Reconstructing the Black-Litterman Model

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Abstract

During the past 20 years several urban legends about the Black-Litterman model have appeared in the literature. These urban legends lie in wait for authors who do not consider the canon of Black-Litterman literature written by the primary authors of the model.

Generally these urban legends were created when authors attempted to simplify the model or introduce extensions to the model which they felt were more tractable than the canonical model. At times other authors have used these urban legends attempting to convince investors that the Black-Litterman model does not have a rigorous theoretical basis and should not be used.

In this paper we will provide a short review of the Black-Litterman model as provided by the original authors of the model. We will then provide a taxonomy/literature survey over several important papers in the Black-Litterman literature. Finally we will address the urban legends in detail.

JEL Classification: C1, G11

Introduction

The Black-Litterman Model provides an elegant way for investors to blend their views on returns with a market based prior to estimate expected excess returns, and from these excess returns to identify their optimal portfolio. It is accessible to anyone with a basic understanding of Bayesian Statistics and Markowitz's Mean-Variance work, though its' basis comes from much deeper Economic concepts. The Black-Litterman Model was the first model to wed Equilibrium theory with Bayesian Statistics (and Theil's Mixed Estimation model) in the service of finding a more optimal portfolio.

Since the Black-Litterman model was first published, a body of urban legends has gathered about the model. These urban legends have continued to propagate through the research because of subsequent authors' failure to read and cite primary references, and instead rely on secondary references. Rekdal (2014), and Simkin and Roychowdhury (2003) provide a thorough discussion on the prevalence and impact of these bad research habits.

In this paper we will review the model as described by the original authors. Will then introduce each of the urban legends and show why they are false. We will then construct a historical taxonomy of the literature on the Black-Litterman model which will allow us to trace the course of the most significant urban legends.

Black-Litterman Model Review

In this section of the paper we will review the Black-Litterman model as described by the original authors in Black and Litterman (1991), Black and Litterman (1992) and He and Litterman (1999). For complete derivations see Walters (2011).

Markowitz (1952) defines portfolio selection as a two stage process

> The first stage starts with observation and experience and ends with beliefs about future performances of available securities. The second stage starts with the relevant beliefs about the future performances and ends with the choice of portfolio.[1]

Sharpe (2010) describes the preferred approach for market forecasting using the following formula:

> $\textit{Market forecasts}_{>t} = \mathrm{f}(\textit{History}_{<=t}, \textit{Economic Theory}, \textit{Market Values}_{t})$[2]

The Black-Litterman model addresses Sharpe's forecasting problem and the first stage of Markowitz's approach by providing robust beliefs about the performance of available securities. The Black-Litterman model is relatively flexible when it comes to the method used to choose the portfolio.

> Up to this point we have not specified how we will pick one of the frontier portfolios as "optimal." In order to do this we need somehow to specify a set of preferences between risk and expected return. Throughout the examples in this paper we use a utility function to determine the frontier portfolio that we call "optimal." The extreme sensitivity of portfolio weights to

1 Markowitz (1952), p 77.

2 Sharpe (2010), formula 11, p 53.

expected returns that we focus on here is itself not sensitive to how we make this choice, as long as it reflects a reasonably smooth trade-off between risk and expected return.[3]

In the following formulas the variables are as follows

r	Distribution of unknown returns
δ	Risk aversion for the neutral portfolio
Π	Prior estimate of mean return (implied from neutral portfolio)
$\tau\Sigma$	Covariance of unknown mean about the prior estimated mean
Σ	Known covariance of returns about the unknown mean
w_{eq}	Equilibrium weights of assets
P	View matrix (k x n) with weights of assets in views
Ω	Covariance of unknown view mean returns about estimated view mean returns
Q	Estimated view mean returns
μ	Posterior estimate of the unknown mean return

Equilibrium Prior

The Black-Litterman model is based on General Equilibrium theory which provides the tools to model the prior estimate of returns. It is assumed the market is in equilibrium, and when the market is in equilibrium the market portfolio describes return expectations for all investors.

The investor has wide discretion in what portfolio they use as the neutral portfolio (the portfolio they will hold in the absence of views). They may use the market portfolio appropriate for their utility function, or they could use a benchmark or another so-called "normal portfolio"[4] which is relevant to their situation. Ultimately the Black-Litterman model provides a shrinkage estimator around the excess returns to the neutral portfolio. There is a significant body of work around the use of shrinkage to improve return estimates, for example Jorion (1986).

