Market Manipulation, Bubbles, Corners, and Short Squeezes

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Abstract

This paper investigates market manipulation trading strategies by large traders in a securities market. A large trader is defined as any investor whose trades change prices. A market manipulation trading strategy is one that generates positive real wealth with no risk. Market manipulation trading strategies are shown to exist under reasonable hypotheses on the equilibrium price process. Sufficient conditions for their nonexistence are also provided.

I. Introduction

Famous market manipulations, corners, and short squeezes form an important part of American securities industry folklore. Colorful episodes include the collapse of a gold corner on Black Friday, September 24, 1869,[1] corners on the Northern Pacific Railroad (1901),[2] Stultz Motor Car Company (1920),[3] and the Radio Corporation of America (1928).[4] More recent alleged corners include the soy bean market (1977 and 1989),[5] silver market (1979–1980),[6] tin market (1981–1982 and 1984–1985),[7] and the Treasury bond market (1986).[8] All of these episodes were characterized by extraordinary price increases followed by dramatic collapses, called bubbles. Another commonality in these corners is that they were orchestrated by one individual or a group of individuals acting in concert. These illustrations provide vivid counterexamples to an unqualified

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*Johnson Graduate School of Management, Cornell University, Ithaca, NY 14853. Helpful comments from Peter Carr, Mark Ready, Seymour Smidt, JFQA referee Franklin Allen, and participants at the finance workshops at Cornell University, Indiana University, Harvard University, Syracuse University, University of Wisconsin, Virginia Polytechnic Institute of Technology, The Wharton School, and the University of Southern California are gratefully acknowledged.

[1] See Stedman (1905).
[2] See Wyckoff (1968).
[3] See Brooks (1969).
[4] See Thomas (1967).
[5] See *The Economist,* May 7, 1977, "Hunt the Soya Bean," *Wall Street Journal,* July 13, 1989, "Soybeans Sink as CBOT Acts to Avert Squeeze."
[6] See *Fortune,* July 28, 1980, "Who Guards Whom at the Commodity Exchange?"
[7] See Bailey and Ng (1988).
[8] See Cornell and Shapiro (1989).

and uniform application of the competitive markets paradigm to security pricing. This paper studies large traders and market manipulation.

A "large trader" is one whose trades change prices. To differentiate information traders from manipulators, we assume that the large trader has no information. His trades move prices only because of size or because the "other side" of the market believes (with some probability) that the large trader is informed. We study conditions (necessary and sufficient) on the price process such that when trading strategically, the large trader (called a speculator) generates profits at no risk, i.e., creates arbitrage opportunities. These are market manipulation trading strategies. For example, a market corner followed by a short squeeze is one such market manipulation strategy, but there are others. The existence or nonexistence of these trading strategies depends crucially on the sensitivity of the equilibrium price process to the history of the speculator's holdings. Under very general conditions, we show that if the price process depends on the past sequence of the large trader's holdings (as opposed to only his current holdings), then market manipulation is possible. Otherwise, it is not.

This paper generalizes and extends Hart (1977) who investigated similar questions in an infinite horizon, deterministic economy, with a time homogenous price process. Hart showed that manipulation is possible if the economy is dynamically unstable, and under certain cases even when the economy is stable. We generalize Hart to a stochastic economy, either finite or infinite horizon, with time dependent price processes. This study also provides insights into alternative, but related topics of current interest. First, market manipulation strategies provide additional examples of price destabilizing speculation, distinct from the models of Hart and Kreps (1986), Stein (1987), and DeLong, Shleifer, Summers, and Waldmann (1988). Second, it provides an alternative example to Allen and Gorton (1988) of a rational bubble in a finite horizon economy. Third, it provides a new perspective on arbitrage pricing theory.

Arbitrage pricing theory invokes the price taking paradigm. The theory of market manipulation, however, studies arbitrage when traders affect prices. This generalization requires distinguishing between "paper" wealth and "real" wealth when valuing a trader's position. Paper wealth is defined as the value of the speculator's position evaluated at the prices supported by the large trader. Real wealth, on the other hand, is the value of the large trader's position after liquidation (i.e., return to zero holdings). For a price taker, these values are identical; but for a large trader they are distinct. It is shown below that this difference is important for interpreting the conclusions of existing studies testing for the existence of arbitrage opportunities.

An outline of this paper is as follows. Section II provides the preliminaries of the model, i.e., the notation and terminology. Section III provides the basic structure of the economy studied through the introduction of four assumptions. The first assumption is frictionless trading by the large trader. The second and third assumptions imply that the large trader's holdings influence prices. The fourth assumption is that, given the large trader's information, there are no arbitrage opportunities for a price taker. Next, Section IV defines the concepts of paper wealth, real wealth, and market manipulation trading strategies. Section V shows that the previous four assumptions do not preclude the existence of market

manipulation trading strategies. Two examples are provided. The first involves a corner and a short squeeze. The second involves the trader generating a trend in prices and then trading against it. These examples provide illustrations of finite horizon bubbles and price destabilizing speculation. Section VI provides a sufficient condition on the stock price process that excludes market manipulation trading strategies. The sufficient condition is that the stock price process depends only on the large trader's aggregate holdings and not the particular sequence of trades which attained it. This section's analysis generates a criticism of the existing empirical tests for arbitrage opportunities. Section VII extends the previous analysis to infinite horizon traders. We show that if doubling strategies are excluded, then under the hypotheses of Section VI, no new market manipulation trading strategies arise. A summary section completes the paper.

II.   The Model

This section presents the basic model structure. A partial equilibrium approach is taken, common to option valuation, where the properties of the equilibrium price process itself are exogenously specified. We depart here from the option pricing literature, however, in that the large trader's actions influence prices. Consequently, exogenously determined is the functional relationship between the equilibrium price process and the large trader's trades. This approach is robust if the specified properties on the price process are not too restrictive, and they are consistent with numerous different economic constructs (economies and equilibrium notions). We will subsequently argue, through examples, that the assumptions imposed satisfy this desired robustness property.

We consider a discrete trading economy with trading dates denoted by the set $\tau = \{0, 1, 2, \ldots, T\}$. The uncertainty in the economy at date T is represented by the pair (Ω, F) where Ω is the state space and F is a σ-algebra of subsets of Ω. Before date T, partial information about the "true" state is available and represented by the filtration $\{F_t: t\epsilon\tau\}$, which is a nondecreasing sequence of σ-algebras where $\cup_{t\epsilon\tau}F_t = F$.

Two assets trade. The first is a limited liability risky asset, called a *stock*. Its price is represented by a stochastic process[9] $\{S_t: t\epsilon\tau\}$ adapted to $\{F_t: t\epsilon\tau\}$. Adapted means that the stock price at time t is part of the information available to the market at date t, i.e., S_t is F_t measurable for all $t\epsilon\tau$. In addition, we assume that $S_t \geq 0$ for all $t\epsilon\tau$ and $\omega\epsilon\Omega$. This captures the notion of limited liability. For simplicity, we assume that the stock pays no dividends before date T.

Finally, we let the risky asset be in positive supply with $N_t > 0$ total shares outstanding at each trading date $t\epsilon\tau$. We assume that $\{N_t: t\epsilon\tau\}$ is a stochastic process predictable with respect to $\{F_t: t\epsilon\tau\}$. Predictability means that the number of shares outstanding at time t, N_t, is known to the market at time $t-1$ for all $t\epsilon\tau$, i.e., N_t is F_{t-1} measurable. Changes in shares outstanding must be announced at least one period before the date of change.

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[9]A stochastic process is a mapping $S: \tau \times R \rightarrow R$.

The second asset that trades provides a riskless return over the "next" trading interval and is called a *money market fund*. Its price is represented by a predictable stochastic process $\{B_t : t\epsilon\tau\}$ initialized with a dollar investment, i.e., $B_0 \equiv 1$ for all $\omega\epsilon\Omega$. Furthermore, "interest rates" are assumed to be nonnegative in that $B_t \geq B_{t-1}$ for all $t\epsilon\tau$ and $\omega\epsilon\Omega$.

We let the money market fund be the numeraire, and subsequently utilize relative prices for the remaining analysis. This implies that the relative price of the money market fund is the constant, 1, for all $t\epsilon\tau$ and $\omega\epsilon\Omega$. Define

$$Z_t(\omega) \equiv S_t(\omega)/B_t(\omega) \text{ for all } t\epsilon\tau \text{ and } \omega\epsilon\Omega$$

as the stock's relative price. By implication, the stochastic process $\{Z_t : t\epsilon\tau\}$ is F_t-adapted and satisfies $Z_t(\omega) \geq 0$ for all $t\epsilon\tau$ and $\omega\epsilon\Omega$.[10]

Two types of traders exist. The first type of trader is represented by a singleton, called the *large trader* or *speculator*. This could be a single trader or a coalition of traders acting in unison. This trader is endowed with a probability belief $P : F \rightarrow [0, 1]$, a probability measure over (Ω, F).[11] The terminology "large trader" is justified in the next section when we characterize the speculator's trades as influencing relative asset prices.

The second type of trader is represented by an index set I. It could consist of a singleton (a single trader) or an interval (a continuum of atomistic traders). Each trader $i\epsilon I$ is endowed with a probability belief $P^i : F \rightarrow [0, 1]$, a probability measure over (Ω, F). No additional *explicit* assumptions concerning the trading behavior of this class of traders are imposed at this time. Additional *implicit* assumptions concerning their behavior will be introduced in the next section when additional structure is imposed on market prices.

The large trader's holdings of the stock and money market fund are given by a two dimensional $\{F_t : t\epsilon\tau\}$ adapted stochastic process $\{\alpha_t, \beta_t : t\epsilon\tau\}$, where α_t is the number of stocks held at time t and β_t is the number of money market fund units held at time t. The measurability condition implies that, to formulate a trading strategy, the speculator can only use the information available in the set $\{F_t : t\epsilon\tau\}$. The information sets, consequently, have the interpretation of being the large trader's information. We emphasize that the other traders in the market can have information sets distinct from $\{F_t : t\epsilon\tau\}$.

III.   The Market Structure

This section provides the formal assumptions imposed upon the economy constructed in Section II.

A1. Frictionless Markets. There are no transaction costs or short sale restrictions imposed upon the large trader's holdings $\{\alpha_t, \beta_t : t\epsilon\tau\}$.

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[10]In fact, this assumption is the starting place for the analysis, see Assumption A2 in the next section.

[11]We assume for technical considerations and without loss of generality, that the information set at time 0, F_0, contains all P-null events.

This assumption is self-explanatory. Its relaxation could easily be included into the subsequent analysis. Note, however, that we do not impose this assumption upon the other traders in the economy.

