Robust Leveraged ETF Portfolios Extending Classic 40/60 Portfolios and Portfolio Insurance

Mikhail Smirnov

Columbia University

Alexander Smirnov

Stuyvesant High School

Abstract.

Leveraged ETFs provide a convenient mechanism to dynamically change portfolio exposure and can be successfully used to construct robust portfolios that perform well during equity market drops. We start with a classical 60 percent Bonds/ 40 percent Stocks portfolio with monthly rebalancing that delivered 9.4 percent annually over since 1986. Its 120 percent leveraged cousin that is 72 percent Bonds/ 48 percent Stocks delivered 10.4 percent annually since 1986, same as stocks but with lower volatility and drawdowns. Instead of leveraging with borrowing at portfolio level we can use a portfolio of leveraged ETFs.

We consider several balanced stocks/bonds portfolios created with leveraged ETFs but without borrowing money at portfolio level and show that they present a very attractive risk adjusted alternative to just stock index and classical stocks/ bonds portfolios without leverage. In particular portfolio of 40 percent TQQQ, 20 percent TMF, 40 percent TLT with monthly rebalancing proposed by us in 2017 paper as a leveraged ETF alternative to classical stocks/bonds portfolios performed well in 2018 and through beginning of Coronavirus crisis up to February 28, 2020. What happens to this portfolio as crisis continues still to be seen.

A classical portfolio insurance strategy of Black-Jones-Perold can be easily implemented with leveraged ETFs. More complex dynamic portfolio strategies can also be implemented using leveraged ETFs.

We will analyze following ETF portfolios with monthly rebalancing

1. Net 100%   60% Bonds VUSTX
   40% SP500 as VFINX (Not ETFs but Vanguard Mutual Funds)

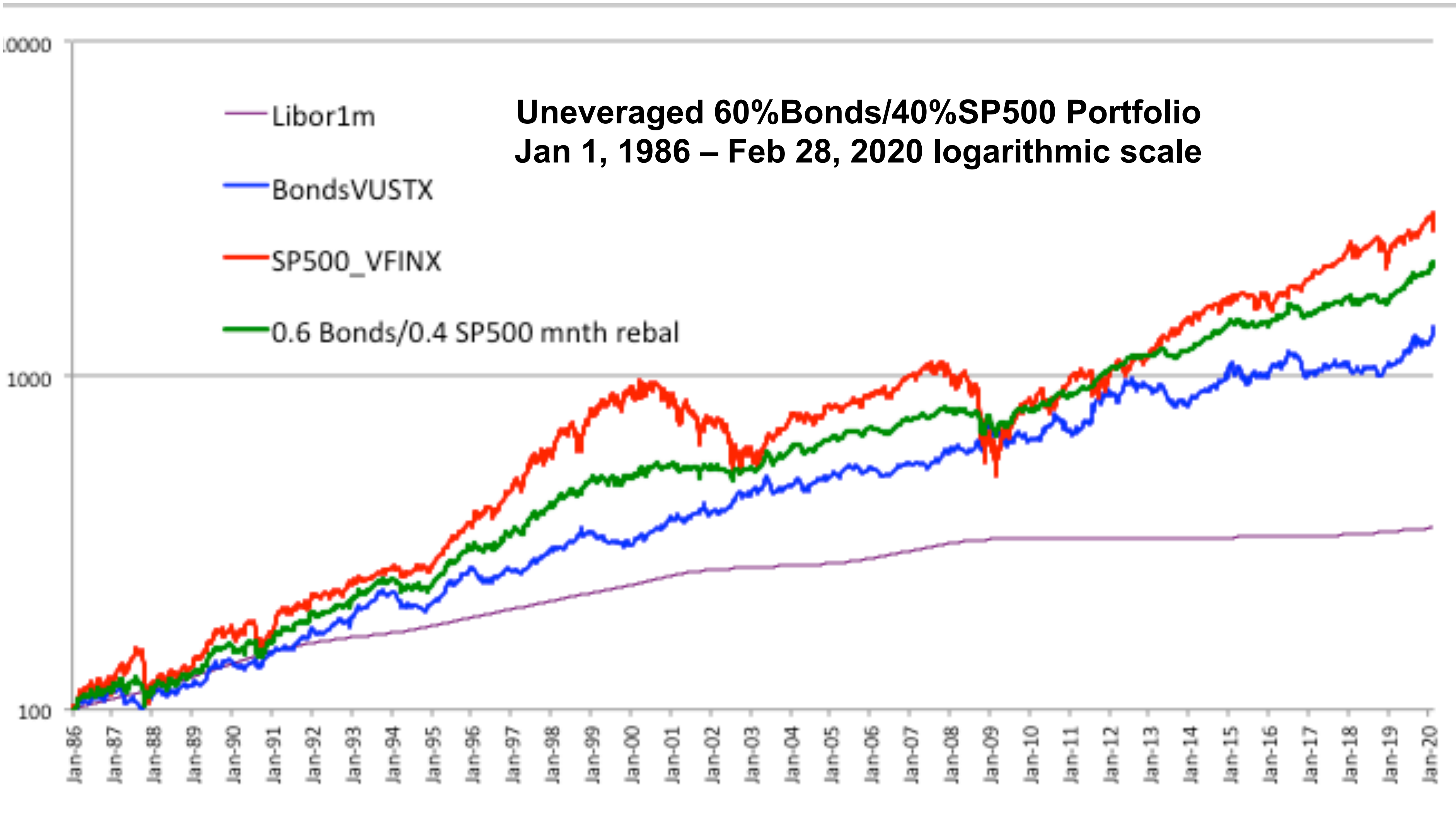
2. Net 120%   72% Bonds VUSTX
   48% SP500 as VFINX,
   20% Borrowed at 1mLibor+25bp

3. 40 % TQQQ (Triple Leveraged Nasdaq 100 ETF)
   20 % TMF (Triple Leveraged Long U.S. Government Bonds Index ETF)
   40% TLT (Single Leveraged Long U.S. Government Bonds Index ETF)

4. 20 % TQQQ (Triple Leveraged Nasdaq 100 ETF)
   0 % TMF (Triple Leveraged Long U.S. Government Bonds Index ETF)
   80% TLT (Single Leveraged Long U.S. Government Bonds Index ETF)

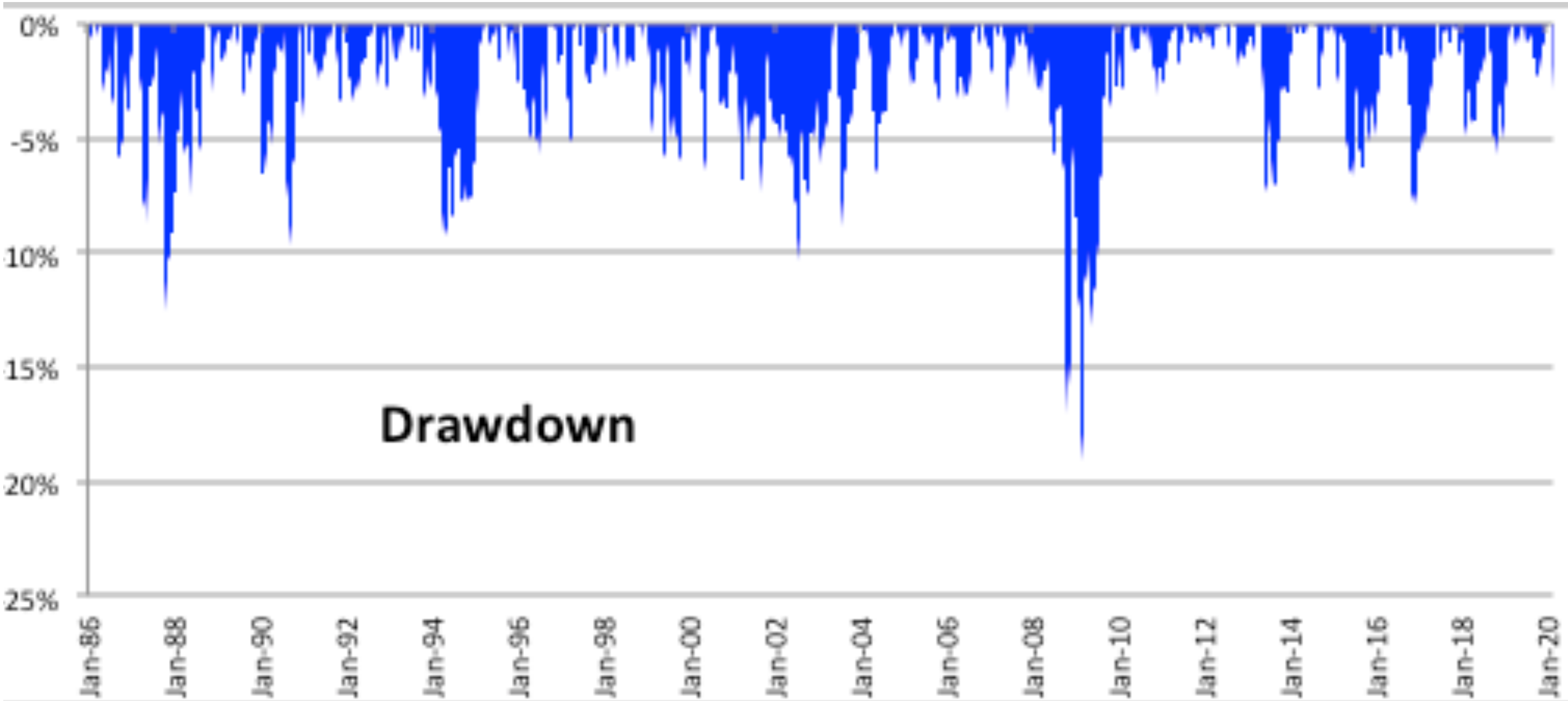
Portfolios of ETFs.

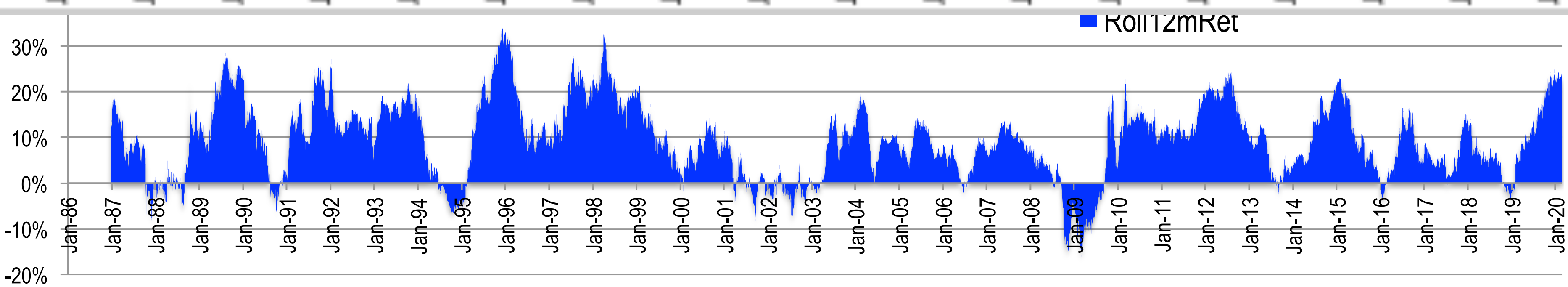
We start with classic balanced portfolios of mutual funds

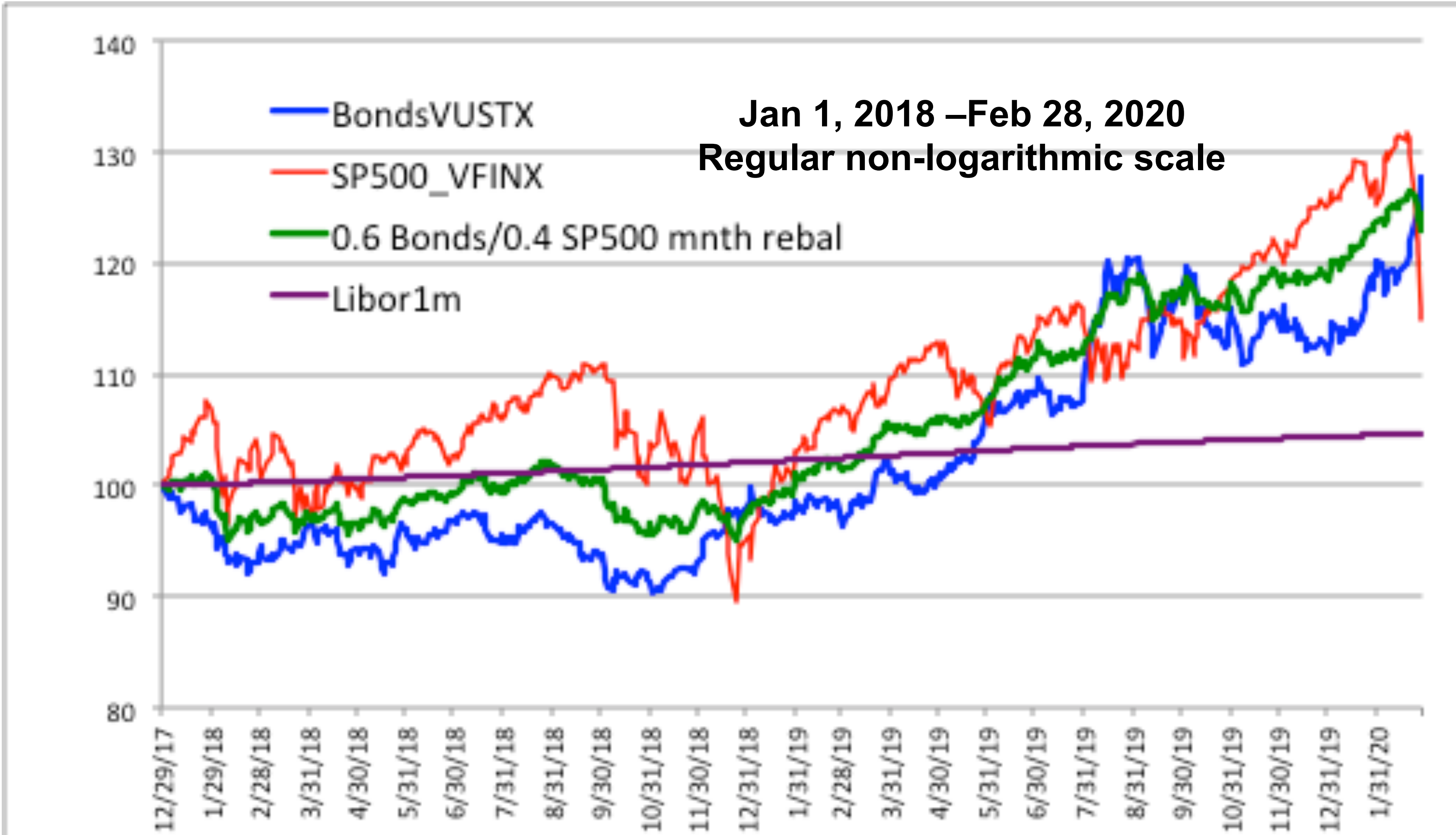


60% BondsVUSTX 40%SP500_VFINX monthly rebalancing total return				100% SP500_VFINX total return			
AnnRet 1986-2020	9.4%	AnnRet 2009-2020	9.7%	AnnRet 1986-2020	10.1%	AnnRet 2009-2020	13.4%
AnnVol 1986-2020	8.5%	AnnVol 2009-2020	7.4%	AnnVol 1986-2020	17.8%	AnnVol 2009-2020	16.3%
AnnRet/AnnVol	1.10	AnnRet/AnnVol	1.31	AnnRet/AnnVol	0.57	AnnRet/AnnVol	0.82
SharpeLibor1m	0.64	SharpeLibor1m	1.20	SharpeLibor1m	0.35	SharpeLibor1m	0.77

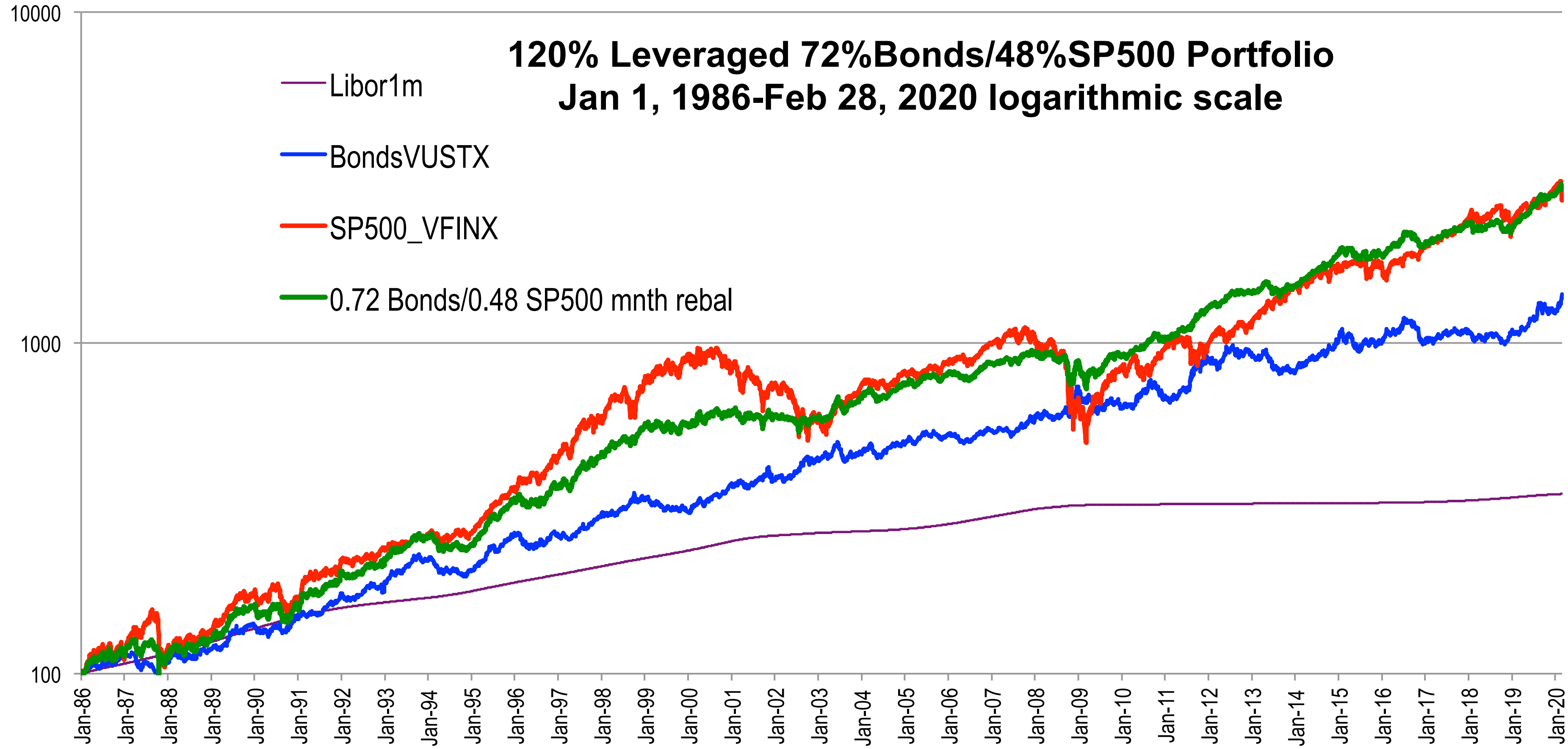
60% BondsVUSTX 40%SP500_VFINX dividends reinvested monthly rebalancing







