MTL101::Linear Algebra and Differential Equations

Tutorial 4

Department of Mathematics
Indian Institute of Technology Delhi

Question 1

Verify that f is a particular solution of the given (IVPs) Initial Value Problems:

a $\frac{dy}{dx} = y - x,\ y(0) = 3,\ f(x) = 2e^x + x + 1$

b $\frac{dy}{dx} = y \tan x,\ y(0) = \frac{\pi}{2},\ f(x) = \frac{\pi}{2} \sec x$

Question 1(a)

Solution: Given IVP is

$$\frac{dy}{dx} = y - x, \; y(0) = 3$$

- We need to verify that $f(x) = 2e^x + x + 1$ is a particular solution of the given IVP.
- Now, $\dfrac{df}{dx} = 2e^x + 1 = 2e^x + x + 1 - x = y - x$.
- Also $f(0) = 3$.
- This shows that f is a particular solution of given IVP.

Question 1(b)

Solution: Given IVP is

$$\frac{dy}{dx} = y \tan x, \;\; y(0) = \frac{\pi}{2}$$

- We need to verify that $f(x) = \dfrac{\pi}{2} \sec x$ is a particular solution of given IVP.
- Now $\dfrac{df}{dx} = \dfrac{\pi}{2} \sec x \tan x = y \tan x$.
- Also $f(0) = \frac{\pi}{2}$
- Thus f is a particular solution of given IVP.

Question 2

Solve the following seperable equations:

(a) $(x-4)y^4dx - x^3(y^2-3)dy = 0$; (b) $x\sin y dx + (x^2+1)\cos y dy = 0$

(c)$y' = \dfrac{x^2+y^2}{xy}$ $\left(\text{Substitute } \frac{y}{x} = u\right)$; (d) $y' + \csc y = 0$

Solution: Given ODE is

$$(x-4)y^4dx - x^3(y^2-3)dy = 0$$

- The equation is seperable: seperating the variables by dividing by x^3y^4, we obtain

$$\frac{(x-4)dx}{x^3} - \frac{(y^2-3)dy}{y^4} = 0$$

- or

$$(x^{-2}-4x^{-3})dx - (y^{-2}-3y^{-4})dy = 0.$$

Question 2(a)contd.

- Integrating, we have the one parameter family of solutions

$$\frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

- where c is the arbitrary constant.

Question 2(b)

Solution: Given ODE is

$$x \sin y dx + (x^2+1) \cos y dy = 0.$$

- Seperating the variables by dividing by $(x^2+1)\sin y$, we obtain

$$\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0.$$

- Thus

$$\int \frac{x}{x^2+1}dx + \int \frac{\cos y}{\sin y}dy = \ln|c|$$

- which implies

$$\frac{1}{2}\ln(x^2+1) + \ln|siny| = \ln|c|$$

Question 2(b) contd.

- Multipling by 2, we have

$$\ln(x^2+1) + \ln\sin^2 y = \ln c_1,$$

- where

$$c_1 = c^2 > 0,$$

- which implies

$$(x^2+1)\sin^2 y = c_1.$$

Question 2(c)

Solution: Given ODE is

$$y' = \frac{x^2 + y^2}{xy}. \tag{1}$$

- Substitute

$$\frac{y}{x} = u.$$

- Then

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

- and (1) becomes

$$u + x\frac{du}{dx} = \frac{1 + u^2}{u}$$

- which implies

$$udu - \frac{dx}{x} = 0$$

- On integrating, we get
$$u^2 - \ln|x| = c$$
- where c is an arbitrary constant.
- Replacing u by $\frac{y}{x}$ we obtain the solution in the form

$$\frac{y^2}{x^2} - \ln|x| = c.$$

Question 2(d)

Solution: Given ODE is

$$y' + \csc y = 0.$$

- This implies

$$\sin y dy + dx = 0.$$

- On integrating, we get

$$-\cos y + x = c,$$

- where c is an arbitrary constant.

Question 3

Question 3

Solve the following equation by reducing it to a separable equation:

$$(x^2 - 3y^2)dx + 2xydy = 0.$$

Question 3

Solution: Given ODE is

$$(x^2 - 3y^2)dx + 2xydy = 0.$$

- or
$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}. \tag{2}$$
- Substitute
$$\frac{y}{x} = u.$$
- Then
$$\frac{dy}{dx} = u + x\frac{du}{dx}$$
- and (2) becomes
$$u + \frac{xdu}{dx} = \frac{3u^2 - 1}{2u}$$

- which implies

$$\frac{xdu}{dx} = \frac{u^2 - 1}{2u}.$$

- Seperating the variables, we get

$$\frac{2udu}{u^2 - 1} - \frac{dx}{x} = 0$$

- On integrating, we get

$$\ln |u^2 - 1| - \ln |x| = \ln |c|,$$

- where c is an arbitrary constant.
- Replacing u by $\frac{y}{x}$, we obtain

$$\ln \left| \frac{y^2 - x^2}{x^2} \right| - \ln |x| = \ln |c|$$

Question 3 contd.

- which implies

$$\frac{|y^2 - x^2|}{x^2|x|} = |c|.$$

Question 4

Determine whether or not each of the given equations is exact; solve those that are exact.

