Optimal Probabilistic Market Timing using Bull Bear Cycle Statistics

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Abstract

Under the historical market view of binary bull and bear cycles, what is an *ex ante* optimal trading strategy? Similar to an original optimal stopping time model (Dai, *et al* 2010, 2011) to maximize long term investment returns, we introduce a market timing strategy based on the *probability* that the market is currently in a bull or bear regime. In doing so, we adjust the Wonham filter (1965) for the regime *conditional probability* calculation to allow *time-varying* volatilities and utilize expected bull/bear regime *risk-adjusted average returns* as statistically estimated parameter inputs. Further, we derive explicit formula to approximate the optimal *probabilistic* buying or selling *thresholds* and extend the analytical formulation to allow both long only and long/flat/short positions for a *hysteresis* trading implementation.

Considering the fractal characteristics of stock market indexes and other asset class index data at different frequencies, we devise a systematic procedure to separate historical bull bear regimes and obtain detailed market cycle statistics. This allows using different index gain, loss and duration thresholds to define historical bull/bear cycles in the same manner as an earlier study (Wagner, 1992). We back test the methodology on two broad stock indexes: S&P 500 Index (SPX 1928-2014) and Shanghai Stock Exchange Composite Index (SSEC 1995-2014). The results, including that of an out-of-sample test case on the SPX since 1940 using 10-cycle look-back rolling statistics, indicate out-performance over the buy-and-hold approach and a popular trend following technical trading strategy, the Golden Cross.

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Introduction

Bull and bear market cycle characteristics have been of great interest not only to market historians, but also to a wide range of investors, from passive indexers to market timers. What seems a paradox is that, although both consider the same data about bull and bear regimes over market cycles, proponents of active or passive investing reach different conclusion. For example, index mutual fund giant Vanguard (2002 and 2003) advised "Stay Calm during Bear Market" to avoid regretful selling at market bottom due to emotional timing decisions. In contrast, from simple *ex post* statistical analysis of one-hundred plus years of stock market's bull bear cycle history, Wagner (1992) argued "Market timing works" based on the relative large tolerance available for a market timing strategy to breakeven with a "buy-and-hold" approach. In practice; however, can one systematically learn from the historical bull/bear market cycle statistics in order to make optimal *ex ante* investment timing decisions?

For market technicians, bull and bear market cycles are intermediate term trends that should be tradable by a "trend following" methodology (Morris, 2014). The ideal for effective or optimal "trend following" is to catch a bull market at its early stage, ride the trend, and then liquidate to cash or a risk free position at the earliest sign of subsequent bear market. However, while the moving average trading rule known such as the Golden Cross[2] is a popular way of trend following implementation, trading it yields results that are far short of optimal. Making matters worse, often the empirical choice of the look-back period in a moving average rule (50 and 200 days in the Golden Cross approach) is subject to the criticism of *data mining bias*.

Based on the rationale that average returns are stable over the long term but that short term returns are not predictable, followers of the Random Walk market hypothesis are led logically to the "buy-and-hold" approach. In contrast, stock market models (Stambaugh, 1999) of *return cyclicality* driven by a fundamental variable such as dividend yield, interest rate spread, or dividend payout ratio, have been applied to address the issue of an optimal moving average based trending following rule (Zhu & Zhou, 2009).

[2] Golden Cross refers to the trading rule that when the 50-day simple moving average price of S&P 500 Index (SPX) crosses above its 200-day moving average price, a bullish market trend is identified and a long position is established. Conversely, a bearish trend is recognized when the opposite occurs. The rule is applicable to other broad based stock market indexes. We extend it later in this study to the Shanghai Stock Exchange Composite Index (SSEC).

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Still there is little doubt that over a century of stock market history testifies to the *cyclicality* of stock market prices and that these can be described as simple bull and bear binary market states. Based on a Hidden Markov Chain bull-bear regime switching model, Dai, Zhang & Zhu (2010) solved for the first time an optimal stopping time problem to maximize long term investment total returns. Their stochastic control model formulation led to optimal trend following rules.

Alternatively, Dai *et al's* (2011) *probabilistic trading* model to the same end utilized quantitative input parameters, including stock market bull/bear regime average returns, average volatilities, average bull/bear cycle frequencies, and risk free interest rate. However, return volatilities over bull and bear cycles were (unrealistically) specified as the same, and the 10-year US Treasury yield was applied as the risk free rate[3] in the simulations and back testing for several broad based US stock indexes. Extensions were also made to allow shorting beyond the long/flat market timing positions. In either cases, the threshold values of market regime probabilities in the *hysteresis* trading rule required numerical solution of a couple of difficult partial differential equations.

Dai *et al* (2010, 2011) demonstrated their theoretical formulations through both Monte Carlo simulations and actual historical data based testing. Using prescribed model parameters to generate test data under controlled distribution, the Monte Carlo simulations turned out better in testing performance. Besides theoretical simplification, our optimal probabilistic market timing proposal intends to focus on practical issues in implementation. We will demonstrate our methodology primarily through historical data based back tests rather than a randomized simulation approach.

One common challenge in developing a model-driven technical trading approach is the choice of model parameters and how to calibrate them effectively. Whether the tradable market trend is driven by econometric factors or statistical metrics, an effective trading strategy only exists if the model is reasonably predictive and model parameters are robust throughout the investment time span. Bull/bear market model calibration is challenging as it requires decades if not a century in time to make statistical sense. In practice, the cycle statistics may not be stationary going forward, not allowing the trading method to work out of sample. Market data is also fractal in nature. The frequency of market index data input, such as monthly, weekly or daily, might make a significant

[3] As the proposed tactical trading scheme moves in or out of the stock market index investment frequently – more than once a year on average, using the 10-year US Treasury yield as the return for the risk free asset is inappropriate. Investment in the 10-year US Treasury for a short time period, say less than one year, might even incur a loss.

difference in estimating model parameters used such as stock market return volatilities or risk-adjusted returns. We have sought to design our back test examples to cover these pragmatic aspects.

This rest of the paper is organized as follows: we will start with describing the bull/bear regime dependent model of market index price dynamics. To back test our analytical formulation with historical market data, we present a systematic procedure to characterize bull/bear market cycles and to calibrate bull/bear regime statistics or parameters. We then outline the probabilistic view of the market regime and adjust the Wonham filter (1965) in Dai *et al* (2010)'s hysteresis trading rule. Under the low transaction cost asymptotic limit, we derive an approximate formula of algebraic expression to calculate directly the threshold values for the tactical rule. Further, we implement a number of long-term back tests of the resulting probabilistic market timing rule with multiple market indexes both in- and out-of-sample.

Finally, we discuss the process of systematic learning from bull/bear market statistics for practical market timing implementation, and conclude. Along with presenting methodologies, numerical results are given as Tables and Figures. Bull/bear cycle individual and summary statistics are listed as Exhibits. Technical and procedural details are included in the Appendices.

Bull /Bear Regime-dependent Market Model

For a market index whose expected return is bull or bear regime-dependent, we describe the price dynamics through a stochastic differential equation as:

$$\frac{dS}{S} = \begin{cases} \mu_{bull} \cdot dt + \sigma(t) \cdot dW, & t \in bull\_regime \\ \mu_{bear} \cdot dt + \sigma(t) \cdot dW, & t \in bear\_regime \end{cases}$$

Here W employs standard Brownian motion which is assumed to be the same for both the bull and bear regimes. For each regime, re-write as $d(\log S) = (\mu_{bull/bear} - \sigma_t^2/2) \cdot dt + \sigma_t \cdot dW$, such that the assumption of time-varying volatility σ in the Wonham filter (Guo and Yin, 2006) can be applied. To use historical market data as samples to calibrate the model input on an *ex post* basis, the time range of a "*bull regime*" is defined as from a market index price bottom to the next top; a "*bear regime*" is the time range from a market index top to the next bottom. For each bull or bear cycle, volatility can be statistically measured as the standard deviation of period returns. By definition, the average period return is positive in a bull regime ($\mu_{bull} > 0$) and negative in a bear regime ($\mu_{bear} < 0$). Thus

the standard deviation estimators of volatility in bull and bear regimes are based on different mean return levels, and can have different magnitudes therein.

Definition and Statistics of Bull/Bear Market Cycles

A chartist can subjectively mark peaks and troughs on a market index price historical chart to identify bull and bear cycles. We use a quantitative approach to define bull or bear cycle time spans, using broad market index data at a particular frequency, daily (which is most common), weekly or monthly. Three limit-type parameters are involved in the process: the *minimum* peak-to-trough loss percentage (*L*) of a bear market, the *minimum* trough-to-peak gain percentage (*G*) of a bull market, and the *minimum* duration (*D*) of a bull or bear cycle in a number of observation periods such as days, weeks, or months.

Due to the fractal and discontinuous characteristics of market index prices, the concept of a minimum duration limit of a bull or bear cycle is introduced to disqualify a short spike or crash (such as the US market's "flash crash" on May 6, 2010) that later quickly dissipated or recovered, but had little long term effect. The steps to separate bull and bear cycles are detailed in Exhibit 1.

As a baseline case, Exhibit 1(a) (b) lists detailed bull and bear markets and cycle statistics for the S&P 500 Index, using the price index's daily close for the past 86 years (since its beginning in 1928). The data source is Bloomberg and the parameters used are *G=20%, L=19%* and *D=25 trading days*[4]. The important parameter "*risk adjusted average return*" ($\lambda=\mu/\sigma$) is calculated (in Exhibit 1(a)'s shaded column) for each bull or bear period. The resulting *risk adjusted average* returns are smaller and more stable than the *Sharpe Ratios* (at zero risk free rate $r = 0$) across bull markets (the other shaded column in Exhibit 1(a)), but more negative than the *Shape Ratios* (shown in Exhibit 1(b)) for bear cycles.

Considering the fractal characteristics of market index time series data, we then study, similar to an early work (Wagner, 1992), the effect of the bull/bear cycle separation parameters. Minimum Gain/Loss parameters (*G=L*) are varied from 10% to 40% in a 5% increment, and minimum cycle duration (*D*) changes from 15 days to 45 days in a 5-day increment accordingly. Summary statistics

[4] The bear market defining threshold $D = 19\%$ is used instead of the more commonly used 20%. This results in the declining periods of 1990 (post first Gulf War) and 1998 (Asian Financial Crisis) are also bear markets as in Vanguard (2003). By the same logic, the summer (May to September) of 2011 correction which centered US Credit Rating downgrade by Standard and Poor, is singled out as a bear period for SPX as well.

of applying the varied $G/L/D$ parameters to S&P 500 Index are presented in Exhibit 2. As G/L and D parameters increase, longer but less numerous bull/bear cycles are identified; average volatility over bull markets remains relatively stable while the higher average volatility over bear markets increases substantially; average total gain or loss over bull/bear cycles increases, while the magnitude of average "*risk adjust annual return*" ($\lambda=\mu/\sigma$) of bull/bear cycles decreases.

For the Shanghai Stock Exchange Composite (SSEC) Index[5] with a shorter history of about 25 years, Exhibit 3 and 4 shows the bull bear cycle statistics with the same G=20% and L=19%, and different D=25 days and 50 days respectively. The shaded rows in Exhibit 3 indicate short market cycles are detected mostly in volatile early years of the less mature market before 1995 with outlier ν and λ values. By restricting bull or bear cycle to least the length of D = 50 market days, Exhibit 4 shows more reasonable levels of parameters measured. In total 11 pairs of bull and bear cycles after 1995/2/7 defined in Exhibit 4 are the samples used to get market cycle average statistics.

Bull/bear cycles can also be defined with data of lower observation frequency by applying the same time stepping procedure of Appendix 1. Exhibit 5 shows weekly bull and bear market statistics of the S&P 500 Index over the time span of 1928-2014, where G=20%, L=15% and D=5 weeks are used. A similar number of cycles were recorded in Exhibit 5 for weekly data frequency, while the same are provided for daily data in Exhibit 1. Further, with monthly data, G=20%, L=15%, and D=2 months, Exhibit 6 shows the S&P 500 Index (1926-2013) bull bear market statistics[6] and Exhibit 7 for the Dow Jones Industrial Average (DJIA) over a longer 129-year history since 1885.

Probabilistic Bull/Bear Market View and a Hysteresis Trading Rule

After finding the market's interim peaks and troughs to separate the bull/bear cycles identified by the previously described methodology, the binary bull/bear regime switching probabilities can be defined and calibrated. The simple model used to accomplish this is a *Markov chain*: the bull or bear state of today only depends on that of yesterday, not any day further back. For

[5] This refers to the Yuan (RMB) based price index that started on 12/19/1990.

[6] When bull/bear cycle statistics are based on only two monthly data points for the 1932 bull cycle and the 1933 bear cycle in Exhibit 6-7, their outlier values of *risk-adjusted average return* λ and cycle frequency ν are excluded from the monthly model input parameter estimation. The computed volatility σ in a 2-month bull or bear cycle is not statistically valid, and thus, is not used to compute averages across the cycles either.

example, we observe *ex post* that a bull cycle lasted N days. This means the bull state continued next day in (N-1) days out of the N days, and only switched to a bear state on the last day of the bull market cycle. So the observed bull-to-bear switching probability or "state transition" probability is one out of N, i.e. $P_{bull\text{-}to\text{-}bear} = 1/N$. The observed probability that a bull market regime is likely to continue without any change is $P_{continue\text{-}bull} = 1 - 1/N$.

In general, we describe the transitional probabilities as:

$$P_{bull-to-bear} = \nu_{bull} \cdot h \qquad P_{continue-bull} = 1 - \nu_{bull} \cdot h$$
$$P_{bear-to-bull} = \nu_{bear} \cdot h \qquad P_{continue-bear} = 1 - \nu_{bear} \cdot h$$

If we use a yearly time unit ($h = 1/252$[7], or 1/52 or 1/12 for daily, weekly or monthly price data), for ν, the frequency of a bull or bear cycle, or the inverse of cycle duration, has the unit of year^{-1}. The values for ν are listed in a shaded column for each bull/bear cycle in Exhibit 1, and Exhibit 3 through 7. As a model input parameter, ν can be calibrated as the arithmetic average over values for all bull or bear cycles – the inverse of the harmonic average of bull or bear cycle durations.

Since the market state of a bull or a bear regime is not known or observable at the time, what is important is the *conditional probability* of whether the market is currently in a bull or bear regime must be determined solely from the price history up to the calculation date. In practice, we have to settle for the determined level of certainty of when to expect the appearance of a bull market or the end of a bear market, and the associated return volatilities around those expectations. In the Hidden Markov Chain binary bull/bear market regime switching model, the *conditional probabilities* of a bull and a bear market at a given time are complementary: $p_{bear} = 1 - p_{bull}$.

The classic Wonham filter (1965) specifies the *conditional probability* dynamics, assuming the *constant* annualized gain of a bull cycle, the *constant* annualized loss of a bear cycle, and the same *constant return* volatility around the mean bull cycle gain and bear cycle loss. Yet from the historical statistics in Exhibit 1(a) (b), we know that return volatilities were neither constant nor the same during and across bull and bear cycles, and they turned out to be higher on average in bear cycles than in bull cycles[8]. Alternatively, the analytical formulation of the Wonham filter allows us to combine average cycle return μ and average cycle volatility σ into a single model input parameter:

[7] For SSEC index, $h = 1/244$ due to about 244 trading days a year for Shanghai Stock Exchange.
[8] See Exhibit 1's Note 6.

the *risk-adjusted average return* $\lambda=\mu/\sigma$. The *risk-adjusted average return* was found to be more stable across bull and bear cycles than calibrating average annual return and volatility separately[9].

To reflect these practicalities we adjusted the Wonham filter to the following discrete form to calculate the *conditional probability* of whether the market at the current time ($t + \Delta t$) is in a bull regime:

$$p_{t+\Delta t} = \min\left[\max\left(p_t + f(p_t)\Delta t + (\lambda_{bull} - \lambda_{bear}) p_t (1 - p_t) \log\left(\frac{S_{t+\Delta t}}{S_t} \right) \Big/ \sigma_t , 0 \right), 1 \right] \qquad (1)$$

where p_t is the conditional probability of bull regime at last time step level t, and $f(p)$ is a cubic polynomial function that can be written as:

$$f(p_t) = -\left[\nu_{bull} \cdot p_t - \nu_{bear} \cdot (1 - p_t) \right] - p_t (1 - p_t)(\lambda_{bull} - \lambda_{bear}) \cdot \left[\left(\lambda_{bull} - \frac{\sigma_t}{2} \right) p_t + \left(\lambda_{bear} - \frac{\sigma_t}{2} \right)(1 - p_t) \right] \qquad (2)$$

Equation (1) is essentially a discrete time stepping procedure. Besides p_t, other inputs include the current and last market index prices $S_{t+\Delta t}$ and S_t, time step size Δt, market cycle frequency parameters ν_{bull} and ν_{bear}, and "risk adjusted average return" parameters λ_{bull} and λ_{bear}. Due to complementary relation $p_{bear}(t) = 1 - p_t$, Equation (1) and (2) are also valid for bear regime conditional probability when subscript "*bull*" and "*bear*" are switched.

As an important extension of standard Wonham filter model, we treat volatility σ_t in Equation (1) as time varying rather than constant. In practice, we will update the value of σ_t at every time step with the standard deviation of return, which is calculated by looking back a limited time window to keep a balance of concurrency and statistical sample size. The σ in Equation (2), whether evaluated on a time varying or constant basis, is less important, as half of the σ is usually much smaller than the λ as shown in Exhibit 1. Because index price S can have a discontinuous jump (e.g. Oct. 19, 1987), the *Min* and *Max* functions in Equation (1) are utilized to ensure that conditional probability is bounded between zero and one.

