MTL101::Linear Algebra and Differential Equations

Tutorial 6

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Question 1

Find the real general solution of the following systems.

(a) $y_1' = -8y_1 - 2y_2,\ y_2' = 2y_1 - 4y_2,$

(b) $y_1' = -3y_1 - y_2 + 2y_3,\ y_2' = -4y_2 + 2y_3, y_3' = y_2 - 5y_3,$

(c) $y_1' = -y_1 - 4y_2 + 2y_3,\ y_2' = 2y_1 + 5y_2 - y_3,\ y_3' = 2y_1 + 2y_2 + 2y_3.$

Question 1(a)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -8 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} -8 & -2 \\ 2 & -4 \end{pmatrix}$ are $\lambda := \lambda_1 = \lambda_2 = -6$
- The eigenvector corresponding to λ : $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Let us solve: $(A - \lambda I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for x_1, x_2.
- That is: $\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- It gives $x_1 + x_2 = 0$

Question 1(a) contd...

- Choose $x_1 = 0, x_2 = \frac{1}{2}$.
- General solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left(te^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-6t} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right).$$

Question 1(b)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 0 & -4 & 2 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} -3 & -1 & 2 \\ 0 & -4 & 2 \\ 0 & 1 & -5 \end{pmatrix}$ are $\lambda_1 = -6, \lambda := \lambda_2 = \lambda_3 = -3$
- The eigenvector corresponding to λ_1 : $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$
- Eigenvectors corresponding to λ : $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

Question 1(b) contd...

- General solution:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 e^{-6t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Question 1(c)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} -1 & -4 & 2 \\ 2 & 5 & -1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} -3 & -1 & 2 \\ 0 & -4 & 2 \\ 0 & 1 & -5 \end{pmatrix}$ are $\lambda_1 = 0, \lambda := \lambda_2 = \lambda_3 = 3$
- The eigenvector corresponding to λ_1 : $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
- Eigenvectors corresponding to λ : $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$

Question 1(c) contd...

- General solution:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}.$$

Question 2

Question 2

Solve the following IVPs.

(a) $y_1' = 2y_1 + 2y_2,\ y_2' = 5y_1 - y_2,\ \ y_1(0) = 0, y_2(0) = -7.$

(b) $y_1' = -14y_1 + 10y_2,\ y_2' = -5y_1 + y_2,\ y_1(0) = -1, y_2(0) = 1.$

Question 2(a)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ are $\lambda_1 = 4,\ \lambda_2 = 3$
- The eigenvectors corresponding to λ_1 : $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- The eigenvectors corresponding to λ_2 : $\begin{pmatrix} -2/5 \\ 1 \end{pmatrix}$
- General solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -2/5 \\ 1 \end{pmatrix}$

Question 2(a) contd...

- Plug in the initial conditions:

$$c_1 - 2/5c_2 = 0;\ c_1 + c_2 = -7$$

- On solving the equations:

$$c_1 = -2, c_2 = -5$$

- Solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -2e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 5e^{3t}\begin{pmatrix} -2/5 \\ 1 \end{pmatrix}.$$

Question 2(b)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -14 & 10 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} -14 & 10 \\ -5 & 1 \end{pmatrix}$ are $\lambda_1 = -4,\ \lambda_2 = -9$
- The eigenvectors corresponding to λ_1 : $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- The eigenvectors corresponding to λ_2 : $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- General solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-9t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Question 2(b) contd...

- Plug in the initial conditions:

$$c_1 + 2c_2 = -1;\ c_1 + c_2 = 1$$

- On solving the equations:

$$c_1 = 3, c_2 = -2$$

- Solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 3e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2e^{-9t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Question 3

Solve the following system of equations:

(a) $y_1' = y_2 + e^{3t},\ y_2' = y_1 - 3e^{3t}$

(b) $y_1' = 3y_1 + y_2 - 3\sin 3t,\ y_2' = 7y_1 - 3y_2 + 9\cos 3t - 16\sin 3t$

(c) $y_1' = -2y_1 + y_2,\ y_2' = -y_1 + e^t$.

Question 3(a)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\lambda_1 = -1,\ \lambda_2 = 1$
- The eigenvectors corresponding to λ_1 : $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- The eigenvectors corresponding to λ_2 : $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- General solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Question 3(a) contd...

- Fundamental matrix $\Phi(t) := \begin{pmatrix} -e^{-t} & e^{t} \\ e^{-t} & e^{t} \end{pmatrix}$
- $\Phi^{-1}(t) := -\frac{1}{2}\begin{pmatrix} e^{t} & -e^{t} \\ -e^{-t} & -e^{-t} \end{pmatrix}$, $\Phi^{-1}(0) := -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$
- Solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \Phi(t)\int \Phi^{-1}(t)\begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t}\,dt$$

- On simplifying:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \Phi(t)\begin{pmatrix} -1/2\ e^{4t} \\ -1/2\ e^{2t} \end{pmatrix},$$

c_1, c_2 are some constants.

Question 3(b)

- From the first equation take: $7 \times [(D-3)y_1 - y_2 = -3\sin 3t]$
- From the second equation take:
  $(D-3) \times [-7y_1 + (D+3)y_2 = -16\sin 3t + 9\cos 3t]$
- Add both the equations, on simplifying we get

$$(D^2 - 16)y_2 = -75\cos 3t$$

- On solving this ODE,

$$y_2(t) = c_1 e^{4t} + c_2 e^{-4t} + 75\frac{1}{D^2-16}\cos 3t = c_1 e^{4t} + c_2 e^{-4t} + 3\cos 3t$$

- So, in the first equation plugging y_2:

$$y_1(t) = \frac{1}{D-3}(y_2 - 3\sin 3t) = \frac{1}{D-3}(c_1 e^{4t} + c_2 e^{-4t} + 3\cos 3t - 3\sin 3t)$$

Question 3(b) contd...

- $\frac{1}{D-3}(3\sin 3t) = 3\frac{(D+3)}{D^2-9}\sin 3t = \frac{1}{6}\{3\cos 3t + 3\sin 3t\}$
- Similarly, we get
  $\frac{1}{D-3}(3\cos 3t) = 3\frac{(D+3)}{D^2-9}\cos 3t = \frac{1}{6}\{3\cos 3t + 3\sin 3t\}$
- Finally, $y_1(t) = c_1 e^{4t} - c_2/7\ e^{-4t} + \sin t$.

Question 3(c)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

- Eigenvalues of the matrix $A := \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ are $\lambda := \lambda_1 = \lambda_2 = -1$
- The eigenvectors corresponding to λ : $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- Let us find x_1, x_2 such that

$$(A + I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Solving this we get: $-x_1 + x_2 = 1$
- Choose : $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Question 3(c) contd...

- General Solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left(te^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$
- Fundamental matrix $\Phi(t) := \begin{pmatrix} e^{-t} & (t+1)e^{-t} \\ e^{-t} & (t+2)e^{-t} \end{pmatrix}$
- $\Phi^{-1}(t) := -\frac{1}{2} \begin{pmatrix} (t+2)e^{t} & -(t+1)e^{t} \\ -e^{t} & e^{t} \end{pmatrix}$,
- Solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \Phi(t) \int \Phi^{-1}(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} dt$$

- On simplifying:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \Phi(t) \begin{pmatrix} -te^{-t} \\ e^{t} \end{pmatrix},$$

c_1, c_2 are some constants.

Question 4

Solve the following IVP:

(a) $y_1' = y_2 - 5\sin t, y_2' = -4y_1 + 17\cos t,\ y_1(0) = 5, y_2(0) = 2.$

(b) $y_1' = y_1 + 4y_2 - t^2 + 6t,\ y_2' = y_1 + y_2 - t^2 + t - 1, y_1(0) = 2, y_2(0) = -1.$

(c) $y_1' = 5y_1 + 4y_2 - 5t^2 + 6t + 25, y_2' = y_1 + 2y_2 - t^2 + 2t + 4, y_1(0) = 0, y_2(0) = 0.$

Question 4 (a)

- Differentiating the first equation:

$$y_1'' = y_2' - 5\cos t$$

- Plug in the second equation:

$$y_1'' = -4y_1' + 12\cos t$$

- On solving this:

$$\begin{aligned} y_1(t) &= c_1\cos 2t + c_2\sin 2t + \frac{1}{D^2+4}(12\cos t) \\ &= c_1\cos 2t + c_2\sin 2t + 6\cos t \end{aligned}$$

- Plug in the initial condition: $c_1 = -4$.

Question 4 (a) contd...

- Plug in y_1, c_1 in the second equation:

$$y_2' = 16\cos 2t - 4c_2 \sin 2t + 17\cos t$$

- Therefore,

$$y_2(t) = -8\sin 2t - 2c_2 \cos 2t - 17\sin t$$

- Plug in the initial condition:

$$c_2 = -1.$$

- Solution :

$$y_1(t) = -4\cos 2t - \sin 2t + 6\cos t$$
$$y_2(t) = -8\sin 2t + \cos 2t - 17\sin t$$

Question 4(b)

- From the first equation: $(D-1)y_1 - 4y_2 = -t^2 + 6t$
- From the second equation take:
  $(D-1) \times [-y_1 + (D-1)y_2 = -t^2 + t - 1]$
- Add both the equations, on simplifying we get

$$((D-1)^2 - 4)y_2 = 3t + 2$$

- On solving this ODE,

$$y_2(t) = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{(D-1)^2 - 4}(3t+2) = c_1 e^{3t} + c_2 e^{-t} - t$$

- So, in the first equation plugging y_2:

$$y_1(t) = \frac{1}{D-1}(4c_1 e^{3t} + 4c_2 e^{-t} - t^2 + 2t)$$

Question 4(b) contd...

