RESEARCH

Swiss Finance Institute Research Paper Series N°21-76

Multi-Asset Financial Bubbles in an Agent-Based Model with Noise Traders' Herding Described by an N-Vector Ising Model

Davide Cividino
Politecnico di Torino

Rebecca Westphal
ETH Zurich

Didier Sornette
ETH Zurich, Southern University of Science and Technology (SUSTech) Shenzhen, and Swiss Finance Institute

Multi-asset financial bubbles in an agent-based model with noise traders' herding described by an n-vector Ising model

Davide Cividino[*]
Politecnico di Torino, Department of Applied Science and Technology, Corso Duca Degli Abruzzi 24, 10129, Torino, Italy

Rebecca Westphal[†]
ETH Zürich, Department of Management, Technology and Economics, Scheuchzerstrasse 7, CH-8092 Zürich, Switzerland

Didier Sornette
ETH Zürich, Department of Management, Technology and Economics, Scheuchzerstrasse 7, CH-8092 Zürich, Switzerland and Institute of Risk Analysis, Prediction and Management, Academy for Advanced Interdisciplinary Studies, Southern University of Science and Technology (SUSTech), Shenzhen, 518055, China
(Dated: November 9, 2021)

We present an agent-based model (ABM) of a financial market with $n > 1$ risky assets, whose price dynamics result from the interaction between rational fundamentalists and trend following imitative noise traders. The interactions and opinion formation of the noise traders are described by an extended $O(n)$ vector model, which generalise the Ising model used previously in ABMs with a single risky asset. Efficient rejection-free transition probabilities are derived to describe realistic investment decisions at the micro level of individual noise traders. The ABM is validated by testing for several characteristics of financial markets such as volatility clustering and fat-tails of the distribution of returns. Furthermore, the model is able to account for the development of endogenous bubbles and crashes. We distinguish three different regimes depending on the traders' propensity to imitate others. In the subcritical regime of the $O(n)$ vector model, the traders' opinions are idiosyncratic and no bubbles emerge. Around the critical value of the $O(n)$ vector model, cross-sectionally asynchronous bubbles emerge. Above the critical value, small random price fluctuations may be amplified by noise traders herding into a given asset, which then impels fundamentalists to re-equilibrate their more valuable portfolios that have become unbalanced, thus pushing the prices of the other assets upward. The resulting transient increase of the momenta of these assets triggers a reorientation of the noise traders' portfolios that further amplifies the burgeoning bubbles. We have thus identified a mechanism by which the cautious risk-adverse contrarian rebalancing strategy of fundamentalists leads to systemic risks in the form of cascades of bubbles spreading the whole financial market.

Usage: Manuscript for submission to Physical Review Research
keywords: financial bubbles; agent-based model; arbitrageurs; noise traders; fundamentalists; multi-assets; $O(n)$ vector model; synchronisation
JEL: C63; G01; G17

I. INTRODUCTION

One of the most important concepts in complex systems theory is the emergence of highly non-trivial collective phenomena from the repetitive interactions between a large number of agents. A clear analogy can be visualised between the interactions of the spins in a ferromagnetic material, which tend to align their orientations while the thermal agitation tends to push the system towards a disordered state, with the social imitation between agents that tends to polarize the class of agents towards a common preference while stochastic idiosyncratic opinions among agents favor lack of consensus. These two systems illustrate the ubiquitous fight between order and disorder, be it in a ferromagnetic material or in a group of investors, which leads to a rich phenomenology and dynamics associated with transitions between different regimes.

To describe decision making within social groups and specifically their polarization, in the 1970s, Weidlich [1] introduced the idea of representing them as a physical ensemble of interacting spins. The idea to use the Ising model to represent opinion dynamics was further developed for example by Galam *et al.* [2] who applied it to a strike process in a plant containing satisfied and dissatisfied workers and by Grabowski and Kosiński [3] who took the spatial location of individuals on a complex network into account. Another application of Ising-based decision models is the voter model by Holley and Liggett [4], in which the opinion of a voter is a binary variable stochastically changing under the influence of its neighbors' opinions. Roehner *et al.* [5] have shown that, un-

[*] davide.cividino@gmail.com
[†] rwestphal@ethz.ch

der partial information, the rational optimization of expected payoffs under a utility function that considers cultural norm, as well as herding, can be described by the Ising model. Many more Ising-like models have been developed to describe collective behavior of animal and human societies. See e.g. Ref.[6] for a review and references therein.

The approaches to model opinion formation in social systems are also useful to understand the investment decisions of traders in a financial market. Neoclassical economic theory is based on the assumption of agents' rationality and helped to describe many macroeconomic phenomena. However, the assumptions of rational representative agents and general equilibrium are hard pushed to explain extreme events such as bubbles and crashes. For instance, the dynamical stochastic general equilibrium (DSGE) models used by central banks to inform their monetary policies were impotent during the great financial crisis of 2008, as bubbles and crashes were by construction assumed impossible. This realisation has motivated renewed interest in agent-based models [7, 8], which emphasise the existence of many interactive decision makers with bounded rationality and subjected to limited and possibly asymmetric information. To better understand financial markets, it is crucial to embrace the fact that the world economy is a constantly evolving multi-agent complex system, that can be studied with agent-based models (ABM). In ABMs, the asset prices are endogenously defined by the agents' investment decisions. De Long *et al.* [9, 10] show that irrational traders with stochastic beliefs, so-called noise traders, can create endogenous financial bubbles from positive feedback in an ABM. Especially, the interaction of agents with heterogeneous beliefs can reproduce some of the characteristic features of financial markets known as "stylized facts". For example, Brock and Hommes [11] introduced an ABM in which traders switch between several available predictors of the future return based on the past performance of these predictors. In Lux and Marchesi [12], the traders switch between being rational fundamentalists and noise traders by comparing the expected returns of the strategies. Furthermore, the noise traders switch between optimistic and pessimistic mood influenced by the other traders' opinions. The model can reproduce some of the "stylized facts" such as excess kurtosis and volatility clustering.

The vast majority of ABMs have been concerned with modeling the dynamics of one risky asset, for instance a financial index or the stock of a firm traded in an organised exchange market, coexisting with a riskless asset such as a treasury bill. But a fundamental characteristic of investing is the possibility to diversify one's wealth among many assets. The typical investor, especially the large institutional investors, mutual funds, pension funds and the like that dominate the markets in terms of asset value under management, is in large part focused on diversifying his portfolio. Diversification of investments over the whole universe of assets has a vast literature in modern portfolio theory and practice [13–17]. The realisation that idiosyncratic risks could be diversified away by optimally constructed portfolios has led to many fundamental developments in asset pricing, starting with the Capital Asset Pricing model [18], the Arbitrage Pricing Theory [19], and the whole "industry" of factor models [20].

Some ABMs do model several assets. Xu *et al.* [21] developed an ABM in which both fundamentalists and noise traders maximise their expected utility under a CRRA utility function differing only in the construction of the expected return. Similarly, Chiarella *et al.* [22] introduced a model of heterogeneous agents that maximise a CARA utility function and build their expectation of the future return based on past observations. Here, we extend this literature and address some of the limitations of previous works. In the aforementioned ABMs, the traders individually optimise their investment strategy without considering the other traders' decisions. However, phenomena such as bubbles and crashes occur due to the tendency for traders to synchronise their trades, either in their buying pattern during a bubble or in their panic selling during a crash. In order to describe realistic price dynamics, we consider the opinion formation among a group of investors. In particular, our goal is to derive a multi-asset market model of interacting agents that is prone to develop bubbles and crashes in order to understand bubble formation among multiple assets. We are especially interested in the formation of time-synchronous and asynchronous bubbles, because the synchronous emergence of a bubble among multiple assets creates systemic risk and the crash of one asset can trigger the other assets to crash as well, potentially resulting in a systemic crash of the whole stock market.

Our model is an extension of the ABM with Ising-like characteristics introduced by Kaizoji *et al.* [23]. Fundamentalists and noise traders co-exist in this ABM of a financial market with fixed strategies (investors of a given type are characterised by a fixed strategy and there is no switching between the strategies as in some other models mentioned above). The rational fundamentalists maximize their expected utility whereas the noise traders invest based on momentum following and social imitation. The noise traders decisions are modelled by an underlying Ising model in which each spin represents one trader invested either in the risky or in the risk-free asset. The noise traders' decision to switch between the assets is influenced by the other traders' opinion, their idiosyncratic opinion, as well as the price momentum, which plays a role analogous to an external field of the Ising model. The collective opinion of the noise traders exhibit a phase transition that underpins the emergence of bubbles in the asset's price. Building on the ability of the Agent-based model by Kaizoji *et al.* [23] to create endogenous super-exponential bubbles in one asset, while being able to reproduce the most important features of financial markets, we present a multi-asset extension in which the social imitation and momentum following of traders are represented by an $O(n)$ vector model. We propose this formulation of the interactions between agents deciding to invest among several assets as an $O(n)$ vector model because it appears as the most elegant and rich description that naturally generalises the Ising model.

The paper is organised as follows. In the next section, the extension of the original market model by Kaizoji et al. [23] to a multi-asset framework consisting of one risk-free and multiple risky assets is introduced. This includes the generalization of the dividend process, the wealth dynamics, and the traders' decision process. A special emphasis is put on the noise traders' stochastic dynamics for the investment derived from an $O(n)$ model. The price equation is derived from the market-clearing conditions according to Walras' theory of general equilibrium [24]. In section III, the time series resulting from the traders' interactions are analyzed and the "stylized-facts" of financial markets are tested. Section IV examines the emergence of bubbles among the risky assets focusing on three different regimes of the noise traders resulting from the phase diagram of the $O(n)$ model. Section V concludes.

II. THE AGENT-BASED MARKET MODEL

The market model evolves according to a discrete-time dynamics, where each time-step represents one trading day. The market is constituted of two types of agents, the fundamentalists and the noise traders. They invest according to different strategies into one risk-free and n risky assets. Their investment decisions enter in the price equations, which govern the dynamics of the prices of the risky assets. In the following, we explain in detail the different components of the model.

A. The assets and the wealth dynamics

The model features one risk-free asset, representing a zero-coupon government bond with constant unitary price, yielding a constant rate of return r_f, and n risky assets, representing n stocks with time-varying prices $P_{k,t}$, paying stochastic dividends $d_{k,t}$ with $k \in \{1,\dots,n\}$. Allowing for possible correlations of their dynamics, we describe the time evolution of the dividends through n standard multiplicative growth processes (Geometric Brownian Motions)

$$\begin{cases} d_{1,t} = (1+r_t^{d,1})d_{1,t-1} \\ d_{2,t} = (1+r_t^{d,2})d_{2,t-1} \\ \quad\vdots \\ d_{n,t} = (1+r_t^{d,n})d_{n,t-1} \end{cases} \tag{1}$$

where the stochastic growth factors follow a multivariate normal distribution

$$(r_t^{d,1}, r_t^{d,2}, \dots, r_t^{d,n}) \sim \mathcal{N}(\vec{\mu}, \Sigma_d), \tag{2}$$

with mean $\vec{\mu} = (r_{d,1}, r_{d,2}, \dots, r_{d,n})$ and covariance matrix Σ_d. The stochastic dividend processes represent the impact of the real economy on the values of the stocks. Consequently, the covariance matrix embodies the exogenous effect of both volatilities and correlations coming from the real economy on the price formation mechanism. This will also have an effect on the presence or absence of synchronization between bubbles that can develop in different assets. In the following, the variances and correlations of the dividend processes are taken to be small in order to focus on the endogenous dynamics resulting from the traders' decisions and interactions.

The vector containing the total returns of the risky assets is given by

$$\vec{r}_t^{\,tot} = \vec{y}_t^{\,d} + \vec{r}_t^{\,p} , \tag{3}$$

where the dividend yields $\vec{y}_t^d$ are defined as

$$\vec{y}_t^{\,d} = \left(\frac{d_{1,t}}{P_{1,t-1}}, \dots, \frac{d_{n,t}}{P_{n,t-1}} \right) , \tag{4}$$

$P_{k,t}$ is the price of risky asset k at time t and the price returns are defined as

$$\vec{r}_t^{\,p} = \left(\frac{P_{1,t}}{P_{1,t-1}} - 1, \dots, \frac{P_{n,t}}{P_{n,t-1}} - 1 \right). \tag{5}$$

The traders' investment decisions are described in terms of the fraction of their wealth they invest into each asset. Thus, the portfolio of a trader i is constituted by n risky fractions $(x_{1,t}^i, \dots, x_{n,t}^i)$ and one risk-free fraction $x_{r_f,t}^i$. Borrowing and short-selling are not admitted in the market model, hence the wealth fractions including the risk-free fraction are constrained to $x_{k,t}^i \in [0,1]$. Moreover, the fractions must sum to one at each time step t: $x_{r_f,t}^i + \sum_{k=1}^n x_{k,t}^i = 1$. The wealth of trader i evolves according to

$$W_t^i = W_{t-1}^i \left[1 + r_f + \sum_{k=1}^n x_{k,t-1}^i r_t^{k,excess} \right], \tag{6}$$

where

$$r_t^{k,excess} = \left(\frac{d_{k,t}}{P_{k,t-1}} + \frac{P_{k,t}}{P_{k,t-1}} - 1 - r_f \right) \tag{7}$$

represents the excess return of the risky asset k with respect to the risk-free return r_f.

