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Optimal Trend Following Trading Rules

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This paper is concerned with the optimality of a trend following trading rule. The underlying market is modeled as a bull-bear switching market in which the drift of the stock price switches between two states: the uptrend (bull market) and the downtrend (bear market). We consider that case when the market mode is not directly observable and model the switching process as a hidden Markov chain. This is a continuation of our earlier study reported in Dai et al. [5] where a trend following rule is obtained in terms of a sequence of stopping times. Nevertheless, a severe restriction imposed in [5] is that only a single share can be traded over time. As a result, the corresponding wealth process is *not* self-financing. In this paper, we relax this restriction. Our objective is to maximize the expected log-utility of the terminal wealth. We show, via a thorough theoretical analysis, that the optimal trading strategy is trend-following. Numerical simulations and backtesting, in support of our theoretical findings, are also reported.

Key words: Trend following trading rule, bull-bear switching model, partial information, HJB equations
MSC2000 subject classification: 91B28, 93E11, 93E20
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History:

1. Introduction Trading strategies can be classified into three categories: i) buy and hold; ii) contra-trend, and iii) trend following. The buy-and-hold strategy is desirable when the average stock return is higher than the risk-free interest rate. Recently Shiryaev et al. [16] provided a theoretical justification of the buy and hold strategy from the angle of maximizing the expected relative error between the stock selling price and the aforementioned maximum price. The contra-trend strategy, on the other hand, focuses on taking advantages of mean reversion type of market behaviors. A contra-trend trader purchases a stock when its price falls to some low level and bets an eventual rebound. The trend following strategy tries to capture market trends. In contrast to the contra-trend investors, a trend following believer often purchases shares when prices advance to a certain level and closes the position at the first sign of upcoming bear market.

There is an extensive literature devoted to contra-trend strategies. For instance, Merton [14] pioneered the continuous-time portfolio selection with utility maximization, which was subsequently extended to incorporate transaction costs by Magil and Constantinidies [13] (see also Davis and Norman [6], Shreve and Soner [17], Liu and Loeweinstein [12], Dai and Yi [4], and references therein). Assuming that there is no leverage or short-selling, the resulting strategies turn out to be contra-trend because the investor is risk averse and the stock market is assumed to follow a geometric Brownian motion with constant drift and volatility. Recently Zhang and Zhang [21] showed that the optimal trading strategy in a mean reverting market is also contra-trend. Other

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work relevant to the contra-trend strategy includes Dai et al. [2], Song et al. [18], Zervors et al. [20], among others.

This paper is concerned with a trend following trading rule. In practice, a trend following trader often uses moving averages to determine the general direction of the market and generate trading signals. Related research along the line of statistical analysis in connection with moving averages can be found in, for example, Faber [7] among others. Nevertheless, rigorous mathematical analysis is absent. Recently, Dai et al. [5] provided a theoretical justification of the trend following strategy in a bull-bear switching market and employed the conditional probability in the bull market to generate trade signals. However, the work imposed a less realistic assumption widely used in existing literature (e.g. [18], [20], and [21]): only one share of stock is allowed to be traded, so the resulting wealth process is *not* self-financing. It is important to address how relevant the trading rule is to practice. It is the purpose of this paper to deal with more realistic self-financing trading strategies. Here we adopt an objective function emphasizing the percentage gains. As a result the corresponding payoff has to account for the gain/loss percentage of each trade, which is also desirable in actual trading. On the other hand, these more realistic considerations make the model more technically involved than in the 'single share' transaction considered in Dai et al. [5].

Most existing literature in trading strategies assumes that the investor can observe full market information (e.g. Jang et al. [8] and Dai et al. [3]). In contrast, we follow Dai et al. [5] to model the trends in the markets using a geometric Brownian motion with regime switching and partial information. More precisely, two regimes are considered: the uptrend (bull market) and downtrend (bear market), and the switching process is modeled as a two-state Markov chain which is not directly observable. We consider a finite horizon investment problem and aim to maximize the percentage gains. We assume that the investor trades all available funds in the form of either "all-in" (long) or "all-out" (flat). That is, when buying, one fills the position with the entire account balance and when selling, one closes the entire position. We will show again that the optimal trading strategy is a trend following system characterized by the conditional probability over time and its up and down crossings of two threshold curves. These thresholds can be obtained through solving a system of associated HJB equations. Such a trading strategy naturally generates entry time and exit time which can be mathematically described by stopping times. We also carry out numerical simulations and market tests to demonstrate how the method works.