Litterman describes the rationale for picking the equilibrium market portfolio as the prior.

> We need not assume that markets are always in equilibrium to find an equilibrium approach useful. Rather, we view the world as a complex, highly random system in which there is a constant barrage of new data and shocks to existing valuations that as often as not knock the system away from equilibrium. However, although we anticipate that these shocks constantly create deviations from equilibrium in financial markets, and we recognize that frictions prevent those deviations from disappearing immediately, we also assume that these deviations represent opportunities. Wise investors attempting to take advantage of these opportunities take actions that create the forces which continuously push the system back toward equilibrium. Thus, we

[3] Black and Litterman (1991), pg 17, end note 6.

[4] Normal Portfolio : The Portfolio that an investor feels comfortable with when he has no views. He can use the normal portfolio to infer a benchmark when no explicit benchmark exists, Black and Litterman, (1992), page 2.

view the financial markets as having a center of gravity that is defined by the equilibrium between supply and demand.[5]

Litterman does not assert that the equilibrium market portfolio is the best estimator; just that it is a reasonable choice for most investors. Most investors will be able to intuitively identify their neutral portfolio; usually it is the ICAPM equilibrium market portfolio, but in some circumstances it might be a benchmark or other portfolio.

The estimated prior distribution of the unknown mean about the estimate is shown in (1) below.

(1) $$r \sim N(\Pi, \tau\Sigma)$$

The investor uses a process which Sharpe (1974) calls reverse optimization to compute the implied returns for the neutral portfolio which correspond to the equilibrium weights. Most commonly the investor uses the quadratic utility function and thus the reverse optimization has a closed form as shown in formula (2).

(2) $$\Pi = \delta\Sigma w_{eq}$$

The inputs to (2) are a known covariance matrix of the distribution of returns about the true (unknowable) mean (Σ), the weights of the assets in the neutral portfolio (w_{eq}), and the risk aversion parameter (δ). Several methods have been described in the literature for selecting a specific value of δ, we will not address this topic here.

Note that the model does not require any specific process to create the covariance matrix. There are no requirements on the method used to create the covariance matrix other than that it is assumed that all investors are using the same one. Black and Litterman (1991) and Litterman (2003) describe their approaches which include using daily data and exponential weighting schemes. Because the approach used to calculate the covariance matrix is not specified by the model, the investor is free to try other complementary approaches as well, such as Ledoit and Wolf (2001).

Reference Model

The canonical reference model uses Theil's mixed estimation (1971) as the mixing model. It is equivalent to the multivariate Bayes problem of unknown mean and known variance, DeGroot, (1970). A significant feature of this model is that the investor is uncertain in their mean estimates. They are working with a distribution of the true known mean about their estimate, rather than point estimates of the mean. Confusion about this point leads to the most common error when using the Black-Litterman model, the mis-specification of τ.

When using the canonical reference model the investor specifies $(\tau\Sigma)^{-1}$ the precision of the prior estimate, and Ω^{-1} which is the precision of the views' estimates. These quantities will typically be small as they measure the investor's uncertainty in their estimate of the mean (similar to the standard error of

[5] Litterman, et al (2003), page 3-4.

the mean), not the variance of the distribution about the mean. The precision of the posterior is $((\tau\Sigma)^{-1} + \Omega^{-1})$ which is an improvement over the precision of the prior and views' estimates.

He and Litterman (1999) is the first paper to illustrate the entire model, it includes all the formulas required to implement the model. They state very clearly that the unknown mean of the expected return distribution is a random variable distributed as shown in (3) below[6].

(3) $$r \sim N(\bar{\mu}, \tilde{\Sigma})$$

Black and Litterman (1992) contains a discussion of the methodology and a selection of the formulas involved in the model. It includes formula (4) below for the expected returns.[7]

(4) $$E(r) = [(\tau\Sigma)^{-1} + P\Omega^{-1}P']^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'^{\Omega^{-1}}Q\right]$$

This is consistent with formula (5) which shows the distribution of the unknown mean about the posterior estimate.[8]

(5) $$r \sim N\left([(\tau\Sigma)^{-1} + P\Omega^{-1}P']^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'^{\Omega^{-1}}Q\right], [(\tau\Sigma)^{-1} + P\Omega^{-1}P']^{-1}\right)$$

He and Litterman (1999) also shows the formulas below which can be used to compute the posterior covariance of returns about the estimated mean, allowing the investor to use the posterior estimate of the mean directly in the Markowitz portfolio choice model. The first, formula (6), is the covariance of the unknown mean about the posterior estimate from (5).[9] The second, formula, (7), is just the well known sum of two independent covariances.[10] We use formula (7) in order to convert the distribution of the unknown mean about the posterior estimate, and the known covariance of the distribution of returns about the unknown mean, into a single covariance of the returns about the posterior estimate so that the estimated posterior mean can be used in mean-variance optimization. If our investor is using a model for portfolio choice which operates on a distribution of the returns then there is no need for formula (7).