A2. The Relative Stock Price Process. There exists a sequence of functions $\{G_t\}_{t\in\tau}$ with $G_t: \Omega \times [R]^{2(t+1)} \rightarrow R$ for all $t\in\tau$ such that for any trading strategy $\{\alpha_t, \beta_t: t\in\tau\}$ of the large trader, the composition mapping $Z: \Omega \times \tau \rightarrow R$ defined by

(1) $Z_t(\omega)$ = $G_t(\omega, \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega), \beta_t(\omega), \beta_{t-1}(\omega), \ldots, \beta_0(\omega))$

          for all $\omega\in\Omega$, $t\in\tau$,

represents the stochastic process for the relative stock price. It is nonnegative with P probability one and $\{F_t: t\in\tau\}$ adapted.

This assumption summarizes the relevant exogenous restrictions imposed upon the economy in Section II. The functions $\{G_t\}_{t\in\tau}$ provide the *reduced form* equilibrium relationship between relative prices and the speculator's trades. The forms that the functions $\{G_t\}_{t\in\tau}$ assume are determined by both the specification of the economy and the equilibrium paradigm employed. The function $G_t(\cdot)$ can be interpreted as a *reaction* function (in market prices) to the speculator's trades by the remaining participants in the market. From the general theory of stochastic games, it is known that in equilibrium, the strategy of an individual player may depend upon the entire history of the strategies of the remaining players in the game (see Friedman (1986), Chapters 3 and 4). Consequently, in its most unrestricted form, the reaction function should allow this potential dependence. Therefore, at time t, the relative risky asset price Z_t is assumed to be some function of the large trader's *past and current* holdings $\{(\alpha_t, \alpha_{t-1}, \ldots, \alpha_0), (\beta_t, \beta_{t-1}, \ldots, \beta_0)\}$, as opposed to his future holdings.[12] This function also preserves limited liability and measurability.

This assumption is very mild and consistent with numerous different types of economies and different types of equilibrium constructs. Two examples are worth discussing since they help to clarify the content of the preceding assumption. The first example is similar to the model used by Hart (1977). It is based on a competitive market model with a continuum of "atomistic" traders ($I = [0, 1]$) with the exception of one large trader, an "atom," with discrete wealth. There is symmetric information, so all traders possess the same information flows over time $\{F_t: t\in\tau\}$. The beliefs are as given in Section II. The equilibrium construct is the standard Walrasian equilibrium adjusted to reflect the large trader. At the equilibrium price, the large trader's demands at each date (α_t) plus the atomistic traders' aggregate demands $(d_t(Z_t, \alpha_t))$ must equal aggregate supply (N_t), i.e., $\alpha_t + d_t(Z_t, \alpha_t) = N_t$. If there exists a unique equilibrium price and $d_t(Z_t, \alpha_t)$ is invertible in Z_t for all α_t, then $Z_t = d_t^{-1}(N_t - \alpha_t, \alpha_t)$ gives the price function in Assumption A2.

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[12]For subsequent analyses, we will often suppress the dependence on ω in the notation for prices, i.e., we will write Expression (1) as

$$G_t(\alpha_t, \alpha_{t-1}, \ldots, \alpha_0; \beta_t, \beta_{t-1}, \ldots, \beta_0).$$

Although similar to Hart (1977), Assumption A2 generalizes Hart's model in two significant ways. First, the function $G_t(\cdot)$ depends on the current date, while in Hart it does not. Second, the function $G_t(\cdot)$ depends on an exogenous random state $(\omega \epsilon \Omega)$, while in Hart it does not. Unlike Hart's analysis, this last difference allows Assumption A2 to be consistent with other equilibrium constructs.

In the above justification for Assumption A2, the speculator has market power because he has significant wealth. A speculator could also have market power because the other traders believe he may be informed. Consider an economy as in Glosten and Milgrom (1985) or Easley and O'Hara (1987) where there is the speculator, an informed trader, noise traders (a continuum), and a specialist, who clears markets. Receiving an order (α_t), the specialist does not know whether it is the informed trader, the speculator, or a subset of the "noise traders" submitting the demands. Given his priors over these possibilities, the specialist determines bid/ask prices to obtain zero expected profits (a competitive market condition). The resulting equilibrium bid or ask prices have the functional form satisfying Assumption A2. It is important to emphasize that, due to the dependence of $G_t(\cdot)$ on $\omega \epsilon \Omega$, the specialist and the other traders need not know the large trader's position (α_t, β_t). A modified version of this economy can be found in Allen and Gale (1990).

Before introducing the next assumption, we need to define the concept of a self-financing trading strategy for the large trader. For convenience, we endow the speculator with zero initial holdings in both the stock and money market fund, i.e., $\alpha_{-1} \equiv 0$, $\beta_{-1} \equiv 0$.

A trading strategy $\{\alpha_t, \beta_t : t \epsilon \tau\}$ is said to be *self-financing* if

$$(2) \qquad \beta_{t-1}(\omega) + \alpha_{t-1}(\omega)Z_t(\omega) \;=\; \beta_t(\omega) + \alpha_t(\omega)Z_t(\omega)$$

$$\text{with } P \text{ probability one for all } t \epsilon \tau.$$

The left side of Equation (2) represents the time t value of the speculator's position taken at time $t - 1$. The right side represents the time t value of the speculator's holdings at time t. The equality implies that no cash inflows or outflows are generated by the portfolio.

Denote by $R^{\Omega \times \tau}$ the set of all functions mapping $\Omega \times \tau$ into the real line R. We define Φ to be the set of all self-financing trading strategies with $\alpha_{-1} \equiv 0$ and $\beta_{-1} \equiv 0$, i.e.,

$$\Phi \;\equiv\; \Big\{ \alpha, \beta \epsilon R^{\Omega \times \tau} \colon \alpha, \beta \text{ are } \{F_t : t \epsilon \tau\} \text{ adapted, self-financing, and}$$

$$\text{satisfy } \alpha_{-1} \;=\; 0, \beta_{-1} \;=\; 0 \Big\}.$$

Given any self-financing trading strategies $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$, Expression (2) combined with Assumption A2 implies that β_0 is a function of $(\alpha_0, \alpha_{-1}, \beta_{-1})$. Since α_{-1}, β_{-1} are fixed constants, we see that β_0 is in fact a function of α_0 alone. Continuing, by induction on t, we see that β_t is a function of $(\alpha_t, \alpha_{t-1}, \ldots, \alpha_0)$. Hence, for self-financing trading strategies, we can define a new function $g_t \colon \Omega \times [R]^{t+1} \to R$ such that

$$(3) \qquad g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega)) \;=\; G_t(\omega; \alpha_t(\omega), \ldots, \alpha_0(\omega);$$
$$\beta_t(\omega), \ldots, \beta_0(\omega)),$$

for all $\omega \in \Omega$, $t \in \tau$, and $\{\alpha_t, \beta_t : t \in \tau\} \in \Phi$. This simplified notation will subsequently be employed.

In Assumption A2, although it was assumed that prices are a function of the speculator's trades, the constant function with respect to the speculator's trades characterizing a price taker, was not excluded. The next assumption completes the characterization of the speculator as a "large trader" by excluding this possibility.

A3. Speculator as a "Large Trader"

For all $t \in \tau$, $\{\alpha_t, \beta_t : t \in \tau\} \in \Phi$, and almost every $\omega \in \Omega$,

(a) if $\alpha(t, \omega) > \alpha(t - 1, \omega)$

   then $g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega)) > g_t(\omega; \alpha_{t-1}(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega))$,

(b) if $\alpha(t, \omega) < \alpha(t - 1, \omega)$ and $[\alpha(t - 1, \omega) \le N_t$ or $\alpha(t, \omega) < N_t]$,

   then $g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega)) < g_t(\omega; \alpha_{t-1}(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega))$.

Condition A3 (a) states that relative prices increase with increases in the speculator's demands, everything else constant. Condition A3 (b) states that relative prices decrease with decreases in the speculator's demands, everything else constant *except* when

$$(4) \qquad \alpha(t - 1, \omega) \;>\; \alpha(t, \omega) \;\ge\; N_t, \quad \text{i.e.,}$$

there is a *market corner* and a *short squeeze* at time $t \in \tau$ under state $\omega \in \Omega$.

For a *market corner,* the shares the speculator brings into time $t(\alpha_{t-1})$ must exceed the total supply (N_t). This is possible, since by construction, the total shares outstanding at time t, N_t, is known at time $t - 1$. For the speculator's position to exceed the total supply, short interest must be strictly positive. This means that some traders have shorted the risky asset, and effectively borrowed them from the speculator.

A *short squeeze* occurs at time t when the speculator reduces his holdings (α_{t-1} to α_t) by calling in the shorts. The shorts are called in when the speculator requires his stockbroker to provide him with the physical delivery of all his outstanding shares. This process, however, keeps his holdings greater than the total supply ($\alpha_t \ge N_t$). The shorts must return the borrowed shares, and because of the corner, they need to purchase them from the speculator. The speculator, however, can arbitrarily determine the price. Consequently, the market condition A3 (b) does not apply. These trading strategies, market corners, and short squeezes, are shown to be market manipulation trading strategies as defined in the next section *if* the price set by the speculator on the shorts squeezed is sufficiently large.

The New York Stock Exchange constitution Article VIII, section 1354, explicitly deals with corners. This section effectively enables the NYSE Board to neutralize a short squeeze by determining a "fair price" on the shares in question; but it gives no indication of what a "fair price" should be. The subsequent analysis provides an answer to this policy question.

The next assumption is designed to capture the condition that, given the speculator's information set, the market contains no arbitrage opportunities. Thus, the speculator is not trading based on information.

A4. No Arbitrage Opportunities Based on the Speculator's Information
(a) $P^i(A) = 0$ if and only if $P(A) = 0$ for all $A \epsilon F$ and $i \epsilon I$.
(b) There exists a probability measure $\overline{P}: F \rightarrow [0, 1]$ equivalent to P (i.e., $\overline{P}(A) = 0$ if and only if $P(A) = 0$ for all $A \epsilon F$) such that for all $t \epsilon \tau$ and adapted $\{\alpha_t, \beta_t : t \epsilon \tau\}$, if $\alpha_{t+1} = \alpha_t$ and $\beta_{t+1} = \beta_t$ with P probability one then,

(5) $\overline{E}\{G_{t+1}(\alpha_{t+1}, \alpha_t, \dots, \alpha_0; \beta_{t+1}, \beta_t, \dots, \beta_0) | F_t\}$

   $= G_t(\alpha_t, \alpha_{t-1}, \dots, \alpha_0; \beta_t, \beta_{t-1}, \dots, \beta_0)$ with $\overline{P}$ probability one.

