 0$.
- A finite subset $X = \{v_1, v_2, ..., v_m\}$ is linearly independent if and only if $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0 \implies a_1 = a_2 = \;\; = a_m = 0$.

Solution:

- Consider $a_1 v_1 + a_2 v_2 = (0,0)$ for $a_1, a_2 \in \mathbb{R}$.
- We have

$$\begin{aligned} av_1 + bv_2 &= a(1,2) + b(0,1) = (0,0) \\ &\Rightarrow (a, 2a+b) = (0,0) \\ &\Rightarrow a = 0 \quad \& \quad 2a+b = 0 \\ &\Rightarrow a = 0 \quad \& \quad b = 0. \end{aligned}$$

- Hence $\{v_1, v_2\}$ is linearly independent.

Question 1(d)

Question 1(d)

Suppose $v_3 = (2, 3)$. Is $\{v_1, v_2, v_3\}$ linearly independent?

Question 1(d)

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Suppose $v_3 = (2, 3)$. Is $\{v_1, v_2, v_3\}$ linearly independent?

Solution: Observe that

$$2v_1 - v_2 - v_3 = (0, 0)$$

Hence, the set $\{v_1, v_2, v_3\}$ is linearly dependent.

Question 2

Suppose $V := \mathbb{C}^2$ is the complex vector space (over $\mathbb{C}$) under component-wise addition.

(a) Show that $\{(1+i, 2), (2, 1)\}$ is linearly independent.

(b) Show that $\{(1, 2), (0, i), (i, 1-i)\}$ is linearly dependent.

(c) Show that every ordered pair can be written as a linear combination of $v_1 = (1+i, 2)$ and $v_2 = (2, 1)$. Also show that up to change of order (of v_1 and v_2) such a linear combination is unique (for each ordered pair).

(d) Show that every ordered pair can be written as a linear combination of $v_1 = (1, 2)$, $v_2 = (0, i)$, $v_3 = (i, 1-i)$ in more than one ways.

Question 2(a)

Show that $\{(1+i, 2), (2, 1)\}$ is linearly independent.

Question 2(a)

Show that $\{(1+i,2),(2,1)\}$ is linearly independent.

Solution:

- Consider $a_1(1+i,2)+a_2(2,1)=(0,0)$, for $a_1,a_2\in\mathbb{C}$.
- We have,

$$\begin{aligned} & a_1(1+i,2)+a_2(2,1)=(0,0)\\ \Rightarrow & ((1+i)a_1+2a_2, 2a_1+a_2)=(0,0)\\ \Rightarrow & (1+i)a_1+2a_2=0 \quad \& \quad 2a_1+a_2=0\\ \Rightarrow & (1+i)a_1+2a_2=0 \quad \& \quad a_2=-2a_1 \end{aligned}$$

- Solving for a_1

$$\begin{aligned} & (1+i)a_1-4a_1=(-3+i)a_1=0.\\ \Rightarrow & a_1=0,\ a_2=0. \end{aligned}$$

- Hence, $\{(1+i,2),(2,1)\}$ is linearly independent.

Question 2(b)

Show that $\{(1,2),(0,i),(i,1-i)\}$ is linearly dependent.

Question 2(b)

Show that $\{(1,2),(0,i),(i,1-i)\}$ is linearly dependent.

Solution:

- Let $a_1, a_2, a_3 \in \mathbb{C}$ and $a_1(1,2)+a_2(0,i)+a_3(i,1-i)=(0,0)$.
- Consider

$$\begin{aligned} & a_1(1,2)+a_2(0,i)+a_3(i,1-i)=(0,0) \\ \implies & a_1+ia_3=0, \\ & 2a_1+ia_2+(1-i)a_3=0. \end{aligned}$$

- We get the system of two linear equations in three unknowns, gives, more than one non-zero solution i.e., not all scalars are zero. One such solution is $a_1=-i+3, a_2=10i, a_3=1+3i$.
- Hence, $\{(1,2),(0,i),(i,1-i)\}$ is linearly dependent.

Question 2(c)

Question 2(c)

Show that every ordered pair can be written as a linear combination of $v_1 = (1 + i, 2)$ and $v_2 = (2, 1)$. Also show that up to change of order (of v_1 and v_2) such a linear combination is unique (for each ordered pair).

Question 2(c)

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Show that every ordered pair can be written as a linear combination of $v_1 = (1 + i, 2)$ and $v_2 = (2, 1)$. Also show that up to change of order (of v_1 and v_2) such a linear combination is unique (for each ordered pair).