(6) $$M = [(\tau\Sigma)^{-1} + P\Omega^{-1}P']^{-1}$$

(7) $$\tilde{\Sigma} = \Sigma + M^{-1}$$

Formulas (2), (5), (6) and (7) comprise the core formulas of the canonical Black-Litterman model when used with the ICAPM model.

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6 He and Litterman (1999) formula 10.

7 Black and Litterman (1992), Appendix section 8.

8 He and Litterman (1999), formula 8.

9 He and Litterman (1999) formula 9.

10 He and Litterman (1999) formula 15.

Portfolio Choice

The final feature of the Black-Litterman model considers how the investor makes their portfolio choice. The model as originally specified allows for a wide range of portfolio choice models. Black and Litterman (1991) state

> Up to this point we have not specified how we will pick one of the frontier portfolios as "optimal." In order to do this we need somehow to specify a set of preferences between risk and expected return. Throughout the examples in this paper we use a utility function to determine the frontier portfolio that we call "optimal." The extreme sensitivity of portfolio weights to expected returns that we focus on here is itself not sensitive to how we make this choice, as long as it reflects a reasonably smooth trade-off between risk and expected return.[11]

In considering the question of constraints, Black and Litterman (1991) warns against the use of *artificial* constraints but allows for the use of *real* constraints such as a budget constraint or a constraint on maximum and minimum asset weights.[12] This is in response to the common approach to mean variance optimization where an investor will apply constraints to the objective function until they get an acceptable portfolio result. The Black-Litterman model with its shrinkage approach is much less prone to require artificial constraints.

When illustrating the concepts of the Black-Litterman model, authors sometimes use views with 100% confidence which can generate extreme portfolios. In practice investors would seldom be 100% confident in a view. We expect that an investor would learn to specify their views with an intensity (Q) and confidence (Ω) that would lead to appropriate levels of shrinkage and more reasonable results.

Along similar lines, usually for reasons of transparency and ease of understanding, authors typically use an unconstrained Mean-Variance optimization to illustrate their results. Because of the shrinkage approach it is not hard to get reasonable results in the context of an example in a research article. However this should not be read as a limit of the model. As shown in the original specification of the model, Black and Litterman leave quite a lot of freedom to the investor in terms of their portfolio choice model. For example, constraints, other risk measures or resampling techniques can easily be used for the portfolio choice step in a non-Bayesian approach.

Urban Legends of the Black-Litterman Model

This section of the document will enumerate the top four urban legends about the Black-Litterman model. The taxonomy/literature survey which follows will trace the origin and lineage of each of the urban legends over time. The urban legends are:

- The prior, conditional (views) or posterior estimate is a point estimate
- τ is just a scaling factor
- τ is often 1

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11 Black and Litterman (1991), pg 17, end note 6.

12 Black and Litterman (1991) pg 16.

- The investor is limited to the use of Mean-Variance techniques

In the following sections we will present each urban legend and then references to the canonical Black-Litterman papers illustrating the fallacy of the urban legend.

The Prior, Conditional (Views) or Posterior Estimate is a Point Estimate

This urban legend arises from investors who do not understand formula (5) or neglect formula (6). The derivation of the model from both Bayes law and Theil's mixed estimation requires that all estimates are distributions. We can find clarity of this point in each of the canonical papers. Black and Litterman (1991) states:

> Formally, we generate expected excess returns as the mean of a posterior distribution that incorporates the information in the equilibrium and the investor's views. We assume a prior distribution based on the ICAPM equilibrium, with the covariances of the prior proportional to the historical covariances of excess returns. The constant of proportionality reflects the relative weight given to the equilibrium versus the views. We translate individual views into "observations" on a set of linear combinations of the expectations for individual securities, again with covariances proportional to historical values (here the constants of proportionality reflect the relative degrees of confidence in the individual views), and combine the results with the prior in a procedure that Theil [1971] describes **as** "mixed estimation".[13]

Black and Litterman (1992) states:

> The expected excess return, E[R] is unobservable. It is assumed to have a probability distribution that is a product of two normal distributions.[14]

He and Litterman (1999) states:

> The Black-Litterman asset allocation model uses the Bayesian approach to infer the assets' expected returns[Black and Litterman, 1990, 1992]. With the Bayesian approach, the expected returns are random variables themselves. They are not observable. One can only infer their probability distribution. The inference starts with a prior belief. Additional information is used along with the prior to infer the posterior distribution.[15]

As we can see, any author who references any of the canonical papers should not be propagating this urban legend as each paper clearly describes each estimate (prior, conditional and posterior) as a distribution and not a point estimate.