[9] See Exhibit 1's Note 7.

From Equation (1)'s right hand side, the time stepping process has two forces to drive the change in the conditional probability of the market being in a bull regime:

1. The $f(p_t)\,\Delta t$ term is an "attractor" for the bull probability to approach an equilibrium level p^* (see Appendix 1 for an explicit formulae to calculate p^*) over time, if the index price S remains stable;

2. The $(\lambda_{bull}-\lambda_{bear})p_t(1-p_t)\log(S_{t+\Delta t}/S_t)/\sigma_t$ term indicates that price change causes the conditional probability to move in the same direction – the accumulating logarithmic return (gain or loss) scaled by the ongoing volatility, drives the bull probability higher or lower.

Dai *et al* (2010) translated the bull regime conditional probability from the Wonham filter into an optimal trading methodology of a hysteresis tactical loop. Their optimal entry and exit market timing proposal relies on two threshold probabilities, p_{buy} and p_{sell}. However, when finite transaction cost and/or trading friction are considered, it requires solving a couple of difficult partial differential equations numerically to get their values.

While not opposed to difficult equations, to simplify the process operationally, we consider the asymptotic limit of zero trading friction and obtain the following approximate formula to compute the threshold probabilities (see Appendix 2 for the assumptions and derivations):

$$p_{sell} = p^* \cdot \frac{(\nu_{bull}+\nu_{bear})}{\nu_{bear}(\lambda_{bull}-\lambda_{bear})/(r/\sigma+\sigma/2-\lambda_{bear})+\nu_{bull}} \tag{3}$$

$$p_{buy} = 2p^* - p_{sell} \tag{4}$$

where r is the risk free interest rate. We take the annualized total return of the US 30-day T-bill [10] to specify the risk free interest rate r. All other inputs (ν, λ and σ) into Equation (3) are specified from the bull and bear market cycle statistics. As $(r/\sigma+\sigma/2)$ is usually considerably smaller than $(-\lambda_{bear})$ in Equation (3), we take a constant, rather than a time varying, value for the annualized average volatility σ through bull/bear cycles to evaluate the probabilistic timing trading thresholds.

[10] We use monthly return data of the US 30-day T-bill from the Morningstar/Ibbotson Associates' SBBI database. Covering 88 years (1926-2013), it averages about a 3.4% annualized return over the 15 negative monthly returns. For example, -0.062% over Nov.1938 is a relatively large negative monthly return. We use variable risk free interest rate later in the back-test examples to calculate the trading thresholds.

Dai *et al* (2011) applied the optimal stopping time formulation to an "inverse" stock index investment $1/S$ to generate a superimposed shorting signal to the original stock index investment. Similarly, we can define for the "inverse" market index $1/S$,

$$\nu'_{bull} = \nu_{bear}; \qquad \nu'_{bear} = \nu_{bull}$$

$$\lambda'_{bull} = \sigma - \lambda_{bear}; \qquad \lambda'_{bear} = \sigma - \lambda_{bull}$$

Then an "inverse" neutral probability p^*' can be calculated according to the cubic equation $f(p', \nu', \lambda') = 0$ which has the functional form of Equation (2). Further, replace p^*, ν and λ with p^*', ν' and λ' respectively in Equation (3) and (4) to calculate two additional threshold probabilities p_{short} and p_{cover} for the original S stock index investment. It is straightforward and necessary to verify that $p^*' \leq p^*$, and the threshold probabilities follow $p_{short} < p_{sell} < p_{cover} < p_{buy}$.

The steps of the hysteresis *Probabilistic Timing Signal* (PTS)-based trading rule are:

1. At the market close of the end date of a past known bear market, set bull probability $p = 1$ and initial signal as a long position in the market index investment S;
2. Calculate the threshold probabilities p_{buy} and p_{sell} from Equation (3) and (4), and additional threshold probabilities p_{short} and p_{cover} accordingly only if a shorting position is considered for the market index investment S;
3. Stepping forward in time for one period Δt, at the market close of the market day, update the bull regime probability p according to Equation (1);
    a. When previous day signal is long, if $p < p_{sell}$ set the current signal as flat; and after the *starting date of trading,* we sell all stock index investment position S and enter a 100% position into a risk free investment, e.g. 30-day US T-bill; otherwise stay 100% in stock index investment S and continue the signal as long position.
    b. When previous day signal is flat, if $p > p_{buy}$ set the current signal as long and after the *starting date of trading,* we sell all risk free investment position and buy 100% into stock index investment S; else if we allow shorting position in S and $p < p_{short}$, set the current signal as short, and after the *starting date of trading* we sell all risk free

investment position and short 100% the stock index investment *S*; otherwise stay 100% in risk free investment and continue the signal as flat position.

c. When the previous day signal is short that we allowed shorting stock index investment *S*, if $p > p_{cover}$ set the current signal as flat; and after the *starting date of trading,* we cover all shorted stock index investment position *S* and enter a 100% position into a risk free investment; otherwise stay 100% in the shorting stock index investment *S* position and continue the signal as short position.

Step 1 initiates the trading process during a bull cycle start. This also can be alternatively initiated during a bear cycle start with $p = 0$.

According to Equation (1)'s adjusted Wonham filter, the bull/bear *probabilistic timing* signal is *trend following* in nature. Rising index prices or the accumulating *market gains* scaled by ongoing market volatility, pushes the bull probability above the buying threshold and triggers a 100% long exposure to the stock market index. Falling index prices or accumulated market losses, scaled by volatility, pulls the bull probability below the selling threshold and causes a zero exposure to the market.

As a probabilistic timing signal, an "Aged Bull" or "Prolonged Bear" time accumulation effect by historical standards alone will not cause the approach to "call a top" or "catch a bottom" in a mean–reversion type of trade. The reason is that the "attractor" mechanism $f(p_i)\ \Delta t$ in Equation (1) only leads the bull regime probability to approach a neutral equilibrium level p^*, right half way between the two trading threshold probabilities according to Equation (4). To cross the threshold probabilities of buying or selling, it needs the impetus of price movement. Price needs to form a strong enough trend to confirm a switching of the market regime.

A common challenge for trend following approaches is the latency of the switch to a bearish stance as one waits for a declining trend to be formed, or to the bullish side as a rising trend is taking shape. To partially counter this, our *probabilistic timing signal* (PTS) follows the market price trend according to the historical bull/bear cycle statistics, with the following additional characteristics:

1. By using the hysteresis trading loop with a finite gap between the buying and selling thresholds, rather than a single threshold probability level, we can avoid some market whip-saws.
2. The bull/bear-neutral probability "attractor" is designed to reduce the latency gap, and trigger the trade to follow a new trend relatively earlier.

3. Using current returns scaled by on-going market volatility in the probability calculation may lead the probabilistic timing signal to respond quicker to a rising or declining volatility before a big market rally or crash hits.
4. The time stepping calculation of bull conditional probability starts retroactively as 1 from the last bear market low (Step 1-3 of the trading rule). This sets the time span from the last bear market low to its confirmation as the limit of the trend following delay.

Unlike the moving average type trend following rule, the probabilistic timing signal (PTS) does not have a pre-fixed time scale or look back period in the model, rather it takes a more adaptive approach by systematically applying historical bull/bear market cycle statistics. However, the optimality sought from the probabilistic market timing strategy may not be achieved in practice, as demonstrated in Dai *et al* (2010) where randomized simulation performed better than actual market data based testing. The model's predictability of bull/bear market cycles and the robust choice of model input parameters is the key. We address this through back-tests.

Back Test of Bull/Bear Probabilistic Market Timing

In-Sample Testing (Long Only)

The historical statistics of each bull/bear cycle for different indexes in Exhibit 1, 4-7 need to be further processed to generate five model input parameters, ν_{bull}, ν_{bear}, λ_{bull}, λ_{bear}, and the average through-cycle volatility σ. We can then calculate the two market timing threshold probabilities of Equation (3) and (4). Only the first four are needed for our adjusted Wonham filter Equation (1), where we evaluate on-going volatility σ_t by return standard deviation in a 15-day look-back time window, or a 12-month window for the back tests of monthly data frequency presented.

As in Dai *et al* (2010), the use of a simple arithmetic average estimator, justified mathematically by Guo and Yin (2006), is utilized to compute ν_{bull}, ν_{bear}, λ_{bull}, λ_{bear} from their columns in Exhibit 1. We include S&P 500 Index data since 1928 in order to have 15 to 30 pairs of bull/bear cycles to estimate model input parameters statistically, while Dai *et al* (2010) was limited to a maximum of 10 cycles of the index data in their test beginning in 1962. Also the extremely volatile decade of 1929-1939 in historical market data can also then be used to test our market timing model to its currently available limits.

In our testing we will apply the model input parameters both *in-sample* and *out-of-sample*. With the in-sample back test we calculate the parameters from the bull/bear cycle statistics over a time range and then apply them using the *probabilistic timing signal* (Equation (1)-(4) and trading rule steps 1-4) over the *same* time range. No optimization to a return or performance target is used to decide the model parameters, The performance output is then compared with results of another well-known trend following method, the Golden Cross Rule[11] to gauge the *ex post* value added.

Table 1: In-sample Back Test: Input Parameters and Computed Threshold Probabilities

(a) Calibrated Parameters for Long/Flat Timing Model

Index	Data Freq.	Test Date Range	ν_{bull}	ν_{bear}	λ_{bull}	λ_{bear}	σ	r	p_{sell}	p^*	p_{buy}
S&P 500	Daily	12/31/1927-1/15/2014	1.639	1.986	2.228	-2.601	18.63%	3.50%	0.404	0.554	0.704
SSEC	Daily	2/8/1995-06/06/2014	1.729	1.844	2.666	-2.749	27.92%	0.72%	0.366	0.528	0.690
S&P 500	Daily	9/5/1940-1/15/2014	1.040	2.004	1.853	-2.557	15.05%	3.92%	0.462	0.621	0.781
S&P 500	Weekly	12/31/1927-1/10/2014	1.5	2	2.5	-2.5	17.98%	3.50%	0.369	0.537	0.705
S&P 500	Monthly	12/31/1927-12/31/2013	1.5	1.5	2	-2.25	16.46%	3.50%	0.396	0.529	0.662
DJIA	Monthly	2/28/1885-12/31/2013	0.7	1.2	1	-2.75	16.39%	3.50%	0.616	0.706	0.796

(b) Additional Calibrated Parameters for Long/Flat/Short Timing Model

Index	Data Freq.	Test Date Range	ν'_{bull}	ν'_{bear}	λ'_{bull}	λ'_{bear}	σ	r	p_{short}	$p^{*\prime}$	p_{cover}
S&P 500	Daily	12/31/1927-1/15/2014	1.986	1.639	2.787	-2.042	18.63%	3.50%	0.300	0.446	0.592
SSEC	Daily	2/8/1995-06/06/2014	1.844	1.729	3.028	-2.387	27.92%	0.72%	0.306	0.472	0.638

The above table lists the calibrated model parameters used for in-sample back tests. Parameters of Table 1 are based on the bull/bear cycles and their statistics in Exhibit 1, and 4-7, where market cycle separation control thresholds G/L and D are also listed. The S&P 500 "Daily" test parameters in row 1 and 3 of Table 1(a) are strict arithmetic averages from Exhibit 1, and the "Weekly" and "Monthly" parameters in Table 1(a) are rounded average values. We exclude the extreme samples of 1930's US Index cycles in *monthly* Exhibits 6-7 (see Footnote 6).

To compute buy/sell threshold probabilities, the risk free interest rate r uses the annualized return of the US 30-day T-bill over the long-term, about 3.5% (1928-2014) or 3.92% (1940-2014).

[11] For daily data, Golden Cross enters 100% long stock index position at a market day's close when the 50-day simple moving average (SMA) of the index crosses above the 200-day SMA (inclusive of the same day); sell 100% stock index position and enter 100% risk free position (e.g. 30-day US T-bill or PBC RMB Deposit rate) when the 50-day SMA crosses below the 200-day SMA. Weekly Golden Cross is the 10-week SMA versus the 40-week SMA. For monthly trading, the 2-month SMA versus the 10-month SMA is used. We also compare results from trading signal of month end price simply crossing the 10-month SMA, and denoted as "10-Month MAC"(Moving Average Cross). Besides this Long/Flat positioning by Golden Cross signal, we can also have a Long/Short implementation where a 100% shorting position is taken when 50-day SMA crosses below 200-day SMA.

For SSEC Index market cycles since 1995, the People's Bank of China (PBC)'s Yuan (RMB) demand deposit yield is used as the risk free rate for China. It averaged about 0.72% in annualized return for 19.3 years period of Feb. 1995 through April 2014. All through-cycle average volatility σ in Table 1 is calculated as the square root of duration based weighted average of return variances for all the bull and bear cycles. Table 1(b) lists additional calibrated parameters for long/flat/short PTS on S&P 500 Index (1928-2014) and SSEC Index (1995-2014) with daily data frequency.

To study the effects of alternative bull/bear cycle definitions (with different G/L and D), we list in Table 2 the computed threshold probabilities, and input parameters for S&P 500 Index Long/Flat PTS only. Since this is an in-sample test, Exhibit 2 has already presented the cycle-averaged ν_{bull}, ν_{bear}, λ_{bull}, and λ_{bear}.

Table 2: In-sample PTS Input Parameters and Computed Threshold Probabilities for Alternative Bull/Bear Cycle Definitions: S&P Index Daily Data (1927/12/31-1/15/2014)

$G=L$	D (days)	ν_{bull}	ν_{bear}	λ_{bull}	λ_{bear}	σ	r	p_{sell}	p^*	p_{buy}
10%	15	3.81	4.97	3.59	-4.21	18.51%	3.50%	0.393	0.557	0.721
15%	20	2.72	3.05	2.54	-3.45	18.61%	3.50%	0.429	0.566	0.704
20%	25	1.80	1.71	2.30	-2.50	18.65%	3.50%	0.385	0.520	0.656
25%	30	1.18	1.48	2.09	-2.23	18.67%	3.50%	0.388	0.544	0.700
30%	35	0.92	1.39	2.03	-2.08	18.70%	3.50%	0.384	0.555	0.725
35%	40	0.84	1.24	1.87	-2.16	18.72%	3.50%	0.412	0.572	0.732
40%	45	0.89	1.31	1.86	-2.10	18.72%	3.50%	0.408	0.569	0.730

Table 3: Probabilistic Timing S&P 500 Index (**Daily**) 12/31/1927-1/15/2014, Varied $G/L/D$

$G=L$	D (days)	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
10%	15	162.54	6.10%	42.65%	55.13%	360
15%	20	269.18	6.72%	47.61%	57.38%	288
20%	25	463.94	7.40%	34.39%	60.06%	182
25%	30	620.07	7.76%	40.62%	64.30%	102
30%	35	347.47	7.04%	43.40%	64.57%	94
35%	40	398.51	7.21%	44.87%	64.88%	94
40%	45	358.77	7.08%	45.73%	64.84%	92
Buy-and-Hold (BAH)		104.66	5.55%	86.19%	100%	0
Golden Cross (GCR)		344.10	7.02%	71.63%	65.94%	102

Even when the bull/bear cycles targeted are defined differently, the effect on probabilistic timing performance is evident from the results in Table 3. When bull/bear cycle defining thresholds (gain/loss and duration) are increased, the probabilistic timing signal is (as logic would suggest)

firing less frequently. Interestingly though, it is less frequent than the linear rate of reduction in number of trades. However, the "Percentage of Time in the Market" results in Table 3 are relatively stable.

Around the bull/bear gain/loss threshold of 20% to 25%, the in-sample PTS seems to perform the best, with higher return and less max drawdown losses. Our conventional choices of *G=20%, L=19%* and *D*=25 days (Exhibit 1) as the baseline case (see Footnote 4) are within the "best" range found. All PTS results in Table 3 out-performed the buy-and-hold (BAH) approach. Most PTS in Table 3 (with *G=L*>15%) also out-performed the Golden Cross rule (GCR) in total return.

Tables 4, 6-7 show that the Probabilistic Timing Signal (PTS) on the S&P 500 Index (1928-2013) has averaged 1 to 2 trades per year. It outperforms the index's buy-and-hold (BAH) by at least 2% in annualized returns, enough to cover transaction costs[12] and dividend income shortfall due to partial time in the stock market index investment[13]. Long only PTS with daily data out-performs the Golden Cross (*GCR*), as shown in Table 5 for SSEC Index with much higher margins. PTS on S&P 500 with daily and weekly data (Table 6) both out-perform but with more trades.

Table 4: Timing S&P 500 Index (**Daily**) 12/31/1927-1/15/2014: Long/Flat PTS & GCR vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	1066.37	8.44%	38.91%	62.36%	140
Buy-and-Hold (BAH)	104.66	5.55%	86.19%	100%	0
Golden Cross (GCR)	344.10	7.02%	71.63%	65.94%	102

Table 5: Timing SSEC Index (**Daily**) 2/8/1995-6/6/2014: Long/Flat PTS & GCR vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	13.79	14.53%	31.67%	48.86%	27
Buy-and-Hold (BAH)	3.81	7.17%	71.98%	100%	0
Golden Cross (GCR)	4.13	7.62%	42.18%	51.94%	27

[12] Dai *et al* (2010) considered $K = 0.1\%$ as the normal case of proportional cost per trade for stock index investment.