Date	Libor1m	BondsVUSTX	SP500_VFINX	0.6 Bonds/0.4 SP500 mnth rebal
2/28/20	104.7	127.8	115.0	124.4
2/27/20	104.7	125.0	115.9	123.0
2/26/20	104.7	123.8	121.2	124.4
2/25/20	104.7	124.5	121.7	125.0
2/24/20	104.7	123.7	125.5	126.0
2/21/20	104.7	121.9	129.8	126.6
2/20/20	104.7	120.8	131.2	126.4
2/19/20	104.7	119.9	131.7	126.1
2/18/20	104.6	119.9	131.1	125.8

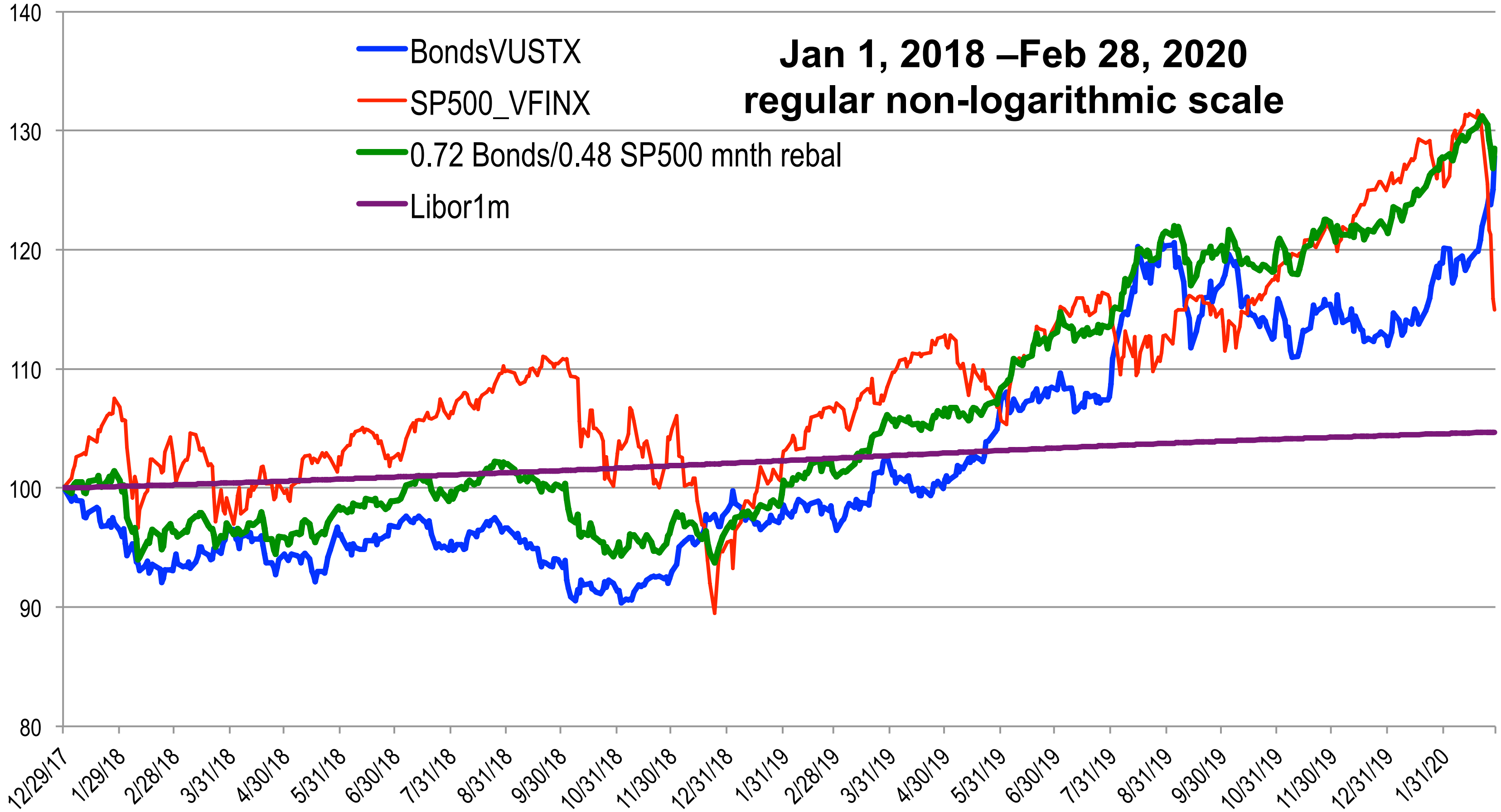


72% BondsVUSTX 48%SP500_VFINX
20% borrowed at 1m Libor +25bp
dividends reinvested, allocation monthly rebalanced
Total return

AnnRet 1986-2020	10.4%	AnnRet 2009-2020	11.5%
AnnVol 1986-2020	10.2%	AnnVol 2009-2020	8.9%
AnnRet/AnnVol	1.02	AnnRet/AnnVol	1.29
SharpeLibor1m	0.63	SharpeLibor1m	1.20

100% SP500_VFINX Total return

AnnRet 1986-2020	10.1%	AnnRet 2009-2020	13.4%
AnnVol 1986-2020	17.8%	AnnVol 2009-2020	16.3%
AnnRet/AnnVol	0.57	AnnRet/AnnVol	0.82
SharpeLibor1m	0.35	SharpeLibor1m	0.77



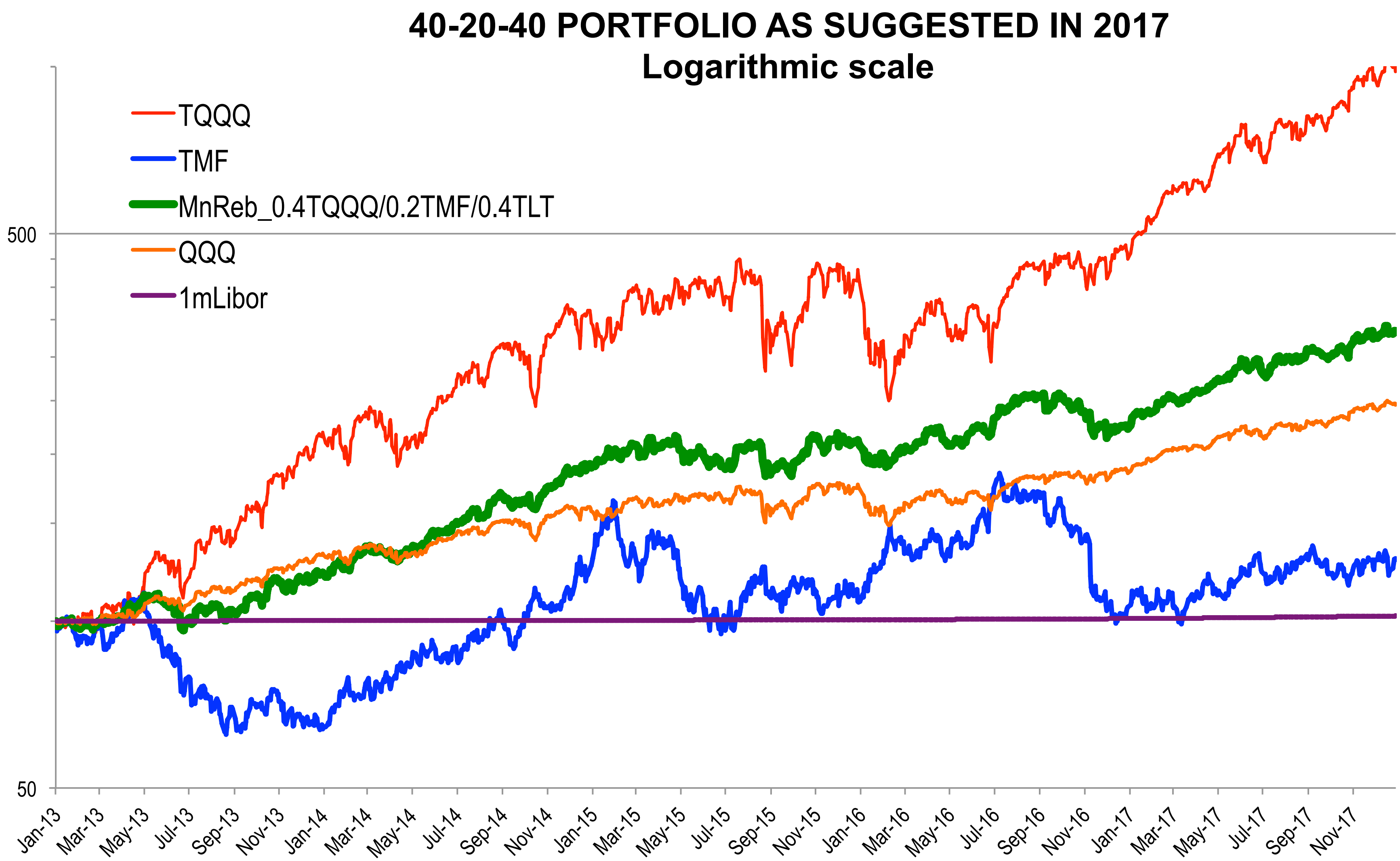
Date	Libor1m	BondsVUSTX	SP500_VFINX	0.72 Bonds/0.48 SP500 mnth rebal
2/28/20	104.7	127.8	115.0	128.5
2/27/20	104.7	125.0	115.9	126.8
2/26/20	104.7	123.8	121.2	128.5
2/25/20	104.7	124.5	121.7	129.2
2/24/20	104.7	123.7	125.5	130.5
2/21/20	104.7	121.9	129.8	131.2
2/20/20	104.7	120.8	131.2	131.0
2/19/20	104.7	119.9	131.7	130.6
2/18/20	104.6	119.9	131.1	130.3

Aggressive 40-20-40 Portfolio of ETFs

40 percent TQQQ (Triple Leveraged Nasdaq),
20 percent TMF (Triple Leveraged Bonds)
40 percent TLT (Single Leveraged Bonds)

with monthly rebalancing was proposed by us in 2017 paper as a leveraged ETF alternative to classical stocks/bonds portfolios.

Next slide shows that portfolio as it was proposed in the end of 2017 as an aggressive alternative to classic Stocks/Bonds.

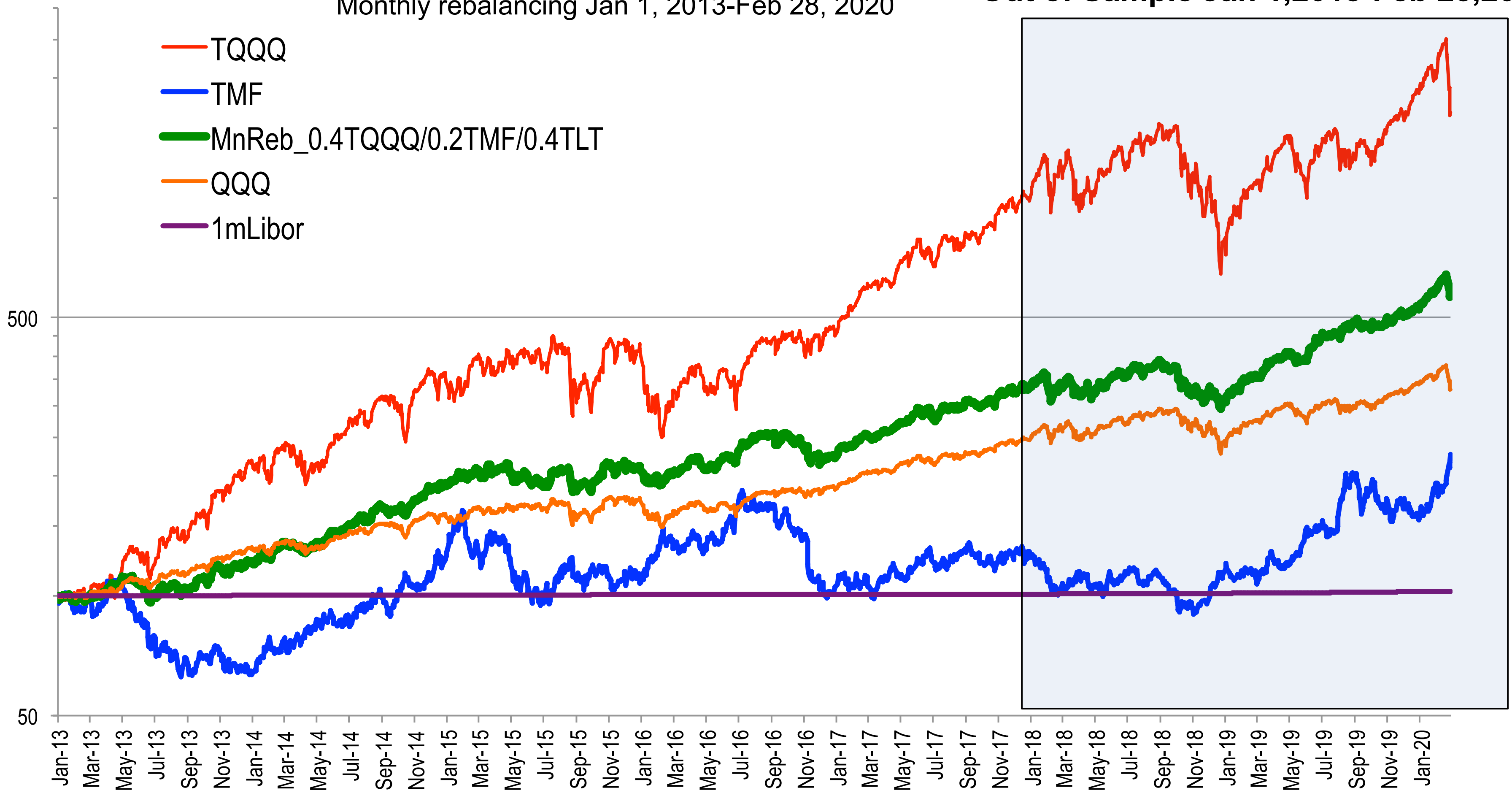


AnnRet Jan2013-Dec2017	**27.2%**
AnnVol Jan2013-Dec2017	**17.5%**
AnnRet/AnnVol	**1.55**
SharpeLibor1m	**1.52**

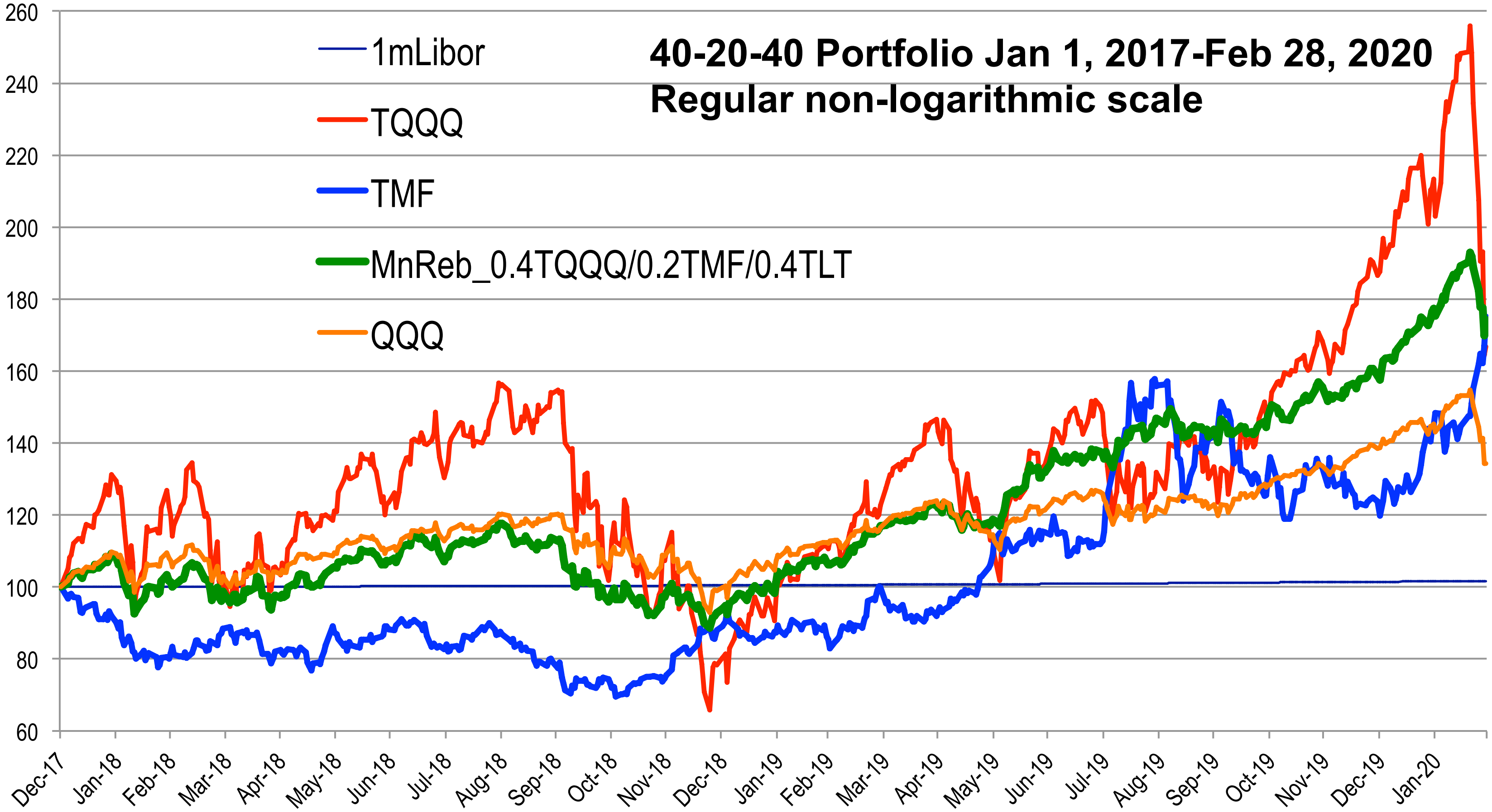
Thick Green Line Second From The Top 40-20-40 Portfolio   Logarithmic Scale

