Gold's Value as an Investment

Urban J. Jermann*

Wharton School of the University of Pennsylvania and NBER

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Abstract

For investors, gold is an asset without a yield that is attractive in times of low and negative real interest rates. Gold also has an embedded put option because investors can sell it to those who value its use as jewelry or as a productive input. This paper presents an approach for pricing gold from investors' perspective. The model is based on no-arbitrage principles with minimal structural assumptions. There is no need to specify investor preferences. When fitted to match 10-year real US Treasury rates the model can replicate the salient fluctuations in the time series of gold prices since 2007. The model implies that the majority of the value of gold is due to its role as an investment asset.

Keywords: Gold, real interest rates, bubbles. JEL codes: G12, G13.

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1 Introduction

For investors, gold is an asset without a yield. This raises the question how they should value gold. Because gold is used as an investment asset, it is believed to be worth more than its fundamental value as jewelry or as productive input. But how much more? This paper tries to answer these questions with an approach based on no-arbitrage principles that can be tightly disciplined by empirical evidence. While inspired by quantitative general equilibrium modeling, implementing the approach is similar to applying methods for pricing fixed income instruments.

Gold is attractive as a store of value in times of low and negative real interest rates. In these times, investors looking for stores of value bid up its price. When interest rates go up again, the price of gold declines again. However, this is not the whole story. Even in times when investors are not holding gold, the prospect that they could enter the market bids up the valuation for those who use it as jewelry or a conductor. In addition, when investors are holding gold, they take into account that gold's valuation by users who derive utility from its good looks and conductivity puts a lower floor to the price at which they can sell their gold. In other words, gold can be viewed as a substitute for real bonds and like such bonds its value moves with real interest rates. However, gold also has option value. Because of gold's potentially long duration, this option value could be very high.

To be more specific, gold is an asset whose value moves negatively with real interest rates, just like a bond. From investors' perspective, gold has an embedded put option because they can sell to users at some floor price. This floor depends on the state of the economy, but is higher than the investors own valuation in that state. This floor or strike price also takes into account the probability that investors will want to buy gold again in the future. In this paper, I formalize this claim. The key feature that distinguishes gold from a tradable asset such as a stock or a bond is that some buyers are not getting a yield or a service flow from the asset beyond the return due to the change in the price.

My approach combines two popular valuation equations: present discounted value of cash flows and rational bubbles. Each one taken separately has serious shortcomings for valuing an asset like gold. Measuring the value of gold's utility flows is elusive. Rational bubbles have typically non-unique values and are appealed to as a residual relative to some fundamental value. Combined however, the two equations become a pricing tool that has similar properties as fixed income derivative pricing models. I implement the approach quantitatively with a one-factor term structure model for real yields. Many term structure models with useful properties have been developed and can be used to implement this approach for high levels of precision.

As a starting point, I document the empirical relation between gold prices and real US Treasury yields since the late 1970s. The real price of gold and real yields are strongly negatively correlated, but this correlation is driven by periods with low rates. When rates were high, the correlation has been weak. My model is naturally suited to display this nonlinear relation because investors only enter the gold market when rates are low enough.

Basic model properties are derived for a two-state example that can be solved in closed form. For the general Markov chain specification, I show how the equilibrium can be represented as a dynamic operator and prove that under some conditions there exist a unique fixed point. This allows for a computationally efficient solution algorithm that is applicable to rich specifications.

For a quantitative analysis of the model, I use a version of the Vasicek term structure model calibrated to real US Treasury yields. For the benchmark calibration, the expected value of the price of gold is more than trice the value gold would have in the absence of investor participation in the gold market. When the model is fitted to match 10-year real yields for 1980-2020, the price of gold reproduces well the salient fluctuations since about 2007. An extension of the model to include the effects of the real USD exchange rate improves the model fit for the period before 2007.

In the literature, there are not many equilibrium models specifically designed for pricing gold. Barro and Misra (2015) and Huang and Kilic (2019) present equilibrium models with a representative agent and rare disasters. More closely related, Jermann (2021) features a fully specified general equilibrium model with two types of agents. The modelling here is distinct in that investor preferences do not need to be specified and that gold is priced based on an exogenously given stochastic discount factor. While more limited in scope than a fully specified general equilibrium model, the model presented here offers the flexibility to closely match empirical interest rate behavior and requires only minimal structural assumptions.

There is a long tradition for pricing assets with stochastic discount factors, for instance fixed income securities (Vasicek (1977), Cox, Ingersoll, and Ross (1985) or currencies (Backus, Foresi, and Telmer (2001)). My model starts from a stochastic discount factor and adds minimal economic structure to capture the interaction between different types of agents.

Some recent empirical studies on gold prices have emphasized the link between real gold prices and real Treasury rates, in particular Johnson (2014) and Erb, Harvey and Viskanta (2020). I show that for a more extended sample period the link between real gold prices and real rates is weaker unconditionally and that it is driven by periods with low real rates.

Bubbles have been put forth to explain differences between the prices and the fundamental values of assets, see for instance Blanchard and Watson (1982), Abreu and Brunnermeier (2003), and Allen, Morris, and Postlewaite (1993). In my model, gold has no fundamental value to investors except as a store of value. Gold has a fundamental value to those who use it as jewelry or productive input, and this value increases as investors' buy up some of gold's supply.

The next section documents empirical properties of gold prices and real interest rates. Section 3 presents the model, followed by a simple example that can be solved and characterized in closed form. Section 5 considers the general model. The quantitative analysis is in section 6.

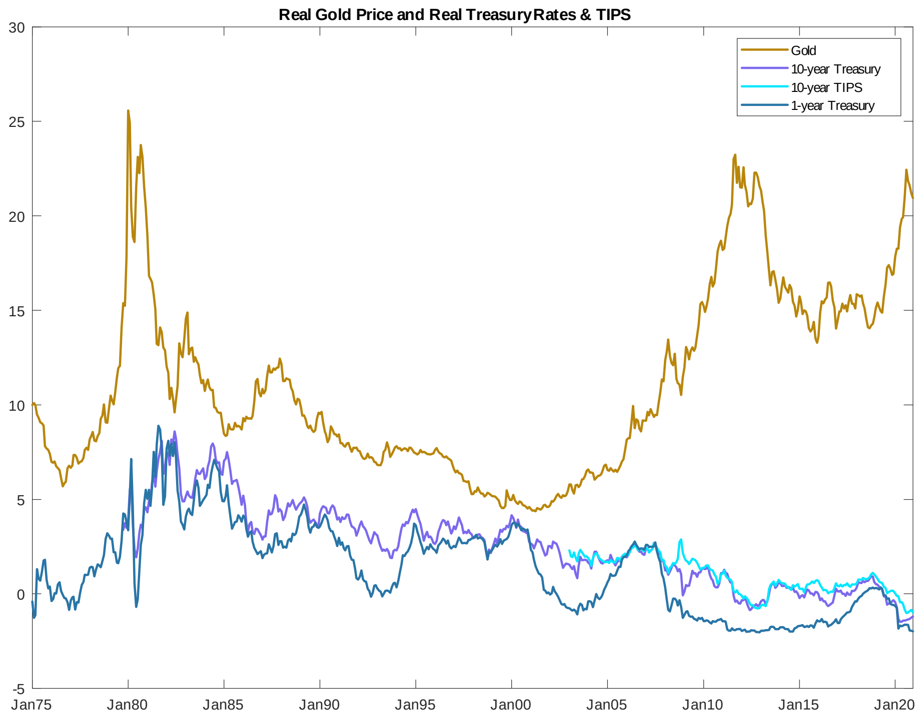


Figure 1: Gold and real Treasury yields.

2 Gold prices and real treasury rates

This section presents empirical properties of the relation between gold prices and real interest rates. I confirm the negative relation documented in Johnson (2014) and Erb, Harvey and Viskanta (2020) for an extended sample. However, the negative relation is not very strong over the longer sample periods considered here, namely 1975-2020 and 1980-2020 for 1-year and 10-year maturities, respectively.[1] I document a new stylized fact: the negative relation between the price of gold and real rates is produced in the periods when rates are low. When rates are high, the relation is very weak.