(a) $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0;$

(b) $(y^2 + 1)\cos x dx + 2y \sin x dy = 0;$

(c) $(2xy + 1)dx + (x^2 + 4y)dy = 0;$

(d) $(3x^2y + 2)dx - (x^3 + y)dy = 0;$

(e) $-2xy\sin(x^2)dx + \cos(x^2)dy = 0;$

(f) $(e^{(x+y)} - y)dx + (xe^{(x+y)} + 1)dy = 0$

Recall: Exact differential equation

Recall:

- Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy$$

- The differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an exact differential if there exists a function F of two variables such that

$$dF = Mdx + Ndy,$$

i.e., $M = F_x$ and $N = F_y$

Recall: Exact differential equation contd.

- Assume that M and N are continuous with continuous partial derivatives w.r.t. x and y. Then
  1. If $Mdx + Ndy$ is exact then $M_y = N_x$
  2. If $M_y = N_x$, then $Mdx + Ndy$ is exact.
- If $Mdx + Ndy = 0$ is exact, then F is given by

$$F(x, y) = \int (\text{terms in M independent of y})dx + \int Ndy + c,$$

  where c is an arbitrary constant.
- Hence a one-parameter family of solution is $F(x, y) = c_1$.

Question 4(a)

Solution: Given ODE is

$$(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$$

- Here $M = 6xy + 2y^2 - 5$ and $N = 3x^2 + 4xy - 6$.
- Now
$$\frac{\partial M}{\partial y} = 6x + 4y = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is exact and F is given by
$$F = \int -5dx + \int 3x^2 + 4xy - 6dy + c$$
- which implies
$$F = 3x^2y + 2xy^2 - 5x - 6y + c$$

Question 4(a) contd.

- Hence a one-parameter family of solution is $F(x, y) = c_1$, or

$$3x^2y + 2xy^2 - 5x - 6y + c = c_1$$

- or

$$3x^2y + 2xy^2 - 5x - 6y = c_0$$

- where $c_0 = c_1 - c$ is an arbitrary constant.

Question 4(b)

Solution: Given ODE is

$$(y^2+1)\cos x dx + 2y \sin x dy = 0$$

- Here $M = (y^2+1)\cos x$ and $N = 2y\sin x$.
- Now
$$\frac{\partial M}{\partial y} = 2y\cos x = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is exact and F is given by
$$F = \int \cos x dx + \int 2y \sin x dy + c$$
- which implies
$$F = \sin x + y^2 \sin x + c$$

Question 4(b) contd.

- Hence a one-parameter family of solution is $F(x, y) = c_1$, or

$$\sin x + y^2 \sin x + c = c_1$$

- or

$$\sin x + y^2 \sin x = c_0$$

- where $c_0 = c_1 - c$ is an arbitrary constant.

Solution: Given ODE is

$$(2xy + 1)dx + (x^2 + 4y)dy = 0$$

- Here $M = 2xy + 1$ and $N = x^2 + 4y$.
- Now
$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is exact and F is given by
$$F = \int dx + \int (x^2 + 4y)dy + c$$
- which implies
$$F = x + x^2y + 2y^2 + c$$

Question 4(c) contd.

- Hence a one-parameter family of solution is $F(x, y) = c_1$, or

$$x + x^2 y + 2y^2 + c = c_1$$

- or

$$x + x^2 y + 2y^2 = c_0$$

- where $c_0 = c_1 - c$ is an arbitrary constant.

Question 4(d)

Solution: Given ODE is

$$(3x^2y + 2)dx - (x^3 + y)dy = 0$$

- Here $M = 3x^2y + 2$ and $N = -(x^3 + y)$.
- Now
$$\frac{\partial M}{\partial y} = 3x^2 \neq -3x^2 = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is not exact.

Question 4(e)

Solution: Given ODE is

$$-2xy\sin(x^2)dx + \cos(x^2)dy = 0$$

- Here $M = -2xy\sin(x^2)$ and $N = \cos(x^2)$.
- Now
$$\frac{\partial M}{\partial y} = -2x\sin(x^2) = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is exact and F is given by
$$F = \int 0dx + \int \cos(x^2)dy + c$$
- which implies
$$F = y\cos(x^2) + c$$

Question 4(e) contd.

- Hence a one-parameter family of solution is $F(x, y) = c_1$, or

$$y \cos(x^2) + c = c_1$$

- or

$$y \cos(x^2) = c_0$$

- where $c_0 = c_1 - c$ is an arbitrary constant.

Question 4(f)

Solution: Given ODE is

$$(e^{(x+y)} - y)dx + (xe^{(x+y)} + 1)dy = 0$$

- Here $M = e^{(x+y)} - y$ and $N = xe^{(x+y)} + 1$.
- Now
$$\frac{\partial M}{\partial y} = e^{(x+y)} - 1 \neq xe^{(x+y)} + e^{(x+y)} = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is not exact.

Question 5

Solve the IVPs.

(a) $(2xy-3)dx+(x^2+4y)dy=0, y(1)=2;$

(b) $(3x^2y^2-y^3+2x)dx+(2x^3y-3xy^2+1)dy=0, y(-2)=1.$

(c) $y\frac{dy}{dx}+4x=0, y(0)=2.$

(d) $\frac{dr}{d\theta}=b\left(\frac{dr}{d\theta}\cos\theta+r\sin\theta\right), r(\frac{\pi}{2})=\pi.$

Question 5(a)

Solution: Given IVP is

$$(2xy-3)dx+(x^2+4y)dy=0, y(1)=2.$$

- Here $M=2xy-3$ and $N=x^2+4y$.
- Now
$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}.$$
- Therefore given ODE is exact and F is given by
$$F=\int -3dx+\int (x^2+4y)dy+c$$
- which implies
$$F=x^2y+2y^2-3x+c$$

Question 5(a) contd.

