MTL101:: Tutorial 2 :: Linear Algebra

Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{P}_n := \{f \in \mathbb{F}[x] : \deg f < n\}$

(1) Suppose $v_1 = (1, 2)$ $v_2 = (0, 1) \in \mathbb{R}^2$.
  (a) Describe geometrically the subsets $W_1 := \{tv_1 : t \in \mathbb{R}\}$, $W_2 := \{tv_2 : t \in \mathbb{R}\}$, $W_3 := \{sv_1 + tv_2 : s, t \in \mathbb{R}\}$ and $W_4 := \{sv_1 + tv_2 : 0 \le s, t \le 1\}$.
  (b) Which of W_1, W_2, W_3, W_4 are subspaces of $\mathbb{R}^2$? Justify your answer in each case.
  (c) Show that $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^2$.
  (d) Suppose $v_3 = (2, 3)$. Is $\{v_1, v_2, v_3\}$ linearly independent?

(2) Suppose $V := \mathbb{C}^2$ is the complex vector space (over $\mathbb{C}$) under component-wise addition.
  (a) Show that $\{(1 + i, 2),\ (2, 1)\}$ is linearly independent.
  (b) Show that $\{(1, 2), (0, i), (i, 1 - i)\}$ is linearly dependent.
  (c) Show that every ordered pair can be written as a linear combination of $v_1 = (1 + i, 2)$ and $v_2 = (2, 1)$. Also show that up to change of order (of v_1 and v_2) such a linear combination is unique (for each ordered pair).
  (d) Show that every ordered pair can be written as a linear combination of $v_1 = (1, 2)$, $v_2 = (0, i)$, $v_3 = (i, 1 - i)$ in more than one ways.

(3) Show that $X = \{(1 + i, 1 - i), (1 - i, 1 + i), (2, i), (3, 2i)\}$ is linearly independent in $\mathbb{C}^2(\mathbb{R})$. Express $(a + ib, c + id)$ as an $\mathbb{R}$-linear combination of vectors belonging to X.

(4) Let V be a vector space over $\mathbb{F}$. Show that $u, v, w \in V$ are linearly independent if and only if $u + v, v + w, w + u$ are linearly independent.

(5) (a) Find the coordinates of $(a, b, c) \in \mathbb{R}^3$ relative to the ordered basis $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.
  (b) Find the coordinates of $a + bX + cX^2$ relative to the ordered basis $\{1, 1 + X, 1 + X^2\}$ in the space $\mathcal{P}_3$ of polynomials of degree at most 2 with coefficients from $\mathbb{R}$.
  (c) Find the coordinate vectors of an element of $\mathbb{R}^3$ with respect to the following bases $B_1 = \{(1, 2, 1), (1, 2, 3), (0, 1, 1)\}$ and $B_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$. Also write the change of coordinate matrix.

(6) (a) Show that if $v \in V$ then $\mathbb{F}v := \{\lambda v : \lambda \in \mathbb{F}\}$ is a subspace of any vector space V over $\mathbb{F}$.
  (b) Show that if W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.
  (c) Show that the intersection of any collection of subspaces of a vector space is a subspace.
  (d) Suppose W_1 and W_2 are subspaces of a vector space V. Show that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_1 \supset W_2$.
  (e) Let X be a nonempty subset of a vector space V over $\mathbb{F}$. Let $\text{span}(X) := \{\sum_{i=1}^{n} a_i v_i : n \in \mathbb{N}, a_i \in \mathbb{F}, v_i \in X\}$ and let $\langle X \rangle$ be the intersection of all the subspaces of V which contain X. Show that $\text{span}(X)$ and $\langle X \rangle$ are subspaces of V. Also show that $\text{span}(X) = \langle X \rangle$.

(7) In each case show that $W_1 + W_2 = V$ (directly) and find $\dim(W_1 \cap W_2)$. Verify the formula $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.
  (a) $V = \mathbb{R}^2$, W_1 is the X-axis, W_2 is the Y-axis.
  (b) $V = \mathbb{R}^2$, W_1 and W_2 are distinct lines through the origin.
  (c) $V = \mathbb{R}^3$, W_1 is the XY plane and W_2 is the YZ plane.
  (d) $V = M_n(\mathbb{R})$, $W_1 = \{A \in M_n(\mathbb{R})\ :\ A$ is upper triangular $\}$, $W_2 := \{A \in M_n(\mathbb{R})\ :\ A$ is lower triangular $\}$.
  (e) $V = M_n(\mathbb{R})$, where W_1 is the space of $n \times n$ symmetric matrices and W_2 is the space of $n \times n$ skew-symmetric matrices.

(8) Which of the following is a linear transformation? Justify.
  (a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_1(x, y) = (x^2 + y^2, x - y)$ over $\mathbb{R}$.
  (b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_2(x, y) = (x + y + 1, x - y)$ over $\mathbb{R}$.
  (c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_3(x, y) = (ax + by, cx + dy)$ over $\mathbb{R}$.
  (d) $T_4 : \mathbb{C} \to \mathbb{C}$ define by $T(z) = \bar{z}$ over $\mathbb{C}$.
  (e) $\mathbb{R}^2$ to $\mathbb{R}^2$ the rotation about the origin by an angle θ. (Write an expression for rotation.)
  (f) $T_5 : M_{m \times n}(\mathbb{F}) \to M_{n \times m}(\mathbb{F})$ defined by $T_5(A) = A^t$ (A^t is the transpose of A).
  (g) $T_6 : M_n(\mathbb{F}) \to \mathbb{F}$ defined by $T_6(A) = \text{tr}(A)$.
  (h) $T_7 : \mathcal{P}_n \to \mathcal{P}_n$ such that $T_7(p)(x) = p(x - 1)$.
  (i) $T_8 : \mathcal{P}_n \to \mathcal{P}_{n+1}$ such that $T_8(p)(x) = xp(x) + p(1)$.

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