MTL101::Linear Algebra and Differential Equations

Tutorial 3

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Question 1

Question 1

Find a basis of the row space of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{pmatrix}^t.$$

Recall:

- Let $A \in M_{m\times n}(\mathbb{F}), R_i$. Denote the i-th row $(1 \leq i \leq m)$ by $R_i \in M_{1\times n}(\mathbb{F})$. Then $Span(R_1, R_2, \ldots R_m)$ is called row space of A.
- Dimension of row space of $A =$ row rank of A.
- If a matrix is row echelon matrix, then non zero rows form a basis of the row space.

Solution:

- $$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix}$$
$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$
- Hence, the required basis of row space is $\{(1,0,-1),(0,1,2)\}$.

Question 1(b)

Solution:

- $$\begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_4 \to R_4 - 5R_1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 1 & -6 \\ 0 & 1 & -14 \end{pmatrix}$$
$$\xrightarrow[R_4 \to R_4 - R_3]{R_2 \to R_2 - 3R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 19 \\ 0 & 1 & -6 \\ 0 & 0 & -8 \end{pmatrix} \xrightarrow{R_4 \to R_4 + \frac{8}{19} R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 19 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 19 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{19} R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
- Hence, the required basis is $\{(1, 0, 3), (0, 1, -6), (0, 0, 1)\}$.

Question 2

For $i \in \{1, 2, \ldots, n\}$, define $p_i : \mathbb{F}^n \to \mathbb{F}$ by $p_i(x_1, x_2, \ldots, x_n) = x_i$ (the i-th projection).

(a) Show that it is a linear transformation.

(b) If $T : \mathbb{F}^n \to \mathbb{F}$ is a linear transformation then it is an $\mathbb{F}$ -linear combination of the projections, that is, $T = a_1 p_1 + a_2 p_2 \cdots + a_n p_n$ for $a_1, \ldots a_n \in \mathbb{F}$.

(c) Further, show that $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation if and only if for each $i \in \{1, 2, \ldots, n\}$, the composition $p_i \circ S : \mathbb{F}^m \to \mathbb{F}$ is a linear transformation.

(d) If $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation then $S(x_1, x_2, \ldots x_m)$ $= (y_1, y_2, \ldots y_n)$ where $y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$ *for* $a_{ij} \in \mathbb{F}$ with $(1 \leq i \leq n, 1 \leq j \leq m)$.

Question 2(a)

Question 2(a)

For $i \in \{1, 2, \ldots, n\}$, define $p_i : \mathbb{F}^n \to \mathbb{F}$ by $p_i(x_1, x_2, \ldots, x_n) = x_i$ (the i-th projection).
(a) Show that it is a linear transformation.

Recall:

Suppose V and W are vector spaces over the same field $\mathbb{F}$. A map $T : V \to W$ is called a linear transformation if

$$T(au + bv) = aT(u) + bT(v)$$

for any $a, b \in \mathbb{F}$ and any $u, v \in V$.

Solution:

- Let $u = (x_1, x_2, \ldots, x_n), v = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$, then

$$\begin{aligned} p_i(au + bv) &= p_i(a(x_1, x_2, \ldots x_n) + b(y_1, y_2, \ldots y_n)) \\ &= p_i(ax_1 + by_1, ax_2 + by_2, \ldots, ax_n + by_n) \\ &= ax_i + by_i \\ &= ap_i(x_1, x_2, \ldots x_n) + bp_i(y_1, y_2, \ldots y_n) \\ &= ap_i(u) + bp_i(v). \end{aligned}$$

- Hence, p_i is a linear transformation for $i \in \{1, 2, \ldots, n\}$.

Question 2(b)

Question 2(b)

For $i \in \{1, 2, \ldots, n\}$, define $p_i : \mathbb{F}^n \to \mathbb{F}$ by $p_i(x_1, x_2 \ldots, x_n) = x_i$ (the i-th projection).
(b) If $T : \mathbb{F}^n \to \mathbb{F}$ is a linear transformation then it is an $\mathbb{F}$ -linear combination of the projections, that is, $T = a_1 p_1 + a_2 p_2 \cdots + a_n p_n$ for $a_1, \ldots a_n \in \mathbb{F}$.

Question 2(b)

Solution:

- Let $u = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$.
- For the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{F}^n$, we have $u = (x_1, x_2, \ldots, x_n) = (x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$.
- Since, T is a linear transformation,

$$\begin{aligned} T(x_1, x_2, \ldots, x_n) &= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \in \mathbb{F}, \end{aligned}$$

- This implies, $T(e_1), T(e_2), \ldots, T(e_n) \in \mathbb{F}$.
- For $a_i \in \mathbb{F}$ $(1 \leq i \leq n)$, suppose $T(e_i) = a_i$, then

$$\begin{aligned} T(x_1, x_2, \ldots, x_n) &= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \\ &= a_1 p_1(x_1, x_2, \ldots, x_n) + \cdots + a_n p_n(x_1, x_2, \ldots, x_n) \\ &= (a_1 p_1 + a_2 p_2 + \cdots + a_n p_n)(x_1, x_2, \ldots, x_n). \end{aligned}$$

- Hence, $T = a_1 p_1 + a_2 p_2 \cdots + a_n p_n$ for $a_1, \ldots a_n \in \mathbb{F}$.

Question 2(c)

For $i \in \{1, 2, \ldots, n\}$, define $p_i : \mathbb{F}^n \to \mathbb{F}$ by $p_i(x_1, x_2 \ldots, x_n) = x_i$ (the i-th projection).
(c) Further, show that $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation if and only if for each $i \in \{1, 2, \ldots n\}$, the composition $p_i \circ S : \mathbb{F}^m \to \mathbb{F}$ is a linear transformation.

Question 2(c)

Solutions:

- Let S is a linear transformation.
- Let $u, v \in \mathbb{F}^m$ and $a, b \in \mathbb{F}$. Then,

$$\begin{aligned}
(p_i \circ S)(au + bv) &= p_i(S(au + bv)) \\
&= p_i(aS(u) + bS(v)) \\
&= ap_i(S(u)) + bp_i(S(v)) \\
&= a(p_i \circ S)(u) + b(p_i \circ S)(u)
\end{aligned}$$

- Hence, the composition $p_i \circ S$ is a linear transformation for $i \in \{1, 2, \ldots, n\}$.

Question 2(c) contd...

Converse Part

- Let the composition $p_i \circ S$ is a linear transformation for each $i \in \{1, 2, \dots n\}$.
- For $v \in \mathbb{R}^n, v = (p_1(v), p_2(v), \dots, p_n(v))$.
- Let $a, b \in \mathbb{F}$ and $u, w \in \mathbb{F}^m$, Then

$$\begin{aligned} S(au + bv) &= (p_1(S(au + bv)), p_2(S(au + bv)), \dots, p_n(S(au + bv))) \\ &= (p_1 \circ S(au + bv), p_2 \circ S(au + bv), \dots, p_n \circ S(au + bv)) \\ &= (a(p_1 \circ S))(u), a(p_2 \circ S))(u), \dots, a(p_n \circ S))(u)) \\ &+ ((b(p_1 \circ S))(v), b(p_2 \circ S))(v), \dots, b(p_n \circ S))(v))) \\ &= aS(u) + bS(v) \end{aligned}$$

- Hence, S is a linear transformation.