Solution:

- Let $(x, y) \in \mathbb{C}^2$.
- Suppose, $(x, y) = \alpha(1 + i, 2) + \beta(2, 1)$ for some $\alpha, \beta \in \mathbb{C}$.
- Solving above equation for α and β, we get $\alpha = \frac{x-2y}{-3+i}, \beta = \frac{-2x+(1+i)y}{-3+i}$.
- Hence, every ordered pair (x, y) can be written as a linear combination of $v_1 = (1 + i, 2)$ and $v_2 = (2, 1)$.

Uniqueness:

- Suppose there exists α, β and $\alpha', \beta' \in \mathbb{C}$ such that

$$(x, y) = \alpha(1 + i, 2) + \beta(2, 1) = \alpha'(1 + i, 2) + \beta'(2, 1),$$
$$(\alpha - \alpha')(1 + i, 2) + (\beta - \beta')(2, 1) = (0, 0)$$
$$\Rightarrow \alpha - \alpha' = \beta - \beta' = 0, \text{ since } \{(1 + i, 2), (2, 1)\} \text{ is linearly independent}$$
$$\Rightarrow \alpha = \alpha', \beta = \beta'$$

Question 2(d)

Question 2(d)

Show that every ordered pair can be written as a linear combination of $v_1 = (1, 2), v_2 = (0, i), v_3 = (i, 1 - i)$ in more than one ways.

Question 2(d)

Show that every ordered pair can be written as a linear combination of $v_1 = (1, 2), v_2 = (0, i), v_3 = (i, 1 - i)$ in more than one ways.

Solution:

- Let $(x, y) \in \mathbb{C}^2$.
- Suppose $(x, y) = a_1(1, 2) + a_2(0, i) + a_3(i, 1 - i)$.
- This implies

$$a_1 + ia_3 = x,$$
$$2a_1 + ia_2 + (1 - i)a_3 = y$$

- Which is the system of two linear equations in three unknowns (a_1, a_2, a_3), hence, there exists more than one solution.
- Two such solutions are, $(x, y) = x(1, 2) + i(2x - y)(0, i) + 0(i, 1 - i)$, $(x, y) = (x + 1)(1, 2) + (-1 + i(2x - y + 3))(0, i) + i(i, 1 - i)$.

Question 3

Show that $X = \{(1+i, 1-i), (1-i, 1+i), (2, i), (3, 2i)\}$ is linearly independent in $\mathbb{C}^2(\mathbb{R})$. Express $(a+ib, c+id)$ as an $\mathbb{R}$-linear combination of vectors belonging to X.

Question 3

Show that $X = \{(1+i, 1-i), (1-i, 1+i), (2, i), (3, 2i)\}$ is linearly independent in $\mathbb{C}^2(\mathbb{R})$. Express $(a+ib, c+id)$ as an $\mathbb{R}$-linear combination of vectors belonging to X.

Solution:

- Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha(1+i, 1-i) + \beta(1-i, 1+i) + \gamma(2, i) + \delta(3, 2i) = (0, 0),$$
$$\Rightarrow (\alpha+\beta+2\gamma+3\delta + i(\alpha-\beta), \alpha+\beta+i(-\alpha+\beta+\gamma+2\delta)) = (0, 0)$$

- Comparing the real and imaginary part, we get,

$$\begin{aligned}
&\alpha+\beta+2\gamma+3\delta = 0,\\
&\alpha-\beta = 0\\
&\alpha+\beta = 0,\\
&-\alpha+\beta+\gamma+2\delta = 0\\
&\implies \alpha=\beta=\gamma=\delta=0.
\end{aligned}$$

Question 3 Cont.

- Suppose,

$$\alpha(1+i,1-i)+\beta(1-i,1+i)+\gamma(2,i)+\delta(3,2i)=(a+ib,c+id),$$
$$\Rightarrow(\alpha+\beta+2\gamma+3\delta+i(\alpha-\beta),\alpha+\beta+i(-\alpha+\beta+\gamma+2\delta))$$
$$=(a+ib,c+id)$$

- Comparing the real and imaginary part, we get,

$$\alpha+\beta+2\gamma+3\delta=a,$$
$$\alpha-\beta=b$$
$$\alpha+\beta=c,$$
$$-\alpha+\beta+\gamma+2\delta=d$$

- On solving, we get,
$\alpha=\frac{b+c}{2},\beta=\frac{c-b}{2},\gamma=2a-3b-2c-3d,\delta=2b+c+2d-a.$

Question 4

Question 4:

Let V be a vector space over $\mathbb{F}$. Show that $u, v, w \in V$ are linearly independent if and only if $u+v, v+w, w+u$ are linearly independent.

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Solution:

- Let $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$\alpha(u+v)+\beta(v+w)+\gamma(w+u)=0$$
$$(\alpha+\gamma)u+(\alpha+\beta)v+(\beta+\gamma)w=0.$$

- Since u, v, w are linearly independent, hence,

$$\alpha+\gamma=0 \quad \& \quad \alpha+\beta=0 \quad \& \quad \beta+\gamma=0$$
$$\implies \alpha=\beta=\gamma=0.$$

Question 4 Cont.