[13] The dividend income shortfall depends on the "Percentage of Time in Market" of PTS. By estimating at most a 2.5% annual dividend yield margin of the S&P 500 over the US 30-day T-bill return, and being 40% of time out of the stock market index investment indicates an upper limit of 40% × 2.5% or a 1% dividend income shortfall for the bull/bear cycle based PTS.

Table 6: Timing S&P 500 Index (**Weekly)** 12/31/1927-1/10/2014: PTS & GCR vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	672.00	7.86%	35.35%	66.43%	110
Buy-and-Hold (BAH)	104.32	5.55%	85.95%	100%	0
Golden Cross (GCR)	499.21	7.49%	57.52%	66.41%	98

Table 7: Timing S&P 500 Index **Monthly** 12/31/1927-12/31/2013: PTS & MAC vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	626.26	7.78%	48.30%	74.61%	88
Buy-and-Hold (BAH)	104.66	5.56%	86.03%	100%	0
10-Month MAC	571.93	7.66%	48.61%	65.99%	122
2-M/10-M MAC	768.15	8.03%	53.36%	66.76%	82

The S&P 500 Index Monthly PTS back test results (Table 7) underperformed the 2-month/10-month Moving Average Cross (2M/10M MAC) signal with less trading. Unlike the Golden Cross or 2M/10M MAC signal, PTS performance declines as the frequency of the data used changes from daily to monthly (Table 4, 6-7). More frequent trades allowed the PTS approach to get in and out of the market quickly, while less frequent monthly decisions with the moving average signal may help to avoid some of the market whipsaws. However, monthly PTS trading on the Dow Jones Industrial Average (DJIA) over 128 years (1885-2013), returned more than both the 2M/10M MAC and 10M MAC with less trades, but had somewhat higher maximum drawdowns (see Table 8).

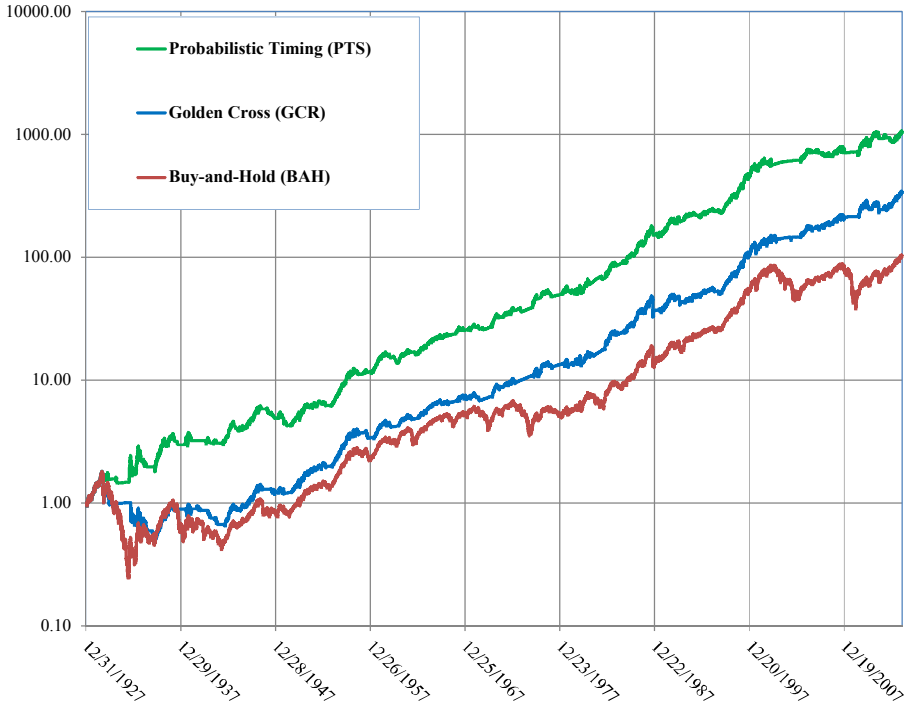
Table 8: Timing the *Dow Jones Industrial Average* Monthly 03/1885-12/2013: PTS & MAC vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	3371.18	6.51%	53.89%	80.80%	120
Buy-and-Hold (BAH)	605.70	5.10%	88.73%	100%	0
10-Month MAC	2244.18	6.17%	45.10%	63.75%	224
2-M/10-M MAC	1411.64	5.79%	49.69%	63.62%	166

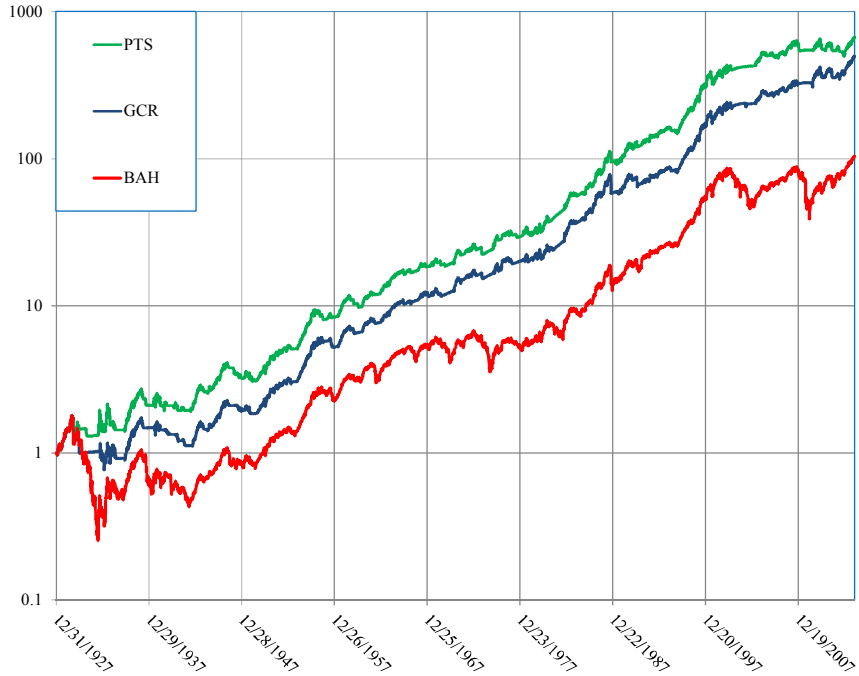
Corresponding to Table 4 to 8, Figures 1-4 show the equity curves of the probabilistic timing signal on different indexes, using the in-sample parameters in Table 1(a) obtained from the bull/bear cycle statistics.

Figure 1: Probabilistic Timing (Long Only) S&P 500 Index with In-sample Cycle Statistics

(a) Daily PTS for S&P 500 Index (12/31/1927-1/15/2014)



(b) Weekly PTS for S&P 500 Index (12/31/1927-1/10/2014)



(c) Monthly PTS for S&P 500 Index (12/31/1927-12/31/2013)

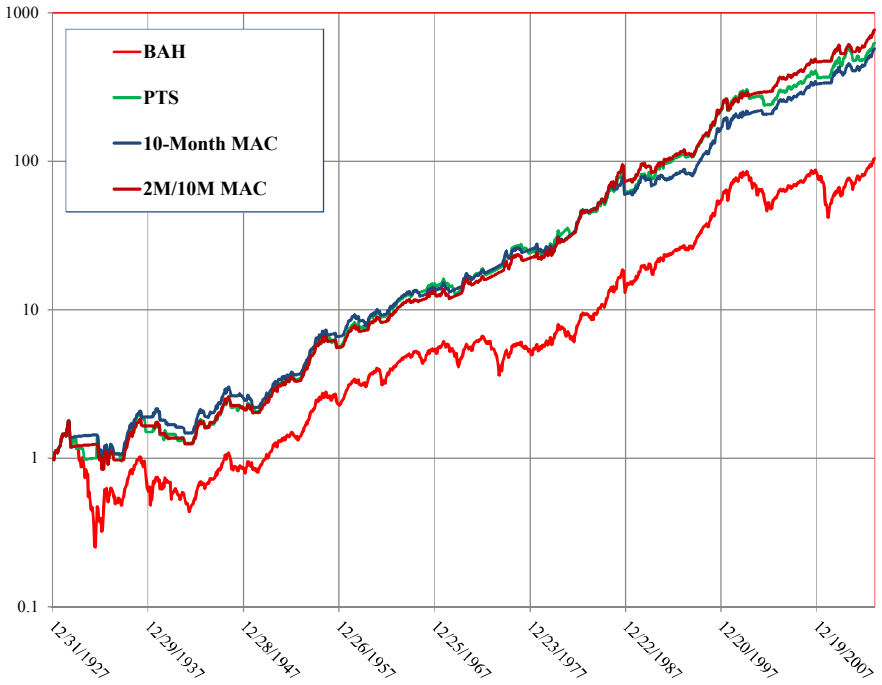


Figure 2: Probabilistic Timing on Shanghai Stock Exchange Composite (SSEC) Index (2/8/1995 – 6/6/2014) Long Only PTS

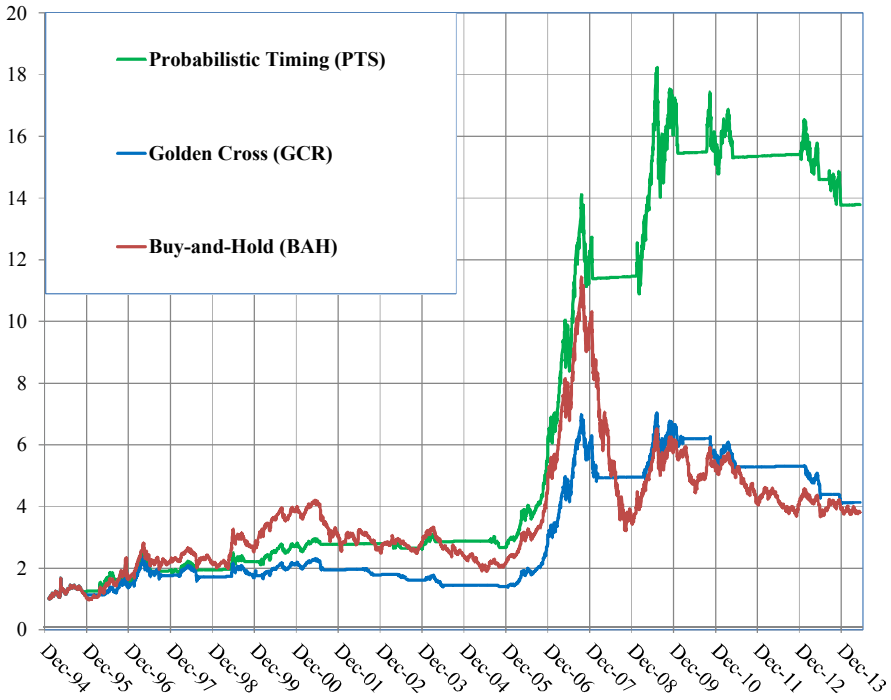


Figure 3: Probabilistic Timing Monthly on Dow Jones Industrial Average (DJIA)

(Mar. 1885 – Dec. 2013)

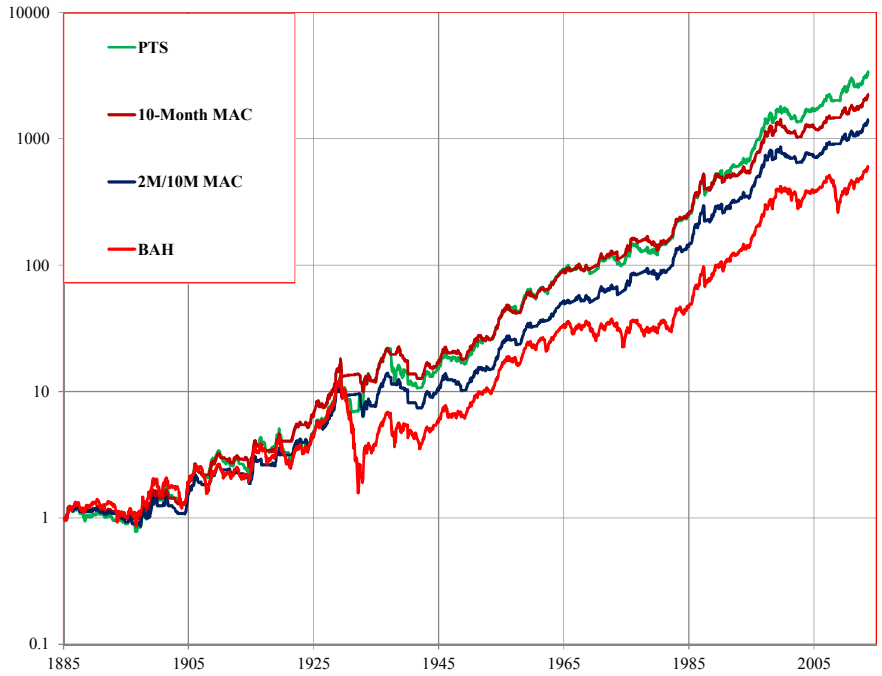


Figure 4: Probabilistic Timing of the S&P 500 Index (12/31/1927-9/5/1940)

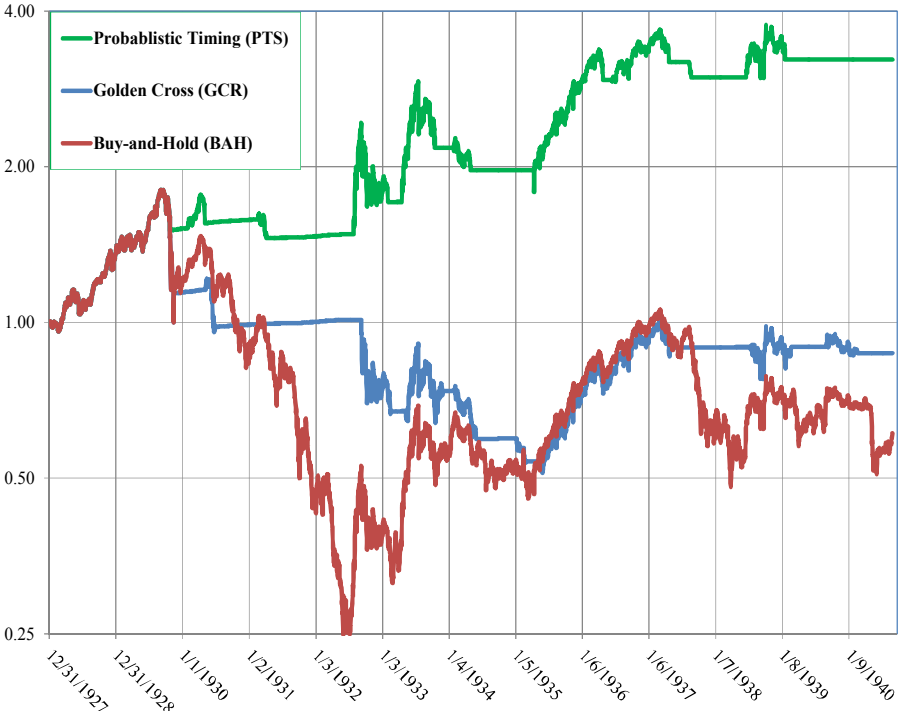
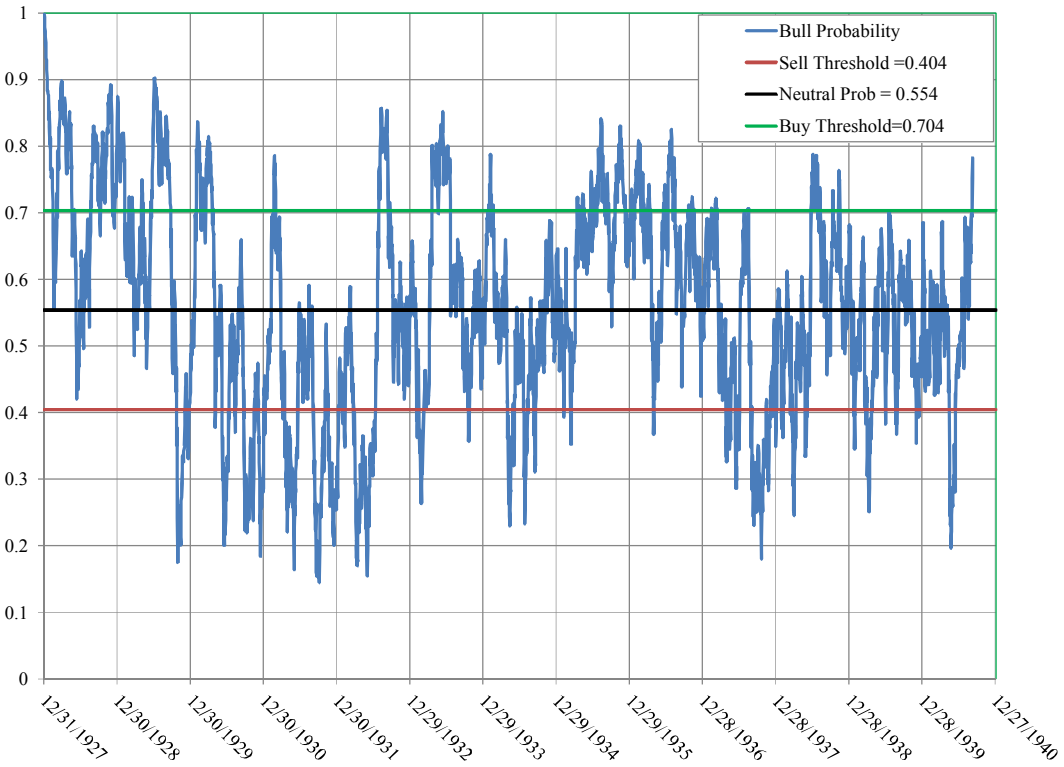
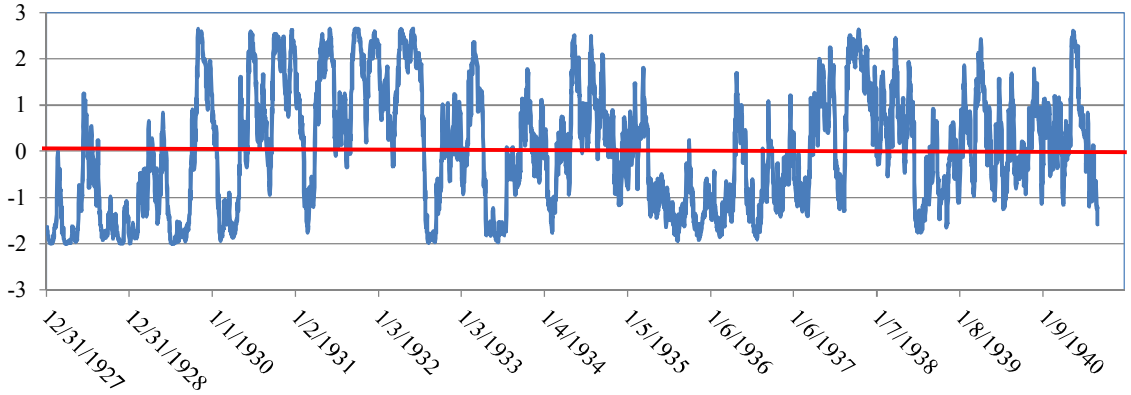


Figure 5: Probabilistic Timing of the S&P 500 Index (12/31/1927-9/5/1940)

(a) History of Bull Regime Conditional Probability and Selling/Buying Thresholds



(b) "Attractor" $f(p)$ component of Bull Regime probability



From Exhibit 1, bull/bear cycles in the decade of 1929-1939 had larger ν and λ values than other cycles. Thus they have a larger impact on the averaged parameters in Table 1 for in-sample

PTS. As a result, the daily S&P 500 PTS parameters in Table 1 for the whole back test period (1928-2014) work better during the first 10 pairs of bull/bear cycles from 12/31/1927 to 9/5/1940[14].