Question 2(d)

Question 2(d)

For $i \in \{1, 2, \ldots, n\}$, define $p_i : \mathbb{F}^n \to \mathbb{F}$ by $p_i(x_1, x_2 \ldots, x_n) = x_i$ (the i-th projection).

(d) If $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation then $S(x_1, x_2, \ldots x_m)$ $= (y_1, y_2, \ldots y_n)$ where $y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$ *for* $a_{ij} \in \mathbb{F}$ with $(1 \leq i \leq n, 1 \leq j \leq m)$.

Question 2(d)

Solution:

- Let $u = (x_1, x_2, \ldots, x_m) \in \mathbb{F}^m$.
- For the standard basis $\{e_1, e_2, \ldots, e_m\}$ of $\mathbb{F}^m$, we know $u = (x_1, x_2, \ldots, x_m) = (x_1 e_1 + x_2 e_2 + \cdots + x_m e_m)$.
- Since, S is a linear transformation,

$$\begin{aligned} S(x_1, x_2, \ldots, x_m) &= S(x_1 e_1 + x_2 e_2 + \cdots + x_m e_m) \\ &= x_1 S(e_1) + x_2 S(e_2) + \cdots + x_m S(e_m) \end{aligned}$$

- This implies, $S(e_1), \ldots, S(e_m) \in \mathbb{F}^n$.
- Let

$$\begin{aligned} S(e_1) &= (a_{11}, \ldots, a_{n1}), \\ S(e_2) &= (a_{12}, \ldots, a_{n2}), \\ &\cdots \\ S(e_m) &= (a_{1m}, \ldots, a_{nm}) \end{aligned}$$

Question 2(d) contd...

- Hence,

$$
\begin{aligned}
S(x_1, x_2, \ldots, x_m) &= x_1 S(e_1) + x_2 S(e_2) + \cdots + x_m S(e_m) \\
&= x_1(a_{11}, \ldots, a_{n1}) + \cdots + x_m(a_{1m}, \ldots, a_{nm}) \\
&= (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m, \\
&\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m, \\
&\quad \ldots, a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m).
\end{aligned}
$$

- Hence, $S(x_1, x_2, \ldots, x_m) = (y_1, y_2, \ldots y_n)$, where, $y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$ *for* $a_{ij} \in \mathbb{F}$.

Question 3

Find the rank and nullity of the following linear transformations. Also write a basis of the range space in each case.

(a) $T : \mathbb{F}^3 \to \mathbb{F}^3$ defined by $T(x, y, z) = (x + y + z, x - y + z, x + z)$.

(b) Assume that $0 \leq m \leq n$. $T : \mathbb{F}^n \to \mathbb{F}^m$ defined by $T(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_m)$.

Question 3(a)

(a) $T : \mathbb{F}^3 \to \mathbb{F}^3$ defined by $T(x, y, z) = (x + y + z, x - y + z, x + z)$.

Recall: Suppose $T : V \to W$ is a linear transformation. Then

- The set $ker(T) := \{v \in V : T(v) = 0\}$ is called the null space of T.
- The set $T(V) := \{T(v) : v \in V\}$ is called the range space of T.
- The dimension of $ker(T)$ is called the nullity of T and the dimension of $T(V)$ is called the rank of T.
- Rank-Nullity theorem: rank(T)+nullity(T)=dim(V).

Question 3(a)

Solution:

- The null space of T is defined as $\{(x, y, z) : x + y + z = 0, x - y + z = 0, x + z = 0\}$, which is the solution space of certain homogeneous system of linear equations,
- The null space can be expressed as the solution space of $Ax = b$, i.e.,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- On solving the above system for (x, y, z), we get,

$$x + y + z = 0, \ y = 0, \ x + z = 0$$

Question 3(a) contd...

- Hence, null space is given by $\{(-a, 0, a), a \in \mathbb{R}\}$ and $(1, 0, -1)$ is a basis of Ker(T).
- Hence, nullity$(T) = 1$.
- By Rank-nullity theorem,

$$\begin{aligned} Rank(T) &= dim(\mathbb{R}^3) - Nullity(T) \\ &= 3 - 1 = 2. \end{aligned}$$

Question 3(a) contd...

- Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Then, $T(\mathbb{R}^3)$ is generated by $\{T(e_1), T(e_2), T(e_3)\}$ given by,

$$\begin{aligned} T(1,0,0) &= (1,1,1), \\ T(0,1,0) &= (1,-1,0), \\ T(0,0,1) &= (1,1,1). \end{aligned}$$

- Consider matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ which is obtained by arranging the images of standard basis row-wise.

- Hence, the row space of A generates range space of T i.e., basis of row space of A is a basis of range space.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 + \frac{1}{2} R_2}$$

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{-1}{2} R_2} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

- Hence, the basis of range space of T is $\{(1, 0, \frac{1}{2}), (0, 1, \frac{1}{2})\}$.

Question 3(b)

Find the rank and nullity of the following linear transformations. Also write a basis of the range space in each case.
(b) Assume that $0 \leq m \leq n$. $T : \mathbb{F}^n \to \mathbb{F}^m$ defined by $T(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_m)$.

Question 3(b)

Solution:

- Let $\{e_1, e_2, \ldots, e_n\}$ be standard basis of $\mathbb{F}^n$ and $\{e_1', e_2', \ldots, e_m'\}$ be standard basis of $\mathbb{F}^m$ where $m \leq n$.
- **Case(i).** Let $m < n$.
  - Then, by definition of T, we have

$$
\begin{aligned}
T(e_1) &= e_1', \\
T(e_2) &= e_2', \\
&\cdots \\
T(e_m) &= e_m', \\
T(e_{m+1}) &= \cdots = T(e_n) = 0.
\end{aligned}
$$

Question 3(b) contd...

- - The matrix representation of T is given by,

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}_{m\times n}$$

  - Rank(T)=Rank(A)= Number of non-zero rows in row-echelon form.
  - Hence, Rank(T)=m and using Rank-Nullity theorem nullity(T)=n-m.
  - Also, $\{e_1', e_2', \dots, e_m'\}$ is the basis for range space.
- **Case(ii).** Let $m = n$.
  - Then, matrix representation of T is given by identity matrix.
  - Hence, rank(T)=n, nullity(T)=0 and $\{e_1', e_2', \dots, e_n'\}$ is the basis for range space.

Question 4

Write the matrix representations of the linear operators with respect to the ordered basis $\mathcal{B}$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x, y), \mathcal{B} = \{(1, 1), (1, -1)\}$

(b) $\mathcal{D} : \mathcal{P}_{n+1} \to \mathcal{P}_{n+1}$ such that
$D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + na_n x^{n-1}, \mathcal{B} = \{1, x, \ldots, x^n\}$.

(c) $T : M_2(\mathcal{F}) \to M_2(\mathcal{F}), T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+w & z \\ z+w & x \end{pmatrix}, \mathcal{B} =$
$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

Question 4(a)

Write the matrix representations of the linear operators with respect to the ordered basis $\mathcal{B}$.
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x, y), \mathcal{B} = \{(1, 1), (1, -1)\}$

Question 4(a)

Solution:

- Let $\mathcal{B} = \{(1,1),(1,-1)\}$ is the ordered basis of $\mathbb{R}^2$ and $T(x,y) = (x,y)$.
- As,

$$T(1,1) = (1,1) = 1(1,1) + 0(1,-1)$$
$$T(1,-1) = (1,-1) = 0(1,1) + 1(1,-1)$$

- Hence, the matrix representation of T is given by,

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 4(b)

Write the matrix representations of the linear operators with respect to the ordered basis $\mathcal{B}$.