40%TQQQ (Triple Leveraged Nasdaq100)
20% TMF   (Triple Leveraged Bonds)
40% TLT   (Single Leveraged Bonds)
Monthly rebalancing Jan 1, 2013-Feb 28, 2020

Out of Sample Jan 1,2018-Feb 28,2020



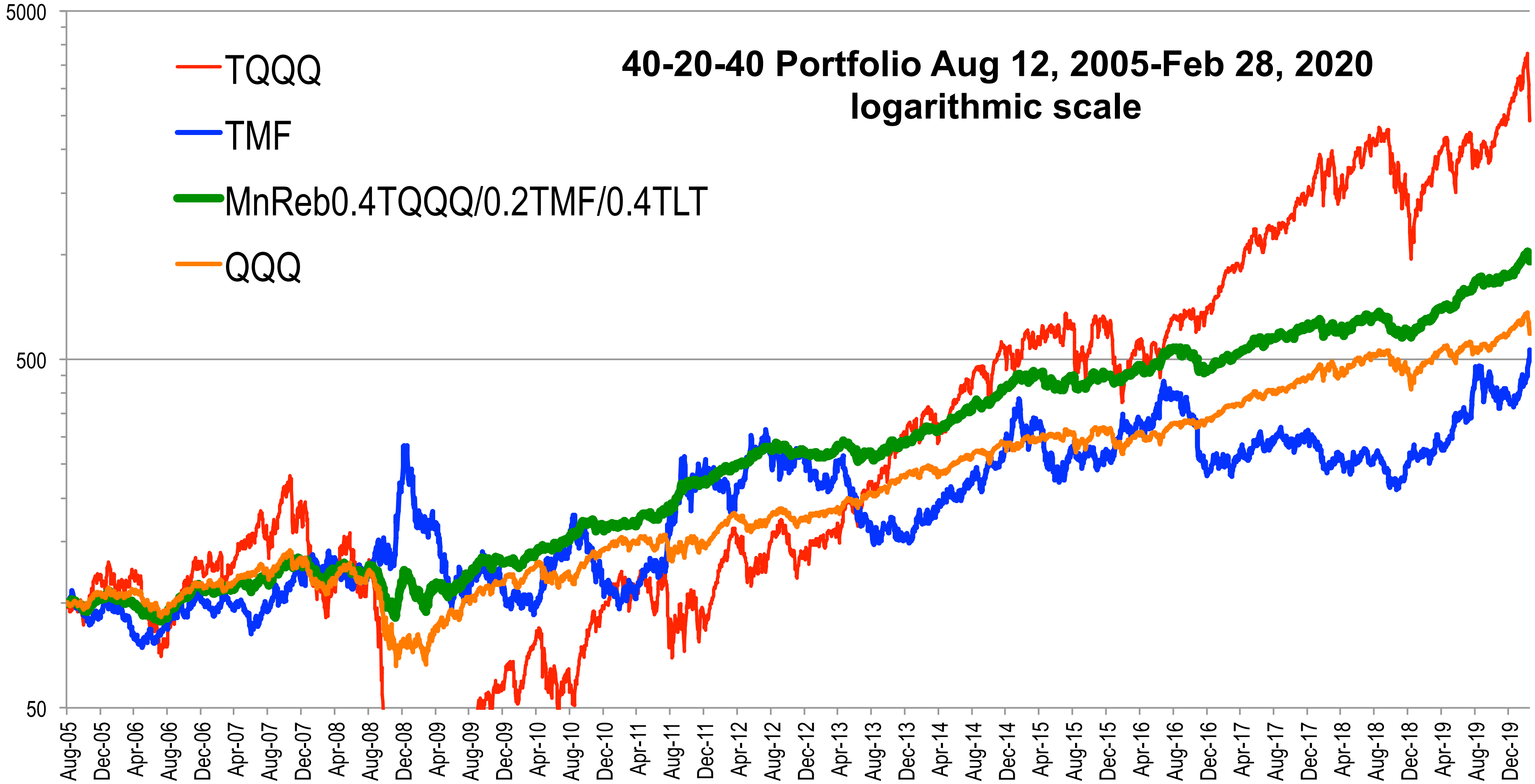
AnnRet Jan2013-Feb2020	**27.8%**
AnnVol Jan2013-Dec2020	**18.8%**
AnnRet/AnnVol	**1.48**
SharpeLibor1m	**1.45**



	1mLibor	TQQQ	TMF	TLT	QQQ	MnReb_0.4TQQQ/ 0.2TMF/0.4TLT
2/28/20	101.7	166.9	175.0	128.9	134.3	174.3
2/27/20	101.7	164.3	167.1	126.1	134.2	169.9
2/26/20	101.7	193.2	162.2	124.7	141.3	177.6
2/25/20	101.7	190.6	164.8	125.4	140.6	177.8
2/24/20	101.7	207.3	162.0	124.7	144.5	182.3
2/21/20	101.6	234.4	155.1	122.9	150.3	188.7
2/20/20	101.6	248.6	151.0	121.8	153.3	191.8
2/19/20	101.6	255.8	147.5	120.8	154.7	192.8
2/18/20	101.6	248.7	147.4	120.8	153.2	190.4

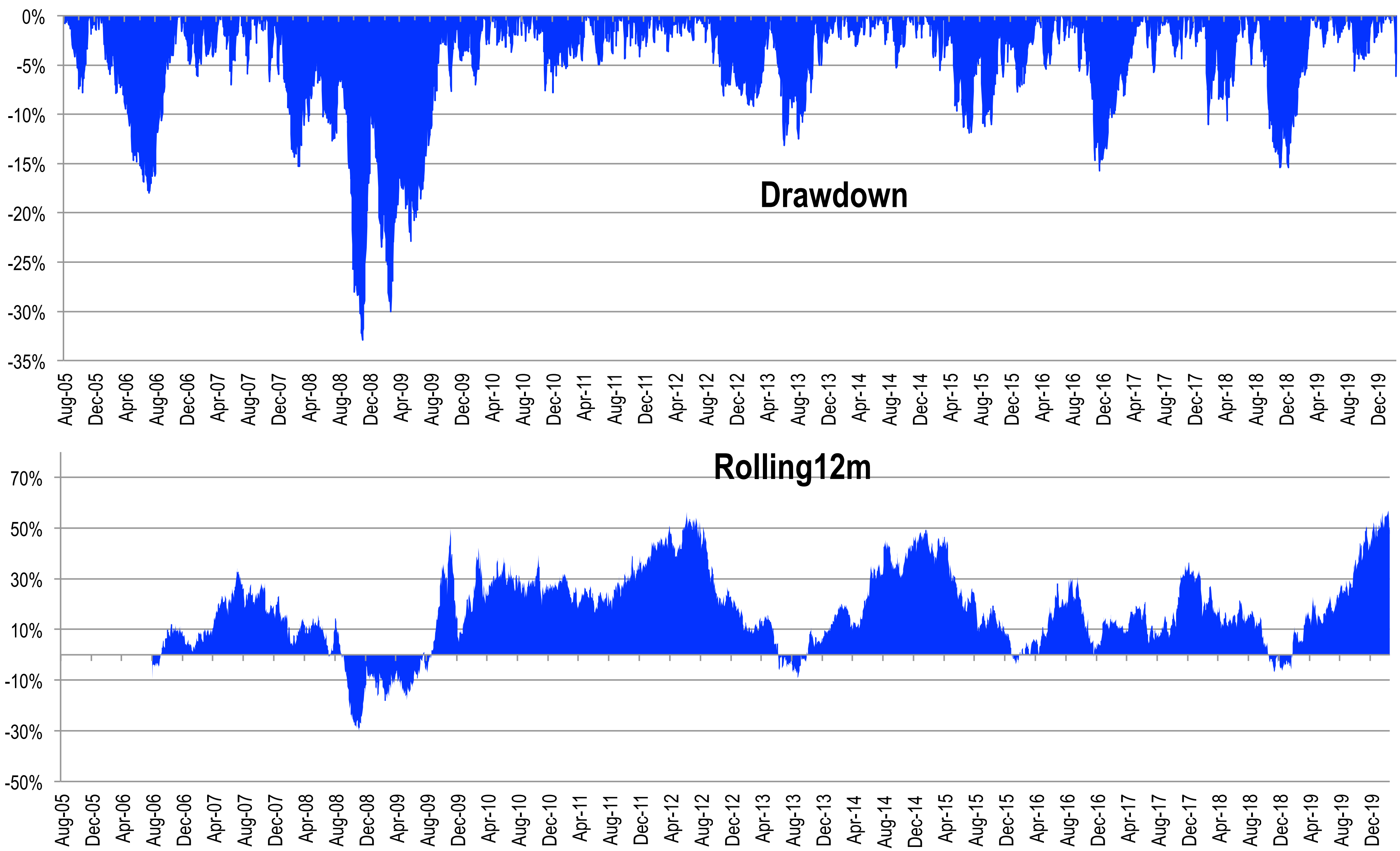
Thick Green Line Second From The Top Portfolio
40%TQQQ (Triple Leveraged Nasdaq100 ETF)
20% TMF (Triple Leveraged Bonds ETF)
40% TLT (Single Leveraged Bonds ETF)
Monthly rebalancing Jan 1, 2018-Feb 28, 2020

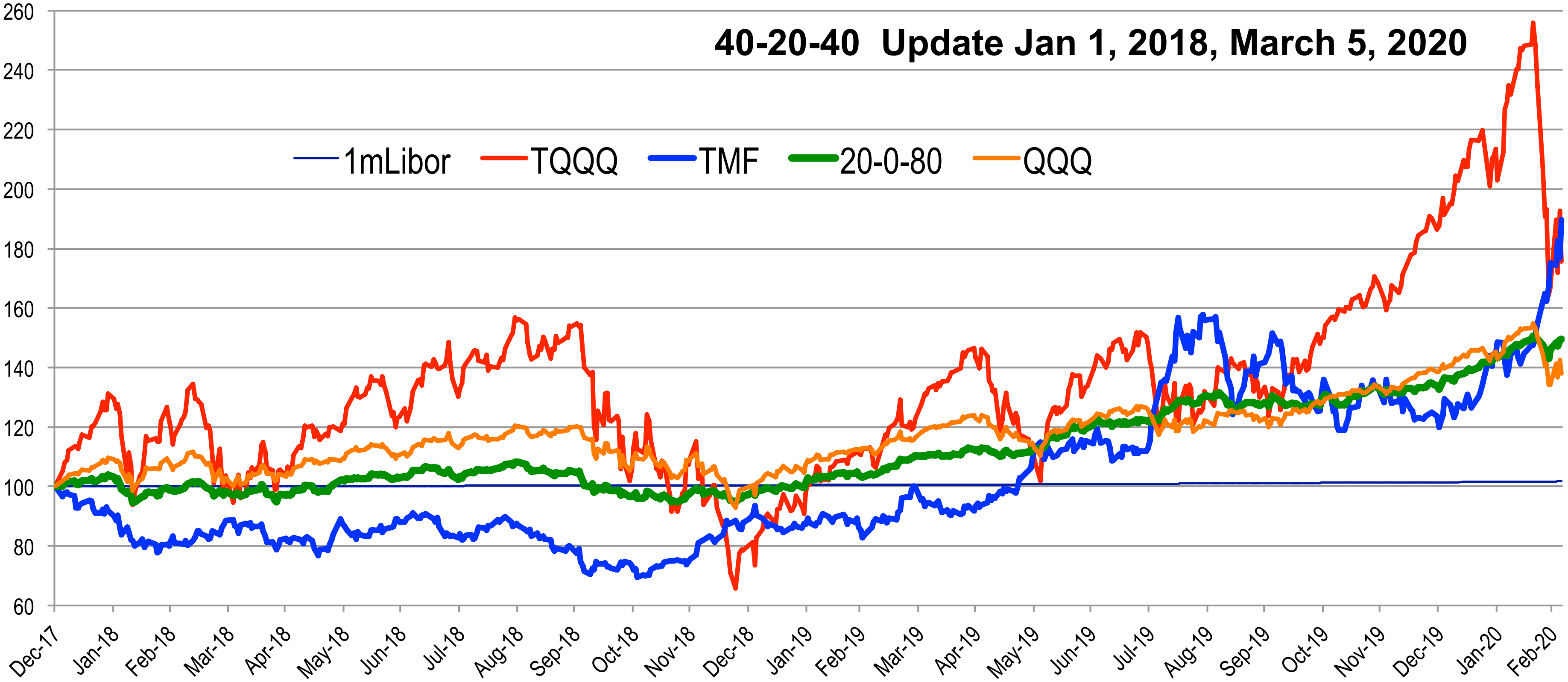
AnnRet Jan2018-Feb2020	**29.2%**
AnnVol Jan2013-Dec2020	**21.6%**
AnnRet/AnnVol	**1.35**
SharpeLibor1m	**1.31**



40-20-40 Portfolio Total Return		QQQ Total Return		SPY Total Return	
AnnRet Aug05-Feb20	22.3%	AnnRet Aug05-Feb20	13.0%	AnnRet Aug05-Feb20	8.4%
AnnVol Aug05-Feb20	21.5%	AnnVol Aug05-Feb20	20.2%	AnnVol Aug05-Feb20	18.5%
AnnRet/AnnVol	1.04	AnnRet/AnnVol	0.65	AnnRet/AnnVol	0.45
SharpeLibor1m	0.95	SharpeLibor1m	0.56	SharpeLibor1m	0.37

40-20-40 Portfolio Aug 12, 2005-Feb 28, 2020





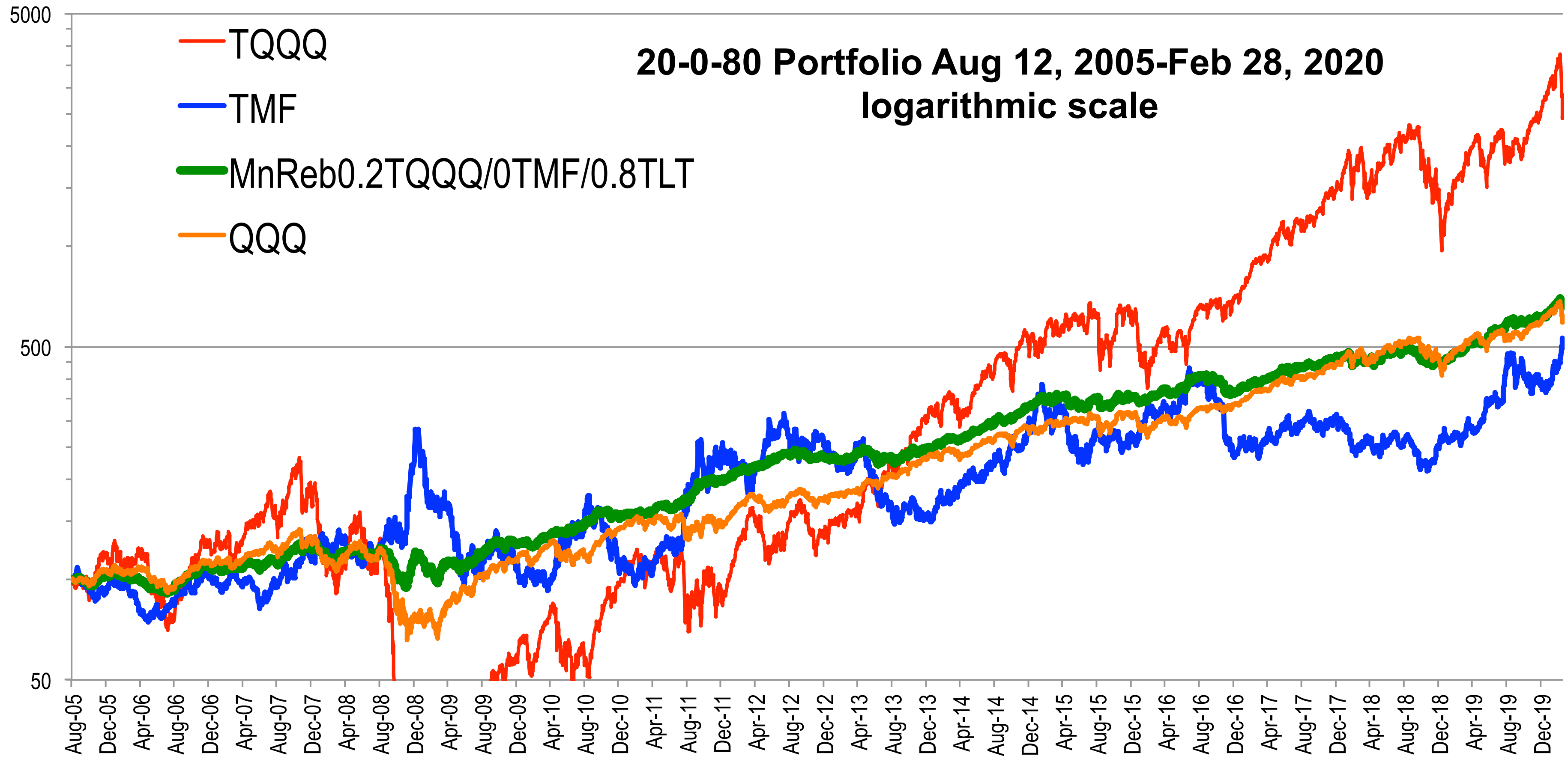
	1mLibor	TQQQ	TMF	TLT	QQQ	40-20-40	DrwdQQQ	Drwd40-20-40
3/5/20	101.8	175.7	189.7	131.8	138.1	181.3	-10.7%	-6.0%
3/4/20	101.8	192.6	176.9	128.6	142.4	183.3	-7.9%	-5.0%
3/3/20	101.8	171.8	182.8	130.0	136.7	177.3	-11.6%	-8.0%
3/2/20	101.7	189.6	174.4	128.0	141.3	181.3	-8.7%	-6.0%
2/28/20	101.7	166.9	175.0	128.9	134.3	174.3	-13.2%	-9.6%
2/27/20	101.7	164.3	167.1	126.1	134.2	169.9	-13.2%	-11.9%
2/26/20	101.7	193.2	162.2	124.7	141.3	177.6	-8.7%	-7.9%
2/25/20	101.7	190.6	164.8	125.4	140.6	177.8	-9.1%	-7.8%
2/24/20	101.7	207.3	162.0	124.7	144.5	182.3	-6.6%	-5.4%
2/21/20	101.6	234.4	155.1	122.9	150.3	188.7	-2.8%	-2.2%
2/20/20	101.6	248.6	151.0	121.8	153.3	191.8	-0.9%	-0.6%
2/19/20	101.6	255.8	147.5	120.8	154.7	192.8	0.0%	0.0%

More conservative 20-0-80 Portfolio of ETFs

20 percent TQQQ (Triple Leveraged Nasdaq),
0 percent TMF (Triple Leveraged Bonds)
80 percent TLT (Single Leveraged Bonds)

with monthly rebalancing is a leveraged ETF alternative to classical stocks/bonds portfolios. It is a version of a Constant Proportion Portfolio Insurance with coefficient of proportionality 3.