- Conversely, Let $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$\alpha u + \beta v + \gamma w = 0. \tag{1}$$

- Then,

$$\alpha u + \beta v + \gamma w = \alpha(u+v) + \beta(v+w) + \gamma(w+u) - \left[\alpha v + \beta w + \gamma u\right]$$
$$= \alpha(u+v) + \beta(v+w) + \gamma(w+u)$$
$$- \left[\alpha(v+w) + \beta(w+u) + \gamma(u+v) - \left[\alpha w + \beta u + \gamma v\right]\right]$$
$$= (\alpha - \gamma)(u+v) + (\beta - \alpha)(v+w) + (\gamma - \beta)(w+u)$$
$$+ \left[\alpha w + \beta u + \gamma v\right]$$
$$= (\alpha - \gamma)(u+v) + (\beta - \alpha)(v+w) + (\gamma - \beta)(w+u)$$
$$+ \left[\alpha(w+u) + \beta(u+v) + \gamma(v+w) - \alpha u + \beta v + \gamma w\right]$$
$$= (\alpha + \beta - \gamma)(u+v) + (\beta + \gamma - \alpha)(v+w) + (\gamma + \alpha - \beta)(w+u)$$
$$- \left[\alpha u + \beta v + \gamma w\right].$$

Question 4 Cont.

- Since $u+v, v+w, w+u$ are linearly independent. Hence,

$$\begin{aligned} \alpha + \beta - \gamma &= 0, \\ -\alpha + \beta + \gamma &= 0, \\ \alpha - \beta + \gamma &= 0, \qquad \implies \alpha = \beta = \gamma = 0. \end{aligned} \tag{2}$$

- Hence u, v, w are linearly independent.

Question 5

Question 5

(a) Find the coordinates of $(a, b, c) \in R^3$ relative to ordered basis $\{(1,0,0),(1,1,0),(1,1,1)\}$.

(b) Find the coordinates of $a + bx + cx^2$ relative to ordered basis $\{1, 1+x, 1+x^2\}$ in the space $\mathcal{P}_3$ of polynomials of degree at most 2 with coefficients from $\mathbb{R}$.

(c) Find the coordinate vector of an element $\in R^3$ with respect to following ordered bases $\mathcal{B}_1 = \{(1,2,1),(1,2,3),(0,1,1)\}$ and $\mathcal{B}_2 = \{(1,0,0),(1,1,0),(1,1,1)$. Also write the change of coordinate matrix.

Question 5

Recall:

Suppose $B = \{v_1, v_2, ..., v_n\}$ be a basis of a vector space V over the field F. Fix the ordering of elements in B as they are listed. Then we know that $v = a_1 v_1 + a_2 v_2 + \ + a_n v_n$ (recall that the coefficients are uniquely determined). We write $[v]_B = (a_1, a_2, \cdots, a_n)^T$ which is in F and called the coordinate vector of v with respect to the basis B.

Question 5

Question 5(a)

Find the coordinates of $(a, b, c) \in R^3$ relative to ordered basis $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

Question 5

Question 5(a)

Find the coordinates of $(a, b, c) \in R^3$ relative to ordered basis $\{(1,0,0),(1,1,0),(1,1,1)\}$.

Solution:

- Let

$$\begin{aligned} &(a, b, c) = \alpha(1,0,0) + \beta(1,1,0) + \gamma(1,1,1), \\ \implies &a = \alpha + \beta + \gamma, \\ &b = \beta + \gamma, \\ &c = \gamma. \end{aligned}$$

- After solving above set of equations, we get, $\alpha = (a-b), \beta = (b-c), \gamma = c$.
- Hence, coordinates are given by, $(a, b, c)_B = (a-b, b-c, c)^T$.

Question 5

Question 5(b)

Find the coordinates of $a + bx + cx^2$ relative to ordered basis $\{1, 1 + x, 1 + x^2\}$ in the space $\mathcal{P}_3$ of polynomials of degree at most 2 with coefficients from $\mathbb{R}$.

Question 5

Question 5(b)

Find the coordinates of $a + bx + cx^2$ relative to ordered basis $\{1, 1 + x, 1 + x^2\}$ in the space $\mathcal{P}_3$ of polynomials of degree at most 2 with coefficients from $\mathbb{R}$.

Solution:

- Let

$$\begin{aligned} a + bx + cx^2 &= \alpha(1) + \beta(1 + x) + \gamma(1 + x^2), \\ \implies a &= \alpha + \beta + \gamma, \\ b &= \beta, \\ c &= \gamma. \end{aligned}$$

- Hence, $\alpha = (a - b - c), \beta = b, \gamma = c$.
- Hence, coordinates are given by, $(a + bx + cx^2)_B = (a - b - c, b, c)^T$.