Table 9: Trades Comparison: Probabilistic Timing Signal vs. Golden Cross Rule* [1, 2]

	Prob. Timing (PTS)			Golden Cross (GCR)			PTS over GCR Relative P&L
Trade Type	Trade Date	Index Price	Standalone Trade P&L	Trade Date	Index Price	Standalone Trade P&L	
Buy	12/31/1927	17.66		12/31/1927	17.66		0
Sell	10/23/1929	26.60		11/18/1929	20.09		6.51
Buy	1/29/1930	22.35		5/8/1930	23.40		1.05
Sell	5/3/1930	22.84		6/27/1930	19.81		3.03
Buy	2/20/1931	17.71					
Sell	4/7/1931	16.28	-1.43				
Buy	7/27/1932	5.67		9/7/1932	9.31		3.64
Sell	2/2/1933	6.55		2/18/1933	6.19		0.36
Buy	4/19/1933	7.14		5/12/1933	9.03		1.89
Sell	10/18/1933	9.07		11/29/1933	9.88		-0.81
Buy	1/30/1934	11.33		2/3/1934	11.58		0.25
Sell	5/2/1934	10.25		6/2/1934	9.35		0.9
Buy				1/10/1935	9.44		
Sell				1/15/1935	8.96	-0.48	
Buy				1/16/1935	9.00		
Sell				1/31/1935	9.10	0.1	
Buy				2/18/1935	9.34		
Sell				3/2/1935	8.79	-0.55	
Buy	4/13/1935	9.07		5/17/1935	10.01		0.94
Sell	4/29/1936	13.53	-1.39				
Buy	6/16/1936	14.92					
Sell	4/26/1937	16.16		5/8/1937	16.60		-0.44
Buy	8/14/1937	17.32					
Sell	8/26/1937	16.15	-1.17				
Buy	6/23/1938	11.03		7/19/1938	12.67		1.64
Sell	1/23/1939	11.93		3/1/1939	12.70		-0.77
Buy				9/7/1939	12.50		
Sell				2/28/1940	12.15	-0.35	
Buy	9/5/1940	10.93					
Total Sum			**-3.99**			**-1.28**	**18.19**

*Notes (to follow on next page):

[14] The first 10 pairs of bull/bear cycles (Exhibit 1) would have ended on 6/10/1940 according to Exhibit 1(b). 9/5/1940 is the first date one can confirm *ex post* the end of the first ten bull/bear cycles. Thus 9/5/1940 is later taken as the first date to apply the probabilistic timing signal using the first ten bull/bear cycle statistics as an out-of-sample application.

1. Values in the green column "PTS over GCR Relative P&L", are the result of PTS subtracting GCR if it is a "Sell" row or GCR subtracting PTS if is a "Buy" row. A positive number means that PTS outperformed GCR for the trade.

2. The "Standalone Trade P&L" Columns (Light Blue Column for PTS and the Orange column for GCR) are calculated in the "Sell" row for each trade, and equal the "Sell" index price value minus the "Buy" index price on the previous row. A positive value is a winning trade.

Focusing on the early time period (1928-1940) of Figure 1(a), Figure 4 shows the historical equity curve of how PTS out-performed Golden Cross rule and Buy-and-Hold. Figure 5(a) displays the history of conditional probability in a bull regime, crossing the buy or sell thresholds in hysteresis loops to trigger trades. Figure 5(b) demonstrates the "attractor" mechanism $f(p)$, oscillates around zero in the time span of ten bull/bear cycles. We have $|f(p)\,\Delta t| \leq 1\%$, as daily time step $\Delta t \sim 1/300$ year [15] and $|f(p)| < 3$ as shown in Figure 5(c). Thus the major force causing a crossing of the selling or buying threshold probability, each at a gap of about 15% away from neutral probability p^* as shown in Figure 5(b), has to be the "trend following" second term in Equation (1).

A side by side comparison of the trades from PTS and Golden Cross is shown in Table 9. During this volatile period (1928-1940), PTS indeed out-performs Golden Cross by buying into a bullish period or exiting a bearish period at an earlier time. The green column of Table 9 indicates an earlier trade date for PTS than the Golden Cross date.

On ten out of thirteen occasions, PTS was able to buy at a lower price or sell at a higher price on the S&P 500 Index than GCR. The GCR had separate losing trades during Jan. – March of 1935 due to market whipsaw (pink column of Table 9). PTS also had stand-alone misfires in 1931, 1936 and 1937 (blue column of Table 9). Overall, PTS had a substantial P&L advantage over GCR as shown in bottom line of Table 9. After the 12.7 years ending on 9/5/1940, PTS had a total return multiple of 3.27 after starting at 1.0 on 12/31/1927, while GCR and Buy-and-Hold both had a losing ending multiple of 0.87 and 0.62 respectively, according to the end points on their equity curves in Figure 4.

After the out-performance in the first ten bull/bear cycles (12/31/1927-9/5/1940), PTS using in-sample parameters for whole time range (first row of Table 1 for 1928-2014), actually under-performed the Golden Cross rule. PTS has a total return multiple of 326.29 by 1/15/2014

[15] Before June 1950, Saturday was also a market day resulting in about 300 trading days in a year.

from starting at 1.0 on 9/5/1940, compared to the Golden Cross rule's 394.38, yet it still beat the total return of Buy-and-Hold at 169.11. The explanation for GCR's outperformance could be that just as the full time range (1928-2014) based parameters for PTS biases its application to the earlier, shorter period, the full time range parameters are skewed by the bull/bear cycle characteristics of the first ten bull/bear cycles (1928-1940) and are, thus, less reflective of those in average cycles over the rest of time range (1940-2014).

Figure 6: In-sample Probabilistic Timing on S&P 500 Index (9/5/1940-1/15/2014)

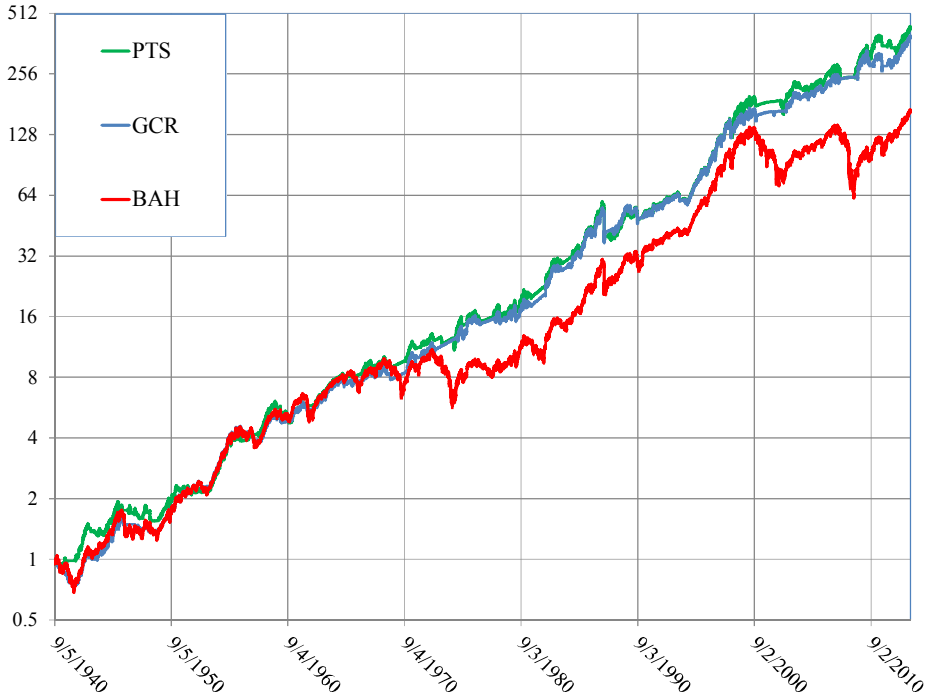


Table 10: Probabilistic Timing In-sample on S&P 500 Index (9/5/1940-1/15/2014)

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	437.90	8.64%	35.93%	67.93%	108
Buy-and-Hold (BAH)	169.11	7.24%	56.78%	100%	0
Golden Cross (GCR)	394.38	8.49%	33.17%	68.90%	81

Indeed, a set of in-sample parameters based on the later time range (1940-2014), i.e., the third row of Table 1, causes the S&P 500 Index's Long Only PTS to out-perform GCR slightly, as shown in Figure 6 and Table 10.

In-Sample Testing (Long/Flat/Short Model)

With the calibrated model parameters in Table 1(a)'s first and second row and Table 1(b) according to market cycle statistics in Exhibits 1 and 4, we are able to implement a full long/flat/short position-allowed PTS back tests, for both the S&P 500 Index (1928-2014) and the SSEC Index (1995-2014).

Table 11: Timing S&P 500 Index (**Daily**) 12/31/1927-1/15/2014

Long/Flat/Short PTS & GCR vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Time % Long	Time % Flat	Time % Short	No.Trades B/Sell/Short/C	No.Trades Total
PTS - L/F/S	252.13	6.64%	44.71%	62.36%	20.23%	17.41%	70/70/57/57	254
BAH	104.66	5.55%	86.19%	100%	0%	0%	0	0
GCR - L/S	31.01	4.07%	91.29%	65.94%	0%	34.06%	51/-/51/-	102

Table 12: Timing SSEC Index (**Daily**) 2/8/1995-6/6/2014

Long/Flat/Short PTS & GCR vs. BAH

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Time % Long	Time % Flat	Time % Short	No.Trades B/Sell/Short/C	No.Trades Total
PTS - L/F/S	19.83	16.71%	56.15%	48.86%	11.86%	39.28%	13/14/15/15	57
BAH	3.81	7.17%	71.98%	100%	0%	0%	0	0
GCR - L/S	2.08	3.86%	70.77%	51.94%	0%	48.06%	13/-/14/-	27

Table 11 indicates that the short positions of S&P 500 Index in PTS did not add value: the returns are worse than those of long-only PTS in Table 4's first row; Maximum drawdown also increases from the level of long-only PTS; and the number of shorting/covering trades is also substantial. Still, it outperforms Buy-and-Hold approach and the long/short Golden Cross rule, which performs poorly – worse off than BAH in most measures of Table 10. This matches the observation of Dai *et al* (2011) that the shorting side of PTS may have difficulties in the long term strong upward trending markets. Figure 7 shows equity curve history of S&P 500 long/short PTS.

In contrast, Table 12 shows that Long/Flat/Short PTS for SSEC (1995-2014) outperforms the long-only PTS (see Table 4 first row) in returns at the expense of higher Maximum Drawdown and a doubled number of trades due to short and covering. Shorting the SSEC index through exchange traded futures is only possible since 2010. Since then, long/short PTS would have incurred larger losses than the long-only PTS when comparing equity curves in Figure 8 to those of Figure 2.

Figure 7: Timing S&P 500 Index (9/5/1940-1/15/2014) Long/Short PTS & GCR vs. BAH

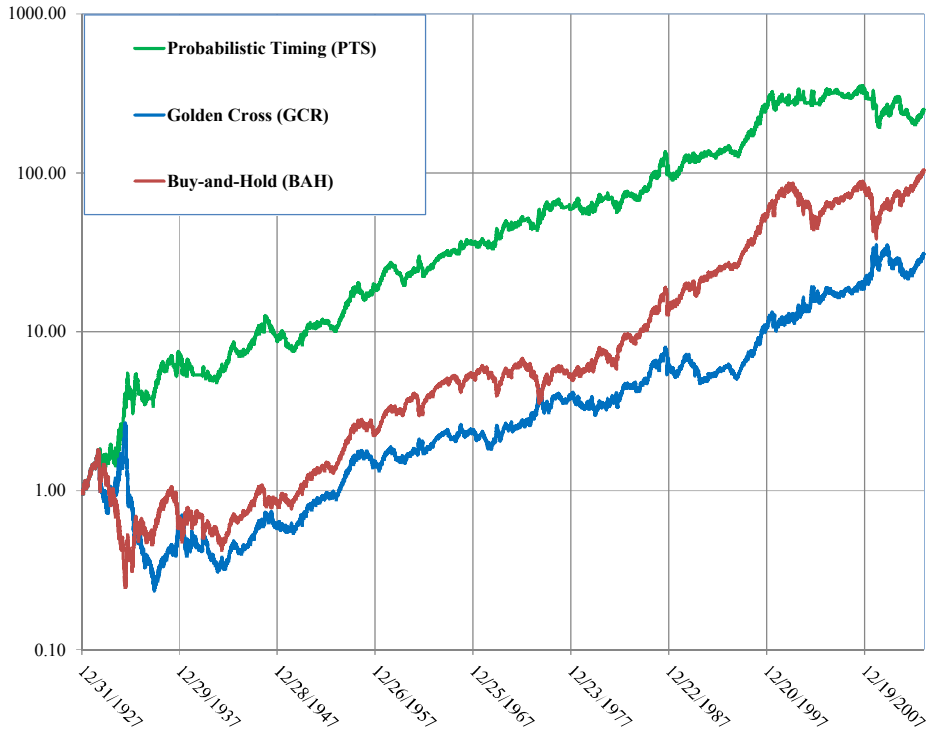


Figure 8: Timing SSEC Index (2/8/1995-6/6/2014): Long/Short PTS & GCR vs. BAH

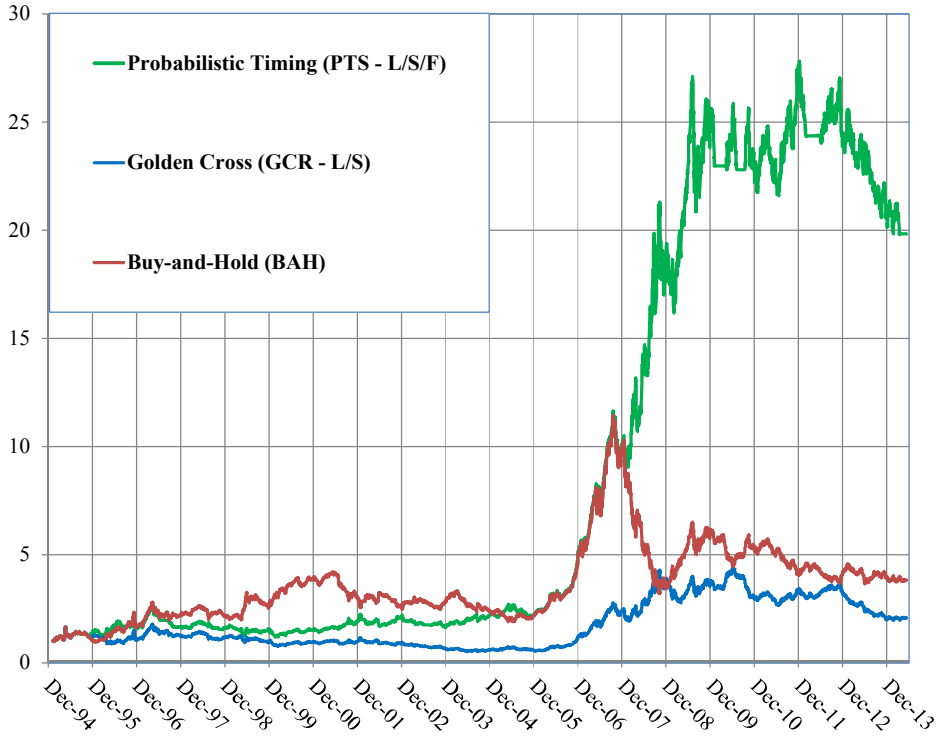


Figure 9: Long/Flat/Short PTS for SSEC Index (2/8/1995-6/6/2014): Probability Hysteresis