Next slides show portfolio delivering respectable returns of 14% in the last 15 years and a Sharpe ratio of 1.



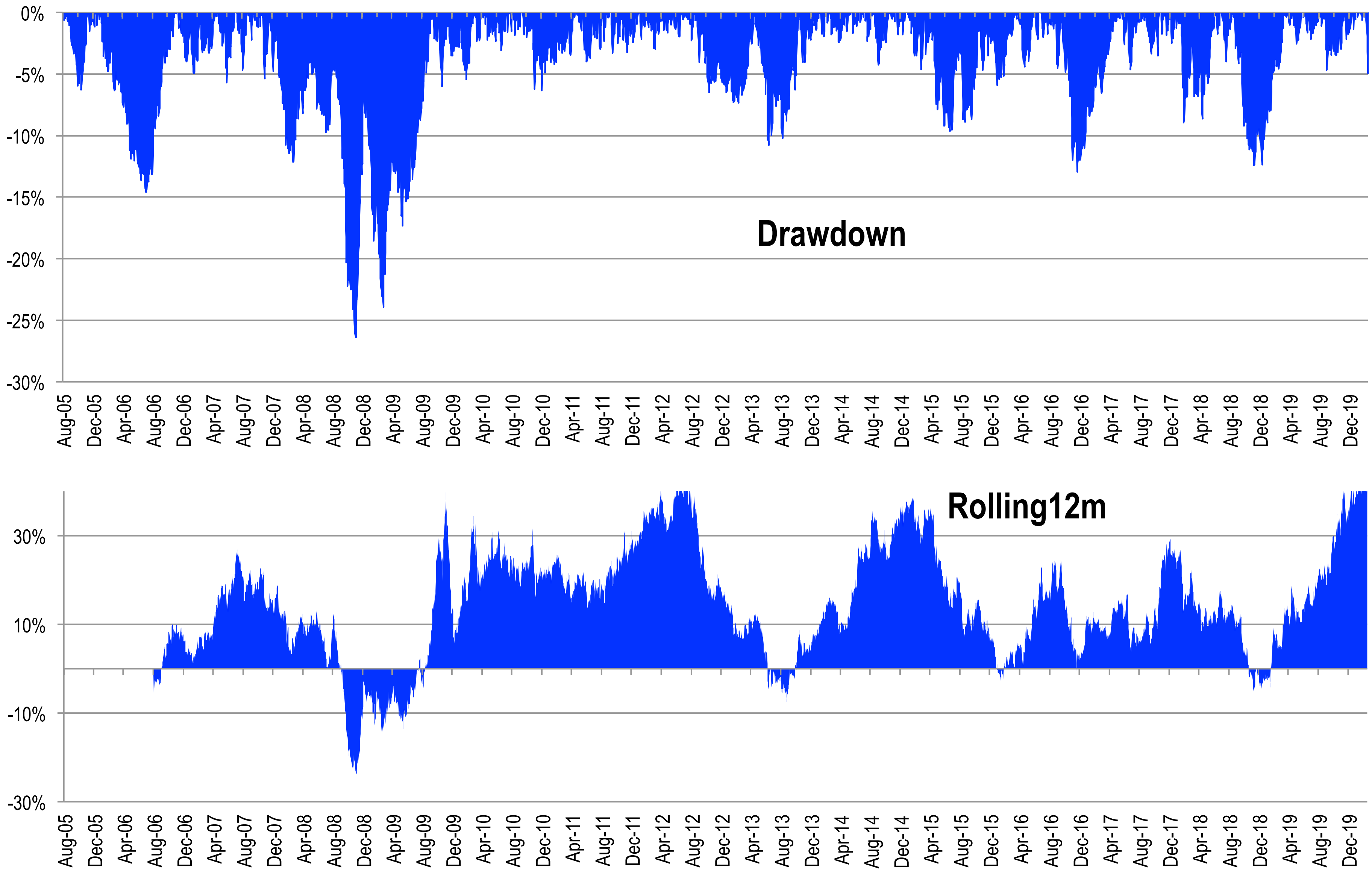
20-0-80 Portfolio Monthly Rebalanced Total Return

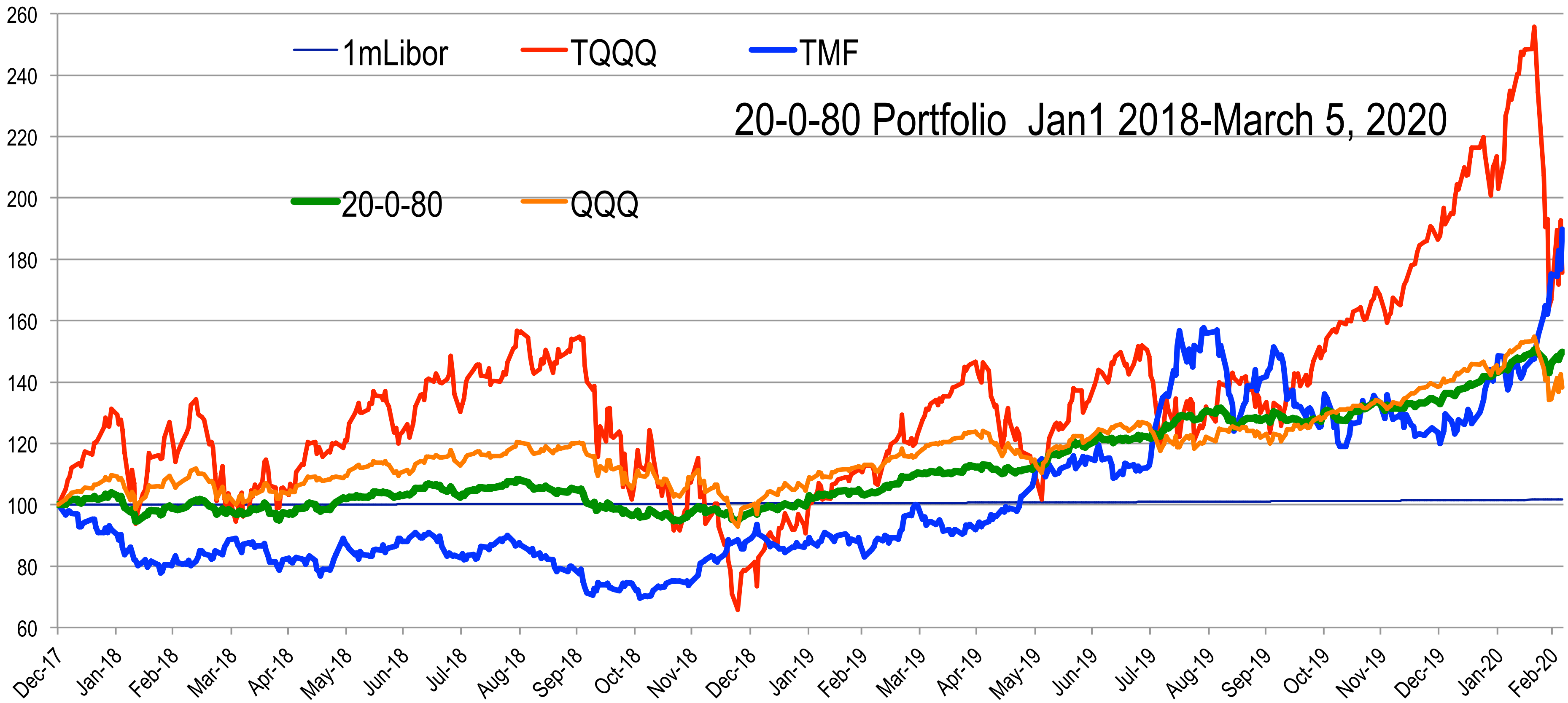
AnnRet Aug05-Feb20	14.0%	AnnRet Jan09-Feb20	16.9%
AnnVol Aug05-Feb20	12.4%	AnnVol Jan09-Feb20	12.1%
AnnRet/AnnVol	1.13	AnnRet/AnnVol	1.40
SharpeLibor1m	1.00	SharpeLibor1m	1.29

QQQ Total Return

AnnRet Aug05-Feb20	13.0%
AnnVol Aug05-Feb20	20.2%
AnnRet/AnnVol	0.65
SharpeLibor1m	0.56

20-0-80 Portfolio Aug 12, 2005-Feb 28, 2020





	1mLibor	TQQQ	TMF	TLT	QQQ	20-0-80	DrwdQQQ	Drwd20-0-80
3/5/20	101.8	175.7	189.7	131.8	138.1	149.7	-10.7%	-0.5%
3/4/20	101.8	192.6	176.9	128.6	142.4	149.3	-7.9%	-0.7%
3/3/20	101.8	171.8	182.8	130.0	136.7	147.3	-11.6%	-2.1%
3/2/20	101.7	189.6	174.4	128.0	141.3	148.3	-8.7%	-1.4%
2/28/20	101.7	166.9	175.0	128.9	134.3	146.1	-13.2%	-2.9%
2/27/20	101.7	164.3	167.1	126.1	134.2	143.0	-13.2%	-4.9%
2/26/20	101.7	193.2	162.2	124.7	141.3	145.7	-8.7%	-3.2%
2/25/20	101.7	190.6	164.8	125.4	140.6	146.0	-9.1%	-3.0%
2/24/20	101.7	207.3	162.0	124.7	144.5	147.6	-6.6%	-1.9%
2/21/20	101.6	234.4	155.1	122.9	150.3	149.5	-2.8%	-0.6%
2/20/20	101.6	248.6	151.0	121.8	153.3	150.4	-0.9%	-0.1%
2/19/20	101.6	255.8	147.5	120.8	154.7	150.5	0.0%	0.0%

ETF or Exchange-traded fund is an investment fund that is traded on stock exchange intraday like stock

ETF holds assets (stocks, bonds, commodities) and is traded intraday close to its net asset value

Attractiveness: low costs, tax efficiency, and similarity to stock in trading, margining and clearing.

The ETF holder owns a piece of a basket of underlying securities.

If the ETF provider go bankrupt then the holder will get cash for the market value of his piece of basket. Holders of large number (like 50,000) of ETF shares can take distribution of the securities from the basket.

Authorized participants are broker-dealers with special agreements with the ETF.

They can buy or sell ETF shares in creation units - large blocks that can be exchanged with the ETF provider in-kind for the underlying securities baskets.

Authorized participants can act as market makers on exchange.

The possibility to exchange creation units for the underlying basket makes intraday ETF market price close to the net asset value of the underlying basket.

ETF Arbitrage Example, Creation and Redemption: Authorized Participant buy the underlying shares of index of ETF at lower prices, short ETF, create ETF creation unit with shares of index bought and close short.

SPDR S&P 500 ETF, NYSE Arca: SPY

Expense Ratio (net) 0.09%

Net Assets 308 Billion

Avg. Volume 78,000,000

Vanguard S&P 500 ETF, NYSE Arca: VOO

Expense Ratio (net) 0.03%

Net Assets 538 Billion

Avg. Volume 3,600,000

iShares S&P 500 Index, NYSE Arca: IVV

Expense Ratio (net) 0.04%

Net Assets 203 Billion

Avg. Volume 5,000,000

Invesco QQQ Trust, Nasdaq GM: QQQ

Nasdaq100

Expense Ratio (net) 0.20%

Net Assets 89 Billion

Avg. Volume 32,000,000

As of February 2020

iShares 20+ Year Treasury Bond ETF, Nasdaq GM: TLT

Expense Ratio (net) 0.15%

Net Assets 19 Billion

Avg. Volume 10,000,000

Direxion Daily 20+ Year Treasury Bull 3X Shares, NYSE Arca: TMF

Expense Ratio (net) 1.05%

Net Assets 256 Million

Avg. Volume 1,300,000

ProShares UltraPro QQQ Nasdaq GM: TQQQ

Expense Ratio (net) 0.95%

Net Assets Billion 4.7 Billion

Avg. Volume 19,000,000

ProShares Ultra S&P500 NYSE Arca: SSO

Expense Ratio (net) 0.90%

Net Assets Billion 2.9 Billion

Avg. Volume 1,900,000

As of February 2020

ETN Exchange-traded note:

Senior, unsecured, unsubordinated debt security issued by a bank. Have a maturity date, is backed by the credit of the bank.

Returns linked to performance of an index or strategy minus fees.

For ETN at maturity investor will receive cash payment based on performance of index from trade date to maturity minus fees.

Typically no principal protection.

ETNs could also be sold before maturity on exchange. ETN holder can redeem large block of ETN securities directly to the issuing bank usually weekly.

ETN Advantages

Tax efficiency (may be disallowed by IRS in future)

Treated as a prepaid contract (forward contract) for tax purposes. Buyer pays initial amount to receive a future payment based on benchmark at a future time.

Mutual funds and ETFs must make annual income and capital gains distributions. When a fund is rebalanced investors pay capital gains tax.

ETNs have no interest payment or dividends, no annual tax.

Capital gain/loss is realized when an ETN is sold or matures.

Long-term capital gains are treated more favorably than short-term capital gains and interest in the US (> 1 year holdings are taxed at a capital gains rate of 20%).

Impossible to avoid capital gains tax, but advantage delaying it.

ETNs are **unsecured debt instruments**. Do not own underlying assets but a promise an index based return.

If ETN provider go bankrupt, the investor may not receive investment back.

Lehman Brothers that went bankrupt in 2008 had three failed ETNs: Opta Lehman Commodity, Agriculture and Private Equity.

In September 2008 when Lehman went bankrupt these ETNs halted trading.

ETN holders were eventually getting single cents on each dollar invested.

XIV exchange-traded note disaster.

XIV was an exchange-traded note that tracked the VIX Volatility Index and promised to pay the daily return inverse of the VIX.

ETNs are unsecured debt instruments issued by a bank that promises to pay the performance of an index. If VIX went up, XIV would go down that same percent amount and if VIX went down, XIV would go up that percent amount. To manage that note XIV manager Credit Suisse was shorting VIX futures.

Over the five years prior to February 2018 XIV returned 2,200%
On February 5, 2018 VIX index went up 100%. XIV price was 72.59 at 4p.m. Eastern and fell to 4.22 an hour later.

Because the note had lost more than 90% of its value, Credit Suisse enacted a provision the next day to close the ETN to prevent further losses. The provision was that if ETN lost 70% of its value, ETN liquidates.

Leveraged ETFs. Most give 2x or 3x of daily performance of the index.

Largest providers: Direxion, ProShares

Effectively are path dependent options. Have Gamma.

Lose money in oscillating markets.

Make more money than a 2x or 3x constant leverage in trending markets.

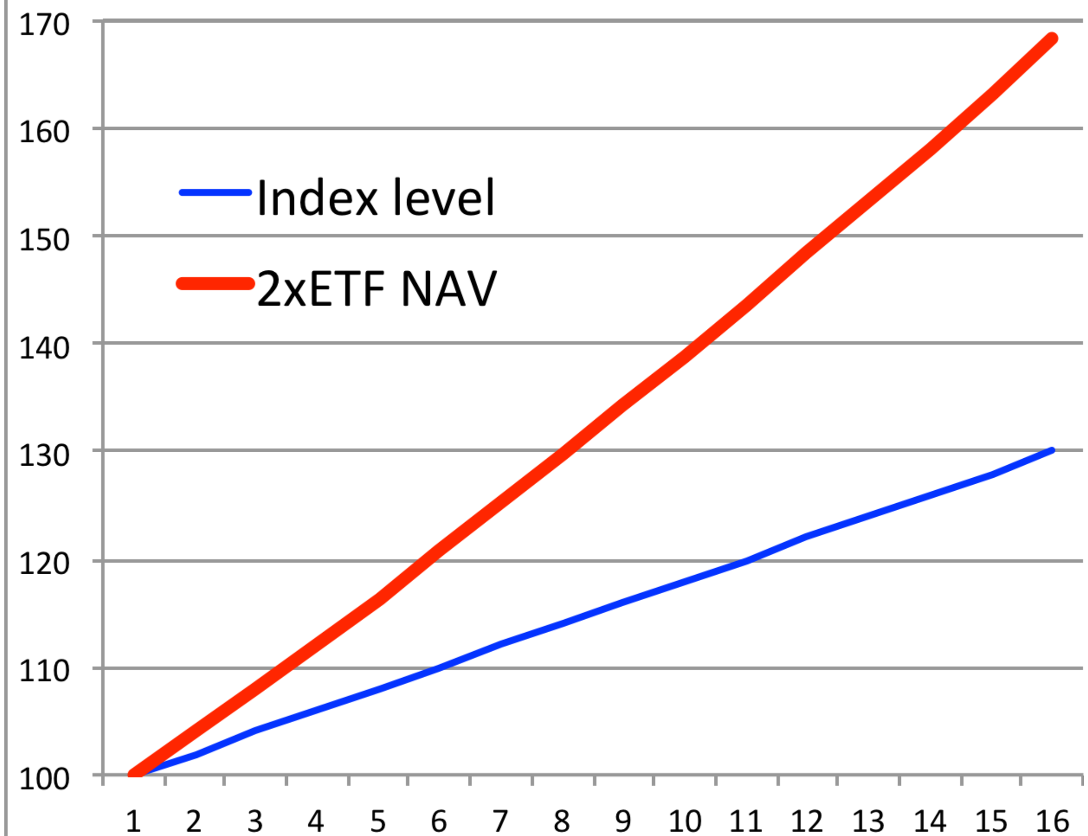
ETF manager must adjust leverage exposure daily to keep a 2x or 3x daily performance.