- That is,
$$y_1(t) = -c_1 e^{3t} - 2c_2\ e^{-t} + t^2.$$
- Plug in the initial conditions:
$$2 = -c_1 - 2c_2;\ \ 1 = c_1 + c_2$$
- On solving the equations: $c_1 = 4, c_2 = 3.$

Question 4(c)

- The given problem can be written in the matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -5t^2 + 6t + 25 \\ -t^2 + 2t + 4 \end{pmatrix}$$

- Eigenvalues of the matrix $A := \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ are $\lambda_1 = 6, \lambda_2 = 1$
- The eigenvectors corresponding to λ_1 : $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
- The eigenvectors corresponding to λ_2 : $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Question 4(c) contd...

- General Solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{6t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Fundamental matrix $\Phi(t) := \begin{pmatrix} 4e^{6t} & -e^{t} \\ e^{6t} & e^{t} \end{pmatrix}$
- $\Phi^{-1}(t) := \frac{1}{5} \begin{pmatrix} e^{-6t} & e^{-6t} \\ e^{-t} & 4e^{-t} \end{pmatrix}, \quad \Phi^{-1}(0) := \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix},$
- Solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi(t)\phi^{-1}(0) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Phi(t) + \int_0^t \Phi^{-1}(t) \begin{pmatrix} -5t^2 + 6t + 25 \\ -t^2 + 2t + 4 \end{pmatrix} dt$$

- On simplifying:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Phi(t)\frac{1}{5} \int_0^t \begin{pmatrix} (-6t^2 + 12t + 29)e^{-6t} \\ (-10t^2 + 20t + 41)e^{-t} \end{pmatrix} dt.$$

Question 5

Question 5

Find the Laplace transform of the following functions:

$$\cos^2 \omega t, e^t \cosh 3t, \sin 2t \cos 2t, e^{-\alpha t} \cos \beta t, \sinh t \cos t, 2e^{-t} \cos^2 \frac{1}{2} t.$$

Recall:

- $\mathcal{L}\{(f(t) + g(t)); s\} = \mathcal{L}\{f(t; s)\} + \mathcal{L}\{g(t); s\}$.
- $\mathcal{L}\{af(t); s\} = a\mathcal{L}\{f(t); s\}\}$.
- **Shifting property of Laplace transform**
  - If $\mathcal{L}\{f(t); s\} = F(s)$ then $\mathcal{L}\{e^{-at}f(t); s\} = F(s + a) = \mathcal{L}\{f(t); s + a\}$.
  - If $\mathcal{L}\{f(t); s\} = F(s)$ and $g(t) = f(t - a)u(t - a)$ then $\mathcal{L}\{g(t); s\} = e^{-as}F(s)$.
- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$.

Question 5

Solution:

$$
\begin{aligned}
\mathcal{L}\{\cos^2\omega t\} &= \mathcal{L}\left\{\frac{1}{2}(\cos 2\omega t + 1)\right\} \\
&= \frac{1}{2}\mathcal{L}\left\{\cos 2\omega t\right\} + \frac{1}{2}\mathcal{L}\left\{1\right\} \\
&= \frac{s}{2(s^2+4\omega^2)} + \frac{1}{2s} \\
&= \frac{s^2+2\omega^2}{s(s^2+4\omega^2)}.
\end{aligned}
$$

Question 5 contd...

Solution:

$$\begin{aligned}\mathcal{L}\{e^t \cosh 3t\} &= \mathcal{L}\left\{\cosh(3t);(s-1)\right\} \\ &= \frac{(s-1)}{(s-1)^2-9} \\ &= \frac{s-1}{(s-1)^2-9}.\end{aligned}$$

Solution:

$$\begin{aligned}\mathcal{L}\{\sin 2t\cos 2t\} &= \mathcal{L}\left\{\frac{1}{2}\sin(4t) + \frac{1}{2}\sin(0)\right\}\\ &= \frac{1}{2}\cdot\frac{4}{s^2+16}\\ &= \frac{2}{s^2+16}.\end{aligned}$$

Question 5 contd...

Solution:

$$\mathcal{L}\{e^{-\alpha t}\cos\beta t\} = \mathcal{L}\left\{\cos\left(\beta t\right); \left(s+\alpha\right)\right\}$$
$$= \frac{\left(s+\alpha\right)}{\left(s+\alpha\right)^2+\beta^2}.$$

Solution:

$$\begin{aligned}
\mathcal{L}\{\sinh t \cos t\} &= \mathcal{L}\left\{\frac{e^t - e^{-t}}{2}\cos t\right\} \\
&= \frac{1}{2}\mathcal{L}\left\{e^t \cos t\right\} - \frac{1}{2}\mathcal{L}\left\{e^{-t}\cos t\right\} \\
&= \frac{1}{2}\left[\frac{(s-1)}{(s-1)^2+1} - \frac{(s+1)}{(s+1)^2+1}\right] \\
&= \frac{s^2-2}{s^4+4}.
\end{aligned}$$

Solution:

$$\begin{aligned}
\mathcal{L}\{2e^{-t}\cos^2\frac{1}{2}t\} &= \mathcal{L}\left\{2e^{-t}\left(\frac{1}{2}+\cos\left(2\left\{\frac{t}{2}\right\}\right)\frac{1}{2}\right)\right\} \\
&= \mathcal{L}\left\{e^{-t}+e^{-t}\cos(t)\right\} \\
&= \mathcal{L}\left\{e^{-t}\right\}+\mathcal{L}\left\{e^{-t}\cos(t)\right\} \\
&= \frac{1}{s+1}+\frac{s+1}{(s+1)^2+1}.
\end{aligned}$$

Question 6

Question 6

Find inverse Laplace transform of the following functions:

$$\frac{5s}{(s^2-25)}, \frac{1-7s}{(s-3)(s-1)(s+2)}, \frac{2s^3}{(s^4-1)}, \frac{2}{s^2+s+\frac{1}{2}}$$

.

Recall: Some properties of inverse Laplace transform

- **Shifting property of inverse Laplace transform**
  If $\mathcal{L}^{-1}\{F(s);t\} = f(t)$ then
  - $\mathcal{L}^{-1}\{F(s+a);t\} = e^{-at}f(t) = \mathcal{L}^{-1}\{F(s);t\}$.
  - $\mathcal{L}^{-1}\{e^{-as}F(s);t\} = f(t-a)u(t-a)$.
- $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1,\ \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at},\ \mathcal{L}^{-1}\{\frac{a}{s^2+a^2}\} = \sin at$.

Question 6

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{5s}{(s^2-25)}\right\} &= \mathcal{L}^{-1}\left\{\frac{5}{2\,(s+5)}+\frac{5}{2\,(s-5)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{5}{2\,(s+5)}\right\}+\mathcal{L}^{-1}\left\{\frac{5}{2\,(s-5)}\right\} \\
&= \frac{5}{2}e^{-5t}+\frac{5}{2}e^{5t} \\
&= 5\cosh 5t.
\end{aligned}$$

Solution:

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1-7s}{(s-3)(s-1)(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{-\frac{2}{s-3}+\frac{1}{s-1}+\frac{1}{s+2}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\
&= -2e^{3t}+e^{t}+e^{-2t}.
\end{aligned}$$

Question 6 contd...

Solution:

$$\mathcal{L}^{-1}\left\{\frac{2s^3}{(s^4-1)}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}+\frac{1}{2\,(s+1)}+\frac{1}{2\,(s-1)}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{2\,(s+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{2\,(s-1)}\right\}$$
$$=\cos t+\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}$$
$$=\cos t+\cosh t.$$

Solution:

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{2}{s^2+s+\frac{1}{2}}\right\}\\
&=\mathcal{L}^{-1}\left\{2\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{1}{4}}\right\}\\
&=2\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right\}\\
&=2e^{-\frac{t}{2}}\frac{\sin\left(\frac{t}{2}\right)}{\frac{1}{2}}=4e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right).
\end{aligned}$$

Question 7

Solve the following IVP using Laplace transform:

(a) $y'' - y' - 2y = 0, y(0) = 8; y'(0) = 7.$

(b) $y'' + 2y' - 3y = 6e^{-2t}; y(0) = 2, y'(0) = -14.$

Question 7(a)

$y'' - y' - 2y = 0, y(0) = 8; y'(0) = 7.$

Solution:

- Take Laplace transform of both sides of the equation

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - (s\mathcal{L}\{y\} - y(0)) - 2\mathcal{L}\{y\} = 0.$$

- Plugging in the initial conditions

$$s^2\mathcal{L}\{y\} - 8s - 7 - (s\mathcal{L}\{y\} - 8) - 2\mathcal{L}\{y\} = 0.$$

- On simplifying this, we get

$$\mathcal{L}\{y\} = \frac{8s - 1}{s^2 - s - 2}.$$

Question 7(a) contd...

- Taking the inverse Laplace transform, we obtain

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{8s-1}{s^2-s-2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{8s-1}{s^2-2s+s-2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{8s-1}{(s-2)(s+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{5}{s-2}+\frac{3}{s+1}\right\} \\
&= 5e^{2t}+3e^{-t}.
\end{aligned}
$$

Question 7(b)

$y'' + 2y' - 3y = 6e^{-2t}; y(0) = 2, y'(0) = -14.$

Solution:

- Take Laplace transform of both sides of the equation

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0)) - 3\mathcal{L}\{y\} = \mathcal{L}\{6e^{-2t}\}.$$

- Plugging in the initial conditions

$$s^2\mathcal{L}\{y\} - 2s + 14 + 2(s\mathcal{L}\{y\} - 2) - 3\mathcal{L}\{y\} = \frac{6}{s+2}.$$

- On simplifying this, we get

$$\mathcal{L}\{y\} = \frac{2s^2 - 6s - 14}{(s+2)(s^2+2s-3)}.$$

- Taking the inverse Laplace transform, we obtain

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{2s^2-6s-14}{(s+2)(s^2+2s-3)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s^2-6s-14}{(s+2)(s+3)(s-1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{-2}{s+2}-\frac{3}{2(s-1)}+\frac{11}{2(s+3)}\right\} \\
&= -2e^{-2t}-\frac{3}{2}e^{t}+\frac{11}{2}e^{-3t}.
\end{aligned}$$

Question 8

Find the Laplace transform of the following functions (where u is the unit step function):

$$tu(t-1), e^{-2t}u(t-3), 4u(t-\pi)\cos t.$$

Recall:

- If $\mathcal{L}\{f(t); s\} = F(s)$ then $\mathcal{L}\{t^n f(t); s\} = (-1)^n F^{(n)}(s)$.
- $\mathcal{L}\left\{\mathrm{u}\left(t-a\right); s\right\} = e^{-as}/s$.