Average monthly prices of gold for 1975.1 to 2020.12 are computed from the daily gold

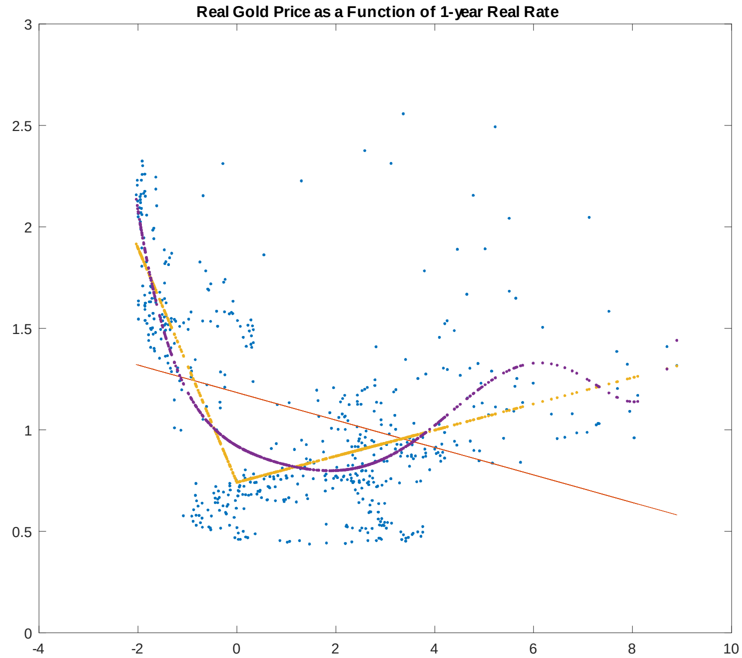
[1] 1975 marks the end of restrictions on US private gold investments' in place since 1933. The official US gold peg was ended in 1971.

fixing in the London Bullion Market. The price of gold is deflated by the CPI-U. 1-year and 10-year constant maturity Treasury rates are combined with inflation forecasts from the Survey of Professional Forecaster extended with additional data for the 10-year horizon from Blue Chip Economic Indicators for 1979.10 to 1991.9.[2] Starting with 2003.1, 10-year TIPS rates are used for the 10-year real rates.

Figure 1 displays the time-series of the real price of gold (scaled to 10 at 1975.1) alongside the 1-year and 10-year real rates, including TIPS, each for their available sample periods. Yields are in general lower after 2000 and gold prices higher. It can be visually detected that the negative relation is also present at higher frequencies. For instance, starting around 2019, gold and yields have sharply moved in opposite directions. Around 2012-13, gold prices are at very high levels and 10-year rates are at their lowest since the beginning of the sample period. Before and after 1985 there are periods of approximately three years each where gold and yields clearly move in opposite directions.

Figure 2 shows the scatter plots of real gold prices (scaled to 1 at 1975.1) against 1-year and 10-year rates, respectively. For the 10-year rates, TIPS are used after 2003. From this perspective, it is clear that the negative relation between the price of gold and yields is situated in the low yield region. Linear regression lines as well as piece-wise linear and 6th-degree polynomials models are included in the plots.

[2]Data is from the Federal Reserve Bank of St. Louis and the Federal Reserve Bank of Philadelphia. The sample length is limited by the availability of inflation forecasts.



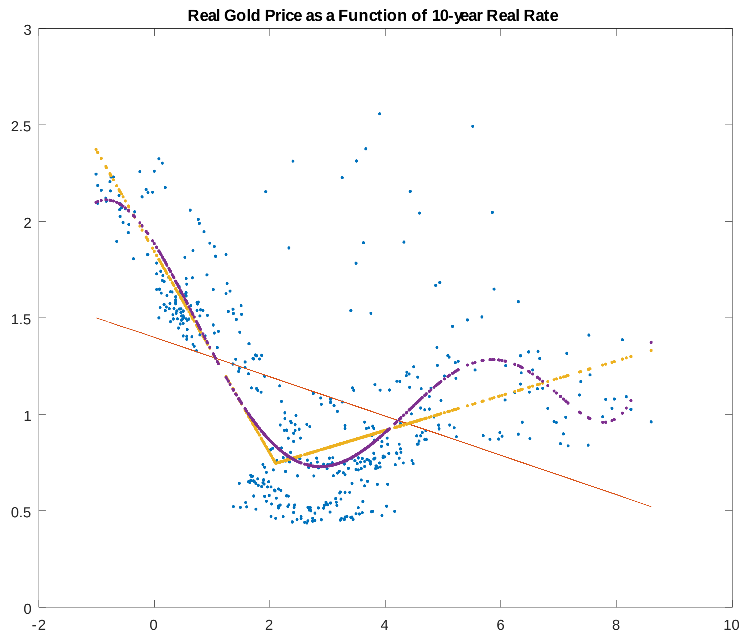


Figure 2. Real Gold Prices against Real Yields. 1975.1-2020.12 and 1979.10-2020.12 for 1-year and 10-year yields, respectively.

Treasury maturity	1-year				10-year			
	monthly		annual		monthly		annual	
33.3% cutoff								
$\Delta y_t,\, y_t < y^*$	-3.2^{**}		-8.7^{***}		-14.1^{***}		-19.3^{***}	
$\Delta y_t,\, y_t \geq y^*$		-0.8^{*}		-1.1		0.2		-4.7^{**}
R^2_{adj}	.03	.01	.21	.00	.35	$-.00$	.65	.11
50% cutoff								
$\Delta y_t,\, y_t < y^*$	-1.1^{*}		-3.4^{*}		-6.3^{***}		-10.0^{**}	
$\Delta y_t,\, y_t \geq y^*$		-1		-1.6		0.6		-5.1^{***}
R^2_{adj}	.01	.01	.04	.01	.10	$-.00$	.19	.19
Full sample								
$\Delta y_t,\, \forall y_t$	-1.0^{**}		-2.5		-0.9		-6.8^{***}	
R^2_{adj}	.01		.03		.00		.19	

Table 1: Regression of log-percentage changes in real gold prices on yield changes conditional on low or high yield levels. Results for monthly and annual changes are reported. The yield cutoffs for the 1-year rate are are -0.1 percent (33.3% cutoff) and 1.35 percent (50% cutoff), for the 10-year rate 1.73 percent (33.3% cutoff) and 2.5 percent (50% cutoff). Significance levels are given as *** (p<.01), ** (p <.05), and * (p<.1).

To isolate high-frequency behavior, I run regressions of percentage changes in the gold price against changes in yields. The regressions are run separately for low-yield periods and high-yield periods. The cutoffs are set so as to produce samples with the lowest 33.3% or 50% of the yields, respectively. Table 1 reports the slope coefficients and adjusted $R^{2\prime}s$. Most striking are the high $R^{2\prime}s$ for the 10-year rates in the regressions based on the periods with the lowest 33.3% of yields. For the monthly changes the R^2 is 0.35, for annual changes the R^2 is 0.65. Such high numbers make it clear that long-term real rates can be considered the main driver of gold prices during the low-rate periods. During high-rate periods the correlation between changes in gold prices and yields is typically lower, for monthly changes the correlation disappears. The slope coefficients reported in the table confirm the visual impression from Figure 2 with the highest sensitivities when rates were the lowest.