Some of Pro Shares leveraged and inverse ETFs

	Index/Benchmark	UltraPro Sh -3x	UltraShort -2x	Short -1x	Ultra 2x	UltraPro 3x
MarketCap	S&P 500	SPXU	SDS	SH	SSO	UPRO
	NASDAQ-100	SQQQ	QID	PSQ	QLD	TQQQ
	Dow Jones	SDOW	DXD	DOG	DDM	UDOW
	S&P MidCap 400	SMDD	MZZ	MYY	MVV	UMDD
	S&P SmallCap 600	--	SDD	SBB	SAA	--
	Russell 2000	SRTY	TWM	RWM	UWM	URTY
International	MSCI EAFE	--	EFU	EFZ	EFO	--
	MSCI Emerging Markets	--	EEV	EUM	EET	--
	FTSE Developed Europe All Cap	--	EPV	--	UPV	--
	MSCI Brazil 25/50 Capped	--	BZQ	--	UBR	--
	FTSE China 50	--	FXP	YXI	XPP	--
	MSCI Japan	--	EWV	--	EZJ	--
Fixed Income	ICE U.S. Treasury 20+ Year Bond	TTT	TBT	TBF	UBT	--
	ICE U.S. Treasury 7-10 Year Bond	--	PST	TBX	UST	--
	Markit iBoxx $ Liquid High Yield	--	--	SJB	UJB	--

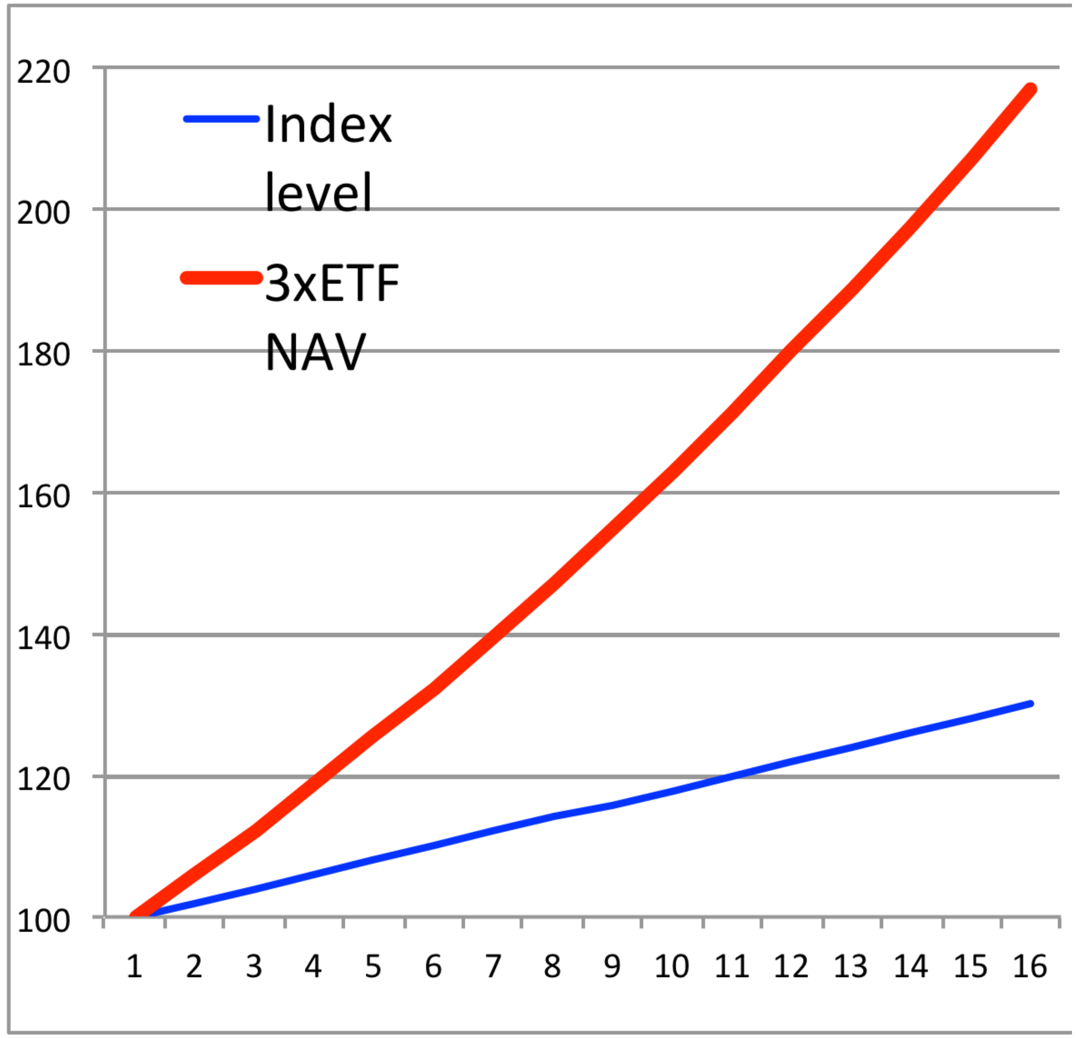
2x Leveraged Fund, Index Goes Straight Up. Fund has Gamma

Day	Index level	2xETF NAV	2xETF Expsr	Mrkt% Change	2xETF %Chnge
0	100	100.00	200.00		
1	102	104.00	208.00	2.00%	4.00%
2	104	108.08	216.16	1.96%	3.92%
3	106	112.24	224.47	1.92%	3.85%
4	108	116.47	232.94	1.89%	3.77%
5	110	120.78	241.57	1.85%	3.70%
6	112	125.18	250.35	1.82%	3.64%
7	114	129.65	259.29	1.79%	3.57%
8	116	134.20	268.39	1.75%	3.51%
9	118	138.82	277.65	1.72%	3.45%
10	120	143.53	287.06	1.69%	3.39%
11	122	148.31	296.63	1.67%	3.33%
12	124	153.18	306.35	1.64%	3.28%
13	126	158.12	316.24	1.61%	3.23%
14	128	163.14	326.27	1.59%	3.17%
15	130	168.24	336.47	1.56%	3.13%



3x Leveraged Fund, Index Goes Straight Up. Fund has Gamma

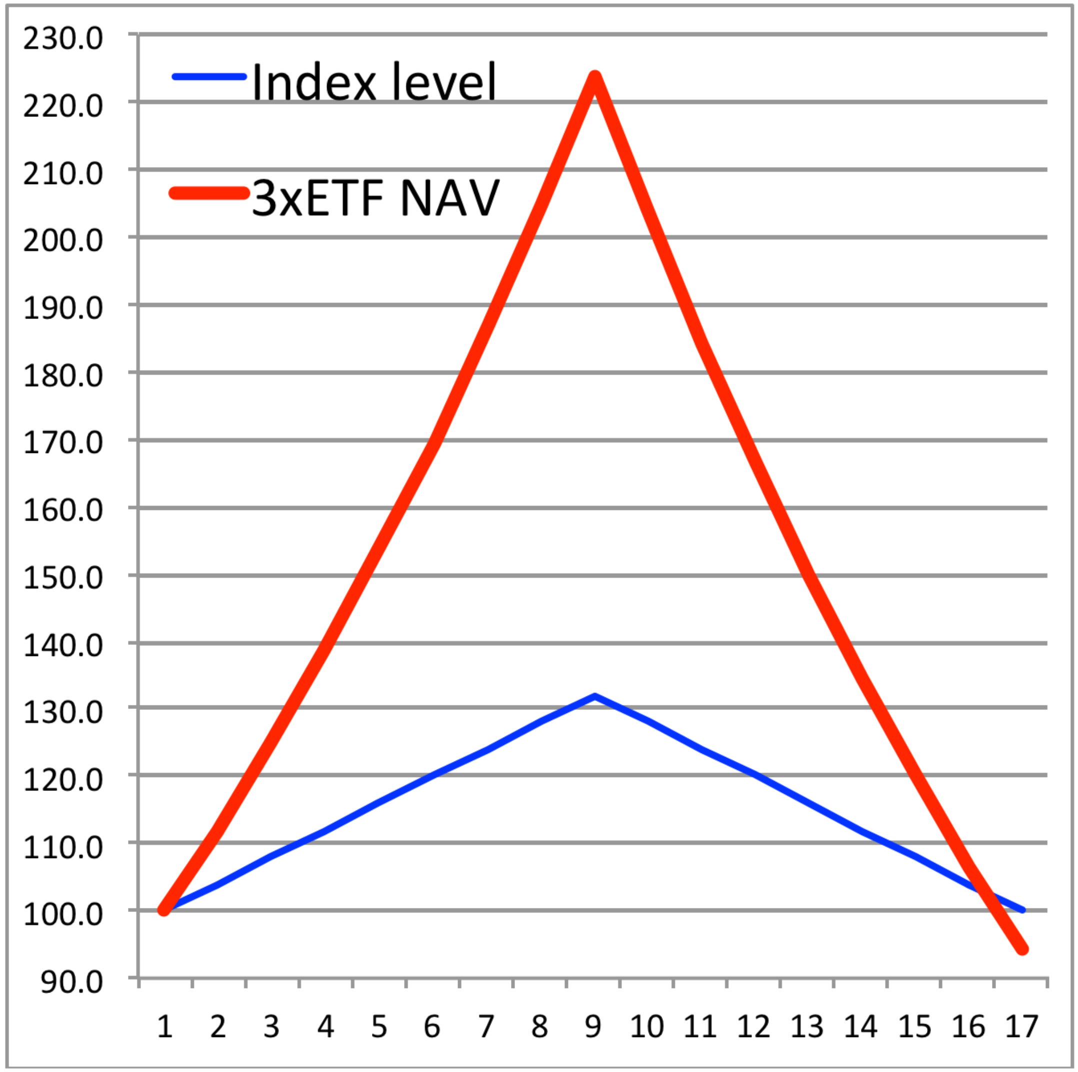
Day	Index level	3xETF NAV	3xETF Expsr	Mrkt% Change	3xETF %Chnge
0	100	100.00	300.00		
1	102	106.00	318.00	2.00%	6.00%
2	104	112.24	336.71	1.96%	5.88%
3	106	118.71	356.13	1.92%	5.77%
4	108	125.43	376.29	1.89%	5.66%
5	110	132.40	397.19	1.85%	5.56%
6	112	139.62	418.86	1.82%	5.45%
7	114	147.10	441.30	1.79%	5.36%
8	116	154.84	464.52	1.75%	5.26%
9	118	162.85	488.55	1.72%	5.17%
10	120	171.13	513.39	1.69%	5.08%
11	122	179.69	539.06	1.67%	5.00%
12	124	188.52	565.57	1.64%	4.92%
13	126	197.65	592.94	1.61%	4.84%
14	128	207.06	621.18	1.59%	4.76%
15	130	216.76	650.29	1.56%	4.69%



3x Leveraged Fund. Index goes straight up then straight down staying unchanged at the end.

3x Leveraged Fund losing 5.67 percent at the end.

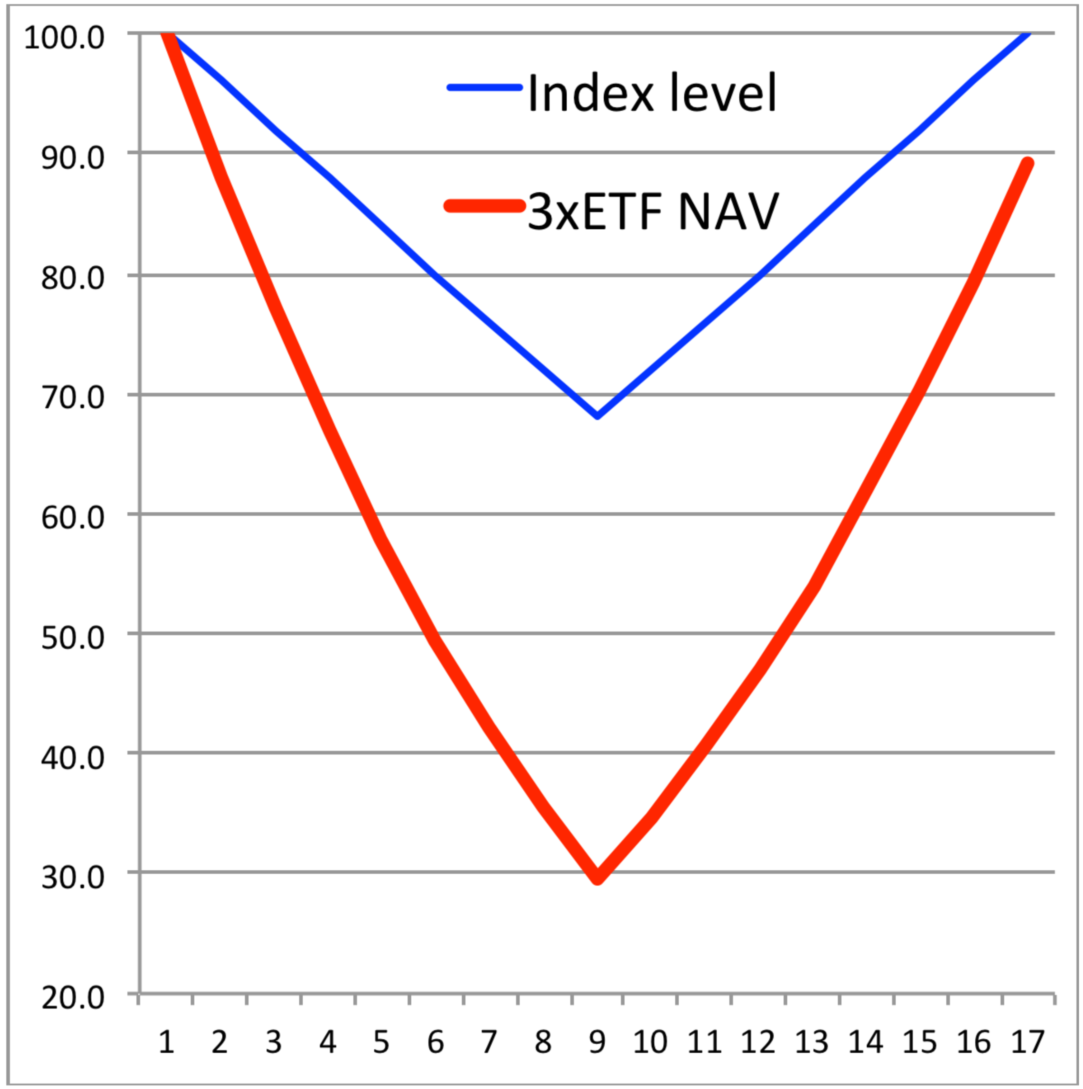
Day	Index level	3xETF NAV	3xETF Expsr	Mrkt% Change	3xETF %Chnge
0	100.0	100.00	300.00		
1	104.0	112.00	336.00	4.00%	12.00%
2	108.0	124.92	374.77	3.85%	11.54%
3	112.0	138.80	416.41	3.70%	11.11%
4	116.0	153.68	461.03	3.57%	10.71%
5	120.0	169.57	508.72	3.45%	10.34%
6	124.0	186.53	559.59	3.33%	10.00%
7	128.0	204.58	613.74	3.23%	9.68%
8	132.0	223.76	671.28	3.13%	9.38%
9	128.0	203.42	610.26	-3.03%	-9.09%
10	124.0	184.35	553.04	-3.13%	-9.38%
11	120.0	166.51	499.52	-3.23%	-9.68%
12	116.0	149.86	449.57	-3.33%	-10.00%
13	112.0	134.35	403.06	-3.45%	-10.34%
14	108.0	119.96	359.88	-3.57%	-10.71%
15	104.0	106.63	319.89	-3.70%	-11.11%
16	100.0	94.33	282.98	-3.85%	-11.54%



3x Leveraged Fund. Index goes straight down then straight up staying unchanged at the end.

3x Leveraged Fund losing 10.76 percent at the end.

Day	Index level	3xETF NAV	3xETF Expsr	Mrkt% Change	3xETF %Chnge
0	100.0	100.00	300.00		
1	96.0	88.00	264.00	-4.00%	-12.00%
2	92.0	77.00	231.00	-4.17%	-12.50%
3	88.0	66.96	200.87	-4.35%	-13.04%
4	84.0	57.83	173.48	-4.55%	-13.64%
5	80.0	49.57	148.70	-4.76%	-14.29%
6	76.0	42.13	126.39	-5.00%	-15.00%
7	72.0	35.48	106.43	-5.26%	-15.79%
8	68.0	29.57	88.70	-5.56%	-16.67%
9	72.0	34.78	104.35	5.88%	17.65%
10	76.0	40.58	121.74	5.56%	16.67%
11	80.0	46.99	140.96	5.26%	15.79%
12	84.0	54.04	162.11	5.00%	15.00%
13	88.0	61.75	185.26	4.76%	14.29%
14	92.0	70.18	210.53	4.55%	13.64%
15	96.0	79.33	237.99	4.35%	13.04%
16	100.0	89.24	267.73	4.17%	12.50%



Consider a portfolio consisting of a combination of 1 risky asset and cash. Risky asset $X(t)$ follows a standard geometric Brownian motion process

$$dX(t)=\mu X(t)dt+\sigma X(t)dW(t) \qquad (1)$$

with constant growth rate μ and constant volatility σ.

We dynamically change asset allocation in portfolio between cash and risky asset.

We assume cash having interest rate 0. (We may always change numeraire to units of bank account and then cash would have interest rate 0 in this numeraire.)

Let us denote by $E(t)$ the value of a portfolio that at time t (E stays for Equity of portfolio). At time t portfolio $E(t)$ has exposure to $f(t, X(t), E(t))$ units of risky asset X where $f(t, X, E)$ is a function of three variables.

In fact $\boldsymbol{f(t, X, E)}$ is a "Portfolio Manager" this function tells how much exposure portfolio must have to a risky asset, at a time t when the risky asset price is X and portfolio level is E.

The *f*-managed portfolio *E(t)* follows the process

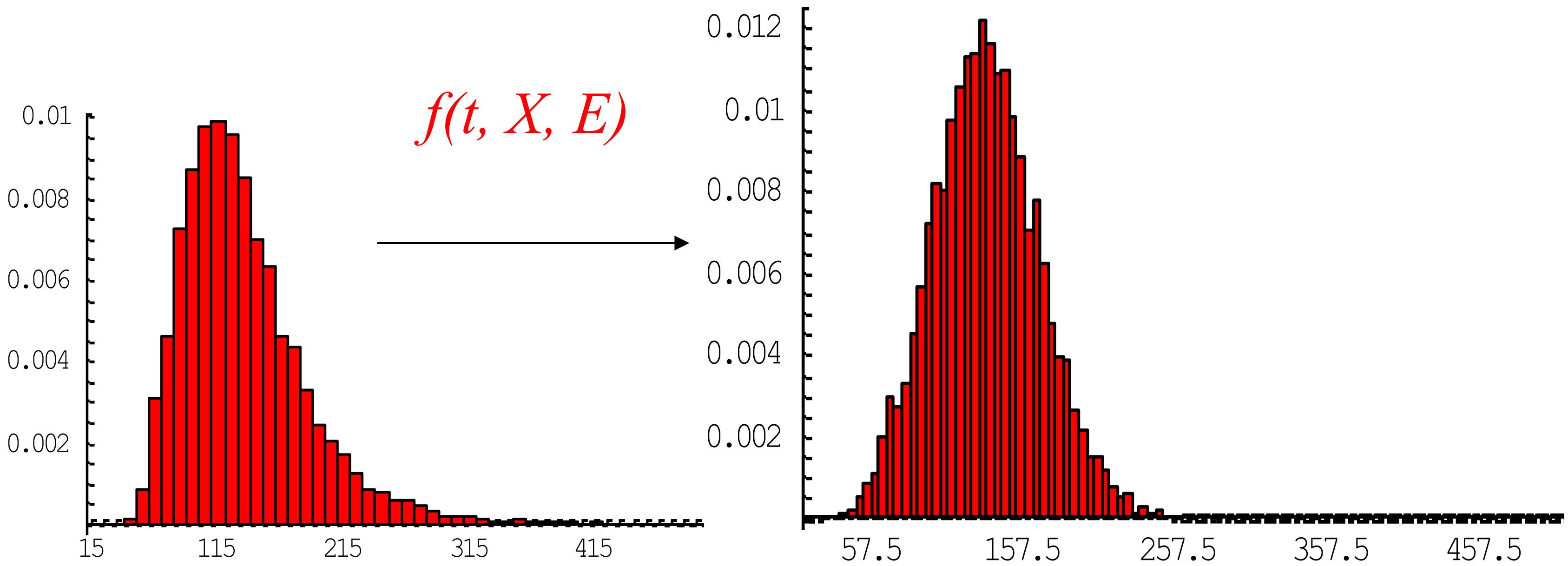
$$dE(t)=\mu f(t, X(t), E(t))\, dt +\sigma f(t, X(t\,),E(t))dW(t)$$

or in short notation

$$dE= \mu\ f\ dt+ \sigma\ f\ dW(t) \qquad \textbf{(2)}$$

The original index *X* has a terminal distribution at time *T* and also has a distribution of "risk" prior to time *T* . The resulting portfolio *E* also has a terminal distribution at T and prior "risk" distribution.