Question 8

Solution:

$$\begin{aligned}
\mathcal{L}\{tu(t-1)\} &= (-1)^1 \frac{d}{ds}\left(\mathcal{L}\left\{\mathrm{u}\left(t-1\right)\right\}\right) \\
&= -1.\frac{d}{ds}(e^{-s}/s) \\
&= (-1)^1 \frac{-e^{-s}s - e^{-s}}{s^2} \\
&= \frac{e^{-s}s + e^{-s}}{s^2}.
\end{aligned}$$

Question 8 contd...

Solution:

$$\begin{aligned}\mathcal{L}\{e^{-2t}u(t-3)\} &= \mathcal{L}\left\{\mathrm{u}\left(t-3\right);(s+2)\right\}\\ &= \frac{e^{-3(s+2)}}{(s+2)}.\end{aligned}$$

Solution:

- Since, $4u(t-\pi) = 1$ when $t \geq \pi$ and 0 otherwise, hence,

$$\begin{aligned}\mathcal{L}\{4u(t-\pi)\cos t\} &= 4\int_0^\infty e^{-st}u(t-\pi)\cos t dt \\ &= 4\int_\pi^\infty e^{-st}\cos t dt \\ &:= 4L_c.\end{aligned}$$

- Define

$$\begin{aligned}L_s := \int_\pi^\infty e^{-st}\sin t dt &= \left[-\frac{e^{-st}}{s}\sin t - \int_{2\pi}^\infty \cos t\frac{e^{-st}}{-s}ds\right]_\pi^\infty \\ &= \frac{1}{s}\int_\pi^\infty e^{-st}\cos t dt \\ &= \frac{1}{s}L_c, \qquad (1)\end{aligned}$$

- and

$$L_c := \int_\pi^\infty e^{-st}\cos t dt = \left[\frac{e^{-st}}{-s}\cos t + \int_\pi^\infty \sin t \frac{e^{-st}}{-s} ds\right]_\pi^\infty$$
$$= \frac{1}{s}(e^{-\pi s} - L_s). \qquad (2)$$

- From (1) and (2), we find L_c,

$$L_c = \frac{e^{-\pi s}s}{s^2+1}.$$

- Hence,

$$\mathcal{L}\{4u(t-\pi)\cos t\} = 4\frac{e^{-\pi s}s}{s^2+1}.$$

Question 9

Question 9

Find inverse Laplace transform of the following functions:

$$\frac{e^{-3s}}{s^3}, \frac{3(1-e^{-\pi s})}{s^2+9}, \frac{se^{-2s}}{s^2+\pi^2}.$$

Question 9

Solution

$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\}$

- Apply inverse transform rule:
  if $\mathcal{L}^{-1}\left\{F\left(s\right)\right\} = f\left(t\right)$ then $\mathcal{L}^{-1}\left\{e^{-as}F\left(s\right)\right\} = u\left(t-a\right)f\left(t-a\right)$,
  where $u(t)$ is unit step function.
- For $\frac{e^{-3s}}{s^3}$: $\quad F\left(s\right) = \frac{1}{s^3}$, $\quad a = 3$, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\} &= \mathrm{u}\left(t-3\right)\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}\left(t-3\right) \\
&= \mathrm{u}\left(t-3\right)\frac{\left(t-3\right)^2}{2} \\
&= \frac{\mathrm{u}\left(t-3\right)\left(t-3\right)^2}{2}.
\end{aligned}$$

Question 9 contd...

Solution

$\mathcal{L}^{-1}\left\{\frac{3(1-e^{-\pi s})}{s^2+9}\right\}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3\left(-e^{-\pi s}+1\right)}{s^2+9}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}-\frac{3e^{-\pi s}}{s^2+9}\right\} \\
&= 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}-3\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+9}\right\} \\
&= 3\cdot\frac{1}{3}\sin\left(3t\right)-3\mathrm{u}\left(t-\pi\right)\frac{1}{3}\sin\left(3\left(t-\pi\right)\right) \\
&= \sin\left(3t\right)-\mathrm{u}\left(t-\pi\right)\sin\left(3\left(t-\pi\right)\right).
\end{aligned}$$

Question 9 contd...

Solution

$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+\pi^2}\right\}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+\pi^2}\right\} &= \mathrm{u}\left(t-2\right)\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}\left(t-2\right)\\ &= \mathrm{u}\left(t-2\right)\cos\left(\pi\left(t-2\right)\right).\end{aligned}$$

Question 10

Solve the following IVP.

(a) $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, y(0) = 0; y'(0) = 0.$

(b) $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0, y'(0) = 0.$

(c) $y'' + 4y' + 5y = \delta(t-1)$, ($\delta$ is the Dirac's Delta) $y(0) = 0, y'(0) = 3.$

(d) $y'' + 5y' + 6y = u(t-1) + \delta(t-2)$ (where u, δ are the step function and the Dirac's Delta function), $y(0) = 0$ and $y'(0) = 1.$

Solution:

- Take Laplace transform of the both the sides of the equation

$$\mathcal{L}\left\{y'' + 6y' + 8y\right\} = \mathcal{L}\left\{e^{-3t} - e^{-5t}\right\}.$$

- On simplifying, we get

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 6(s\mathcal{L}\{y\} - y(0)) + 8\mathcal{L}\{y\} = \frac{1}{s+3} - \frac{1}{s+5}.$$

- Plug in the initial conditions,

$$s^2\mathcal{L}\{y\} - s \cdot 0 - 0 + 6(s\mathcal{L}\{y\} - 0) + 8\mathcal{L}\{y\} = \frac{1}{s+3} - \frac{1}{s+5}.$$

- On simplifying and taking the inverse Laplace transform, we obtain

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{2}{(s^2+8s+15)\,(s^2+6s+8)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{(s+5)\,(s+3)\,(s+4)(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{-\frac{1}{s+3}-\frac{1}{3(s+5)}+\frac{1}{3(s+2)}+\frac{1}{s+4}\right\} \\
&= -e^{-3t}-\frac{1}{3}e^{-5t}+\frac{1}{3}e^{-2t}+e^{-4t}.
\end{aligned}
$$

- $y=-e^{-3t}-\frac{1}{3}e^{-5t}+\frac{1}{3}e^{-2t}+e^{-4t}$.

Question 10(b)

Solution:

- Take Laplace transform of the both the sides of the equation

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\}$$
$$= 4(e^{-s}/s + 1/s^2 + e^{-s}/s^2)$$

- Plugging the initial conditions, we have

$$s^2\mathcal{L}\{y\} - s\cdot 0 - 0 + 3(s\mathcal{L}\{y\} - 0) + 2\mathcal{L}\{y\} = 4(e^{-s}/s + 1/s^2 + e^{-s}/s^2)$$

- On simplifying, we get

$$\mathcal{L}\{y\} = \frac{1}{s^2 + 3s + 2} 4(e^{-s}/s + 1/s^2 + e^{-s}/s^2)$$

Question 10(b) contd...

- Taking the inverse Laplace transform, we obtain

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}4(e^{-s}/s + 1/s^2 + e^{-s}/s^2)\right\}.$$

- $y = [(2t + (2t + e^{2-2t} - 3)u(t-1) - 3)e^{2t} + 4e^t - 1]e^{-2t}.$

Question 10(c)

Solution:

- Take Laplace transform of the both the sides of the equation

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} = e^{-s}$$

- Plugging the initial conditions, we have

$$s^2\mathcal{L}\{y\} - s\cdot\ 0 - 3 + 4(s\mathcal{L}\{y\} - 0) + 5\mathcal{L}\{y\} = e^{-s}$$

- On simplifying, we get

$$\mathcal{L}\{y\} = \frac{3 + e^{-s}}{s^2 + 4s + 5}$$

Question 10(c) contd...

- Taking the inverse Laplace transform, we obtain

$$y = \mathcal{L}^{-1}\left\{\frac{3+e^{-s}}{s^2+4s+5}\right\}.$$

- That is,

$$y = 3e^{-2t}\sin t + \mathrm{u}\left(t-1\right)\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}\left(t-1\right)$$

- $y = 3e^{-2t}\sin t + \mathrm{u}\left(t-1\right)e^{-2(t-1)}\sin\left(t-1\right).$

Question 10(d)

Solution:

- Take Laplace transform of the both the sides of the equation

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 5(s\mathcal{L}\{y\} - y(0)) + 6\mathcal{L}\{y\} = \frac{e^{-s}}{s} + e^{-2s}$$

- Plugging the initial conditions, we have

$$s^2\mathcal{L}\{y\} - s\cdot\ 0 - 0 + 5(s\mathcal{L}\{y\} - 0) + 6\mathcal{L}\{y\} = \frac{e^{-s}}{s} + e^{-2s}$$

- On simplifying, we get

$$\mathcal{L}\{y\} = \frac{e^{-s}}{s(s+2)(s+3)} + \frac{e^{-2s}}{(s+2)(s+3)} + \frac{1}{(s+2)(s+3)}$$

Question 10(d) contd...