B. Fundamental value traders

The fundamentalists are rational risk-averse traders who at each time-step maximize the expected constant relative risk aversion (CRRA) utility function of their future wealth in terms of the risky fractions and for a given level of risk. The derivation of the wealth allocation of fundamentalists follows the description for one risky asset in [23] whereas the multi-asset extension follows Chiarella *et al.* [25]. At each time step, each fundamentalist trader constructs its portfolio $(x^f_{1,t}, \ldots, x^f_{n,t})$ solving the maximization problem

$$\underset{(x^f_{1,t},\ldots,x^f_{n,t})}{\arg\max} \; E_t[U(W^f_{t+1}(x^f_{1,t}, \ldots, x^f_{n,t})], \tag{8}$$

where U represents the CRRA utility function with constant risk aversion γ

$$U(W) = \begin{cases} \log(W) & \gamma = 1 \\ \frac{W^{1-\gamma}}{1-\gamma} & \gamma \neq 1. \end{cases} \tag{9}$$

Each fundamentalist trader solves the same optimization problem. Hence, the investment impact on prices can be considered at the aggregate level through a representative agent, whose wealth is equal to the sum of the wealth of all fundamentalists. Expanding the CRRA utility function to quadratic order, the maximization problem (8) has been solved in Xu *et al.* [21], and we report here the final solution. The fundamentalist portfolio allocation strategy is given by

$$\begin{pmatrix} x^f_{1,t} \\ \vdots \\ x^f_{n,t} \end{pmatrix} = \frac{1}{\gamma}\mathrm{Cov}^{-1} \begin{pmatrix} E_{r,1} + \frac{d_{1,t}(1+r^{d,1})}{P_{1,t}} - r_f \\ \vdots \\ E_{r,n} + \frac{d_{n,t}(1+r^{d,n})}{P_{n,t}} - r_f \end{pmatrix} \tag{10}$$

where Cov^{-1} is the inverse matrix of the expected covariances of the future price returns estimated by the fundamentalist traders. Furthermore, the fundamentalists build an expectation $(E_{r,1}, \ldots, E_{r,n})$ of the price returns of the risky assets. These expectations could in principle depend on time, but are assumed time-independent in the following for simplicity.

C. Noise traders

1. Structure of the model of noise traders

As in Kaizoji *et al.* [23], the intrinsically stochastic investment strategy of the noise traders is driven by social imitation and trend following, see also Lux and Marchesi [12]. The central feature characterizing the noise traders is an Ising-like structure of interactions between them, which accounts for the competition between the ordering force of social imitation and the disordering impact of idiosyncratic opinion. It is the existence of such an Ising-like structure that explains the emergence of bubbles and governs their dynamics. As shown in [23, 26], bubbles appear as the result of the progressive polarisation of opinions among noise traders via a genuine phase transition from a disordered state where all noise traders place independent random orders to an ordered state where they herd. In the following, we extend the Ising-like structure of the noise traders described in Kaizoji *et al.* [23] and Westphal and Sornette [26] to a multi-asset framework. Specifically, to model the noise traders, we introduce an n-vector Ising model (also known as the $O(n)$ model) on the fully connected graph with an external field whose components represent the price momenta.

Each of the N noise traders is associated with a spin vector

$$\vec{S}_i = (s_{i1}, \ldots, s_{in}) \in \mathbb{S}^{n-1} \tag{11}$$

representing its portfolio allocation. The positive components of the spin vector represent investments in the risky assets and the negative components represent investments in the risk-free asset. More precisely, the risky fraction invested in a, if s_{ia} is non-negative is

$$x_{ia} = s^2_{ia} \quad \text{if } s_{ia} \geq 0. \tag{12}$$

The sum of all the negative components squared represents at the aggregate level the risk-free fraction

$$\sum_{a: s_{ia}<0}^{n} s^2_{ia} = x_{r_f}. \tag{13}$$

This definition ensures that the condition of wealth conservation

$$\sum_{a=1}^{n} x_{ia} = 1. \tag{14}$$

is always satisfied as a consequence of the normalisation of the spin vector.

Each noise trader, i.e. spin, interacts with all the others. Additionally, a vectorial external field $\vec{H}$ acts on each spin, modeling the trend following attitude of the noise investors. In particular, each component of the vector $\vec{H}$ corresponds to the price momentum H_k

$$H_{k,t} = \theta H_{k,t-1} + (1-\theta)\left(\frac{P_{k,t}}{P_{k,t-1}} - 1\right), \tag{15}$$

associated with the respective risky asset k. This expression (15) defines an exponential moving average of the past price changes, and is an indicator of the asset performance. In (15), $\theta \in [0, 1)$ controls the characteristic time scale over which past returns are shaping the price momentum.

The interactions between the spins are defined via the Hamiltonian

$$\mathcal{H}(\{\vec{S}_1,\ldots,\vec{S}_N\}) = -\frac{1}{2N}\sum_{i\neq j=1}^{N}\vec{S}_i\cdot\vec{S}_j - \sum_{i=1}^{N}\vec{H}\cdot\vec{S}_i, \tag{16}$$

which assumes the same uniform tendency for imitation among any pairs $\{i,j\}$ of traders. A given configuration of spins, i.e., of portfolio allocations of all the traders, is assumed to occur with a probability given by the standard Boltzmann weight (or Logit function)

$$P(\{\vec{S}_1,\ldots,\vec{S}_N\}) = e^{-\kappa\mathcal{H}}\ . \tag{17}$$

In this expression (17), the parameter κ quantifies the propensity for herding: indeed, the larger κ is, the larger is the likelihood that the traders choose the same portfolio allocation corresponding to the minimum of $\mathcal{H}(\{\vec{S}_1,\ldots,\vec{S}_N\})$; in contrast, for small or vanishing κ's, all portfolio allocations become equally probable and noise traders allocate randomly their wealth to the assets. The parameter κ was introduced in Kaizoji *et al.* [23]. In the language of spin systems, it corresponds to the inverse temperature β of the standard $O(n)$ model and governs the relative importance of the common investment preferences shared by the noise traders with respect to the idiosyncratic opinion of each agent. See e.g. Sornette [6] for a justification from and parallel with the random utility formalism.

2. *Derivation of the transition probabilities*

We are interested in modeling the investments' dynamics of the noise traders and not just their average properties, hence in the following the transition probabilities characterizing the stochastic dynamics of the model are derived. Our goal is to construct a Markov-chain Monte Carlo (MCMC) having (17) as its equilibrium distribution, which defines realistic dynamics for the traders' investments. The latter constitutes a crucial point of the following derivation, because we cannot rely on standard methods to generate a stochastic dynamics for the $O(n)$ model unless they give rise to a realistic description of the investment strategy of the noise traders from the finance point of view. No standard method was found that well fitted the task, hence in this and in the following two sections, we derive an original method to generate a stochastic dynamics for the $O(n)$ model that allows for a realistic investment description from the finance perspective.

On each trading day, each noise trader updates her investment decision based on the information available up to the previous trading day. She first decides whether she wants to hold her previous portfolio or actively trade in the market, which will be defined in the following. The portfolio update is encoded mathematically by the time-dependent conditional transition probability

$$P(\vec{S}_l^t \mid \{\vec{S}_1^{t-1},\ldots,\vec{S}_{l-1}^{t-1},\vec{S}_{l+1}^{t-1},\ldots,\vec{S}_N^{t-1}\}) \tag{18}$$

that the agent l chooses the portfolio allocation $\vec{S}_l^t$ at time t, given the allocations $\{\vec{S}_1^{t-1},\ldots,\vec{S}_{l-1}^{t-1},\vec{S}_{l+1}^{t-1},\ldots,\vec{S}_N^{t-1}\}$ of all other agents at time $t-1$, which can be represented as a point on the $(n-1)$-sphere $\mathbb{S}^{n-1}$. For ease of notations, we denote $P(\vec{A},t)$ the specific value of the transition probability (18) for $\vec{S}_l^t = \vec{A}$. $P(\vec{A},t)$ is a solution of the general discrete time master equation $\frac{P(\vec{A},t)-P(\vec{A},t-\Delta t)}{\Delta t} = \int_{\vec{B}\in\mathbb{S}^{n-1}} W(\vec{B}\to\vec{A})P(\vec{B},t-\Delta t) - W(\vec{A}\to\vec{B})P(\vec{A},t-\Delta t)$, where the unit time increment $\Delta t = 1$ corresponds to one trading day.

Noise traders update their portfolio allocations according to the transition rates $W(\vec{A}\to\vec{B})$, which are specified in the following. We assume that the transition rates obey the condition of detailed balance, which reads

$$\frac{W(\vec{A}\to\vec{B})}{W(\vec{B}\to\vec{A})} = \frac{P(\vec{B})}{P(\vec{A})} \quad \forall \vec{A},\vec{B}\in\mathbb{S}^{n-1}\ , \tag{19}$$

where $P(\vec{A}) = \lim_{t\to+\infty}P(\vec{A},t)$ is the equilibrium conditional probability distribution that the set of noise traders adopt a given set of portfolio allocations $\{\vec{S}_l = \vec{A} \mid \vec{S}_1,\ldots,\vec{S}_{l-1},\vec{S}_{l+1},\ldots,\vec{S}_N\}$. Since the conditional probabilities $P(\vec{A})$ and $P(\vec{B})$ are defined on the same conditioning sets $\{\vec{S}_1,\ldots,\vec{S}_{l-1},\vec{S}_{l+1},\ldots,\vec{S}_N\}$ in expression (19), we have $\frac{P(\vec{B})}{P(\vec{A})} = \frac{P(\{\vec{S}_1,\ldots,\vec{S}_{l-1},\vec{S}_l=\vec{B},\vec{S}_{l+1},\ldots,\vec{S}_N\})}{P(\{\vec{S}_1,\ldots,\vec{S}_{l-1},\vec{S}_l=\vec{A},\vec{S}_{l+1},\ldots,\vec{S}_N\})}$. Using (17) with (16), we obtain $\frac{P(\vec{B})}{P(\vec{A})} = \frac{\frac{1}{Z}e^{-\kappa(-\frac{1}{2N}\sum_{i\neq j\neq l}\vec{S}_i\cdot\vec{S}_j-\frac{1}{N}\sum_{i\neq l}\vec{S}_i\cdot\vec{B}-\sum_{i\neq l}\vec{H}\cdot\vec{S}_i-\vec{H}\cdot\vec{B})}}{\frac{1}{Z}e^{-\kappa(-\frac{1}{2N}\sum_{i\neq j\neq l}\vec{S}_i\cdot\vec{S}_j-\frac{1}{N}\sum_{i\neq l}\vec{S}_i\cdot\vec{A}-\sum_{i\neq l}\vec{H}\cdot\vec{S}_i-\vec{H}\cdot\vec{A})}}$. Simplifying the common factors in the numerator and denominator, we get

$$\begin{aligned}\frac{P(\vec{B})}{P(\vec{A})} &= e^{-\kappa(-\frac{1}{N}\sum_i\vec{S}_i\cdot\vec{B}-\vec{H}\cdot\vec{B}+\frac{1}{N}\sum_i\vec{S}_i\cdot\vec{A}-\vec{H}\cdot\vec{A})} \\ &= e^{\kappa(\frac{\sum_i\vec{S}_i\cdot(\vec{B}-\vec{A})}{N}+\vec{H}\cdot(\vec{B}-\vec{A}))}\end{aligned} \tag{20}$$

up to a constant term of order $\frac{1}{N}$ negligible for large N. From (19), we obtain the ratio of the transition probabilities as

$$\frac{W(\vec{A}\to\vec{B})}{W(\vec{B}\to\vec{A})} = \frac{P(\vec{B})}{P(\vec{A})} = e^{\kappa(\frac{\sum_i\vec{S}_i\cdot(\vec{B}-\vec{A})}{N}+\vec{H}\cdot(\vec{B}-\vec{A}))}. \tag{21}$$