3 Model

Time is discrete with an infinite horizon. The model features two types of buyers of gold: Investors and users.[3] From investors' perspective, the price of gold p_t satisfies the following no-arbitrage relation

$$p_t \geq E_t M_{t,t+1} p_{t+1}, \tag{1}$$

with $M_{t,t+1}$ the stochastic discount factor (SDF). Equation (1) holds with equality for periods when investors are holding the asset; with inequality when investors are not holding the asset. It is assumed that investors cannot short gold with users. Investors face no restrictions on the financial contracts they trade with each other. Equation (1) is a valuation equation for an asset without a yield. If the equation always holds with equality, this would be the standard equation for a rational bubble. The problem with a bubble is that it should be worth zero, or an arbitrary number if one accepts the belief that in the long run the price is increasing exactly at the right rate. In this model, there is a unique value for the price because the investors' pricing equation sometimes holds with inequality. I assume that investors are active in other financial markets, in particular, in fixed income markets. The SDF connects the price of gold to the prices in other financial markets.

Users derive utility from their gold holdings in each period, and the marginal valuation of this utility in terms of the numeraire good is given by v_t. From their perspective, the price satisfies the present value relation

$$p_t = v_t + \beta E_t p_{t+1}. \tag{2}$$

Users are assumed to be risk neutral with a constant discount factor $0 < \beta < 1$. Knowing the process for v_t and β, the price can be determined from users' valuation alone. However,

[3] The notion that gold has very distinct groups of buyers is consistent with the Gold Value Framework, a regression-based approach disseminated by the World Gold Council (2021), WGC. According to WGC, total above-ground stocks of gold (end 2021) are 205,238 tonnes, with jewellery 46%, bar and coins (incl. ETFs) 22%, central banks 17%, and other 15%.

it is not clear how v_t could be measured.

Equilibrium requires that both equations hold. A key insight is that when the two equations are combined, the model can determine the users' utility value v_t for the times and states of nature when both buyers are in the market. Therefore, we do not need a priori knowledge of all values of v_t. Values for v_t are only needed for those times and states when users are alone holding the asset. As a starting point, we can set the marginal utility when users are holding the entire supply of gold to a constant $\underline{v} > 0$. If we are interested in how much the store of value property adds to the fundamental value, then $\underline{v}$ is just a normalization parameter.

As a requirement for an equilibrium we assume decreasing marginal utility for gold. Specifically, we require that equilibrium values are bounded from below by the autarchy value

$$v_t \geq \underline{v}. \tag{3}$$

In addition to the economic reason for this constraint, it also serves to eliminate possible multiple equilibriums.

To accommodate long-run growth in the real price of gold we can let users autarchy utility grow at a deterministic rate γ. With utility evolving as $\gamma^t \underline{v}$ with time t the price of gold inherits the same deterministic trend. The pricing equations can be rewritten for a detrended price of gold which is stationary and therefore numerically tractable. In this case, the discount factor in equation (2) becomes $\beta\gamma$, and in equation (1) the parameter γ multiplies p_{t+1} which is then interpreted as the detrended price level.

To summarize, an equilibrium is an exogenously given process for $M_{t,t+1}$, parameters $(\beta, \gamma, \underline{v})$, and processes p_t and v_t for which equation (1), (2) and (3) hold. In Jermann (2021), I present a fully specified general equilibrium model within which these equilibrium equations hold if one assumes that users have linear utility in consumption. There are many ways to represent the process $M_{t,t+1}$. I study the model with process specified as a finite-state Markov chain.

Rearranging the pricing equations offers intuition about key model mechanisms. Based on this, the model suggests an interpretation of gold as a real bond with a put option. The investors' pricing equation (1) can be rewritten as

$$p_t \geq \frac{E_t p_{t+1}}{Y_t^{(1)}} + cov_t\left(M_{t,t+1}, p_{t+1}\right)$$

with $Y_t^{(1)}$ the one-period gross real interest rate. A lower current interest rate $Y_t^{(1)}$ and lower future interest rates raise the current price of gold, like for a zero-coupon bond. The sensitivity of the price of gold to the interest rate depends on how long the equation is expected to hold with equality. This can be viewed as gold's duration. From a physical perspective, this duration infinite. With the investors' pricing equation at times holding with inequality, the effective duration is finite.

Combining the two valuation equations for a period when investors are in the gold market, the price of gold can be written as

$$p_t = E_t M_{t,t+1} \max\left[E_{t+1} M_{t+1,t+2} p_{t+2}, v_{t+1} + \beta E_{t+1} p_{t+2}\right]. \tag{4}$$

The equation shows that gold has an option-like payoff. From investors' perspective, the ability to sell gold to users sets a floor on the price of gold, similarly to a put option. The floor price $v_{t+1} + \beta E_{t+1} p_{t+2}$ depends on users' future marginal utility and on future equilibrium prices. As suggested by equation (4), increasing volatility in the discount factor $M_{t+1,t+2}$, through the convexity of the max-operator, has a positive impact on the current price of gold, just like the volatility of the underlying asset price contributes positively to the value of an option.

4 Two-state example

To illustrate model properties, I start with a simple two-state Markov chain model. This model can be solved analytically and uniqueness of the equilibrium can easily be shown.

In every period t, one of two states of nature is realized. Log real interest rates in state 1 and 2 are such that $r_1 < r_2$. A possible equilibrium would be such that in state 1, with a low interest rate, investors are holding gold. In state 2, investors are out of the gold market.

The SDF is represented by a 2 by 2 matrix

$$M = \begin{bmatrix} \exp(-r_1) - \Delta, & \exp(-r_1) + \Delta\left(\frac{\pi_{11}}{1-\pi_{11}}\right) \\ \exp(-r_2) - \Delta, & \exp(-r_2) + \Delta\left(\frac{\pi_{21}}{1-\pi_{21}}\right) \end{bmatrix}.$$

Element $m_{i,j}$ is the discount factor if at time t we are in state i and in time $t+1$ in state j. The innovation in the SDF, $\Delta > 0$, determines the slope of the term structure.[4] In this case, the SDF has a negative innovation going into state 1 and this is the state with the low interest rate which produces a capital gain for a two-period bond. This implies a positive risk premium and the unconditional term structure is upward sloping. The transition probability matrix is given as

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ \pi_{21} & 1-\pi_{21} \end{bmatrix}.$$

Clearly

$$E_t\left(M_{t,t+1}\right) = [\Pi. * M] \cdot \mathbf{1} = \begin{bmatrix} \exp(-r_1) \\ \exp(-r_2) \end{bmatrix}.$$

This is a property of many term structure models which conveniently allows one to directly specify a process for the short rate.

Given that $r_1 < r_2$, I conjecture that investors' pricing equation holds with equality in state 1 and with inequality in state 2. In this case, we have 3 unknowns (p_1, p_2, v_1) and 3

[4] The requirement that the elements of M are positive limits admissible values for Δ. In the more general case studied below the elements of M are specified through the exponential function to guarantee positivity.

equations

$$
\begin{aligned}
p_1 &= v_1 + \beta\gamma\pi_{11}p_1 + \beta\gamma\left(1-\pi_{11}\right)p_2 \\
p_2 &= \underline{v} + \beta\gamma\pi_{21}p_1 + \beta\gamma\left(1-\pi_{21}\right)p_2 \\
p_1 &= \pi_{11}m_{11}\gamma p_1 + \left(1-\pi_{11}\right)m_{12}\gamma p_2
\end{aligned}
$$

for given parameters $(\beta, \gamma, \pi_{ij}, m_{ij}, \underline{v})$. It is easy to see that this linear system of equations has in general a unique solution (except for knife-edge cases). Note the variables p and v are detrended by γ^t, for conciseness the notation does not acknowledge that.

For this solution to be an equilibrium, the initial conjecture that investors are not in the market in state 2 needs to be verified, that is

$$p_2 > \pi_{21}m_{21}\gamma p_1 + \pi_{22}m_{22}\gamma p_2.$$

We also need to check that users' marginal utility in state 1 satisfies

$$v_1 > \underline{v}.$$

Note also that with $v_1 > \underline{v}$, values for $\underline{v}$ in state 1 that are slightly higher would also produce the same equilibrium outcome.