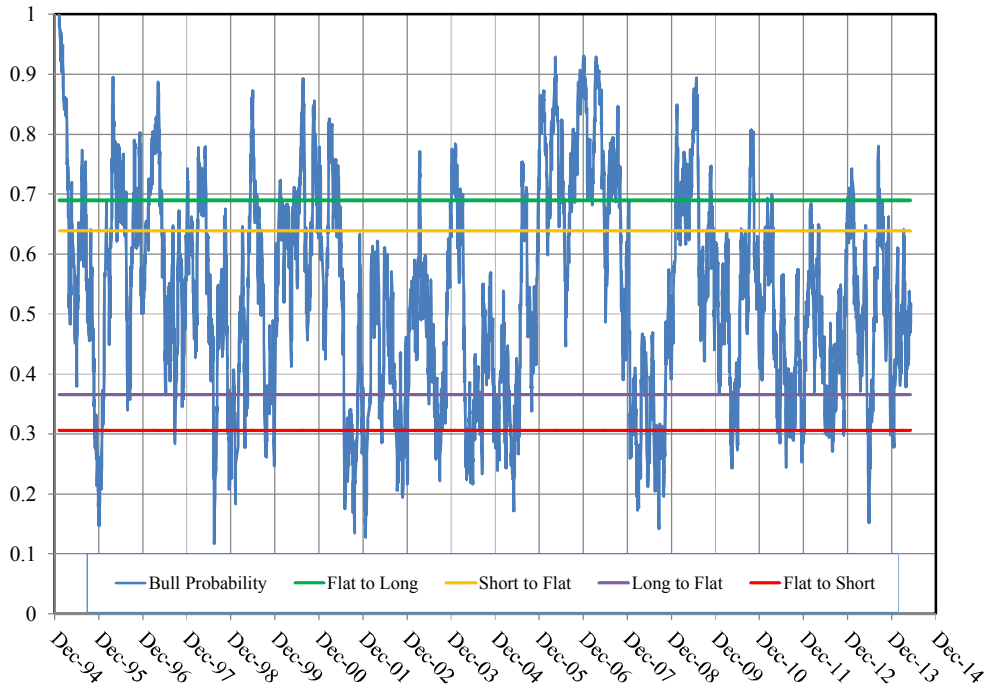


Figure 9 demonstrates a history of the bull regime conditional probability for the long/short PTS on SSEC Index (1995-2014), similar to Figure 5(a) for the long-only PTS on S&P 500 Index (1928-1940). The difference is Figure 5(a) only had two line levels of threshold probabilities triggering trades from long to flat, or flat to long, rather Figure 9 has four lines of threshold probability level that triggers trades from flat to long, from short to flat, from long to flat and from flat to short positions, in a descending order.

Out of Sample Testing

In practice, bull-bear cycle statistics over a time range can only be *ex post* observed and apply PTS for a time period afterwards. One way to determine the methodologies practical feasibility is to do out-of-sample back testing. To demonstrate the non-stationary of the parameters, we can take the statistics for every bull or bear cycle in Exhibit 1(a) and (b) and calculate a 10-cycle rolling average. This we have done in Table 13, along with the computed PTS threshold probabilities calculated in accord with Equation (3) and (4).

Table 13*:* S&P 500 Index 10-cycle rolling statistics of bull/bear regime and threshold probabilities

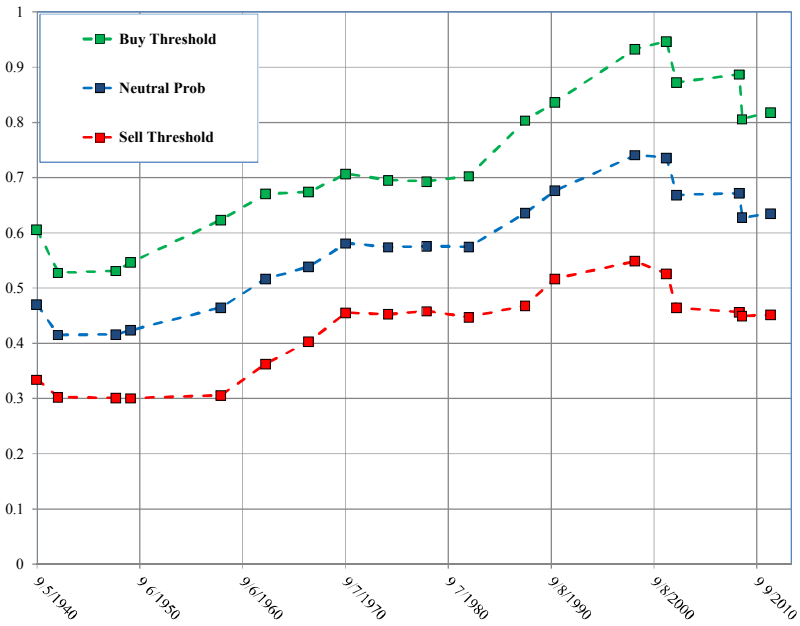
Last Bear End Date	First PTS Date	ν_{bull}	ν_{bear}	λ_{bull}	λ_{bear}	σ	r	p_{sell}	p_*	p_{buy}
6/10/1940	9/5/1940	2.776	1.954	2.941	-2.679	31.04%	0.02%	0.334	0.470	0.606
4/28/1942	10/2/1942	2.925	1.452	2.969	-2.335	30.33%	0.38%	0.303	0.415	0.528
5/17/1947	5/15/1948	2.732	1.420	2.829	-2.218	26.64%	0.92%	0.301	0.416	0.531
6/13/1949	11/2/1949	2.381	1.372	2.654	-2.061	25.20%	1.01%	0.301	0.424	0.547
10/22/1957	7/25/1958	1.494	1.304	2.389	-1.870	21.07%	0.81%	0.306	0.465	0.624
6/26/1962	12/6/1962	1.207	1.311	2.086	-2.016	18.62%	2.81%	0.363	0.517	0.671
10/7/1966	2/14/1967	1.004	1.122	1.893	-2.081	16.59%	4.37%	0.403	0.539	0.674
5/26/1970	9/28/1970	0.766	1.113	1.778	-2.230	15.59%	6.64%	0.455	0.581	0.707
10/3/1974	11/11/1974	0.762	1.092	1.835	-2.249	13.61%	6.66%	0.453	0.574	0.696
3/6/1978	8/9/1978	0.656	0.989	1.801	-2.196	12.65%	6.87%	0.458	0.576	0.693
8/12/1982	9/21/1982	0.541	0.913	1.767	-2.053	12.38%	6.32%	0.447	0.575	0.703
12/4/1987	3/8/1988	0.363	1.210	1.699	-2.144	13.08%	5.42%	0.468	0.636	0.804
10/11/1990	2/6/1991	0.380	1.524	1.649	-2.439	13.20%	5.87%	0.517	0.677	0.837
8/31/1998	11/19/1998	0.310	2.247	1.648	-2.905	13.11%	3.74%	0.549	0.741	0.933
9/21/2001	12/5/2001	0.360	2.232	1.634	-2.897	14.14%	1.77%	0.526	0.736	0.947
10/9/2002	11/21/2002	0.686	2.177	1.858	-2.754	14.81%	1.42%	0.465	0.669	0.873
11/20/2008	12/31/2008	0.679	2.115	1.768	-2.627	16.07%	0.01%	0.457	0.672	0.887
3/9/2009	4/14/2009	1.470	2.647	1.926	-2.870	16.86%	0.16%	0.449	0.628	0.806
10/3/2011	1/25/2012	1.478	2.822	1.897	-2.840	17.50%	0.00%	0.452	0.635	0.819

Using these parameters we can create a *walk-forward* process for a PTS back-test. In this back test, we begin with the first 10 pairs of bull/bear cycles for the S&P 500 Index starting on 12/31/1927 and ending on the bear market low on 6/10/1940 as shown in Exhibit 1. But since a market observer would not have realized that a new bull market had started until 9/5/1940, we have designated the "first date to confirm bull market" (*FDPEL*) in Exhibit 1 as that date. The first set of bull/bear cycle statistics, the first row of Table 13, are applied along with initial probability condition p=1 as calculated at the time of the market close of 6/10/1940, and then utilized according to Equation (1) and (2), to calculate bull regime conditional probability until 10/2/1942 (the next *FDPEL* in Exhibit 1(a)). The daily long stock index or risk free investment position from PTS is applied from 9/5/1940 to 10/2/1942.

Next we roll one bull/bear cycle forward. The next 10-cycle statistics in the 2nd row of Table 12 cover the period from 11/14/1929 to 4/28/1942 (see Exhibit 1). They are used in the daily walk forward process starting from 4/28/1942 (with p=1) until 5/15/1948 (next *FDPEL*), then we invest according to PTS for each day from 10/3/1942 to 5/15/1948, and so forth. By the 1/15/2014 market high, this walk forward process rolls forward the 10-cycle bull/bear statistics 19 times in 86

years. The most recent known 10-cyle bull/bear statistics (last row of Table 13) was applied as the market timing signal since 1/25/2012.

Figure 10: History of 10-cycle Rolling Threshold Probabilities (SPX: 1940/9/5-2012/1/25)



Buy and sell threshold probabilities also depend on the risk free interest rate, according to Equation (3). They reflect the balance of expected risk free interest income in the next bull/bear cycle and the potential opportunity loss from missing a more profitable bull market entry. We list the annualized 30-day US T-bill return on the first date[16] of the probabilistic timing signal application for each cycle, as ***r*** in Table 13. Higher risk free interest rates, over 4%, existed from the late 1960's to early 1990's. We apply the higher of the T-bill rate on the first date of PTS application for the current rolling cycle and long term average T-bill rate (3.92% over 1940-2014) as the risk free rate to calculate the buying or selling threshold probabilities. Figure 7 shows the history of PTS buying and selling threshold probabilities. Both the level and gap between them change over the past 73.4 years.

Performance of out-of-sample back-tests with the 10-cycle rolling cycle statistics is shown in Figure 11 and Table 14. Over 73.4 years (1940-2014), not only does the out-of-sample PTS out-perform Buy-and-Hold (*BAH*) and the Golden Cross rule (*GCR*) especially after early 1990's, but it

[16] This is the "first date to confirm bull (*FDPEL*)" for market cycles as listed in the last column of Exhibit 1(a).

also beats the in-sample PTS results listed in Table 10, with fewer trades. This indicates that on average, systematic learning gleaned from the rolling 10-cycle bull/bear statistics has some predictive value during the market's trajectory over its next cycle.

Figure 11: Probabilistic Timing on S&P 500 Index Daily (1940/9/5-2014/1/15)

Equity Curves of Out-of-Sample Back-test with Rolling10-Cycle Bull/Bear Statistics

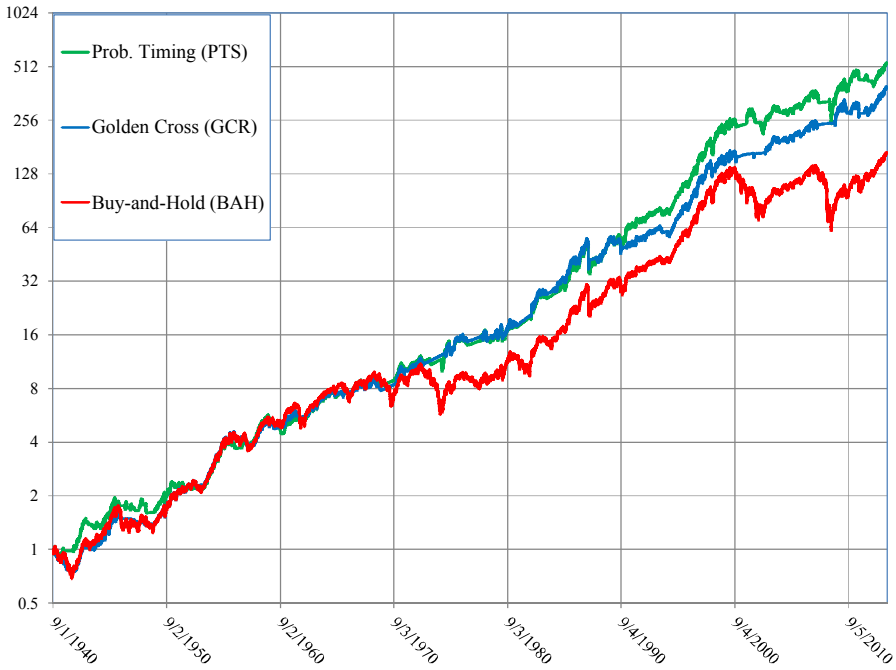


Table 14: Performance of Probabilistic Timing on S&P 500 Index Daily

Out-of-Sample Back-test with Rolling10-Cycle Bull/Bear Statistics (1940/9/5-2014/1/15)

	Total Return (Ending Multiple)	Annualized Return	Max Drawdown	Percentage of Time in Market	Number of Trades
Prob. Timing (PTS)	538.05	8.95%	35.30%	74.19%	96
Buy-and-Hold (BAH)	169.11	7.24%	56.78%	100%	0
Golden Cross (GCR)	394.38	8.49%	33.17%	68.90%	81

The probabilistic market timing methodology suffered a pitfall near the end of the bear cycle when a strong but brief bullish period occurred. PTS got back into market on 12/31/2008 after a

"bear bounce" rising period[17] in late 2008. Despite this early entry, PTS stayed invested through the market low on 3/9/2009 and then benefited from the strong market recovery afterwards.

It is noteworthy that although the current method requires a restart in the walk forward period to identify the bull probability, in every cycle following a bullish trend the methodology was able to do so at an early stage. In the earliest example of this, PTS moved back into the market quickly on 6/3/1942, shortly after the bear market low on 4/28/1942. Although the methodology still falls short of identifying the optimal high and low of each market cycle, it appears to be a significant improvement over a classic moving average crossover system, like the Golden Cross.

Summary and Discussion

The belief in bull and bear market cycles has been one of the important underpinnings of market timing. By extending an early quantitative framework, we have applied a probabilistic view of bull bear cycle dynamics for market timing implementation. A simple hysteresis trading rule is given along with explicit formula to calculate current bull/bear market probability and its threshold values to trigger buying or selling trades. The resultant timing signal is trend following in nature; and mathematically it seeks to maximize long term expected returns by probabilistically indicating investment at an early stage into a bull market or an exit from a bear market at the first sign of foul play.

In order to calibrate bull/bear market model parameters, we developed a systematic procedure to separate bull and bear market cycles, considering the fractal characteristics of market price time series. From long term historical data of the S&P 500 Index since 1928, we find that risk adjusted returns for separate bull or bear cycles are more stable than using optimized cycle data gleaned from the entirety of the market cycle data series. Furthermore, we find timely estimation of variable return volatility, as compared to past practices using a constant volatility value, may help to scale the dynamic market trend correctly in the trading rule.

Return data of broad market indexes, like the S&P 500, appear to deviate significantly from normal distribution even in a separate bull or bear regime. In addition, time-varying return volatility is a more realistic model for index price dynamics. Therefore, conventional simulations with

[17] According to Exhibit 1(a), 30 trading days from 11/21/2008 to 1/6/2009 qualified as a bull market period.

randomly generated Gaussian distributed return series or even Markov Chain Monte Carlo (MCMC) simulation may be less useful in testing the current probabilistic timing formulation. Instead, we extensively back-tested the probabilistic market timing method with historical market data.

Our back tests included both in-sample (since data inception) and out-of-sample (since 1940) back tests performed on the S&P 500 index daily data. We utilized parameter specifications testing and derivation from historical bull/bear cycle statistics stemming back to 1928. The in-sample performance changes smoothly in general when varying the parameters used to define the bull/bear cycles are applied, as shown in Table 3. Nevertheless a closer examination indicates that parameter sensitivity or performance fluctuation still exists, e.g. the baseline case of Table 4 outperforms the case of row three in Table 3 despite only a small difference in the cycle definition parameters (L=19% versus 20%).

As most of the broad market indexes considered here for bull/bear cycle definition can have short market exposure through derivatives, such as exchange traded futures contracts. We extend the current long-only market timing model to a full scale long/short/flat positioning implementation, as Dai et al (2011) did. As a component of probabilistic market timing, shorting was found in back tests more effective for the more volatile and less mature Shanghai Stock Exchange Composite Index (SSEC) than the S&P 500 Index. As an extreme position in a theoretically optimal trend following system, shorting should at most be cautiously implemented in practice with an appropriate position size and risk control.

On the other hand, systematic learning from a 10-cycle rolling walk forward process appears to have some predictive value out-of-sample. Even accounting for reasonable transaction costs and dividend income arising from active trading and out-of-market periods, the current probabilistic market timing approach outperforms both a Buy-and-Hold strategy and a popular, empirically derived moving average technical rule in the long-term back tests.