τ is Just a Scaling Factor

The urban legend builds upon "The prior, conditional (views) or posterior estimate is a point estimate". If the investor neglects formula (6), then the scaling is based only on the relative values of τ and Ω, not on the absolute value of either. As we have seen above, formula (6) is a critical part of the model and

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[13] Black and Litterman (1991), Note 10, page 17.
[14] Black and Litterman (1992), Note 7, page 42.
[15] He and Litterman (1999), page 2.

thus τ is not just scaling $\tau\Sigma$, the covariance of the unknown mean about the prior estimate, it also impacts covariance of the unknown mean about the posterior estimate.

τ is Often 1

This urban legend also requires the investor to have previously adopted the both "The Prior, Conditional (Views) or Posterior Estimate is a Point Estimate" and "τ is Just a Scaling Factor". Black and Littermanm (1992) states:

> Because the uncertainty in the mean is much smaller than uncertainty in the return itself, τ will be close to zero.[16]

Selecting a large value for τ or Ω corresponds to a very diffuse estimate. This is best illustrated by considering the confidence intervals represented by the uncertainty. If we consider view 1 from He and Litterman (1999), the σ_{prior} is 3.26% and the π is 1.79%, thus we can expect the unknown mean to be on the interval (-1.47%, 5.05%) 70% of the time. If we instead set τ =1, we would find the 70% confidence interval (-12.81%, 14.39%) and the 95% confidence interval (-23.40%, 26.99%) which is so diffuse as to be useless.

The Investor is Limited to the Use of Mean-Variance Techniques

Investors use Mean-Variance portfolio choice models because they are well understood. Most authors use them because of their transparency and simplicity. We have already seen from Black and Litterman (1991) that the investor is free to use any portfolio choice model so long as it has a smooth tradeoff between risk and return.

Furthermore, in Black and Litterman (1991), they also state:

> "that the user can specify the objective of their optimization process as a particular level of risk, maximizing the Sharpe ratio, or specifying a utility function".[17]

This gives the investor using the Black-Litterman model quite a lot of flexibility in terms of their model for portfolio choice and clearly does not limit them to Mean-Variance techniques.

Historical Taxonomy

Here we construct a brief historical taxonomy of the literature on the Black-Litterman model to aid in the tracing the origin and course of the most significant urban legends. For each paper we also review the references used to further illuminate our observations.

The definitive references for the Black-Litterman model, as cited in Litterman, et al, (2003) are Black and Litterman (1991, 1992), and He and Litterman (1999). These are the only publications on the model by the primary authors and each provides significant information in forming an understanding of the model. We will add Lee (2000), Satchell and Scowcroft (2000), Blamont and Firoozye (2003), Idzorek

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[16] Black and Litterman (1992), page 34.

[17] Black and Litterman (1991) pg 16.

(2005) and the work of Fusai and Meucci (2003) and Meucci (2006), Michaud, et al (2013) to our taxonomy as each of these papers creates an inflection point in the understanding of the model. The reader is directed to Walters (2011) for a more complete literature survey.

The Black-Litterman model originated in two papers by Black and Litterman (1991, 1992) and was further described in the paper He and Litterman (1999). These three papers each with at least one of the original model authors offer a consistent vision, which we will call the canonical Black-Litterman model. Modern Investment Management: An Equilibrium Approach, Litterman et al (2003), describes the asset allocation process built around the Black-Litterman model used by Goldman Sachs Asset Management. Further Goldman Sachs papers, e.g. Bevan and Winkelmann (1998), have stayed true to the canonical Black-Litterman model. The canonical Black-Litterman model can be viewed as either using Theil's Mixed Estimation model, Theil (1971), or the Bayes problem of unknown mean and known covariance, DeGroot (1970).

Lee (2000) appears to be the first unrelated author to write significantly about the Black-Litterman model. Lee describes the model in detail clearing describing the prior estimate and views as distributions.