The existence of a solution with these properties depends on parameter values. Uniqueness of this equilibrium can be checked by ruling out other candidate equilibriums. In this two-state case, only two other equilibrium configurations are possible. Either investors are out of the gold market in both states or they are in the market only in state 2. As part of the following proposition, which solves this example in closed form, I show that these alternative equilibrium configurations can be ruled out. An equilibrium where investors are always pricing gold is ruled out by assuming enough discounting (see the $N-$state case below for the exact condition).

Proposition 1 *Assuming $r_1 < r_2$ and for a range of parameter values $(\beta, \gamma, \Delta, r_1, r_2)$ for which there exists an equilibrium with investors in the gold market in state 1, the price of gold in the two states is given by*

$$p_1 = \left(\frac{(1-\pi_{11}) \exp(-r_1) + \pi_{11}\Delta}{1/\gamma - \pi_{11} \exp(-r_1) + \pi_{11}\Delta} \right) p_2,$$

$$p_2 = \frac{\underline{v}}{1 - \beta\gamma \left[1 - \pi_{21} \left(1 - \frac{(1-\pi_{11}) \exp(-r_1) + \pi_{11}\Delta}{1/\gamma - \pi_{11} \exp(-r_1) + \pi_{11}\Delta} \right) \right]},$$

$r_1 < \ln(\gamma) < r_2$, $p_1 > p_2 > \underline{p}$ *(with $\underline{p} = \underline{v}/(1-\beta\gamma)$ the price of gold for users in autarchy forever), and this is the unique equilibrium (for a given set of parameter values).*

The proof solves the equilibrium equalities algebraically and checks the inequality conditions. See the appendix for details.

For marginal utility of gold to be higher in state 1, $v_1 > \underline{v}$, the interest rate is required to be smaller than the trend growth rate in that state, $r_1 < \ln(\gamma)$. The two other equilibrium configurations would require $r_1 > \ln(\gamma)$, and therefore with $r_1 < \ln(\gamma)$ the equilibrium computed here is unique. Without the requirement that $v_1 > \underline{v}$, uniqueness would not be guaranteed.

Interestingly, gold prices in both states exceed what they would be if users are in autarchy forever: $p_1 > p_2 > \underline{p}$. In other words, the presence of investors buying and selling gold does not only raise the average gold price but it raises gold prices in all states of nature. This illustrates that even though users hold the entire supply of gold in state 2 they forecast that investors will drive up the price of gold in state 1. This can also be explained by the assumption of declining marginal utility of gold. Indeed, when investors buy up gold, this reduces the supply to users and raises users' marginal utility. Investors' presence cannot reduce marginal utilities below autarchy values.

Note also that the exact value for r_2 is not relevant in equilibrium as long as it is high enough so that investors do not want to buy gold in that state.

5 N-state Markov chain model

This section shows how an equilibrium can be defined as the fixed point of an operator. The model can be solved numerically by iterating on this operator. Extending the guess and verify approach used for the 2-state example for solving the N-state case is possible, but the operator approach is more efficient for rich specifications.

The environment is defined by $(\beta, \underline{v}, \Pi, M)$ with $0 < \beta < 1$ a scalar, $\underline{v}$ a $N \times 1$ vector of users utility values in autarchy when they hold the entire supply, and Π and M the $N \times N$ transition probability and SDF matrices. For notational compactness define the element by element multiplication $\Pi. * M \equiv \tilde{M}$.[5]

Define the operator T mapping $N \times 1$ vectors p of positive real numbers into themselves as

$$Tp = \max\left(\underline{v} + \beta\Pi p, \tilde{M}p\right). \tag{5}$$

A fixed point of this operator, $p^* = Tp^*$, satisfies equation (1) and can be used to compute the users' equilibrium marginal utility values

$$v^* = (I - \beta\Pi)\, p^* \geq \underline{v} \tag{6}$$

to satisfy the users pricing equation (2) and the decreasing marginal utility assumption, equation (3). Vice-versa, every equilibrium p is a fixed point of the operator T.

The operator T is monotone because all elements of Π and M are positive; that is, for $p_1 \geq p_2$ (defined as $p_{1,i} \geq p_{2,i}$ for all $i = 1..N$), $Tp_1 \geq Tp_2$. T is not a contraction because one-period interest rates are allowed to be negative in some states, and this implies $\tilde{M}_j \mathbf{1} > 1$ for some j, with $\tilde{M}_j$ defined as a row vector of $\tilde{M}$. The following proposition presents a sufficient condition for T to have a unique fixed point.

Proposition 2 *For the space of N-dimensional vectors $P \subseteq \mathbb{R}_+^N$ and the operator T defined*

[5] The deterministic growth trend γ can be introduced by replacing β by $\beta\gamma$, and $\tilde{M}$ by $\gamma\tilde{M}$.

in equation (5) mapping elements from P into P, the K-stage operator T^K is a contraction if

$$\max\left(\Xi_K \mathbf{1}\right) < 1$$

with

$$\Xi_K \equiv \arg\max_{\text{row}} \left[\tilde{M}\Xi_{K-1}, \beta\Pi\Xi_{K-1}\right] \mathbf{1}$$

starting with

$$\Xi_0 = I,$$

and its fixed point is the unique fixed point of T.

The proof combines standard steps with a discounting condition that is specific to the operator T, see the appendix for details. The nonstandard discounting condition is needed to deal with the max function and to allow for some negative short-term interest rates. This discounting property is embedded in the condition on the matrix Ξ_K defined in the proposition. Intuitively, there needs to be enough discounting in the effective discount factors that are implied by the max function. The condition rules out explosive paths and "bubble" solutions where investors are always pricing gold. This discounting condition can easily be checked numerically even for large N, and the fixed point is computed by iterating on T starting from some initial condition.

An alternative way to solve the model extends the guess and verify approach illustrated with the two-state example to the N-state case. Partition states into a set I for which investors are holding the asset and its complement I^c where users are in autarchy holding all the gold. Solve for the candidate price and utility flow vectors (p, v) satisfying

$$\begin{aligned} p &= v + \beta\Pi p \\ p_j &= \tilde{M}_j p \text{ for } j \in I \\ v_j &= \underline{v}_j \text{ for } j \in I^c \end{aligned}$$

through matrix inversion. Verify that

$$
\begin{aligned}
p_j &> \tilde{M}_j p \text{ for } j \in I^c \\
v_j &> \underline{v}_j \text{ for } j \in I.
\end{aligned}
$$

For N states, there are 2^N possible partitions. Practically, for the one-factor model solved in the quantitative section, fewer partitions are required. For instance, if states are ordered by interest rates, defining I by a cutoff rate is typically sufficient. For richer specifications and with a larger number of states, iterating on the Bellman operator is more efficient because there is no need to customize the selection of partitions corresponding to candidate solutions.

6 Quantitative analysis

The model is evaluated with respect to its ability to produce the empirical properties documented in Section 2, specifically, a strong negative relation between long-term real rates and gold prices which is limited to periods of low rates. After calibrating the model to a one-factor term structure model and documenting some properties, a second factor is introduced, the real exchange rate. The time series of gold prices produced by the model when driven by these factors is compared to the historical series of gold prices.

The SDF M_{t+1} is specified as in a discrete-time version of the Vasicek term structure model following Backus, Foresi and Telmer (1998). In that model, $\ln M_{t+1} = -y_t^{(1)} - \sigma_m \varepsilon_{t+1} - a_t$ and $y_{t+1}^{(1)} = \rho_y y_t^{(1)} + (1 - \rho_r)\,\bar{y}^{(1)} + \sigma_y \varepsilon_{t+1}$ with ε_{t+1} IID normally distributed with mean zero and variance one. This model has four free parameters $\left(\bar{y}^{(1)}, \sigma_y, \rho_y, \sigma_m\right)$; a_t is used to make sure that $E_t M_{t+1} = \exp(r_t)$. The first three parameters determine dynamics of the short rate. The fourth, σ_m, is the price of risk parameter. This process is approximated by a discrete Markov chain with 100 states following the Tauchen method. As calibration targets, I choose the mean, standard deviation and first autocorrelation of the ten-year rate. The price of risk parameter is set to match the unconditional spread between the ten-year

Table 2: Calibration and Implied Real Interest Rates. Data are annual averages for 1980-2020 based on U.S. Treasury bonds and TIPS.