- Taking the inverse Laplace transform, we obtain

$$y = \mathcal{L}^{-1}\left\{e^{-s}.\left[\frac{1}{2}\left(\frac{1}{s}-\frac{1}{s+2}\right)-\frac{1}{3}\left(\frac{1}{s}-\frac{1}{s+3}\right)\right]\right\}$$
$$= \mathcal{L}^{-1}\left\{e^{-2s}.\left(\frac{1}{s+2}-\frac{1}{s+3}\right)\right\}+\mathcal{L}^{-1}\left\{\left(\frac{1}{s+2}-\frac{1}{s+3}\right)\right\}$$

- That is,

$$y = \left[\frac{1}{2}(1-e^{-2(t-1)})-\frac{1}{3}(1-e^{-3(t-1)})\right]u(t-1)$$
$$+[e^{-2(t-2)}-e^{-3(t-2)}]u(t-2)+e^{-2t}-e^{-3t}.$$

Question 11

Question 11

Find the Laplace transform (by differentiation) of the following functions: $t^2 \cosh \pi t$, $te^{-t} \sin t$, $t^2 \cos wt$.

Recall:

- If $\mathcal{L}\{f(t); s\} = F(s)$ then $\mathcal{L}\{t^n f(t); s\} = (-1)^n F^{(n)}(s)$.
- **Shifting property of Laplace transform**
  - If $\mathcal{L}\{f(t); s\} = F(s)$ then $\mathcal{L}\{e^{-at} f(t); s\} = F(s+a) = \mathcal{L}\{f(t); s+a\}$.
  - If $\mathcal{L}\{f(t); s\} = F(s)$ and $g(t) = f(t-a)u(t-a)$ then $\mathcal{L}\{g(t); s\} = e^{-as} F(s)$.
- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$.

Question 11

$t^2 \cosh \pi t$.

Solution:

- Since, $\mathcal{L}\{\cosh \pi t\} = F(s) = \frac{s}{s^2-\pi^2}$.
- Hence, by differentiation,

$$\begin{aligned}
\mathcal{L}\{t^2 \cosh \pi t; s\} &= (-1)^2 \frac{d^2 F(s)}{ds^2} \\
&= \frac{d^2}{ds^2}\left(\frac{s}{s^2 - \pi^2}\right) \\
&= -\frac{d}{ds}\left(\frac{\pi^2 + s^2}{(s^2 - \pi^2)^2}\right).
\end{aligned}$$

Question 11 contd...

- Again differentiating with respect to s, we get

$$\begin{aligned}\mathcal{L}\{t^2\cosh \pi t; s\} &= \frac{1}{(s^2-\pi^2)^3}\Big[-2s(s^2-\pi^2)+4s(\pi^2+s^2)\Big]\\ &= \frac{2s(s^2+\pi^2)}{(s^2-\pi^2)^3}.\end{aligned}$$

- Hence,

$$\mathcal{L}\{t^2\cosh \pi t\} = \frac{2s(s^2+\pi^2)}{(s^2-\pi^2)^3}.$$

Question 11

$te^{-t}\sin t$.

Solution:

- Since, $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$.
- Also,

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

- Hence, using the shifting property of Laplace transform,

$$\mathcal{L}\{te^{-t}\sin t; s\} = \frac{2(s+1)}{((s+1)^2+1)^2}.$$

Question 11

$t^2 \cos wt$.

Solution:

- Since, $\mathcal{L}\{\cos wt\} = \frac{s}{s^2+w^2}$.
- Hence, by differentiation,

$$\begin{aligned}
\mathcal{L}\{t^2 \cos wt; s\} &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+w^2}\right) \\
&= \frac{d}{ds}\left(\frac{w^2-s^2}{(s^2+w^2)^2}\right) \\
&= \frac{-2s(s^2+w^2)^2 - 4s(w^2-s^2)(s^2+w^2)}{(s^2+w^2)^4}.
\end{aligned}$$

Question 11 contd...

- On simplification, we get

$$\mathcal{L}\{t^2 \cos wt; s\} = \frac{-2s}{(s^2+w^2)^3}\left(s^2 + w^2 + 2w^2 - 2s^2)\right)$$
$$= -\frac{2s}{(s^2+w^2)^3}(3w^2 - s^2).$$

- Hence,

$$\mathcal{L}\{t^2 \cos wt; s\} = -\frac{2s}{(s^2+w^2)^3}(3w^2 - s^2).$$

Question 12

Question 12

Find inverse Laplace transform of the following functions by differentiation or integration:

$$\frac{1}{(s-3)^3}, \frac{2s+6}{(s^2+6s+10)^2}, \ln\left(\frac{s+a}{s+b}\right), \cot^{-1}\frac{s}{\pi}$$

.

Question 12

Recall: Some properties of inverse Laplace transform

- **Shifting property of inverse Laplace transform**
  If $\mathcal{L}^{-1}\{F(s); t\} = f(t)$ then
  - $\mathcal{L}^{-1}\{F(s+a); t\} = e^{-at} f(t) = \mathcal{L}^{-1}\{F(s); t\}$.
  - $\mathcal{L}^{-1}\{e^{-as} F(s); t\} = f(t-a)u(t-a)$.

Question 12

$\frac{1}{(s-3)^3}$.

Solution:

- Using the shifting property of inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^3}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}.$$

- Above equation can be written as,

$$\begin{aligned} e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} &= \frac{e^{3t}}{2}\mathcal{L}^{-1}\left\{\frac{d^2\left(\frac{1}{s}\right)}{ds^2}\right\} \\ &= \frac{e^{3t}}{2}t^2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= \frac{e^{3t}}{2}t^2.1 \end{aligned}$$

Question 12 contd...

- Hence, $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^3}\right\} = \frac{e^{3t}}{2}t^2$.

Question 12

$\frac{2s+6}{(s^2+6s+10)^2}$.

Solution:

- Since, $s^2+6s+10 = s^2+3s+3s+9+1 = (s+3)^2+1$.
- Hence,

$$\frac{2s+6}{(s^2+6s+10)^2} = \frac{2(s+3)}{(s+3)^2+1}.$$

- Using the shifting property of inverse Laplace transform, the above expression can be solved as

$$\mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+1}\right\} = 2e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}.$$

Question 12 contd...

- Now,

$$2e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = -e^{-3t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$
$$= -e^{-3t}(-1)t\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\}$$
$$= e^{-3t}t\sin t.$$

- Hence,

$$\mathcal{L}^{-1}\left\{\frac{2s+6}{(s^2+6s+10)^2}\right\} = e^{-3t}t\sin t.$$

Question 12 contd...

$\ln\left(\frac{s+a}{s+b}\right)$.

Solution:

- $\ln\left(\frac{s+a}{s+b}\right) = \ln(s+a) - \ln(s+b)$.
- Let $\mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} = f(t)$.
- We want to find $f(t)$ such that, $\mathcal{L}\{f(t)\} = \ln(s+a) - \ln(s+b)$.
- Then,

$$\begin{aligned}
\mathcal{L}\{tf(t)\} &= -\frac{d}{ds}(\ln(s+a) - \ln(s+b)) \\
&= -\left[\frac{1}{s+a} - \frac{1}{s+b}\right] \\
&= -\mathcal{L}\{e^{-at} + e^{-bt}\} \qquad (3)
\end{aligned}$$

- Taking inverse Laplace transform both sides of (3), we get

$$f(t) = \frac{e^{-bt} - e^{-at}}{t}.$$

- Hence,

$$\mathcal{L}^{-1}\left\{ \ln\left(\frac{s+a}{s+b}\right)\right\} = f(t) = \frac{e^{-bt} - e^{-at}}{t}.$$

Question 12 contd...

$\cot^{-1} \frac{s}{\pi}$.

Solution:

- Let $\cot^{-1} \frac{s}{\pi} = F(s)$.
- And let $\mathcal{L}^{-1}\left\{ \cot^{-1} \frac{s}{\pi} \right\} = f(t)$.
- Since,

$$F^{'}(s) = \frac{-1}{\frac{s^2}{\pi^2} + 1} \frac{1}{\pi} = \frac{-\pi}{s^2 + \pi^2}.$$

- Hence,

$$\mathcal{L}^{-1}\{-F^{'}(s)\} = \mathcal{L}^{-1}\left\{ \frac{\pi}{s^2 + \pi^2} \right\}$$
$$\implies tf(t) = \sin \pi t.$$

Question 12 contd...

- Hence,

$$\mathcal{L}^{-1}\left\{\cot^{-1}\frac{s}{\pi}\right\} = \frac{\sin \pi t}{t}.$$

Question 13

Question 13

Compute convolution of the following:
$1 * \sin wt, e^t * e^{-1}, \cos wt * \sin wt, u(t-1) * t^2, u(t-3) * e^{2t}.$

Question 13

Recall:

- Convolution of two integrable functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = \int_0^t f(t-u)g(u)du = \int_0^t f(u)g(t-u)du.$$

- **Unit Step function or Heavyside function:**

$$u(t-a) = \begin{cases} 1, & \text{if } t \geq a \\ 0, & \text{if } t < a \end{cases}$$

- The Laplace transform of $u(t-a) = \frac{e^{-as}}{s}, \ (s > 0)$.

Question 13

Question 13

$1 * \sin wt$.

Solution:

- The convolution of 1 and $\sin wt$ is defined as,

$$
\begin{aligned}
1 * \sin wt &= \int_0^t f(t-u)g(u)du \\
&= \int_0^t 1.\sin wu \; du \\
&= \left[\frac{-\cos wu}{w}\right]_0^t \\
&= \frac{1}{w}(1-\cos wt).
\end{aligned}
$$

Question 13 contd...

Question 13

$e^t * e^{-1}$.

Solution:

- The convolution of e^t and e^{-1} is defined as,

$$\begin{aligned} e^t * e^{-1} t &= \int_0^t e^{t-u} . e^{-1} du \\ &= -e^{-1} [e^{t-u}]_0^t \\ &= e^{-1}(e^t - 1). \end{aligned}$$

Question 13

$\cos wt * \sin wt$.