This work and Dai et al. [5] were initialized by an attempt to justify the technical analysis with moving average. A moving average trading strategy is generally in "all in - all out" form but is difficult to justify theoretically. This motivates us to design and justify an alternative "all in - all out" strategy that is analogous to the moving average trading strategy. This work has been recently extended to the Merton's portfolio optimization problem by Chen et al. [1], where the investor may choose an optimal fraction of wealth invested in stock.

In contrast to [5], the present paper provides not only a more reasonable modeling but also a more thorough theoretical analysis. First, we remove a technical condition imposed in [5] when proving the verification theorem. The key step is to show that the optimal trading strategy incurs only a finite number of trades almost surely (Lemma 2). Second, since the solution to the resulting HJB equation is not smooth enough to use the Itô lemma, we employ an approximation approach to prove the verification theorem (Theorem 4). Third, we show that for the optimal trading strategy, the upper limit involved in defining the reward function is, in fact, a limit (Theorem 5). Hence, the definition of the reward function makes sense in practice. Last but not least, we find that the theoretical characterization on the optimal trading strategy obtained in [5] remains valid for the present model (Theorem 1). We further present sufficient conditions to examine whether or not the optimal trading boundaries are attainable (Theorem 2 and Theorem 3). In spite that these conditions are not sharp, our result reveals that under certain scenario, the optimal trading boundaries are always attainable for sufficiently small transaction costs.

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The rest of the paper is arranged as follows. Following the problem formulation in the next section, Section 3 is devoted to a theoretical characterization of the optimal trading strategy in a regime switching market. We report our simulation results and market tests in Section 4. We conclude in Section 5. Some technical proofs are given in Appendix.

2. Problem Formulation Consider a complete probability space $(\Omega, \mathcal{F}, P)$. Let S_r denote the stock price at time r satisfying the equation

$$dS_r = S_r[\mu(\alpha_r)dr + \sigma dB_r],\ S_t = S,\ t \le r \le T < \infty,$$

where $\alpha_r \in \{1,2\}$ is a two-state Markov chain, $\mu(i) \equiv \mu_i$ is the expected return rate in regime $i = 1,2$, $\sigma > 0$ is the constant volatility, B_r is a standard Brownian motion, and t and T are the initial and terminal times, respectively. We assume that the stock does not pay any dividends. No generality is lost because dividends, if exist, can be re-invested in the stock and, thus, reflected in the stock price.

The process α_r represents the market mode at each time r: $\alpha_r = 1$ indicates a bull market (uptrend) and $\alpha_r = 2$ a bear market (downtrend). In this paper, we make a realistic assumption that α_r is not directly observable. Let $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, $(\lambda_1 > 0,\ \lambda_2 > 0)$, denote the generator of α_r. So, λ_1 (λ_2) stands for the switching intensity from bull to bear (from bear to bull). We assume that $\{\alpha_r\}$ and $\{B_r\}$ are independent.

Due to the non-observability of α_r, the decisions (of buying and selling) have to base purely on the stock prices. Let $\mathcal{F}_t = \sigma\{S_r :\ r \le t\}$ denote the σ-algebra generated by the stock price. Let

$$t \le \tau_1^0 \le v_1^0 \le \tau_2^0 \le v_2^0 \cdots \le \tau_n^0 \le v_n^0 \le \cdots,\ a.s.,$$

denote a sequence of $\mathcal{F}_t$-stopping times. For each n, define

$$\tau_n = \min\{\tau_n^0, T\} \text{ and } v_n = \min\{v_n^0, T\}.$$

A buying decision is made at τ_n if $\tau_n < T$ and a selling decision is at v_n if $v_n < T$, $n = 1, 2, \ldots$. In addition, we require the liquidation of all long positions (if any) at the terminal time T.

We assume that the investor is taking an "all in - all out" strategy. This means that she is either long so that her entire wealth is invested in the stock or flat so that all of her wealth is in a bank account that draws the riskfree interest rate. We use indicator $i = 0$ or 1 to signify the initial position to be flat or long, respectively. If initially the position is long (i.e, $i = 1$), the corresponding sequence of stopping times is denoted by $\Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots)$.