Since expression (21) provides only the ratio $\frac{W(\vec{A}\to\vec{B})}{W(\vec{B}\to\vec{A})}$, we need another equation or condition to specify the transition rate $W(\vec{A} \rightarrow \vec{B})$ (and thus $W(\vec{B} \rightarrow \vec{A})$). By analogy with spin system dynamics in Physics, one could imagine to use the standard Metropolis-Hastings rule [27], which assumes that the transition probabilities can be decomposed into a uniform move proposal probability $\mathcal{P}$ and a move acceptance probability $\mathcal{A}$:

$$W(\vec{A} \rightarrow \vec{B}) = \mathcal{P}(\vec{A} \rightarrow \vec{B})\mathcal{A}(\vec{A} \rightarrow \vec{B}), \tag{22}$$

Using (21), this would yield

$$\mathcal{A}(\vec{A} \rightarrow \vec{B}) = \begin{cases} 1 & \text{if } \left(\frac{\sum_i \vec{S}_i \cdot (\vec{B}-\vec{A})}{N} + \vec{H} \cdot (\vec{B} - \vec{A})\right) > 0 \\ e^{\kappa\left(\frac{\sum_i \vec{S}_i \cdot (\vec{B}-\vec{A})}{N} + \vec{H} \cdot (\vec{B}-\vec{A})\right)} & \text{otherwise.} \end{cases} \tag{23}$$

The uniform proposal probability together with equation (23) define an unrealistic dynamics where, at each time-step, each noise trader chooses at random a new investment portfolio to switch to and this trade decision is accepted according to the probability $\mathcal{A}(\vec{A} \rightarrow \vec{B})$ given by expression (23).

In order to obtain a more realistic investment allocation dynamics for the noise traders, we impose

$$\mathcal{A}(\vec{A} \rightarrow \vec{B}) = \mathcal{A}(\vec{B} \rightarrow \vec{A}) = 1 \quad \forall \vec{A}, \vec{B} \in \mathbb{S}^{n-1}. \tag{24}$$

Hence, from the detailed balance condition (19), it follows that

$$\frac{\mathcal{P}(\vec{A} \rightarrow \vec{B})}{\mathcal{P}(\vec{B} \rightarrow \vec{A})} = \frac{P(\vec{B})}{P(\vec{A})} = e^{\kappa\left(\frac{\sum_i \vec{S}_i \cdot (\vec{B}-\vec{A})}{N} + \vec{H} \cdot (\vec{B}-\vec{A})\right)}. \tag{25}$$

This equation is satisfied by setting

$$\mathcal{P}(\vec{A} \rightarrow \vec{B}) = \frac{1}{Z} e^{\kappa_t\left(\frac{\sum_i \vec{S}_i \cdot \vec{B}}{N} + \vec{H} \cdot \vec{B}\right)}. \tag{26}$$

where Z is a normalising constant. Normalizing, we finally get the following expression for the transition probabilities

$$W(\vec{A} \rightarrow \vec{B}) = \frac{e^{\kappa\left(\frac{\sum_i \vec{S}_i \cdot \vec{B}}{N} + \vec{H} \cdot \vec{B}\right)}}{\int_{\vec{K} \in \mathbb{S}^{n-1}} e^{\kappa\left(\frac{\sum_i \vec{S}_i \cdot \vec{K}}{N} + \vec{H} \cdot \vec{K}\right)}}. \tag{27}$$

Note that the transition probability from state $\vec{A}$ to state $\vec{B}$ is independent of the initial state $\vec{A}$, which is a typical property of mean-field models (indeed, the fully-connected $O(n)$ model is equivalent to the correspondent mean-field version in the large N regime). From a financial perspective, the independence of $W(\vec{A} \rightarrow \vec{B})$ with respect to $\vec{A}$ makes the noise traders impervious to the so-called "disposition effect" [28, 29]. In order to account for the disposition effect and the status-quo bias, we reintroduce the path dependence in the portfolio dynamics through a Bernoulli random number.

Each trading day, before proceeding with her trading decisions, the noise trader first decides to adjust her portfolio allocation or to be inactive on the financial market for that day and hold the previous portfolio allocation which she considers solid and profitable.

The trader decides to be active in the financial market to modify her portfolio allocation with a probability

$$P(active) = \min\left\{1, \frac{1}{t_h} e^{\kappa\|\vec{M}\|}\right\}, \tag{28}$$

where we have defined $\vec{M} = (1/N)\sum_i \vec{S}_i + \vec{H}$, and holds her previous portfolio allocation otherwise. When there is no clear momentum and the noise traders have heterogenous opinions, $P(active)$ reduces to $\frac{1}{t_h}$, showing that the parameter t_h represents the average number of trading days the noise trader keeps her assets in absence of herding behaviour. In this way, we can directly control the trading frequency of the time series characterizing the resulting market dynamics while allowing for a realistic behavior of the agents at the micro level. When noise traders herd, they tend to be constantly active. Expression in (28) constitutes a natural generalization to the multi-asset framework of the parameter p controlling the average holding time in the single risky asset case [23].

Summarizing, the investment decision of each noise trader boils down to a two-step decision process. First, the investor decides to be active for that trading day or to just hold its previous portfolio, according to a Bernoulli trial described by equation (28). In case the trader decides to be active, then her new portfolio is constructed according to expression (27). In the following two sections we focus in particular on the second step of the decision process, when the trader actively reallocates her investment.

3. *Decision-theoretic interpretation of the transition probabilities*

In this section, we show that equation (27) defines realistic investment strategies for noise traders by connecting the Ising-like $O(n)$ model with the framework of decision theory.

The form of the probability distribution (27) coincides with a continuous version of the Logit probability distribution for the arrival states. The Logit distribution is throughout used and studied in discrete choice theory. McFadden has shown that the Logit probability distribution models individuals who maximize a utility function which has an implicit random idiosyncratic part [30]. If each agent makes a choice s^* according to the following maximisation program

$$s^* = \arg\max_s \{\beta u_s + \eta_s\}, \tag{29}$$

where s represents a possible choice taken from a finite set, u_s is the deterministic part of the utility function, η_s is a random variable and β plays the role of the inverse temperature, McFadden proved that $\Pr(s^* = s)$ coincides with the Logit distribution if the random variable η_s is distributed according to the Gumbel distribution with cumulative distribution function

$$F_\eta(x) = e^{-e^{-\frac{x-\mu}{\lambda}}}. \tag{30}$$

The result of McFadden has been extended to continuous spaces of choices, see e.g. [6, 31], and therefore applies to our set-up with a continuous space of choices $\vec{B}$, located on the hypersphere $\mathbb{S}^{n-1}$. Hence, modelling the investment decisions with the multivariate distribution (27) amounts to considering traders who solve the maximisation problem

$$\vec{B}^* = \arg\max_{\vec{B}\in\mathbb{S}^{n-1}} \{\kappa u_{\vec{B}} + \eta_{\vec{B}}\}, \tag{31}$$

where the deterministic part of the utility function is given by

$$u_{\vec{B}} = \left(\frac{\sum_i \vec{S}_i}{N} + \vec{H}\right) \cdot \vec{B}. \tag{32}$$

This expression (32) implies that the decision $\vec{B}$ chosen by a trader tends to align with (i.e. imitate) the average decision of all the other traders (first term in the parenthesis in the r.h.s.) and is also favouring stocks that have higher momentum (second term in the parenthesis in the r.h.s.). Imitation is often the rational, or boundedly rational, attitude in the case where a single individual does not have access to sufficient information, and when there are reasons to believe that other individuals might have complementary knowledge from different sources. Imitation can also become the optimal strategy in the case where the stock prices are dominated by endogeneity [5], such as during self-fulfilling prophecies. The second term $\eta_{\vec{B}}$ in (31) represents the random idiosyncratic part, specific to each trader, which enters in the decision process. Formula (31) exemplifies how the herding propensity κ governs the relative importance of the deterministic part of the utility function common to all traders, hence pushing traders towards the same investments, in comparison with the random part modelled by the Gumbel distribution representing the importance of the idiosyncratic opinion of each individual.

This interpretation is useful to construct an algorithm to simulate the noise traders' investment decisions. Computationally, we need to sample realizations of the random vector $\vec{B} = (B_1, \ldots, B_n)$ from the multivariate probability distribution (27). This is unfeasible due to the high dimensionality of the distribution and the relation $\sum_{k=1}^n B_k^2 = 1$ between the components of the random vector. We propose to replace this direct sampling problem by discretizing the space of choices $\mathbb{S}^{n-1}$ and considering the discrete version of the maximization problem (31).

At the beginning of the simulation, we discretize the space of choices $\mathbb{S}^{n-1}$ once and for all in such a way that the detailed balance condition (19) holds. Then, at each time-step, the trading decision of each noise trader is generated according to the following algorithm. The conditional statement accounts for the first step of the trader's decision process discussed in the previous section, when the investor decides to actively trade or to just hold her previous portfolio.

Algorithm 1: Simulation of the noise traders' investments (decision-theoretic approach)

```
for each noise trader i do
    if Uniform(0,1) < min{1, (1/t_h) e^{κ‖M‖}}
    then
        for p ∈ {1, ..., N_points} do
            generate i.i.d. Gumbel RV η_p;
        end
        B*_i = argmax_{B_p} { κ_t( Σ_i S_i / N + H ) · B_p + η_p }
    else
        B* = previous time-step portfolio;
    end
end
```

Notwithstanding the fact that the deterministic quantities entering the decision process refer to the previous trading day and are common to all traders, the method suffers from the curse of dimensionality. The computational cost of the simulation is exponential in the number N_{points} used to discretize the hypersphere. As a large number of points are needed to ensure a good discretization and a realistic simulation, this constitutes a significant bottleneck. Moreover, the problem aggravates when augmenting the investment universe, i.e., when increasing the number n of risky assets. In order to have a realistic description of financial markets, one would like to eventually consider tens to hundreds of stocks. Thus, the simulation algorithm should be scalable as the number of risky assets n increases. To overcome the computational problems of the present method, in the next section, we derive a different algorithm to generate the stochastic dynamics of the noise traders' investments.

4. *Symmetry-based approach*

To sample efficiently from the probability density function (27), we develop a method that exploits its symmetry.

Let us denote θ the angle between the two vectors $\vec{M} = (1/N)\sum_i \vec{S}_i + \vec{H}$ and $\vec{B}$, which belong to the

Euclidean space $\mathbb{R}^n$. Hence, the dot products in the exponential terms in expression (27) can be written

$$\kappa_t \vec{M} \cdot \vec{B} = \kappa_t \|\vec{M}\| \cos\theta, \tag{33}$$

accounting for the normalisation of $\vec{B}$ to unity. This normalisation condition means that the tip of the vector $\vec{B}$ needs to be sampled on the hypersphere $\mathbb{S}^{n-1}$. In other words, the sets of equiprobable choices, i.e. equiprobable vectors $\vec{B}$, are defined by the conditions

$$\begin{cases} \frac{1}{\|\vec{M}\|}\vec{M} \cdot \vec{B} = \cos\theta \\ \|\vec{B}\| = 1 \end{cases} \tag{34}$$

The first condition defines an hyperplane in $\mathbb{R}^n$. In fact it can be written as

$$m_1 b_1 + m_2 b_2 + \ldots + m_n b_n = \cos\theta, \tag{35}$$

where $m_1, \ldots, m_n$ are fixed coefficients. The second condition enforces the choice vectors to belong to the hypersphere $\mathbb{S}^{n-1}$. The intersection of a hypersphere $\mathbb{S}^{n-1}$ and an n-dimensional hyperplane is a hypersphere $\mathbb{S}^{n-2}$. Indeed, the system (34) defines a hypersphere $\mathbb{S}^{n-2}$ in $\mathbb{R}^n$, with center

$$\vec{C} = \cos\theta \frac{1}{\|\vec{M}\|}\vec{M} \tag{36}$$

and radius

$$r = \sqrt{1-\cos^2\theta} = \sin\theta. \tag{37}$$

Then, sampling the transition rate $W(\vec{A} \to \vec{B})$ amounts to sampling an angle θ with the probability

$$P(\theta) \propto e^{\kappa_t \|\vec{M}\| \cos\theta} (\sin\theta)^{n-2}, \tag{38}$$

where the term $(\sin\theta)^{n-2}$ stems from the Jacobian over the degrees of freedom on the hypersphere $\mathbb{S}^{n-2}$ orthogonal to $\vec{M}$. We have thus reduced the sampling of the hypersphere $\mathbb{S}^{n-1}$ to the sampling of a single variable $\theta \in [0, 2\pi)$.