Parameter			Target	
Mean $y_t^{(1)}$	$\bar{y}^{(1)}$	.0137	$\mathrm{E}\left(y_t^{(1)}\right)$	
Innovation Std $y_t^{(1)}$	σ_y	.0085	$\mathrm{Std}\left(y_t^{(10)}\right)$	
AR(1) $y_t^{(1)}$	ρ_y	.95	$\mathrm{AR}(1)\left(y_t^{(10)}\right)$	
Price of risk	σ_m	$-.323$	$\mathrm{E}\left(y_t^{(10)} - y_t^{(1)}\right)$	
Interest rates	Data		Model	
	$y^{(1)}$	$y^{(10)}$	$y^{(1)}$	$y^{(10)}$
Mean	1.37	2.67	1.37	2.67
Std	2.38	1.93	2.73	1.93
AR(1)	.90	.95	.95	.95

rate relative to the one-year rate.

As shown in Table 2, the model can match these four calibration targets perfectly. The volatility of the short rate and its autocorrelation which were not separately targeted are only slightly off.

In addition to the SDF, the model has two parameters that need to be calibrated, the users' discount factor β and the trend growth rate of the price of gold γ. I set $\beta = \exp\left(-\bar{y}^{(1)}\right)$, implying the users discount at the same rate as investors on average. The trend growth rate of the gold price γ plays a crucial role for model properties. Barro and Misra (2015) find an average growth rate of 1.1% for 1836-2011 with a one-standard deviation band of $(0.1\%, 2.1\%)$. In my model, with all other parameters as described, the highest growth rate before gold prices tend to infinity is somewhat above $\gamma = 1.009$. Based on this, a plausible range for this parameter is $(1.001, 1.009)$. To more tightly identify this parameter, I consider the sensitivity of the log growth price changes with respect to changes in the 10-year yields for periods with relatively low 10-year yields. As documented Table 1 gold price changes are closely related to yield changes but only during periods of low rates, which mostly correspond to the time after about 2003. For the 50% of realizations for the 10-year rate that are below the average, this sensitivity equals -10.0 in the data. The model can

Table 3: Model implications. γ is the equilibrium trend gross growth rate of the price of gold. β is the discount factor of the users.

	$E\left(p/\underline{p}\right)$	Std($\ln p$)	Std($\ln p'/p$)	Prob(I^c)	Sensitivity low $y^{(10)}$
$\gamma = 1.0084$	3.8	12.5	4.3	0.71	-10.0
$\gamma = 1.001$	1.3	9.1	3.2	0.78	-7.7
$\gamma = 1.009$	6.6	12.8	4.4	0.71	-10.2
Lower $\beta = \exp\left(-\bar{y}^{(1)}\right)/1.005$	1.7	12.3	4.3	0.71	-9.7
Lower price of risk, $\sigma_m/1.2$	8.2	14.4	4.9	0.69	-10.7
More volatile rates, $\sigma_y \times 1.1$	6.3	14.1	4.8	0.71	-10.1
More persist., AR(1) $y_t^{(1)} = 0.98$	2.3	22.4	4.9	0.78	-12.4

match this value with $\gamma = 1.0084$, which is taken as the benchmark value.

Table 3 displays model properties for these parameter values and for various alternative calibrations. For the benchmark calibration displayed in the first row, the standard deviation of the logarithmic price changes, Std($\ln p'/p$), is 4.3%. This represents the fluctuations in gold prices that come from real interest rate movements. Empirically, the unconditional standard deviation in annual gold prices 1980-2020 is 14.8%. This implies that interest rate movements transmitted to gold by investors were responsible for somewhat more than a quarter of the gold price volatility. As documented below, during the more recent years characterized by relatively lower interest rates, the share of gold price movements coming from interest rate changes has been significantly larger.

As shown by the second to last column, 71% of the time users are holding the entire supply of gold with investors out of the gold market.

In the model, interest rate changes have a nonlinear impact on the price of gold. As shown in Table 3, the average level of the price of gold is 3.8 times the level of the users' *autarchy* value $\underline{p}$, which would be the value if users would hold all the gold forever (and investor would be forever out of the gold market). In other words, the fact that gold is an investment asset and not just a commodity is responsible for more than two thirds of its value.

The average level of the price of gold is sensitive to the selected values for γ. At $\gamma = 1.001$, which is the low end of my plausible range, investors add on average only 30% to the value of gold. At the other end, with $\gamma = 1.009$ gold is on average worth more than 6 times its value without investors. As γ increases, gold prices become more sensitive to interest rates. Relatively moderate increases in the standard deviation of the gold price produce very large increases in the level of the gold price. This illustrates the strong nonlinearity in the model; declines in yields not only produce valuation effects but also increase the sensitivity of the price to yields, that is, the effective duration increases.

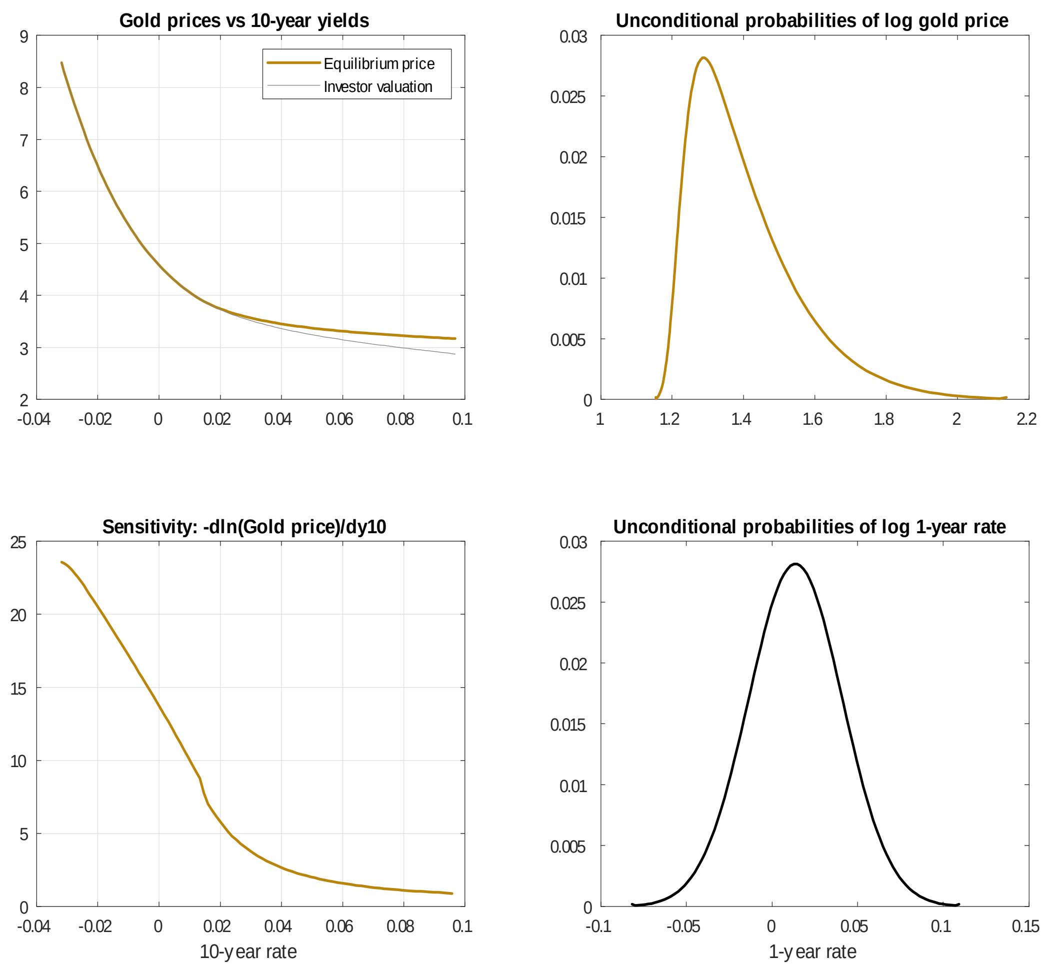


Figure 3: Model properties.

