6.13

(a) We have to find $\hat{J}_z \Psi_{21m_\ell m_s}$ and $\hat{J}_z$ is $\hat{L}_z + \hat{S}_z$ so $\hat{J}_z \Psi_{21m_\ell m_s} = \hat{L}_z \Psi_{21m_\ell m_s} + \hat{S}_z \Psi_{21m_\ell m_s}$

$$\hat{L}_z \Psi_{21m_\ell m_s} = R_{21}(r)\Big[\frac{1}{\sqrt{3}}(\hat{L}_z Y_{10}(\theta,\varphi))|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+ \frac{2}{\sqrt{3}}(\hat{L}_z Y_{11}(\theta,\varphi))|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$$

$$= R_{21}\frac{2}{\sqrt{3}}\hbar Y_{11}(\theta,\varphi)|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\hat{S}_z \Psi_{21m_\ell m_s} = R_{21}(r)\Big[\frac{1}{\sqrt{3}}Y_{10}(\theta,\varphi)\hat{S}_z|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+ \frac{2}{\sqrt{3}}Y_{11}(\theta,\varphi)\hat{S}_z|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$$

$$= R_{21}(r)\Big[\frac{1}{\sqrt{3}}Y_{10}(\theta,\varphi)\frac{\hbar}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+ \frac{2}{\sqrt{3}}Y_{11}(\theta,\varphi)\left(-\frac{\hbar}{2}\right)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$$

$$\Rightarrow \hat{J}_z \Psi_{21m_\ell m_s} = R_{21}(r)\frac{\hbar}{2}\Big[\frac{1}{\sqrt{3}}Y_{10}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+ \frac{2}{\sqrt{3}}Y_{11}(\theta,\varphi)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$$

$$= \frac{\hbar}{2}\Psi_{21m_\ell m_s}$$

(b) In measuring of z component of spin angular momentum of electron we will get $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ with probability of P_+ or P_- accordingly

$$P_+ = |\langle \tfrac{1}{2}, \tfrac{1}{2}|\psi\rangle|^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$P_- = |\langle \tfrac{1}{2}, \tfrac{-1}{2}|\psi\rangle|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$$

(c) $\hat{J}^2 = (\hat{L}+\hat{S})^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{L}_z\hat{S}_z + \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+$

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$

$$\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$$

$$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$$

$$\hat{L}^2\psi_{21m_\ell m_s} = R_{21}(r)\left[\frac{1}{\sqrt{3}}(\hat{L}^2Y_{10}(\theta,\varphi))|\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{2}{\sqrt{3}}(\hat{L}^2Y_{11}(\theta,\varphi))|\tfrac{1}{2},\tfrac{-1}{2}\rangle\right.$$

$$= R_{21}(r)\left[\frac{1}{\sqrt{3}}2Y_{10}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{2}{\sqrt{3}}2Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{-1}{2}\rangle\right]\hbar^2$$

$$= 2\hbar^2\psi_{21m_\ell m_s}$$

$$\hat{S}^2 \psi_{21m_\ell m_s} = R_{21}(r)\Big[\frac{1}{\sqrt{3}} Y_{10}(\theta,\varphi)\hat{S}^2 |\tfrac{1}{2},\tfrac{1}{2}>$$

$$+\frac{2}{\sqrt{3}} Y_{11}(\theta,\varphi)\hat{S}^2 |\tfrac{1}{2},\tfrac{-1}{2}>\Big]$$

$$= \frac{3}{4}\hbar^2 \psi_{21m_\ell m_s}$$

$$\hat{L}_z \psi_{21m_\ell m_s} = R_{21}(r)\,\frac{2}{\sqrt{3}}\,\hat{L}_z Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{-1}{2}>$$

$$= \frac{2}{\sqrt{3}}\hbar R_{21}(r) Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{-1}{2}>$$