Question 5

Question 5(c)

Find the coordinate vector of an element $\in R^3$ with respect to following ordered bases $\mathcal{B}_1 = \{(1,2,1),(1,2,3),(0,1,1)\}$ and $\mathcal{B}_2 = \{(1,0,0),(1,1,0),(1,1,1)$. Also write the change of coordinate matrix.

Question 5

Question 5(c)

Find the coordinate vector of an element $\in R^3$ with respect to following ordered bases $\mathcal{B}_1 = \{(1,2,1),(1,2,3),(0,1,1)\}$ and $\mathcal{B}_2 = \{(1,0,0),(1,1,0),(1,1,1)$. Also write the change of coordinate matrix.

Solution:

- Let $(a,b,c) \in \mathbb{R}^3$.
- Also,

$$\begin{aligned}
&(a,b,c) = \alpha(1,2,1) + \beta(1,2,3) + \gamma(0,1,1),\\
\implies &a = \alpha + \beta,\\
&b = 2\alpha + 2\beta + \gamma,\\
&c = \alpha + 3\beta + \gamma\\
\implies &\alpha = \frac{a+b-c}{2},\ \beta = \frac{a-b+c}{2},\ \gamma = (b-2a).
\end{aligned}$$

Question 5(c) cont.

- Hence, the coordinates with respect to B_1 are given by $(\frac{a+b-c}{2}, \frac{a-b+c}{2}, (b-2a))^T$.
- Similarly, for the coordinate with respect to ordered basis $\mathcal{B}_2$,

$$(a, b, c) = \alpha(1, 0, 0) + \beta(1, 1, 0) + \gamma(1, 1, 1).$$

- After solving, we get, the coordinates with respect to B_2 $(a-b, b-c, c)^T$.

Question 5(c) cont.

Change of coordinate matrix,

- First express the vectors of B_1 in linear combinations of vectors of B_2

$$\begin{aligned}(1,2,1) &= a_{11}(1,0,0) + a_{12}(1,1,0) + a_{13}(1,1,1),\\ (1,2,3) &= a_{21}(1,0,0) + a_{22}(1,1,0) + a_{23}(1,1,1),\\ (0,1,1) &= a_{31}(1,0,0) + a_{32}(1,1,0) + a_{33}(1,1,1)\end{aligned}$$

- Solve the obtained system of equation for the scalars

$$\begin{aligned}a_{11} &= -1, a_{12} = 1, a_{13} = 1,\\ a_{21} &= -1, a_{22} = -1, a_{23} = 3,\\ a_{31} &= -1, a_{32} = 0, a_{33} = 1\end{aligned}$$

- Arrange the scalars column wise to get the change of coordinate matrix,

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Question 6

Question 6

(a) Show that if $v \in V$ then $\mathbb{F}v := \{\lambda v : \lambda \in \mathbb{F}\}$ is a subspace of any vector space V over $\mathbb{F}$.

(b) Show that if W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

(c) Show that the intersection of any collection of subspaces of a vector space is a subspace.

(d) Suppose W_1 and W_2 are subspaces of a vector space V. Show that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

(e) Let X be a non empty subset of a vector space V over $\mathbb{F}$. Let $\text{span}(X) := \{\sum_{i=1}^{n} a_i v_i : n \in \mathbb{N}, a_i \in \mathbb{F}, v_i \in X\}$ and let $< X >$ be the intersection of all the subspace of V which contain X. Show that $\text{span}(X)$ and $< X >$ are subspaces of V. Also show that $\text{span}(X) =< X >$.

Question 6

Question 6(a)

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Question 6

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Show that if $v \in V$ then $\mathbb{F}v := \{\lambda v : \lambda \in \mathbb{F}\}$ is a subspace of any vector space V over $\mathbb{F}$.

Solution:

- Let $\lambda = 0$, then $\lambda v = 0 \in W \;\Rightarrow W \neq \phi$.
- Let $x = \lambda_1 v, y = \lambda_2 v \in W$ and $\alpha, \beta \in \mathbb{F}$, then

$$\alpha x + \beta y = (\alpha\lambda_1 + \beta\lambda_2)v = \lambda^{'} v \in W \quad \text{for some } \lambda^{'} \in \mathbb{F}$$

- Hence, W is a subspace.

