MTL101::LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
TUTORIAL 1

(1) Suppose we have a system of three linear equations in real coefficients and in two unknowns:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has a unique solution.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i \in \mathbb{R}$, $(i = 1, 2, 3)$, so that the above statements hold.

(2) Suppose we have a system of three linear equations in real coefficients and in three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has a unique solution.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i, d_i \in \mathbb{R}, (i = 1, 2, 3)$, so that the above statements hold.

(3) Suppose we have a system of three linear equations in real coefficients and three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

Interpret geometrically the following statements:
(a) the system has no solutions.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i, d_i \in \mathbb{R}, (i = 1, 2, 3)$, so that the above statements hold. Can you have unique solution ?

(4) Let $\vec{y} = A\vec{x}$ and $\vec{x} = B\vec{w}$ with A and B being 2×2 matrices and $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^2$. If $\vec{y} = C\vec{w}$, then find the relation between A, B and C.

(5) Let L_1, L_2 are lower triangular and U_1, U_2 are upper triangular, then which of the following matrices are lower triangular and upper triangular ?
(a) $L_1 + L_2$ (b) $U_1 + L_2$ (c) U_1^2 (d) L_1U_1 (e)U_1L_2

(6) Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ and } B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

Show that $\vec{y} = A\vec{x}$ is the rotation of vector $\vec{x}$ counter clockwise by angle α. Also compute $\vec{y} = A^n\vec{x}$ and $\vec{y} = AB\vec{x}$. Interpret the results geometrically.

(7) Which of the following matrices are row echelon and row reduced echelon matrix. Give a reason when the matrix is not row reduced echelon.

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(8) Find row reduce echelon matrix which row equivalent to the matrices in the previous question and their transposes.

(9) Show that every elementary matrix is invertible, and the inverse is an elementary matrix.

(10) Compute the rank of the following matrices. Using the rank determine which of these matrices are invertible.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}.$$

(11) Find the inverse of the invertible matrices in the previous question by reducing the matrix to row reduced echelon form (identity matrix).

(12) Write down the following matrices as the product of elementary matrices (whenever possible).

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}.$$

(13) Solve the following systems of equations by reducing the augmented matrix to the row reduced echelon form:
a) $x_1 - x_2 + 2x_3 = 1, 2x_1 + 2x_3 = 1, x_1 - 3x_2 + 4x_3 = 2,$
b) $x_1 + 7x_2 + x_3 = 4, x_1 - 2x_2 + x_3 = 0, -4x_1 + 5x_2 + 9x_3 = -9,$
c) $x_2 + 5x_3 = -4, x_1 + 4x_2 + 3x_3 = -2, 2x_1 + 7x_2 + x_3 = -1,$
d) $-2x_1 - 3x_2 + 4x_3 = 5, x_2 - x_3 = 4, x_1 + 3x_2 - x_3 = 2.$

(14) Consider the following system of equations:

$$x + 2y + z = 3, ay + 5z = 10, 2x + 7y + az = b.$$

a) Find all values of a for which the following system of equations has a unique solution. b) Find all pairs (a, b) for which the system has more than one solution.

(15) Find $a, b, c, p, q \in \mathbb{R}$ such that the following system has a solution:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & p \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

(16) Assume A, B are square matrices os same size, then,
(a) $\mathrm{trace}(A+B) = \mathrm{trace}(A) + \mathrm{trace}(B)$ and $\mathrm{trace}(AB) = \mathrm{trace}(BA)$.
(b) Let A, B are 2×2 matrices, then $\det(AB) = \det(A)\det(B)$.
(c) Find A, B such that $\det(A+B) \neq \det(A) + \det(B)$

(17) Show that computation of nth order determinant using expansion need $n!$ multiplication. If a multiplication takes 10^{-9} sec on a computer, compute the time needed for computing determinant of a 25×25 matrix.

END

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