- Hence a one-parameter family of solution is $F(x, y) = c_1$, or
$$x^2y + 2y^2 - 3x + c = c_1$$
- or
$$x^2y + 2y^2 - 3x = c_0$$
- where $c_0 = c_1 - c$ is an arbitrary constant.
- Using initial condition $y(1) = 2$, we get $c_0 = 7$.
- Hence particular solution of given IVP is
$$x^2y + 2y^2 - 3x = 7$$

Question 5(b)

Solution: Given IVP is

$$(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0, y(-2) = 1.$$

- Here $M = 3x^2y^2 - y^3 + 2x$ and $N = 2x^3y - 3xy^2 + 1$.
- Now
$$\frac{\partial M}{\partial y} = 6x^2y - 3y^2 = \frac{\partial N}{\partial x}.$$
- Therefore given ODE is exact and F is given by
$$F = \int 2xdx + \int (2x^3y - 3xy^2 + 1)dy + c$$
- which implies
$$F = x^2 + x^3y^2 - xy^3 + y + c$$

Question 5(b) contd.

- Hence a one-parameter family of solution is $F(x, y) = c_1$, or

$$x^2 + x^3y^2 - xy^3 + y + c = c_1$$

- or

$$x^2 + x^3y^2 - xy^3 + y = c_0$$

- where $c_0 = c_1 - c$ is an arbitrary constant.
- Using initial condition $y(-2) = 1$, we get $c_0 = 1$.
- Hence particular solution of given IVP is

$$x^2 + x^3y^2 - xy^3 + y = 1$$

Question 5(c)

Solution: Given IVP is

$$y\frac{dy}{dx} + 4x = 0, y(0) = 2.$$

- The equation is seperable: seperating the variables, we obtain

$$ydy + 4xdx = 0$$

- Integrating, we have the one parameter family of solutions

$$\frac{y^2}{2} + 2x^2 = c$$

- where c is the arbitrary constant.
- Using the initial condition $y(0) = 2$, we get $c = 2$.
- Hence a particular solution of given IVP is given by

$$\frac{y^2}{2} + 2x^2 = 2$$

Question 5(d)

Solution: Given initial value problem is

$$\frac{dr}{d\theta} = b\left(\frac{dr}{d\theta}\cos\theta + r\sin\theta\right), r(\frac{\pi}{2}) = \pi$$

.

- or

$$(1 - b\cos\theta)\frac{dr}{d\theta} - r\sin\theta = 0, r(\frac{\pi}{2}) = \pi$$

- The equation is seperable: seperating the variables, we obtain

$$\frac{dr}{r} - \frac{\sin\theta d\theta}{1 - b\cos\theta} = 0$$

- Integrating, we get

$$\ln|r| - \frac{1}{b}\ln|1 - b\cos\theta| = \ln|c|$$

Question 5(d) contd.

- which implies
$$\frac{|r|^b}{1 - b\cos\theta} = |c|^b.$$
- Using initial condition, we get
$$|c|^b = \pi^b$$
- which gives
$$c = \pm\pi.$$

Question 6

Question 6

Consider a stone falling freely through the air. Assuming that the air resistance is negligible and the acceleration due to gravity $g = 9.8m/s^2$, construct the resulting ODE and find its solution, if the initial position is $h_0$and the initial velocity is v_0.

Question 6

Solution:

- Let m be the mass of the stone and w be its weight.
- We choose the positive $x-$axis vertically downward with the origin where the fall began.
- Since the air resistance is negligible, only force working upon the stone is the weight of the stone.
- By Newton's second law of motion

$$F = ma$$

- From this, we get

$$\frac{dv}{dt} = 9.8$$

Question 6 contd.

- Also we have

$$v(0) = v_0$$

- Integrating, we get

$$v = 9.8t + c$$

- Using initial condition, we get

$$v = 9.8t + v_0$$

- Now to find the distance fallen at time t, we write

$$\frac{dx}{dt} = 9.8t + v_0$$

- and given the initial height is

$$x(0) = h_0$$

Question 6 contd.

- On integrating,we get

$$x = 9.8t^2 + v_0 t + c_0$$

- Using the initial condition $x(0) = h_0$, we get

$$x = 9.8t^2 + v_0 t + h_0$$

Question 7

Question 7

The efficiency of engines of subsonic airplanes depends on the air pressure and (usually) is maximum at $36000ft$. The rate of change of air pressure $y'(x)$ is proportional to air pressure $y(x)$ at height x. If y_0 is the pressure at sea level and the pressure decreases to half at $18000ft$, then find the air pressure at $36000ft$.

Question 7

Solution:

- Given $y'(x)$ is proportional to $y(x)$, we have

$$y'(x) = -ky(x)$$

- where $k > 0$ is the constant of proportionality and negative sign represent the decrease of air pressure with height.
- Also given $y(0) = y_0$ and $y(18000) = \frac{1}{2}y_0$
- On integrating, we get

$$\ln(y(x)) = -kx + \ln c$$

- where c is an arbitrary constant.
- This implies

$$y(x) = ce^{-kx}$$

Question 7 contd.