Solution:

- The convolution of $\cos wt$ and $\sin wt$ is defined as,

$$\begin{aligned}
\cos wt * \sin wt &= \int_0^t \cos w(t-u).\sin wu \ du \\
&= \int_0^t \frac{\sin(wt) - \sin(wt - 2wu)}{2} du \\
&= \frac{1}{2}\left[\sin wt.u + \frac{\cos w(t-2u)}{-2w}\right]_0^t \\
&= \frac{1}{2}(t \sin wt).
\end{aligned}$$

Question 13

$u(t-1) * t^2$.

Solution:

- The convolution of $u(t-1)$ and t^2 is defined as,

$$\begin{aligned} u(t-1) * t^2 &= \int_0^t u(t-1).u^2 du \\ &= \int_1^t 1.u^2 du \\ &= \left[\frac{u^3}{3}\right]_1^t \\ &= \frac{1}{3}(t^3 - 1). \end{aligned}$$

Question 13 contd...

Question 13

$u(t-3) * e^{2t}$.

Solution:

- The convolution of $u(t-3)$ and e^{2t} is defined as,

$$
\begin{aligned}
u(t-3) * e^{2t} &= \int_0^t u(t-3).e^{2u} du \\
&= \int_3^t 1.e^{2u} du \\
&= \left[\frac{e^{2u}}{2}\right]_3^t \\
&= \frac{1}{2}(e^{2t} - e^6).
\end{aligned}
$$

Question 14

Question 14

Use convolution theorem to compute the inverse transform:

$$\frac{6}{s(s+3)}, \frac{s^2}{(s^2+w^2)^2}, \frac{e^{-as}}{s(s^2+s-2)}, \frac{w}{s^2(s^2+w^2)}, \frac{1}{(s+3)(s-2)}.$$

Recall: Convolution theorem

Let $f(t) = \mathcal{L}^{-1}\{F(s); t\}$ and $g(t) = \mathcal{L}^{-1}\{g(s); t\}$ then

$$\mathcal{L}^{-1}\{F(s).G(s); t\} = \int_0^t f(t-u)g(u)du = (f * g)(t),$$

$$\mathcal{L}\{(f * g)(t); s\} = \mathcal{L}\{\int_0^t f(t-u)g(u)du; s\} = F(s).G(s).$$

Question 14

$\frac{6}{s(s+3)}$.

Solution:

- Since, $\frac{6}{s(s+3)} = 6\left(\frac{1}{s}.\frac{1}{s+3}\right)$.
- Hence, using convolution theorem for inverse transform, $\mathcal{L}^{-1}\{\frac{6}{s(s+3)}\} = 6\mathcal{L}^{-1}\left\{\frac{1}{s}.\frac{1}{s+3}\right\} = 6(1 * e^{-3t})$.
- where,

$$\begin{aligned} 1 * e^{-3t} &= \int_0^t f(t-u)g(u)du \\ &= \int_0^t 1.e^{-3u}\ du \\ &= \left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{1}{3}(1-e^{-3t}). \end{aligned}$$

Question 14 contd...

Question 14

$\frac{s^2}{(s^2+w^2)^2}$.

Solution:

- Since, we can write,

$$\frac{s^2}{(s^2+w^2)^2} = \left(\frac{s}{s^2+w^2} . \frac{s}{s^2+w^2}\right).$$

- Hence, using convolution theorem for inverse transform,

$$\mathcal{L}^{-1}\{\frac{s^2}{(s^2+w^2)^2}\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+w^2} . \frac{s}{s^2+w^2}\right\} = (\cos wt * \cos wt).$$

- Further, we can calculate the convolution of $\cos wt$ with $\cos wt$ as done in previous part.

Question 14

$\frac{e^{-as}}{s(s^2+s-2)}$.

Solution:

- Since,

$$\frac{e^{-as}}{s(s^2+s-2)} = \left(\frac{e^{-as}}{s} \cdot \frac{1}{s^2+s-2}\right).$$

- Here,

$$\frac{1}{(s^2+s-2)} = \left(\frac{1}{(s-1)(s+2)}\right) = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right).$$

- Also,

$$\mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)\right\} = \frac{1}{3}(e^{t} - e^{-2t}).$$

Question 14 contd...

- Hence, using second shifting property for inverse Laplace transform and convolution theorem, we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s(s^2+s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}.\frac{1}{s^2+s-2}\right\}u(t-a)$$
$$= \left(1 * \frac{1}{3}(e^{(t-a)} - e^{-2(t-a)})\right)u(t-a).$$

- Further, we can calculate the convolution of above functions.

Question 14 contd...

Question 14

$\frac{w}{s^2(s^2+w^2)}$.

Solution:

- Since,

$$\frac{w}{s^2(s^2+w^2)} = \left(\frac{1}{s^2}.\frac{w}{s^2+w^2}\right).$$

- Hence, using convolution theorem for inverse transform,

$$\mathcal{L}^{-1}\{\frac{w}{s^2(s^2+w^2)}\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}.\frac{w}{s^2+w^2}\right\} = (t * \sin wt).$$

- Further, we can calculate the convolution of t and $\sin wt$ as done in first part.

Question 14

$\frac{1}{(s+3)(s-2)}$.

Solution:

- Since,

$$\frac{1}{(s+3)(s-2)} = \left(\frac{1}{s+3}.\frac{1}{s-2}\right).$$

- Hence, using convolution theorem for inverse transform,

$$\mathcal{L}^{-1}\{\frac{1}{(s+3)(s-2)}\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}.\frac{1}{s-2}\right\} = (e^{-3t} * e^{2t}).$$

- Further, we can calculate the convolution of t and $\sin wt$ as done in first part.

Question 15

Question 15

Solve IVP by using convolution.

(a) $y'' + y = 3\cos 2t$; $y(0) = 0,\ y'(0) = 0$.

(b) $y'' + 2y' + 2y = 5u(t - 2\pi)\sin t$; $y(0) = 1,\ y'(0) = 0$.

(c) $y'' + y = r(t)$, $r(t) = 4t$ if $1 < t < 2$ and 0 otherwise; $y(0) = 0,\ y'(0) = 0$.

(d) $y'' + 3y' + 2y = r(t)$, $r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0,\ y'(0) = 0$.

Question 15

Recall: Laplace transform of derivative $f^{(n)}$ of any order

Let $f, f^{'}, ..., f^{(n-1)}$ be continuous for all $t \geq 0$ and satisfy the growth restriction, $|f(t)| \leq Me^{kt}$, for constant M and k. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - f^{(n-1)}(0).$$

Question 15(a)

Solve IVP by using convolution.

(a) $y^{''} + y = 3\cos 2t;\ y(0) = 0,\ y^{'}(0) = 0.$

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Consider the IVP: $y^{''} + y = 3\cos 2t;\ \ y(0) = 0,\ \ y^{'}(0) = 0.$
- Applying the Laplace transform both sides of above IVP, we get,

$$s^2 Y(s) + Y(s) = \frac{3s}{s^2+4}$$

$$\implies Y(s) = \frac{3s}{(s^2+1)(s^2+4)}. \tag{4}$$

Question 15(a) contd...

- Taking inverse Laplace transform both sides of (4) and using Convolution theorem, we get

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s); t\} &= \mathcal{L}^{-1}\left\{\frac{3s}{(s^2+1)(s^2+4)}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}.\frac{s}{s^2+4}; t\right\} \\ &= 3\sin t * \cos 2t. \end{aligned}$$

- Here,

$$\begin{aligned} 3\sin t * \cos 2t &= 3\int_0^t \sin(t-u).\cos 2u \; du \\ &= 3\int_0^t \frac{\sin(t+u)+\sin(t-3u)}{2} du \\ &= \frac{3}{2}\left[-\cos(t+u) - \frac{\cos(t-3u)}{-3}\right]_0^t \\ &= \cos t - \cos 2t. \end{aligned}$$

Question 15(b)

(b) $y^{''} + 2y^{'} + 2y = 5u(t-2\pi)\sin t$; $y(0) = 1$, $y^{'}(0) = 0$.

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Consider the IVP: $y^{''} + 2y^{'} + 2y = 5u(t-2\pi)\sin t$; $y(0) = 1$, $y^{'}(0) = 0$.
- Applying the Laplace transform both sides of above IVP, we get,

$$s^2Y(s) - s + 2sY(s) - 1 + 2Y(s) = \mathcal{L}\{5u(t-2\pi)\sin t\}.$$

- Since, $5u(t-2\pi) = 1$ when $t \geq 2\pi$ and 0 otherwise, hence,

$$\begin{aligned}\mathcal{L}\{5u(t-2\pi)\sin t\} &= 5\int_0^\infty e^{-st}u(t-2\pi)\sin t dt\\ &= 5\int_{2\pi}^\infty e^{-st}\sin t dt\\ &:= 5L_s.\end{aligned}$$

Question 15(b) contd...

- Where,

$$L_s := \int_{2\pi}^{\infty} e^{-st} \sin t dt = \left[-\frac{e^{-st}}{s} \sin t - \int_{2\pi}^{\infty} \cos t \frac{e^{-st}}{-s} dt \right]_{2\pi}^{\infty}$$
$$= \frac{1}{s} \int_{2\pi}^{\infty} e^{-st} \cos t dt$$
$$= \frac{1}{s} L_c, \tag{5}$$

- and

$$L_c := \int_{2\pi}^{\infty} e^{-st} \cos t dt = \left[\frac{e^{-st}}{-s} \cos t + \int_{2\pi}^{\infty} \sin t \frac{e^{-st}}{-s} dt \right]_{2\pi}^{\infty}$$
$$= -\frac{1}{s}(e^{-2\pi s} + L_s). \tag{6}$$

- From (5) and (6), we find L_s,

$$L_s = -\frac{e^{-2\pi s}}{s^2 + 1}.$$

Question 15(b) contd...