Condition A4 (b) states that if the large trader's holdings are held constant over the time period $[t, t + 1]$, then there exists an equivalent probability belief that makes relative stock prices a martingale *with respect to the speculator's information*. All traders, including the speculator, agree on zero probability events by Condition A4 (a). Hence, there would be no arbitrage opportunities available to the other market participants if they had the trader's information and if they act as price takers (see Harrison and Pliska (1981) or Heath and Jarrow (1987)).[13] In the market microstructure models (e.g., Easley and O'Hara (1987)), Assumption A4 could be interpreted as a condition satisfied by the risk-neutral, zero-profit earning specialist (I is a singleton). Alternatively, in the Allen and Gale (1990) model, this assumption is satisfied by the risk-adjusted beliefs held by the investors (at time 0) and the beliefs of the arbitrageurs (at time 1).

Using the notation of Expression (3), we can rewrite Expression (5) in Assumption A4 as[14]

(6)   if $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$ and $\alpha_t = \alpha_{t+1}$ almost everywhere,

          then $\overline{E}\{g_{t+1}(\alpha_{t+1}, \alpha_t, \dots, \alpha_0) | F_t\} = g_t(\alpha_t, \alpha_{t-1}, \dots, \alpha_0)$

          with $\overline{P}$ probability one.

We utilize this formulation below.

Assumptions A1–A4 characterize the structure on the stock and bond markets studied. For subsequent analysis, we provide an example of an exogenously specified price process satisfying Assumptions A1–A4.

Example 1. (A Price Process Satisfying Assumptions A1–A4)
Let $c: \Omega \times \tau \rightarrow R$ and $Y: \Omega \times \tau \rightarrow R$ be F-measurable. Define

(7) $g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \dots, \alpha_0(\omega)) = \exp\left\{ \sum_{j=0}^{t} c_j(\omega) \left[\alpha_j(\omega) - \alpha_{j-1}(\omega) \right] \right\} + Y_t(\omega),$

for $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$ such that $\alpha_{t-1} \leq N_t$ almost everywhere for all $t \epsilon \tau$,

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[13]This assumption is the generalization of Hart's (1977) steady state condition.
[14]If $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$ and $\alpha_t = \alpha_{t+1}$ a.e., then by (2), $\beta_t = \beta_{t+1}$ almost everywhere.

where $\{c_t: t \in \tau\}$ is adapted and strictly positive with P probability one, and $\{Y_t: t \in \tau\}$ is adapted and nonnegative with P probability one.

The coefficient process $\{c_t: t \in \tau\}$ determines the sensitivity of market prices to changes in the large trader's holdings. Since these coefficients are strictly positive, Assumption A3 is satisfied.

Consider the price process when the speculator remains out of the market for all times $t \in \tau$. In this case, given $\alpha_t \equiv 0$ for all $t \in \tau$, the price process is

$$(8) \qquad g_t(\omega, 0, 0, \dots, 0) \quad = \quad 1 + Y_t(\omega).$$

One should interpret this price process as being determined by fundamentals, i.e., future cash dividends (beyond T). To generate Assumption A4, we assume first that the speculator and the other traders agree on zero probability events. Second, we assume that there exists a probability measure $\overline{P}: F \to R$ satisfying $[\overline{P}(A) = 0$ if and only if $P(A) = 0$ for all $A \in F]$ and making Y_t a $\overline{P}$-martingale with respect to $\{F_t: t \in \tau\}$, i.e., $\overline{E}(|Y_t|) < +\infty$ for all $t \in \tau$ and

$$(9) \quad \overline{E}(Y_t(\omega)|F_s) \quad = \quad Y_s(\omega) \text{ with } \overline{P} \text{ probability one for all } 0 \le s \le t \le T.$$

We note that the variance of $Y_s(\omega)$ with respect to both P and $\overline{P}$ (if it exists) is unrestricted and could be any F_s-measurable function of $\{\alpha_t: t \in \tau\}$.

To see that this price process satisfies Assumption A4(b), we show that Condition (6) holds. Indeed,

$$\overline{E}\left\{g_{t+1}(\alpha_t, \alpha_t, \alpha_{t-1}, \dots, \alpha_0)|F_t\right\} \quad = \quad \overline{E}\left\{\exp\left\{\sum_{j=0}^{t} c_j(\alpha_j - \alpha_{j-1})\right\} + Y_{t+1}(\omega)|F_t\right\}$$

$$= \quad \exp\left\{\sum_{j=0}^{t} c_j(\alpha_j - \alpha_{j-1})\right\} + \overline{E}\left\{Y_{t+1}(\omega)|F_t\right\} \quad = \quad g_t(\alpha_t, \alpha_{t-1}, \dots, \alpha_0).$$

This completes the discussion of the example. ///

IV. Paper Wealth, Real Wealth, and Market Manipulation Trading Strategies

This section defines the concept of a *market manipulation trading strategy* for the speculator. Before that, however, we need to make a distinction between *paper wealth* and *real wealth*. The speculator's paper wealth is defined to be the value of his portfolio position when relative prices are evaluated using his current holdings. Real wealth, on the other hand, is defined to be the value of the speculator's position when relative prices are evaluated as if his stock holdings were liquidated. Real wealth is the amount of wealth the speculator could "consume" in terms of the numeraire (money market account).

Definition. (Paper Wealth)

Time t *paper wealth* at state $\omega \in \Omega$ of the portfolio position $\{\alpha_t, \beta_t: t \in \tau\} \in \Phi$ is defined by

$$(10) \qquad W_t(\omega) \quad \equiv \quad \alpha_{t-1}(\omega)g_t(\omega; \alpha_{t-1}(\omega), \alpha_{t-1}(\omega), \alpha_{t-2}(\omega), \dots, \alpha_0(\omega))$$

$$+ \beta_{t-1}(\omega) \text{ for all } t \in \tau.$$

This corresponds to the time t value of the speculator's position taken at time $t-1$.

Definition. (Real Wealth)

Time t *real wealth* at state $\omega\epsilon\Omega$ of the portfolio position $\{\alpha_t, \beta_t : t\epsilon\tau\}\epsilon\Phi$ is defined by

(11) $V_t(\omega) \equiv \alpha_{t-1}(\omega)g_t(\omega; 0, \alpha_{t-1}(\omega), \alpha_{t-2}(\omega), \ldots, \alpha_0(\omega)) + \beta_{t-1}$ for all $t\epsilon\tau$.

This corresponds to the time t value of the speculator's position after liquidating his stock holdings ($\alpha_t(\omega) = 0$).

The relationship between real wealth and paper wealth follows easily by solving for β_{t-1} in Expression (10) and substituting the result into Expression (11).

(12) $V_t(\omega) \quad = \quad W_t(\omega) + \alpha_{t-1}(\omega) \left[g_t(\omega; 0, \alpha_{t-1}(\omega), \alpha_{t-2}(\omega), \ldots, \alpha_0(\omega)) \right.$
$\left. - g_t(\omega; \alpha_{t-1}(\omega), \alpha_{t-1}(\omega), \alpha_{t-2}(\omega), \ldots, \alpha_0(\omega)) \right]$

for all $t\epsilon\tau$ and almost every $\omega\epsilon\Omega$.

The difference between real and paper wealth is due to the difference in the price of the stock at time $t-1$ under two different portfolio positions: those supported by the large trader ($g_t(\alpha_{t-1}, \alpha_{t-1}, \alpha_{t-2}, \ldots, \alpha_0)$), and those not ($g_t(0, \alpha_{t-1}, \ldots, \alpha_0)$).

The following lemma follows directly from (12) and Assumption A3. It states that real wealth is strictly less than paper wealth if and only if the large trader's risky asset position is nonzero.

Lemma 1. Given Assumptions A1–A3 and $\{\alpha_t, \beta_t : t\epsilon\tau\}\epsilon\Phi$,

$$V_t(\omega) \quad < \quad W_t(\omega) \text{ if and only if } \alpha_{t-1}(\omega) \neq 0.$$

This is true for almost every $\omega\epsilon\Omega$.

This distinction between paper wealth and real wealth is important in defining market manipulation trading strategies.

Definition. (A Market Manipulation Trading Strategy)

A *market manipulation trading strategy* is defined to be any zero initial wealth self-financing trading strategy $\{\alpha_t, \beta_t : t\epsilon\tau\}\epsilon\Phi$, such that

(13a) $V_T \geq 0$ with P probability one, and

(13b) $P(V_T > 0) > 0$.

Since the trading strategy $\{\alpha_t, \beta_t : t\epsilon\tau\}\epsilon\Phi$, it has zero initial wealth, i.e., $V_0 = 0$. Conditions (13a) and (13b) require that the real wealth of the trading strategy at liquidation (when $\alpha_T = 0$) is nonnegative for sure, and strictly positive with positive probability. Hence, a market manipulation trading strategy has a positive probability of generating positive real wealth with no losses from a zero initial investment. As such, this definition generalizes the standard concept of an arbitrage opportunity given price takers. Indeed, the only distinction between

the two definitions is the replacement of paper wealth, W_T, by real wealth, V_T, as in Expression (13).[15]

The primary purpose of the remaining analysis is to investigate whether market manipulation trading strategies exist under Assumptions A1–A4. Given the absence of arbitrage opportunities with the large trader's information (Assumption A4), market manipulation trading strategies exist, if at all in the above economy, because of the speculator's market power. Indeed, for a price-taking speculator, Assumption A3 would be replaced by the condition that $g_t(\alpha_t, \alpha_{t-1}, \ldots, \alpha_0)$ is independent of $(\alpha_t, \alpha_{t-1}, \ldots, \alpha_0)$. Then, by Expression (12), real wealth equals paper wealth, i.e., $V_T = W_T$. This in turn implies, by Assumption A4 and the analysis in Harrison and Pliska (1981), that no market manipulation trading strategies exist. We state this result as our first proposition.

Proposition 1. (Nonexistence of Market Manipulation Trading Strategies for Price Takers)

Under Assumptions A1, A2, and A4 where for all $t\epsilon\tau$, $\{\alpha_t, \beta_t : t\epsilon\tau\}\epsilon\Phi$, and almost every $\omega\epsilon\Omega$,

$$g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega)) \;=\; g_t(\omega),$$

then no market manipulation trading strategies exist.

V. The Existence of Market Manipulation Trading Strategies

This section of the paper studies the existence of market manipulation trading strategies under Assumptions A1–A4. The analysis proceeds through a series of propositions and examples, demonstrating the existence of market manipulation trading strategies.

The next proposition studies a simple two-period economy with $T = 1$. In such an economy, as Proposition 2 shows, no market manipulation strategies exist under Assumptions A1–A4; not even market corners and short squeezes.

Proposition 2. (Two Period Trading Strategies)

Under Assumptions A1–A4, if $T = 1$ then no market manipulation trading strategies exist.