Question 6

Question 6(b)

Show that if W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

Question 6(b)

Show that if W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

Solution:

- Since $0 \in W_1 \cap W_2$, we have $W_1 \cap W_2 \neq \phi$
- Let $x, y \in W_1 \cap W_2$ and $\alpha, \beta \in \mathbb{F}$.
- Then

$$\begin{aligned} & x, y \in W_1 \ \& \ x, y \in W_2 \\ & \Rightarrow \alpha x + \beta y \in W_1 \ \& \ \alpha x + \beta y \in W_2 \\ & \Rightarrow \alpha x + \beta y \in W_1 \cap W_2. \end{aligned}$$

- Hence, $W_1 \cap W_2$ is a subspace of V.

Question 6

Question 6(c)

Show that the intersection of any collection of subspaces of a vector space is a subspace.

Question 6(c)

Show that the intersection of any collection of subspaces of a vector space is a subspace.

Solution: Let V be a vector space over a field $\mathbb{F}$, and $W = \cap_{\lambda \in \wedge} W_\lambda$, where W_λ is a subspace of V.

- Since $0 \in W$, $W \neq \phi$.
- Let $x, y \in W$ and $\alpha, \beta \in \mathbb{F}$.
- Then

$$\begin{aligned} & x, y \in W_\lambda \ \forall \ \lambda \in \wedge \\ & \Rightarrow \alpha x + \beta y \in W_\lambda \ \forall \ \lambda \in \wedge \\ & \Rightarrow \alpha x + \beta y \in W. \end{aligned}$$

- Hence intersection of any collection of subspaces of a vector space is a subspace.

Question 6

Question 6(d)

Suppose W_1 and W_2 are subspaces of a vector space V. Show that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Question 6(d)

Suppose W_1 and W_2 are subspaces of a vector space V. Show that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Solution:

- Suppose $W_1 \cup W_2$ is a subspace of V. Let $x \in W_1$ and $y \in W_2$.
- Then

$$\begin{aligned}
&x + y \in W_1 \cup W_2 \\
&\Rightarrow x + y \in W_1 \text{ or } x + y \in W_2 \\
&\Rightarrow x + y - x = y \in W_1 \text{ or } \; x + y - y = x \in W_2 \\
&\Rightarrow W_1 \subset W_2 \text{ or } W_2 \subset W_1.
\end{aligned}$$

- Conversely, suppose $W_1 \subset W_2$ or $W_2 \subset W_1$ then $W_1 \cup W_2 \subset W_2$ or $\subset W_1$, Hence, it is a subspace of V.

Question 6

Question 6(e)

Let X be a non empty subset of a vector space V over $\mathbb{F}$. Let $\text{span}(X) := \{\sum_{i=1}^{n} a_i v_i : n \in \mathbb{N}, a_i \in \mathbb{F}, v_i \in X\}$ and let $< X >$ be the intersection of all the subspace of V which contain X. Show that span(X) and $< X >$ are subspaces of V. Also show that $\text{span}(X) =< X >$.

Question 6(e)

Let X be a non empty subset of a vector space V over $\mathbb{F}$. Let span$(X) := \{\sum_{i=1}^{n} a_i v_i : n \in \mathbb{N}, a_i \in \mathbb{F}, v_i \in X\}$ and let $< X >$ be the intersection of all the subspace of V which contain X. Show that span(X) and $< X >$ are subspaces of V. Also show that span$(X) =< X >$.

Solution: First prove that span(X) and $< X >$ are subspaces of V.

- Let $a_i = 0\ \forall i$, then $\sum_{i=1}^{n} a_i v_i = 0 \in$ span(X) $\Rightarrow$ span(X) $\neq \phi$.
- Let $x, y \in$ span(X), then $x = \sum_{i=1}^{n} c_i v_i$ and $y = \sum_{i=1}^{n} d_i v_i$ and let $\alpha, \beta \in \mathbb{F}$

$$\Rightarrow \alpha x + \beta y = \sum_{i=1}^{n} (\alpha b_i + \beta c_i) v_i \in \text{span}(X).$$

- Hence, span(X) is a subspace of V.

- Since, the intersection of subspaces of V is a subspace of V, Hence, $< X >$ is also a subspace of V.

To show: span(X)=$< X >$.

- Clearly, $X \subset$ span(X), hence $< X > \subset$ span(X)
- Let $x = \sum_{i=1}^{n} a_i v_i \in$ span(X).
- Since, $v_i \in < X >$,

$$\sum_{i=1}^{n} a_i v_i \in < X >$$
$$\Rightarrow x \in < X >$$
$$\Rightarrow \text{span}(X) \subset < X >$$

- Hence span(X)=$< X >$.

Question 7

Question 7

In each case show that $W_1 + W_2 = V$ (directly) and find $\dim(W_1 \cap W_2)$. Verify the formula $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

(a) $V = \mathbb{R}^2$, W_1 is the X-axis, W_2 Y-axis.

(b) $V = \mathbb{R}^2$, W_1 and W_2 are distinct line passing through the origin.