(b) $\mathcal{D} : \mathcal{P}_n \to \mathcal{P}_n$ such that $D(a_0 + a_1 x + \cdots + a_n x^n)$
$= a_1 + 2a_2 x + \cdots + n a_n x^{n-1}, \mathcal{B} = \{1, x, \ldots, x^n\}$.

Question 4(b)

Solution:

- Let $\mathcal{B} = \{1, x, \ldots, x^n\}$ is the ordered basis of $\mathbb{R}^2$ and $D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$.
- As,

$$\begin{aligned}
\mathcal{D}(1) &= 0 = 0.1 + 0.x + \cdots + 0.x^n \\
\mathcal{D}(x) &= 1 = 1.1 + 0.x + \cdots + 0.x^n \\
\mathcal{D}(x^2) &= 2x = 0.1 + 2.x + \cdots + 0.x^n \\
&\vdots \\
\mathcal{D}(x^n) &= nx^{n-1} = 0.1 + 0.x + \cdots + n.x^{n-1} + 0.x^n.
\end{aligned}$$

Question 4(b)

- Hence, the matrix representation of T is given by,

$$[\mathcal{D}]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n+1 \times n+1}$$

Question 4(c)

Question 4(c)

Write the matrix representations of the linear operators with respect to the ordered basis B.

(c) $T : M_2(\mathcal{F}) \to M_2(\mathcal{F})$, $T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+w & z \\ z+w & x \end{pmatrix}$,

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Question 4(c)

Solution:

- Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is the ordered basis and $T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+w & z \\ z+w & x \end{pmatrix}$.
-

$$
\begin{aligned}
T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Question 4(c)

- $$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- $$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 4(c)

- $$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$
  $$= 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
- Hence, the matrix representation of T is given by, $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Question 5

Suppose $dimV = dimW < \infty$ and $T : V \to W$ is a linear transformation. Show that the following statements are equivalent

(a) T is an isomorphism.

(b) T is injective (i.e., one to one).

(c) $kerT = 0$.

(d) T is surjective (i.e., onto).

Question 5

Recall:

Let $T : V \to W$ is a linear transformation. Then

- T is injective if its null space is the zero space.
- T is surjective if range space=W.
- If U is a subspace of V such that dim(U)=dim(V), then U=V.
- A linear transformation T is said to be an isomorphism if it is one-one and onto.

Question 5

Solution:

- $(a) \implies (b)$, obvious.
- $(b) \implies (c)$
  - Suppose T is injective.
  - This implies $T(x) = T(y) \implies x = y$ for $x, y \in V$
  - If possible, suppose $KerT \neq 0$, then $\exists$ a non-zero vector $x \in KerT$ such that $T(x) = 0 = T(0) \implies x = 0$, A CONTRADICTION.
  - Hence, $KerT = 0$.
- $(c) \implies (d)$
  - Suppose $KerT = 0$, hence, nullity(T)=0.
  - Since dim(V)=dim(W), by Rank-nullity theorem, we get rank(T)=dim(V)=dim(W).
  - Since, range(T) is a subspace of W and dim(range(T))=dim(W),
  - This implies range space of T is W itself.
  - Hence, T is surjective.

Question 5

Solution:

- $(d) \implies (a)$
  - Let $T : V \to W$ is a surjective linear transformation and dim(V)=dim(W).
  - Hence, rank(T)=dim(W).
  - By Rank-nullity theorem, we have nullity(T)=0, i.e. ker(T)={0}
  - Thus for any x,y $\in$ V such that T(x)=T(y), we have
    - T(x) - T(y) = 0
    - i.e., T(x-y) = 0,
    - $\implies$ x-y $\in$ ker(T)={0}
    - $\implies$ $x - y = 0$, and hence x=y.
  - This implies, T is one to one.
  - Hence, T is a one-one, onto linear transformation. Hence, T is an isomorphism.

Question 6

Suppose $m > n$. Justify the following statements:
(a) There is no one to one (injective) $\mathbb{R}$-linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.
(b) There is no onto (surjective) $\mathbb{R}$-linear tranformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Question 6(a)

Suppose $m > n$. Justify the following statements:
(a) There is no one to one (injective) $\mathbb{R}$-linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Solution:

- If possible, suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a one-one linear transformation and $m > n$.
- Hence, nullity(T)=0.
- By Rank-nullity theorem, we have

$$\begin{aligned} Rank(T) + Nullity(T) &= dim(\mathbb{R}^m) \\ Rank(T) + 0 &= m > n \end{aligned}$$

- Since, Range(T) is a subspace of $\mathbb{R}^n$. Hence, rank(T)$\leq n$.
- Hence, the assumption was wrong, there is no one to one $\mathbb{R}$-linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Question 6(b)

Suppose $m > n$. Justify the following statements:
(b) There is no onto (surjective) $\mathbb{R}$-linear tranformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Solution:

- If possible, suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a onto linear transformation and $m > n$.
- Hence, rank(T)=m.
- By Rank-nullity theorem, we have

$$\begin{aligned} Rank(T) + Nullity(T) &= dim(\mathbb{R}^n) \\ m + nullity(T) &= n < m \\ nullity(T) &= n - m < 0 \end{aligned}$$

- Since, null-space(T) is a subspace of $\mathbb{R}^n$. Hence, $0 \leq$nullity(T)$\leq n$.
- Hence, the assumption was wrong, there is no onto $\mathbb{R}$-linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Question 7

Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (x + y, x)$,
(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, x)$,
(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, -x)$
(d) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$ with $T(x, y) = (y, -x)$.
(e) $T : \mathbb{C}^n \to \mathbb{C}^n$ with $T(x_1, x_2, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$.
(f) $T : \mathbb{C}^2 \to \mathbb{C}^2$ with $T(z_1, z_2) = (z_1 - 2z_2, z_1 + 2z_2)$.

Question 7

Recall:

Suppose $T : V \to V$ is a linear operator on a vector space V .

- A scalar λ is said to be an eigenvalue of T if there is a nonzero vector $v \in V$ such that $T(v) = \lambda v$ and v is called an eigenvector of T associated to the eigenvalue λ.
- Collection of all eigen-vectors v corresponding to λ, along with the 0 vector, is called the eigen space of λ.
- Eigen-values of T are the root of the characteristic polynomial $|\lambda I - A|$, where A is the matrix representation of T with respect to the standard basis.

Question 7(a)

Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operator. $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x; y) = (x + y, x)$,

Solution:

- Consider the standard ordered basis $\mathcal{B} = \{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$.
- The matrix representation of T with respect of $\mathcal{B}$ is $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
- The characteristic polynomial of A is given by $det(\lambda I - A) = 0$,

$$\begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 0 \end{vmatrix} = 0 \implies \lambda^2 - \lambda - 1 = 0$$

$$\implies \lambda = \frac{1 \pm \sqrt{5}}{2}$$

Question 7(a) contd...