Proof. By (12), $V_1 = \alpha_0[g_1(0, \alpha_0) - g_1(\alpha_0, \alpha_0)] + W_1$ where $W_1 = \alpha_0 g_1(\alpha_0, \alpha_0) + \beta_0$. But by Expression (2), $\beta_0 = -\alpha_0 g_0(\alpha_0)$, hence,

$$W_1 \;=\; \alpha_0 \left[g_1(\alpha_0, \alpha_0) - g_0(\alpha_0)\right].$$

By Assumption A4, $\overline{E}(W_1) = \alpha_0 \overline{E}(g_1(\alpha_0, \alpha_0) - g_0(\alpha_0)) = 0 = W_0$. Hence, paper wealth is a $\overline{P}$-martingale. Furthermore, by Assumption A3,

$$\alpha_0 \left[g_1(0, \alpha_0) - g_1(\alpha_0, \alpha_0)\right] \;\leq\; 0 \text{ for all } \alpha_0 \epsilon R.$$

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[15]An alternative, weaker definition of a market manipulation trading strategy is explored in Section VII.

Thus, V_1 is a supermartingale, i.e., $\overline{E}(V_1) \leq 0$ for all $\{\alpha_0, \beta_0\}\epsilon\Phi$. No market manipulation trading strategies can exist since they must satisfy $\overline{E}(V_1) > 0$ by Expression (13). □

The reasoning underlying this proposition is straightforward. In a two-period economy, a market manipulation trading strategy requires either a purchase followed by a liquidation or a short followed by a cover. In either case, the large trader is in the market only for one time period. Prior to and after his entry, prices are fair (i.e., they follow a $\overline{P}$-martingale by Assumption A4). Due to Assumption A3, when the speculator purchases, he buys at too high a price relative to what's fair. When he sells, he sells at too low a price relative to what's fair. The speculator is playing in an unfair game, hence, no market manipulation strategies can exist.

In fact, in Proposition 2 above, the speculator cannot even create a market manipulation strategy by cornering the market and then squeezing the shorts. Although he can corner the market by purchasing $\alpha_0 > N_1$ shares (recall that N_1 is known at time 0), he cannot squeeze the shorts since $\alpha_1 = 0$ (see Condition (4)). To squeeze the shorts, the large trader needs at least one more time period in which to trade.[16]

The next proposition extends the results of Proposition 2 to three trading periods, however, the hypotheses of the proposition are more restrictive. No market manipulation trading strategies can only be assured for a *proper subset* of all possible trading strategies.

Proposition 3. (Three-Period Trading Strategies)
Under Assumptions A1–A4, if $T = 2$, then no market manipulation trading strategies exist with

$$[N_1 \geq \alpha_0 \geq \alpha_1 \geq \alpha_2 \ = \ 0 \text{ almost everywhere } P]$$

$$\text{or} \quad [\alpha_0 \leq \alpha_1 \leq \alpha_2 \ = \ 0 \text{ almost everywhere } P].$$

Proof. By (12), $V_2 \ = \ \alpha_1[g_2(0, \alpha_1, \alpha_0) - g_2(\alpha_1, \alpha_1, \alpha_0)] + W_2$ where $W_2 = \alpha_1 g_2(\alpha_1, \alpha_1, \alpha_0) + \beta_1$. Using Lemma A.1 in the Appendix yields

$$W_2 \ = \ \alpha_1 g_2(\alpha_1, \alpha_1, \alpha_0) - \alpha_0 g_0(\alpha_0) - [\alpha_1 - \alpha_0] g_1(\alpha_1, \alpha_0).$$

By algebra and adding and subtracting $\alpha_0 g_1(\alpha_0, \alpha_0)$, we get

$$W_2 \ = \ \alpha_0 \left[g_1(\alpha_1, \alpha_0) - g_1(\alpha_0, \alpha_0)\right] + \alpha_1 \left[g_2(\alpha_1, \alpha_1, \alpha_0) - g_1(\alpha_1, \alpha_0)\right]$$
$$+ \alpha_0 \left[g_1(\alpha_0, \alpha_0) - g_0(\alpha_0)\right].$$

By Assumption A3, the first term in W_2 is nonpositive since either

$$\left[N_1 \geq \alpha_0 \geq \alpha_1 \geq 0 \text{ and } g_1(\alpha_1, \alpha_0) - g_1(\alpha_0, \alpha_0) \leq 0\right]$$

$$\text{or} \quad \left[\alpha_0 \leq \alpha_1 \leq 0 \text{ and } g_1(\alpha_1, \alpha_0) - g_1(\alpha_0, \alpha_0) \geq 0\right].$$

---

[16]This assertion follows directly from Assumption A3 (b), which implies that a single price is obtained on all shares traded when going from α_{t-1} to α_t. If one price could be obtained for the squeezed shorts, and another for the remaining shares, then this proposition would not hold. However, inserting a fictitious time interval between times 0 and 1 would incorporate this type of price discrimination into the above model.

Hence, $\overline{E}(W_2) \le 0$, because the expectation of the last two terms in W_2 is zero by Assumption A4. Using Assumption A3 again on the first term in V_2 yields

$$\alpha_1 \left[g_2(0, \alpha_1, \alpha_0) - g_1(\alpha_1, \alpha_1, \alpha_0) \right] \quad \le \quad 0.$$

This implies $\overline{E}(V_2) \le 0$ for all $\{\alpha_1, \alpha_0, \beta_1, \beta_0\} \epsilon \Phi$ such that

$$[N_1 \ge \alpha_0 \ge \alpha_1 \ge \alpha_2 \quad = \quad 0 \quad \text{almost everywhere} \quad P]$$

$$\text{or} \quad [\alpha_0 \le \alpha_1 \le \alpha_2 \quad = \quad 0 \text{ almost everywhere } P].$$

No market manipulation strategies can exist in this subclass since they must satisfy $\overline{E}(V_2) > 0$ by Expression (13). □

Proposition 3 provides additional trading strategies that will not manipulate the market: large purchases followed by "slow" liquidations (without a market corner), or large shorts followed by "slow" covers. The reasoning behind this result is the same as in Proposition 2. The speculator, by his trading strategy, guarantees that he is playing in an unfair game. He is always either purchasing too high relative to what's fair or he is selling too low relative to what's fair. To manipulate the market, the speculator needs to either corner it (get $\alpha_0 > N_1$) or generate a trend in prices and the sell against it ([$0 < \alpha_0 < \alpha_1$ then $\alpha_2 = 0$] or [$0 > \alpha_0 > \alpha_1$ then $\alpha_2 = 0$]). Both situations are possible, without additional assumptions on the price process, as the following two examples show.

Example 2. (A Market Corner and Short Squeeze)
This example demonstrates that market manipulation trading strategies exist under Assumptions A1–A4. One such trading strategy exists because the speculator can avoid the market equilibrium process dictated by Assumption A3 (b). He does this by cornering the market and then squeezing the shorts. Indeed, consider the following trading strategy

(14) $\alpha_0 \quad = \quad N_1 + 1 \quad$ (Corner plus 1 share),

$\alpha_1 \quad = \quad N_1 \quad$ (Squeeze the shorted share),

$\alpha_2 \quad = \quad 0 \quad$ (Liquidate).

This strategy is feasible since the total shares outstanding, N_1, is known at time 0.

The real wealth of the speculator under this strategy at time 2 is given by Expression (11), rewritten here as

(15) $V_2 \quad = \quad N_1 g_2(0, N_1, N_1 + 1) + \beta_1.$

By the self-financing Condition (2) or Lemma A1 in the Appendix,

(16) $\beta_1 \quad = \quad -N_1 g_1(N_1, N_1 + 1) + (N_1 + 1)g_1(N_1, N_1 + 1) - (N_1 + 1)g_0(N_1 + 1).$

Substitution and algebra generate

(17) $V_2 \quad = \quad N_1 g_2(0, N_1, N_1 + 1) + g_1(N_1, N_1 + 1) - (N_1 + 1)g_0(N_1 + 1).$

The first term on the right side of Expression (17) is the real wealth received at final liquidation. The second term is the real wealth obtained on the short squeeze (the 1 share called in), and the third term is the real wealth paid out at time 0 to obtain the initial position. The sum of these three quantities is the speculator's real wealth at time 2.

Because the market equilibrium process is avoided by a short squeeze, the large trader can *arbitrarily* choose a value for $g_1(N_1, N_1 + 1)$ that the short must pay. Given any positive number, $M > 0$ since $N_1 g_2(0, N_1, N_1 + 1) \geq 0$ almost everywhere P by limited liability, the speculator, by requiring

$$(18) \qquad g_1(N_1, N_1 + 1) \quad \geq \quad (N_1 + 1)g_0(N_1 + 1) + M,$$

guarantees $V_2 \geq M > 0$ with P probability one. This proves that the trading strategy (14) under this choice for $g_1(N_1, N_1 + 1)$ satisfies Condition (13) and is thus a market manipulation trading strategy.

One characteristic of this trading strategy is important to emphasize. It generates a sequence of market prices starting from $g_0(N_1 + 1)$ moving up to $g_1(N_1, N_1 + 1)$ and then down to $g_2(0, N_1, N_1 + 1)$. This manifests itself as a *bubble* followed by a *crash*. Hence, this market manipulation strategy provides an alternative example of a finite horizon bubble (see Allen and Gorton (1988)) and an additional example of destabilizing price speculation (see Hart and Kreps (1986) or Stein (1987)).

Two considerations thus arise. Given rational traders, why would investors ever allow a short squeeze to take place? First, the short trader may not realize the market is cornered, because he cannot observe the speculator's trades. In this circumstance, however, because of the additional risk involved in the short, the trader may require an added risk premium. Nonetheless, short positions are still taken. Second, it may be that the speculator has special information about a *technical corner,* rather than an *actual corner,* which the other traders do not share. A technical corner occurs when the speculator's holdings exceed the floating supply, those shares available for sale, and the floating supply is less than the actual supply of shares outstanding. This can occur, for example, because shares (or supplies) may sit in trusts or escrow accounts that cannot (or will not) be sold. Such situations appear possible, see Cornell and Shapiro (1989).

As this trading strategy makes clear, the profits come at the expense of the single trader whose short is called in. This is possible because the market equilibrium process (Assumption A3) is avoided. Adding additional restrictions upon the price process itself will not exclude these strategies. As discussed in Section 3, NYSE legislation sends a short squeeze to arbitration if agreement cannot be reached by the parties involved. The arbitrator is the NYSE Board, and they have the power to determine a "fair price." For policy considerations, what should the fair price be?

To eliminate manipulation profits, it is sufficient to "set" $g_1(N_1, N_1 + 1)$, the fair price, equal to $g_0(N_1 + 1)$ or $g_2(0, N_1, N_1 + 1)$. Given this determination, ex ante real wealth under the previous Strategy (14) changes to

(19) $\tilde{V}_2$ = $N_1 \left[g_2(0, N_1, N_1 + 1) - g_0(N_1 + 1) \right]$ or

= $[N_1 + 1] \left[g_2(0, N_1, N_1 + 1) - g_0(N_1 + 1) \right]$, respectively.