(c) $V = \mathbb{R}^3$, W_1 is XY plane and W_2 is YZ plane.

(d) $V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R}) : A \text{ is upper triangular } \}$ and $W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is lower triangular } \}$.

(e) $V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R}) : A \text{ is a symmetric } \}$ and $W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is skew-symmetric } \}$.

Question 7(a)

$V = \mathbb{R}^2$, W_1 is the X-axis, W_2 Y-axis.

Question 7(a)

$V = \mathbb{R}^2$, W_1 is the X-axis, W_2 Y-axis.

Solution:

- Let $(x, y) \in \mathbb{R}^2$, then $(x, y) = (x, 0) + (0, y) \;\Rightarrow V = W_1 + W_2$
- $\dim W_1 = \dim W_2 = 1$
- Clearly, $W_1 \cap W_2 = (0, 0) \Rightarrow \dim(W_1 \cap W_2) = 0.$
- Hence, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$

Question 7

Question 7(b)

$V = \mathbb{R}^2$, W_1 and W_2 are distinct line passing through the origin.

Question 7(b)

$V = \mathbb{R}^2$, W_1 and W_2 are distinct line passing through the origin.

Solution:

- Since, $(x, y) \in \mathbb{R}^2$ can be expressed as sum of the points lies on two distinct lines(Using parallelogram law). So, $\Rightarrow V = W_1 + W_2$
- $\dim V = 2$, $\dim W_1 = \dim W_2 = 1$
- Since lines are distinct and passing through origin $W_1 \cap W_2 = (0, 0) \Rightarrow \dim(W_1 \cap W_2) = 0$.
- Hence, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

Question 7

Question 7(c)

$V = \mathbb{R}^3$, W_1 is XY plane and W_2 is YZ plane.

Question 7(c)

$V = \mathbb{R}^3$, W_1 is XY plane and W_2 is YZ plane.

Solution:

- $V = \mathbb{R}^3 \implies \dim V = 3.$
- $W_1 = \{(x, y, 0); x, y \in \mathbb{R}\} \implies \dim W_1 = 2.$
- $W_2 = \{(0, y, z); y, z \in \mathbb{R}\} \implies \dim W_2 = 2.$
- $W_1 \cap W_2 = \{(0, y, 0); y \in \mathbb{R}\} \implies \dim W_1 \cap W_2 = 1.$
- Let $(x, y, z) \in \mathbb{R}^3$, then
  $(x, y, z) = (x, y, 0) + (0, 0, z) = (x, 0, 0) + (0, y, z) \Rightarrow V = W_1 + W_2.$
- Hence, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$- $\dim(W_1 \cap W_2)$.

Question 7(d)

$V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R}) : A \text{ is upper triangular }\}$ and $W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is lower triangular }\}$.

Question 7(d)

$V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R}) : A \text{ is upper triangular } \}$ and $W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is lower triangular } \}$.

Solution:

- Since, any matrix can be written as sum of an upper triangular matrix and a lower triangular matrix, so $\Rightarrow V = W_1 + W_2$.
- $\dim V = n^2$, $\dim W_1 = \frac{n(n+1)}{2}$ and $\dim W_2 = \frac{n(n+1)}{2}$. (How?)
- Since, $W_1 \cap W_2 = \{A \in M_n(\mathbb{R}) : A \text{ is a diagonal matrix } \}$ so, $\dim(W_1 \cap W_2) = n$.
- Hence, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

Question 7

Question 7(e)

$V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R}) : A$ is a symmetric $\}$ and
$W_2 = \{A \in M_n(\mathbb{R}) : A$ is skew-symmetric $\}$.

Question 7

Question 7(e)

$V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R}) : A$ is a symmetric $\}$ and $W_2 = \{A \in M_n(\mathbb{R}) : A$ is skew-symmetric $\}$.

Solution:

- Let $A \in M_n(\mathbb{R})$, $A = \frac{A+A'}{2} + \frac{A-A'}{2}$, where $\frac{A+A'}{2}$ is symmetric and $\frac{A-A'}{2}$ is skew-symmetric, so $\Rightarrow V = W_1 + W_2$
- $\dim V = n^2$, $\dim W_1 = \frac{n(n+1)}{2}$ and $\dim W_2 = \frac{n(n-1)}{2}$
- Since, only zero matrice is both symmetric and skew-symmetric, so $W_1 \cap W_2 = \{$ zero matrix $\}$ so, $\dim(W_1 \cap W_2) = 0$.
- Hence, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$- $\dim(W_1 \cap W_2)$.

Question 8

Which of the following is a linear transformation? Justify.

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_1(x, y) = (x^2 + y^2, x - y)$ over $\mathbb{R}$.

(b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_2(x, y) = (x + y + 1, x - y)$ over $\mathbb{R}$.