- Hence,

$$Y(s) = \frac{s+1}{(s+1)^2+1} - \frac{5e^{-2\pi s}}{(s^2+1)((s+1)^2+1)}$$

- Taking inverse Laplace transform both sides of (4) and using Convolution theorem, we get

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s);t\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{5e^{-2\pi s}}{(s^2+1)((s+1)^2+1)}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 5\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)((s+1)^2+1)}\right\} \\ &= e^{-t}\cos t \\ &- 5(\sin(t-2\pi) * e^{-(t-2\pi)}\sin(t-2\pi))u(t-2\pi). \end{aligned}$$

Question 15(c)

(c) $y^{''} + y = r(t)$, $r(t) = 4t$ if $1 < t < 2$ and 0 otherwise; $y(0) = 0,\ y'(0) = 0$.

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Consider the IVP: $y^{''} + y = r(t)$; $r(t) = 4t$ if $1 < t < 2$ and 0 otherwise; $y(0) = 0,\ y'(0) = 0$.
- Applying the Laplace transform both sides of above IVP, we get,

$$s^2 Y(s) + Y(s) = \mathcal{L}\{r(t)\},$$

Question 15(c) contd...

- where,

$$\mathcal{L}\{r(t)\} = \int_0^\infty e^{-st} r(t)dt = \int_1^2 e^{-st} 4t\ dt$$
$$= \left[\frac{4te^{-st}}{-s} - \frac{4e^{-st}}{s^2} dt\right]_1^2$$
$$= \frac{-8}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} + \frac{4}{s} e^{-s} + \frac{4}{s^2} e^{-s}.$$

- Hence,

$$Y(s) = \frac{-8}{s(s^2+1)} e^{-2s} - \frac{4}{s^2(s^2+1)} e^{-2s} + \frac{4}{s(s^2+1)} e^{-s} + \frac{4}{s^2(s^2+1)} e^{-s}.$$

- Taking inverse Laplace transform both sides of (4) and using Convolution theorem, we get

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s); t\} &= -8\{1 * \sin(t-2)\}u(t-2) \\ &- 4\{(t-2) * \sin(t-2)\}u(t-2) \\ &+ 4\{1 * \sin(t-1)\}u(t-1) + 4\{(t-1) * \sin(t-1)\}u(t-1). \end{aligned}$$

Question 15(d)

(d) $y'' + 3y' + 2y = r(t)$, $r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$;
$y(0) = 0,\ y'(0) = 0$.

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Consider the IVP: $y'' + 3y' + 2y = r(t)$; $r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0,\ y'(0) = 0$.
- Applying the Laplace transform both sides of above IVP, we get,

$$
\begin{aligned}
s^2 Y(s) + 3Y(s) + 2Y(s) &= \mathcal{L}\{r(t)\},\\
\implies Y(s)((s+1)(s+2)) &= \mathcal{L}\{r(t)\}.
\end{aligned}
$$

- where,

$$\mathcal{L}\{r(t)\} = \int_0^\infty e^{-st} r(t) dt = \int_0^1 e^{-st} 4t \; dt + \int_1^\infty 8e^{-st} \; dt$$
$$= \frac{-4}{s} e^{-s} - \frac{4}{s^2} e^{-s} + \frac{4}{s^2}.$$

- Hence,

$$Y(s) = \frac{4(s-1)}{s^2((s+\frac{3}{2})^2 - \frac{1}{4})} e^{-s} + \frac{4}{s^2((s+\frac{3}{2})^2 - \frac{1}{4})}.$$

- Taking inverse Laplace transform both sides of (4) and using Convolution theorem, we get

$$y(t) = \mathcal{L}^{-1}\{Y(s); t\} = A + B.$$

- where,

$$A = \mathcal{L}^{-1}\left\{ \frac{4s}{s^2((s+\frac{3}{2})^2-\frac{1}{4})}e^{-s} - \frac{4}{s^2((s+\frac{3}{2})^2-\frac{1}{4})}e^{-s} \right\}$$

$$= 4\left\{ (t-1) * e^{-\frac{3}{2}(t-1)} \cosh\frac{(t-1)}{2} \right\} u(t-1)$$

$$-4\left\{ (t-1) * e^{-\frac{3}{2}(t-1)} \frac{\sinh\frac{(t-1)}{2}}{\frac{1}{2}} \right\} u(t-1),$$

- and,

$$B = \mathcal{L}^{-1}\left\{ \frac{4}{s^2((s+\frac{3}{2})^2-\frac{1}{4})} \right\}$$

$$= 4\left\{ t * e^{-\frac{3}{2}t} \frac{\sinh\frac{t}{2}}{\frac{1}{2}} \right\}.$$

Question 16

Solve the integral equations using Laplace transform.

$$y(t) = 1 + \int_0^t y(r)dr, \;\; y(t) = 2t - 4\int_0^t y(r)(t-r)dr,$$

$$y(t) = 1 - \sinh t + \int_0^t (1+r)y(t-r)dr.$$

Question 16

$$y(t) = 1 + \int_0^t y(r)dr.$$

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Let $y(t) = 1 + \int_0^t y(r)dr$.
- Also, $\int_0^t y(r)dr = \int_0^t 1.y(r)dr = 1 * y(t)$.
- Taking Laplace transform both sides of above equation, we get

$$Y(s) = \frac{1}{s} + \frac{1}{s}.Y(s)$$

Question 16 contd...

- On solving, we get

$$Y(s) = \frac{1}{s-1}.$$

- Taking inverse Laplace transform, we get,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}.$$

Question 16 contd...

Question 16

$$y(t) = 2t - 4\int_0^t y(r)(t-r)dr.$$

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Let $y(t) = 2t - 4\int_0^t y(r)(t-r)dr$.
- Also, $\int_0^t y(r)(t-r)dr = y(t) * t$.
- Taking Laplace transform both sides of above equation, we get

$$Y(s) = \frac{2}{s^2} - 4Y(s)\frac{1}{s^2}$$

Question 16 contd...

- On solving, we get

$$Y(s) = \frac{2}{s^2+4}.$$

- Taking inverse Laplace transform, we get,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t.$$

Question 16

$$y(t) = 1 - \sinh t + \int_0^t (1+r)y(t-r)dr.$$

Solution:

- Let $\mathcal{L}\{y(t)\} = Y(s)$.
- Let $y(t) - 1 + \sinh t = \int_0^t (1+r)y(t-r)dr$.
- Also, $\int_0^t (1+r)y(t-r)dr = (1+t) * y(t)$.
- Taking Laplace transform both sides of above equation, we get

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}$$
$$\implies Y(s)\left(\frac{s^2-s-1}{s^2}\right) = \frac{s^2-s-1}{s(s^2-1)}$$

Question 16 contd...

- On solving, we get

$$Y(s) = \frac{s}{s^2 - 1}.$$

- Taking inverse Laplace transform, we get,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 1}\right\} = \cosh t.$$

Question 17

Question 17

Use partial fraction method to find the inverse Laplace transform of the following:

$$\frac{6}{(s+2)(s-4)}, \frac{s^2+9s-9}{s^3-9s}, \frac{s^3+6s^2+14s}{(s+2)^4}.$$

Question 17

$$\frac{6}{(s+2)(s-4)}$$

Solution:

- Doing partial fractions, we get,

$$\frac{6}{(s+2)(s-4)} = \frac{1}{s-4} - \frac{1}{s+2}.$$

- Hence, the inverse Laplace transform is,

$$\mathcal{L}^{-1}\left\{\frac{6}{(s+2)(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4} - \frac{1}{s+2}\right\} = e^{4t} - e^{-2t}.$$

Question 17 contd...

Question 17

$$\frac{s^2+9s-9}{s^3-9s}$$

Solution:

- The denominator can be factorised as, $s^3-9s=s(s-3)(s+3)$.
- Doing partial fractions,

$$\frac{s^2+9s-9}{s^3-9s}=\frac{A}{s}+\frac{B}{s-3}+\frac{C}{s+3},$$

- Comparing the coefficients both sides, we get

$$\begin{aligned} A+B+C&=1\\ 3B-3C&=9\\ -9A&=-9. \end{aligned}$$

Question 17 contd...

- On solving we get, $A = 1,\ B = \frac{3}{2},\ C = \frac{-3}{2}$.
- Hence, taking the inverse Laplace transform both sides,

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s-3}+\frac{C}{s+3}\right\}$$
$$= 1+\frac{3}{2}e^{3t}-\frac{3}{2}e^{-3t}.$$

Question 17 contd...

Question 17

$$\frac{s^3 + 6s^2 + 14s}{(s+2)^4}$$

Solution:

- Doing partial fractions,

$$\frac{s^3 + 6s^2 + 14s}{(s+2)^4} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4},$$

- Comparing the coefficients both sides, we get

$$\begin{aligned}
&8A + 4B + 2C + D = 0 \\
&4A + 4B + C = 14 \\
&2A + B = 6 \\
&A = 1.
\end{aligned}$$

Question 17 contd...

- On solving we get, $A = 1,\ B = 4,\ C = -6,\ D = -12.$
- Hence, taking the inverse Laplace transform both sides and using shifting property, we get,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^3+6s^2+14s}{(s+2)^4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}+\frac{4}{(s+2)^2}-\frac{6}{(s+2)^3}-\frac{12}{(s+2)^4}\right\}\\ &= e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{4}{s^2}-\frac{6}{s^3}-\frac{12}{s^4}\right\}\\ &= e^{-2t}(1+4t-3t^2-2t^3).\end{aligned}$$

- Since,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}.$$

Question 18

Derive the following formulae.

(a) $\mathcal{L}^{-1}\{\frac{1}{s^4+4a^4}\} = \frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$,

(b) $\mathcal{L}^{-1}\{\frac{s}{s^4+4a^4}\} = \frac{1}{2a^2}(\sinh at \sin at)$.

Question 18(a)

Question 18(a)

(a) $\mathcal{L}^{-1}\{\frac{1}{s^4+4a^4}\} = \frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$

Solution:

- $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2as)^2 = (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$.
- By method of partial fractions, we can write

$$\frac{1}{s^4 + 4a^4} = \frac{As + B}{s^2 + 2as + 2a^2} + \frac{Cs + D}{s^2 - 2as + 2a^2}$$

- Multiplying both sides by $(s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$,

$$1 = (As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2).$$

Question 18(a) contd...