- Using the initial condition $y(0) = y_0$, we get

$$y(x) = y_0 e^{-kx}$$

- Now using the initial condition $y(18000) = \frac{y_0}{2}$, we get

$$\frac{y_0}{2} = y_0 e^{-18000k}$$

- which implies

$$e^{-k} = \left(\frac{1}{2}\right)^{\frac{1}{18000}}$$

- Now

$$y(36000) = y_0 e^{-36000k} = \frac{y_0}{4}$$

Question 8

Question 8

In a laundry dryer loss of moisture is directly proportional to moisture content of the laundry. If wet laundry loses one fourth of its moisture in the first 10 minutes, when will the laundry be 95% dry.

Solution:

- Let $y(t)$ be the moisture content of the laundry at time t.
- Then by the given condition

$$\frac{dy}{dt} = -ky(t)$$

- where $k > 0$ is the constant of proportionality and the negative sign represent the decrease in moisture content.
- Let y_0 be the initial moisture content in the laundry room.
- Then

$$y(10) = \frac{y_0}{4}$$

- On integrating, we get

$$y(t) = ce^{-kt},$$

Question 8 contd.

- where c is an arbitrary constant.
- Using initial condition $y(0) = y_0$, we get

$$y(t) = y_0 e^{-kt}$$

- Again using the initial condition $y(10) = \frac{y_0}{4}$, we get

$$\frac{y_0}{4} = y_0 e^{-k10}$$

- which implies

$$e^{-k} = \left(\frac{1}{4}\right)^{\frac{1}{10}}$$

Question 8 contd.

- Now we want to find t at which the laundry is 95% dry,
- i.e., the time at which 5% of the initial moisture is left,
- which gives
$$y(t) = \frac{y_0}{20}$$
- and so
$$y_0 \left(\frac{1}{4}\right)^{\frac{t}{10}} = \frac{y_0}{20}$$
- which on simplification gives $t = 21$ min approx.

Question 9

Question 9

Find all the curves in the $xy-$plane whose tangents pass through the point (a, b).

Question 9

Solution:

Question 10

Question 10

Under what conditions on constants A, B, C, and D, is $(Ax + By)dx + (Cx + Dy)dy = 0$ is exact. Solve the equation when it is exact.

Question 10

Solution: Given ODE is

$$(Ax + By)dx + (Cx + Dy)dy = 0$$

- Here $M = Ax + By$ and $N = Cx + Dy$.
- Now given ODE is exact iff

$$M_y = N_x,$$

- which gives

$$B = C$$

.

- In this case

$$F = \int Axdx + \int (Cx + Dy)dy + c_0$$

Question 10 contd.

- which implies

$$F = \frac{Ax^2}{2} + Cxy + \frac{Dy^2}{2} + c_0$$

- Hence the general solution of given ODE is given by

$$F(x, y) = c$$

- where c is an arbitrary constant.

Question 11

Determine the constant A such that the equation is exact, and solve the resulting exact equation:

(a) $(x^2 + 3xy)dx + (Ax^2 + 4y)dy = 0;$

(b) $\left(\frac{1}{x^2} + \frac{1}{y^2}\right) dx + \left(\frac{Ax+1}{y^3}\right) dy = 0.$

Question 11(a)

Solution: Given ODE is

$$(x^2 + 3xy)dx + (Ax^2 + 4y)dy = 0$$

- Here $M = x^2 + 3xy$ and $N = Ax^2 + 4y$.
- Now given ODE is exact iff

$$M_y = N_x$$

- which implies

$$3x = 2Ax$$

- and so

$$A = \frac{3}{2}.$$

Question 11(a) contd.

- When $A = \frac{3}{2}$, F is given by

$$F = \int x^2 dx + \int \left(\frac{3}{2}x^2 + 4y\right) dy + c_0$$

- where c_0 is an arbitrary constant.
- which implies

$$F = \frac{x^3}{3} + \frac{x^3}{2} + 2y^2 + c_0$$

- Hence the general solution of given ODE is given by

$$F(x, y) = c$$

- where c is an arbitrary constant.

Question 11(b)

Solution: Given ODE is

$$\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx+\left(\frac{Ax+1}{y^3}\right)dy=0$$

- Here $M=\frac{1}{x^2}+\frac{1}{y^2}$ and $N=\left(\frac{Ax+1}{y^3}\right)$.
- Now given ODE is exact iff

$$M_y=N_x$$

- which implies

$$\frac{-3}{y^3}=\frac{A}{y^3}$$

- and so

$$A=-3.$$

Question 11(b) contd.

- When $A = -3$, F is given by

$$F = \int \frac{1}{x^2} dx + \int \left(\frac{-3x+1}{y^3} \right) dy + c_0$$

- where c_0 is an arbitrary constant.
- which implies

$$F = \frac{-1}{x} - \frac{-3x+1}{2y^2} + c_0$$

- Hence the general solution of given ODE is given by

$$F(x, y) = c$$

- where c is an arbitrary constant.