(c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_3(x, y) = (ax + by, cx + dy)$ over $\mathbb{R}$.

(d) $T_4 : \mathbb{C} \to \mathbb{C}$ define by $T_4(z) = \bar{z}$ over $\mathbb{C}$.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation about the origin by an angle θ.(Write the expression for rotation)

(f) $T_5 : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ define by $T_5(A) = A'$ (A' transpose of A)

(g) $T_6 : M_n(\mathbb{F}) \to (\mathbb{F})$ define by $T_6(A) =$tr(A)

(h) $T_7 : \mathcal{P}_n \to \mathcal{P}_n$ such that $T_7(p)(x) = p(x - 1)$.

(i) $T_8 : \mathcal{P}_n \to \mathcal{P}_{n+1}$ such that $T_8(p)(x) = xp(x) + p(1)$.

Question 8

Recall:

Suppose V and W are vector spaces over the same field $\mathbb{F}$. A map $T : V \to W$ is called a linear transformation if

$$T(au + bv) = aT(u) + bT(v)$$

for any $a, b \in \mathbb{F}$ and any $u, v \in V$.
In particular,

- $T(0_V) = 0_W$ where 0_V is the zero vector of vector space V and 0_W is the zero vector space of W.
- $T(au) = aT(u)$.

Question 8 cont.

Question 8(a)

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_1(x, y) = (x^2 + y^2, x - y)$ over $\mathbb{R}$.

Question 8 cont.

Question 8(a)

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_1(x, y) = (x^2 + y^2, x - y)$ over $\mathbb{R}$.

Solution:

- Since,
  $T_1(a(x, y)) = T_1(ax, ay) = (a^2(x^2 + y^2), a(x - y)) \neq aT_1(x, y)$,
- Hence, T_1 is not a linear transformation.

Question 8 cont.

Question 8(a)

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_1(x, y) = (x^2 + y^2, x - y)$ over $\mathbb{R}$.

Solution:

- Since,
  $T_1(a(x, y)) = T_1(ax, ay) = (a^2(x^2 + y^2), a(x - y)) \neq aT_1(x, y)$,
- Hence, T_1 is not a linear transformation.

Question 8(b)

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_2(x, y) = (x + y + 1, x - y)$ over $\mathbb{R}$.

Question 8 cont.

Question 8(a)

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_1(x, y) = (x^2 + y^2, x - y)$ over $\mathbb{R}$.

Solution:

- Since,
  $T_1(a(x, y)) = T_1(ax, ay) = (a^2(x^2 + y^2), a(x - y)) \neq aT_1(x, y)$,
- Hence, T_1 is not a linear transformation.

Question 8(b)

$T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_2(x, y) = (x + y + 1, x - y)$ over $\mathbb{R}$.

Solution:

- Since, $T_2(0, 0) \neq (0, 0)$,
- Hence, T_1 is not a linear transformation.

Question 8 cont.

Question 8(c)

$T_3 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_3(x, y) = (ax + by, cx + dy)$ over $\mathbb{R}$.

Question 8 cont.

Question 8(c)

$T_3 : \mathbb{R}^2 \to \mathbb{R}^2$, define by $T_3(x, y) = (ax + by, cx + dy)$ over $\mathbb{R}$.

Solution:

- Let $u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$, then

$$
\begin{aligned}
T_3(\alpha u + \beta v) &= T_3(\alpha(x_1, y_1) + \beta(x_2, y_2)) \\
&= T_3(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\
&= \Big(a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2), c(\alpha x_1 + \beta x_2) + d(\alpha y_1 + \beta y_2)\Big) \\
&= \Big(\alpha(ax_1 + by_1) + \beta(ax_2 + by_2), \alpha(cx_1 + dy_1) + \beta(cx_2 + dy_2)\Big) \\
&= \Big(\alpha(ax_1 + by_1, cx_1 + dy_1) + \beta(ax_2 + by_2, cx_2 + dy_2)\Big) \\
&= \alpha T_3(u) + \beta T_3(v).
\end{aligned}
$$

- Hence, T_3 is a linear transformation.

Question 8 cont.

Question 8(d)

$T_4 : \mathbb{C} \to \mathbb{C}$ define by $T_4(z) = \bar{z}$ over $\mathbb{C}$.

Question 8 cont.

Question 8(d)

$T_4 : \mathbb{C} \to \mathbb{C}$ define by $T_4(z) = \bar{z}$ over $\mathbb{C}$.

Solution:

- Since, $T_4(az) = \bar{az} = \bar{a}\bar{z} \neq aT_4(z)$
- Hence, T_4 is not a linear transformation.

