3-4

(a) To investigate whether $\psi_0(x)$ and $\psi_1(x)$ are eigen functions of Hamiltonian or not we act $\widehat{H}$ on them:

$$\widehat{H}\psi_0(x) = \left(\frac{\widehat{P}^2}{2} + \frac{\widehat{X}^2}{2}\right)\psi_0(x) = \left(\frac{1}{2}\left(-i\frac{d}{dx}\right)^2 + \frac{\widehat{X}^2}{2}\right)\psi_0(x) = \left(\frac{-1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}\right)e^{-\frac{x^2}{2}} = \frac{1}{2}\psi_0(x)$$

$$\widehat{H}\psi_1(x) = \left(\frac{-1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}\right)xe^{-\frac{x^2}{2}} = \frac{-1}{2}\frac{d^2}{dx^2}\left(xe^{-\frac{x^2}{2}}\right) + \frac{x^3}{2}e^{-\frac{x^2}{2}} = \frac{-1}{2}\frac{d}{dx}\left(e^{-\frac{x^2}{2}} - x^2e^{-\frac{x^2}{2}}\right) + \frac{x^3}{2}e^{-\frac{x^2}{2}}$$
$$= \frac{3}{2}\psi_1(x)$$

Thus $\psi_0(x)$ and $\psi_1(x)$ are eigen functions of Hamiltonian with eigen values $\frac{1}{2}$ and $\frac{3}{2}$ respectively.

(b) orthogonality condition for $\psi_0(x)$ and $\psi_2(x)$ is $\langle\psi_0|\psi_2\rangle = 0$

So

$$\langle\psi_0|\psi_2\rangle = \int_{-\infty}^{\infty} dx\psi_0{}^*(x)\psi_2(x) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}(1+\alpha x^2)e^{-\frac{x^2}{2}}dx = \int_{-\infty}^{\infty}(1+\alpha x^2)e^{-x^2}dx = \sqrt{\pi}\left(1+\frac{\alpha}{2}\right)$$
$$= 0$$

$$\Rightarrow \quad \alpha = -2$$

Now we can calculate $\widehat{H}\psi_2(x)$

$$\widehat{H}\psi_2(x) = \left(\frac{-1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}\right)(1-2x^2)e^{-\frac{x^2}{2}} = \frac{-1}{2}\frac{d}{dx}\left[(2x^3-5x)e^{-\frac{x^2}{2}}\right] + \frac{x^2}{2}(1-2x^2)e^{-\frac{x^2}{2}} = \frac{5}{2}\psi_2(x)$$

Thus $\psi_2(x)$ is also eigen function of Hamiltonian with eigen value equal to $\frac{5}{2}$.

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Translation :