Having effectively overcome the computational problem related to the dimensionality, we can directly sample the angle θ with the rejection sampling method, which is presented in the following algorithm.

Algorithm 2: Rejection sampling from the univariate $P(\theta)$ distribution

```
while (u * e^{κ_t ||M||}) > e^{κ_t ||M|| cos θ}(sin θ)^{n-2} do
    θ = Uniform(0,1) * 2π;
    u = Uniform(0,1);
    (the uniform RVs are sampled i.i.d.)
end
return θ;
```

After sampling the angle θ, we choose a vector uniformly at random from the equiprobable set defined by that angle. This set of choices is the hypersphere $\mathbb{S}^{n-2}$ defined by equations (36) and (37), resulting from the conditions in (34). We can rely on an efficient algorithm to perform the sampling. Due to the spherical symmetry property of the multivariate normal distribution, the normalized random vector whose components are sampled in an i.i.d. manner from the standard normal distribution $\mathcal{N}(0,1)$

$$\vec{\mathcal{B}}_{unnorm} = \big(\mathcal{N}_1(0,1), \ldots, \mathcal{N}_{n-1}(0,1)\big) \tag{39}$$

$$\vec{\mathcal{B}}^*_{n-1} = \frac{\vec{\mathcal{B}}_{unnorm}}{\|\vec{\mathcal{B}}_{unnorm}\|} \tag{40}$$

is uniformly distributed on $\mathbb{S}^{n-2}$. To immerse the vector in $\mathbb{R}^n$, we have to add one extra zero component, for example at the beginning of the vector, effectively increasing its dimensionality by one. The new vector is

$$\vec{\mathcal{B}}^*_n = \big(0, \vec{\mathcal{B}}^*_{n-1}\big). \tag{41}$$

Moreover, the hypersphere has to be translated and its radius rescaled according to (36) and (37). Finally, in order to correctly represent the intersection between the higher dimensional $\mathbb{S}^{n-1}$ hypersphere and the hyperplane, the $\mathbb{S}^{n-2}$ hypersphere needs to be rotated in such a way that the unit versor, corresponding to the extra component added in (41), is put along the direction of the normalized vector $\frac{1}{\|\vec{M}\|}\vec{M}$.

We construct the orthogonal matrix R representing the rotation of the unit versor

$$\vec{X} = (1, 0, 0, \ldots, 0), \tag{42}$$

to the direction of the vector $\frac{1}{\|\vec{M}\|}\vec{M}$. The matrix R has to satisfy

$$\frac{1}{\|\vec{M}\|}\vec{M} = R\vec{X}. \tag{43}$$

In two or three dimensions, such a rotation is given by the standard matrices containing sine and cosine functions. In the general case of n dimensions, we use an approach based on Givens rotations to construct an efficient and numerically stable algorithm. In order to construct the rotation, we thoroughly refer to Zhelezov [32].

Eventually, the choice vector $\vec{B}$ sampled from the distribution (27), representing the noise trader's portfolio reallocation, is given by

$$\vec{B}^* = \sin\theta R \vec{\mathcal{B}}^*_n + \cos\theta \frac{1}{\|\vec{M}\|}\vec{M}. \tag{44}$$

In summary, the noise trader either decides to hold his previous portfolio as described in section II C 2 or decides on a new portfolio allocation according to the probability density function (27) following a symmetry-based sampling approach. This results in the following overall sampling algorithm.

Algorithm 3: Simulation of the noise traders' investments (symmetry-based approach)

if $Uniform(0,1) < \min\left\{1, \frac{1}{t_h} e^{\kappa \|\vec{M}\|}\right\}$ **then**
  sample angle θ from $P(\theta)$ with algorithm 2;
  sample uniformly a vector $\vec{\mathcal{B}}^*_{n-1} \in \mathbb{S}^{n-2}$;
  $\vec{\mathcal{B}}^*_n = (0, \vec{\mathcal{B}}^*_{n-1}) \in \mathbb{S}^{n-1}$;
  construct the rotation matrix R following [32];
  $\vec{B}^* = \sin\theta R \vec{\mathcal{B}}^*_n + \cos\theta \frac{1}{\|\vec{M}\|}\vec{M}$;
else
  $\vec{B}^*$ = previous time-step portfolio;
end
return $\vec{B}^*$;

D. Market clearing and price equations

The market price is set according to a Walrasian auction, i.e. at each time-step, supply and demand must equilibrate [24].

Setting the aggregate excess demand to zero in the model with only one risky asset leads to a second-order algebraic equation in the unknown P_t [23, 26]. The equation can be solved explicitly giving a unique positive price, which represents the new price of the risky asset resulting from the new set of demands. Extending the model to n risky assets, the equilibrium condition has to hold simultaneously for each asset.

Defining the excess demand from time $t-1$ to t for each risky asset k for the trader i with $i \in \{f, n\}$ as

$$\Delta D^{i,k}_{t-1\to t} = W^i_t x^i_{k,t} - W^i_{t-1} x^i_{k,t-1} \frac{P_{k,t}}{P_{k,t-1}} \quad (45)$$

together with the risky fractions defined in (10) for the fundamentalists and in (12) for the noise traders, the equilibrium condition translates into the system

$$\begin{cases} \Delta D^{f,1}_{t-1\to t} + \Delta D^{n,1}_{t-1\to t} = 0 \\ \Delta D^{f,2}_{t-1\to t} + \Delta D^{n,2}_{t-1\to t} = 0 \\ \quad \vdots \\ \Delta D^{f,n}_{t-1\to t} + \Delta D^{n,n}_{t-1\to t} = 0. \end{cases} \quad (46)$$

The system (46) is a non-linear system in the n unknowns $P_{1,t}, \ldots, P_{n,t}$, where each equation is a polynomial equation of degree $n+1$ in all the unknowns. The system is solved numerically with an iterative method, based on the Hybrid algorithm proposed in Powell [33, 34], derived from the classical Newton-Raphson algorithm. Using the prices at the previous time-step as initial condition for the numerical solver, the method consistently converges to the correct solution with positive prices, for a wide range of parameters compatible with real markets.

Parameters

Assets	$n = 4$	$r_f = 4 \times 10^{-5}$
	$r_{d,i} = 1.6 \times 10^{-4}\ \forall i$	$d_{i,0} = 1.6 \times 10^{-4}\ \forall i$
	$P_{i,0} = 1\ \forall i$	$\Sigma^d_{i,i} = 1.6 \times 10^{-5}\ \forall i$
Fundamentalist traders	$W^f_0 = 10^9$	$E_{r,i} = 1.6 \times 10^{-4}\ \forall i$
	$\Sigma^f_{i,i} = 0.0004\ \forall i$	
Noise traders	$W^n_0 = 10^9$	$N = 1000$
	$\theta = 0.99$	$H_{i,0} = 1.6 \times 10^{-4}\ \forall i$
Market	$T = 5000$	

TABLE I. Set of parameters and initial values used in the simulations. These are values that can be interpreted from our proposed correspondence between one time step and one trading day. An average holding time of ten trading days is imposed ($t_h = 10$).

E. Parameters

The market parameters and initial values are chosen such that each time-step represents a typical trading day. This translates into the fact that the standard deviation of returns over one time step is approximately equal to the daily volatility of 1-2% observed in real markets, here following the logic of Kaizoji *et al.* [23].

The complete set of parameters is reported in table I. All rates such as the risk-free return r_f, the initial dividend $d_{i,0}$, the dividend growth rate $r_{d,i}$, and the expected returns $E_{r,i}$, as well as variances are reported as daily values. Note that we focus our attention here on a market with four risky assets ($n = 4$).

Fundamentalists and noise traders are initialized with equal wealth $W^n_0 = 10^9$ and the simulations are conducted over $T = 5000$ time-steps, which corresponds to 20 years assuming that one year contains approximately 250 trading days. No correlation among the dividend processes is assumed.

Regarding the fundamentalist traders, the expected covariance matrix is determined by combining a vector of expected variances and a matrix of expected correlations. The variances are

$$\Sigma^f_{i,i} = 0.0004 \quad i = 1, \ldots, n = 4 \quad (47)$$

and the correlation matrix C^f is assumed to be diagonal (therefore with 1's along the diagonal and 0's elsewhere).

The initial investment decisions for fundamentalists and noise traders are as follows

$$\vec{x}^f_0 = (0.075, 0.075, 0.075, 0.075) \quad (48)$$

$$\vec{x}^n_0 = (0.125, 0.125, 0.125, 0.125), \quad (49)$$

where each component represents the investment into one of the $n = 4$ risky assets. The remaining fraction of each traders' wealth is invested into the risk-free asset. This means that each fundamentalist (resp. noise) trader puts initially 70% (resp. 50%) of her wealth in the riskless asset and the remaining 30% (resp. 50%) in the four risky assets.

The constant risk aversion is endogenously computed at the beginning of the simulation from the initial conditions as

$$\gamma = \frac{E_{r,1} + \frac{d_{1,0}(1+r^{d,1})}{P_{1,0}} - r_f}{\mathrm{Cov}_{1,1}x^f_{1,0} + \ldots + \mathrm{Cov}_{1,n}x^f_{n,0}}, \tag{50}$$

which constitutes a natural generalization of the original model's formula in Kaizoji *et al.* [23], also adopted in Damiani [35] and Kopp [36].

Following Kaizoji *et al.* [23] and Westphal and Sornette [26], we analyze the impact of both constant herding propensity κ and time-varying κ_t, in several ranges of values. The time-varying herding propensity parameter models the impact of a changing geopolitical and economical situation on the tendency of the noise traders to herd and is defined by an Ornstein-Uhlenbeck stochastic process as

$$\kappa_t = \kappa_{t-1} + \eta_\kappa(\mu_\kappa - \kappa_{t-1}) + \sigma_\kappa \nu_t, \tag{51}$$

with $\nu_t \sim \mathcal{N}\big(0,1\big)$. The mean reversion strength η_κ and the standard deviation σ_κ are explicitly indicated for each different simulation.

The core of the model is implemented in *C++*, following an object-oriented programming paradigm. To have reproducible results, a pseudo-random number generator with a random seed specified as a run-time parameter is used.

III. TIME SERIES ANALYSIS AND STYLIZED FACTS OF FINANCIAL MARKETS

A. Time series analysis

In this section, we present the time series resulting from a simulation characterized by the set of parameters introduced in the previous section. Figure 1 shows characteristic price time series of the four assets resulting from the traders' investment decisions. In this simulation, the herding propensity follows an Ornstein-Uhlenbeck process (51) to represent time-varying susceptibility to herding and momentum. The mean value $\mu_k = 0.98 \cdot \kappa_c$ in expression (51) is defined slightly below the critical value $\kappa_c = 4$ at which a transition of the underlying $O(n)$ model occurs between a disordered phase for $\kappa < \kappa_c$ to an ordered regime for $\kappa > \kappa_c$ of noise traders converging to the same opinion, as explained in the introduction. Given $\mu_k = 0.98 \cdot \kappa_c$ in expression (51), κ_t transiently fluctuates above κ_c and thus the system of noise traders temporarily enters into the ordered regime. The mean reversion strength $\eta = 0.013$ and the standard deviation $\sigma_\kappa = 0.25\kappa_c\sqrt{2\eta}$ are defined such that κ_t returns from two standard deviations above the mean to the subcritical regime in approximately $\Delta T = 250$ time steps, see Kaizoji *et al.* [23] for the derivation.

The price time series exhibit volatile and stable regimes while demonstrating a long-term growth rate that is similar for all four assets. The long-term growth rate is equal to the growth rate of the dividend process as verified in the case of a single risky asset [26]. Figure 2 exemplarily presents the dynamics of relevant variables for one of the assets in more detail. Comparing the time series of the price and of κ_t shows that the noise traders create bubbles by polarizing their opinion when κ_t is above the critical value. This is for example the case between $t = 4000$ and $t = 4500$ where the increase of the noise traders' fraction of wealth invested in the asset results in a bubble that grows over a time interval of two years. When κ_t is below the critical value, for example between $t = 1000$ and $t = 1500$, the noise traders' opinions are heterogeneous (corresponding to the disordered regime of the underlying $O(n)$ model), the price is stable and mainly controlled by the growth of the dividend. One can also observe bursts of increased as well as reduced return amplitudes, corresponding to the well-documented phenomenon of volatility clustering.