- Hence, $\lambda = \frac{1 \pm \sqrt{5}}{2}$ are the eigen-values of T.
- If X is eigen vector corresponding to eigen value λ, then

$$AX = \lambda X$$
$$(A - \lambda I)X = 0$$
$$\Longrightarrow \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- For $\lambda_1 = \frac{1+\sqrt{5}}{2}$, the eigen-vector X_1 is the non-zero solution of

$$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Question 7(a) contd...

- On solving for x, y, we get $X_1 = \begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_1} = \{aX_1; a \in \mathbb{R}\}$. Hence, dimension of eigen space for λ_1 is 1.
- For $\lambda_2 = \frac{1-\sqrt{5}}{2}$, the eigen-vector X_2 is the solution of

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- On solving for x, y, we get $X_2 = \begin{pmatrix} 1-\sqrt{5} \\ 2 \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_2} = \{bX_2; b \in \mathbb{R}\}$. Hence, dimension of eigen space for λ_2 is 1.

Question 7(b)

Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators. (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x;y) = (y,x)$,

Solution:

- Consider the standard basis $B = \{(1,0),(0,1)\}$ for $\mathbb{R}^2$.
- The matrix representation of T with respect to $\mathcal{B}$ is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- Solving $det(\lambda I - A) = 0$, we get the characteristic polynomial $\lambda^2 - 1 = 0$.
- Hence, $\lambda = \pm 1$ are the eigen values of T.
- If X is eigen vector corresponding to eigen value λ, then

$$(A - \lambda I)X = 0$$

$$\implies \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Question 7(b) contd...

- For $\lambda = 1$, the eigen-vector X_1 is the solution of

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$-x + y = 0, \; x - y = 0$$

$$\implies x - y = 0 \implies x = y.$$

- Hence, $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigen vector corresponding to eigen value $\lambda = 1$.
- Hence, eigen space $W_{\lambda_1} = \{aX_1; a \in \mathbb{R}\}$. Hence, dimension of eigen space for λ_1 is 1.
- Similarly, the eigen-vector corresponding to $\lambda_2 = -1$ is $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_2} = \{bX_2; b \in \mathbb{R}\}$. Hence, dimension of eigen space for λ_2 is 1.

Question 7(c)

Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators.
(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x; y) = (y, -x)$

Solution:

- The matrix representation of T with respect of standard basis $\mathcal{B}$ is $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- Hence, the characteristic polynomial is $\lambda^2 + 1 = 0$.
- Above polynomial does not have real root. Hence, the linear transformation does not have real eigen values.

Question 7(d)

Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators. (d) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$ with $T(x; y) = (y, -x)$.

Solution:

- Consider the standard basis $\mathcal{B} = (1, 0), (0, 1)$ of $\mathbb{C}^2(\mathbb{C})$.
- Then, matrix representation of T with respect to standard basis is $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- The characteristic polynomial of T is $\lambda^2 + 1 = 0$.
- Hence, the eigen values of T are $\lambda = \pm i$.

Question 7(d) contd...

- For $\lambda = i$, the eigen-vector X_1 is given by

$$(A - iI)X_1 = 0$$
$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\implies -ix + y = 0, \; -x - iy = 0$$
$$\implies ix = y$$

- Hence eigen vector $X_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_1} = \{aX_1; a \in \mathbb{C}\}$. Hence, dimension of eigen space for λ_1 is 1.
- Similarly, the eigen-vector corresponding to $\lambda_2 = -i$ is $X_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_2} = \{bX_2; b \in \mathbb{C}\}$. Hence, dimension of eigen space for λ_2 is 1.

Question 7(e)

$T : \mathbb{C}^n \to \mathbb{C}^n$ with $T(x_1, x_2, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$.

Solution:

- The standard basis of $\mathbb{C}^n(\mathbb{C})$ is given by $\mathcal{B} = \{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$.
- Then, matrix representation of T with respect to $\mathcal{B}$ is
$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$
(This type of matrix is known as companion matrix)

Question 7(e)

- The characteristic polynomial is given by

$$det(T-\lambda I) = \begin{vmatrix} -\lambda & 0 & \cdots & 0 & 1 \\ 1 & -\lambda & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^n - 1 = 0.$$

- Hence, the eigen values of T are the $n-$th root of unity.
- Note that, $\lambda^n = 1$ for any eigenvalue λ of the above matrix.

Question 7(e)

- Let $X = (x_1, x_2, \ldots, x_n)^t$ be an eigen-vector for the general eigenvalue λ.
- Then $X \neq 0$, i.e., atleast one of x_i is non-zero. For simplicity, we are assuming $x_1 \neq 0$ (as each λ is non-zero, we can divide by λ in case of need and can go ahead with the similar proof)
- and $TX = \lambda X$. On comparing both sides we get

$$x_n = \lambda x_1,\ x_1 = \lambda x_2,\ x_2 = \lambda x_3, \ldots, x_{n-1} = \lambda x_n.$$

- Writing each variable in terms of x_1 to get

$$x_n = \lambda x_1,\ x_{n-1} = \lambda^2 x_1,\ x_{n-2} = \lambda^3 x_1,\ \ldots,$$

$$x_2 = x_{n-(n-2)} = \lambda^{n-2+1} x_1 = \lambda^{n-1} x_1,\quad x_1 = \lambda^n x_1 = 1x_1 = x_1.$$

- Thus, $X = x_1(\lambda^n, \lambda^{n-1}, \ldots, \lambda^2, \lambda)^t$ is an eigenvector corresponding to eigen value λ. (as $x_1 \neq 0$, we can divide by x_1 to get a vector free from x_1)

Question 7(f)

Question 7(f)

Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators. (f) $T : \mathbb{C}^2 \to \mathbb{C}^2$ with $T(z_1, z_2) = (z_1 - 2z_2, z_1 + 2z_2)$.

Solution:

- The matrix representation of T with respect to $\mathcal{B}$ is
  $\implies [T] = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$.
- Hence, the eigen values of T are given by

$$det(\lambda I - A) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda + 4 = 0$$

$$\implies \lambda = \frac{3 \pm i\sqrt{7}}{2}$$

.

- For $\lambda_1 = \frac{3+i\sqrt{7}}{2}$, the eigen-vector X_1 is given by,

$$
\begin{aligned}
(A - \lambda_1 I)X_1 &= 0 \\
\begin{pmatrix} -\frac{1+i\sqrt{7}}{2} & -2 \\ 1 & \frac{1-i\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\implies -\frac{1+i\sqrt{7}}{2}x - 2y &= 0.
\end{aligned}
$$

- Hence, eigen-vector $X_1 = \begin{pmatrix} -4 \\ 1+i\sqrt{7} \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_1} = \{aX_1; a \in \mathbb{C}\}$. Hence, dimension of eigen space for λ_1 is 1.

Question 7(f)

- Similarly, the eigen-vector corresponding to $\lambda_2 = \frac{3-i\sqrt{7}}{2}$ is $X_2 = \begin{pmatrix} 1+i\sqrt{7} \\ -2 \end{pmatrix}$.
- Hence, eigen space $W_{\lambda_2} = \{bX_2; b \in \mathbb{C}\}$. Hence, dimension of eigen space for λ_2 is 1.