Question 12

Determine the most general function $N(x, y)$ such that the equation is exact:

(a) $(x^2 + xy^2)dx + N(x, y)dy = 0;$

(b) $(x^{-2}y^{-2} + xy^{-3})dx + N(x, y)dy = 0.$

Questtion 12(a)

Solution: Given ODE is

$$(x^2 + xy^2)dx + N(x,y)dy = 0$$

- Here $M(x,y) = x^2 + xy^2$.
- Now given equation is exact iff

$$M_y = N_x$$

- which gives

$$N_x = 2xy$$

- and so

$$N(x,y) = x^2y + h(y).$$

Questtion 12(b)

Solution: Given ODE is

$$(x^{-2}y^{-2} + xy^{-3})dx + N(x, y)dy = 0.$$

- Here $M(x, y) = x^{-2}y^{-2} + xy^{-3}$.
- Now given equation is exact iff

$$M_y = N_x$$

- which gives

$$N_x = -2x^{-2}y^{-3} - 3xy^{-4}$$

- and so

$$N(x, y) = 2x^{-1}y^{-3} - \frac{3}{2}x^2y^{-4} + h(y).$$

Question 13

Question 13

Consider the differential equation $(y^2 + 2xy)dx - x^2 dy = 0$.

(a) Observe that this equation is not exact. Multiply the given equation through by y^n, where n is an integer, and then determine n so that y^n is an integrating factor of the given equation. Solve the resulting exact equation.

(b) Show that $y = 0$ is a solution of the original nonexact equation but is not a solution of the essentially equivalent exact equation found in (a).

Question 13(a) contd.

Solution: Given ODE is

$$(y^2 + 2xy)dx - x^2 dy = 0.$$

- Clealy given ODE is not exact.
- **Integrating factor of the form** y^n
- Multiplying the equation $Mdx + Ndy = 0$ with y^n,
- we get

$$y^n Mdx + y^n Ndy = 0$$

- This equation is exact if and only if

$$\frac{\partial(y^n M)}{\partial y} = \frac{\partial(y^n N)}{\partial x}$$

Question 13 (a) contd.

- i.e.,
$$ny^{n-1}M + y^nM_y = y^nN_x.$$
- This implies
$$\frac{nM}{y} = N_x - M_y$$
- For given ODE this implies
$$ny + 2nx = 4x - 2y$$
- which implies
$$n = -2$$

  .
- Hence the integrating factor is y^{-2}. (Check !)

Question 13(a) contd.

- Thus the resulting exact equation is

$$(1 + 2xy^{-1})dx - x^2y^{-2}dy = 0$$

- It's general solution is given by

$$F(x, y) = c$$

- where

$$F = \int dx - \int x^2y^{-2}dy + c_0$$

- which implies

$$F = x + x^2y^{-1} + c_0$$

Question 13(b)

Solution:

- From given ODE , we get

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}.$$

- Clearly $y = 0$ is a solution of this ODE.
- Also the one parameter family of solutions of the essentially equivalent exact equation is given

$$x + x^2 y^{-1} = c$$

- Clearly this family of solutions does not include $y = 0$.

Quesion 14

Question 14

Consider a differential equation of the form

$$[y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0.$$

(a) Show that an equation of this form is not exact.

(b) Show that $\frac{1}{x^2+y^2}$ is an integrating factor of an equation of this form.

Question 14(a)

Solution:

- Let $M := y + xf(x^2 + y^2)$ and $N := yf(x^2 + y^2) - x$.
- Then $\frac{\partial M}{\partial y} = 1 + 2xyf'(x^2 + y^2) \neq \frac{\partial N}{\partial x} = 2xyf'(x^2 + y^2) - 1$.
- So given ODE is not exact

Question 14(b)

Solution: Multiplying by the integrating factor $\frac{1}{x^2+y^2}$ in the both side of the ODE, we have

- $M' := [y + xf(x^2 + y^2)]/(x^2 + y^2)$ and
  $N' := [yf(x^2 + y^2) - x]/(x^2 + y^2)$.
- Now
  $\frac{\partial M'}{\partial y} = \frac{(x^2+y^2)(1+2xyf'(x^2+y^2)-2y(y+xf)}{(x^2+y^2)^2} = \frac{(x^2+y^2)2xyf'(x^2+y^2)+x^2-y^2-2xyf}{(x^2+y^2)^2}$
- and $\frac{\partial N'}{\partial x} = \frac{(x^2+y^2)(2xyf'(x^2+y^2)-1)-2x(yf-x)}{(x^2+y^2)^2} =$
  $\frac{(x^2+y^2)2xyf'(x^2+y^2)+x^2-y^2-2xyf}{(x^2+y^2)^2}$.
- Here
  $$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$
  and thus, the equation is exact.

Question 15

Question 15

Use the result of the above exercise to solve the equation

$$[y + x(x^2 + y^2)^2]dx + [y(x^2 + y^2)^2 - x]dy = 0.$$

Question 15

Solution: Multiplying by the integrating factor $\frac{1}{x^2+y^2}$ in the both side of the ODE and clubbing the terms appropriately, we have

- $$\frac{ydx - xdy}{x^2+y^2} + (x^2+y^2)(xdx+ydy) = 0$$
- which implies
$$\frac{y^2}{x^2+y^2}\frac{ydx - xdy}{y^2} + \frac{1}{2}(x^2+y^2)\,d(x^2+y^2) = 0$$
- which implies
$$\frac{1}{1+\frac{x^2}{y^2}}d(\frac{x}{y}) + \frac{1}{2}(x^2+y^2)\,d(x^2+y^2) = 0$$
- Integrating the above, we finally get the general solution
$$tan^{-1}(x/y) + \frac{1}{4}(x^2+y^2)^2 = C,$$
- where C is some constant.