> Of course, to be realistic, the investor must have some uncertainty about these views. Therefore, both sources of information, the equilibrium expected returns and the investor's views, are expressed as distributions. The relative weights put on the equilibrium versus the views then depend on the investor's relative degree of confidence in this information.[18]

He also shows the formula for the posterior estimate of the returns, formula (4), but does not mention the updated uncertainty of said returns caused by the fact that the posterior estimate is actually a distribution. This a a weak beginning for the first urban legend, "The Prior, Conditional (Views) or Posterior Estimate is a Point Estimate." This is more clearly the beginning of the second urban legend that "τ is just a scaling factor" and that one can select any value for τ as the formula for the mean estimate of returns scales with τ and Ω such that τ could indeed be anything if Ω is chosen properly. Actually, both the prior and posterior estimate's covariances vary with τ, so the value selected for τ becomes important when properly using the model. Lee cites Black and Litterman (1991, 1992), but does not cite He and Litterman (1999) perhaps because it was being written concurrently with his book.

Satchell and Scowcroft (2000) attempted to demystify the Black-Litterman model. They noted that the two original Black-Litterman papers, Black and Litterman (1991 and 1992) did not include all the formulas or derivations of the formulas. They also seem to have been writing their paper concurrently with He and Litterman (1999) as they only referenced Black and Litterman (1991) and (1992). Satchell and Scowcroft set out to illustrate the model by deriving and showing all the key formulas. In their paper they present the Black-Litterman model as a Bayesian model. They comment that the Bayesian formulation of the Black-Litterman model is well known.

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[18] Lee (2000), page 174.

> We emphasize that Theorem 1 is a result known to Bayesian econometricians and the BL, although they did not report the variance formula in their papers.[19]

Unfortunately they also introduced the third urban legend that "τ should be 1", even stating "τ is a (known) scaling factor often set to 1."[20] It is unclear from which of their references this assertion is made. The authors show the posterior as a distribution (explicitly referencing the variance of the posterior estimate). They do not provide enough raw data to reproduce their results so one is not able to determine if they used an adjusted posterior covariance of returns, formula (7). They introduce a new model which includes estimates for the volatility of assets by using a stochastic τ with a value close to 1. Their model is different from the standard Bayesian problem of unknown mean and unknown covariance which we can find in (for example) DeGroot (1970).

Blamont and Firoozye (2003) describe the Black-Litterman model and make the important contribution that τΣ is analogous to the standard error concept from classical statistics. This supports the idea that τ << 1 consistent with the original model's authors. They provide illustrations (see exhibits 1 and 2 in their document) which show that the prior, conditional and posterior estimates are all distributions. Their work pushs the literature back towards the canonical model as they do not explicitly present any of the urban legends. They work an example, but do not provide the raw data to show whether they treat the posterior estimate as a point estimate or as a distribution, or how to use the posterior estimate with mean variance optimization.

Fusai and Meucci (2003) describe a new model similar to the Black-Litterman model and cite only Black and Litterman (1992) as a reference.. Unfortunately they link their model to the Black-Litterman model as if it is the same. In their model they replace the usual parameter τ with a new parameter α. This reinforces the first urban legend that τ is just a scaling factor which regulates the shrinkage.. They do not tie α to anything but the amount of shrinkage, showing a graph where α ranges from 0 to 5. Meucci (2006) followed up on this paper and coined the phrase, "Beyond Black-Litterman". The "Beyond Black-Litterman" model can be viewed as a standard shrinkage model, where the amount of shrinkage toward the prior is controlled by the magnitude of Ω. In this usage of the formulas, the posterior covariance has no meaning.

Idzorek (2005) introduces a new method of calibrating the shrinkage to the prior. He cites essentially all of the papers in this taxonomy. He chooses a value for τ << 1, but he does not use an adjusted posterior covariance of returns. He introduces a classic graphic representation of the blending process which shows the posterior as a distribution. He treats the posterior returns as point estimates and does not use an adjusted posterior covariance of returns. Idzorek's approach to selecting Ω has no dependence on whether the posterior estimate is a distribution or a point estimate, thus his method of calibrating Ω can be used with the canonical Black-Litterman model as well as the model described in his paper.

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[19] Satchell and Scowcroft (1999), p 141.

[20] Satchell and Scowcroft (1999), p140.