Question 8

Question 8

Find a basis B such that $[T]_B$ is a diagonal matrix in case T is diagonalizable. Find P such that $[T]_B = P[T]_S P^{-1}$ where S is the standard basis in each case.

(a) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (y, -x)$.

(b) $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by
$T(x, y, z) = (5x - 6y - 6z, -x + 4y + 2z, 3x - 6y - 4z)$.

(c) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (xcos\theta + ysin\theta, -xsin\theta + ycos\theta)$.

Question 8

Recall:

A linear operator $T : V \to V$ is diagonalizable (over $\mathbb{F}$)

- iff V has a basis B with respect to which the matrix of T is diagonal.
- if and only if $dimV$ is equal to the sum of the dimensions of the eigen spaces of T.
- if all eigen values are distinct.

Let T is diagonalizable.

- Let B is the collection of distinct eigen-vector corresponding to different eigen values.
- Then, $[T]_B = P[T]_S P^{-1}$ where S is the standard basis and P^{-1} is the change of basis matrix from S to B.

Question 8(a)

Question 8(a)

$T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (y, -x)$.

Solution:

- The matrix representation of T with respect to standard basis S is,
  $[T]_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- The characteristic polynomial of $[T]_S$ is, $\lambda^2 + 1 = 0$, it has distinct roots $\lambda = \pm i$.
- Hence, T is diagonalizable.

Question 8(a) contd...

- Define, $B = \{(1, i), (1, -i)\}$ the collection of distinct eigen-vectors. Then matrix representation of T with respect to B is,

$$T(1, i) = (i, -1) = a(1, i) + b(1, -i)$$

$$T(1, -i) = (-i, -1) = c(1, i) + d(1, -i)$$

$$\implies [T]_B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

- The matrix P^{-1} is the change of basis matrix, given by

$$(1, i) = 1.(1, 0) + i.(0, 1)$$
$$(1, -i) = 1.(1, 0) - i(0, 1)$$

- Hence, $P^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$

Question 8(a) contd...

Verification:

- $$P[T]_S P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{i}{2} \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$
  $$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = [T]_B$$

Question 8(b)

$T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T(x, y, z) = (5x - 6y - 6z, -x + 4y + 2z, 3x - 6y - 4z)$.

Solution:

- The matrix representation of T with respect to standard basis S is, $[T]_S = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$.
- The characteristic polynomial of $[T]_S$ is, $(\lambda - 2)^2(\lambda - 1) = 0$.
- The eigen vector corresponding to $\lambda = 1$ is given by,

$$AX = X$$

$$\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Question 8(b) contd...

- Solving the above system for x, y, x,

$$\begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \xrightarrow[R_1 \to R_1 + 4R_2]{R_3 \to R_3 + 3R_2} \begin{pmatrix} 0 & 6 & 2 \\ -1 & 3 & 2 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{-1}{3} R_1}$$

$$\begin{pmatrix} 0 & 6 & 2 \\ -1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{pmatrix} 0 & 3 & 1 \\ -1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_2} \begin{pmatrix} -1 & 3 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- We get, $x - 3y - 2z = 0 \; 3y + z = 0$
- Hence, The eigen space correponding to $\lambda = 1$ is spanned by $\langle (3, -1, 3) \rangle$.

Question 8(b) contd...

- The eigen vector corresponding to $\lambda = 2$ is given by,

$$(A - 2I)X = 0$$

$$\begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Solving the above system for x, y, x, we get $-x + 2y + 2z = 0$.
- The eigen space correponding to $\lambda = 2$ is spanned by $\langle (2, 1, 0), (2, 0, 1) \rangle$.
- Hence, dim(eigen-spaces)$= 3 =$dim(V). So, V is diagonalizable.
- Define $B = \{(3, -1, 3), (2, 1, 0), (2, 0, 1)\}$. Then, $[T]_B$ is diagonal matrix and change of basis matrix $P^{-1} = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}^{-1}$.

Question 8(c)

$T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (xcos\theta + ysin\theta, -xsin\theta + ycos\theta)$.

Solution:

- The matrix representation of T with respect to standard basis S is, $[T]_S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.
- The characteristic polynomial of $[T]_S$ is,

$$\lambda^2 - 2\lambda\cos\theta + 1) = 0$$
$$\implies \lambda = \cos\theta \pm i\sin\theta$$

- Since, both the eigen values are distinct. Hence, T is diagonalizable.

Question 8(c) contd...

- The eigen vector corresponding to $\lambda = \cos\theta + i\sin\theta$ is given by,

$$det(A - \lambda I) = \begin{vmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{vmatrix} = 0$$

$$\implies -i\sin\theta x + \sin\theta y = 0$$

- Hence, the eigen space is spanned by $\langle(1, i)\rangle$.
- The eigen vector corresponding to $\lambda = \cos\theta - i\sin\theta$ is given by,

$$det(A - \lambda I) = \begin{vmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{vmatrix} = 0$$

$$\implies ix + y = 0$$

- Hence, the eigen space is spanned by $\langle(1, -i)\rangle$.

- Define $B = \{(1, i), (1, -i)\}$. Then, $[T]_B$ is diagonal matrix and change of basis matrix $P^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}$.

Question 9

Characteristic polynomial of a matrix is satisfied by the matrix(Cayley Hamilton). Use it to find(invertibility and) the inverse of the following operators.
(a) $(x, y, z) \to (x + y + z, x + z, -x + y)$.
(b) $(x, y, z) \to (x, x + 2y, x + 2y + 3z)$.

Recall:

- If p(x) is the Characteristic polynomial of a matrix A then p(0) $=$ det(A)
- A square matrix A is invertible iff $det(A) \neq 0$

Question 9(a)

Solution:

- Let T denotes the given linear operator.
- Then, the matrix representation of T with respect to the standard ordered basis is,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

- Let $p(X) = det(XI - A)$ is the characteristic polynomial of A.
- Here

$$p(X) = (X-1)^2(X+1) = X^3 - X^2 - X + 1$$

  Since $p(0) = 1 \neq 0$, hence, A is invertible.
- Thus, by Cayley-Hamilton theorem,

$$A^3 - A^2 - A + I = O \implies A^2 - A - I + A^{-1} = O$$

Question 9(a) contd...

- Hence

$$
\begin{aligned}
A^{-1} &= I + A - A^2 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}
\end{aligned}
$$

  which is matrix representation of inverse of given operator with respect to the standard ordered basis.

- Hence, $T^{-1}(x, y, z) = (x - y - z, x - y, -x + 2y + z)$.

Question 9(b)

(b) $(x, y, z) \to (x, x + 2y, x + 2y + 3z)$.

Solution:

- Let T denotes the given linear operator.
- Then, the matrix representation of T with respect to the standard ordered basis is,
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$
.
- Let $p(X) = det(XI - A)$ is the characteristic polynomial of A.
- Here, $p(X) = (X - 1)(X - 2)(X - 3) = X^3 - 6X^2 + 11X - 6$, since, $p(0) = -6 \neq 0$, hence, A is invertible.
- Thus, by Cayley-Hamilton theorem, $A^3 - 6A^2 + 11A - 6I = 0$.

Question 9(b) contd...