Dynamic risk allocation strategy *f(t, X, E)* transforms the probability distribution of *X(T)* into probability distribution of portfolio equity *E(T)* by

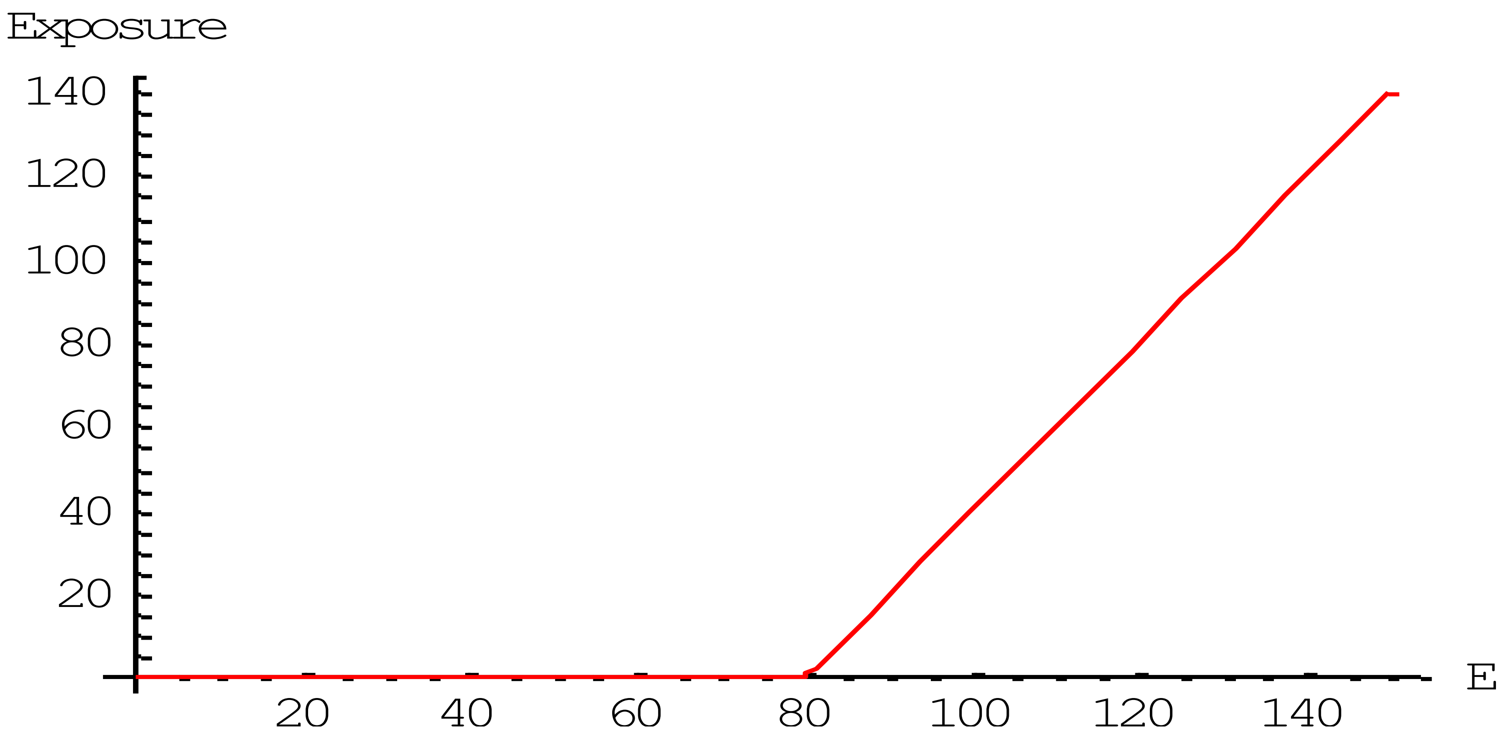


Distribution of *X(T)* Distribution of *E(T)*

Example: Constant Proportion Portfolio Insurance (Black-Jones-Perold). Let us fix a protective floor *F* and a coefficient *C* of proportionality of exposure:

$$f(t, X, E) = \begin{cases} C\,(E-F) & \text{if } E>F \\ 0 & \text{otherwise} \end{cases}$$

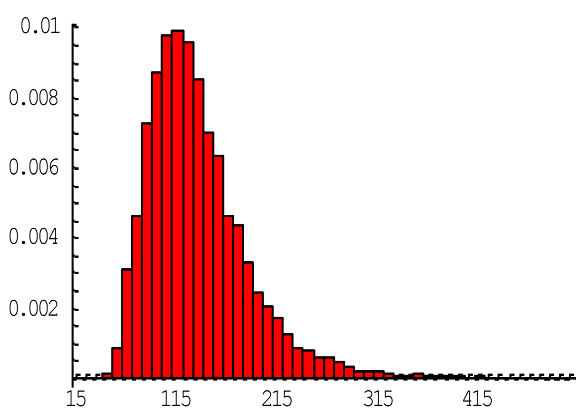
Constant Proportion Portfolio Insurance *Floor=80, C=2*



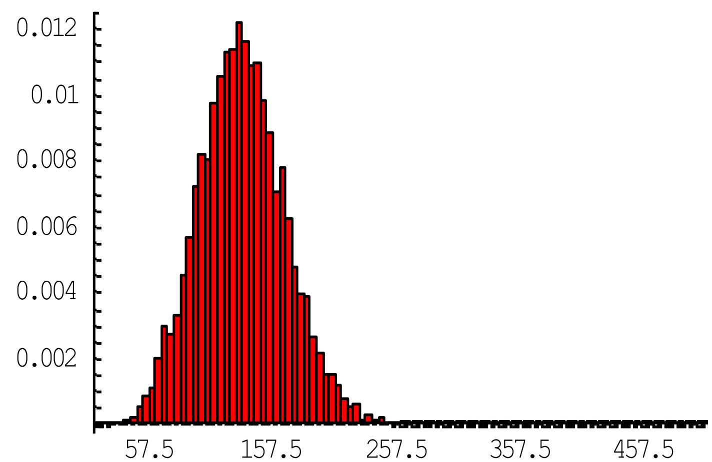
Direct problem of portfolio insurance:

1) We are given a process for X(t)
2) We are given a dynamic allocation strategy *f(t, X, E)*

Find distribution of E(T) at time T
(and also find some associated risk measures of E(t) $t<T$
For example distribution of max drawdowns of E for $t<T$)



f(t, X, E)
is given



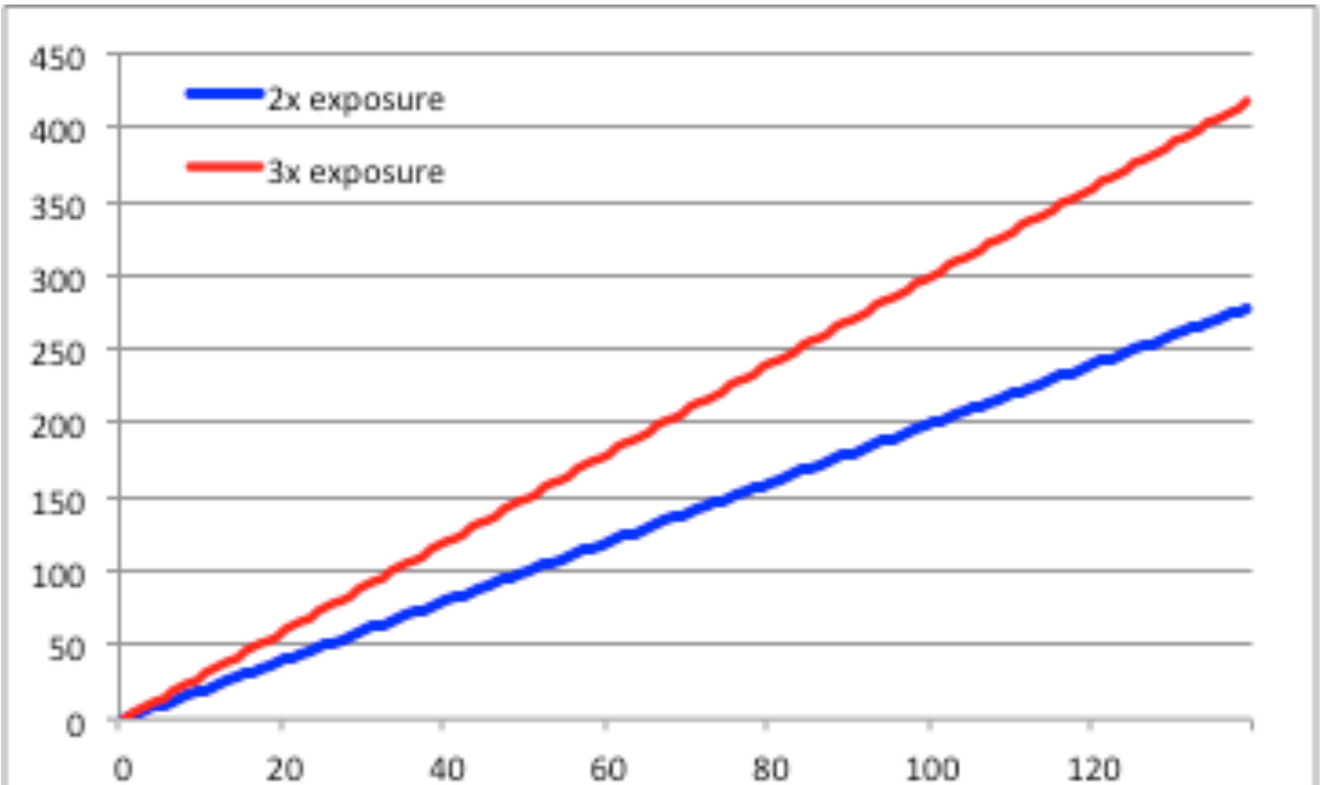
Process for of *X(t)* is given

Find Distribution of *E(T)*

Example: 2 times leveraged ETF:

Constant Proportion Portfolio Insurance with a protective floor $F=0$ and a coefficient $C=2$ of proportionality of exposure:

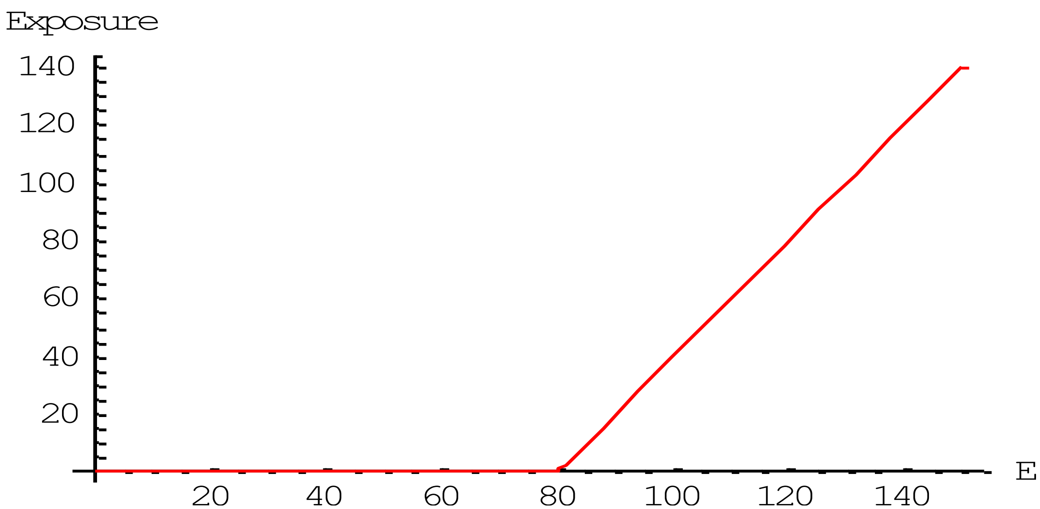
$$f(t, X, E)=2\ (E-0)=2\ E$$



Constant Proportion Portfolio Insurance *Floor=0, C=2*

Implementation of Constant Proportion Portfolio Insurance with a protective floor $F=80$ and a coefficient of proportionality of exposure $C=2$:

$$f(t, X, E) = \begin{cases} 2\,(E-80) & \text{if } E>80 \\ 0 & \text{otherwise} \end{cases}$$



Constant Proportion Portfolio Insurance *Floor=80, C=2*

Implementation with double leveraged ETF

Start with 100$. Put 20$ in 2x leveraged ETF tracking 2 times return of index X and 80$ in cash account

In general case there are

for stocks:

ETFs with 1x leverage exposure to stock index, have no Gamma, for example SPY for SP500 index or QQQ for Nasdaq 100 index.
ETFs with 2x leverage exposure to stock index, have small Gamma and small decay in volatile oscillating market, for example SSO for SP500 index QLD for Nasdaq 100.
ETFs with 3x leverage exposure to stock index, have larger Gamma and larger decay in volatile oscillating market, for example UPRO for SP500 index, TQQQ for Nasdaq 100.

For bonds

ETFs with 1x leverage exposure to bond index, have no Gamma, for example TLT for ICE U.S. Treasury 20+ Year Bond Index.
ETFs with 2x leverage exposure to bond index, have small Gamma and small decay in volatile oscillating market, for example UBT.
ETFs with 3x leverage exposure to bond index, have larger Gamma and larger decay in volatile oscillating market, for example TMF.

It is possible to use these ETFs as building blocks to create dynamic portfolios Where dynamics would come from changing exposures of ETFs itself plus periodic (annual or quarterly) rebalancing to designated mix.

We used leveraged ETFs as instruments to create portfolios with useful properties in particular Constant Proportion Portfolio Insurance portfolios.

In the next section we introduce a new measure of risk generalizing VAR for a case of investment having a critical liquidation level.

This risk measure is called Dynamic Leverage and formally defined in the next section.

Portfolio insurance strategy can be modified using that measure.
We call it Constant Dynamic Leverage Portfolio Insurance.
We describe that strategy below.

Dynamic Leverage - Definition

We propose as a new measure of risk or “leverage”.

It is the probability that a fund will encounter a critical liquidation point within a given time horizon. We call this risk measure a Dynamic Leverage to differentiate it from the more conventional static measures in common use.

This critical liquidation point level i.e. liquidation barrier level is an external input of the model.

Dynamic Leverage: Formula

Let x_0 be the fund NAV at time t_0 where t_0 is the time when the leverage is calculated, let t_1 be the end of the period, μ the fund's annual growth rate, σ its annual volatility and b critical liquidation barrier. **Dynamic Leverage** in a 1 – barrier model framework is defined as probability of fund to touch the barrier any time before t_1 (also called Absorption probability). It is given by the following formula:

$$Lev(x_0, b, \mu, \sigma, t_0, t_1) = N(A) + C \bullet N(B)$$

Where

$$A = \frac{\ln b - \ln x_0 - \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} \quad , \quad B = \frac{\ln b - \ln x_0 + \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} \quad , \quad C = \exp\left(\frac{2\left((\ln b - \ln x_0)\left(\mu - \frac{\sigma^2}{2}\right)\right)}{\sigma^2}\right)$$

And $N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$ the cumulative normal distribution function

Dynamic Leverage: Sensitivity to Volatility and Distance to Barrier

Dynamic Leverage is dependent on time elapsed into the period, NAV distance from a critical barrier, the return, and the volatility of the current portfolio.

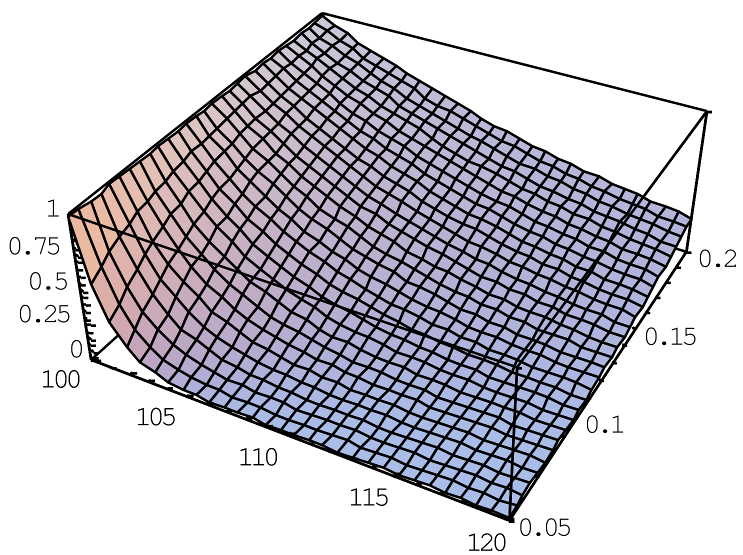


Fig. 1 Plot of Dynamic Leverage on vertical axis vs. current fund NAV (horizontal axis with range 100 to 120) and Volatility (horizontal axis with range 0.05 to 0.2). Time frame 1 year.

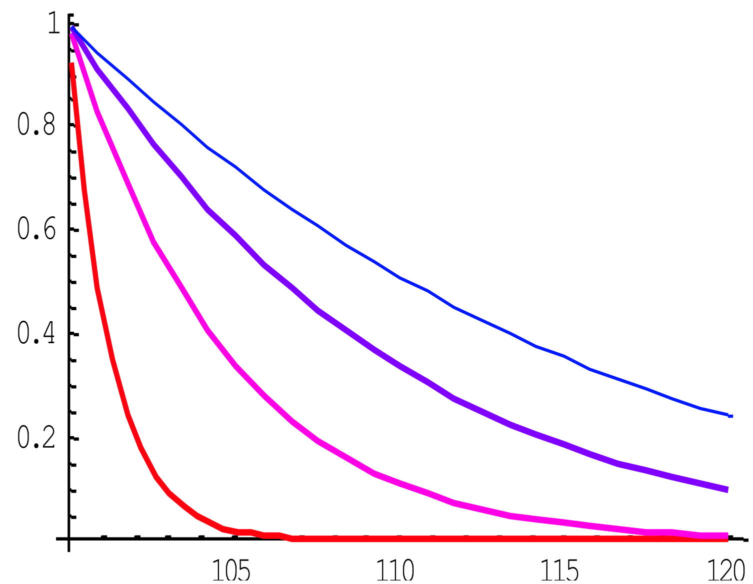


Fig. 2. Plot of Dynamic Leverage on vertical axis vs. current fund NAV on horizontal axis. Curves from bottom to top correspond to volatilities 5%, 10%, 15%, 20%. Time frame 1 year

Dynamic Leverage: Characteristics

The following features also emerge as a result of this model of dynamic leverage.