Michaud et al (2013) is included in this list only because it is essentially a compilation of the urban legends around the Black-Litterman model, though we can credit them with the origin of the fourth urban legend, the dependency on Mean-Variance techniques for portfolio choice. In their paper the authors cite Black and Litterman (1992) as a reference, but clearly ignore much of it given that it disproves several of the urban legends they promote. They proved that the model they derived is not Bayesian, but given it is also not Black-Litterman it is unclear how valuable this conclusion is. These authors have significant undisclosed conflicts of interest, their company produces an optimizer which is often compared to the Black-Litterman model and they would likely see economic benefit from any negativity written about the Black-Litterman model. We can gather from this paper's faults that the Journal of Investment Management does not adequately review all published papers. Rather than divert the reader further, a specific critique of the ideas in Michaud et al (2013) is provided in Appendix A.

In summary, several significant urban legends about the Black-Litterman model have been introduced by authors over the last 15 years. Unfortunately these urban legends propagate as authors cite primary references, but do not read them; or cite secondary references which propagate the urban legends.

Summary

In summary the literature on the Black-Litterman model is a mixed bag. Many authors reference the original authors' papers and illustrate the canonical Black-Litterman model. Unfortunately many other authors cite the canonical papers, but illustrate subtly different models while calling it Black-Litterman, even when they have dropped formulas or redefined parameters of the model. Their changes breaking the linkage to the formal methods from which the canonical model was derived.

We can certainly refer to the paper Simkin and Roychowdhury (2003) and ask prospective authors to "Read Before You Cite!".

Most authors show arbitrary examples and it is largely impossible to test the Black-Litterman model because the test would jointly test the model and the views. As a result we will likely never know with certainty whether it is superior or inferior to other simple optimization models.

References

The two references, Black and Litterman (1991) and Black and Litterman (1992) are based on Goldman Sachs Fixed Income Research papers which are also available on the internet. We are citing the peer reviewed journal publications.

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Appendix A

This section of the document will discuss specific points in the Michaud et al (2013) paper. As previously noted they cited Black and Litterman (1992) but specifically excluded many important points from that paper in their discussion. Their second Black-Litterman reference comes from the “Beyond Black-Litterman” lineage of pure shrinkage models with formal like to Bayes law or to Theils’s mixed estimation.

This leave them with their proposed version of the Black-Litterman model which does have theoretical linkage to Bayes law, or Theil’s mixed estimation, nor does it have a theoretical linkage to the canonical Black Litterman model as described in their reference. Given this basis, their paper does not really address many issues relevant to the canonical Black-Litterman model.

In general it seems that many of their critiques come from applying classical statistical techniques to their non-Bayesian specification of the Bayesian model. Most students of statistics or economics will note that this approach should not be expected to be successful. The Black-Litterman model is built on a foundation of uncertain estimates, but they have attempted to tie it to point estimates with their choice of the improperly specified model.

They do not formally describe their reference model, but it does not have the characteristics of the canonical Black-Litterman reference model. For some reason they have specified their model to use point estimates, not distributions though they later show it is non-Bayesian. In further support of the conclusion that they are not using the canonical model we have both their arguments that τ can be close to 1 in magnitude and formula (2) in their paper which does not show an updated posterior covariance. The scale of their Ω is also a clue that they are not properly using the Black-Litterman model, Bayes law or Theil’s mixed estimation technique. We could call this, a trifecta of naivete on their part. They dutifully show their model has no theoretical basis, though it is unclear the value of this proposition. Substantially all of their arguments in sections 2.1-2.4 apply only to the Black-Litterman model as described by Satchell and Scowcroft (2000), and not to the canonical model of Black and Litterman (1991,1992).

The following sections provide detailed comments on the various assertions in their paper.

Comments on Note 5

Note 5 in their paper is the backbone of their assertion that the reverse optimization technique used to derive the Black-Litterman prior excess return estimate is not theoretically sound. They include six assertions, five of which we will address in this section, and one which will be addressed in a later section. They assert the following set of issues with implied returns:

- Inverse returns are a function of the covariance matrix, which by definition is devoid of return information.
- Inverse returns function solely to reverse engineer the unconstrained MV optimization and negate any optimality properties bestowed by that optimization.
- Inverse returns are not unique and not on the same scale as actual forecast returns: a positive scalar multiple is also an inverse return.

- Inverse returns require MSR optimality of the market portfolio, which is unknown and highly unlikely a priori.
- Inverse returns require an unconstrained MV optimization framework, which is unrealistic for practical investment.[21]

Here we will address these issues one at a time, and show that none of these are valid issues with the Black-Litterman model.