- Hence,

$$A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$$
$$= \frac{1}{6}\left(\begin{pmatrix}1 & 0 & 0\\ 3 & 4 & 0\\ 6 & 10 & 9\end{pmatrix} - \begin{pmatrix}6 & 0 & 0\\ 6 & 12 & 0\\ 6 & 12 & 18\end{pmatrix} - \begin{pmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{pmatrix}\right)$$
$$= \begin{pmatrix}1 & 0 & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & -\frac{1}{3} & \frac{1}{3}\end{pmatrix}$$

  which is matrix representation of inverse of given operator with respect to standard ordered basis.

- Hence, $T^{-1}(x, y, z) = (x, \frac{y-x}{2}, \frac{z-y}{3})$.

Question 10

Which of the following is an inner product.

(a) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2 + 3$ on $\mathbb{R}^2$ over $\mathbb{R}$.

(b) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 - y_1y_2$ on $\mathbb{R}^2$ over $\mathbb{R}$.

(c) $\langle (x_1, y_1), (x_2, y_2) \rangle = y_1(x_1 + 2x_2) + y_2(2x_1 + 5x_2)$ on $\mathbb{R}^2$ over $\mathbb{R}$.

(d) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$ on $\mathbb{C}^2$ over $\mathbb{C}$.

(e) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1\bar{x_2} - y_1\bar{y_2}$ on $\mathbb{C}^2$ over $\mathbb{C}$.

(f) If $A, B \in \mathbb{M}_n(C)$ define $\langle A, B \rangle = Trace(A\bar{B})$.

(g) Suppose $C[0, 1]$ is the space of continuous complex valued functions on the interval $[0, 1]$ and for $f, g \in C[0, 1]$, $\langle f, g \rangle := \int_0^1 f(t)\overline{g(t)}\, dt$.

Question 10

Recall

Let V be a vector space over F (where $F = \mathbb{R}$ or $\mathbb{C}$). A map $V \times V \to F$ denoted by $(u, v) \to \langle u, v\rangle$ is called an inner product on V if the following properties hold:
(a) $\langle u, u\rangle \in \mathbb{F}$ and ≥ 0 for each $u \in V$;
(b) $\langle u, u\rangle = 0$ if and only if $u = 0$;
(c) $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$;
(d) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ (the complex conjugate).
A vector space together with an inner product is called an inner product space.

Question 10

Question 10(a)

Which of the following is an inner product.
(a) $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2 + 3$ on $\mathbb{R}^2$ over $\mathbb{R}$.

Solution:

- For $u = (0, 0)$, $\langle u, u \rangle = 3 \neq 0$.
- Hence, it is not an inner product.

Question 10(b)

$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 - y_1y_2$ on $\mathbb{R}^2$ over $\mathbb{R}$.

Solution:

- For $u = (1, -1)$, $\langle u, u \rangle = \langle (1, -1), (1, -1) \rangle = 1.1 - (-1)(-1) = 0$.
- Hence, it is not an inner product.

Question 10

Question 10(c)

$\langle (x_1, y_1), (x_2, y_2) \rangle = y_1(x_1 + 2x_2) + y_2(2x_1 + 5x_2)$ on $\mathbb{R}^2$ over $\mathbb{R}$.

Solution:

- For $u = (1, -1)$,
  $\langle u, u \rangle = \langle (1, -1), (1, -1) \rangle = (-1).(3) + (-1)(7) = -10 \not\geq 0$.
- Hence, it is not an inner product.

Question 10(d)

$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + y_1 y_2$ on $\mathbb{C}^2$ over $\mathbb{C}$.

Solution:

- For $u = (i, 0)$, $\langle u, u \rangle = \langle (i, 0), (i, 0) \rangle = i.i + 0 = -1 \not\geq 0$.
- Hence, it is not an inner product.

Question 10

Question 10(e)

$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 \bar{x_2} - y_1 \bar{y_2}$ on $\mathbb{C}^2$ over $\mathbb{C}$.

Solution:

- For $u = (i, i)$, $\langle u, u \rangle = \langle (i, i), (i, i) \rangle = i.(-i) - i(-i) = 0$.
- Hence, it is not an inner product.

Question 10(f)

If $A, B \in \mathbb{M}_n(C)$ define $\langle A, B \rangle = Trace(A\bar{B})$.

Solution:

- For $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

  $\langle A, A \rangle = \text{Trace}(A\bar{A}) = \text{Trace} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$
- Hence, it is not an inner product.

Question 10(g)

Suppose $C[0,1]$ is the space of continuous complex valued functions on the interval $[0,1]$ and for $f, g \in C[0,1]$, $\langle f, g\rangle := \int_0^1 f(t)\overline{g(t)}\, dt$.

Solution:

- Let $f(t), g(t) \in C[0,1]$, then,

$$\begin{aligned}\overline{\langle g(t), f(t)\rangle} :&= \overline{\int_0^1 g(t)\overline{f(t)}\, dt}\\ &= \int_0^1 \overline{g(t)\overline{f(t)}\, dt}\\ &= \int_0^1 \overline{g(t)}f(t)\, dt\\ &= \int_0^1 f(t)\overline{g(t)}\, dt = \langle f(t), g(t)\rangle.\end{aligned}$$

Question 10(g) contd...

- $$\langle f(t), f(t)\rangle = \int_0^1 f(t)\overline{f(t)}\, dt = \int_0^1 |f(t)|^2\, dt \geq 0$$
  $$\text{and} \quad \langle f(t), f(t)\rangle = 0 \text{ iff } |f(t)|^2 = 0 \text{ iff } f(t) = 0$$

- For $a, b \in \mathbb{C}$ and $f(t), g(t), h(t) \in C[0,1]$

$$\begin{aligned}\langle af(t) + bg(t), h(t)\rangle &= \int_0^1 [af(t) + bg(t)]\overline{h(t)}\, dt \\ &= a\int_0^1 f(t)\overline{h(t)}\, dt + b\int_0^1 g(t)\overline{h(t)}\, dt \\ &= a\langle f(t), h(t)\rangle + b\langle g(t), h(t)\rangle\end{aligned}$$

- Hence, V is an inner product space.

Question 11

Suppose $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{R})$ is such that $a > 0$ and $det(A) = ad - b^2 > 0$. Show that $\langle X, Y \rangle = X^t AY$ is an inner product on $\mathbb{R}^2$.

Question 11

Solution:

- Let $X, Y \in \mathbb{R}^2$, then,

$$\begin{aligned}\langle Y, X\rangle = Y^t AX &= (Y^t AX)^t \text{ as } Y^t AX \in \mathbb{R}\\ &= X^t A^t Y = X^t AY = \langle X, Y\rangle.\end{aligned}$$

- Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$

$$\begin{aligned}\langle X, X\rangle = X^t AX &= a{x_1}^2 + 2bx_1x_2 + d{x_2}^2\\ &= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(\frac{ad-b^2}{a}\right){x_2}^2 > 0\end{aligned}$$

- For $m, n \in \mathbb{R}$

$$\begin{aligned}\langle mX + nY, Z\rangle &= (mX + nY)^t AZ\\ &= m(X^t AZ) + n(Y^t AZ)\\ &= m\langle X, Z\rangle + n\langle Y, Z\rangle\end{aligned}$$

- Hence, it is an inner product space.

Question 12

Suppose V is an inner product space. Define $\|v\| = \sqrt{\langle v, v\rangle}$. Show the following statements.