1. It is apparent that Dynamic Leverage, unlike most conventional measures of leverage, is not a linear function of risk.
2. The approach to a critical liquidation point as defined by this model is in fact similar to that of a short gamma position in a put (i.e the increase in leverage as equity depreciates).
3. This feature serves to explain why funds can unconsciously find themselves in critical condition faster than expected.

Dynamic Leverage: Sensitivity to Volatility and Distance to Barrier

Vol

NAV	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%
110	0.06	0.12	0.18	0.24	0.30	0.36	0.40	0.45	0.49	0.52	0.55
109	0.09	0.16	0.23	0.29	0.35	0.41	0.45	0.49	0.53	0.56	0.59
108	0.13	0.21	0.28	0.35	0.41	0.46	0.50	0.54	0.58	0.60	0.63
107	0.18	0.27	0.35	0.41	0.47	0.52	0.56	0.59	0.62	0.65	0.67
106	0.25	0.34	0.42	0.48	0.53	0.58	0.61	0.65	0.67	0.70	0.72
105	0.34	0.43	0.50	0.56	0.60	0.64	0.67	0.70	0.72	0.74	0.76
104	0.44	0.52	0.59	0.64	0.68	0.71	0.74	0.76	0.78	0.79	0.81
103	0.56	0.63	0.68	0.72	0.75	0.78	0.80	0.82	0.83	0.84	0.86
102	0.70	0.75	0.78	0.81	0.83	0.85	0.87	0.88	0.89	0.90	0.90
101	0.85	0.87	0.89	0.91	0.92	0.93	0.93	0.94	0.94	0.95	0.95
100	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 1. Dynamic leverage in a One Barrier Model as a function of NAV and Volatility.
The barrier level is at NAV=100, time frame 1 year, NAV growth rate 0.

Dynamic Leverage: Derivatives $\frac{\partial \,\mathrm{Lev}}{\partial \sigma}$

1. First derivative with respect to volatility demonstrates a maximum at varying distances from the barrier.
2. In essence this measure captures how quickly a fund manager can reduce the probability of liquidation by a de-leveraging of the portfolio.
3. For more volatile funds at some critical NAV the ability to reduce dynamic leverage by reduction in volatility (through liquidation) becomes effectively smaller and smaller – again relating to the ease with which a critical liquidation point can be reached.

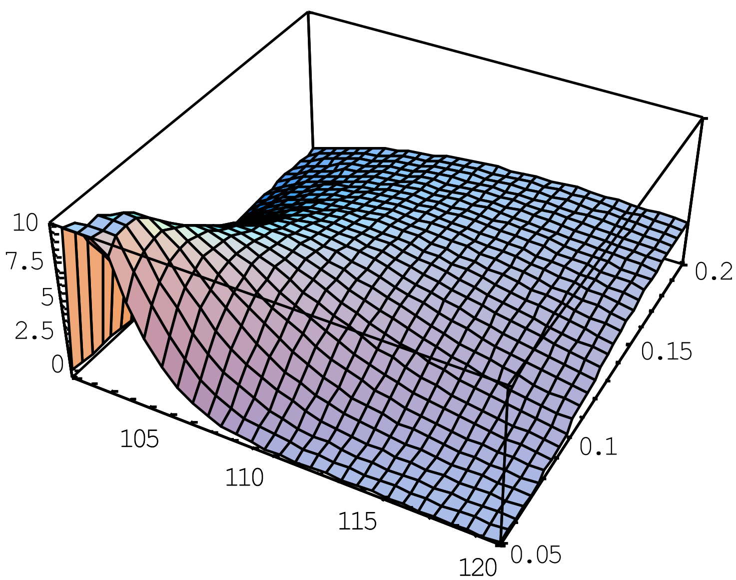


Fig.5 Plot of $\frac{\partial \,\mathrm{Lev}}{\partial \sigma}$

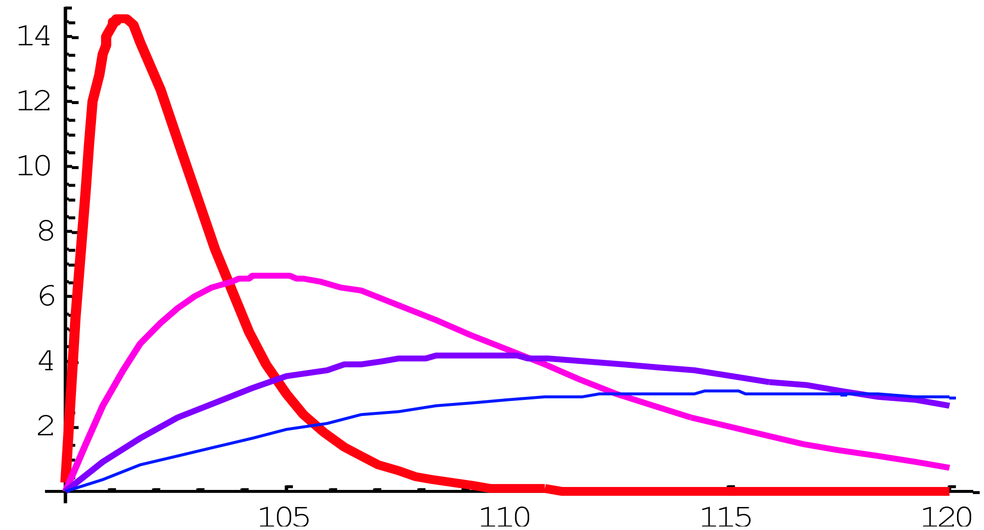


Fig. 6 Plot of $\frac{\partial \,\mathrm{Lev}}{\partial \sigma}$ on vertical axis vs. current fund NAV on horizontal axis. Curves at NAV=102 from top to bottom correspond to volatilities 5%, 10%, 15%, 20%.

Dynamic Leverage: Simplified Normal Model

In this model we assume that asset price $X(t)$ follows arithmetic Brownian motion with no drift

$$dX(t) = C dW(t)$$

Where $W(t)$ is a standard Brownian Motion and constant C is dollar volatility of the price. We assume that asset has a price x_0 at initial time $t_0 = 0$.

In this situation the distribution of prices at time t is normal with mean x_0 and standard deviation $C\sqrt{t}$.

Probability starting from point x_0 at time $t_0 = 0$ to hit a barrier $b < x_0$ any time before time t is

$$\text{Leverage}(x_0, b, t, C) = \text{Prob}_{\text{Hit b}} = 1 - \text{Prob}_{\text{Aove b}} = 2N\left(\frac{b - x_0}{C\sqrt{t}}\right)$$

Where and $N(y)$ is the cumulative normal distribution function. Dollar volatility can be expressed through percent volatility σ as $C = \sigma b$

Dynamic Leverage: Simplified Normal Model (Cont'd)

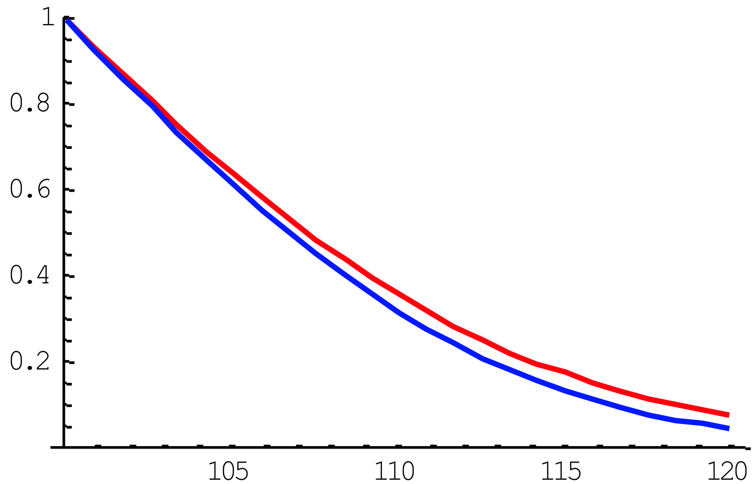
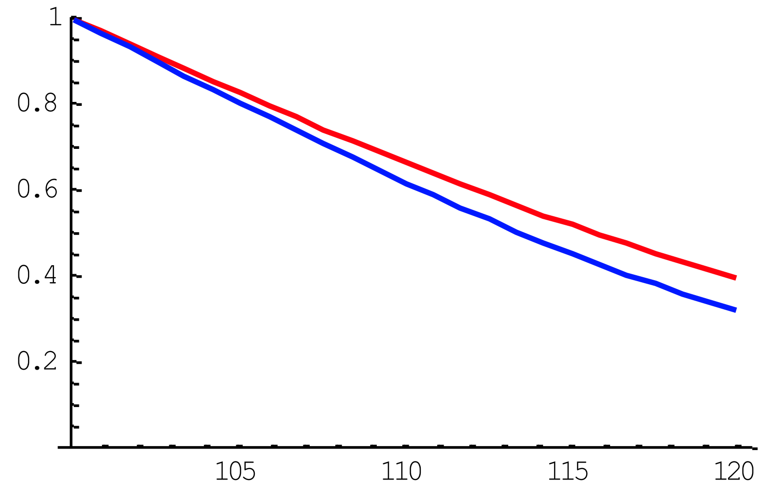


Fig 1. Red line is dynamic leverage using 1 barrier leverage model as in section 3 with barrier b=100, m=0, s=10%, blue line is simplified normal model with same barrier and volatility. Time left 1 year.



Red line is dynamic leverage using 1 barrier leverage model as in section 3 with barrier *b=100, m=0*, *s*=20%, blue line is

simplified normal model with same barrier and volatility. Time left 1 year.

Dynamic Leverage: Simplified Normal Model (Cont' d)

We can measure distance from barrier in terms of standard deviations of the NAV. In this case formula for dynamic leverage will become "universal".

$$\text{Leverage} = 2\,N(-z)$$

Where $z = \left(\dfrac{x_0 - b}{b\sigma\sqrt{t}}\right)$ is a distance between barrier *b* and the fund NAV *x* expressed in terms of standard deviation of NAV for time frame *t*. That standard deviation is $b\sigma\sqrt{t}$ where σ is percentage standard deviation near level *b*.

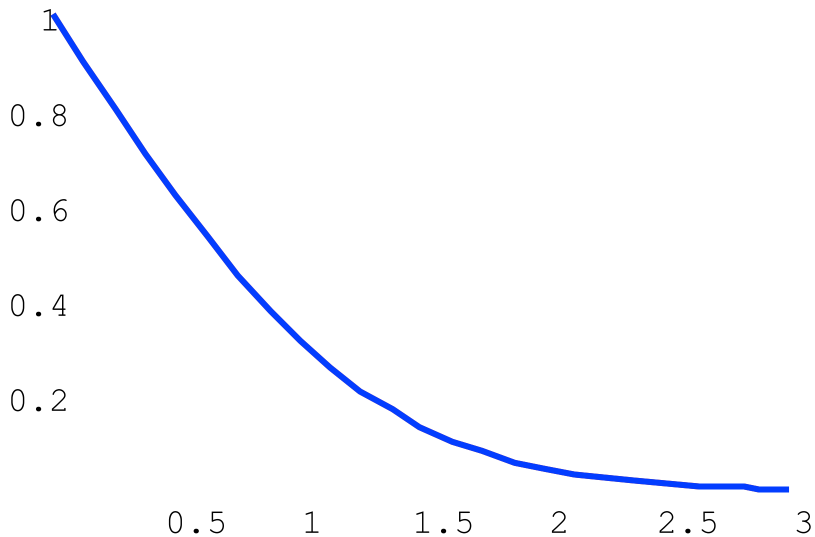
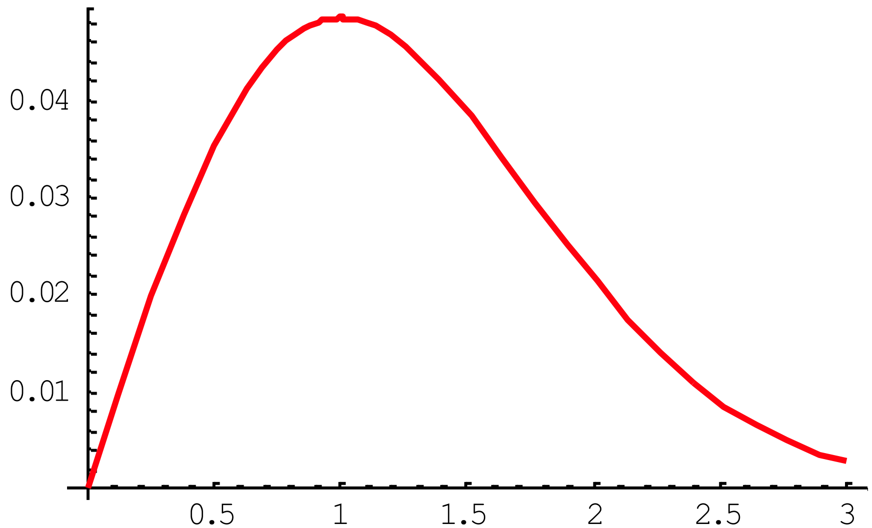


Fig 1. Dynamic Leverage as function of number of standard deviations from the barrier using simplified normal model *Lev=2 N(-z)*

Dynamic Leverage: Simplified Normal Model (Cont’d)

This simple model in fact captures many important effects. Let us consider sensitivity of dynamic leverage to volatility.



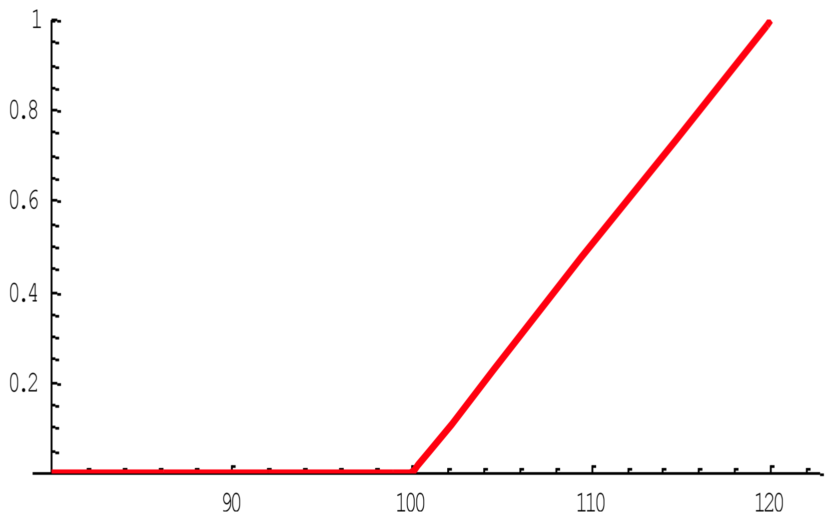
Plot of 0.01*(D Leverage/D Vol) . Here b=100, s=10%, t=1year vertical axis shows the effect on dynamic leverage of 1%, volatility increase (from 10% to 11%) and horizontal axis is distance from barrier measured in standard deviations of NAV scaled for 1 year

This curve has maximum at 1 standard deviation from barrier. That shows that portfolio manager who can control volatility of the fund would have greatest ability to reduce dynamic leverage at 1 standard deviation from barrier.

Dynamic Leverage and Portfolio Insurance

Portfolio insurance is a rule-based system of dynamically allocating portfolio between a risky asset and cash.

$$\textit{Exposure} = \begin{cases} C\,(E-F) & \text{if } E>F \\ 0 & \text{otherwise} \end{cases}$$

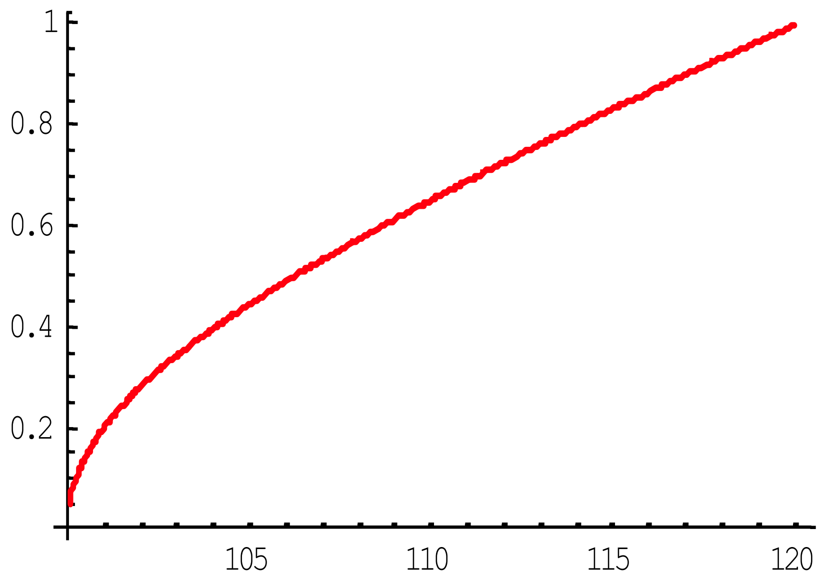


Constant Proportion Portfolio Insurance. Floor=100, C=4.

CPPI has some similarities to dynamic leverage with an implicit concept of a barrier and a requirement to stay above it. It differs in that the rule for reducing risk is defined *a priori* and with no regard for the dynamics of the fund under consideration.

Constant Dynamic Leverage Portfolio Insurance or CDLPI

The Portfolio is de-levered as it approaches the barrier in such a way so that portfolio's dynamic leverage is kept constant. (i.e. probability of reaching the barrier within a given period is kept constant). Takes into account the loss in equity approaching the barrier and factors in the probability of hitting the barrier as a function of volatility for a given time period.