Question 8

Question 8(e)

$T : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation about the origin by an angle θ.(Write the expression for rotation)

Solution:

- $T(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)\ \forall\ (x, y) \in \mathbb{R}$ (from question 6 tutsheet 1).
- Let $u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} T(\alpha u + \beta v) &= T(\alpha(x_1, y_1) + \beta(x_2, y_2)) \\ &= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\ &= \Big((\alpha x_1 + \beta x_2)\cos\theta - (\alpha y_1 + \beta y_2)\sin\theta \\ &\quad , (\alpha x_1 + \beta x_2)\sin\theta + (\alpha y_1 + \beta y_2)\cos\theta\Big) \end{aligned}$$

- $$= \Big(\alpha(x_1 \cos\theta - y_1 \sin\theta) + \beta(x_2 \cos\theta - y_2 \sin\theta)$$
  $$, \alpha(x_1 \sin\theta + y_1 \cos\theta) + \beta(x_2 \sin\theta + y_2 \cos\theta)\Big)$$
  $$= \Big(\alpha(x_1 \cos\theta - y_1 \sin\theta, x_1 \sin\theta + y_1 \cos\theta)$$
  $$+ \beta(x_2 \cos\theta - y_2 \sin\theta, x_2 \sin\theta + y_2 \cos\theta)\Big)$$
  $$= \alpha T(u) + \beta T(v).$$
- Hence, T is a linear transformation.

Question 8

Question 8(f)

$T_5 : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ define by $T_5(A) = A'$ (A' transpose of A)

Question 8(f)

$T_5 : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ define by $T_5(A) = A'$ (A' transpose of A)

Solution:

- Since, $T_5(aA + bB) = (aA + bB)' = aA' + bB' = aT_5(A) + bT_5(B)$ for all $A, B \in M_{m\times n}$ and $a, b \in \mathbb{F}$.
- Hence, T_5 is a linear transformation.

Question 8

Question 8(f)

$T_5 : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ define by $T_5(A) = A'$ (A' transpose of A)

Solution:

- Since, $T_5(aA + bB) = (aA + bB)' = aA' + bB' = aT_5(A) + bT_5(B)$ for all $A, B \in M_{m\times n}$ and $a, b \in \mathbb{F}$.
- Hence, T_5 is a linear transformation.

Question 8(g)

$T_6 : M_n(\mathbb{F}) \to (\mathbb{F})$ define by $T_6(A) =\text{tr}(A)$

Question 8

Question 8(f)

$T_5 : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ define by $T_5(A) = A'$ (A' transpose of A)

Solution:

- Since, $T_5(aA + bB) = (aA + bB)' = aA' + bB' = aT_5(A) + bT_5(B)$ for all $A, B \in M_{m\times n}$ and $a, b \in \mathbb{F}$.
- Hence, T_5 is a linear transformation.

Question 8(g)

$T_6 : M_n(\mathbb{F}) \to (\mathbb{F})$ define by $T_6(A) =\text{tr}(A)$

Solution:

- Since, $T_6(aA + bB) =$ $\text{tr}(aA + bB) = a\,\text{tr}A + b\,\text{tr}B = aT_6(A) + bT_6(B)$ for all $A, B \in M_n$ and $a, b \in \mathbb{F}$.
- Hence, T_6 is a linear transformation.

Question 8

Question 8(h)

$T_7 : \mathcal{P}_n \to \mathcal{P}_n$ such that $T_7(p)(x) = p(x-1)$.

Question 8(h)

$T_7 : \mathcal{P}_n \to \mathcal{P}_n$ such that $T_7(p)(x) = p(x-1)$.

Solution:

- Since,

$$\begin{aligned} T_7(ap+bq)(x) &= (ap+bq)(x-1) \\ &= ap(x-1) + bq(x-1) \\ &= aT_7(p)(x) + \ bT_7(q)(x) \ \forall p, q \in \mathcal{P}_n, \ a, b \in \mathbb{F}. \end{aligned}$$

- Hence, T_7 is a linear transformation.

Question 8

Question 8(i)

$T_8 : \mathcal{P}_n \to \mathcal{P}_{n+1}$ such that $T_8(p)(x) = xp(x) + p(1)$.

Question 8

Question 8(i)

$T_8 : \mathcal{P}_n \to \mathcal{P}_{n+1}$ such that $T_8(p)(x) = xp(x) + p(1)$.

Solution:

- Since,

$$\begin{aligned} T_8(ap+bq)(x) &= x(ap+bq)(x) + (ap+bq)(1) \\ &= a(xp(x)+p(1)) \ + \ b(xq(x)+q(1)) \\ &= aT_8(p)(x) + bT_8(q)(x) \ \forall p, q \in \mathcal{P}_n, \ a, b \in \mathbb{F}. \end{aligned}$$

- Hence, T_8 is a linear transformation.