MTL101:: Tutorial-3:: Linear Algebra

Notation: $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}, \mathcal{P}_n := \{f \in F[x] : degf < n\}$

(1) Find a basis of the row space of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{pmatrix}^t.$$

(2) For $i \in \{1, 2, \ldots, n\}$, define $p_i : \mathbb{F}^n \to \mathbb{F}$ by $p_i(x_1, x_2 \ldots, x_n) = x_i$ (the i-th projection).

(a) Show that it is a linear transformation.

(b) If $T : \mathbb{F}^n \to \mathbb{F}$ is a linear transformation then it is an $\mathbb{F}$ -linear combination of the projections, that is, $T = a_1p_1 + a_2p_2 \ldots + a_np_n$ for $a_1, \ldots a_n \in \mathbb{F}$.

(c) Further, show that $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation if and only if for each $i \in \{1, 2, \ldots n\}$, the composition $pi \circ S : \mathbb{F}^m \to \mathbb{F}$ is a linear transformation.

(d) If $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation then $S(x_1, x_2, \ldots x_m)$ $= (y_1, y_2, \ldots y_n)$ where $y_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_m$ *for* $a_{ij} \in \mathbb{F}$ with $(1 \leq i \leq n, 1 \leq j \leq m)$.

(3) Find the rank and nullity of the following linear transformations. Also write a basis of the range space in each case.

(a) $T : \mathbb{F}^3 \to \mathbb{F}^3$ defined by $T(x, y, z) = (x + y + z, x - y + z, x + z)$.

(b) Assume that $0 \leq m \leq n$. $T : \mathbb{F}^n \to \mathbb{F}^m$ defined by $T(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$.

(4) Write the matrix representations of the linear operators with respect to the ordered basis B.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x, y), \mathcal{B} = \{(1, 1), (1, -1)\}$

(b) $\mathcal{D} : \mathcal{P}_{n+1} \to \mathcal{P}_{n+1}$ such that $D(a_0 + a_1x + \ldots + a_nx^n)$ $= a_1 + 2a_2x + \ldots + na_nx^{n-1}, \mathcal{B} = \{1, x, \ldots, x^n\}$.

(c) $T : M_2(\mathcal{F}) \to M_2(\mathcal{F}), T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+w & z \\ z+w & x \end{pmatrix}$,
$\mathcal{B} = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$

(5) Suppose $dimV = dimW < 1$ and $T : V \to W$ is a linear transformation. Show that the following statements are equivalent

(i) T is an isomorphism.

(ii) T is injective (i.e., one to one).

(iii) $kerT = 0$.

(iv) T is surjective (i.e., onto).

(6) Suppose $m > n$. Justify the following statements:

(a) There is no one to one (injective) $\mathbb{R}$-linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

(b) There is no onto (surjective) $\mathbb{R}$-linear tranformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(7) Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x; y) = (x + y, x)$,

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x; y) = (y, x)$,

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x; y) = (y, -x)$

(d) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$ with $T(x; y) = (y, -x)$.

(e) $T : \mathbb{C}^n \to \mathbb{C}^n$ with $T(x_1, x_2, \dots, x_n) = (x_n, x_1, \dots, x_{n-1})$.

(f) $T : \mathbb{C}^2 \to \mathbb{C}^2$ with $T(z_1, z_2) = (z_1 - 2z_2, z_1 + 2z_2)$.

(8) Find a basis B such that $[T]_B$ is a diagonal matrix in case T is diagonalizable. Find P such that $[T]_B = P[T]_S P^{-1}$where S is the standard basis in each case.

(a) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (y, -x)$.

(b) $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T(x, y, z) = (5x - 6y - 6z, -x + 4y + 2z, 3x - 6y - 4z)$.

(c) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (xcos\theta + ysin\theta, -xsin\theta + ycos\theta)$.

(9) Characteristic polynomial of a matrix is satisfied by the matrix(Cayley Hamilton). Use it to find(invertibility and) the inverse of the following operators.

(a) $(x, y, z) \to (x + y + z, x + z, -x + y)$.
(b) $(x, y, z) \to (x, x + 2y, x + 2y + 3z)$.

(10) Which of the following is an inner product.

(a) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2 + 3$ on $\mathbb{R}^2$ over $\mathbb{R}$.
(b) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 - y_1y_2$ on $\mathbb{R}^2$ over $\mathbb{R}$.
(c) $\langle (x_1, y_1), (x_2, y_2) \rangle = y_1(x_1 + 2x_2) + y_2(2x_1 + 5x_2)$ on $\mathbb{R}^2$ over $\mathbb{R}$.
(d) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$ on $\mathbb{C}^2$ over $\mathbb{C}$.
(e) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1\bar{x_2} - y_1\bar{y_2}$ on $\mathbb{C}^2$ over $\mathbb{C}$.
(f) If $A, B \in \mathbb{M}_n(C)$ define $\langle A, B \rangle = Trace(A\bar{B})$.
(g) Suppose $C[0, 1]$ is the space of continuous complex valued functions on the interval $[0, 1]$ and for $f, g \in C[0, 1]$, $\langle f, g \rangle := \int_0^1 f(t)\overline{g(t)}\, dt$.

(11) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{R})$ is such that $a > 0$ and $det(A) = ad - b^2 > 0$. Show that $\langle X, Y \rangle = X^tAY$ is an inner product on $\mathbb{R}^2$.

(12) Suppose V is an inner product space. Define $\|v\| = \sqrt{\langle v, v \rangle}$. Show the following statements.

(a) $\|v\| = 0$ if and only if $v = 0$.
(b) For $a \in F$, $\|av\| = |a|\|v\|$.
(c) $\|u + v\| \leq \|u\| + \|v\|$.
(d) $|\|v\| - \|w\|| \leq \|v - w\|$.
(e) $\langle u, v \rangle = 0$ then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

(13) Use standard inner product on $\mathbb{R}^2$ over $\mathbb{R}$ to prove the following statement: "A parallelogram is a rhombus if and only if its diagonals are perpendicular to each other."

(14) Find with respect to the standard inner product of $\mathbb{R}^3$, an orthonormal basis containing $(1, 1, 1)$.

(15) Find an orthonormal basis of $\mathcal{P}_n = \{f(x) \in \mathbb{R}[x] : \deg f(x) < 3\}$ with respect to the inner product defined by $\langle f, g \rangle := \int_0^1 f(t)g(t)\ dt$.

(16) Suppose W is a subspace of the finite dimensional inner product space. Define $W^\perp := \{v \in V : \langle w, v \rangle = 0 \text{ for all } w \in W\}$. Show the following statements.

(a) $W^\perp$ is a subspace of V.

(b) $W \cap W^\perp = 0$.

(c) $V = W \bigoplus W^\perp$.

(d) $(W^\perp)^\perp = W$.

(17) Suppose $W = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. Find the shortest distance of $(a, b) \in \mathbb{R}^2$ from W with respect to (i) the standard inner product, (ii) the inner product defined by $\langle (x_1, y_1), (x_2, y_2) \rangle = 2x_1x_2 + y_1y_2$.