This does not guarantee manipulative profits. In fact, if $g_2(0, N_1, N_1 + 1) \leq g_2(N_1+1, N_1+1, N_1+1)$ almost everywhere, then $\overline{E}(\tilde{V}_2) \leq 0$ under either condition since

(20) $\overline{E} \left[g_2(N_1 + 1, N_1 + 1, N_1 + 1) - g_1(N_1 + 1, N_1 + 1) \right.$

$\left. + g_1(N_1 + 1, N_1 + 1) - g_0(N_1 + 1) \right]$ = 0,

by Assumption A4. Here, the trading strategy given by (14) is an unfair gamble. The condition that $g_2(0, N_1, N_1 + 1) \leq g_2(N_1 + 1, N_1 + 1, N_1 + 1)$ with P probability one is consistent with Assumption A3 (b) and Assumption A5, which is introduced in the next section. This completes Example 2. ///

Example 3. (Establishing a Trend and Trading against It)

This example provides an alternative market manipulation trading strategy under Assumptions A1–A4 that does not involve a market corner. As such, it is perhaps a more interesting example. It is characterized by the speculator using his market power to create a trend (a bubble), and then selling against the trend before it collapses.

The price process utilized is that provided in Example 1, Expression (7), where it was shown that the price process satisfies Assumptions A1–A4. For this market manipulation strategy, it is sufficient to consider only a three-period economy $(T = 2)$. We further specialize the price process of Expression (7) by setting $c_0 \equiv c_1 \equiv 1$, i.e., for any $(\alpha_2, \alpha_1, \alpha_0; \beta_2, \beta_1, \beta_0) \epsilon \Phi$ with $\alpha_t \leq N_{t+1}$ for all $t \epsilon \{0, 1\}$,

(21) $g_0(\omega, \alpha_0)$ = $e^{\alpha_0} + Y_0(\omega),$

$g_1(\omega, \alpha_1, \alpha_0)$ = $e^{\alpha_1} + Y_1(\omega),$ and

$g_2(\omega, \alpha_2, \alpha_1, \alpha_0)$ = $e^{c_2(\omega)(\alpha_2 - \alpha_1) + \alpha_1} + Y_2(\omega)$

where $0 < c_2(\omega) < 1$ with P probability one.

By construction, the sensitivity of time 2 prices to the speculator's trades at time 2, $0 < c_2(\omega) < 1$, is strictly less than the price sensitivity $(c_1 = c_0 = 1)$ in the prior periods.

Given this scenario, the speculator can potentially generate manipulative profits by bidding up the price $(0 < \alpha_0 < \alpha_1)$ then selling at time 2 $(\alpha_2 = 0)$. Following a self-financing trading strategy, the speculator's real wealth at time 2 is (see the proof of Proposition 3)

(22) V_2 = $\alpha_1 g_2(0, \alpha_1, \alpha_0) - [\alpha_1 - \alpha_0] g_1(\alpha_1, \alpha_0) - \alpha_0 g_0(\alpha_0).$

The first term on the right side of Expression (22) is the gain from the trading strategy. The second and third terms represent the total costs of attaining the position α_1 at time 1 by purchasing α_0 shares at time 0 and $[\alpha_1 - \alpha_0]$ additional shares at time 1. Substituting Expression (21) into Expression (22) yields

(23) V_2 = $\alpha_1 \left[e^{\alpha_1(1 - c_2(\omega))} + Y_2(\omega) \right] - (\alpha_1 - \alpha_0) \left[e^{\alpha_1} + Y_1(\omega) \right] - \alpha_0 \left[e^{\alpha_0} + Y_0 \right].$

If at time 1, α_1 can be chosen so that

$$(24a) \quad P\left\{\alpha_1(1 - c_2(\omega)) \geq \text{Log}\left[\left(1 - \frac{\alpha_0}{\alpha_1}\right)(e^{\alpha_1} + Y_1(\omega)) + \frac{\alpha_0}{\alpha_1}(e^{\alpha_0} + Y_0)\right]\right\} = 1,$$

and

$$(24b) \quad P\left\{\alpha_1(1 - c_2(\omega)) > \text{Log}\left[\left(1 - \frac{\alpha_0}{\alpha_1}\right)(e^{\alpha_1} + Y_1(\omega)) + \frac{\alpha_0}{\alpha_1}(e^{\alpha_0} + Y_0)\right]\right\} > 0,$$

then market manipulation profits exist. Indeed,

$$(25) \quad V_2 \geq \alpha_1 e^{\alpha_1(1 - c_2(\omega))} + (-\alpha_1 + \alpha_0)(e^{\alpha_1} + Y_1(\omega)) - \alpha_0(e^{\alpha_0} + Y_0),$$

with P probability one, since $\alpha_1 > 0$ and $Y_2(\omega) \geq 0$ almost everywhere P. But by Expression (24), with P probability 1, the right side of Expression (25) is nonnegative. Further, with positive probability, it is strictly positive. Hence, the conditions for a market manipulation trading strategy (Condition (13)) are satisfied.

To show that Condition (24) is not vacuous, let $Y_2 \equiv Y_1 \equiv Y_0 \equiv 0$, and $c_2 = (1/30)$. This corresponds to a deterministic economy. The choice $\alpha_0 = 1$, $\alpha_1 = 3$, $\alpha_2 = 0$ satisfies Condition (24) since

$$3(1 - 1/30) > \text{Log}\left[\left(1 - \frac{1}{3}\right)e^3 + \left(\frac{1}{3}\right)e^1\right] \quad \text{if and only if}$$

$$e^{29/10} = 18.174 > (2/3)e^3 + (1/3)e^1 = 14.30.$$

In this deterministic example, prices go from $e^1 = 2.7183$ at time 0 to $e^3 = 20.086$ at time 1, and crash to $e^{29/10} = 18.174$ at time 2. This completes Example 3. ///

This example provides another illustration of a finite horizon bubble and price destabilizing speculation distinct from Allen and Gorton (1988), Hart and Kreps (1986), or Stein (1987). It also captures the essence of the price desta-bilizing speculation in Delong, Shleifer, Summers, and Waldmann (1988). The price process in their model exhibits differences in the intertemporal price sensitivity as illustrated above, due to noise traders ($i\epsilon I$) following positive feedback investment strategies. That is, as the price rises, noise traders buy with a lag (as price falls, they sell with a lag).

This trend-creating strategy can, in fact, be the outcome from a rational equilibrium. This is the essence of the manipulation equilibrium contained in the takeover model of Bagnoli and Lipman (1989) and the raider model of Allen and Gale (1990). In the Bagnoli and Lipman (1989) model, a monopolistic bidder imitates a serious bidder only to increase the stock's price, after which he sells his shares and drops the bid, making manipulative profits. Similarly, in the Allen and Gale (1990) model, an uninformed raider mimics an informed raider to raise the stock price, and then sells his shares at a profit.

More broadly, numerous market phenomena could potentially generate differences in the price sensitivity coefficients ($c_t: t\epsilon\tau$) of Example 3 favorable for market manipulation. For example, program trading caused by portfolio insurers could induce similar patterns. Alternatively, these could be caused by large

anticipated changes in aggregate demand or supply like the increase in demand due to (in the money) equity call options being exercised. The critical condition for potential market manipulation, as shown in the next section, is the dependence of prices on the past sequence of the speculator's holdings as opposed to just his current holdings.

VI. Sufficient Conditions for the Nonexistence of Market Manipulation Trading Strategies

This section studies conditions sufficient to guarantee the nonexistence of market manipulation trading conditions with the exception of corners and short squeezes. The condition is motivated by Example 3 of the previous section. In Example 3, market manipulation trading strategies exist because the price process depends on the past sequence of the speculator's holdings, and not just his current position.

A5. Price Process Independence of the Speculator's Past Holdings

For all $t \epsilon \tau$, almost every $\omega \epsilon \Omega$, and $\{\alpha_t, \beta_t : t \epsilon \tau\}$, $\{\overline{\alpha}_t, \overline{\beta}_t : t \epsilon \tau\} \epsilon \Phi$,

$$(26) \quad \text{if} \quad \alpha(t, \omega) = \overline{\alpha}(t, \omega) \quad \text{then} \quad g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \dots, \alpha_0(\omega))$$
$$= g_t(\omega; \overline{\alpha}_t(\omega), \overline{\alpha}_{t-1}(\omega), \dots, \overline{\alpha}_0(\omega)).$$

This assumption states that time t prices are "independent" of the path of the speculator's previous holdings. Equivalently, the price process is a function of α_t alone, i.e.,

$$(27) \quad g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \dots, \alpha_0(\omega)) = g_t(\omega; \alpha_t(\omega)).$$

To simplify notation, whenever Assumption A5 holds, we will utilize this form of the price process.

Assumption A5 has a strong implication.

Lemma 2. (Prices Are Martingales for Constant Speculator Holdings)

Given Assumptions A1–A5 and any self-financing trading strategy $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$ such that $\alpha_{t-1} \leq N_t$ almost everywhere P for all $t \epsilon \tau$ then,

$$\overline{E}(g_t(\alpha_s)|F_s) = g_s(\alpha_s) \quad \text{with } \overline{P} \text{ probability one for all } 0 \leq s \leq t \leq T.$$

Proof.

$\overline{E}(g_t(\alpha_s)|F_s)$

$= \overline{E}(g_t(\alpha_s, \alpha_s, \dots, \alpha_s, \alpha_{s-1}, \alpha_{s-2}, \dots, \alpha_0)|F_s)$ by Assumption A5

$= \overline{E}(\overline{E}(g_t(\alpha_s, \alpha_s, \dots, \alpha_s, \alpha_{s-1}, \dots, \alpha_0)|F_{t-1})|F_s)$

$= \overline{E}(g_{t-1}(\alpha_s, \dots, \alpha_s, \alpha_{s-1}, \dots, \alpha_0)|F_s)$ by Assumption A4

$\vdots$

$= \overline{E}(g_{s+1}(\alpha_s, \alpha_s, \alpha_{s-1}, \dots, \alpha_0)|F_s)$

$= g_s(\alpha_s, \alpha_{s-1}, \dots, \alpha_0)$ by Assumption A4

$= g_s(\alpha_s)$ by Assumption A5. $\square$

This lemma states that the expected future price (with respect to the "risk-neutral" probability measure) for constant holdings of the speculator, is the same as the current price. Recall that the speculator starts out with $\alpha_{-1} = 0$ shares at time 0 at a price of $g_0(0)$. In a market manipulation strategy, he must return to zero shares at time T at a price of $g_T(0)$. This lemma asserts that the expected price at time T (with respect to the "risk-neutral" probability $\overline{P}$) is the same as the current price, i.e.,

$$\overline{E}(g_t(0)) \quad = \quad g_0(0).$$

Consequently, the market appears to provide a fair gamble to the speculator. This should preclude the existence of market manipulation trading strategies.[17] This is indeed the case as the following proposition shows.