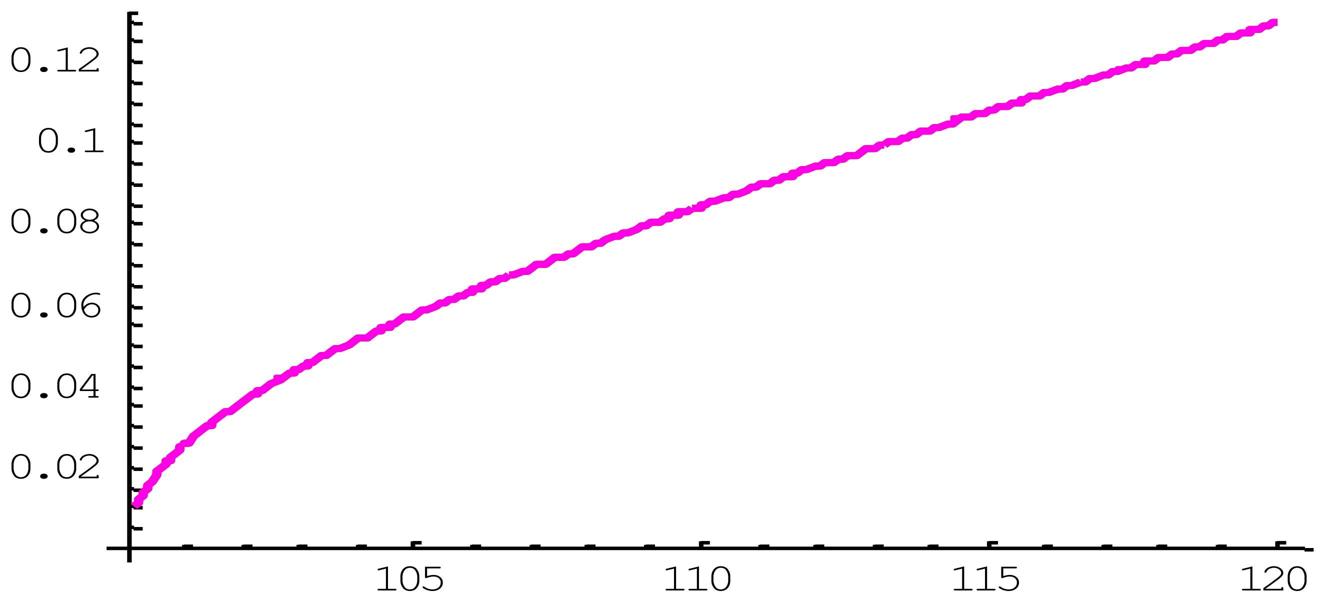


This graph shows how much a fund with a fixed 13% volatility must allocate in a risky asset to keep 5% chances of hitting the barrier.

NAV	Amt in Fund	Amt in Cash
120	100%	0%
118	93%	7%
116	86%	14%
114	79%	21%
112	72%	28%
110	65%	35%
109	61%	39%
108	57%	43%
107	53%	47%
106	49%	51%
105	44%	56%
104	39%	61%
103	34%	66%
102	28%	72%
101	20%	80%

Portfolio insurance table. Fund volatility = 13 %. Probability of hitting the barrier of 100 is fixed at 5%

CDLPI using a one-barrier log-normal model



This graph shows volatility necessary to keep fixed 5% chances of hitting the barrier as a function of NAV. We fix a probability of 5% of hitting the 100 level NAV barrier within a year. The fund with a growth rate 10% per year and NAV=120 must have a 13% volatility to have this 5% chance of hitting the barrier within a year. If NAV decreases then volatility must decrease to have the same 5% chance of hitting the barrier

So we chose a level for the barrier between minus 15% and minus 30% of fund’s initial NAV depending on category and specific information about the fund. The time frequency with which the fund allows to redeem (usually monthly, quarterly or annually) influence the choice of time frame in leverage calculation. In many cases 1 year is reasonable even for more liquid funds.

Appendix Formulas for Dynamic Leverage.

1. Probability of Absorption: One Barrier in a Log-Normal Model

Let us consider a standard log-normal model of evolution of asset price *X(t):*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

where *W(t)* is a standard Brownian Motion. Here μ is a constant growth rate and σ is a constant volatility. The distribution of prices at time t_1 of an asset that has a price x_0 at time t_0 is given by the following probability density function:

$$p(x_0, t_0, x_1, t_1) = \frac{1}{x_1 \sigma \sqrt{2\pi(t_1 - t_0)}} \exp\left(-\frac{\ln x_1 - \ln x_0 - \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{2\sigma^2 (t_1 - t_0)} \right)$$

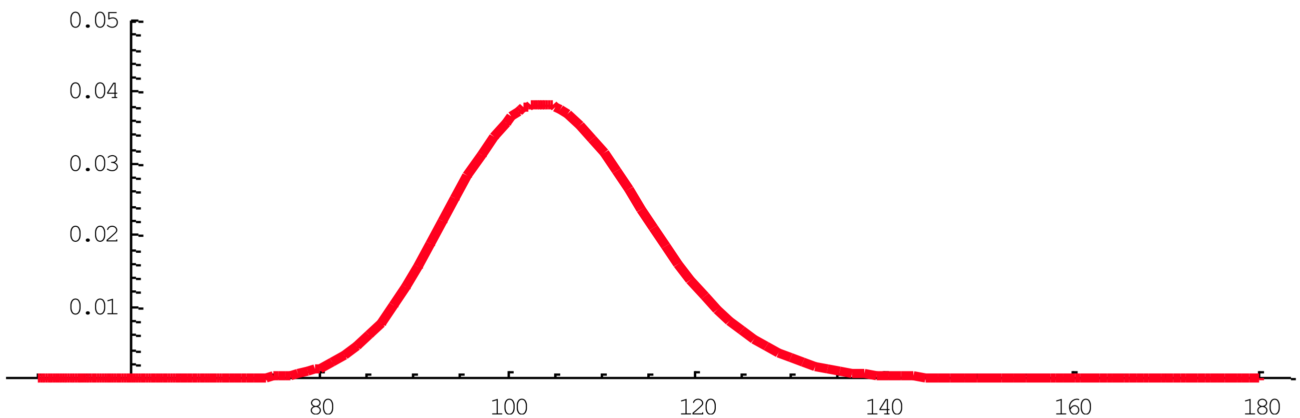


Fig. 1. Log-Normal Probability density function with μ, σ, =100, x_0=0, t_0=1.

1.1 Probability of Absorption (Cont'd)

The probability of the asset starting at x_0 at time t_0 to be in the interval [a, b] at time t_1 is just

$$\int_a^b p(x_0, t_0, x_1, t_1)\,dx_1$$

Now let us consider an absorbing barrier at level $b<x_0$. The probability density function for an asset that starts at x_0 at time t_0 and follows the standard log-normal model with constant growth rate μ and volatility σ and always remains above level *b* at all times is given by

$$P_{Barrier}(x_0, t_0, x_1, t_1) = \begin{cases} 0 \ \text{ for } x_1 < b \\ \dfrac{1}{x_1\sigma\sqrt{2\pi(t_1-t_0)}}\exp\left(-\dfrac{\left(\ln x_1 - \ln x_0 - \left(\mu-\dfrac{\sigma^2}{2}\right)(t_1-t_0)\right)^2}{2\sigma^2(t_1-t_0)}\right) - C\dfrac{1}{x_1\sigma\sqrt{2\pi(t_1-t_0)}}\exp\left(-\dfrac{\left(\ln x_1 - \ln\left(\dfrac{b^2}{x_0}\right) - \left(\mu-\dfrac{\sigma^2}{2}\right)(t_1-t_0)\right)^2}{2\sigma^2(t_1-t_0)}\right) \end{cases}$$

for $x_1 > b$, where $C = \exp\left(\dfrac{2\left(\left(\ln b - \ln x_0\right)\left(\mu - \dfrac{\sigma^2}{2}\right)\right)}{\sigma^2}\right)$

1.2 Probability of Absorption (Cont' d)

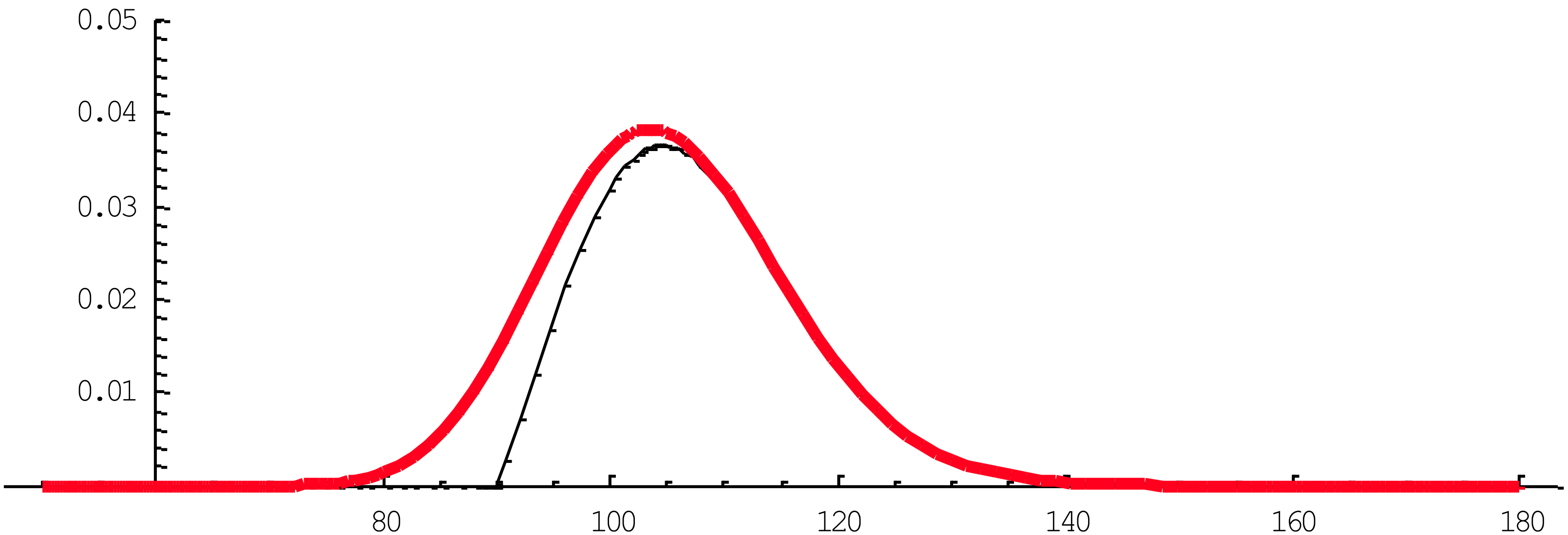


Fig. 2. Thin line: Log-Normal Probability density function with barrier *b*=90. Thick line: Probability density function without barrier. Here μ=0.05,σ=0.1 , x_0=100, t_0=0, t_1=1.

This result is obtained using reflection principle for a geometric Brownian motion see for example [2]. The probability of the asset starting at x_0 at time t_0 to be above barrier at all times before t_1 is

$$\text{Prob}_{\,Above\ B} = \int_{-\infty}^{\infty} P_{Barrier}\left(x_0, t_0, x_1, t_1\right) dx_1 = \int_{b}^{\infty} P_{Barrier}\left(x_0, t_0, x_1, t_1\right) dx_1$$

1.3 Probability of Absorption (Cont’d)

Doing the standard transformation y = ln x_1, the last integral can be calculated as

$$\text{Prob}_{Above\ B} = 1 - N(A) - C \cdot N(B)$$

where

$$A = \frac{\ln b - \ln x_0 - \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} \qquad C = \exp\left(\frac{2\left((\ln b - \ln x_0)\ \left(\mu - \frac{\sigma^2}{2}\right)\right)}{\sigma^2}\right)$$

$$B = \frac{\ln b - \ln x_0 + \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} = - \frac{\ln b - \ln\frac{b^2}{x_0} - \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}}$$

The probability of absorption (i.e hitting the barrier at any time between t_0 and t_1) is

$$\text{Prob}_{Hit\ b} = 1 - \text{Prob}_{Aove\ b} = N(A) + C \cdot N(B)$$

where

$N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$ the cumulative normal distribution function.

1.4 Probability of Absorption (Cont’d)

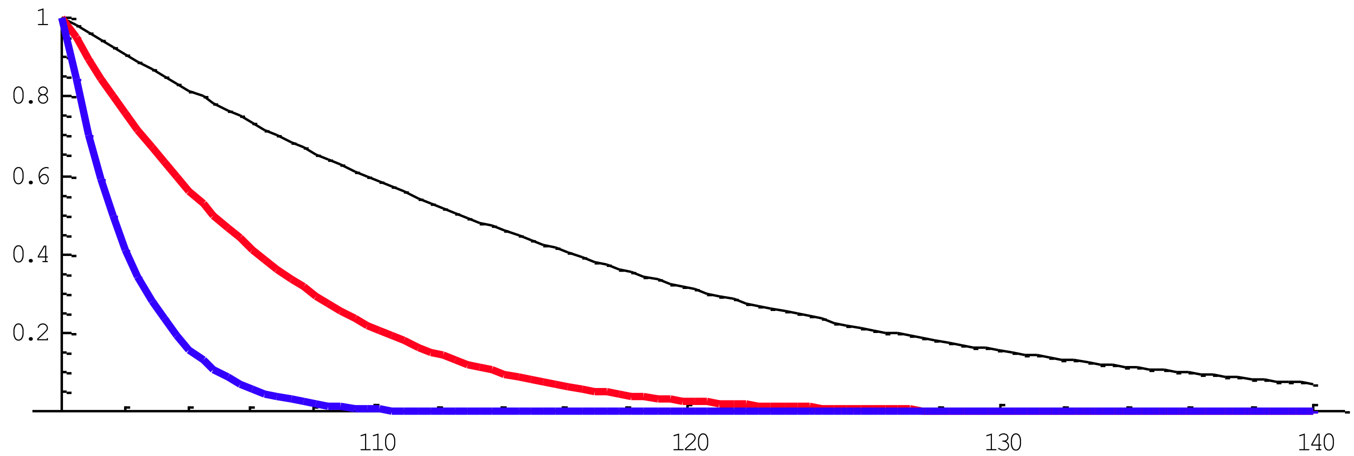


Fig. 3.Probability of absorption (hitting the barrier *b=100*) as a function of current asset price x_0 for three different volatilities σ=20%, 10% and 5%. Where t_1-t_0=1 year, μ=5%. Lower curves correspond to lower volatility.

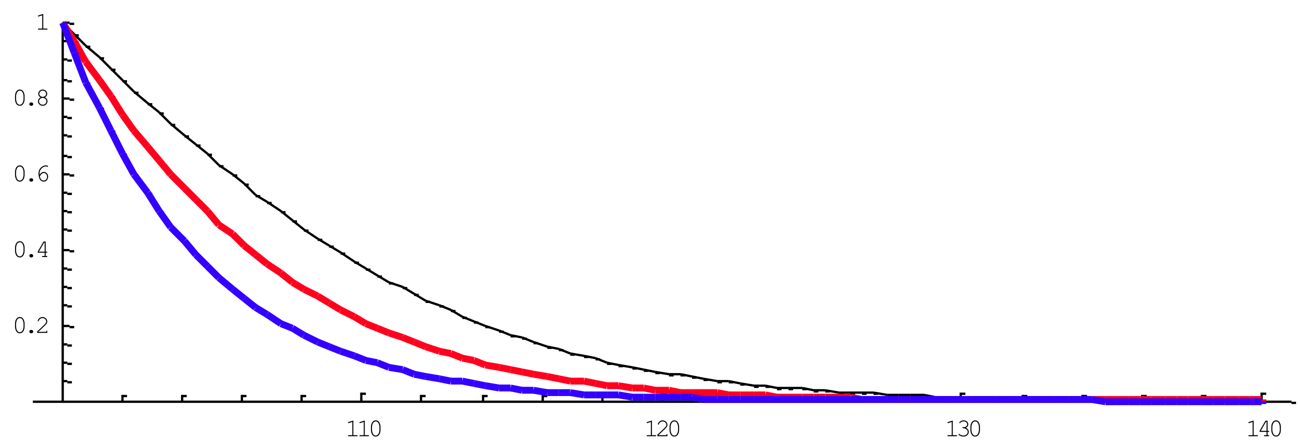


Fig. 4.Probability of absorption (hitting the barrier *b=100*) as a function of current asset price x_0 for three different growth rates μ=0%, 5% and 10%. Where t_1-t_0=1 year, σ=10%. Lower curves correspond to higher growth rate.

1.5 Probability of Absorption (Cont' d)

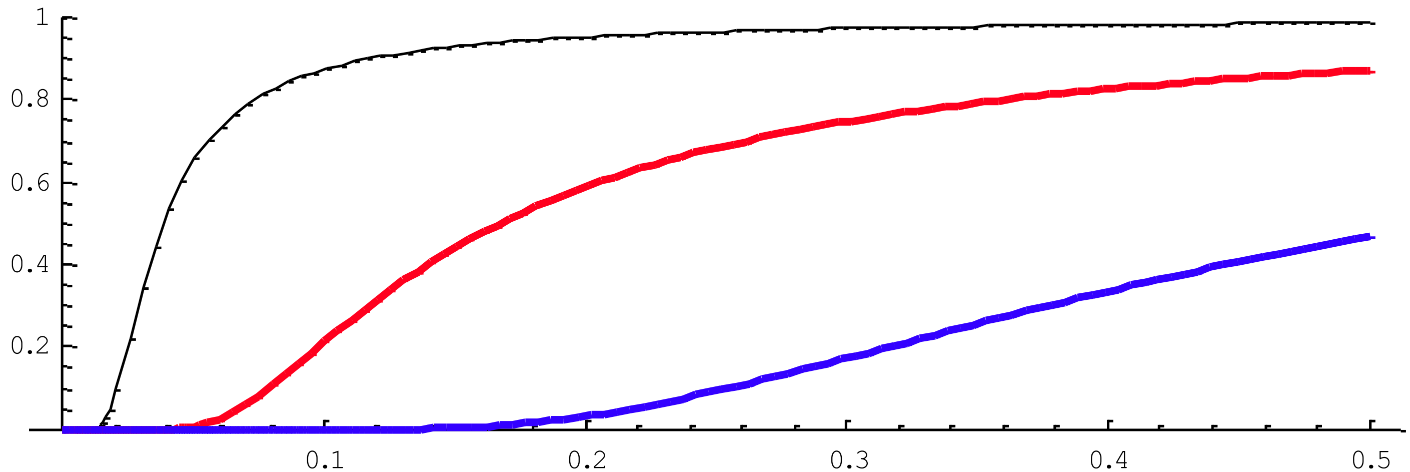


Fig.5. Probability of absorption (hitting the barrier *b=100*) as a function of current volatility σ. Where t_1-t_0=1 year, μ=5%, initial asset value x_0=101, 110 and 150. Lower curves correspond to higher x_0.

If we present the barrier as a percentage of the initial asset value , then

$$\text{Prob}_{Hit\ b} = 1 - \text{Prob}_{Aove\ b} = N(A) + C \cdot N(B)$$

where

$$A = \frac{\ln k - \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} \qquad B = \frac{\ln k + \left(\mu - \frac{\sigma^2}{2}\right)(t_1 - t_0)}{\sigma\sqrt{t_1 - t_0}} \qquad C = \exp\left(\frac{2\left((\ln k)\left(\mu - \frac{\sigma^2}{2}\right)\right)}{\sigma^2}\right)$$

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