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(a)

$$\Psi(x,t) = e^{\frac{-i\hat{H}t}{\hbar}}\Psi(x,0) = \frac{1}{\sqrt{8\pi}} e^{\frac{-i\hat{H}t}{\hbar}}\left(e^{-\frac{x^2}{2}}\right)$$

$$+ \frac{1}{\sqrt{18\pi}} e^{\frac{-i\hat{H}t}{\hbar}}\left[\left(1-2x^2\right)e^{-\frac{x^2}{2}}\right]$$

$$= \frac{1}{\sqrt{8\pi}} e^{\frac{-it}{\hbar}\left(\frac{1}{2}\right)} e^{-\frac{x^2}{2}} + \frac{1}{\sqrt{18\pi}} e^{\frac{-it}{\hbar}\left(\frac{5}{2}\right)} (1-2x^2) e^{-\frac{x^2}{2}}$$

(b)

Normalization of $\psi_0(x)$ and $\psi_2(x)$ yields

$$\psi_{0N}(x) = \frac{1}{\pi^{1/4}} e^{-\frac{x^2}{2}} , \quad \psi_{2N}(x) = \frac{2}{3}\frac{1}{\pi^{1/4}}(1-2x^2)e^{-\frac{x^2}{2}}$$

but $\psi(x,0)$ is not normalize

$$\psi(x,0) = \frac{1}{\pi^{1/4}}\left(\frac{1}{\sqrt{8}}\psi_{0N}(x) + \frac{3}{2\sqrt{18}}\psi_{2N}(x)\right)$$

if we normalize it

$$\psi_N(x,0) = \frac{1}{\sqrt{2}}\psi_{0N}(x) + \frac{1}{\sqrt{2}}\psi_{2N}(x)$$

$$\Rightarrow P_0 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$P_2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

P_0 and P_2 are not going to change over time because ψ_0 and ψ_2 are stable states.

(c)

$$\psi_N(x,t) = \frac{1}{\sqrt{2}} e^{\frac{-it}{2\hbar}} \psi_{0N}(x) + \frac{1}{\sqrt{2}} e^{\frac{-5it}{2\hbar}} \psi_{2N}(x)$$

$$\rho(x,t) = \psi_N^*(x,t)\,\psi_N(x,t)$$

$$= \left(\frac{1}{\sqrt{2}} e^{\frac{it}{2\hbar}} \psi_{0N}(x) + \frac{1}{\sqrt{2}} e^{\frac{5it}{2\hbar}} \psi_{2N}(x)\right)$$

$$\left(\frac{1}{\sqrt{2}} e^{\frac{-it}{2\hbar}} \psi_{0N}(x) + \frac{1}{\sqrt{2}} e^{\frac{-5it}{2\hbar}} \psi_{2N}(x)\right)$$

$$= \frac{1}{2} \psi_{0N}^2(x) + \frac{1}{2} \psi_{2N}^2(x) + \frac{1}{2}\left(e^{\frac{4it}{2\hbar}} + e^{\frac{-4it}{\hbar}}\right) \psi_{0N}(x)\,\psi_{2N}(x)$$

$$= \frac{1}{2}\left(\psi_{0N}^2(x) + \psi_{2N}^2(x)\right) + \cos^2\left(\frac{\cancel{4}t}{\cancel{2}\hbar}\right) \psi_{0N}(x)\,\psi_{2N}(x)$$

$$\vec{J}(x,t) = \frac{\hbar}{m} \operatorname{Im}\left(\psi_N^*(x,t)\,\vec{\nabla}\psi_N(x,t)\right)$$

$$\vec{\nabla}\psi_N(x,t) = \frac{1}{\sqrt{2}} e^{\frac{-it}{2\hbar}} \vec{\nabla}\psi_{0N}(x) + \frac{1}{\sqrt{2}} e^{\frac{5it}{2\hbar}} \vec{\nabla}\psi_{2N}(x)$$

$$\vec{\nabla}\psi_{0N}(x) = -\hat{i}\,\pi^{\frac{-1}{4}}\, x e^{-\frac{x^2}{2}}$$

$$\vec{\nabla}\psi_{2N}(x) = \hat{i}\,\frac{2}{3}\pi^{-1/4}(2x^3 - 5x)\,e^{-\frac{x^2}{2}}$$

$$\Rightarrow \vec{\nabla}\psi_N(x,t) = \frac{1}{\sqrt{2}} e^{\frac{-it}{2\hbar}} \left(\frac{-\hat{i}}{\pi^{1/4}}\right) x e^{-\frac{x^2}{2}}$$

$$+ \frac{1}{\sqrt{2}} e^{\frac{-5it}{2\hbar}} \frac{\hat{i}\,2}{3\pi^{1/4}} (2x^3 - 5x)\, e^{-\frac{x^2}{2}}$$

$$\psi^*_{\sim}\vec{\nabla}\psi_{\sim} = \frac{1}{2}\left(\frac{-\hat{i}}{\pi^{1/2}}\right)x e^{-\frac{x^2}{2}} + \frac{1}{2}e^{\frac{-4it}{2\hbar}}\,\hat{i}\,\frac{2}{3\pi^{1/2}}(2x^3-5x)e^{-x^2}$$

$$-\frac{1}{2}e^{\frac{4it}{2\hbar}}\,\hat{i}\,\frac{2}{3\pi^{1/2}}x(1-2x^2)e^{-x^2} + \frac{1}{2}\frac{4\hat{i}}{9\pi^{1/2}}(2x^3-5x)(1-2x^2)e^{-x^2}$$

$$\Rightarrow \vec{j}(x,t) = \frac{\hbar}{m}\left[\frac{-\hat{i}}{3\pi^{1/2}}\sin\left(\frac{2t}{\hbar}\right)(2x^3-5x)e^{-x^2}\right.$$

$$\left.\frac{-\hat{i}}{3\pi^{1/2}}\sin\left(\frac{4t}{2\hbar}\right)x(1-2x^2)e^{-x^2}\right]$$

$$= \frac{\hat{i}\hbar\,4x}{m\,3\sqrt{\pi}}\sin\left(\frac{2t}{\hbar}\right)e^{-x^2}$$

(d)

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = \frac{-4}{2\hbar}\sin\left(\frac{2t}{\hbar}\right)\frac{2}{3\sqrt{\pi}}(1-2x^2)e^{-x^2}$$

$$+\frac{\hbar}{m}\frac{4x}{3\sqrt{\pi}}\sin\left(\frac{2t}{\hbar}\right)(1-2x^2)e^{-x^2} = 0$$

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