The two panels on the left in Figure 3 further illustrate this mechanism. The first panel on the left shows gold prices against 10-year real yields. The price of gold is scaled by gold's autarchy value (that is, if users were holding all gold forever). As is clear in the graph, with low rates gold prices increase strongly, and even during periods of very high interest rates, gold would be worth significantly more than this autarchy value. The plot nicely illustrates the option-like value profile from investors' perspective. The lower panel on the left displays the sensitivity of the price of gold with respect to the 10-year real yield. This can be viewed as gold's duration. For instance, if the 10-year rate is at 0, gold's duration is about 13 years and it is strongly increasing as rates decline. The distribution of gold prices in the model is far from normal or log-normal. The probability density function for the logarithms of gold prices in the right upper panel displays strong positive skewness. This nonlinearity is entirely endogenous to the model, as short rates are log-normally distributed.

Table 4 compares slope coefficients from regressing annual changes in log gold prices on changes in yields. Model implied gold prices are computed for the sequence of US 10-year real yields 1980-2020. Because the state space is discrete, I am directly interpolating the vector of gold prices relative to the vector of model implied 10-year yields. Consistent with the data, the sensitivity of model implied gold prices to 10-year real yields is substantially lower for periods with high yields. The model is calibrated to match the slope coefficient of -10.0 for the lower 50% of yields, for the corresponding higher 50% the model's sensitivity of -2.4 is somewhat lower then its empirical counterpart of -5.1.

The table also includes gold's interest sensitivity for a representative agent (RA) model. In this case, gold is priced with a constant utility flow v based on

$$p_t = v + E_t M_{t,t+1} p_{t+1}.$$

The RA model cannot produce the strong nonlinearity in the sensitivity of gold prices to interest rates. The sensitivity when yields are high is excessively high. Nevertheless, the RA also features a sensitivity of gold prices to interest rates that is declining in the level of

Maturity		10-year	
Data	$\Delta y_t,\, y_t < y^*$	-19.3^{***}	
33.3%	$\Delta y_t,\, y_t \geq y^*$		-4.7^{**}
50%		-10.0^{**}	
			-5.1^{***}
Model	$\Delta y_t,\, y_t < y^*$	-12.1	
33.3%	$\Delta y_t,\, y_t \geq y^*$		-2.6
50%	$\Delta y_t,\, y_t < y^*$	-10.0	
	$\Delta y_t,\, y_t \geq y^*$		-2.4
RA Model	$\Delta y_t,\, y_t < y^*$	-19.2	
33.3%	$\Delta y_t,\, y_t \geq y^*$		-13.9
50%	$\Delta y_t,\, y_t < y^*$	-18.4	
	$\Delta y_t,\, y_t \geq y^*$		-13.7

Table 4: Regression of log-percentage changes in real gold prices on yields conditional on low or high yield levels. Slope coefficients are reported. RA stands for representative agent economy. The cutoff levels y* are set to assign 33.3% or 50% of the observations to the low yield sample, respectively. Significance levels are given as *** (p<.01), ** (p <.05), and * (p<.1).

interest rates. In other words, the representative agent model features nonnegligble convexity. Recalibrating the RA model with a lower trend growth rate γ only weakly reduces the slope coefficients in the RA model. For instance, with $\gamma = 1$, the slope coefficients are -17.0 and -12.6 with the 50% cutoff compared to -18.4 and -13.7 in the table corresponding to the benchmark calibration $\gamma = 1.0084$.

6.1 How well does the model explain gold prices since 1980?

The only driving factor considered so far is the investors' SDF which has been specified to match the term structure of real interest rates. Independently measuring variations in the users' utility flows, which could be another factor, would seem to be a hopeless task. However, considering that gold users are to a large extent non-US residents who are not using the USD as their numeraire creates a role for the real exchange to affect the price of gold.

6.1.1 Real exchange rates

Assume that users are non-US residents. For pricing gold in real USD terms, we need to convert their utility flow value measured in their own numeraire to real USD. In the model, the users' autarchy utility flow $\underline{v}$ is multiplied by the real exchange rate S_t, defined as real USD per unit of the foreign good. The permanent autarchy valuation becomes

$$\underline{p}_t = S_t \underline{v} + \beta E_t \underline{p}_{t+1}.$$

Represent the exchange rate as

$$S_{t+1} = \Gamma_{t+1} S_t$$

with Γ_{t+1} the gross growth rate, which I assume follows an exogenous stochastic process. Divide both sides by S_t, and denote gold prices scaled by the exchange rate as $\underline{\hat{p}_t} \equiv \underline{p}_t / S_t$,

$$\widehat{\underline{p}}_t = \underline{v} + \beta E_t \Gamma_{t+1} \widehat{\underline{p}}_{t+1}. \tag{7}$$

The pricing equation of the scaled gold price now includes the stochastic growth rates of exchange rate Γ_{t+1} but is otherwise identical to the pricing equation in the model without the exchange rate. The two equilibrium pricing equations, equations (1) and (2), are equally unchanged except for the exchange rate changes Γ_{t+1} multiplying the scaled gold prices dated $t+1$.

This model can be solved once a process for Γ_{t+1} is specified. Here I consider a special case. Assume exchange rate changes are uncorrelated with real interest rates, identically and independently distributed with $E_{t+1}\Gamma_{t+1} = 1$. It is easy to see that in this case Γ_{t+1} drops out of equation (7) and the other equilibrium pricing equations of the model. This implies that the solution for the scaled gold price $\widehat{p}_t$ is identical to the solution for p_t in the model without the exchange rate. The only effect of introducing the exchange rate is that we need to multiply the scaled gold price by the exchange rate, so that the equilibrium gold price in this two-factor model is $S_t \widehat{p}_t$.

While this specification is convenient, it is debatable whether empirically the real exchange follows exactly this multiplicative random walk and whether the growth rates are exactly uncorrelated. In particular, there is a large literature on whether real exchange rates have units roots which does not seem to have reached a consensus; for prominent examples see Engel (2000) and Rogoff (1996). If the real exchange rate is stationary in the model, its impact on the price of gold would be weaker. The random walk is a special case where the present discounted value changes by the same percentage as the dividend growth (here the utility flow).

6.1.2 Gold prices 1980-2020

As the real exchange rate, I use the Real Broad Dollar Index from the Board of Governors of the Federal Reserve System (US) retrieved from FRED, (RTWEXBGS, 2006-2020, TWEXBPA 1980-2006).

Figure 4 displays model implications with and without the exchange rate. The average of each series is normalized to 1. Clearly, the model driven by 10-year yields can capture quite well the price movements in the later part of the sample from about 2007 onwards. This is consistent with the regression evidence in Table 1 that shows a tight negative relation between yields and gold prices when yields are low, which is predominantly in the later part of the sample. For the period prior to about 2007, the model-implied gold price explains little of the actual gold price fluctuations. This is also consistent with the regression evidence showing only a weak connection between gold prices and real rates during these times when rates were high.

Incorporating the real exchange rate helps the model in 2012 but not in 2020. For the time period 1995- 2007, including the exchange clearly helps the model at medium frequencies. Overall, the mechanisms and the factors represented in this model can reproduce a significant part of the gold price fluctuations since 2007.