Question 16

Find all solutions of the following equations:

(a) $y' - 2y = 1$;

(b) $y' + y = e^x$;

(c) $y' - 2y = x^2 + x$;

(d) $3y' + y = 2e^{-x}$;

(e) $y' + 3y = e^x$.

Question 16(a)

Solution:

- The ODE:

$$\frac{dy}{1+2y} = dx$$

- Integrating in the both side,
- we get

$$\ln[(1+2y)] = 2x + C$$

- which implies

$$y = \frac{1}{2}e^{2x+C} - \frac{1}{2},$$

- where C is some arbitrary constant.

Question 16(b)

Solution:

- Here $p(x) := 1, q(x) := e^x$.
- Then the solution is given by:

$$y(x) = e^{-\int dx}\left[C + \int e^{\int dx} e^x dx\right]$$

- which gives

$$y(x) = e^{-x}\left[C + \int e^{2x} dx\right]$$

- and so

$$y(x) = e^{-x}\left[C + \frac{1}{2}e^{2x}\right],$$

- where C is some arbitrary constant.

Question 16(c)

Solution:

- Here
$$p(x) := -2, q(x) := x^2 + x.$$
- Then the solution is given by:
$$y(x) = e^{\int 2dx}\left[C + \int e^{\int -2dx}(x^2 + x)dx\right]$$
- which gives
$$y(x) = e^{2x}\left[C + \int e^{-2x}(x^2 + x)dx\right]$$
- and so
$$y(x) = e^{2x}\left[C - \frac{1}{2}(x^2 + 2x)e^{-2x}\right] \text{(integration by parts formula is used)}$$
- where C is some arbitrary constant.

Question 16(d)

Solution:

- Here
$$p(x) := \frac{1}{3}, q(x) := \frac{2}{3}e^{-x}.$$
- Then the solution is given by:
$$y(x) = e^{-\int \frac{1}{3}dx}\left[C + \int e^{\int \frac{1}{3}dx}.\frac{2}{3}e^{-x}dx\right]$$
- which gives
$$y(x) = e^{-\frac{1}{3}x}\left[C + \frac{2}{3}\int e^{-\frac{2}{3}x}dx\right]$$
- and so
$$y(x) = e^{-\frac{1}{3}x}\left[C - e^{-\frac{2}{3}x}\right],$$
- where C is some arbitrary constant.

Question 16(e)

Solution:

- Here
$$p(x) := 3, q(x) := e^x.$$
- Then the solution is given by:
$$y(x) = Ce^{-\int 3dx}\left[1 + \int e^{\int 3dx} e^x dx\right]$$
- which implies
$$y(x) = e^{-3x}\left[C + \int e^{4x} dx\right]$$
- and so
$$y(x) = e^{-3x}\left[C + \frac{1}{4}e^{4x}\right],$$
- where C is some arbitrary constant.

Question 17

Consider the equation

$$y' + (\cos x)y = e^{-\sin x}.$$

(a) Find the solution ϕ which satisfies $\phi(\pi) = \pi$.

(b) Show that any solution ϕ has the property that $\phi(\pi k) - \phi(0) = \pi k$, where k is any integer.

Question 17(a)

Solution:

- Here
$$p(x) := \cos x, q(x) := e^{-\sin x}.$$
- Then the solution is given by:
$$\begin{aligned}\phi(x) &= e^{-\int \cos x \, dx}\left[C + \int e^{\int \cos x \, dx} e^{-\sin x} \, dx\right] \\ &= e^{-\sin x}\left[C + \int dx\right] \\ &= e^{-\sin x}\left[C + x\right], \qquad (3)\end{aligned}$$
- where C is some arbitrary constant.
- Now putting $\phi(\pi) = \pi$ in (3), we get $C = \pi$.

Question 17(b)

Solution:

- From (3),
- we have

$$\phi(\pi k) - \phi(0) = 1 + \pi k - 1 = \pi k,$$

- where k is any integer.

Question 18

Solve the Bernoulli's equations:

(a) $y' - 2xy = xy^2$;

(b) $y' + y = xy^3$.

Question 18(a)

Solution:

- Define the transformation $z = \frac{1}{y}$, $y > 0$.
- Then
$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}.$$
- Now from $\frac{dy}{dx} = xy^2 - 2xy$, we get
$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx} = -\frac{1}{y^2}(xy^2 - 2xy) = -x(1+2z).$$
- The above can be written as
$$\frac{dz}{1+2z} = -xdx.$$

Question 18(a) contd.

- Then integrating both the sides, we get

$$\log(1+2z) = -x^2 + C, \text{ where } C \text{ is some arbitrary constant.}$$

- This implies

$$z(x) = \frac{1}{2}[e^{-x^2+C} - 1], \; i.e. \;\; y(x) = \frac{2}{e^{-x^2+C} - 1}.$$

Question 18(b)

Solution:

- Define the transformation

$$z = \frac{1}{y^2},\ y > 0.$$

- Then

$$\frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}.$$

- Now from $\frac{dy}{dx} = xy^3 - y$, we get

$$\frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx} = -\frac{2}{y^3}(xy^3 - y) = -2x + 2z.$$

- Therefore,

$$p(x) = -2, q(x) = -2x.$$

Question 18(b) contd.