Let $0 < K_b < 1$ denote the percentage of slippage (or commission) per transaction with a buying order and $0 < K_s < 1$ that with a selling order.

Let $\rho \ge 0$ denote the risk-free interest rate. Given the initial time t, initial stock price $S_t = S$, initial market trend $\alpha_t = \alpha$, and initial net position $i = 0,1$, the reward functions of the decision sequences, Λ_0 and Λ_1, are the expected return rates of wealth:

$$J_i(S,\alpha,t,\Lambda_i) = \begin{cases} E_t\left\{ \log\left(e^{\rho(\tau_1 - t)} \prod_{n=1}^{\infty} e^{\rho(\tau_{n+1} - v_n)} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n < T\}}} \right) \right\}, & \text{if } i = 0, \\ E_t\left\{ \log\left(\left[\frac{S_{v_1}}{S} e^{\rho(\tau_2 - v_1)}(1-K_s)\right] \prod_{n=2}^{\infty} e^{\rho(\tau_{n+1} - v_n)} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n < T\}}} \right) \right\}, & \text{if } i = 1, \end{cases}$$

where I_A represents the indicator function of A.

REMARK 1. Note that different from the reward functions in [5], the above reward functions account for percentage gain/loss of each trade. Between trades, the entire balance is in a risk-free asset drawing interests at rate ρ. We only consider the control problem in the finite time horizon $[0,T]$. This is signified by involving the indicator function $I_{\{\tau_n<T\}}$ in the payoff function J_i. The meaning of this indicator function is that a buying order at stopping time τ_n will be accounted only when $\tau_n<T$. If a long position remains at $t=T$, then it has to be sold at that time. Transactions at $t>T$ will not affect the payoff J_i.

It is clear that

$$
\begin{aligned}
&J_i(S,\alpha,t,\Lambda_i)\\
&=\begin{cases}
E_t\left\{\rho(\tau_1-t)+\sum_{n=1}^{\infty}\left[\log\frac{S_{v_n}}{S_{\tau_n}}+\rho(\tau_{n+1}-v_n)+\log\left(\frac{1-K_s}{1+K_b}\right)I_{\{\tau_n<T\}}\right]\right\}, & \text{if } i=0,\\
E_t\left\{\left[\log\frac{S_{v_1}}{S}+\log(1-K_s)+\rho(\tau_2-v_1)\right]\right.\\
\quad\left.+\sum_{n=2}^{\infty}\left[\log\frac{S_{v_n}}{S_{\tau_n}}+\rho(\tau_{n+1}-v_n)+\log\left(\frac{1-K_s}{1+K_b}\right)I_{\{\tau_n<T\}}\right]\right\}, & \text{if } i=1,
\end{cases}
\end{aligned}
$$

where the term $E_t\sum_{n=1}^{\infty}\xi_n$ for random variables ξ_n is interpreted as $\limsup_{N\to\infty}E_t\sum_{n=1}^{N}\xi_n$.

REMARK 2. We will show in Section 3 that the optimal strategy can be given in terms of the conditional probability in a bull market and two threshold levels. A buying (selling) decision is triggered when the conditional probability in a bull market crosses these thresholds. Moreover, the optimal strategy involves only a finite number of trades (see Lemma 2).

It is easy to see that one should never buy a stock if the riskfree rate is greater than the log-return rate of stock in the bull market, i.e. $\rho\geq\mu_1-\frac{\sigma^2}{2}$, and never sell the stock if the riskfree rate is lower than the log-return rate of stock in the bear market, i.e., $\rho\leq\mu_2-\frac{\sigma^2}{2}$. To exclude these trivial cases, we assume

$$\mu_2-\frac{\sigma^2}{2}<\rho<\mu_1-\frac{\sigma^2}{2}. \tag{1}$$

Note that the market trend α_r is not directly observable. Thus, it is necessary to convert the problem into one that is observable. One way to accomplish this is to use the Wonham filter [19].