Inverse returns are a function of the covariance matrix

They assert that there is no statistical process for recovering historical means from the historical covariance matrix and they are absolutely correct! Actually we are solving a different problem, which is identifying the instantaneous estimates of the mean and covariance matrix in use by market participants such that the market is in equilibrium. We are all quite aware that summary statistics from historical returns are not the best indicator of future returns. In this case we are using a structured econometric model (Markowitz Mean-Variance) and we are supplying additional information in the form of the normal portfolio weights and a value δ for the risk tolerance of the market.[22] This allows us to use the historical estimate of the covariance matrix to estimate the expected return. This approach is consistent with model econometric models.

While Michaud et al assert that any student using such an approach in their statistics class would receive an F[23], it is clear that students of economics are allowed more arrows in their quiver. This is a fundamental conflict between investors preferring classical statistical techniques and those who are open to modern econometric models.

Inverse returns function solely to reverse engineer the unconstrained MV optimization

The Black-Litterman model draws on the ICAPM model for a theoretical explanation of the market portfolio when using the quadratic utility function. In practice, we know all investors are not using the model. Empirically, the market portfolio is the aggregate of all investors' positions, independent of any theory. We are applying a structured econometric model to observed data to imply an estimate of the unobserved implied mean vector. I am not able to find any other source to confirm their assertion that this use of reverse optimization is overfitted.

As previously noted, the inputs to the reverse optimization procedure are the covariance matrix, the neutral portfolio weights and the market risk aversion, δ. It is true that we can generate different sets of implied returns for different values of δ, but there is exactly one set of implied returns which will satisfy the model for a specific set of input data (weights, covariance and risk aversion). It is in general true of models that changing the inputs will change the outputs, and it is also clear that the relationship in

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21 Michaud et al (2013), endnote 5.

22 Note that because we are starting from portfolio weights which sum to 1, we do not need to apply any constraints at this point in the process.

23 Michaud presentation to BSAS March 24, 2014

many models is linear. It is unclear exactly what Michaud, et al mean by their comment that the scale is incorrect. The example they show in their paper illustrates results with the proper scale.

Inverse returns require MSR optimality of the market portfolio

The use of implied returns does not in fact require MSR optimality of the neutral portfolio. This is another area where the fixation Michaud, et al have with point estimates undermines their analysis. The investor specifies the known covariance matrix, the known portfolio weights and the risk aversion of the neutral portfolio. The reverse optimization process returns the estimated returns that match the inputs. Because the prior estimate is a distribution, there is also the uncertainty in this estimate ($\tau\Sigma$). Thus the neutral portfolio lies in an uncertainty reason around the unknown MSR optimal portfolio because of the uncertainty in the prior estimate. Thus the optimization cannot be run forward without taking into account that the prior estimate is not a point estimate. This is why the Bayesian Black-Litterman investor will not invest 100% of their funds in the prior estimate of the portfolio, they invest the quantity ($1/1+\tau$).

Inverse returns require an unconstrained MV optimization framework

Michaud, et al, assert that using reverse optimization requires an unconstrained mean variance framework. Investors who choose to use the ICAPM for their equilibrium are indeed restricted to using Mean-Variance (which is not quite the devil Michaud claims), however recent research in other distributions and investors who select other utility functions are free to use other equilibrium conditions. The canonical model as generally described assumes no constraints on the market portfolio in accordance with the ICAPM equilibrium. This is typically how reverse optimization is performed because the fully invested constraint is implied in the reverse optimization model, and no other constraints are easily considered to be applied consistently across all investors. The use of an unconstrained reverse optimization does not have any bearing on the portfolio choice model used by the investor.

Comments on Notes 7 and 13

In note 7 Michaud, et al, assert that the implied returns are not normally distributed about the actual mean with a covariance structure proportional to the covariance of the distribution of returns about the mean. For their non-Bayesian approach this is entirely correct because they use point estimates. They errantly use this argument to assert that the use of Theil's mixed estimation is invalid.

The canonical model is described as

> We do assume that the mean, E[RA], is itself an unobservable random variable whose distribution is centered at the equilibrium risk premiums. Our uncertainty about E[RA] is due to our uncertainty about E[Z] and the E[ui]s. Furthermore, we assume the degree of uncertainty about E[Z] and the E[ui]s is proportional to the volatilities of Z and the uis themselves. This implies that E[RA] is distributed with a covariance structure proportional to S.[24]

The canonical model specifies the prior distribution as centered about the implied return estimates with a covariance structure proportional to the covariance of returns about the true unknown mean

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[24] Black and Litterman (1992)

(specifically proportional with constant of proportionality τ). The argument in note 7 is clearly not relevant to the canonical Black-Litterman model.