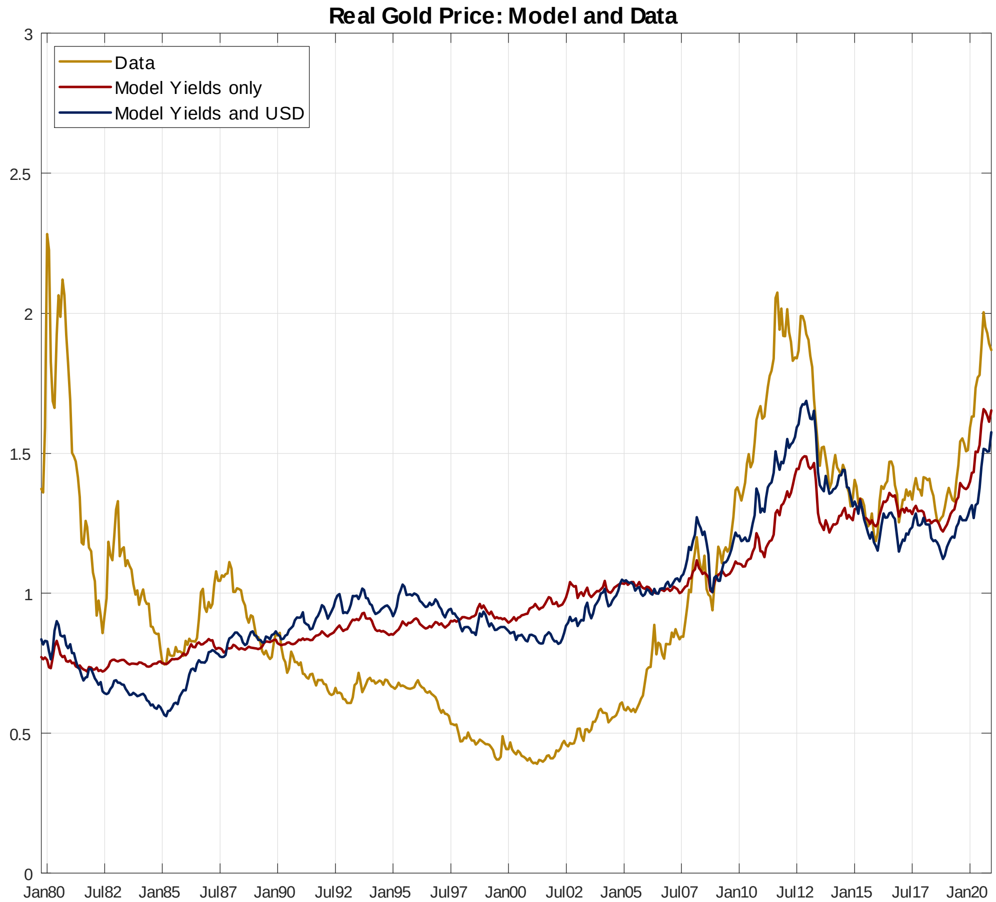


Figure 1: Figure 4

7 Conclusion

This paper values gold from an investor perspective by recognizing that users of gold as jewelry or as a productive input put a lower floor to the price of gold. This embeds an option into gold whose strike price is implicitly derived by the users' price forecasts. A model incorporating these ideas is presented, analytically characterized, and an efficient computational algorithm is developed. Motivated by the empirical evidence on the relation between gold prices and real interest rates, the quantitative analysis is based on a term structure mode calibrated to real US interest rates. The model can go a long way towards replicating the time series of gold prices since about 2007. The approach developed in this paper can be extended to richer term structure models or discount factor specifications.

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Appendix A: Two state example (Proposition 1)

Assume $r_1 < r_2$ and that the investors' pricing equation holds in state 1. Rearranging that equation gives

$$p_1 = \frac{(1-\pi_{11})\exp(-r_1) + \pi_{11}\Delta}{1/\gamma - \pi_{11}\exp(-r_1) + \pi_{11}\Delta} p_2.$$

Combining this with the users' pricing equation in state 2 gives the solutions for p_2 in the proposition.

For an equilibrium we require $v_1 > \underline{v}$. From the users' pricing equations in state 1 and 2 we get

$$\begin{aligned} p_1 &= v_1 + \beta\gamma\pi_{11}p_1 + \beta\gamma(1-\pi_{11})p_2 \\ v_1 &= (1-\beta\gamma\pi_{11})p_1 - \beta\gamma(1-\pi_{11})p_2 \end{aligned}$$

$$\begin{aligned} p_2 &= \underline{v} + \beta\gamma\pi_{21}p_1 + \beta\gamma(1-\pi_{21})p_2 \\ \underline{v} &= (1-\beta\gamma(1-\pi_{21}))p_2 - \beta\gamma\pi_{21}p_1, \end{aligned}$$

the requirement implies

$$\begin{aligned} v_1 &> \underline{v} \\ (1-\beta\gamma\pi_{11})p_1 - \beta\gamma(1-\pi_{11})p_2 &> (1-\beta\gamma(1-\pi_{21}))p_2 - \beta\gamma\pi_{21}p_1 \\ p_1 &> p_2. \end{aligned}$$

Imposing this to the first equation in this appendix implies

$$\begin{aligned} \frac{(1-\pi_{11})\exp(-r_1) + \pi_{11}\Delta}{1/\gamma - \pi_{11}\exp(-r_1) + \pi_{11}\Delta} &> 1 \\ r_1 &< \ln(\gamma). \end{aligned}$$

So, this is an additional requirement for this equilibrium to exist.

For the postulated equilibrium we also need that investors are out of the market in state 2, that is,

$$p_2 > \pi_{21}\gamma m_{21} p_1 + \pi_{22}\gamma m_{22} p_2$$

$$\begin{aligned} p_2 &> \pi_{21}\gamma\left(\exp\left(-r_2\right) - \Delta\right) p_1 + \left(1-\pi_{21}\right)\gamma\left(\exp\left(-r_2\right) + \Delta\frac{\pi_{21}}{\left(1-\pi_{21}\right)}\right) p_2 \\ p_2 &> \frac{\pi_{21}\exp\left(-r_2\right) - \pi_{21}\Delta}{1/\gamma - \left(1-\pi_{21}\right)\exp\left(-r_2\right) - \Delta\pi_{21}} p_1 \end{aligned}$$

combined with the property $1 > \frac{p_2}{p_1}$ shown above,

$$1 > \frac{p_2}{p_1} > \frac{\pi_{21}\exp\left(-r_2\right) - \pi_{21}\Delta}{1/\gamma - \left(1-\pi_{21}\right)\exp\left(-r_2\right) - \Delta\pi_{21}}$$

$$1 > \frac{\pi_{21}\exp\left(-r_2\right) - \pi_{21}\Delta}{1/\gamma - \left(1-\pi_{21}\right)\exp\left(-r_2\right) - \Delta\pi_{21}}$$

$$r_2 > \ln\left(\gamma\right).$$

Ruling out the other two equilibrium configurations.

In an equilibrium where autarchy forever is a solution we have

$$\underline{p}_1 = \underline{p}_2 = \frac{\underline{v}}{1-\beta\gamma} = \underline{p}.$$

To rule out such an equilibrium we need that

$$\begin{aligned} \underline{p} &< \pi_{11} m_{11}\gamma\underline{p} + \pi_{12} m_{12}\gamma\underline{p} \\ \underline{p} &< \pi_{11}\gamma\left(\exp\left(-r_1\right) - \Delta\right)\underline{p} + \left(1-\pi_{11}\right)\gamma\left(\exp\left(-r_1\right) + \Delta\left(\frac{\pi_{11}}{\left(1-\pi_{11}\right)}\right)\right)\underline{p} \\ r_1 &< \ln\gamma, \end{aligned}$$

which we have already assumed for existence of the conjectured equilibrium.

For an equilibrium where investors are holding gold in state 2 but not in state 1, redoing the analysis for the first candidate equilibrium would require $r_2 < r_1$, here we started with the opposite assumption. The assumption of decreasing marginal utility, $v_t \geq \underline{v}_t$, implies that the price in an autarchy equilibrium, $\underline{p} = \frac{\underline{v}}{1-\beta\gamma}$, is a lower bound to any equilibrium price of gold; with strict inequality if investors enter the market in at least one state.