$$2\hat{S}_z\hat{L}_z\psi_{21m_\ell m_s} = \frac{4}{\sqrt{3}}\hbar R_{21}(r) Y_{11}(\theta,\varphi)\hat{S}_z|\tfrac{1}{2},\tfrac{-1}{2}>$$

$$= \frac{-2}{\sqrt{3}}\hbar^2 R_{21}(r) Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{-1}{2}>$$

$$\hat{L}_+\psi_{21m_\ell m_s} = R_{21}(r)\,\frac{1}{\sqrt{3}}\,\hat{L}_+ Y_{10}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}>$$

$$= \sqrt{\frac{2}{3}}\hbar R_{21}(r) Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}>$$

$$\hat{S}_-\hat{L}_+\psi_{21m_\ell m_s} = \sqrt{\frac{2}{3}}\hbar R_{21}(r) Y_{11}(\theta,\varphi)\hat{S}_-|\tfrac{1}{2},\tfrac{1}{2}>$$

$$= \sqrt{\frac{2}{3}}\hbar^2 R_{21}(r) Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{-1}{2}>$$

$$\hat{S}_+\psi_{21m_\ell m_s} = R_{21}(r)\,\frac{2}{\sqrt{3}}\,Y_{11}(\theta,\varphi)\hat{S}_+|\tfrac{1}{2},\tfrac{-1}{2}>$$

$$= \hbar R_{21}(r)\,\frac{2}{\sqrt{3}}\,Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}>$$

$$\hat{S}_+\hat{L}_-\psi_{21m_\ell m_s} = \hbar R_{21}(r)\,\frac{2}{\sqrt{3}}\,\hat{L}_- Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}>$$

$$= \frac{2\sqrt{2}}{\sqrt{3}}\hbar^2 R_{21}(r) Y_{10}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}>$$

$$\Rightarrow \hat{J}^2 \psi_{21m_l m_s} = \left(2\hbar^2 + \frac{3}{4}\hbar^2\right)\psi_{21m_l m_s}$$

$$+ \hbar^2 R_{21}(r) Y_{11}(\theta,\varphi) |\tfrac{1}{2}, \tfrac{-1}{2}\rangle \left(\frac{-2}{\sqrt{3}} + \sqrt{\frac{2}{3}}\right)$$

$$+ \hbar^2 R_{21}(r) Y_{10}(\theta,\varphi) |\tfrac{1}{2}, \tfrac{1}{2}\rangle \, 2\sqrt{\frac{2}{3}}$$

$$= \hbar^2 R_{21}(r) \Big[\left(2\sqrt{2} + \frac{11}{4}\right) Y_{10}(\theta,\varphi) |\tfrac{1}{2}, \tfrac{1}{2}\rangle \frac{1}{\sqrt{3}}$$

$$+ \left(\frac{\sqrt{2}-2}{\sqrt{2}} + \frac{11}{4}\right) Y_{11}(\theta,\varphi) |\tfrac{1}{2}, \tfrac{-1}{2}\rangle \frac{\sqrt{2}}{\sqrt{3}} \Big]$$

$$= \left(\hbar^2 \frac{8\sqrt{2}+11}{4}\right) R_{21}(r) Y_{10}(\theta,\varphi) |\tfrac{1}{2}, \tfrac{1}{2}\rangle \frac{1}{\sqrt{3}}$$

$$+ \left(\hbar^2 \frac{4\sqrt{2}+15}{4}\right) R_{21}(r) Y_{11}(\theta,\varphi) |\tfrac{1}{2}, \tfrac{-1}{2}\rangle \frac{2}{\sqrt{3}}$$

so the probability of finding system in $\alpha Y_{10}(\theta,\varphi)|\tfrac{1}{2},\tfrac{1}{2}\rangle$ is $\frac{1}{3}$ and the Probability of finding system in $\beta Y_{11}(\theta,\varphi)|\tfrac{1}{2},\tfrac{-1}{2}\rangle$ is $\frac{2}{3}$