Proposition 4. (Sufficient Conditions for the Nonexistence of Market Manipulation Trading Strategies)

Given Assumptions A1–A5 and self-financing trading strategies $\{\alpha_t, \beta_t: t\epsilon\tau\}\epsilon\Phi$ such that $\alpha_{t-1} \leq N_t$ with P probability one for all $t\epsilon\tau$, there exist no market manipulation trading strategies.

Proof. In the Appendix.

Due to its size, the proof is delegated to the Appendix. The idea of the proof, however, is simple. We use Lemma 2 and Assumption A5 to reduce the analysis to that of Proposition 2, from which the result follows directly.

This proposition and the previous examples (2 and 3) provide a characterization for the existence of market manipulation trading strategies. First, Example 2 demonstrates that Assumptions A1–A4 are not sufficient to exclude market corners and short squeezes. This occurs because the speculator avoids the market process (Assumption A3) by utilizing such a strategy. Hence, the need for restricting his aggregate holdings to $\alpha_{t-1} \leq N_t$ for all $t\epsilon\tau$ by market regulations is justified. Secondly, Example 3 demonstrates that an additional condition beyond Assumptions A1–A4 is needed to exclude market manipulation trading strategies. The sufficient condition provided in Assumption A5 is useful because it can be subjected to empirical verification by the large trader himself.

Although Assumptions A1–A5 preclude market manipulation trading strategies (with the exception of market corners and short squeezes), they do not preclude bubbles and crashes as the following example shows.

Example 4. (Bubbles, Crashes, and Arbitrage Opportunities)

This example serves two purposes. First, it provides an example of a stochastic price process satisfying Assumptions A1–A5. Second, it clarifies the distinction between paper wealth and real wealth by demonstrating that the speculator can design a trading strategy to generate paper wealth of any desired level, even when this is not possible for real wealth.

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[17]It is interesting to note that the separating nonmanipulation equilibrium price process in Allen and Gale (1990) satisfies Assumption A5.

This example is based on the price process of Example 1 with coefficients $c_t(\omega) \equiv c > 0$, a positive constant for all $t\epsilon\tau$, i.e.,

$$(28) \qquad g_t(\alpha_t, \alpha_{t-1}, \ldots, \alpha_0) = e^{c\alpha_t} + Y_t(\omega) \quad \text{for all } t\epsilon\tau.$$

That this process satisfies Assumptions A1–A4 was proven in Example 1. That it satisfies Assumption A5 is true by inspection.

Paper wealth for this speculator, under Expression (28), (as given by Lemma A2 in the Appendix) can be written as

$$(29) \quad W_t = \alpha_{t-1}\left[e^{c\alpha_{t-1}} + Y_t - e^{c\alpha_{t-1}} - Y_{t-1}\right] + \sum_{j=0}^{t-2} \alpha_j\left[e^{c\alpha_{j+1}} + Y_{j+1} - e^{c\alpha_j} - Y_j\right]$$

$$= \alpha_{t-2}\left[e^{c\alpha_{t-1}} - e^{c\alpha_{t-2}}\right] + \sum_{j=0}^{t-3} \alpha_j\left[e^{c\alpha_{j+1}} - e^{c\alpha_j}\right] + \alpha_{t-1}\left[Y_t - Y_{t-1}\right]$$

$$+ \sum_{j=0}^{t-2} \alpha_j\left[Y_{j+1} - Y_j\right].$$

Under this price process, the speculator can make time t paper wealth at least as large as any arbitrarily specified constant $M > 0$ with P probability one. Indeed, this can be attained as follows. Let α_j for $j = 0, \ldots, t-3$ be arbitrary. Next, let $\alpha_{t-2} > 0$ and $\alpha_{t-1} > 0$. The exact magnitude of α_{t-1} will be determined shortly. Note that for $\alpha_{t-1} > 0$,

$$(30) \quad W_t \geq \alpha_{t-2}e^{c\alpha_{t-1}} - \alpha_{t-1}Y_{t-1}$$

$$+ \left[-\alpha_{t-2}e^{c\alpha_{t-2}} + \sum_{j=0}^{t-3} \alpha_j\left[e^{c\alpha_{j+1}} - e^{c\alpha_j}\right] + \sum_{j=0}^{t-2} \alpha_j\left[Y_{j+1} - Y_j\right]\right],$$

with P probability one since $Y_t \geq 0$ with P probability one.

At time $t-1, Y_{t-1}$ and the last term in Expression (30) are known. Hence, to reach the paper wealth of M, choose α_{t-1} so that

$$(31) \quad \alpha_{t-2}e^{c\alpha_{t-1}} - \alpha_{t-1}Y_{t-1} \geq \left[-\alpha_{t-2}e^{c\alpha_{t-2}} + \sum_{j=0}^{t-3} \alpha_j\left[e^{c\alpha_{j+1}} - e^{c\alpha_j}\right]\right.$$

$$\left. + \sum_{j=0}^{t-2} \alpha_j\left[Y_{j+1} - Y_j\right]\right] + M.$$

This is possible since

$$\lim_{\alpha_{t-1}\to+\infty} \frac{\alpha_{t-2}e^{c\alpha_{t-1}}}{\alpha_{t-1}Y_{t-1}} = +\infty.$$

Thus, we have shown that for this trading strategy, $W_t \geq M > 0$ with P probability one.

Paper wealth, and the trend of market prices, is controlled by the speculator, even when (by Proposition 4) there are no market manipulation trading strategies

available (with the exception of market corners). To illustrate Proposition 4, we next show that market manipulation strategies do not exist for this example. Consider the strategy $\alpha_j > \alpha_{j-1} > 0$ for $j\epsilon\{1, 2, \ldots, t\}$ (any other strategy would work as well, only the algebra becomes more difficult).

By Expression (11),

$$(32) \quad V_t = \alpha_{t-1}\left(1 + Y_t - e^{c\alpha_{t-1}} - Y_t\right) + W_t$$

$$= \alpha_{t-1}\left(1 - e^{c\alpha_{t-1}}\right) + \sum_{j=0}^{t-2}\alpha_j\left[e^{c\alpha_{j+1}} - e^{c\alpha_j}\right] + \sum_{j=0}^{t-2}\alpha_j\left[Y_{j+1} - Y_j\right].$$

The sum of the first two terms in Expression (32) is negative, i.e.,

$$\alpha_{t-1}\left(1 - e^{c\alpha_{t-1}}\right) + \sum_{j=0}^{t-2}\alpha_j\left[e^{c\alpha_{j+1}} - e^{c\alpha_j}\right]$$

$$\leq \alpha_{t-1}\left(1 - e^{c\alpha_{t-1}}\right) + \alpha_{t-1}\sum_{j=0}^{t-2}\left[e^{c\alpha_{j+1}} - e^{c\alpha_j}\right] = \alpha_{t-1}\left(1 - e^{c\alpha_0}\right) < 0,$$

since $c\alpha_0 > 0$. Hence, because $\{Y_t: t\epsilon\tau\}$ is a $\overline{P}$-martingale,

$$\overline{E}[V_t] \leq \alpha_{t-1}\left(1 - e^{c\alpha_0}\right) + \overline{E}\left\{\sum_{j=0}^{t-2}\alpha_j\left[Y_{j+1} - Y_j\right]\right\}$$

$$= \alpha_{t-1}\left(1 - e^{c\alpha_0}\right) + \overline{E}\left\{\sum_{j=0}^{t-2}\alpha_j\left[\overline{E}\left(Y_{j+1} - Y_j|F_j\right)\right]\right\}$$

$$= \alpha_{t-1}\left(1 - e^{c\alpha_0}\right) < 0.$$

This strategy, therefore, cannot be a market manipulation strategy since it violates Condition (13).

Note that for this strategy $\overline{E}(W_t) \geq M > 0$ if α_{t-1} satisfies Expression (31), yet $\overline{E}(V_t) < 0$! Paper wealth and real wealth are distinct. Paper wealth is under the control of the speculator, while real wealth is not. Since paper wealth and observed market prices are related, the speculator can also control market prices (all but Y_t). This completes Example 4. ///

Although Proposition 4 excludes market manipulation trading strategies, it does not necessarily exclude trades that increase the *expected utility* of the speculator above that which he would receive if he did not trade. In fact, however, if one adds an additional assumption concerning the speculator's opportunities, then Assumption A5 does exclude expected utility increasing trades. Paraphrased, the additional assumption is that the market price must be "fair" in the sense that it satisfies the speculator's first order condition for portfolio optimization when he acts like a price taker (see Gastinaeu and Jarrow (1991)).

Example 4 also provides insights into studies testing for market efficiency that search for arbitrage opportunities in observed market prices. These arbitrage opportunities, if discovered, are in terms of paper wealth. If traders

influence market prices with their trades (as the large trader does), then Example 4 provides an example where arbitrage opportunities would be discovered by an econometrician, although by Assumption A4 and Proposition 4, no arbitrage opportunities or market manipulation strategies exist (except corners). One solution has been to see if arbitrage profits are still available after utilizing a trading strategy based on the next observed market price (see Bhattacharya (1983) and Whaley (1982)). Unfortunately, this does not correct the problem as paper wealth is still being examined. The conclusions of these studies need to be interpreted with this example in mind.

VII.   Infinite Trading Horizon Speculators

The previous analysis was restricted to finite horizon speculators (where $T < +\infty$). The purpose of this section is to extend the previous analysis to infinite horizon speculators such as financial institutions. Under Assumptions A1–A5, we investigate whether adding the possibility of trading in perpetuity changes the previous results and adds market manipulation trading strategies that otherwise would not exist.

An issue can be raised regarding the previous definition of a market manipulation trading strategy: whether, in fact, paper wealth should be used rather than real wealth. This line of reasoning argues that the large trader can borrow against paper wealth, and consume from the borrowings, as long as he never has to liquidate his portfolio in finite time (given bequests, this could go on forever). This section can be interpreted as investigating this alternative, weaker definition of a manipulative trading strategy to see if they exist under Assumptions A1–A5.

The major adjustment to the previous structure is a change to the definition of a market manipulation trading strategy. First, let $\tau = \{0, 1, 2, \ldots, +\infty\}$.

Definition. (Infinite Horizon Market Manipulation Trading Strategies)

An *infinite horizon market manipulation trading strategy* is any zero initial wealth self-financing trading strategy $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$, such that

(33a) $\lim_{T \to \infty} V_t \geq 0$ with P probability one and

(33b) $P[\lim_{T \to \infty} V_T > 0] > 0.$

The difference between this definition and that given in Section IV for finite horizon traders is in letting $T \to \infty$. By the definition of the set Φ, we have $\alpha_{-1} \equiv \beta_{-1} \equiv 0$, so that $V_0 = 0$ for any trading strategy satisfying this definition. Hence, our infinite horizon market manipulation trading strategy generates nonnegative real wealth for sure and positive real wealth with positive probability starting from a zero investment.