- Comparing coefficients both sides, we get,

$$\begin{aligned}
&2a^2A + 2a^2C - 2aB + 2aD = 0\\
&-2aA + B + 2aC + D = 0\\
&A + C = 0\\
&B + D = \frac{1}{2a^2}.
\end{aligned}$$

- On solving, we get, $A = \frac{1}{8a^3},\ C = -\frac{1}{8a^3},\ B = \frac{1}{4a^2},\ D = \frac{1}{4a^2}.$
- Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^4+4a^4}\right\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{8a^3}s + \frac{1}{4a^2}}{s^2+2as+2a^2} + \frac{-\frac{1}{8a^3}s + \frac{1}{4a^2}}{s^2-2as+2a^2}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{8a^3}\left(\frac{s+2a}{(s+a)^2+a^2}\right) + \frac{1}{8a^3}\left(\frac{2a-s}{(s-a)^2+a^2}\right)\right\}
\end{aligned}$$

Question 18(a) contd...

- $$\mathcal{L}^{-1}\left\{\frac{1}{s^4+4a^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{8a^3}\left(\frac{s+a}{(s+a)^2+a^2}\right) + \frac{1}{8a^3}\left(\frac{a}{(s+a)^2+a^2}\right) - \frac{1}{8a^3}\left(\frac{s-a}{(s-a)^2+a^2}\right) + \frac{1}{8a^3}\left(\frac{2a-a}{(s-a)^2+a^2}\right)\right\}$$

- Hence,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^4+4a^4}\right\} &= \frac{1}{8a^3}e^{-at}\cos at + \frac{1}{8a^3}e^{-at}\sin at \\ &\quad + \frac{1}{8a^3}e^{at}\sin at - \frac{1}{8a^3}e^{at}\cos at \\ &= \frac{1}{8a^3}\sin at(e^{at}+e^{-at}) - \frac{1}{8a^3}\cos at(e^{at}-e^{-at}) \\ &= \frac{1}{8a^3}(2\sin at\cosh at - 2\cos at\sinh at) \\ &= \frac{1}{4a^3}(\sin at\cosh at - \cos at\sinh at).\end{aligned}$$

Question 18(b)

(b) $\mathcal{L}^{-1}\{\frac{s}{s^4+4a^4}\} = \frac{1}{2a^2}(\sinh at \sin at)$.

Solution:

- $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2as)^2 = (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$.
- By method of partial fractions, we can write

$$\frac{s}{s^4 + 4a^4} = \frac{As + B}{s^2 + 2as + 2a^2} + \frac{Cs + D}{s^2 - 2as + 2a^2}$$

- Multiplying both sides by $(s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$,

$$s = (As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2).$$

Question 18(b) contd...

- Comparing coefficients both sides, we get,

$$\begin{aligned}
&2a^2A - 2aB + 2aD = 1\\
&-2aA + B + 2aC + D = 0\\
&A + C = 0\\
&B + D = 0.
\end{aligned}$$

- On solving, we get, $A = 0 = C,\ B = \frac{-1}{4a},\ D = \frac{1}{4a}$.
- Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{s}{s^4 + 4a^4}\} &= \mathcal{L}^{-1}\left\{-\frac{1}{4a(s^2 + 2as + 2a^2)} + \frac{1}{4a(s^2 - 2as + 2a^2)}\right\}\\
&= -\frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2 + a^2}\right\} + \frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2 + a^2}\right\}\\
&= -\frac{1}{4a}\frac{\sin at}{a}e^{-at} + \frac{1}{4a}\frac{\sin at}{a}e^{at}
\end{aligned}$$

Question 18(b) contd...

- Hence,

$$\mathcal{L}^{-1}\{\frac{s}{s^4+4a^4}\} = \frac{1}{4a^2}\sin at(e^{at}-e^{-at})$$
$$= \frac{1}{2a^2}\sin at \sinh at.$$

- Since, $\frac{(e^{at}-e^{-at})}{2} = \sinh at$.

Question 19

Solve the following IVPs (using Laplace transform).

(a) $y_1^{'} = -y_1 + y_2,\ y_2^{'} = -y_1 - y_2,\ y_1(0) = 1,\ y_2(0) = 0,$

(b) $y_1^{''} + y_2 = -5\cos 2t,\ y_2^{''} + y_1 = 5\cos 2t$
$y_1(0) = 1,\ y_1^{'}(0) = 1,\ y_2(0) = -1,\ y_2^{'}(0) = 1.$

(c) $y_1^{'} = 2y_1 + 4y_2 + 64tu(t-1),\ y_2^{'} = y_1 + 2y_2,;$
$y_1(0) = -4,\ y_2(0) = -4.$

Question 19(a)

(a) $y_1^{'} = -y_1 + y_2,\ y_2^{'} = -y_1 - y_2,\ y_1(0) = 1,\ y_2(0) = 0$

Solution:

- Consider the IVP: $y_1^{'} = -y_1 + y_2,\ y_2^{'} = -y_1 - y_2,\ y_1(0) = 1,\ y_2(0) = 0.$
- Taking the Laplace transform in above equations, we get

$$\begin{aligned} sY_1(s) - 1 &= -Y_1(s) + Y_2(s), \\ sY_2(s) &= -Y_1(s) - Y_2(s). \end{aligned}$$

- On solving,

$$(s+1)Y_1(s) - Y_2(s) = 1, \tag{7}$$
$$Y_1(s) + (s+1)Y_2(s) = 0. \tag{8}$$

Question 19(a) contd...

- By eliminating $Y_1(s)$,

$$Y_2(s) = -\frac{1}{(s+1)^2+1}$$

- Taking inverse Laplace transform,

$$y_2(t) = -e^{-t}\sin t.$$

- By putting expression for $Y_2(s)$ in (8),

$$Y_1(s) = -(s+1)Y_2(s) = \frac{(s+1)}{(s+1)^2+1}$$

- Taking inverse Laplace transform,

$$y_2(t) = e^{-t}\cos t.$$

Question 19(b)

(b) $y_1'' + y_2 = -5\cos 2t,\ y_2'' + y_1 = 5\cos 2t$
$y_1(0) = 1,\ y_1'(0) = 1,\ y_2(0) = -1,\ y_2'(0) = 1.$

Solution:

- Consider the IVP: $y_1'' + y_2 = -5\cos 2t,\ y_2'' + y_1 = 5\cos 2t$
  $y_1(0) = 1,\ y_1'(0) = 1,\ y_2(0) = -1,\ y_2'(0) = 1.$
- Taking the Laplace transform in above equations, we get

$$s^2 Y_1(s) - s - 1 + Y_2(s) = -\frac{5s}{s^2+4},$$
$$s^2 Y_2(s) + s - 1 + Y_1(s) = \frac{5s}{s^2+4}.$$

Question 19(b) contd...

- On solving,

$$s^2 Y_1(s) + Y_2(s) = s + 1 - \frac{5s}{s^2+4}, \tag{9}$$
$$Y_1(s) + s^2 Y_2(s) = -s + 1 + \frac{5s}{s^2+4}. \tag{10}$$

- By eliminating $Y_2(s)$ ((9) $\times s^2 -$ (10)), we get

$$Y_1(s) = \underbrace{\frac{s^3}{s^4-1}}_{:=I_1} + \underbrace{\frac{s^2}{s^4-1}}_{:=I_2} + \underbrace{\frac{s}{s^4-1}}_{:=I_3} - \underbrace{\frac{1}{s^4-1}}_{:=I_4} - \underbrace{\frac{5s}{(s^2-1)(s^2+4)}}_{:=I_5}$$

- Now, by partial fractions,

$$I_1 = \frac{s^3}{s^4-1} = \frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1}.$$

Question 19(b) contd...

- Multiplying both sides by $(s^2-1)(s^2+1)$ and comparing coefficients,

$$\begin{aligned} A + C &= 1, \\ B + D &= 0, \\ A - C &= 0, \\ B - D &= 0. \end{aligned}$$

- On solving, we get, $A = \frac{1}{2} = C, \; B = 0 = D$.
- Hence,

$$I_1 = \frac{s^3}{s^4-1} = \frac{s}{2(s^2-1)} + \frac{s}{2(s^2+1)}.$$

- Taking inverse Laplace transform,

$$\mathcal{L}^{-1}\{I_1\} = \frac{1}{2}(\cosh t + \cos t).$$

Question 19(b) contd...

- Similarly, by partial fractions,

$$I_2 = \frac{s^2}{s^4 - 1} = \frac{As + B}{s^2 - 1} + \frac{Cs + D}{s^2 + 1}.$$

- Multiplying both sides by $(s^2 - 1)(s^2 + 1)$ and comparing coefficients,

$$\begin{aligned} A + C &= 0, \\ B + D &= 1, \\ A - C &= 0, \\ B - D &= 0. \end{aligned}$$

- On solving, we get, $A = 0 = C, \; B = \frac{1}{2} = D$.
- Hence,

$$I_2 = \frac{s^2}{s^4 - 1} = \frac{1}{2(s^2 - 1)} + \frac{1}{2(s^2 + 1)}.$$

Question 19(b) contd...

- Taking inverse Laplace transform,

$$\mathcal{L}^{-1}\{I_2\} = \frac{1}{2}(\sinh t + \sin t).$$

- Similarly, by partial fractions,

$$I_3 = \frac{s}{s^4 - 1} = \frac{As + B}{s^2 - 1} + \frac{Cs + D}{s^2 + 1}.$$

- Multiplying both sides by $(s^2 - 1)(s^2 + 1)$ and comparing coefficients,

$$\begin{aligned} A + C &= 0, \\ B + D &= 0, \\ A - C &= 1, \\ B - D &= 0. \end{aligned}$$

- On solving, we get, $A = \frac{1}{2},\ C = -\frac{1}{2},\ B = 0 = D.$

Question 19(b) contd...