Robustness of the bull/bear cycle statistics used for the conditional probability dynamics and market timing model is also important. Guo and Yin (2006) gave some convergence and error bound results about the estimations of a bull or bear regime's return and volatility. However, we also observe large variability in estimated bull or bear cycle durations or frequencies, as shown in ν columns of Exhibit 1(a) and (b). One approach to address this issue can be using additional empirical filter technique to identify the less trendy periods in a simplified bull or bear binary regime,

and adjust the calibration of bull/bear cycle frequencies and risk-adjusted average return parameters accordingly. In this way, the probabilistic market timing method may better align with the characteristics of real historical market data for future practical applications.

Conclusion

Unlike most tactical trading methodologies, the probabilistic market timing formulation pursues analytically an optimal combination of long term market cycle properties, a current return volatility projection, and the price trend. Although market cycle characteristics would seem to have significantly evolved during the past century, systematic learning about bull/bear cycle statistics seems still capable of adding value through a model driven market timing implementation. It is our belief that an adaptive approach of probabilistic market timing is an important development toward the long sought after optimal timing model, and may bring us one step closer to settling the "passive" versus "active" investing debate.

Exhibit 1 (a) S&P 500 Index Bull Market Statistics (1927/12/31-2014/1/15) with **Daily** Observation: *G* = 20%, D =25 Market Days

Start Date	End Date	Type	Index Price (PUH)	Duration (Days)	Total Return	Annualized Return	Annualized Volatility	Skew	Excess Kurtosis	Sharpe Ratio (r=0)	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Average Return Per Day	First Date to Confirm Bull (FDPEL)
12/31/1927	9/16/1929	Bull	31.86	507	80.41%	34.16%	14.39%	-0.58	2.03	2.37	2.04	0.50	0.16%	
11/14/1929	4/10/1930	Bull	25.92	118	46.77%	126.93%	25.96%	1.20	7.86	4.89	3.16	2.14	0.40%	
12/17/1930	2/26/1931	Bull	18.17	58	25.83%	171.37%	30.98%	0.10	4.46	5.53	3.23	4.34	0.45%	
10/6/1931	11/9/1931	Bull	11.52	28	30.61%	1006.25%	57.04%	1.24	2.91	17.64	4.23	9.00	1.09%	
6/2/1932	9/7/1932	Bull	9.31	81	111.59%	929.57%	51.23%	-0.07	0.13	18.14	4.57	3.11	1.38%	
2/28/1933	7/18/1933	Bull	12.2	109	120.61%	522.96%	47.55%	1.50	6.98	11.00	3.86	2.31	1.11%	
10/23/1933	2/6/1934	Bull	11.82	88	37.92%	151.11%	28.67%	0.68	0.71	5.27	3.22	2.86	0.43%	
3/15/1935	3/6/1937	Bull	18.68	595	131.76%	42.76%	26.15%	0.25	19.02	1.64	1.36	0.42	0.22%	
4/1/1938	10/10/1938	Bull	13.91	161	63.65%	116.18%	36.91%	0.38	2.31	3.15	2.09	1.57	0.40%	
4/10/1939	10/25/1939	Bull	13.21	167	29.76%	48.17%	24.04%	1.56	8.66	2.00	1.64	1.51	0.18%	
6/11/1940	11/9/1940	Bull	11.4	127	26.81%	60.20%	20.26%	0.83	4.90	2.97	2.33	1.98	0.21%	9/5/1940
4/29/1942	5/29/1946	Bull	19.25	1211	157.70%	21.77%	11.22%	-0.69	4.28	1.94	1.76	0.21	0.13%	10/2/1942
5/19/1947	6/15/1948	Bull	17.06	303	24.43%	19.94%	12.27%	-0.61	4.09	1.62	1.48	0.83	0.08%	5/15/1948
6/14/1949	8/2/1956	Bull	49.74	1889	267.08%	18.94%	10.96%	-1.07	8.40	1.73	1.58	0.13	0.14%	11/2/1949
10/23/1957	12/12/1961	Bull	72.64	1042	86.35%	16.25%	9.77%	0.12	3.65	1.66	1.54	0.24	0.08%	7/25/1958
6/27/1962	2/9/1966	Bull	94.06	913	79.78%	17.57%	8.39%	0.44	8.66	2.10	1.93	0.28	0.09%	12/6/1962
10/10/1966	11/29/1968	Bull	108.37	516	48.05%	21.12%	9.28%	0.31	1.85	2.28	2.07	0.49	0.09%	2/14/1967
5/27/1970	1/11/1973	Bull	120.24	665	73.53%	23.23%	10.79%	0.88	5.36	2.15	1.94	0.38	0.11%	9/28/1970
10/4/1974	9/21/1976	Bull	107.83	497	73.14%	32.09%	15.91%	0.38	1.68	2.02	1.75	0.51	0.15%	11/11/1974
3/7/1978	11/28/1980	Bull	140.52	692	61.70%	19.13%	13.51%	0.02	1.35	1.42	1.30	0.36	0.09%	8/9/1978
8/13/1982	8/25/1987	Bull	336.77	1273	228.81%	26.57%	14.26%	0.16	2.24	1.86	1.65	0.20	0.18%	9/21/1982
12/7/1987	7/16/1990	Bull	368.95	659	64.81%	21.05%	15.26%	-0.88	7.27	1.38	1.25	0.38	0.10%	3/8/1988
10/12/1990	7/17/1998	Bull	1186.69	1962	301.64%	19.55%	12.17%	-0.23	5.60	1.61	1.47	0.13	0.15%	2/6/1991
9/1/1998	3/24/2000	Bull	1527.51	395	59.53%	34.71%	20.68%	0.20	0.87	1.68	1.44	0.64	0.15%	11/19/1998
9/24/2001	1/4/2002	Bull	1172.51	72	21.40%	97.15%	17.95%	0.28	0.68	5.41	3.79	3.50	0.30%	12/5/2001
10/10/2002	10/9/2007	Bull	1565.15	1258	101.50%	15.07%	13.60%	0.20	2.37	1.11	1.03	0.20	0.08%	11/21/2002
11/21/2008	1/6/2009	Bull	934.7	30	24.22%	518.42%	50.17%	-0.63	1.86	10.33	3.64	8.40	0.81%	12/31/2008
3/10/2009	4/29/2011	Bull	1363.61	541	101.56%	38.61%	19.82%	0.37	3.59	1.95	1.65	0.47	0.19%	4/14/2009
10/4/2011	1/15/2014	Bull	1848.38	576	68.15%	25.48%	14.13%	0.14	2.38	1.80	1.61	0.44	0.12%	1/25/2012

Exhibit 1 (b) S&P 500 Index Bear Market Statistics (1927/12/31-2014/1/15) **Daily** Data: L = 19%, D =25 Market Days

Start Date	End Date	Type	Index Price (PEL)	Duration (Days)	Total Return	Annualized Return	Annualized Volatility	Skew	Excess Kurtosis	Sharpe Ratio (r=0)	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Average Return Per Day
9/17/1929	11/13/1929	Bear	17.66	45	-44.57%	-96.33%	62.86%	0.40	2.63	-1.53	-5.22	5.60	-0.99%
4/11/1930	12/16/1930	Bear	14.44	203	-44.29%	-51.63%	27.24%	-0.08	0.15	-1.90	-2.66	1.24	-0.22%
2/27/1931	10/5/1931	Bear	8.82	184	-51.46%	-62.84%	32.31%	0.55	2.45	-1.94	-3.06	1.37	-0.28%
11/10/1931	6/1/1932	Bear	4.4	168	-61.81%	-76.40%	42.25%	0.86	1.63	-1.81	-3.41	1.50	-0.37%
9/8/1932	2/27/1933	Bear	5.53	140	-40.60%	-60.84%	46.32%	0.54	2.19	-1.31	-2.02	1.80	-0.29%
7/19/1933	10/21/1933	Bear	8.57	74	-29.75%	-69.96%	49.54%	-0.19	0.91	-1.41	-2.42	3.41	-0.40%
2/7/1934	3/14/1935	Bear	8.06	330	-31.81%	-25.35%	24.45%	0.51	8.46	-1.04	-1.20	0.76	-0.10%
3/8/1937	3/31/1938	Bear	8.5	320	-54.50%	-46.21%	31.99%	-0.10	2.84	-1.44	-1.94	0.79	-0.17%
10/11/1938	4/8/1939	Bear	10.18	146	-26.82%	-41.66%	25.53%	-1.06	3.05	-1.63	-2.11	1.73	-0.18%
10/26/1939	6/10/1940	Bear	8.99	187	-31.95%	-40.47%	18.80%	-3.19	15.13	-2.15	-2.76	1.35	-0.17%
11/12/1940	4/28/1942	Bear	7.47	439	-34.47%	-21.55%	13.56%	-0.11	3.87	-1.59	-1.79	0.57	-0.08%
5/31/1946	5/17/1947	Bear	13.71	273	-28.78%	-26.90%	21.02%	-1.58	11.95	-1.28	-1.49	0.92	-0.11%
6/16/1948	6/13/1949	Bear	13.55	281	-20.57%	-18.66%	13.91%	-1.43	5.61	-1.34	-1.48	0.90	-0.07%
8/3/1956	10/22/1957	Bear	38.98	307	-21.63%	-18.13%	13.35%	-0.21	7.02	-1.36	-1.50	0.82	-0.07%
12/13/1961	6/26/1962	Bear	52.32	135	-27.97%	-45.80%	17.55%	-0.92	11.16	-2.61	-3.49	1.87	-0.21%
2/10/1966	10/7/1966	Bear	73.2	167	-22.18%	-31.50%	12.33%	-0.14	0.65	-2.55	-3.07	1.51	-0.13%
12/2/1968	5/26/1970	Bear	69.29	369	-36.06%	-26.32%	11.37%	-0.17	1.13	-2.31	-2.68	0.68	-0.10%
1/12/1973	10/3/1974	Bear	62.28	436	-48.20%	-31.63%	17.82%	0.31	0.53	-1.77	-2.13	0.58	-0.11%
9/22/1976	3/6/1978	Bear	86.9	366	-19.41%	-13.81%	9.46%	-0.05	-0.17	-1.46	-1.57	0.69	-0.05%
12/1/1980	8/12/1982	Bear	102.42	430	-27.11%	-16.92%	13.93%	-0.04	0.56	-1.21	-1.33	0.59	-0.06%
8/26/1987	12/4/1987	Bear	223.87	71	-33.52%	-76.53%	53.64%	-2.75	17.63	-1.43	-2.69	3.55	-0.47%
7/17/1990	10/11/1990	Bear	295.46	62	-19.92%	-59.46%	20.29%	0.20	0.33	-2.93	-4.44	4.06	-0.32%
7/20/1998	8/31/1998	Bear	957.53	31	-19.31%	-82.52%	28.26%	-1.44	3.65	-2.92	-6.15	8.13	-0.62%
3/27/2000	9/21/2001	Bear	965.8	377	-36.77%	-26.39%	21.71%	0.09	1.69	-1.22	-1.41	0.67	-0.10%
1/7/2002	10/9/2002	Bear	776.76	192	-33.75%	-41.75%	26.26%	0.50	0.99	-1.59	-2.06	1.31	-0.18%
10/10/2007	11/20/2008	Bear	752.44	283	-51.93%	-47.91%	36.19%	0.27	5.80	-1.32	-1.80	0.89	-0.18%
1/7/2009	3/9/2009	Bear	676.53	42	-27.62%	-85.62%	37.80%	0.04	-0.33	-2.27	-5.11	6.00	-0.66%
5/2/2011	10/3/2011	Bear	1099.2	108	-19.39%	-39.52%	27.30%	-0.49	2.43	-1.45	-1.84	2.33	-0.18%

Notes of Exhibit 1:

1. The "Start/End" dates of the Exhibits are all inclusive. We use the daily S&P 500 price index rather than the S&P 500 Total Return Index because the latter was not directly available prior to 1989.

2. The *average annual return* μ equals 252 times the *average daily return*, which is [(1+Total Return of a Bull or Bear Cycle) $^{(1/\text{Cycle Duration in Days})}$-1].

3. *Sharpe ratios* in Exhibit 1 are computed with risk free rate r=0 as: "Annualized Return" divided by "Annualized Volatility".

4. The "*Annualized Volatility*" is the annualized return standard deviation calculated from daily data in each bull or bear period. Since average daily return in a bull period is always positive and that of a bear period always negative, the average volatility σ across bull bear cycles is slightly different from the conventional standard deviation of all sampled daily returns. σ is calculated as the square root of duration based weighted average of variances for all the bull and bear cycles.

5. After the initial ten bull markets since 12/31/1927, Exhibit 1(a) also lists the *first date confirming last bear ending low* (*FDPEL*), the first one being 9/5/1940.

6. The average of bull cycle annualized volatility is 22.1% from averaging the column in Exhibit 1(a), while the average bear cycle annualized volatility is 27.0% from Exhibit 1(b).

7. We can use the simple ratio of standard deviation to the absolute value of the mean to measure sample variability. Exhibit 1(a) bull cycle's λ column has a variability ratio at 0.45, while larger for σ column at 0.61 and "Average Return Daily" column at 1.08. Exhibit 1(b) bear cycle's λ column variability is at 0.48, while larger for σ column at 0.50 and "Average Return Daily" column at 0.87.

8. The non-zero "Skew" and "Excess Kurtosis" values of each bull or bear market indicate significant deviation from normal distribution for daily returns, even after the market regime separations.

Exhibit 2: Bull/Bear Cycle Summary Statistics with Different Gain/Loss and Duration Thresholds: S&P 500 Index (1928-2014/1/15)

(a) Bull Cycles

G	D (Days)	Number of Bull Cycles	Average Duration (Days)	Average Return	Average CAGR	Average STDEV	Average Sharpe	Average Skew	Average Kurtosis	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Average Return Per Day	Average Annual Return
10%	15	73	215	39.90%	166.94%	20.45%	6.48	0.29	2.43	3.59	3.81	0.34%	75.16%
15%	20	37	453	71.47%	188.19%	24.37%	5.26	0.29	3.78	2.54	2.72	0.35%	72.70%
20%	25	25	685	108.08%	162.60%	23.21%	4.42	0.26	4.66	2.30	1.80	0.34%	65.23%
25%	30	19	924	143.37%	129.77%	22.68%	3.76	0.35	5.17	2.09	1.18	0.32%	56.74%
30%	35	12	1507	273.70%	156.77%	23.78%	4.09	0.21	5.80	2.03	0.92	0.38%	61.73%
35%	40	10	1865	459.94%	132.73%	22.90%	3.62	0.01	8.48	1.87	0.84	0.34%	53.11%
40%	45	9	2113	520.53%	144.79%	24.23%	3.77	-0.06	8.73	1.86	0.89	0.37%	56.59%

(b) Bear Cycles

L	D (Days)	Number of Bear Cycles	Average Duration (Days)	Average Return	Average CAGR	Average STDEV	Average Sharpe	Average Skew	Average Kurtosis	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Average Return Per Day	Average Annual Return
10%	15	72	96	-21.14%	-55.76%	23.54%	-2.50	-0.19	2.45	-4.21	4.97	-0.36%	-101.88%
15%	20	36	164	-29.92%	-53.64%	26.63%	-2.07	-0.30	3.70	-3.45	3.05	-0.32%	-96.61%
20%	25	24	231	-36.10%	-45.68%	27.78%	-1.70	-0.34	4.47	-2.50	1.71	-0.24%	-76.69%
25%	30	18	309	-40.16%	-43.90%	29.50%	-1.54	-0.39	5.13	-2.23	1.48	-0.22%	-73.25%
30%	35	11	416	-47.88%	-44.46%	33.63%	-1.34	-0.08	4.53	-2.08	1.39	-0.24%	-80.77%
35%	40	9	446	-51.02%	-43.71%	31.77%	-1.41	0.21	3.04	-2.16	1.24	-0.23%	-80.30%
40%	45	8	456	-52.89%	-45.89%	34.32%	-1.30	0.26	3.27	-2.10	1.31	-0.25%	-86.53%

Note for Exhibit 2:

All average statistics in the above table are an "arithmetic average" from a column of Exhibit 1 format Table for particular *G*/*L* and *D*.