We next investigate the existence of market manipulation trading strategies under Assumptions A1–A5. Adding Assumption A5 (by Lemma 2) implies that the speculator faces a fair gamble in terms of the prices received at entry ($g_0(0)$) and liquidation ($g_T(0)$ as $T \to \infty$). Using the insights of Heath and Jarrow (1987) for this scenario, it seems likely that the only manner in which a market

manipulation trading strategy can be generated is through a "doubling" strategy. The following proposition verifies this intuition.

Proposition 5. (Nonexistence of Infinite Horizon Market Manipulation Strategies under Wealth Constraints)

Given Assumptions A1–A5, $\{\alpha_t, \beta_t: t\epsilon\tau\}\epsilon\Phi$ such that $\alpha_{t-1} \leq N_t$ and $V(t) \geq -K$ almost everywhere for all $t\epsilon\tau$ and some positive constant K, then no infinite horizon market manipulation trading strategies exist.

Proof. Under Assumption A4, we have from the proof of Proposition 4 that

$$\overline{E}(V_T) \leq 0 \text{ for all } T.$$

Hence, $\lim_{T\to\infty} \overline{E}(V_T) \leq 0$. Given a wealth constraint,

$$V_T \geq -K \text{ or equivalently } V_T + K \geq 0 \text{ almost everywhere,}$$

by Fatou's lemma (Bartle (1966), p. 49),

$$\overline{E}(\lim_{T\to\infty} V_T + K) \leq \liminf_{T\to\infty} \overline{E}(V_T + K) = \lim_{T\to\infty} \overline{E}(V_T) + K \leq K,$$

i.e., $\overline{E}(\lim_{T\to\infty} V_T) \leq 0$.

This is true for all strategies satisfying the hypotheses of the proposition. Since this contradicts Expression (34), no infinite horizon market manipulation strategies can exist. □

Proposition 5 proves that the ability to trade for an infinite number of periods does not generate market manipulation strategies, when they otherwise do not exist (for finite horizon investors). Indeed, the hypotheses of this proposition (Assumptions A1–A5) contain the hypotheses of Proposition 4, which states that for finite horizon traders, there exist no market manipulation strategies. For infinite horizon traders, subject to these same hypotheses *and* a wealth constraint ($V_t \geq -K$ for almost everywhere), there also exist no market manipulation strategies. The wealth constraint concerns real wealth since by Example 4 it has no effect if imposed on paper wealth. The wealth constraint is imposed because it excludes doubling strategies, and these create arbitrage opportunities for price takers even under Assumption A5 (for a proof of this statement, see Heath and Jarrow (1987)).

VIII. Conclusion

This paper investigates whether large traders, those with market power, can manipulate prices to their advantage and generate profits at no risk. The answer to this question is shown to depend critically on the properties of the price process as a function of the speculator's trades. The existence of market manipulation trading strategies (with the exclusion of market corners and short squeezes, which always exist) is related to the time asymmetry in the sensitivity of price changes to the speculator's trades. Asymmetries create manipulation opportunities, which otherwise would not exist. Numerous examples are provided supporting this assertion.

This investigation is preliminary in nature, concentrating on characterizing the properties of the price process such that no riskless profit opportunities exist. As such, many directions for future research remain open to investigation. First, we need to determine whether the sufficient conditions for the nonexistence of market manipulation trading strategies are satisfied in practice. Second, theoretical investigations into the types of economies and equilibrium concepts that generate these sufficient conditions are needed. For recent papers along these lines, see Bagnoli and Lipman (1989), Bhattacharya and Spiegel (1989), and Allen and Gale (1990). Together, these investigations will increase our understanding of the robustness and the likelihood that the various assumptions invoked in this paper are satisfied in reality. Finally, the study of how derivative security markets influence the possibility of market manipulation in "large trader" economies needs to be analyzed. This is the subject of a companion paper (Jarrow (1990)).

Appendix

Lemma A1. For $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$, $\beta_t = -\alpha_0 g_0(\alpha_0) - \sum_{j=0}^{t-1} [\alpha_{j+1} - \alpha_j] \, g_{j+1}(\alpha_{j+1}, \alpha_j, \ldots, \alpha_0)$ for all $t \epsilon \tau$, where the series summation is set equal to zero for $t = 0$.

Proof. (By induction)

From (2), $\beta_0 + \alpha_0 g_0(\alpha_0) = \beta_{-1} + \alpha_{-1} g_0(\alpha_0)$. But, $\alpha_{-1} = \beta_{-1} = 0$, implying $\beta_0 = -\alpha_0 g_0(\alpha_0)$.

Suppose the expression holds for $t-1$. Consider (2), $\beta_t = -\alpha_t g_t(\alpha_t, \ldots, \alpha_0) + \alpha_{t-1} g_t(\alpha_t, \ldots, \alpha_0) + \beta_{t-1} = -[\alpha_t - \alpha_{t-1}] \, g_t \, (\alpha_t, \ldots, \alpha_0) - \alpha_0 g_0 \, (\alpha_0) - \sum_{j=0}^{t-2} [\alpha_{j+1} - \alpha_j] \, g_{j+1}(\alpha_{j+1}, \ldots, \alpha_0)$, by the induction hypothesis. Rearranging terms completes the proof. $\square$

Lemma A2. For $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$, $W_t = \alpha_{t-1} [g_t(\alpha_{t-1}, \alpha_{t-1}, \alpha_{t-2}, \ldots, \alpha_0) - g_{t-1}(\alpha_{t-1}, \ldots, \alpha_0)] + \sum_{j=0}^{t-2} \alpha_j [g_{j+1}(\alpha_{j+1}, \ldots, \alpha_0) - g_j(\alpha_j, \ldots, \alpha_0)]$ for all $t \epsilon \tau$, where the series summation is set equal to zero for $t \leq 1$.

Proof.

By Expression (10), $W_t = \alpha_{t-1} g_t(\alpha_{t-1}, \alpha_{t-1}, \alpha_{t-2}, \ldots, \alpha_0) + \beta_{t-1}$. By Lemma A1, $W_t = \alpha_{t-1} g_t \, (\alpha_{t-1}, \alpha_{t-1}, \ldots, \alpha_0) - \sum_{j=0}^{t-2} \alpha_{j+1} g_{j+1}(\alpha_{j+1}, \alpha_j, \ldots, \alpha_0) + \sum_{j=0}^{t-2} \alpha_j g_{j+1} (\alpha_{j+1}, \alpha_j, \ldots, \alpha_0) - \alpha_0 g_0(\alpha_0)$.

The second and fourth terms in this expression equal $-\alpha_{t-1} g_{t-1}(\alpha_{t-1}, \ldots, \alpha_0) - \sum_{j=0}^{t-2} \alpha_j g_j(\alpha_j, \alpha_{j-1}, \ldots, \alpha_0)$. Substitution and rearranging terms gives the result. $\square$

Proof of Proposition 4. By Lemma A1 and Expression (11) we get

$$(\text{B-1}) \quad V_T = \alpha_{T-1} g_T(0) + \beta_{T-1}$$

$$= \alpha_{T-1} g_T(0) - \sum_{j=0}^{T-2} [\alpha_{j+1} - \alpha_j] \, g_{j+1}(\alpha_{j+1}) - \alpha_0 g_0(\alpha_0)$$

$$= -\sum_{j=0}^{T-1} [\alpha_{j+1} - \alpha_j] \, g_{j+1}(\alpha_{j+1}) - \alpha_0 g_0(\alpha_0) \quad \text{with } \alpha_T \equiv 0.$$

To prove this result, we will show that $\overline{E}(V_T) \leq 0$ for all $\{\alpha_t, \beta_t : t \epsilon \tau\} \epsilon \Phi$ satisfying $\alpha_T = 0$ almost everywhere and $\alpha_t \leq N_{t+1}$ almost everywhere for all $t \epsilon \tau$. This

will guarantee the nonexistence of market manipulation trading strategies since Condition (13) implies $\overline{E}(V_T) > 0$.

(STEP 1)   First, we write an equivalent expression for $\overline{E}(V_T)$ utilizing Lemma 2 in the text.

$$
(B\text{-}2) \quad \overline{E}(V_T) \;=\; \overline{E}\left\{-\alpha_0 g_0(\alpha_0) - \sum_{j=0}^{T-1}\left[\alpha_{j+1} - \alpha_j\right]g_{j+1}\left(\alpha_{j+1}\right)\right\}
$$

$$
= \overline{E}\left\{-\alpha_0\overline{E}\left(g_T(\alpha_0)\right) - \sum_{j=0}^{T-1}\left[\alpha_{j+1} - \alpha_j\right]\overline{E}\left[g_T\left(\alpha_{j+1}\right)|F_{j+1}\right]\right\}
$$

$$
= \overline{E}\left\{-\alpha_0 g_T(\alpha_0) - \sum_{j=0}^{T-1}\left[\alpha_{j+1} - \alpha_j\right]g_T\left(\alpha_{j+1}\right)\right\} \quad \text{with} \; \alpha_T \equiv 0.
$$

(STEP 2)   Define

$$
(B\text{-}3) \quad v_T(\omega) \;\equiv\; -\alpha_0 g_T(\omega, \alpha_0) - \sum_{j=0}^{T-1}\left[\alpha_{j+1} - \alpha_j\right]g_T\left(\omega, \alpha_{j+1}\right) \;\; \text{with} \; \alpha_T(\omega) \equiv 0.
$$

This value has all prices indexed by the same date T. Step 1 shows $\overline{E}(V_T) = \overline{E}(v_T)$. We complete the proof by showing $v_T(\omega) \le 0$ for all $\omega \in \Omega$.

For the remainder of the proof, we fix a particular $\omega_0 \in \Omega$ and all random variables have this argument. To further simplify the notation, since all subscripts in Expression (B-3) are identical and equal to T, we write $g_T(\alpha) \equiv g(\alpha)$.

The remaining proof is divided into three cases. Before starting, without loss of generality, we assume that $\alpha_j \ne \alpha_{j-1}$ for all $j \in \tau$. This is without loss of generality since if $\alpha_j = \alpha_{j-1}$ for some j, this term drops out of v_T in Expression (B-3).

Case 1. $\alpha_0 \ge 0$, $\alpha_j > 0$ for all $j \in \{1, \ldots, T-1\}$.

