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(a) considering orthonormality of $|l,m\rangle$s we have

$$1 = \langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{7}}\right)^2 + A^2 + \left(\sqrt{\frac{2}{7}}\right)^2 = \frac{3}{7} + A^2$$

$$\rightarrow A = \pm\frac{2}{\sqrt{7}}$$

(b) $|l,m\rangle$s are eigenstates of both $\hat{\vec{L}}^2$ and $\hat{L}_z$ thus

$$\langle\hat{\vec{L}}^2\rangle = 1(1+1)\hbar^2 = 2\hbar^2$$

$$\langle\hat{L}_z\rangle = \left(\frac{1}{\sqrt{7}}\right)^2(-\hbar) + \frac{4}{7}(0\hbar) + \left(\sqrt{\frac{2}{7}}\right)^2(+\hbar) = \frac{\hbar}{7}$$

To find $\langle\hat{L}_x\rangle$ and $\langle\hat{L}_y\rangle$ first we find $\langle\hat{L}_\pm\rangle$ with this formula

$$\hat{L}_\pm|l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}|l,m\pm 1\rangle$$

$$\hat{L}_\pm|\psi\rangle = \frac{1}{\sqrt{7}}\hat{L}_\pm|1,-1\rangle + A\hat{L}_\pm|1,0\rangle + \sqrt{\frac{2}{7}}\hat{L}_\pm|1,1\rangle$$

$$= \hbar\left[\frac{1}{\sqrt{7}}\sqrt{(1\pm 1)(1\mp 1+1)}|1,-1\pm 1\rangle + A\sqrt{(1\mp 0)(1\pm 0+1)}|1,0\pm 1\rangle \right.$$
$$\left. + \sqrt{\frac{2}{7}}\sqrt{(1\mp 1)(1\pm 1+1)}|1,1\pm 1\rangle\right]$$

$$\langle\hat{L}_+\rangle = \langle\psi|\hat{L}_+|\psi\rangle = \hbar\sqrt{\frac{2}{7}}A\left(1+\sqrt{2}\right)$$

$$\langle\hat{L}_-\rangle = \langle\psi|\hat{L}_-|\psi\rangle = \hbar\sqrt{\frac{2}{7}}A\left(1+\sqrt{2}\right)$$

So

$$\langle\hat{L}_x\rangle = \frac{\langle\hat{L}_+\rangle + \langle\hat{L}_-\rangle}{2} = \hbar\sqrt{\frac{2}{7}}\left(\pm\frac{2}{\sqrt{7}}\right)\left(1+\sqrt{2}\right) = \pm\hbar\frac{2\sqrt{2}}{7}\left(1+\sqrt{2}\right)$$

$$\langle\hat{L}_y\rangle = \frac{\langle\hat{L}_+\rangle - \langle\hat{L}_-\rangle}{2i} = 0$$

(c) $P_{+1} = \left(\sqrt{\frac{2}{7}}\right)^2 = \frac{2}{7}$

(d)

$$\hat{L}_+{}^2|\psi\rangle = \hat{L}_+\hbar\left[\frac{1}{\sqrt{7}}\sqrt{2}|1,0\rangle + A\sqrt{2}|1,1\rangle\right] = \frac{\sqrt{2}}{\sqrt{7}}\hat{L}_+\hbar|1,0\rangle = \frac{2}{\sqrt{7}}\hbar^2|1,1\rangle$$

$$\hat{L}_-{}^2|\psi\rangle = \frac{2\sqrt{2}}{\sqrt{7}}\hbar^2|1,-1\rangle$$

Thus

$$\left\langle 1,m\middle|\hat{L}_+{}^2\middle|\psi\right\rangle = \frac{2}{\sqrt{7}}\hbar^2\delta_{1,m}$$

$$\left\langle 1,m\middle|\hat{L}_-{}^2\middle|\psi\right\rangle = \frac{2\sqrt{2}}{\sqrt{7}}\hbar^2\delta_{-1,m}$$

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