- Then the solution is given as

$$z(x) = e^{\int 2dx}[C + \int e^{-\int 2dx}(-2x)dx]$$
$$= e^{2x}[C + (x + \frac{1}{2})e^{-2x}] \text{(Integration by parts formula is used here)},$$

- where C is some arbitrary constant.
- Hence,

$$y(x) = \frac{e^{-2x}}{C + (x + \frac{1}{2})e^{-2x}}.$$

Question 19

Solve the following nonlinear ODEs.

(a) $y' \sin 2y + x \cos 2y = 2x$

(b) $2yy' + y^2 \sin x = \sin x, y(0) = \sqrt{5}$

Question 18(a)

Soluiton:

- The equation can be written as

$$\frac{\sin 2y}{2+\cos 2y}dy = xdx.$$

- Integrating both the sides, we get

$$\log(2+\cos 2y) = -x^2 + C,$$

- where C is some arbitrary constant.
- Then

$$y = \frac{1}{2}\cos^{-1}(e^{-x^2+C} - 2).$$

Question 19(b)

Solution:

- The equation can be written as

$$\frac{dy}{dx} + \frac{y}{2}\sin x = \frac{\sin x}{2}.\frac{1}{y}.$$

- Setting $z = y^2$, $y \geq 0$, we see that

$$\frac{dz}{dx} = 2y\frac{dy}{dx} = \sin x - z\sin x$$

- The solution is given as

$$z(x) = e^{-\int \sin x dx}[C + \int e^{\int \sin x dx} sinxdx]$$
$$= e^{\cos x}[C - e^{-\cos x}] = Ce^{\cos x} - 1,$$

- where C is some arbitrary constant.

Question 19(b) Cont...

- Now

$$y(x) = \sqrt{Ce^{\cos x} - 1}, y(0) = \sqrt{5} \implies C = 6.$$

- Thus

$$y(x) = \sqrt{6e^{\cos x} - 1}.$$

Question 20

Solve the following IVP:

$$(x-1)y' = 2y, y(1) = 1.$$

Explain the results in view of the theory of existence and uniqueness of IVPs.

Question 20

Solution:

- The equation can be written as
$$\frac{dy}{y} = 2\frac{dx}{x-1}.$$
- Solving this, we get
$$y = C(x-1)^2,$$
- which together with the initial condition $y(1) = 1$ implies $0 = 1$(absurd).
- Hence the ODE has no solution.
- Here $f(x, y) = \frac{2y}{x-1}$ is not even defined on the line $x = 1$. Hence Picard's theorem can not be applied here.

Question 21

Find all the initial conditions, such that corresponding IVP, with ODE,

$$(x^2 - 4x)y' = (2x - 4)y$$

has no solution, unique solution and more than one solutions.

Question 21

Solution:

- The ODE: $(x^2 - 4x)\frac{dy}{dx} = (2x-4)y,\ y(x_0) = y_0$.
- Here
$$f(x,y) = \frac{(2x-4)y}{x^2-4x},\ \frac{\partial f}{\partial y} = \frac{(2x-4)y}{x^2-4x}.$$
- The existence and uniqueness theorem guarantees the existence of unique solution in the vicinity of (x_0, y_0) where f and $\frac{\partial f}{\partial y}$ are continuous and bounded.
- Thus, existence of unique solution is guaranteed at all x_0 for which $x_0(x_0-4) \neq 0$. Hence, unique solution exists when $x_0 \neq 0, 4$.

Question 21, Cont...

- When $x_0 = 0$ or $x_0 = 4$, nothing can be said using the Picard's existence and uniqueness theorem.
- However, we can find the general solution of the ODE, given as

$$y = Cx(x-4).$$

Using initial condition, we get $y_0 = Cx_0(x_0 - 4)$.

- Clearly, the IVP has no solution if $x_0(x_0 - 4) = 0$ and $y_0 \neq 0$.
- If $x_0(x_0 - 4) = 0$ and $y_0 = 0$ then $y = cx(x-2)$ is a solution to the IVP for any real c.

Hence, in summary

(i) No solution for $x_0 = 0$ or $x_0 = 4$ and $y_0 \neq 0$;

(ii) Infinite number of solutions for $x_0 = 0$ or $x_0 = 4$ and $y_0 = 0$;

(iii) Unique solution for $x_0 \neq 0, 4$.

Question 22

Show that the Lipschitz condition is satisfied by the function $|\sin y| + x$ at every point on the $xy-$plane though its partial derivative with respect to y does not exist on the line $y = 0$.

Question 22

Solution:

- $f(x,y) = |\sin y| + x$. Let $y_1 < y_2$ be any two points in $\mathbb{R}$. Then, $|f(x,y_1) - f(x,y_2)| = ||\sin y_1| - |\sin y_2|| \le |\sin y_1 - \sin y_2|$. By MVT, $\exists y_0 \in (y_1,y_2)$, s.t. $|\sin y_1 - \sin y_2| \le |\cos y_0|\,|y_1 - y_2| \le |y_1 - y_2|$. Thus, $|f(x,y_1) - f(x,y_2)| \le |y_1 - y_2|$. This shows that f is Lipschitz with $L = 1$.
- On the line $y = 0$, with $0 < h < \pi$,

$$\begin{aligned}\lim_{h\to 0^+} \frac{|f(x,h) - f(x,0)|}{h} &= \lim_{h\to 0^+} \frac{|\sin h|}{h} \\ &= \lim_{h\to 0^+} \frac{\sin h}{h} = 1.\end{aligned}$$

  Similarly, On the line $y = 0$, with $-\pi < h < 0$,

$$\lim_{h\to 0^-} \frac{|f(x,h) - f(x,0)|}{h} = \lim_{h\to 0^-} \frac{-\sin h}{h} = -1.$$

  Hence, partial derivative of f on $y = 0$ does not exist.