Appendix B: Proposition 2

The proof adapts Theorem 3.2., its Corrollary 2, and Theorem 3.3. in Stokey and Lucas with Prescott (1989), SLP, with a more complex discounting property that is specific to the operator presented in this paper and that is needed because of the max function and the requirement that some one-period real interest rates are negative.

Two properties of the max function are used repeatedly to allow elements to be moved from the inside to the outside of the max function. First, a constant a (positive or negative) on both sides can be taken out of the max function,

$$\max(X-a, Y-a) = \max(X,Y) - a,$$

because the order of X and Y is not affected by the constant.

Second, we can take a positive value a out of one side of the max function with inequality

$$\max(X+a, Y) \leq \max(X,Y) + a.$$

Proof. If $X > Y$ then $X + a > Y$ and the order cannot flip so that the first element stays the larger and

$$\max(X+a, Y) = \max(X,Y) + a$$

because in both cases we had $X + a$.

If $Y > X$ and $Y > X + a$, the order cannot flip, the second element stays larger and we have

$$\max(X + a, Y) < \max(X, Y) + a$$

because now the positive a matters.

If $Y > X$ and $Y < X + a$, the order flips, initially the first element was active and we have $X + a$ and now the second is active and we have $Y + a$, by assumption $Y > X$, so that now a is added to a larger value. QED.

Define the matrix that selects among the two rows the one that produces a higher sum of the elements

$$\Xi = \arg\max_{\text{row}} \left[\tilde{M}, \beta\Pi\right] \mathbf{1}$$

for $\mathbf{1}$ a vector of $1's$. Intuitively, this will include the rows of $\tilde{M}$ for the states for which the interest rate is negative, that is $\tilde{M}_j 1 > 1$.

Define the corresponding selection matrix and its complement

$$I_\Xi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ & & & \dots \\ & & & \end{bmatrix}$$

with 1 if row j of $\tilde{M}_j 1$ is the larger and is included in Ξ

$$I_\Xi . * \tilde{M} = \begin{bmatrix} M_1 & & & \\ 0 & 0 & 0 & 0 \\ & & & \dots \\ & & & \end{bmatrix},$$

and I_Ξ^C the complement which selects the rows for which $\beta\Pi 1$ is larger. With this definition

$$\left[I_\Xi.*\tilde{M}\right]+\left[I_\Xi^C.*\beta\Pi\right]=\Xi.$$

With these preliminaries, consider evaluating the operator for an arbitrary price vector p with the addition of a $N\times 1$ vector a whose elements are an arbitrary positive constant $\in\mathbb{R}_+$,

$$\begin{aligned}
T(p+a) &= \max\left(\underline{v}+\beta\Pi p+\beta\Pi a,\tilde{M}p+\tilde{M}a\right)\\
&= \max\left(\underline{v}+\beta\Pi p+\beta\Pi a-\left[I_\Xi.*\beta\Pi\right]a,\tilde{M}p+\tilde{M}a-\left[I_\Xi.*\beta\Pi\right]a\right) \qquad +\left[I_\Xi.*\beta\Pi\right]a\\
&= \max\left(\underline{v}+\beta\Pi p+\beta\Pi a-\left[I_\Xi.*\beta\Pi\right]a-\left[I_\Xi^C.*\tilde{M}\right]a,\tilde{M}p+\tilde{M}a-\left[I_\Xi.*\beta\Pi\right]a-\left[I_\Xi^C.*\tilde{M}\right]a\right)\\
&\quad +\left[I_\Xi.*\beta\Pi\right]a+\left[I_\Xi^C.*\tilde{M}\right]a\\
&= \max\left(\underline{v}+\beta\Pi p+\left[I_\Xi^C.*\left(\beta\Pi-\tilde{M}\right)\right]a,\tilde{M}p+\left[I_\Xi.*\left(\tilde{M}-\beta\Pi\right)\right]a\right)\\
&\quad +\left[I_\Xi.*\beta\Pi a\right]+\left[I_\Xi^C.*\tilde{M}a\right]\\
&\leq \max\left(\underline{v}+\beta\Pi p,\tilde{M}p\right)\\
&\quad +\left[I_\Xi.*\beta\Pi a\right]+\left[I_\Xi^C.*\tilde{M}a\right]+\left[I_\Xi^C.*\left[\beta\Pi-\tilde{M}\right]\right]a+\left[I_\Xi.*\left[\tilde{M}-\beta\Pi\right]\right]a\\
&= Tp \qquad +\left[I_\Xi.*\tilde{M}\right]a+\left[I_\Xi^C.*\beta\Pi\right]a\\
&= Tp+\Xi a.
\end{aligned}$$

First and second steps add the same constants to both sides of the max function. The third step does some algebra, and the fourth step takes out positive terms on both sides. The inequality applies to each element of the vectors on both sides. The fifth step is algebra, and last step is the definition of Ξ.

For discounting, I need to show that

$$T(p+a)\leq Tp+\Xi a\leq Tp+\delta a$$

for some $0 < \delta < 1$. Because I want to allow negative interest rates, for some rows of $\tilde{M}_j$

$$\Xi_j a > a,$$

which goes against discounting. However, I can apply the operator K times, and for each K the matrix Ξ_K changes (almost like $(\Xi)^K$ but not exactly). As a sufficient condition for discounting, I am looking for a case where for K large enough

$$\max\left(\Xi_K \mathbf{1}\right) < 1,$$

with Ξ_K such that

$$T^K(p+a) \leq T^K p + \Xi_K a.$$

For such a case, discounting applies for the $K-$stage operator and there is a unique fixed point for T^K and T, by Corrollary 2 (to Th. 3.2) in SLP.

To derive Ξ_K, in line with the previous calculations, we have

$$\begin{aligned}
T^2(p+a) &= \max\left(\underline{v} + \beta\Pi\left(T\left[p+a\right]\right), \tilde{M}\left(T\left[p+a\right]\right)\right) \\
&\leq \max\left(\underline{v} + \beta\Pi\left(Tp + \Xi a\right), \tilde{M}\left(Tp + \Xi a\right)\right) \\
&= \max\left(\underline{v} + \beta\Pi Tp + \beta\Pi\Xi a, \tilde{M}Tp + \tilde{M}\Xi a\right) \\
&= \max\left(\underline{v} + \beta\Pi Tp + \left[\beta\Pi\Xi\right]a, \tilde{M}Tp + \left[\tilde{M}\Xi\right]a\right) \\
&\quad ... \\
&\leq T^2 p + \Xi_2 a
\end{aligned}$$

with

$$\Xi_2 \equiv \arg\max_{\text{row}}\left[\tilde{M}\Xi, \beta\Pi\Xi\right]\mathbf{1}$$

and so on for K with

$$\Xi_K \equiv \arg\max_{\text{row}}\left[\tilde{M}\Xi_{K-1}, \beta\Pi\Xi_{K-1}\right]\mathbf{1}$$

starting with

$$\Xi_0 = I.$$

If this discounting property is satisfied, a slightly modified version of Th. 3.3. in SLP can be used to proof that we have a contraction. Specifically, for any p_1 and $p_2 \in P$, for each element j

$$p_{1,j} \leq p_{2\,j} + \|p_1 - p_2\|$$

with $\|.\|$ the sup norm. Moving to vectors

$$p_1 \leq p_2 + \|p_1 - p_2\| \mathbf{1}.$$

Now apply T^K to both sides

$$T^K p_1 \leq T^K \left(p_2 + \|p_1 - p_2\| \mathbf{1}\right) \leq T^K p_2 + \delta_K \|p_1 - p_2\| \mathbf{1}$$

by monotonicity (first inequality) and under my discounting property for $0 < \delta_K < 1$ (second inequality). Exchanging p_1 and p_2 implies

$$T^K p_2 \leq T^K p_1 + \delta_K \|p_1 - p_2\| \mathbf{1}.$$

Combining the last two inequalities

$$\left\|T^K p_1 - T^K p_2\right\| \leq \delta_K \|p_1 - p_2\|,$$

which is the property that defines a contraction.