The price momentum for each asset used in the noise trader's investment decision is calculated as the exponential moving average of returns according to expression (15). Thus, it exhibits similar peaks and troughs as the price time series but lags a few time steps behind. The lag is controlled by the memory length $\sim 1/(1-\theta)$ the noise trader uses to calculate the momentum. Similarly, the fraction of wealth the noise trader invests in the asset contains peaks and valleys. The invested fraction controls the price changes and the price changes introduce a positive feedback on the risky fraction via the momentum. The fundamentalists invest counter-cyclically by investing proportionally to the dividend-price ratio. This means that they decrease their wealth fraction invested in the risky asset during a bubble and return to their normal risky fraction after the crash. Interestingly, one can observe that the fundamentalists become wealthier than the noise traders in the long-term. During bubbles, the noise traders' wealth increases due to their larger exposure to the risky asset. However, during a crash, they lose most of their transiently acquired wealth while the fundamentalists are less affected by the crash and tend to buy the risky asset when it is undervalued.

B. Stylized facts of the financial markets

Financial time series feature the presence of ubiquitous statistical properties independent of the details of the series itself [37, 38]. These emerging empirical properties have been observed across a wide range of instruments, markets, and time periods and they constitute the

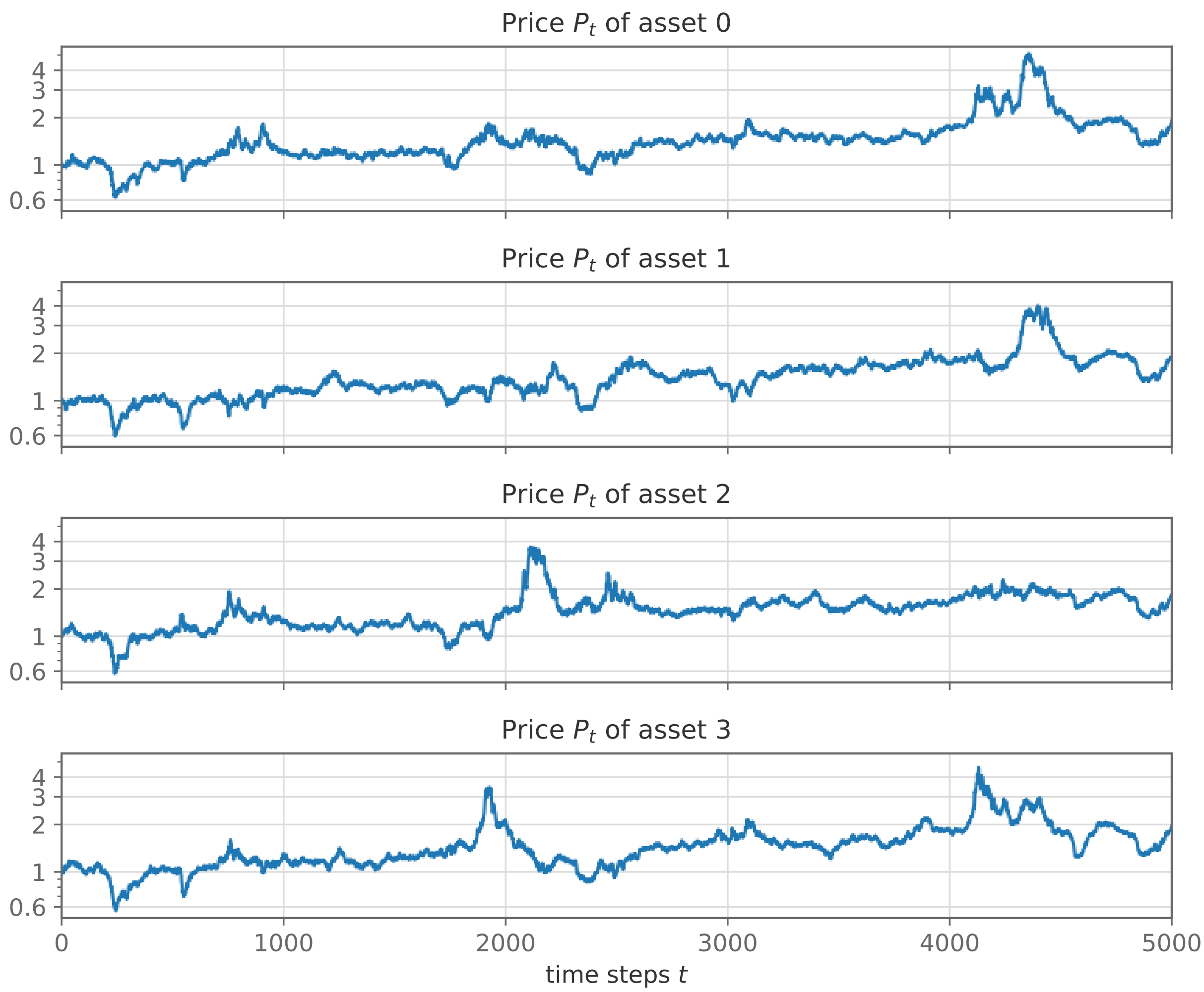

FIG. 1. Price time series obtained by a simulation with four risky and one risk-free assets featuring an Ornstein-Uhlenbeck process given by expression (51) for the imitation strength parameter κ_t, stochastically fluctuating near the critical value κ_c of the underlying $O(n)$ vector model. The simulation parameters are given in table I. Several bubbles are identifiable as super-exponential growths of the prices (note that the vertical axes for the prices are in logarithmic scale).

so-called stylized facts of the financial markets.

As an important step towards validating the model, it is crucial to check if the time series generated by the simulation of our model obey some of these stylized facts. We consider in particular two of them, the fat-tailedness of assets' returns and the long memory in the autocorrelation of the amplitude of returns. The time series analyzed are obtained from the simulation presented in the previous section.

1. The fat-tailedness of the distribution of the amplitude of returns

In this section, we compare the shape of the distribution of the amplitude x of the returns in our simulations of the market model to the observed leptokurtic behavior of their empirical counterparts. Empirical distributions exhibit a power tail

$$p(x) \sim x^{-1-\alpha} \tag{52}$$

with an exponent α in the interval $(2, 4)$ [37, 39–41]. As shown in figure 3, the fitted parameter from the simulated time series shown in figure 1 falls in this interval over a range of approximately one decade corresponding to the 20% largest return amplitudes. We exclude the ten largest values for the calibration as the seemingly faster decay for these ten largest values is compatible with careful comparisons of different families of distributions calibrated on empirical observations, which suggest that the extreme tail of the distribution may be characterised by a

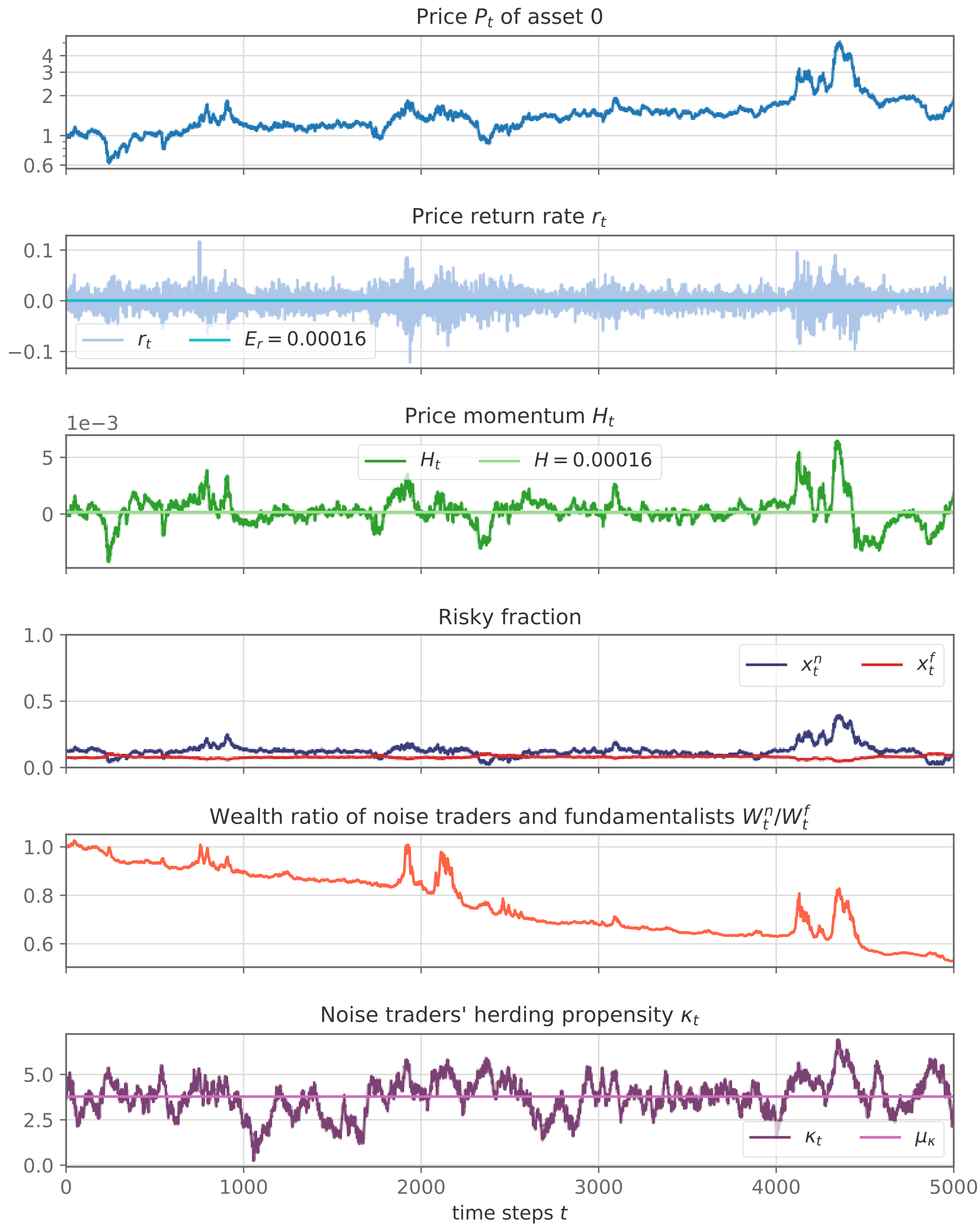

FIG. 2. The figure shows several time series characterizing the risky asset 0. From top to bottom, one can see the price, the actual return and the return expected by the fundamentalists, the price momentum, the fraction of the wealth that fundamentalists and noise traders invest into the asset, the ratio of the wealth of noise traders divided by the wealth of fundamentalists over time and the herding propensity κ_t. The market is simulated over $T = 5000$ time steps with the parameters given in Table I. For a comparison with the prices of the other assets, see figure 1.

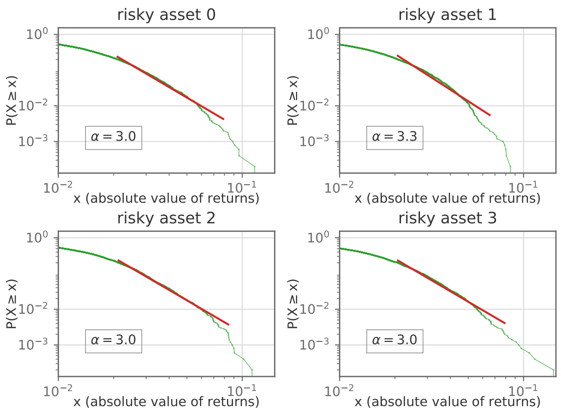

FIG. 3. Log-log plot of the complementary cumulative distribution functions of the amplitude of the returns of the four risky assets from the simulation presented in figure 1. The exponents are determined by fitting data from the last 20th percentile of the cumulative distribution, and disregarding the ten largest values.

decrease faster than a pure power law (see [41] and chapter 2 of [42]).