Exhibit 3: Shanghai Stock Exchange Composite Index (SSEC) Market Cycle Statistics (1990/12/19-2014/6/6) using **Daily** Prices

G = 20%, L = 19%, D =25 Market Days

Start Date	End Date	Regime Type	Ending Index Price	Duration (Days)	Total Return	Annualized Return	Annualized Volatility	Sharpe Ratio (r=0)	Max Drawdown	Skew	Excess Kurtosis	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Average Return Per Day	Average Annual Return
12/19/1990	2/22/1991	Bull	134.87	45	34.90%	455.34%	23.66%	19.24	4.57%	2.13	3.52	7.27	5.42	0.78%	172.04%
5/17/1991	5/25/1992	Bull	1421.57	260	1244.02%	1140.75%	104.18%	10.95	4.61%	15.55	247.45	2.43	0.94	4.78%	253.13%
12/11/1992	2/15/1993	Bull	1536.82	46	145.80%	13694.84%	61.53%	222.56	8.71%	0.80	2.07	8.09	5.30	3.17%	497.70%
10/28/1993	12/7/1993	Bull	1022.34	29	31.34%	968.78%	28.31%	34.22	2.47%	0.55	0.68	8.41	8.41	1.08%	238.06%
8/1/1994	9/13/1994	Bull	1033.47	32	209.50%	730890.29%	148.47%	4922.81	12.67%	1.26	2.38	6.10	7.63	6.55%	906.12%
2/8/1995	5/22/1995	Bull	897.42	74	68.53%	491.50%	71.33%	6.89	16.12%	4.52	28.23	2.50	3.30	0.93%	178.40%
7/4/1995	8/14/1995	Bull	776.13	30	26.57%	623.95%	33.85%	18.44	3.52%	0.52	0.87	5.87	8.13	0.89%	198.76%
1/23/1996	5/12/1997	Bull	1500.40	340	190.52%	120.44%	39.65%	3.04	30.62%	-0.30	3.56	2.00	0.72	0.56%	79.17%
9/24/1997	6/3/1998	Bull	1420.00	181	36.28%	53.87%	20.32%	2.65	9.17%	0.46	5.62	2.12	1.35	0.20%	43.13%
8/18/1998	6/29/1999	Bull	1739.21	226	62.48%	71.81%	24.27%	2.96	18.08%	0.66	1.96	2.23	1.08	0.28%	54.18%
12/28/1999	6/13/2001	Bull	2242.42	382	66.68%	40.08%	18.84%	2.13	11.04%	0.67	9.88	1.79	0.64	0.17%	33.73%
1/23/2002	7/8/2002	Bull	1732.93	119	27.54%	67.40%	29.11%	2.32	13.24%	1.37	7.08	1.77	2.05	0.23%	51.58%
1/6/2003	4/15/2003	Bull	1631.47	72	23.61%	109.98%	19.06%	5.77	4.54%	1.58	5.51	3.90	3.39	0.33%	74.30%
11/19/2003	4/6/2004	Bull	1777.52	100	35.01%	113.08%	19.04%	5.94	4.92%	0.36	0.25	3.98	2.44	0.35%	75.77%
7/12/2005	10/16/2007	Bull	6092.06	573	502.28%	120.27%	25.97%	4.63	16.59%	-0.96	4.10	3.05	0.43	0.88%	79.10%
11/5/2008	11/23/2009	Bull	3338.66	272	95.62%	86.20%	31.82%	2.71	23.15%	-0.33	1.59	1.96	0.90	0.35%	62.24%
7/6/2010	11/8/2010	Bull	3159.51	89	33.65%	127.36%	19.32%	6.59	4.06%	0.07	0.02	4.26	2.74	0.38%	82.27%
12/4/2012	2/6/2013	Bull	2434.48	44	24.22%	246.35%	18.94%	13.01	1.85%	1.08	1.72	6.58	5.55	0.55%	124.54%
2/25/1991	5/16/1991	Bear	105.77	58	-21.58%	-65.22%	5.28%	-12.34	21.30%	-1.19	10.42	-19.96	4.21	-0.37%	-105.38%
5/26/1992	12/10/1992	Bear	625.23	143	-56.02%	-76.48%	81.40%	-0.94	69.76%	1.85	8.77	-1.77	1.71	-0.39%	-144.30%
2/16/1993	10/27/1993	Bear	778.38	182	-49.35%	-61.01%	61.30%	-1.00	48.57%	0.80	3.40	-1.53	1.34	-0.27%	-94.00%
12/8/1993	7/29/1994	Bear	333.92	168	-67.34%	-81.33%	41.76%	-1.95	66.93%	-0.12	4.33	-4.01	1.45	-0.40%	-167.25%
9/14/1994	2/7/1995	Bear	532.49	105	-48.48%	-79.64%	59.80%	-1.33	45.61%	0.30	1.90	-2.65	2.32	-0.46%	-158.64%
5/23/1995	7/3/1995	Bear	613.18	30	-31.67%	-95.92%	58.29%	-1.65	18.28%	-2.52	9.86	-5.45	8.13	-1.06%	-317.82%
8/15/1995	1/22/1996	Bear	516.46	115	-33.46%	-59.04%	24.81%	-2.38	33.40%	0.30	1.04	-3.59	2.12	-0.29%	-89.09%
5/13/1997	9/23/1997	Bear	1041.97	96	-30.55%	-61.60%	39.32%	-1.57	28.84%	-1.00	1.58	-2.43	2.54	-0.32%	-95.52%
6/4/1998	8/17/1998	Bear	1070.41	53	-24.62%	-73.91%	26.71%	-2.77	24.01%	-1.82	7.94	-5.02	4.60	-0.46%	-133.99%
6/30/1999	12/27/1999	Bear	1345.35	129	-22.65%	-39.44%	27.24%	-1.45	20.37%	0.45	5.22	-1.84	1.89	-0.18%	-50.10%
6/14/2001	1/22/2002	Bear	1358.69	159	-39.41%	-54.80%	26.03%	-2.11	39.17%	1.09	9.15	-3.05	1.53	-0.25%	-79.28%
7/9/2002	1/3/2003	Bear	1319.87	129	-23.84%	-41.25%	14.75%	-2.80	23.71%	-0.36	1.16	-3.60	1.89	-0.18%	-53.13%
4/16/2003	11/18/2003	Bear	1316.56	155	-19.30%	-29.44%	15.24%	-1.93	18.41%	0.45	1.81	-2.29	1.57	-0.12%	-34.85%
4/7/2004	7/11/2005	Bear	1011.50	329	-43.09%	-35.07%	21.66%	-1.62	43.00%	1.29	4.87	-1.99	0.74	-0.13%	-43.15%
10/17/2007	11/4/2008	Bear	1706.70	264	-71.98%	-70.32%	42.93%	-1.64	71.73%	0.39	1.41	-2.82	0.92	-0.27%	-121.17%
11/24/2009	7/5/2010	Bear	2363.95	159	-29.19%	-42.14%	23.12%	-1.82	29.05%	-0.53	1.26	-2.36	1.53	-0.18%	-54.65%
11/9/2010	12/3/2012	Bear	1959.77	512	-37.97%	-20.95%	18.07%	-1.16	37.74%	-0.14	1.49	-1.30	0.48	-0.07%	-23.50%
2/7/2013	6/6/2014	Bear	2029.96	317	-16.62%	-13.41%	17.29%	-0.78	19.90%	-0.30	2.30	-0.83	0.77	-0.05%	-14.39%

Exhibit 4: Shanghai Stock Exchange Composite Index (SSEC) Market Cycle Statistics (1990/12/19-2014/6/6) using **Daily** Prices

G = 20%, L = 19%, D =50 Market Days

Start Date	End Date	Regime Type	Ending Index Price	Duration (Days)	Total Return	Annualized Return	Annualized Volatility	Sharpe Ratio (r=0)	Max Drawdown	Skew	Excess Kurtosis	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Average Return Per Day	Average Annual Return
12/19/1990	2/22/1991	Bull	134.87	45	29.20%	365.09%	22.31%	16.36	4.57%	2.36	4.90	6.91	5.42	0.65%	154.19%
5/17/1991	5/25/1992	Bull	1421.57	260	1244.02%	1140.75%	104.18%	10.95	4.61%	15.55	247.45	2.43	0.94	4.78%	253.13%
2/8/1995	8/14/1995	Bull	776.13	133	45.75%	104.18%	63.15%	1.65	31.67%	3.19	28.53	1.13	1.83	0.34%	71.49%
1/23/1996	5/12/1997	Bull	1500.40	323	190.52%	129.80%	40.67%	3.19	30.62%	-0.31	3.25	2.05	0.76	0.59%	83.35%
9/24/1997	6/3/1998	Bull	1420.00	168	36.28%	59.09%	21.08%	2.80	9.17%	0.41	5.02	2.20	1.45	0.22%	46.47%
8/18/1998	6/29/1999	Bull	1739.21	211	62.48%	78.55%	25.11%	3.13	18.08%	0.61	1.62	2.31	1.16	0.30%	58.04%
12/28/1999	6/13/2001	Bull	2242.42	344	66.68%	45.39%	19.84%	2.29	11.04%	0.60	8.61	1.89	0.71	0.19%	37.45%
1/23/2002	7/8/2002	Bull	1732.93	104	27.54%	80.31%	31.12%	2.58	13.24%	1.24	5.80	1.90	2.35	0.26%	59.02%
1/6/2003	4/15/2003	Bull	1631.47	65	23.61%	127.44%	20.01%	6.37	4.54%	1.44	4.67	4.11	3.75	0.36%	82.31%
11/19/2003	4/6/2004	Bull	1777.52	91	35.01%	129.63%	19.90%	6.51	4.92%	0.27	-0.03	4.18	2.68	0.38%	83.27%
7/12/2005	10/16/2007	Bull	6092.06	549	502.28%	128.00%	26.51%	4.83	16.59%	-0.97	3.86	3.11	0.44	0.91%	82.56%
11/5/2008	11/23/2009	Bull	3338.66	257	95.62%	93.08%	32.73%	2.84	23.15%	-0.34	1.36	2.01	0.95	0.37%	65.88%
7/6/2010	11/8/2010	Bull	3159.51	83	33.65%	141.27%	19.96%	7.08	4.06%	0.01	-0.15	4.42	2.94	0.41%	88.23%
2/25/1991	5/16/1991	Bear	105.77	58	-21.58%	-65.22%	5.28%	-12.34	21.30%	-1.19	10.42	-19.96	4.21	-0.37%	-105.38%
5/26/1992	2/7/1995	Bear	532.49	689	-62.54%	-30.17%	70.08%	-0.43	78.27%	1.79	10.17	-0.51	0.35	-0.09%	-35.88%
8/15/1995	1/22/1996	Bear	516.46	114	-33.46%	-59.36%	24.92%	-2.38	33.40%	0.30	1.01	-3.61	2.14	-0.29%	-89.88%
5/13/1997	9/23/1997	Bear	1041.97	94	-30.55%	-62.37%	39.73%	-1.57	28.84%	-0.98	1.48	-2.46	2.60	-0.33%	-97.54%
6/4/1998	8/17/1998	Bear	1070.41	53	-24.62%	-73.91%	26.71%	-2.77	24.01%	-1.82	7.94	-5.02	4.60	-0.46%	-133.99%
6/30/1999	12/27/1999	Bear	1345.35	123	-22.65%	-40.91%	27.90%	-1.47	20.37%	0.46	4.86	-1.88	1.98	-0.18%	-52.55%
6/14/2001	1/22/2002	Bear	1358.69	151	-39.41%	-56.66%	26.69%	-2.12	39.17%	1.09	8.64	-3.13	1.62	-0.26%	-83.47%
7/9/2002	1/3/2003	Bear	1319.87	122	-23.84%	-43.02%	15.15%	-2.84	23.71%	-0.31	0.94	-3.71	2.00	-0.20%	-56.18%
4/16/2003	11/18/2003	Bear	1316.56	143	-19.30%	-31.47%	15.85%	-1.99	18.41%	0.47	1.48	-2.38	1.71	-0.13%	-37.76%
4/7/2004	7/11/2005	Bear	1011.50	306	-43.09%	-37.14%	22.45%	-1.65	43.00%	1.27	4.38	-2.07	0.80	-0.14%	-46.38%
10/17/2007	11/4/2008	Bear	1706.70	259	-71.98%	-71.00%	43.33%	-1.64	71.73%	0.40	1.34	-2.85	0.94	-0.28%	-123.47%
11/24/2009	7/5/2010	Bear	2363.95	151	-29.19%	-43.79%	23.72%	-1.85	29.05%	-0.50	1.04	-2.43	1.62	-0.19%	-57.54%
11/9/2010	6/6/2014	Bear	2029.96	869	-35.75%	-12.03%	17.97%	-0.67	38.28%	-0.11	1.81	-0.71	0.28	-0.04%	-12.81%

Exhibit 5: S&P 500 Index Bull and Bear Market Statistics (1927/12/31-2014/1/10), **Weekly** Data: G = 20%, L = 15%, D =5 Weeks

Start Date	End Date	Type	Index Price (PUH)	Duration (Weeks)	Total Return	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Start Date	End Date	Type	Index Price (PEL)	Duration (Weeks)	Total Return	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})
12/31/1927	9/7/1929	Bull	31.75	89	79.78%	2.13	0.58	9/14/1929	12/21/1929	Bear	20.21	15	-36.35%	-3.62	3.47
12/28/1929	4/12/1930	Bull	25.89	16	28.10%	5.09	3.25	4/17/1930	12/27/1930	Bear	14.76	37	-42.99%	-2.56	1.41
1/3/1931	2/28/1931	Bull	17.93	9	21.48%	4.19	5.78	3/7/1931	10/3/1931	Bear	9.37	31	-47.74%	-3.15	1.68
10/10/1931	11/7/1931	Bull	11.37	5	21.34%	3.92	10.40	11/14/1931	7/9/1932	Bear	4.46	35	-60.77%	-3.26	1.49
7/16/1932	9/3/1932	Bull	9.06	8	103.14%	9.28	6.50	9/10/1932	2/25/1933	Bear	5.59	25	-38.30%	-1.89	2.08
3/18/1933	7/8/1933	Bull	11.97	17	114.13%	4.80	3.06	7/15/1933	10/21/1933	Bear	8.57	15	-28.40%	-2.22	3.47
10/28/1933	2/17/1934	Bull	11.59	17	35.24%	3.29	3.06	2/24/1934	3/23/1935	Bear	8.46	57	-27.01%	-0.97	0.91
3/30/1935	3/6/1937	Bull	18.68	102	120.80%	2.19	0.51	3/13/1937	3/26/1938	Bear	9.2	55	-50.75%	-2.03	0.95
4/2/1938	11/12/1938	Bull	13.78	33	49.78%	1.62	1.58	11/19/1938	4/8/1939	Bear	10.18	21	-26.12%	-2.72	2.48
4/15/1939	10/21/1939	Bull	13.02	28	27.90%	1.86	1.86	10/28/1939	6/1/1940	Bear	9.23	32	-29.11%	-2.24	1.63
6/8/1940	11/9/1940	Bull	11.4	23	23.51%	2.75	2.26	11/16/1940	4/25/1942	Bear	7.57	76	-33.60%	-1.74	0.68
5/2/1942	5/31/1946	Bull	19.18	213	153.37%	1.84	0.24	6/7/1946	5/17/1947	Bear	13.71	50	-28.52%	-1.99	1.04
5/24/1947	6/18/1948	Bull	16.96	57	23.71%	1.37	0.91	6/25/1948	6/10/1949	Bear	13.81	51	-18.57%	-1.50	1.02
6/17/1949	8/3/1956	Bull	49.64	373	259.45%	1.58	0.14	8/10/1956	12/20/1957	Bear	39.48	71	-20.47%	-1.52	0.73
12/27/1957	12/8/1961	Bull	72.04	206	82.47%	1.65	0.25	12/15/1961	6/22/1962	Bear	52.68	28	-26.87%	-3.54	1.86
6/29/1962	2/11/1966	Bull	93.81	190	78.08%	1.78	0.27	2/18/1966	10/7/1966	Bear	73.2	34	-21.97%	-2.60	1.53
10/14/1966	11/29/1968	Bull	108.37	112	48.05%	1.79	0.46	12/6/1968	5/22/1970	Bear	72.25	77	-33.33%	-2.07	0.68
5/29/1970	1/5/1973	Bull	119.87	137	65.91%	1.55	0.38	1/12/1973	10/4/1974	Bear	62.34	91	-47.99%	-1.80	0.57
10/11/1974	12/31/1976	Bull	107.46	117	72.38%	1.34	0.44	1/7/1977	3/3/1978	Bear	87.45	61	-18.62%	-1.56	0.85
3/10/1978	11/28/1980	Bull	140.52	143	60.69%	1.24	0.36	12/5/1980	8/6/1982	Bear	103.71	88	-26.20%	-1.17	0.59
8/13/1982	8/21/1987	Bull	335.9	263	223.88%	1.58	0.20	8/28/1987	12/4/1987	Bear	223.87	15	-33.35%	-4.57	3.47
12/11/1987	7/13/1990	Bull	367.31	136	64.07%	1.28	0.38	7/20/1990	10/12/1990	Bear	300.03	13	-18.32%	-4.98	4.00
10/19/1990	7/17/1998	Bull	1186.69	405	295.52%	1.54	0.13	7/24/1998	9/4/1998	Bear	973.89	7	-17.93%	-8.51	7.43
9/11/1998	3/24/2000	Bull	1527.51	81	56.85%	1.44	0.64	3/31/2000	9/21/2001	Bear	965.8	78	-36.77%	-1.39	0.67
9/28/2001	1/4/2002	Bull	1172.51	15	21.40%	3.65	3.47	1/11/2002	10/4/2002	Bear	800.58	39	-31.72%	-2.55	1.33
10/11/2002	10/12/2007	Bull	1561.8	261	95.08%	1.10	0.20	10/19/2007	3/6/2009	Bear	683.38	73	-56.24%	-1.86	0.71
3/13/2009	4/23/2010	Bull	1217.28	59	78.13%	2.62	0.88	4/30/2010	7/2/2010	Bear	1022.6	10	-15.99%	-3.79	5.20
7/9/2010	4/29/2011	Bull	1363.61	43	33.35%	2.70	1.21	5/6/2011	8/19/2011	Bear	1123.5	16	-17.61%	-3.01	3.25
8/26/2011	1/10/2014	Bull	1842.37	125	63.98%	1.42	0.42								

Exhibit 6: **S&P 500** Bull and Bear Market Statistics (1927/12/31-2013/12/31), Monthly Data: G = 20%, L = 15%, D =2 Months