Define $t^* = \text{argmax}\{\alpha_j : j \in \{0, 1, \ldots, T-1\}\}$. Consider the terms in v_T involving t^*, which are

$$
(B\text{-}4) \qquad -\left[\alpha_{t^*+1} - \alpha_{t^*}\right]g\left(\alpha_{t^*+1}\right) - \left[\alpha_{t^*} - \alpha_{t^*-1}\right]g\left(\alpha_{t^*}\right).
$$

By definition of t^*, $\alpha_{t^*} > \alpha_{t^*+1}$. By Assumption A3, $g(\alpha_{t^*}) > g(\alpha_{t^*+1})$, so,

$$
(B\text{-}5) \quad v_T \;=\; (\text{OTHER TERMS}) - (\alpha_{t^*+1} - \alpha_{t^*})\,g\left(\alpha_{t^*+1}\right) - (\alpha_{t^*} - \alpha_{t^*-1})\,g\left(\alpha_{t^*}\right)
$$

$$
\le (\text{OTHER TERMS}) - (\alpha_{t^*+1} - \alpha_{t^*-1})\,g\left(\alpha_{t^*+1}\right).
$$

The process just described removes the α_{t^*} term from the right side of Expression (B-3). In doing so, it replaces two intervals with one, i.e., $\{(t^* - 1, t^*), (t^*, t^* + 1)\}$ with $\{(t^* - 1, t^* + 1)\}$. Observe that there are no problems at the endpoints:
  (a) if $t^* = T - 1$, then $g(\alpha_{t^*+1}) = g(\alpha_T) = g(0)$ and $t^* + 1$ remains at T,
  (b) if $t^* = 1$, then $g(\alpha_{t^*+1}) = g(\alpha_2)$, so $g(\alpha_1)$ is replaced by $g(\alpha_2)$ and the remaining interval is $[0, 2]$.
Next, if $\alpha_{t^*+1} = \alpha_{t^*-1}$, this term drops out as well. So, without loss of generality, we assume in Expression (B-5) that $\alpha_j \ne \alpha_{j+1}$ for $j \in \{1, \ldots, T-1\}/\{t^*\}$.

We continue the process above inductively, reducing pairs of intervals into single intervals. Eventually, the Expression (B-5) reduces to one point remaining

in $\{1, \ldots, T-1\}$. This point is $s \equiv \mathrm{argmin}\{\alpha_j : j\epsilon\{1, \ldots, T-1\}\}$. Note that $\alpha_s > 0$. After this process, Expression (B-5) becomes

(B-6) $\quad v_T \quad \leq \quad -\alpha_0 g(\alpha_0) - [\alpha_T - \alpha_s] g(\alpha_T) - [\alpha_s - \alpha_0] g(\alpha_s)$ with $\alpha_T = 0$

$\qquad\qquad = \quad -\alpha_0 g(\alpha_0) + \alpha_s g(0) - \alpha_s g(\alpha_s) + \alpha_0 g(\alpha_s)$

(B-7) $\qquad\quad = \quad \alpha_s \left[g(0) - g(\alpha_s)\right] + \alpha_0 \left[g(\alpha_s) - g(\alpha_0)\right]$.

The right side of Expression (B-7) is nonpositive in all cases. Indeed, (recall $\alpha_0 \geq 0$ and $\alpha_s > 0$).

    (i)   if $\alpha_s = \alpha_0$, then (B-7) equals $\alpha_0[g(0) - g(\alpha_0)] \leq 0$ by Assumption A3 since $\alpha_0 \geq 0$.

    (ii) if $\alpha_s < \alpha_0$, then (B-7) $\leq \alpha_s[g(0) - g(\alpha_s)]$ by Assumption A3 on the first term, ≤ 0 by Assumption A3 again.

    (iii) if $\alpha_s > \alpha_0$, then (B-7) $\leq \alpha_s[g(0) - g(\alpha_0)] + \alpha_s[g(\alpha_s) - g(\alpha_0)]$ by Assumption A3, $= \alpha_s[g(0) - g(\alpha_s)] \leq 0$ by Assumption A3 again.

This shows $v_T \leq 0$ for all $\{\alpha_j, \beta_j : t\epsilon\tau\}$ satisfying the hypotheses of Case 1.

Case 2. $\alpha_0 \leq 0$, $\alpha_j < 0$ for all $j\epsilon\{1, \ldots, T-1\}$.

    The argument is symmetric to case 1, so the discussion is more brief.

    Define $t^* = \mathrm{argmin}\{\alpha_j : j\epsilon\{0, 1, \ldots, T-1\}\}$. Consider the terms in v_T involving t^*, these are

(B-8) $\qquad\qquad -[\alpha_{t^*+1} - \alpha_{t^*}] g(\alpha_{t^*+1}) - [\alpha_{t^*} - \alpha_{t^*-1}] g(\alpha_{t^*})$.

By definition of t^*, $\alpha_{t^*} < \alpha_{t^*+1}$ so $g(\alpha_{t^*+1}) > g(\alpha_{t^*})$ by Assumption A3. Hence,

(B-9) $\qquad v_T \quad \leq \quad (\text{OTHER TERMS}) - (\alpha_{t^*+1} - \alpha_{t^*-1})g(\alpha_{t^*+1})$.

This is the same as before. Continuing, we reduce the summation to

(B-10) $\qquad\qquad v_T \quad \leq \quad \alpha_s \left[g(0) - g(\alpha_s)\right] + \alpha_0 \left[g(\alpha_s) - g(\alpha_0)\right]$,

where $s \equiv \mathrm{argmax}\{\alpha_j : j\epsilon\{1, \ldots, T-1\}\}$.

Using Assumption A3 on Expression (B-10) shows that $v_T \leq 0$ for all $\{\alpha_j, \beta_j : j\epsilon\tau\}$ satisfying the hypotheses of Case 2.

Case 3. α_j for $j\epsilon\{0, \ldots, T-1\}$ switches signs a finite number of times over $[0, \ldots, T-1]$.

    For simplicity (the argument easily generalizes), we suppose that it switches once at time δ from positive to negative,

(B-11) $\qquad\qquad\qquad \alpha_0 \quad \geq \quad 0, \quad \alpha_j \quad > \quad 0$ for $j = 1, \ldots, \delta - 1$

$\qquad\qquad\qquad\qquad\qquad \alpha_\delta \quad \leq \quad 0$

$\qquad\qquad\qquad\qquad\qquad \alpha_j \quad < \quad 0$ for $j = \delta + 1, \ldots, T$.

(A symmetric argument applies if it switches from negative to positive.)

$$v_T \quad = \quad -\alpha_0 g(\alpha_0) - \sum_{j=0}^{\delta-2} [\alpha_{j+1} - \alpha_j] g(\alpha_{j+1}) - [\alpha_\delta - \alpha_{\delta-1}] g(\alpha_\delta)$$

$$-\sum_{j=\delta}^{T-1} [\alpha_{j+1} - \alpha_j] g(\alpha_{j+1}) \quad \text{with} \quad \alpha_T \equiv 0.$$

The middle term, $-[\alpha_\delta - \alpha_{\delta-1}]g(\alpha_\delta)$ $= -[0 - \alpha_{\delta-1}]\,g(\alpha_\delta) - \alpha_\delta g(\alpha_\delta)$

$$\le -[0 - \alpha_{\delta-1}]\,g(0) - \alpha_\delta g(\alpha_\delta).$$

Since $\alpha_{\delta-1} > 0$ and $g(0) \ge g(\alpha_\delta)$ by Assumption A3 because $\alpha_\delta \le 0$. So,

$$v_T \le -\alpha_0 g(\alpha_0) - \sum_{j=0}^{\delta-2} [\alpha_{j+1} - \alpha_j]\, g\,(\alpha_{j+1}) - [0 - \alpha_{\delta-1}]\, g(0)$$

$$-\alpha_\delta g(\alpha_\delta) - \sum_{j=\delta}^{T-1} [\alpha_{j+1} - \alpha_j]\, g(\alpha_{j+1}) \quad \text{with} \quad \alpha_T \equiv 0.$$

This decomposition gives two terms, one falls under Case 1 and the second under Case 2. So, $v_T \le 0$ for the trading strategies in Expression (B-11). $\square$

References

Allen, F., and D. Gale. Stock Price Manipulation." Unpubl. Manuscript, The Wharton School (1990).

Allen, F., and G. Gorton. "Rational Finite Bubbles." Unpubl. Manuscript, The Wharton School (1988).

Bagnoli, M., and B. Lipman. "Stock Price Manipulation through Takeover Bids." Unpubl. Manuscript, Univ. of Michigan (1989).

Bailey, W., and E. Ng. "Default Premiums in Commodity Markets: Evidence and Implications." Unpubl. Manuscript, Ohio State Univ. (1988).

Bartle, R. *The Elements of Integration.* New York: John Wiley & Sons, Inc. (1966).

Bhattacharya, M. "Transaction Data Tests of Efficiency of the CBOE." *Journal of Financial Economics,* 12 (Aug. 1983), 161–185.

Bhattacharya, U., and H. Spiegel. "Insiders, Outsiders, and Market Breakdowns." Unpubl. Manuscript, Columbia Univ. (1989).

Brooks, J. *Once in Golconda.* New York: Harper & Row (1969).

Cornell, B., and A. Shapiro. "The Mispricing of U.S. Treasury Bonds: A Case Study." *The Review of Financial Studies,* 2 (No. 3, 1989), 297–310.

DeLong, J. B.; A. Shleifer; L. Summers; and R. Waldmann. "Positive Feedback Investment Strategies and Destabilizing Speculation." Unpubl. Manuscript, Harvard Univ. (1988).

Easley, D., and M. O'Hara. "Price, Trade Size, and Information in Securities Markets." *Journal of Financial Economics,* 19 (Sept. 1987), 69–90.

Friedman, J. *Game Theory with Applications to Economics.* New York: Oxford University Press (1986).

Gastinaeu, G., and R. Jarrow. "Large Trader Impact and Market Regulation." *The Financial Analysts Journal,* 47 (July–Aug. 1991), 40–72.

Glosten, L., and P. Milgrom. "Bid, Ask, and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders." *Journal of Financial Economics,* 13 (1985), 71–100.

Harrison, J. M., and S. Pliska. "Martingales and Stochastic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Their Applications,* 11 (1981), 215–260.

Hart, O. "On the Profitability of Speculation." *Quarterly Journal of Economics,* 91 (Nov. 1977), 579–597.

Hart, O., and D. Kreps. "Price Destabilizing Speculation." *Journal of Political Economy,* 94 (No. 5, 1986), 927–952.

Heath, D., and R. Jarrow. "Arbitrage, Continuous Trading, and Margin Requirements." *Journal of Finance,* 42 (Dec. 1987), 1129–1142.

Jarrow, R. "Derivative Security Markets, Market Manipulation, and Option Pricing Theory." Unpubl. Manuscript, Cornell Univ. (1990).

Stedman, E. C. *The New York Stock Exchange,* Vol. 1. New York Stock Exchange Historical Co. (1905).

Stein, J. "Informational Externalities and Welfare-Reducing Speculation." *Journal of Political Economy,* 95 (No. 6, 1987), 1123–1145.

Thomas, D. L. *The Plungers and the Peacocks.* New York: G. P. Putnam's Sons (1967).

Whaley, R. "Valuation of American Call Options on Dividend Paying Stocks." *Journal of Financial Economics,* 10 (March 1982), 29–59.

Wyckoff, R. *Wall Street Ventures and Adventures.* New York: Greenwood Press (1968).