Question 23

Apply Picard's iteration method to the following problems. Do three steps of the iteration.

(a) $y' = y, y(0) = 1;$

(b) $y' = x + y, y(0) = -1.$

Question 23(a)

Solution: We have,

$$\frac{dy}{dx} = y, \; y(0) = 1.$$

Here $f(x,y) = y$. We start the Picard's iteration as follows:

- $y_1(x) = y(0) + \int_0^x f(t, y(0))dt = 1 + \int_0^x 1dt = 1 + x.$
- $y_2(x) = 1 + \int_0^x f(t, y_1(t))dt = 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2}.$
- $y_3(x) = 1 + \int_0^x f(t, y_2(t))dt = 1 + \int_0^x (1 + t + \frac{t^2}{2})dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}.$
- Therefore, by induction we get,
  $y_n(x) = 1 + \int_0^x f(t, y_{n-1}(t))dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$
- Now as $n \to \infty$, $y_n(x) \to e^x$. So, $y(x) = e^x$ is the exact solution of the ODE.

Question 23(b)

Solution: We have

$$\frac{dy}{dx} = x + y,\ y(0) = -1.$$

Here $f(x, y) = x + y$. We start the Picard's iteration as follows:

- $y_1(x) = y(0) + \int_0^x f(t, y(0))dt = -1 + \int_0^x -1dt = -1 - x.$
- $y_2(x) = -1 + \int_0^x f(t, y_1(t))dx = -1 + \int_0^x (t - 1 - t)dt = -1 - x.$
- $y_3(x) = -1 + \int_0^x f(t, y_2(t))dx = -1 - x.$
- Therefore, by induction we get,
  $y_n(x) = 1 + \int_0^x f(0, y_{n-1}(t))dx = -1 - x.$
- Now as $n \to \infty$, $y_n(x) \to (-1 - x)$. So, $y(x) = -1 - x$ is the exact solution of the ODE.

Question 24

Consider the following IVP: $ydy = xdx,\ y(0) = \beta$. Find all possible $\beta \in \mathbb{R}$ for which the IVP has

(a) a unique solution,

(b) more than one solutions,

(c) no solutions.

Question 24

Solution:
Solving the ODE $ydy = xdx$ and applying the initial condition $y(0) = \beta$, we get

$$y^2 - x^2 = \beta^2.$$

(a) For all $\beta \neq 0$, we get unique solution.
  - When $\beta > 0$ the solution is given as $y = \sqrt{x^2 + \beta^2}$.
  - When $\beta < 0$ the solution is given as $y = -\sqrt{x^2 + \beta^2}$.

(b) For $\beta = 0$, there exist two solutions $y = x,\ y = -x$.

(c) No solution case never occurs for any β.

Question 25

Question 25

Consider the following IVP: $\frac{dy}{dx} = f(x, y).y(0) = 0$ where

$$f(x, y) = \frac{\sin(x + y) + \cos(x + y)}{1 + x^2 + y^2}, forx, y \in [-1, 1].$$

Using existence uniqueness theorem, find the largest interval in which it has a unique solution.

Question 25

Solution:

- IVP:
$$\frac{dy}{dx} = f(x,y),\ y(0) = 0.$$
- Here
$$f(x,y) = \frac{\sin(x+y) + \cos(x+y)}{1+x^2+y^2},\ x, y \in [-1,1].$$
- Now
$$\frac{\partial f}{\partial y} = \frac{(x^2+(y-1)^2)\cos(x+y) - (x^2+(y+1)^2)\sin(x+y)}{(1+x^2+y^2)^2}.$$
- Clearly $\frac{\partial f}{\partial y}$ is bounded for all $x, y \in [-1,1]$.

Question 25 contd.

- Thus f is Lipscitz function and then by Picard's theorem there exists $h_0 = \min\{-1, 1/M_0\}$, where $M_0 = \max_{R_0} f(x, y)$ and $R_0 = \{(x, y) : |x| \leq 1, |y| \leq 1\}$ such that the IVP has a unique solution in $[-h_0, h_0]$.
- Now set
$$y(h_0) = k_0.$$
- Then using the similar argument as above for the IVP:
$$\frac{dy}{dx} = f(x, y),\ y(h_0) = k_0,$$
we get a $h_1 = \min\{-1, 1/M_1\}$, where $M_1 = \max_{R_1} f(x, y)$ and $R_1 = \{(x, y) : |x - h_0| \leq 1, |y - k_0| \leq 1\}$ such that the IVP has a unique solution in $[h_0 - h_1, h_0 + h_1]$.
- Proceeding in this way, we can extend the interval $[-h_0, h_0]$ up to $[-1, 1]$ (since $[-1, 1]$ is compact) where we get a unique solution of the given ODE.