2. *The long memory in the autocorrelation of absolute returns*

The daily returns are not independent random variables. In financial markets, periods of tranquility alternate with periods of high volatility. This is referred to as volatility clustering and intermittency [37, 38, 43]. Figure 4 shows the autocorrelation functions (ACF) of the signed returns and of the absolute returns for the time series of returns for the four risky assets simulated with our model. One can observe that the statistical estimations of the ACF of the signed returns are fluctuating around 0, ensuring an approximate absence of arbitrage using linear factor decomposition. The absence of linear autocorrelation is a prerequisite for the often required "efficient market hypothesis". In contrast, the ACF of the absolute returns are significantly positive and decay very slowly: for most assets, the ACF are still visibly non zero and above their counterparts for the signed returns at time larger than 250 time steps. This later behavior is qualitatively agreeing with empirical facts [37, 38, 43]. However, we are not interested in reproducing a detailed quantitative agreement as we are cognisant of the fact that our agents lack a sufficient large set of time scales for their investments. Moreover, considering four risky assets is still a far cry from a real market made of thousands of stocks.

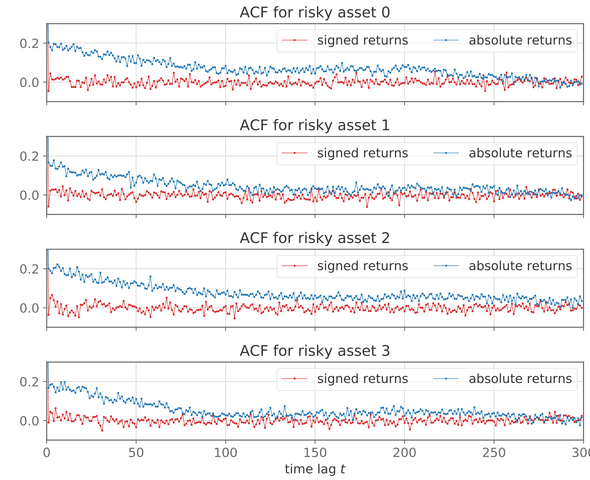

FIG. 4. Autocorrelation functions (ACF) of signed and absolute returns of the four risky assets from the simulation presented in figure 1. The autocorrelation functions are computed for the data after the 500th trading day to minimise the possible biases resulting from the initial conditions.

IV. EMERGENCE AND DEGREE OF SYNCHRONIZATION OF BUBBLES IN THE RISKY ASSETS

The analysis of the simulated time series shows the presence of three distinct regimes characterizing the dynamics of the market model. The three regimes depend on the range of values of the herding propensity κ and are determined by the phases of the underlying $O(n)$ model.

A. The subcritical market regime

The first regime corresponds to small values of κ far below the critical value $\kappa_c = n$. In this subcritical regime, the market model does not produce super-exponential bubbles, as evident in figure 5. This is true for both constant κ and Ornstein-Uhlenbeck κ_t processes, provided that the latter moves stochastically in a range of values far from the critical point.

The origin of this property can to be found in the behavior of the noise traders. Small values of κ correspond to a larger impact of the random component specific to each agent compared with the deterministic utility function common to all traders, as observable in equation (31). A small herding propensity corresponds to noise traders who formulate their investment decisions based in large part on their idiosyncratic personal information, which varies randomly from one trader to the next. Hence, there is no polarisation of opinions, no consensus, but rather a cacophony of views, ensuring that the market prices exhibit a small volatility and grow steadily

FIG. 5. The figure presents the time series of a simulation with four risky and one risk-free assets featuring an Ornstein-Uhlenbeck process (51) for κ_t, characterized by a mean value $\mu_k = 0.98$, far below the critical value $\kappa_c = 4$. The mean reversion strength $\eta = 0.013$ and the standard deviation $\sigma_\kappa = 0.25 \cdot 1 \cdot \sqrt{2\eta}$ are defined analogously to section III A, following [23]. The other parameters are the same as in table I.

at the growth rate of the dividends.

B. The critical market regime

The second regime occurs when the herding propensity κ can approach the critical value $\kappa \approx \kappa_c$. In this regime, we observe the emergence of well-defined bubbles when κ_t follows an Ornstein-Uhlenbeck process (51) that fluctuates by crossing the critical value, as presented in figure 6. Note that κ_t does not need to stay close to κ_c but just have to approach it and stay in its vicinity for a while for bubbles to emerge. Tranquil periods of approximate exponential growth of the prices with low volatility alternate with transient periods of aggressive price growth, which can be associated with the emergence of bubbles, in relation with where κ_t is compared to κ_c.

1. *Phase transition and emergence of bubbles*

The emergence of bubbles can be attributed to the imitation between noise traders and the polarisation of their trading decisions embedded in the $O(n)$ model. This collective behavior is associated with the underlying phase transition of the $O(n)$ model, which becomes critical for $\kappa = \kappa_c$. In the critical regime or close to it, the noise traders exhibit continuously rearranging hierarchical intertwined clusters of the different possible opinions (corresponding to investing in different assets), with critical dynamics and large susceptibility to random fluctuations [44, 45]. The influence of the critical transition of the

FIG. 6. The figure presents the time series of a simulation with four risky and one risk-free assets featuring an Ornstein-Uhlenbeck process (51) for κ_t, characterized by a mean value $\mu_k = 0.98\kappa_c$, with $\kappa_c = 4$. The market is simulated with the same set of parameters used for the simulation in figure 1, see table I and section III A.

$O(n)$ model is indeed undeniable, notwithstanding the added complications resulting from the feedbacks of the noise traders' market impact on the price momentum and from their interaction with the fundamentalists. This is verified in figure 7, which shows that the average opinion $(1/N)\sum_i \vec{S}_i$ of the noise traders presents the well-known properties of a critical phase transition characterizing the $O(n)$ model.

The alternation of tranquil periods of approximate exponential growth of the prices with low volatility that switch to transient periods of aggressive price growth is controlled by the dynamics of κ_t driven by the Ornstein-Uhlenbeck process (51) that propels κ_t from values lower than κ_c to values close to or larger than κ_c at which the strong imitation between noise traders develops, leading to the bubble regimes. When the bubbles develop in one or more risky assets, the price momenta of these assets increase, pushing more and more noise traders to invest into them, hence creating a self-reinforcing loop. When the herding propensity κ_t reverts to the sub-critical regime, the polarization of the noise traders starts to decrease, the idiosyncratic opinion starts to regain importance and the noise traders move to the other assets. The market impact of these sales pushes the prices of the exuberant assets down. Then, the price momenta turn negative, pushing more and more traders to sell the assets with falling prices. This leads to a burst of the bubbles and the prices return quickly close to their fundamental values or overshoot below the fundamental value before reversing to it eventually. The above description applies to "positive bubbles", i.e. overpriced assets. Symmetrically, noise traders can also polarize to sell the risky assets, and "neg-

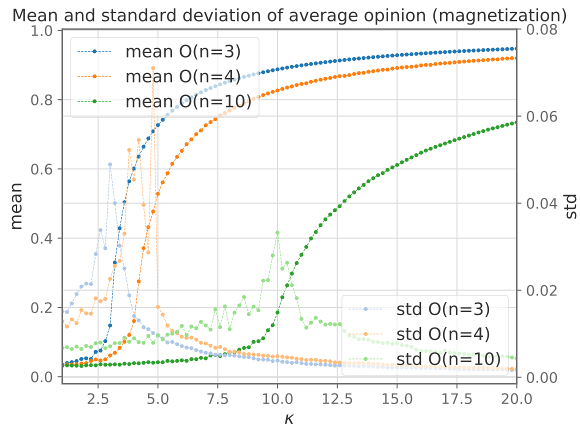

FIG. 7. Mean and standard deviation of the average opinion of the noise traders defined as the norm of the average spin vector over all trading days $\|\frac{\sum_i \vec{S}_i^t}{N}\|$ as a function of κ, the constant value of the propensity to imitate in numerical simulations. The three pairs of curves are obtained from simulations with 3, 4 and 10 risky assets (top to bottom curves for the means and left to right peaks for the standard deviations). One can observe that the means (resp. standard deviations) exhibit an inflection point (resp. peak) at the corresponding critical values $\kappa_c = 3, 4$ and 10 of the underlying $O(3), O(4)$ and $O(10)$ models.

ative bubbles" can thus be created, developing as negative mirrors to the positive bubbles (given their limited duration of typically no more than a few hundreds days, the symmetry breaking positive dividend [46] is of negligible effect for the development of each bubble). The positive dividend just tends to lead to more positive than negative bubbles being nucleated. These alternating regimes of pessimistic and exuberant mood are described for example in [47, 48].

These properties provide additional motivations for our choice to impose the detailed balance condition (19) in deriving the stochastic dynamics of the noise traders, in contrast to the more general global balance rule. Choosing the detailed balance condition corresponds to restricting all the sources of non-equilibrium to the stochastic wandering of the herding propensity parameter κ_t. Since we are interested in the out-of-equilibrium effects deriving from a change in herding propensity κ_t of the noise traders, which itself models the changes in the geopolitical and economical situation, we assume the system is constantly pushed out of equilibrium solely by the parameter κ_t.

2. *Degree of synchronisation of bubbles developing in the different risky assets in the critical regime*

When the herding propensity κ_t approaches κ_c or even overpasses it, one can observe that most risky assets tend to develop almost simultaneously significant deviations from their previous low volatility price trend growing at the dividend growth rate. There is a large degree of variability in the way these large deviations develop across the four different risky assets. For instance, in the first bubble regime in the first few hundred days in figure 6, assets 1 and 3 develop a significant overpricing, asset 2 remains more or less flat while asset 0 develops a negative bubble. After this first phase, a quasi-simultaneous negative bubble develops in all four assets. This is followed by a positive bubble nucleating in asset 0, followed by a smaller bubble in asset 1, while the two other assets remain approximately flat in their price behavior. Around time 1400, assets 0, 1 and 2 develop simultaneous bubbles, while asset 3 exhibits a negative bubble of rather small amplitude. The fifth regime is more complex, with a first negative bubble on asset 2, on top of which a later bubble on asset 1 develops, followed by assets 0 and 4 also entering bubble regimes. And so on. It is informative to compare the timing of these bubble regimes with the periods when κ_t approaches and exceeds κ_c. One can observe that all bubble regimes are indeed associated with the times when κ_t is above κ_c. The complexity of some bubbles can be understood as the result of an interplay between the exogenous dynamics of κ_t, which exhibit stochastic patterns around κ_c and the endogenous collective behavior of noise traders making decisions according to the rules of the $O(n)$ model. There is thus an intricate interplay between the exogenous stochastic driving by the "control parameter" κ_t and the exogenous response of the order parameter, i.e. the average opinion $\|\frac{\sum_i \vec{S}_i^t}{N}\|$ shown in figure 7. This interplay mirrors a typical real financial market whose endogenous reflexivity [49] is also influenced by exogenous economic and geopolitical conditions.

C. The supercritical market regime

1. *Phenomenological description*

The third regime is characterized by the herding propensity κ_t always remaining well above κ_c. For $\kappa_t > \kappa_c$, the vectors of the $O(n)$ model are aligned, barring small fluctuations. This means that the investment decisions of the noise traders are polarised, i.e., are closely aligned across all noise traders. This is an extreme regime of very strong herding towards a common investment preference. We study this regime as an interesting theoretical state of the model, keeping in mind that the extreme values of the herding propensity make this regime unlike what can be observed empirically in real financial markets. Nevertheless, this regime is instructive to understand better the inner working of the model and inform on the underlying mechanisms also at work in settings more relevant to real markets.