(a)$\|v\| = 0$ if and only if $v = 0$.

(b) For $a \in F$, $\|av\| = |a|\|v\|$.

(c) $\|u + v\| \leq \|u\| + \|v\|$.

(d)$|\|v\| - \|w\|| \leq \|v - w\|$.

(e) $\langle u, v\rangle = 0$ then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

Question 12

Solution: **(a)** If $v = 0$ then clearly $\|v\| = 0$. If $\|v\| = 0$ implies that $\langle v, v\rangle = 0$ which further implies that $v = 0$.
(b) $\|av\|^2 = \langle av, av\rangle = a\langle v, av\rangle = a\bar{a}\langle v, v\rangle = |a|^2\|v\|^2$

$$\implies \|av\|^2 = |a|^2\|v\|^2$$

Taking square root both sides, $\|av\| = |a|\|v\|$.

$$\begin{aligned}
\textbf{(c)}\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \|u\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} + \|v\|^2 \\
&= \|u\|^2 + 2Re\langle u, v\rangle + \|v\|^2 \\
&\leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \\
&\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2. \quad \text{(By Cauchy Schwarz Inequality)}
\end{aligned}$$

Question 12 contd...

$$\implies \|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

Taking square root, $\|u+v\| \leq \|u\| + \|v\|$

(d) Consider $\|v\| = \|v - w + w\| \leq \|v - w\| + \|w\|$

$$\implies \quad \|v\| - \|w\| = \|v - w\| \tag{1}$$

Similarly, $\|w\| = \|w - v + v\| = \|(-1)(v - w) + v\| \leq \|v - w\| + \|v\|$

$$\implies \quad \|w\| - \|v\| = \|v - w\| \tag{2}$$

From(1) and (2), $|\|v\| - \|w\|| \leq \|v - w\|$.

(e) Given that $\langle u, v\rangle = 0$ implies that $\langle v, u\rangle = 0$. Now,

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \|u\|^2 + \|v\|^2
\end{aligned}$$

$$\implies \|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Question 13

Use standard inner product on $\mathbb{R}^2$ over $\mathbb{R}$ to prove the following statement: "A parallelogram is a rhombus if and only if its diagonals are perpendicular to each other."

Recall:

The vectors $u, v \in \mathbb{R}^2$ are said to be perpendicular (orthogonal) if and only if $\langle u, v \rangle = 0$.

Question 13

Solution:

- Let u, v are adjacent sides of parallelogram, then, $u + v$ and $u - v$ represent diagonals of parallelogram.
- Consider $\langle u + v, u - v\rangle = \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle$.
- Since, the filed is $\mathbb{R}$, we have $\langle u, v\rangle = \langle v, u\rangle$. Hence,

$$\langle u + v, u - v\rangle = \|u\|^2 - \|v\|^2 = 0 \text{ if and only if } \|u\| = \|v\|.$$

- Hence, A parallelogram is a rhombus if and only if its diagonals are perpendicular to each other.

Question 14

Find with respect to the standard inner product of $\mathbb{R}^3$, an orthonormal basis containing $(1, 1, 1)$.

Question 14

Solution:

- Consider a basis $\mathcal{B} = \{(1,1,1),(1,0,0),(0,1,0)\}$ of $\mathbb{R}^3$ containing $(1,1,1)$.
- Denote $u_1 = (1,1,1)$, $u_2 = (1,0,0)$, $u_3 = (0,1,0)$.
- Now, using Gram Schmidt process, we have, $v_1 = \frac{u_1}{\|u_1\|}$

$$\|u_1\|^2 = \langle u_1, u_1 \rangle = 1.1 + 1.1 + 1.1 = 3$$

- Hence, $v_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
- Let

$$\begin{aligned}
w_2 &= u_2 - \langle u_2, v_1 \rangle v_1 \\
&= (1,0,0) - \langle (1,0,0), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \rangle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right).
\end{aligned}$$

Question 14 contd...

- $$\|w_2\|^2 = \sqrt{\frac{2}{3}.\frac{2}{3} + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)}$$
$$\implies \|w_2\| = \frac{\sqrt{2}}{\sqrt{3}}$$
- Hence, $v_2 = \frac{w_2}{\|w_2\|} = \left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.
- Let
$$\begin{aligned} w_3 &= u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2 \\ &= (0,1,0) - \langle (0,1,0), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\rangle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &- \langle (0,1,0), \left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\rangle \left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right). \end{aligned}$$

- $$= (0,1,0) - \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) + \left(\frac{1}{3},-\frac{1}{6},-\frac{1}{6}\right) = \left(0,\frac{1}{2},-\frac{1}{2}\right).$$
$$\|w_3\|^2 = 0.0 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}$$
$$\implies \|w_3\| = \frac{1}{2\sqrt{2}}.$$
- Hence, $v_3 = \frac{w_3}{\|w_3\|} = \left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$.
- Hence the required orthonormal basis is

$$\left\{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{\sqrt{2}}{\sqrt{3}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right),\left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\right\}$$

Question 15

Find an orthonormal basis of $\mathcal{P}_3 = \{f(x) \in \mathbb{R}[x] : \deg f(x) < 3\}$ with respect to the inner product defined by $\langle f, g \rangle := \int_0^1 f(t)g(t)\, dt$.

Question 15

Solution:

- Let $\mathcal{B} = \{1, x, x^2\}$ is the basis of $\mathcal{P}_n$.
- Denote $u_1 = 1, u_2 = x, u_3 = x^2$.
- Using Gram Schmidt process, we have

$$v_1 = \frac{u_1}{\|u_1\|}$$

- $\|u_1\|^2 = \langle u_1, u_1 \rangle = \int_0^1 1.1 \; dt = 1,$
- Hence, $v_1 = 1$.
- Let

$$\begin{aligned} w_2 &= u_2 - \langle u_2, v_1 \rangle v_1 \\ &= x - \langle x, 1 \rangle 1 \\ &= x - \int_0^1 t.1 \; dt = x - \frac{1}{2}. \end{aligned}$$

- $$\|w_2\|^2 = \int_0^1 (t - \frac{1}{2})(t - \frac{1}{2})\ dt$$
$$= \int_0^1 (t^2 + \frac{1}{4} - t)\ dt = \frac{1}{12}.$$
- Hence, $v_2 = \frac{w_2}{\|w_2\|} = \sqrt{12}(x - \frac{1}{2})$.
- Let

$$\begin{aligned} w_3 &= w_3 - \langle w_3, v_1\rangle v_1 - \langle w_3, v_2\rangle v_2 \\ &= x^2 - \langle x^2, 1\rangle 1 - \langle x^2, \sqrt{12}(x - \frac{1}{2})\rangle \sqrt{12}(x - \frac{1}{2}) \\ &= x^2 - \int_0^1 t^2\ dt - 12(x - \frac{1}{2}) \int_0^1 t^2(t - \frac{1}{2})\ dt \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

- $$\|w_3\|^2 = \int_0^1 (t^2 - t + \frac{1}{6})(t^2 - t + \frac{1}{6})\ dt$$
$$= \int_0^1 (t^4 - 2t^3 + \frac{4}{3}t^2 - \frac{1}{3}t + \frac{1}{36})\ dt = \frac{1}{180}.$$
- Hence, $v_3 = \frac{w_3}{\|w_3\|} = \sqrt{180}(x^2 - x + \frac{1}{6})$.
- Hence, the required orthonormal basis is

$$\left\{1,\ \sqrt{12}(x - \frac{1}{2}),\ \sqrt{180}(x^2 - x + \frac{1}{6})\right\}$$

Question 16

Suppose W is a subspace of the finite dimensional inner product space. Define $W^{\perp} := \{v \in V : \langle w, v \rangle = 0 \text{ for all } w \in W\}$. Show the following statements.