In note 13 they repeat this assertion. The confusion on their part appears to be tied to their use of their non-Bayesian model rather than the Bayesian canonical model. Clearly these two notes are not valid for the canonical model which expresses the prior estimate as a distribution.

Comments on Note 9

Note 9 is tied to text indicating some negative idea about constraints. Black and Litterman (1991) specifically states as shown above that *real* constraints are an acceptable part of the portfolio choice model. This would be consistent with most investors.

Portfolio Choice

This section will address issues raised under the topic of portfolio choice by Michaud, et al (2013).

They use a 60/40 portfolio as their neutral portfolio. They then apply a view which indicates US equities will outperform European equities by a significant amount. Their Black-Litterman results marked as BL or BL* indicate a clear preference for US equities over European equities given this view. This is clearly an intuitive result. Interestingly, the Michaud results as shown in their Table 3 actually lowers the investment in the US Equity market below the neutral portfolio and increases allocations to Europe in order to diversity the portfolio. This portfolio is inconsistent with the investors views and the weights have changed in a non-intuitive fashion.

In section 4.3 they specify that the Black Litterman model does not cater for investor risk aversion. It is unclear how this statement can stand when the canonical Black-Litterman model as shown in formula (2) shows the use of a risk aversion value.

Bootstrapping to Determine τ

The process of generating "statistically equivalent" covariance matrices described in section 3.6 actually makes an interesting positive contribution to the Black-Litterman literature. Using the standard bootstrapping process to determine the standard error of a mean estimate is a basic statistical practice, but as far as this author understands, it has not been applied to this specific circumstance. Bootstrapping can be used to put upper bounds on the precision of the estimates for the parameters τ and Ω. Interestingly it also seems to indicate that an extension to the model where τ would be a vector might be possible.

If we take the distribution from Michaud Table 4 and fit a cumulative normal distribution to the 25% and 75% points, we can then compute the effective Black-Litterman τ for each asset class. This allows us to imply lower limits on τ values. As we would expect these numbers cluster around 1/216 which corresponds to the number of samples in the Michaud, et al data set. If we use the 5% and 95% values instead we get similar results.

What we see is that the bootstrap results indicates for this example the value of τ generally matches the value described in the canonical model literature (1/n). It is very interesting to note that they use this data to show how poor the prior estimates are (their point estimates have no uncertainty), but when compared to the standard inputs to the canonical model we see this procedure generated the expected confidence intervals around the estimated means. In this part of their paper they actually provide an alternative explanation for why $\tau << 1$.

Their table 5 which shows view means is not entirely clear; there was no view on Euro Bonds so how can there be a view mean? It is expected that we could use the same bootstrapping process used to bound τ to also bound Ω.

They assert in this section that because bootstrapping defines a distribution (exactly as the canonical model requires) that there is no unique set of implied returns. This is entirely correct as it matches the specification of the canonical model which specifies that the prior estimate is a distribution not a point estimate.

Summary

In summary much of the analysis in Michaud et al (2013) is irrelevant to the canonical Black-Litterman model. It boggles the mind of this reviewer how muddied their analysis is. Their analysis is limited only to a non-Bayesian use of some of the Black-Litterman formulas which does not strike this author as a good idea in general. They did not properly address the canonical model and many of their critiques actually support the canonical Black-Litterman model.

Some portion of their paper is developed to show how their optimizer is better than a plain Black-Litterman implementation using an unconstrained mean variance optimization. To be fair, it is not possible to easily perform proper out of sample testing of the results from using the Black-Litterman model because of the need for views. The test would jointly test the model and the views. As a result we will likely never know with certainty if inputs to the Michaud Resampled optimization model generated via the Black-Litterman model outperform other mixing models.

Matlab code suitable to reproduce all of the Black-Litterman model results in Michaud et al (2013) is available on the blacklitterman.org website.

We finish the paper with a comment from Michaud (2008),

> While it is of interest to compare the relative value of Bayesian Estimation versus Resampled Efficient™, the procedures are complementary, not exclusive. Resampled optimization, properly implemented is an additional way to improve investment value of optimized portfolios whatever risk-return estimation methods are used.[25]

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[25] Michaud (2008)