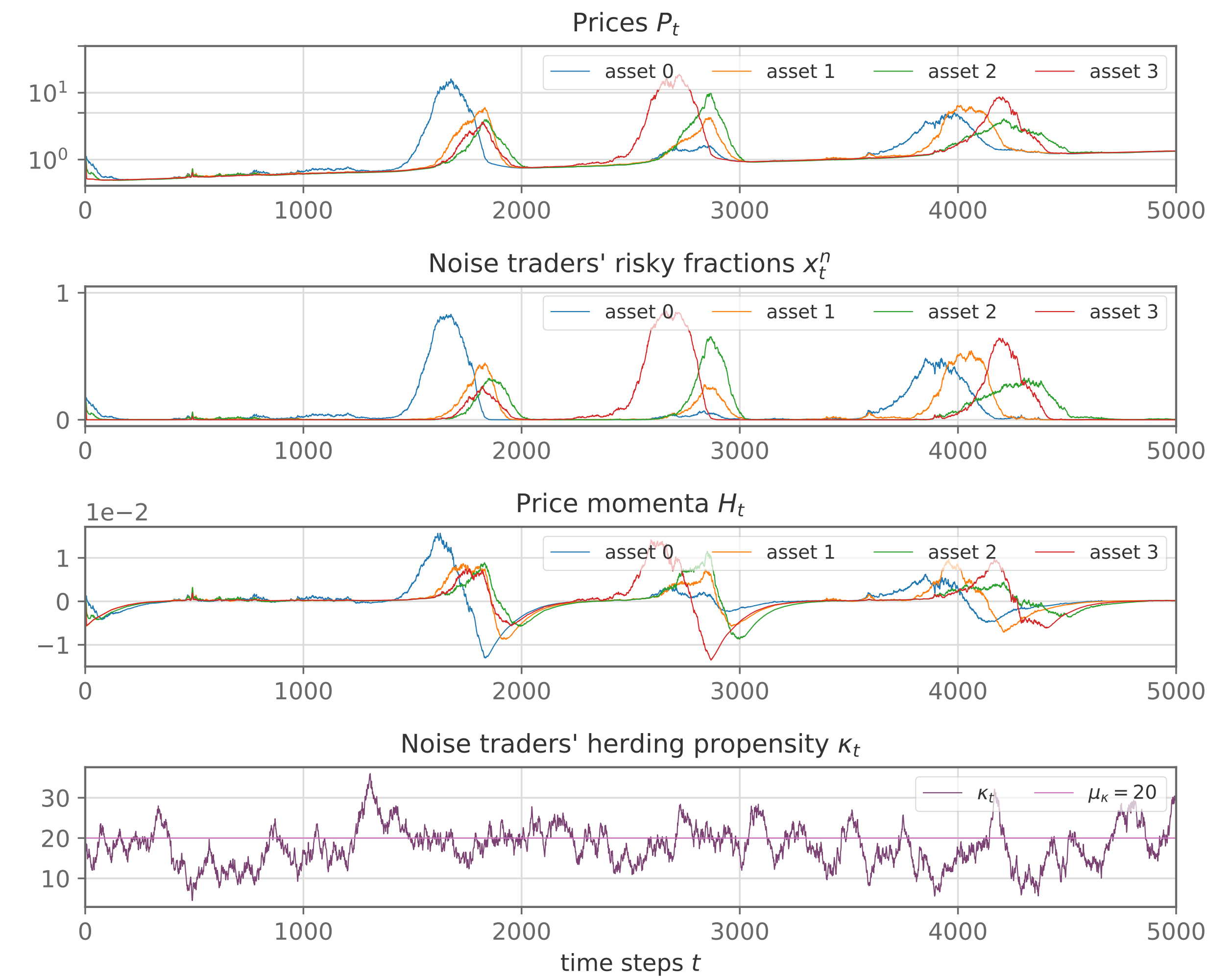

FIG. 8. The figure presents the time series resulting from a simulation with four risky and one risk-free assets featuring an Ornstein-Uhlenbeck process (51) for κ_t, characterized by a mean value $\mu_k = 0.98 \cdot 20$, far above the critical value $\kappa_c = 4$. The noise traders are completely polarized in their decisions. The mean reversion strength $\eta = 0.013$ and the standard deviation $\sigma_\kappa = 0.25 \cdot 20 \cdot \sqrt{2\eta}$ are defined analogously to section III A, following [23]. The other parameters are identical to those presented in table I.

This regime also presents bubbles, but these bubbles are of a different origin than the bubbles in the critical regime. The bubbles in the critical regime previously discussed result from the phase transitions of the underlying $O(n)$ model. In contrast, the bubbles for κ_t always well above κ_c are associated with the strong polarisation that moves along a degenerate valley of minima driven by the momenta (playing the role of effective magnitude fields). More precisely, the macroscopic properties of the $O(n)$ model are well-known to be represented by an effective Mexican hat potential in the space of polarisations for $\kappa > \kappa_c$. This Mexican hat shape succeeds to a paraboloid shape for $\kappa < \kappa_c$ corresponding to the disordered phase. The single minimum of the potential, which represents the macroscopic state of the system of spins, is located at the origin of the paraboloid for $\kappa < \kappa_c$. It then transforms into a degenerate valley of minima for $\kappa > \kappa_c$. The transformation of the potential from paraboloid to a Mexican hat shape is a very effective representation of the transition of the $O(n)$ model occurring at κ_c. In the supercritical regime $\kappa > \kappa_c$, the presence of the degenerate valley of minima governs the occurrence of bubbles. Even very far from the critical point, the system of spins is characterized by a diverging susceptibility for fluctuations of the polarisation that are perpendicular to the non-zero magnetization vector. This makes the noise traders react with a collective behavior in response to small changes in the external field of price momenta. Then price momenta can tilt the common investment preferences of the noise traders at the “macro” level. This emerging collective behavior is governed by the price momenta acting on a system that has a very large susceptibility for fluctuations along the degenerate valley of minima of the representative Mexican hat potential. Thus, in the critical regime, the bubbles are governed by the social imitation attitude, while in the supercritical one, they are dominated by the trend-following attitude. In the first case, it is the transition to the ordered phase triggering the bubbles. In the second case, the noise traders are already polarized and the tilting effect of the external field (momentum) drives them. This leads to a much stronger synchronous behavior (with phase shifts) of the bubbles developing in the four risky assets, which are mainly driven by the trend-following attitude of the noise traders.

Figure 8 presents the resulting time series from a setup with an Ornstein-Uhlenbeck κ_t fluctuating around a mean reversion level $\mu_k = 0.98 \cdot 20$, which is deeply inside the ordered regime. As in the subcritical regime, both constant κ and Ornstein-Uhlenbeck κ_t lead to the same market characteristics as long as κ_t remains far away from the critical value. At odds with the regime where κ_t crosses κ_c repeatedly, the time-varying nature of the herding propensity is no more so important while the strong polarisation of the noise traders becomes the dominent feature governing this regime.

During the first trading days, the noise traders polarize towards the risk-free asset as a result of the initial imbalance towards the risk-free asset defined by the initial condition for the noise traders' investment fractions in equation (49). The noise traders initially invest half of their total wealth in the risk-free asset and the remaining half, equally distributed among the various risky investments. Consequently, the tendency to polarize in this supercritical regime pushes the noise traders to sell the risky assets and invest only in the risk-free asset.

To analyze the bubbles at the phenomenological level, we focus on the first one, while observing that their structure is similar, with a first bubble in one asset followed by a cascade of bubbles in all the other assets. The only difference from one specific bubble to the next is constituted by the velocity and intensity of the phenomenon, which are stochastic. Around trading day 1500, a first bubble develops in asset 0, triggered by a stochastic fluctuation and then amplified by the positive feedback of the action of noise traders implementing momentum trading that acts as a magnetisation leading to a first-order transition of the polarisation on that asset 0. Its price starts to grow super-exponentially. This bubble then cascades, around trading day 1600, into bubbles in the other risky assets. This cascade of bubbles results from investments following the wealth increase of the traders produced by the first bubble and develops due to the strong influence of the momentum. More details of this cascade of bubbles are presented in the following section.

At odds with the original single risky asset case [23], here the noise traders never reach full polarization. This is due to the exponential form of the transition probabilities, which always guarantees a non-zero probability for each trader to go against the tide, even in presence of extreme polarization of the group. When the first bubble reaches its peak around $t \approx 1670$, the other assets' prices have already started to grow resulting in an increase of the price momenta of these assets. This pushes the traders to sell the asset that is already at its price peak to shift their investments towards the other growing risky assets. In addition, the fundamentalist traders tend to sell the asset whose price is skyrocketing, according to their countercyclical strategy (10). They invest a fraction of their wealth that is proportional to the dividend-price ratio. Thus, assuming the dividends remain approximately constant during the bubble, the invested wealth fraction decreases as the asset price increases. Fundamentalists and noise traders both selling the asset explains the decrease in price of the first asset, which triggers a self-reinforcing loop through the price momentum, in turns pushing more and more traders to sell the falling price's asset. This results in the crash of the first bubble. The subsequent synchronous crashes of the cascade of bubbles on the three other risky assets around $t \approx 1850$ can again be traced back to the self-reinforcing loop governed by the price momenta. This time however, the loop is triggered only by the fundamentalists' countercyclical strategy.

Over long times, the relative wealth of the noise traders decreases as shown in figure 2, which progressively decreases their market impact. Thus, their opinion polarisation and abrupt shifts have progressively smaller and smaller effects, leading to smoother and longer lasting bubbles.

2. *Mechanism for the cascade of bubbles across assets in the supercritical regime*

Each of the three bubbles in Figure 8 starts with a single asset deviating from its long-term trend, accelerating in a typical bubbly fashion. The other three assets remain for a while on their long-term trend, approximately until the first bubble has reached a level of the order of its maximum amplitude. Then, one by one or simultaneously, the other assets start to nucleate a bubble in their turn. This clustering or "cascade' of bubbles is one of the striking property of the regime $\kappa > \kappa_c$ and we know elaborate on its origin.

First, we list properties or characteristics of the model that cannot be explanations for these cascades. By construction in the present set-up, the fundamentalists form an expectation that no correlation exists between the assets and consequently allocate their investment according to independent equations for the different risky fractions. The correlations between the exogenous stochastic dividend processes are also set to zero, so this cannot be a source of coordination, clustering or cascade of assets' prices. The components of the spin vectors defining the noise traders' wealth allocation for each asset fluctuate in an uncorrelated manner and thus do not provide a source for the cascades.

To identify the source of the cascades, let us look in more details at the price dynamics of the assets and at the wealth dynamics of the investors. Starting with the first bubble that develops in one asset, triggered by a stochastic fluctuation, it is then amplified by the positive feedback of the momentum-based trading of noise traders that pushes the price to accelerate and grow super-exponentially. As a result of their exposure to this bubbling asset, the investors, both fundamentalists and noise traders, become wealthier as the price of the shares they hold increases massively.

Since the traders' wealth increases, in order for the Walrasian equilibrium to be still satisfied, the other assets' prices must increase. This derives from the market clearing condition characterizing the Walrasian equilibrium

$$\Delta D^{f,k}_{t-1\to t} + \Delta D^{n,k}_{t-1\to t} = 0 \quad \forall k \tag{53}$$

where $\Delta D^{f,k}_{t-1\to t}$ and $D^{n,k}_{t-1\to t}$ represent respectively the aggregate excess demands of each trader type for the risky asset k. Expressing the equation in a more explicit form

$$\begin{aligned} W^f_t x^f_{k,t} - W^f_{t-1} x^f_{k,t-1} \frac{P_{k,t}}{P_{k,t-1}} \\ + W^n_t x^n_{k,t} - W^n_{t-1} x^n_{k,t-1} \frac{P_{k,t}}{P_{k,t-1}} = 0 \end{aligned} \tag{54}$$

and assuming constant all the quantities not explicitly depending on the other assets' prices, the effect of the bubble of a specific asset on the other assets' price equations is mediated solely by an increase of the wealth of fundamentalists and noise traders. Expression (54) shows that the effect of a bubble of an asset $i \neq k$ on the price equation (54) of any other asset k is working via an increase of the wealths W^f_t and W^n_t. As a consequence, in order to satisfy the equilibrium equation, the price $P_{k,t}$ has to be larger than the the price $P_{k,t-1}$ at the previous time-step.

It is important to distinguish the roles of fundamentalists and noise traders. Using different strategies that respond differently to a bubble, their wealth increase differs. Pushed by their social imitation and trend following tendencies, the noise traders invest more and more in the asset developing a bubble as its price rises. Following their risk-averse (or optimal risk-adjusted return) strategy, the fundamentalists decrease their exposure to this bubbling asset as its price rises and reallocate their wealth progressively to the other assets. Hence, while both traders become richer during the build-up of bubbles, the noise traders' wealth increases transiently much more than the fundamentalists' wealth.

In financial intuitive term, the strong price appreciation of the bubbling asset pushes the fundamentalists to sell it and buy the other risky assets to ensure a good diversification of their portfolio. Their increasing wealth tilted strongly towards the bubbling asset requires a strong readjustment and thus demand for the other risky assets, which pushes their price up, given the limited supply provided by the noise traders. In this way, a vigorous increase in one price eventually triggers an increase in all the other prices, which results in a cascade of growth of the prices of all assets. If this increase is strong enough to have a relevant impact on the price momenta associated with the other assets, as in the case of a super-exponential bubble in the supercritical regime, the noise traders will shift their allocations to these new growing bubbles as their increasing price momenta lead to a large shift of their portfolio vector. This triggers the emergence of the cascade of bubbles in all the other risky assets, through the mechanism explained above of the strong polarisation of noise traders that moves along a degenerate valley of minima driven by the momentum.

In summary, the pattern of bubble cascades shown in figures 8 is due to the following three steps.