(a) $W^{\perp}$ is a subspace of V.

(b) $W \cap W^{\perp} = 0$.

(c) $V = W \oplus W^{\perp}$.

(d) $(W^{\perp})^{\perp} = W$.

Question 16(a)

$W^\perp$ is a subspace of V.

Solution:

- Since, $\langle 0, w\rangle = 0\ \forall\ w \in W$. Hence, $0 \in W^\perp \implies W^\perp \neq \phi$.
- Let $v_1, v_2 \in W^\perp$ and $a, b \in F$.
- Then, for $w \in W$,

$$\langle av_1 + bv_2, w\rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle = a.0 + b.0 = 0.$$

- This implies, $av_1 + bv_2 \in W^\perp$.
- Hence, $W^\perp$ is a subspace of V.

Question 16(b)

$W \cap W^{\perp} = 0.$

Solution:

- Let $u \in W \cap W^{\perp}$.
- This implies $u \in W$ and hence, $\langle u, v \rangle = 0 \ \forall \ v \in W^{\perp}$.
- Also, $u \in W^{\perp}$ and hence, $\langle u, w \rangle = 0 \ \forall \ w \in W$.
- i.e., $\langle u, w \rangle = 0 \ \forall \ w \in W U W^{\perp}$.
- Thus, $\langle u, u \rangle = 0$, and hence $u = 0$.
- Also note that $0 \in W \cap W^{\perp}$. Hence, $W \cap W^{\perp} = \{0\}$.

Question 16(c)

$V = W \oplus W^{\perp}$.

Solution:

- Let V is a finite dimensional vector space and W is subspace of V, hence, W is finite dimensional.
- Let $\dim(W) = m$.
- Let $B = \{u_1, u_2, ..u_m\}$ be an orthonormal basis of W, such that

$$\langle u_i, u_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

- Let $v \in V$ and consider

$$w = v - \sum_{i=1}^{n} \langle v, u_i \rangle u_i, \tag{3}$$

Question 16(c) contd...

- then, for $k = 1, 2, ..., n$

$$\begin{aligned}\langle w, u_k \rangle &= \langle v - \sum_{i=1}^{n} \langle v, u_i \rangle u_i, u_k \rangle \\ &= \langle v, u_k \rangle - \sum_{i=1}^{n} \langle v, u_i \rangle \langle u_i, u_k \rangle \\ &= \langle v, u_k \rangle - \langle v, u_k \rangle = 0\end{aligned}$$

- Hence, w is orthogonal to each basis vector of basis of W, this implies, $w \in W^{\perp}$.
- Now, (3) can be written as

$$v = -\sum_{i=1}^{n} \langle v, u_i \rangle u_i + w$$

Question 16(c) contd...

- This implies, $v \in W + W^{\perp}$. Hence, $V \subseteq W + W^{\perp}$, but $W + W^{\perp} \subseteq V$, hence, $V = W + W^{\perp}$.
- On the other hand, $W \cap W^{\perp} = \{0\}$.
- Hence, $V = W \oplus W^{\perp}$.

Question 16(d)

Question 16(d)

$(W^{\perp})^{\perp} = W$.

Solution:

- Let $w \in W$, then $\langle w, v \rangle = 0 \ \forall \ v \in W^{\perp}$.
- Hence, $w \in (W^{\perp})^{\perp}, \implies W \subseteq (W^{\perp})^{\perp}$.
- Since, $V = W \oplus W^{\perp} \implies dimV = dimW + dimW^{\perp}$
- Replacing $W^{\perp}$ instead of W in above equation, we get

$$V = W^{\perp} \oplus (W^{\perp})^{\perp} \implies dimV = dimW^{\perp} + dim(W^{\perp})^{\perp}.$$

- Hence, $dimW = dim(W^{\perp})^{\perp} \implies W = (W^{\perp})^{\perp}$.

Question 17

Suppose $W = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. Find the shortest distance of $(a, b) \in \mathbb{R}^2$ from W with respect to (i) the standard inner product, (ii) the inner product defined by $\langle (x_1, y_1), (x_2, y_2) \rangle = 2x_1x_2 + y_1y_2$.

Question 17

Recall:

- Suppose V is an inner product space and W is a proper subspace of V. Given $v \in V$, a vector $w_0 \in W$ is said to be a best approximation of v if for each $w \in W$ we have

$$||v - w_0|| \leq ||v - w||.$$

- Let W be a finite dimensional subspace of V. Suppose $\{w_1, w_2, \ldots, w_n\}$ be an orthogonal basis of W. Then the best approximation of $v \in V$ is given by

$$w_0 = \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{||w_i||^2} w_i.$$

Question 17(i)

Suppose $W = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. Find the shortest distance of $(a, b) \in \mathbb{R}^2$ from W with respect to (i) the standard inner product.

Solution:

- Consider an orthogonal basis $\mathcal{B} = \{1, -1\}$ of W.
- Let $v = (a, b) \in \mathbb{R}^2$.
- Then, the best approximation of v is,

$$w_0 = \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i = \frac{\langle (a, b), (1, -1) \rangle}{\|(1, -1)\|^2}(1, -1)$$

$$= \frac{a.1 + b.(-1)}{2}(1, -1) = (\frac{a-b}{2}, \frac{b-a}{2}).$$

- Hence, the shortest distance between (a, b) and W is, $\|v - w_0\| = \|(a, b) - (\frac{a-b}{2}, \frac{b-a}{2})\| = \|(\frac{a+b}{2}, \frac{a+b}{2})\| = \frac{(a+b)}{\sqrt{2}}$.

Question 17(ii)

The inner product defined by $\langle (x_1, y_1), (x_2, y_2)\rangle = 2x_1x_2 + y_1y_2$.

Solution:

- Consider an orthogonal basis $\mathcal{B} = \{1, -1\}$ of W.
- Let $v = (a, b) \in \mathbb{R}^2$.
- Then, the best approximation of v is,

$$w_0 = \sum_{i=1}^{n} \frac{\langle v, w_i\rangle}{\|w_i\|^2} w_i = \frac{\langle (a, b), (1, -1)\rangle}{\|(1, -1)\|^2}(1, -1)$$

$$= \frac{2a.1 + b.(-1)}{2.1.1 + (-1)(-1)}(1, -1) = (\frac{2a-b}{3}, \frac{b-2a}{3}).$$

- Hence, the shortest distance between (a, b) and W is,
$\|v - w_0\| = \|(a, b) - (\frac{2a-b}{3}, \frac{b-2a}{3})\| = \|(\frac{a+b}{3}, \frac{2b+2a}{3})\| = \sqrt{\frac{6(a+b)^2}{9}} = \frac{\sqrt{2}(a+b)}{\sqrt{3}}.$