Start Date	End Date	Type	Index Price (PUH)	Duration (Months)	Total Return	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Start Date	End Date	Type	Index Price (PEL)	Duration (Months)	Total Return	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})
12/31/1927	8/30/1929	Bull	31.71	21	79.56%	1.97	0.57	9/30/1929	6/30/1932	Bear	4.43	34	-86.03%	-1.88	0.35
7/30/1932	8/31/1932	Bull	8.39	2	89.39%	1175.40	6.00	9/30/1932	2/28/1933	Bear	5.66	6	-32.54%	-2.48	2.00
3/31/1933	8/31/1933	Bull	11.09	6	95.94%	2.43	2.00	9/30/1933	10/31/1933	Bear	8.96	2	-19.21%	-19.73	6.00
11/29/1933	1/31/1934	Bull	11.17	3	24.67%	5.57	4.00	2/28/1934	3/30/1935	Bear	8.47	14	-24.17%	-1.35	0.86
4/30/1935	2/27/1937	Bull	18.09	23	113.58%	3.12	0.52	3/31/1937	3/31/1938	Bear	8.50	13	-53.01%	-2.17	0.92
4/30/1938	12/31/1938	Bull	13.21	9	55.41%	1.84	1.33	1/31/1939	4/30/1942	Bear	7.66	40	-42.01%	-0.70	0.30
5/29/1942	5/31/1946	Bull	19.18	49	150.39%	1.82	0.24	6/28/1946	2/28/1948	Bear	14.00	21	-27.01%	-1.36	0.57
3/31/1948	7/31/1956	Bull	49.39	101	252.79%	1.15	0.12	8/31/1956	12/31/1957	Bear	39.99	17	-19.03%	-1.29	0.71
1/31/1958	12/29/1961	Bull	71.55	48	78.92%	1.48	0.25	1/31/1962	6/29/1962	Bear	54.75	6	-23.48%	-3.64	2.00
7/31/1962	1/31/1966	Bull	92.88	43	69.64%	1.54	0.28	2/28/1966	9/30/1966	Bear	76.56	8	-17.57%	-2.76	1.50
10/31/1966	11/29/1968	Bull	108.37	26	41.55%	1.37	0.46	12/31/1968	6/30/1970	Bear	72.72	19	-32.90%	-1.67	0.63
7/31/1970	12/29/1972	Bull	118.05	30	62.33%	1.79	0.40	1/31/1973	9/30/1974	Bear	63.54	21	-46.18%	-2.35	0.57
10/31/1974	12/31/1976	Bull	107.46	27	69.12%	1.27	0.44	1/31/1977	2/28/1978	Bear	87.04	14	-19.00%	-1.81	0.86
3/31/1978	11/28/1980	Bull	140.52	33	61.44%	1.12	0.36	12/31/1980	7/30/1982	Bear	107.09	20	-23.79%	-1.39	0.60
8/31/1982	8/31/1987	Bull	329.79	61	207.96%	1.52	0.20	9/30/1987	11/30/1987	Bear	230.31	3	-30.16%	-3.95	4.00
12/31/1987	5/31/1990	Bull	361.23	30	56.85%	1.36	0.40	6/29/1990	10/31/1990	Bear	304.00	5	-15.84%	-3.00	2.40
11/30/1990	6/30/1998	Bull	1133.84	92	272.97%	1.55	0.13	7/31/1998	8/31/1998	Bear	957.53	2	-15.55%	-2.96	6.00
9/30/1998	8/31/2000	Bull	1517.68	24	58.50%	1.53	0.50	9/29/2000	9/30/2002	Bear	815.26	25	-46.28%	-1.66	0.48
10/31/2002	10/31/2007	Bull	1549.38	61	90.05%	1.32	0.20	11/30/2007	2/27/2009	Bear	735.09	16	-52.56%	-2.78	0.75
3/31/2009	4/29/2011	Bull	1363.61	26	85.50%	1.82	0.46	5/31/2011	9/30/2011	Bear	1131.42	5	-17.03%	-4.85	2.40
10/31/2011	12/31/2013	Bull	1848.36	27	63.37%	2.82	0.44								

Exhibit 7: **Dow Jones Industrial Average (DJIA)** Bull/Bear **Monthly** Statistics (Mar.1885-2013): *D*=2Months, *G*=20%, *L*=15%

Start Date	End Date	Type	Duration (Months)	Total Return	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})	Start Date	End Date	Type	Duration (Weeks)	Total Return	$\lambda=\mu/\sigma$	ν Duration^{-1} (year^{-1})
2/28/1885	11/30/1886	Bull	22	33.17%	1.09	0.55	12/31/1886	3/31/1888	Bear	16	-18.05%	-1.26	0.75
4/30/1888	5/31/1890	Bull	26	29.47%	1.02	0.46	6/30/1890	7/31/1893	Bear	38	-34.69%	-0.97	0.32
8/31/1893	8/31/1895	Bull	25	29.05%	0.87	0.48	9/30/1895	8/31/1896	Bear	12	-27.83%	-1.94	1.00
9/30/1896	8/31/1897	Bull	12	71.42%	2.46	1.00	9/30/1897	3/31/1898	Bear	7	-17.12%	-2.17	1.71
4/30/1898	4/29/1899	Bull	13	68.91%	2.22	0.92	5/31/1899	9/29/1900	Bear	17	-29.27%	-1.25	0.71
10/31/1900	6/29/1901	Bull	9	43.63%	2.56	1.33	7/31/1901	11/30/1903	Bear	29	-43.14%	-1.41	0.41
12/31/1903	1/31/1906	Bull	26	127.14%	2.39	0.46	2/28/1906	10/31/1907	Bear	21	-42.70%	-1.59	0.57
11/30/1907	9/30/1909	Bull	23	72.54%	1.97	0.52	10/30/1909	10/31/1911	Bear	25	-23.89%	-1.06	0.48
11/29/1911	9/30/1912	Bull	11	24.21%	2.03	1.09	10/31/1912	7/30/1914	Bear	22	-24.15%	-1.10	0.55
12/31/1914	11/29/1916	Bull	24	102.61%	1.60	0.50	12/29/1916	11/30/1917	Bear	12	-31.45%	-2.08	1.00
12/31/1917	10/31/1919	Bull	23	63.65%	1.77	0.52	11/29/1919	8/31/1921	Bear	22	-43.57%	-1.46	0.55
9/30/1921	2/28/1923	Bull	18	54.82%	2.58	0.67	3/31/1923	7/31/1923	Bear	5	-16.34%	-3.13	2.40
8/31/1923	8/30/1929	Bull	73	337.61%	1.42	0.16	9/30/1929	6/30/1932	Bear	34	-88.73%	-1.94	0.35
7/30/1932	8/31/1932	Bull	2	70.77%	18.41	6.00	9/30/1932	2/28/1933	Bear	6	-29.75%	-2.27	2.00
3/31/1933	1/31/1934	Bull	11	108.64%	1.76	1.09	2/28/1934	7/31/1934	Bear	6	-17.87%	-3.10	2.00
8/31/1934	2/27/1937	Bull	31	112.70%	2.49	0.39	3/31/1937	3/31/1938	Bear	13	-47.17%	-1.91	0.92
4/30/1938	12/31/1938	Bull	9	56.43%	2.05	1.33	1/31/1939	4/30/1942	Bear	40	-38.35%	-0.70	0.30
5/29/1942	5/31/1946	Bull	49	122.54%	1.75	0.24	6/28/1946	2/28/1948	Bear	21	-21.19%	-1.16	0.57
3/31/1948	7/31/1956	Bull	101	209.52%	1.13	0.12	8/31/1956	12/31/1957	Bear	17	-15.85%	-1.01	0.71
1/31/1958	12/29/1961	Bull	48	67.81%	1.25	0.25	1/31/1962	6/29/1962	Bear	6	-23.23%	-3.75	2.00
7/31/1962	1/31/1966	Bull	43	75.22%	1.57	0.28	2/28/1966	9/30/1966	Bear	8	-21.28%	-4.25	1.50
10/31/1966	11/29/1968	Bull	26	27.24%	0.89	0.46	12/31/1968	6/30/1970	Bear	19	-30.62%	-1.65	0.63
7/31/1970	12/29/1972	Bull	30	49.23%	1.37	0.40	1/31/1973	9/30/1974	Bear	21	-40.40%	-1.71	0.57
10/31/1974	12/31/1976	Bull	27	65.27%	1.23	0.44	1/31/1977	2/28/1978	Bear	14	-26.14%	-2.89	0.86
3/31/1978	3/31/1981	Bull	37	35.28%	0.66	0.32	4/30/1981	7/30/1982	Bear	16	-19.45%	-1.63	0.75
8/31/1982	8/31/1987	Bull	61	229.33%	1.59	0.20	9/30/1987	11/30/1987	Bear	3	-31.15%	-3.78	4.00
12/31/1987	7/31/1990	Bull	32	58.45%	1.34	0.38	8/31/1990	10/31/1990	Bear	3	-15.93%	-4.03	4.00
11/30/1990	4/30/1998	Bull	90	271.09%	1.52	0.13	5/29/1998	8/31/1998	Bear	4	-16.82%	-2.13	3.00
9/30/1998	12/31/1999	Bull	16	52.50%	2.22	0.75	1/31/2000	9/30/2002	Bear	33	-33.97%	-0.83	0.36
10/31/2002	10/31/2007	Bull	61	83.48%	1.16	0.20	11/30/2007	2/27/2009	Bear	16	-49.30%	-2.81	0.75
3/31/2009	12/30/2013	Bull	58	133.67%	1.36	0.21							

Symbol List

P	Probability to continue or switch from bull market regime to bear regime, or vice versa
p	Conditional probability that currently market is in a bull regime
h	period of market price observation, such as a day, week or month at period close
Δt	time increment of possible trading interval, such as a daily, weekly or monthly period
S	Price or total return price of a market index
μ	Average annual return during a bull or bear market regime for a market index
σ	Annualized volatility during a bull or bear market regime for a market index
λ	Volatility scaled annual return over a bull or bear cycle for a market index $\lambda = \mu/\sigma$
ν	Frequency of bull or bear market per year or reciprocal of duration of a bull/bear market
r	Risk free annualized interest rate over a bull plus bear market cycle
t	subscript of time count in number of allowable trading period intervals
bull	subscript for bull market regime
bear	subscript for bear market regime
p_{sell}	Bull probability threshold to go from long to flat when crossed below from above
p_{buy}	Bull probability threshold to go from flat to long when crossed above from below
p_{short}	Bull probability threshold to go from flat to short when crossed below from above
p_{cover}	Bull probability threshold to go from short to flat when crossed above from below
SPX	Standard and Poor 500 Composite Index
SSEC	Shanghai Stock Exchange Composite Index

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Appendix 1: Time Stepping Procedure to Define Bull and Bear Market Cycles

0. Start with first point of a market index price time series, and record it as both the *temporary previous bull high* (*TUH)* and the *temporary previous bear low* (*TEL*). Mark the starting date as the *date of temporary bull high* (*DTUH*) and *date of temporary bear low* (*DTEL*).
1. Move in time one step *h* forward, and get market index price *S*.
2. Compare *S* with *TUH* and *TEL*. Update *TUH* or *TEL* with *S* if *S*>*TUH* or *S*<*TEL* and record the date as the new *DTUH* or *DTEL*.
3. If $(S/\ TEL - 1) \geq G\%$ and $(t - DTEL) \geq D$ (we can be sure that a new bull cycle has started but it may not have ended yet), update *TEL* as a *new permanent bear low* (*PEL*) and *DTEL* as a new *date of permanent bear low* (*DPEL*) as needed; further (to address the situation that a previous cycle high has not been surpassed) if there is an update of *PEL* at current time and $S \leq TUH$, make *S* the new *TUH* and the current date the new *DTUH*.
4. If $(1 - S/TUH) \geq L\%$, and $(t - DTUH) \geq D$ (we can be sure a new bear cycle has started but it may not have ended yet), update *TUH* as a new *permanent bull high* (*PUH*) and *DTUH* as a new *date of permanent bull high* (*DPUH*) as needed; further (to address the situation that a previous cycle low has not been breached) if there is an update of *PUH* at current time and $S \geq TEL$, make *S* the new *TEL* and the current date the new *DTEL*.
5. Record updated new *PEL, DPEL, PUH,* and *DPUH* to define a new bull or bear cycle boundary. Go back to Step 1 if time has not reached the end of data range.

In the above Step 3, there is an earliest or *first date* from which one can confirm every *permanent bear low* (*FDPEL*) since the market has taken at least *D* time periods climbing *G* percentage higher from the last *TEL*. This is important in the real time observation of bull/bear market cycles, as *FDPEL* is used as a trading signal connection date for the out-of-sample back-test of with rolling statistics of known previous multiple bull/bear cycles.

Appendix 2: Equilibrium Bull Regime Probability p^* Calculation

The function $f(p)$ in our adjusted Wonham filter has:

$$f(p) = -\left[\nu_{bull} \cdot p - \nu_{bear} \cdot (1-p)\right] - p(1-p)(\lambda_{bull} - \lambda_{bear}) \cdot \left[\left(\lambda_{bull} - \frac{\sigma}{2}\right)p + \left(\lambda_{bear} - \frac{\sigma}{2}\right)(1-p)\right]$$

which can be re-written as a cubic polynomial. The equation $f(p) = 0$ becomes:

$$p^3 + a_2 p^2 + a_1 p + a_0 = 0$$

and $a_2 = \frac{\lambda_{bear} - \sigma/2}{\lambda_{bull} - \lambda_{bear}} - 1;$ $\quad a_1 = \frac{\sigma/2 - \lambda_{bear}}{\lambda_{bull} - \lambda_{bear}} - \frac{(\nu_{bull} + \nu_{bear})}{(\lambda_{bull} - \lambda_{bear})^2};$ $\quad a_0 = \frac{\nu_{bear}}{(\lambda_{bull} - \lambda_{bear})^2}$

We know the cubic equation has a unique solution p^* between zero and one, as $f(1) = -\nu_{bull} < 0$ and $f(0) = \nu_{bear} > 0$. The solution is[18]:

$$p^* = 2\sqrt{\frac{|C|}{3}} \cos\left\{\frac{2\pi}{3} - \frac{1}{3}\arccos\left[\left(\frac{3}{|C|}\right)^{3/2} \frac{q}{2}\right]\right\} - \frac{a_2}{3} \qquad \text{(A.1)}$$

Where

$$C = \left(3a_1 - a_2^2\right)/3 \qquad q = \left(9a_1 a_2 - 27a_0 - 2a_2^3\right)/27$$

We call p^* the equilibrium bull regime probability.

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Appendix 3: Derivation of Selling and Buying Threshold Probabilities:

Dai *et el* (2010)'s Equation (9) (pg. 10) can be re-written as:

$$\frac{\partial Z(t,p)}{\partial t}+LZ(t,p)=0 \qquad if \qquad 1-K<Z(t,p)<1+K \text{ And } Z(t\to\infty,p)=1-K$$

where Z is the difference between bull starting and bear starting value function of HJB (Hamilton-Jacob-Bellman) equation divided by price S; and the differential operator L

$$L=\tfrac{1}{2}(\lambda_{bull}-\lambda_{bear})^2(1-p)^2p^2\frac{\partial^2}{\partial p^2}-[\nu_{bull}\cdot p-\nu_{bear}\cdot(1-p)-p(1-p)(\lambda_{bull}-\lambda_{bear})]\frac{\partial}{\partial p}$$
$$+[(\lambda_{bull}-\sigma/2)p+(\lambda_{bear}-\sigma/2)(1-p)-r/\sigma]$$

Take $K\to 0$ as the asymptotic limit of zero transaction cost, and expand Z about K=0: $Z=1+K\cdot W(\tau,p)+o(K)$ and rescaled time $\tau=t/K$

$$p(W=1,\tau=0)=p_{buy},\ p(W=-1,\tau=0)=p_{sell},\ p(W=0,\tau=0)=p^*$$

When the difference of value functions between starting in a bull regime and starting in a bear regime is always the market price itself (W=0, Z=1), there must be no trading due to stable market price at neutral or equilibrium bull regime probability p^*, defined by $f(p)=0$ in Wonham filter. Assume $W(0,p)$ is a linear function of p, so does $p(0,W)$ against W, then $p_{buy}+p_{sell}=2p^*$. We get Equation (4).

At O (1), we have $\frac{\partial W}{\partial\tau}=-(\lambda_{bull}-\lambda_{bear})(p-\eta)$, where $\eta=(r/\sigma+\sigma/2-\lambda_{bear})/(\lambda_{bull}-\lambda_{bear})$

$$\frac{\partial W}{\partial\tau}=\left.\frac{dW}{dp}\right|_{W=-1}\cdot\frac{dp}{d\tau}=\frac{1}{p^*-p_{sell}}\frac{dp}{d\tau}=-(\lambda_{bull}-\lambda_{bear})(p-\eta),\ \text{leads to}\ p(W=-1,\infty)=\eta$$

Also $p(W=0,\tau)\equiv p^*=p^*\frac{\eta}{1-x(1-\eta)}$ is only possible when $x=1$. At neutral probability there is no trading such that the time partition expectation of x is 100% of the time holding initial position..

For sell threshold probability, $x=\nu_{bull}/(\nu_{bull}+\nu_{bear})$ is the estimated holding time of initial risk free position in a full bull/bear cycle. Assume the same $p(\eta,x)$ functional form for p_{sell}:

$p_{sell}=p(W=-1,\tau=0)=p^*\frac{\eta}{1-(1-\eta)\nu_{bull}/(\nu_{bull}+\nu_{bear})}$. This is Equation (3).