- Hence,

$$I_3 = \frac{s}{s^4 - 1} = \frac{s}{2(s^2 - 1)} - \frac{s}{2(s^2 + 1)}.$$

- Taking inverse Laplace transform,

$$\mathcal{L}^{-1}\{I_3\} = \frac{1}{2}(\cosh t - \cos t).$$

- Similarly, by partial fractions,

$$I_4 = \frac{1}{s^4 - 1} = \frac{As + B}{s^2 - 1} + \frac{Cs + D}{s^2 + 1}.$$

- Multiplying both sides by $(s^2 - 1)(s^2 + 1)$ and comparing coefficients,

$$\begin{aligned} A + C &= 0, \\ B + D &= 0, \\ A - C &= 0, \\ B - D &= 1. \end{aligned}$$

Question 19(b) contd...

- On solving, we get, $A = 0,\ B = \frac{1}{2},\ C = 0,\ D = -\frac{1}{2}$.
- Hence,

$$I_4 = \frac{1}{s^4 - 1} = \frac{1}{2(s^2 - 1)} - \frac{1}{2(s^2 + 1)}.$$

- Taking inverse Laplace transform,

$$\mathcal{L}^{-1}\{I_4\} = \frac{1}{2}(\sinh t - \sin t).$$

- Now, by partial fractions,

$$\begin{aligned} I_5 = \frac{5s}{(s^2 - 1)(s^2 + 4)} &= s\left\{\frac{1}{(s^2 - 1)} - \frac{1}{(s^2 + 4)}\right\} \\ &= \frac{s}{(s^2 - 1)} - \frac{s}{(s^2 + 4)} \end{aligned}$$

Question 19(b) contd...

- Taking inverse Laplace transform,

$$\mathcal{L}^{-1}\{I_5\} = \cosh t - \cos 2t.$$

- Combining all the inverse Laplace transforms, we get $y_1(t)$

$$\begin{aligned} y_1(t) &= \mathcal{L}^{-1}\{I_1\} + \mathcal{L}^{-1}\{I_2\} + \mathcal{L}^{-1}\{I_3\} - \mathcal{L}^{-1}\{I_4\} - \mathcal{L}^{-1}\{I_5\} \\ &= \frac{1}{2}(\cosh t + \cos t) + \frac{1}{2}(\sinh t + \sin t) \\ &+ \frac{1}{2}(\cosh t - \cos t) - \frac{1}{2}(\sinh t - \sin t) - (\cosh t - \cos 2t) \\ &= \sin t + \cos 2t. \end{aligned}$$

- By putting expression for $Y_1(s)$ in (10),

$$\begin{aligned} Y_2(s) &= \frac{5s}{s^2(s^2+4)} - \frac{1}{s} + \frac{1}{s^2} - \frac{Y_1(s)}{s^2} \\ &= \frac{5}{4}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} - \frac{1}{s} + \frac{1}{s^2} - \frac{Y_1(s)}{s^2}. \end{aligned}$$

Question 19(b) contd...

- Taking inverse Laplace transform,

$$y_2(t) = \frac{5}{4}(1 - \cos 2t) - 1 + t - \mathcal{L}^{-1}\left\{\frac{Y_1(s)}{s^2}\right\},$$

- where,

$$\frac{Y_1(s)}{s^2} = \underbrace{\frac{s}{s^4-1}}_{:=P_1} + \underbrace{\frac{1}{s^4-1}}_{:=P_2} + \underbrace{\frac{1}{s(s^4-1)}}_{:=P_3} - \underbrace{\frac{1}{s^2(s^4-1)}}_{:=P_4} - \underbrace{\frac{5}{s(s^2-1)(s^2+4)}}_{:=P_5}.$$

- By partial fraction method,

$$\mathcal{L}^{-1}\{P_1 + P_2\} = \frac{1}{2}(\cosh t - \cos t) + \frac{1}{2}(\sinh t - \sin t).$$

- By Convolution theorem,

$$\mathcal{L}^{-1}\{P_3\} = 1 * \frac{1}{2}(\sinh t - \sin t) = -1 + \frac{1}{2}(\cosh t + \cos t).$$

Question 19(b) contd...

- Again by Convolution theorem,

$$\mathcal{L}^{-1}\{P_4\} = t * \frac{1}{2}(\sinh t - \sin t) = \frac{1}{2}(-2t + \sinh t + \sin t).$$

- Using partial fraction method,

$$\begin{aligned} P_5 &= \frac{5}{s(s^2-1)(s^2+4)} \\ &= \frac{1}{s}\left\{\frac{1}{s^2-1} - \frac{1}{s^2+4}\right\} \\ &= \frac{1}{s(s^2-1)} - \frac{1}{s(s^2+4)}. \end{aligned}$$

- Taking Laplace inverse using Convolution theorem,,

$$\mathcal{L}^{-1}\{P_5\} = 1 * \sinh t - 1 * \frac{\sin 2t}{2} = \frac{-3}{4} + \cosh t - \frac{\cos 2t}{4}.$$

- Combining all the inverse Laplace transforms, we get $y_2(t)$

$$\begin{aligned} y_2(t) &= \frac{5}{4}(1-\cos 2t) - 1 + t - (\mathcal{L}^{-1}\{P_1\} + \mathcal{L}^{-1}\{P_2\} \\ &\quad + \mathcal{L}^{-1}\{P_3\} - \mathcal{L}^{-1}\{P_4\} - \mathcal{L}^{-1}\{P_5\}) \\ &= \frac{1}{2} + \sin t - \frac{3}{2}\cos 2t. \end{aligned}$$

Question 19(c)

(c) $y_1^{'} = 2y_1 + 4y_2 + 64tu(t-1),\ y_2^{'} = y_1 + 2y_2,;\ y_1(0) = -4,\ y_2(0) = -4.$

Solution:

- Consider the IVP: $y_1^{'} = 2y_1 + 4y_2 + 64tu(t-1),\ y_2^{'} = y_1 + 2y_2,;$ $y_1(0) = -4,\ y_2(0) = -4.$
- Now, the Laplace transform of

$$\begin{aligned}
\mathcal{L}\{64tu(t-1)\} &= 6\int_1^{\infty} e^{-st}tdt \\
&= 64\left[\frac{te^{-st}}{-s} - \int_1^{\infty}\frac{e^{-st}}{-s}dt\right]_1^{\infty} \\
&= 64\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^{\infty} \\
&= \frac{64e^{-s}}{s^2}(s+1).
\end{aligned}$$

Question 19(c) contd...

- Taking the Laplace transform both sides in given IVP, we get

$$sY_1(s) + 4 = 2Y_1(s) + 4Y_2(s) + \frac{64e^{-s}}{s^2}(s+1),$$
$$sY_2(s) + 4 = Y_1(s) + 2Y_2(s).$$

- On solving,

$$(s-2)Y_1(s) - 4Y_2(s) = \frac{64e^{-s}}{s^2}(s+1) - 4, \tag{11}$$
$$Y_1(s) + (s-2)Y_2(s) = -4. \tag{12}$$

Question 19(c) contd...

- By eliminating $Y_1(s)$ $((11) - \frac{(s-2)}{4}(12))$,

$$\begin{aligned} Y_2(s) &= \frac{-256e^{-s}(s+1)}{s^2((s-2)^2+16)} + \frac{16}{(s-2)^2+16} - \frac{4(s-2)}{(s-2)^2+16} \\ &= \frac{-256e^{-s}}{s((s-2)^2+16)} + \frac{-256e^{-s}}{s^2((s-2)^2+16)} \\ &+ \frac{16}{(s-2)^2+16} - \frac{4(s-2)}{(s-2)^2+16} \end{aligned}$$

- Taking inverse Laplace transform and using Convolution theorem,

$$\begin{aligned} y_2(t) &= -256\left(1 * e^{2(t-1)}\frac{\sin 4(t-1)}{4}\right)u(t-1) \\ &- 256\left((t-1) * e^{2(t-1)}\frac{\sin 4(t-1)}{4}\right)u(t-1) + 4e^{2t}\sin 4t - 4e^{2t}\cos 4t. \end{aligned}$$

- By putting expression for $Y_2(s)$ in (11),

$$
\begin{aligned}
Y_1(s) &= \frac{64e^{-s}(s+1)}{s^2(s-2)} - \frac{4}{s-2} + \frac{4}{s-2}Y_2(s) \\
&= \frac{64e^{-s}}{s(s-2)} + \frac{64e^{-s}}{s^2(s-2)} - \frac{4}{s-2} - \frac{1024e^{-s}(s+1)}{s^2(s-2)((s-2)^2+16)} \\
&+ \frac{64}{(s-2)((s-2)^2+16)} - \frac{16}{(s-2)^2+16}.
\end{aligned}
$$

- Taking inverse Laplace transform and using Convolution theorem, we get,

$$
\begin{aligned}
y_2(t) = 64(1 * e^{2(t-1)})u(t-1) + 64((t-1) * e^{2(t-1)})u(t-1) \\
- 4e^{2t} - A + 64(e^{2t} * \frac{e^{2t}\sin 4t}{4}) - 4e^{2t}\sin 4t,
\end{aligned}
$$

Question 19(c) contd...

- where,

$$
\begin{aligned}
A &= \mathcal{L}^{-1}\left\{\frac{1024e^{-s}(s+1)}{s^2(s-2)((s-2)^2+16)}\right\} \\
&= 256\mathcal{L}^{-1}\left\{\frac{e^{-s}(s+1)}{s^2(s-2)}.\frac{4}{((s-2)^2+16)}\right\} \\
&= 256\left(\left\{\frac{-3}{4}-\frac{(t-1)}{2}+\frac{3}{4}e^{2(t-1)}\right\} * e^{2(t-1)}\sin 4(t-1)\right)u(t-1).
\end{aligned}
$$