1. some random fluctuation creates a transient momentum fluctuation on one asset price that is amplified by the underlying collective behavior of the noise traders making investment decisions obeying the rules of the $O(n)$ model in the supercritical regime. This leads to the emergence of a bubble in one asset and its super-exponential price growth.

2. As the fundamentalists see their wealth grow via the explosive value of the bubbling asset, their portfolio optimisation requires rebalancing away from this bubbling asset towards the other assets. As their wealth has increased significantly and their demand for the other assets is growing to compensate for the portfolio unbalance due to the bubbling asset, they thus create a strong price increase in the other risky assets.

3. The resulting price momentum on these other assets becomes noticeable to the noise traders who then rotate their portfolio more and more towards them, thus creating the cascade of bubbles via the strong polarisation of noise traders that moves along a degenerate valley of minima driven by the momentum.

.

V. CONCLUSION

We have derived a market model with two types of agents who trade $n > 1$ risky assets and one risk-free asset. The model is a multi-asset extension of an agent-based model with fundamentalists and noise traders introduced by Kaizoji *et al.* [23] and elaborated by Westphal and Sornette [26]. The fundamentalists allocate their portfolio according to a maximization of an expected CRRA utility function. The noise traders' investment decision is described by an $O(n)$ vector model in which the n components represent the different assets in which the traders can invest. This allows us to define realistic stochastic dynamics while having control over the statistical properties of the model. The price momenta influence the traders' investment decisions in the form of the external field. The stochastic dynamics is completely specified by a discrete-time Markov chain, defined by the possible states and the transition rates among them. We derived efficient rejection-free transition probabilities in order to describe a realistic behavior at the "micro" level of the single investor.

The price at each time-step is defined by the traders' demand and supply for each of the assets. The resulting price time series exhibits bubbles and crashes and reproduces several "stylized facts" of financial markets such as volatility clustering or fat-tails of the return distribution. The bubbles emerge when the noise traders polarize their opinions towards one or more assets .

The model has been applied to understand the relationship between bubbles in different co-existing assets. Three regimes were found in which the mechanism responsible for creating bubbles differ. The regimes are defined with respect to the traders' propensity κ to herd. The critical value κ_c in the fully connected mean field regime is known and equal to n in the $O(n)$ model. In the disordered regime with a herding propensity smaller than n, no polarization and consequently no bubbles emerge. Around the critical value, with a herding propensity transiently fluctuating above the critical value, asynchronous bubbles emerge in various assets associated with the critical behavior of the $O(n)$ model. The bubbles are largely driven by the polarization of the traders' opinion. In the third regime, where the herding propensity κ remains above the critical value $\kappa_c = n$ and the noise traders are already polarized, cascades of bubbles emerge. This occurs notwithstanding the fact that there is no phase transition and the noise traders' portfolios are well polarised. The bubbles are instead driven by the momentum following strategies of the noise traders. Small random price fluctuations can trigger the noise traders to herd into a given asset. The cascades of bubbles result from the reallocation of fundamentalists re-equilibrating their portfolios in the face of a bubbling asset that tends to dominate the increasing valuation of their portfolios, followed by a growth of these assets' momenta that then trigger a reorientation of the noise traders' portfolios. Thus, the risk-adverse rebalancing strategy of fundamentalists can be seen as an important process in the creation of systemic risks in the form of cascades of bubbles.

The realistic multi-asset market model that reproduces the main stylized facts of financial markets can be applied to test the market impact of various portfolio optimization strategies in future research. Furthermore, the role of contrarian traders modeled by a negative herding propensity in stabilizing markets or triggering crashes could be analyzed. The market model also provides a framework to test policies intended to decrease systemic risk or to prevent bubbles and crashes. It can help to quantify the effect bursting one bubble has on other assets.

CONTRIBUTIONS

D.C. and R.W. contributed equally to this work.

[1] W. Weidlich, The statistical description of polarization phenomena in society, British Journal of Mathematical and Statistical Psychology **24**, 251 (1971).

[2] S. Galam, Y. Gefen, and Y. Shapir, Sociophysics: A new approach of sociological collective behaviour. I. mean-behaviour description of a strike, Journal of Mathematical Sociology **9**, 1 (1982).

[3] A. Grabowski and R. Kosiński, Ising-based model of opinion formation in a complex network of interpersonal interactions, Physica A: Statistical Mechanics and its Applications **361**, 651 (2006).

[4] R. A. Holley and T. M. Liggett, Ergodic theorems for weakly interacting infinite systems and the voter model, The Annals of Probability , 643 (1975).

[5] B. Roehner, D. Sornette, and J. V. Andersen, Response functions to critical shocks in social sciences: An empirical and numerical study, International Journal of Modern Physics C **15**, 809 (2004).

[6] D. Sornette, Physics and financial economics (1776–2014): puzzles, Ising and agent-based models, Reports on Progress in Physics **77**, 062001 (2014).

[7] S. Gualdi, J.-P. Bouchaud, G. Cencetti, M. Tarzia, and F. Zamponi, Endogenous crisis waves: Stochastic model with synchronized collective behavior, Physical Review Letters **114**, 088701 (2015).

[8] S. Gualdi, M. Tarzia, F. Zamponi, and J.-P. Bouchaud, Tipping points in macroeconomic agent-based models, Journal of Economic Dynamics and Control **50**, 29 (2015).

[9] J. B. De Long, A. Shleifer, L. H. Summers, and R. J. Waldmann, Positive feedback investment strategies and destabilizing rational speculation, The Journal of Finance **45**, 379 (1990).

[10] J. B. De Long, A. Shleifer, L. H. Summers, and R. J. Waldmann, Noise trader risk in financial markets, The Journal of Political Economy **98**, 703 (2007).

[11] W. A. Brock and C. H. Hommes, A rational route to randomness, Econometrica: Journal of the Econometric Society , 1059 (1997).

[12] T. Lux and M. Marchesi, Scaling and criticality in a stochastic multi-agent model of a financial market, Nature **397**, 498 (1999).

[13] H. Markowitz, Mean-variance analysis in portfolio choice and capital markets, Wiley (2000).

[14] F. Fabozzi, P. D. Kolm Petter N., and S. M. Focardi, Robust portfolio optimization and management, Wiley, 1st edition (2007).

[15] J.-P. Bouchaud, D. Sornette, C. Walter, and J.-P. Aguilar, Taming large events : Optimal portfolio theory for strongly fluctuating assets, International Journal of Theoretical and Applied Finance **1**, 25 (1998).

[16] Y. Malevergne and D. Sornette, Var-efficient portfolios for a class of super- and sub-exponentially decaying assets return distributions, Quantitative Finance **4**, 17 (2003).

[17] Y. Malevergne and D. Sornette, Higher-moment portfolio theory (capitalizing on behavioral anomalies of stock markets), Journal of Portfolio Management **31**, 49 (2005).

[18] W. F. Sharpe, Capital asset prices: A theory of market equilibrium under conditions of risk, The journal of finance **19**, 425 (1964).

[19] S. A. Ross, The arbitrage theory of capital asset pricing, in *Handbook of the fundamentals of financial decision making: Part I* (World Scientific, 2013) pp. 11–30.

[20] R. Campbell, Y. Liu, and H. Zhu, ...and the cross-section of expected returns, The Review of Financial Studies **29**, 5 (2016).

[21] H.-C. Xu, W. Zhang, X. Xiong, and W.-X. Zhou, Wealth share analysis with "fundamentalist/chartist" heterogeneous agents, in *Abstract and Applied Analysis*, Vol. 2014 (Hindawi, 2014).

[22] C. Chiarella, R. Dieci, and X.-Z. He, Heterogeneous expectations and speculative behavior in a dynamic multi-asset framework, Journal of Economic Behavior & Organization **62**, 408 (2007).

[23] T. Kaizoji, M. Leiss, A. Saichev, and D. Sornette, Super-exponential endogenous bubbles in an equilibrium model of fundamentalist and chartist traders, Journal of Economic Behavior & Organization **112**, 289 (2015).

[24] L. Walras, Elements of Pure Economics. English translation by W. Jaffe (1954) (1926).

[25] C. Chiarella, R. Dieci, and X.-Z. He, Heterogeneity, market mechanisms, and asset price dynamics, in *Handbook of financial markets: Dynamics and evolution* (Elsevier, 2009) pp. 277–344.

[26] R. Westphal and D. Sornette, Market impact and performance of arbitrageurs of financial bubbles in an agent-based model, Journal of Economic Behavior & Organization **171**, 1 (2020).

[27] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, The journal of chemical physics **21**, 1087 (1953).

[28] H. Shefrin and M. Statman, The disposition to sell winners too early and ride losers too long: Theory and evidence, The Journal of Finance **40**, 777 (1985).

[29] N. Barberis and W. Xiong, What drives the disposition effect? an analysis of a long-standing preference-based explanation, The Journal of Finance **64**, 751 (2009).

[30] D. McFadden, Econometric models of probabilistic choice, Structural analysis of discrete data with econometric applications **198272** (1981).

[31] M. Ben-Akiva, N. Litinas, and K. Tsunokawa, Continuous spatial choice: the continuous logit model and distributions of trips and urban densities, Transportation Research Part A: General **19**, 119 (1985).

[32] O. I. Zhelezov, N-dimensional rotation matrix generation algorithm, American Journal of Computational and Applied Mathematics **7**, 51 (2017).

[33] M. J. Powell, *A Fortran subroutine for solving systems of nonlinear algebraic equations*, Tech. Rep. (Atomic Energy Research Establishment, Harwell, England (United Kingdom), 1968).

[34] M. J. Powell, A hybrid method for nonlinear equations, Numerical methods for nonlinear algebraic equations (1970).

[35] E. Damiani, Equilibrium model of fundamentalist and noise traders in a multi-asset framework, Master's thesis, Politecnico di Torino and ETH Zurich (2019).

[36] A. Kopp, Equilibrium model of a fixed income market with fundamentalist and chartist traders, Master's thesis, ETH Zürich (2020).

[37] A. Chakraborti, I. Toke, M. Patriarca, and F. Abergel, Econophysics review: I. empirical facts, Quantitative Finance **11**, 991 (2011).

[38] Z.-Q. Jiang, W.-J. Xie, W.-X. Zhou, and D. Sornette, Multifractal analyses of financial markets: a review, Reports on Progress in Physics **82**, 125901 (105pp) (2019).

[39] R. Cont, Empirical properties of asset returns: stylized facts and statistical issues, Quantitative finance **1**, 223 (2001).

[40] T. Lux and D. Sornette, On rational bubbles and fat tails, Journal of Money, Credit and Banking, Part 1 **34**, 589 (2002).

[41] Y. Malevergne, V. Pisarenko, and D. Sornette, Empirical distributions of log-returns: between the stretched exponential and the power law?, Quantitative Finance **5**, 379 (2005).

[42] Y. Malevergne and D. Sornette, Extreme financial risks (from dependence to risk management), Springer, Heidelberg (2006).

[43] R. Cont, Volatility clustering in financial markets: empirical facts and agent-based models, in *Long memory in economics* (Springer, 2007) pp. 289–309.

[44] D. Sornette, Critical phenomena in natural sciences (chaos, fractals, self-organization and disorder: Concepts and tools), 2nd ed., Springer Series in Synergetics, Heidelberg (2004).

[45] G. Harras, C. Tessone, and D. Sornette, Noise-induced volatility of collective dynamics, Phys. Rev. E **85**, 011150 (2012).

[46] D. Sornette, Stock market speculation: Spontaneous symmetry breaking of economic valuation, Physica A: Statistical Mechanics and its Applications **284**, 355 (2000).

[47] D. Sornette, *Why stock markets crash: critical events in complex financial systems*, Vol. 49 (Princeton University Press, 2017).

[48] R. J. Shiller, Irrational exuberance: Revised and expanded third edition, (2015).

[49] D. Soros, The alchemy of finance, Wiley; 2007th edition (2015).

Swiss Finance Institute

Swiss Finance Institute (SFI) is the national center for fundamental research, doctoral training, knowledge exchange, and continuing education in the fields of banking and finance. SFI's mission is to grow knowledge capital for the Swiss financial marketplace. Created in 2006 as a public–private partnership, SFI is a common initiative of the Swiss finance industry, leading Swiss universities, and the Swiss Confederation.

swiss:finance:institute

c/o University of Geneva, Bd. Du Pont d'Arve 42, CH-1211 Geneva 4
T +41 22 379 84 71, rps@sfi.ch, www.sfi.ch