Let $p_r=P(\alpha_r=1|\mathcal{S}_r)$ denote the conditional probability in a bull market $(\alpha_r=1)$ given the filtration $\mathcal{S}_r=\sigma\{S_u:\ 0\leq u\leq r\}$. Then we can show (see Wonham [19]) that p_r satisfies the following stochastic differential equation

$$dp_r=[-(\lambda_1+\lambda_2)\,p_r+\lambda_2]\,dr+\frac{(\mu_1-\mu_2)p_r(1-p_r)}{\sigma}d\widehat{B}_r, \tag{2}$$

where $\widehat{B}_r$ is the innovation process (a standard Brownian motion; see e.g., Øksendal [15]) given by

$$d\widehat{B}_r=\frac{d\log(S_r)-[(\mu_1-\mu_2)p_r+\mu_2-\sigma^2/2]dr}{\sigma}. \tag{3}$$

It is easy to see that S_r can be written in terms of $\widehat{B}_r$:

$$dS_r=S_r\left[(\mu_1-\mu_2)\,p_r+\mu_2\right]dr+S_r\sigma d\widehat{B}_r. \tag{4}$$

In view of this, the separation principle holds for the partially observed optimization problem.

The problem is to choose Λ_i to maximize the discounted return J_i subject to (2) and (4). We emphasize that this new problem is completely observable because p_r, the conditional probability in a bull market, can be obtained by using the stock price up to time r.

Note that the reward function J_i only accounts for the percentage gain/loss. For any given τ_n and v_n, we have

$$\log \frac{S_{v_n}}{S_{\tau_n}} = \int_{\tau_n}^{v_n} f(p_r)dr + \int_{\tau_n}^{v_n} \sigma d\widehat{B}_r, \tag{5}$$

where

$$f(p_r) = (\mu_1 - \mu_2)p_r + \mu_2 - \frac{\sigma^2}{2}. \tag{6}$$

Note also that

$$E_t \int_{\tau_n}^{v_n} \sigma d\widehat{B}_r = 0. \tag{7}$$

Therefore, the reward function J_i is independent of the initial stock price S. Consequently, given $p_t = p$, we can rewrite the reward function as

$$J_i = J_i(p, t, \Lambda_i).$$

For $i = 0, 1$, let $V_i(p, t)$ denote the value function with the state p at time t. That is,

$$V_i(p, t) = \sup_{\Lambda_i} J_i(p, t, \Lambda_i). \tag{8}$$

The following lemma gives the bounds of the value functions.

Lemma 1. *Let $V_i(p, t)$, $i = 1, 2$ be the value functions defined in (8). We have*

$$\rho(T - t) \le V_0(p, t) \le \left(\mu_1 - \frac{\sigma^2}{2}\right)(T - t)$$

and

$$\log(1 - K_s) + \rho(T - t) \le V_1(p, t) \le \log(1 - K_s) + \left(\mu_1 - \frac{\sigma^2}{2}\right)(T - t).$$

Proof. It is clear that the lower bounds for V_i follow from their definition. It remains to estimate their upper bounds. Using (5) and (7) and noticing $0 \le p_r \le 1$, we have

$$\begin{aligned} E_t\left(\log \frac{S_{v_n}}{S_{\tau_n}}\right) &= E_t\left[\int_{\tau_n}^{v_n} f(p_r)dr\right] \\ &\le \left(\mu_1 - \frac{\sigma^2}{2}\right)\int_{\tau_n}^{v_n} dr = \left(\mu_1 - \frac{\sigma^2}{2}\right)(v_n - \tau_n). \end{aligned}$$

Note that $\log(1 - K_s) < 0$ and $\log(1 + K_b) > 0$. It follows that

$$\begin{aligned} J_0(p, t, \Lambda_0) &\le E_t\left\{\rho(\tau_1 - t) + \sum_{n=1}^{\infty}\left[\left(\mu_1 - \frac{\sigma^2}{2}\right)(v_n - \tau_n) + \rho(\tau_{n+1} - v_n)\right]\right\} \\ &\le \max\left\{\rho, \mu_1 - \frac{\sigma^2}{2}\right\}(T - t) = \left(\mu_1 - \frac{\sigma^2}{2}\right)(T - t), \end{aligned}$$

where the last equality is due to (1). We then obtain the desired result. An upper bound for V_1 can be established similarly. □

Next, we consider the associated Hamilton-Jacobi-Bellman equations. Using the dynamic programming principle, one has

$$V_0(p,t) = \sup_{\tau_1 \geq t} E_t \left\{ \rho(\tau_1 - t) - \log(1+K_b) + V_1(p_{\tau_1}, \tau_1) \right\}$$

and

$$V_1(p,t) = \sup_{v_1 \geq t} E_t \left\{ \int_t^{v_1} f(p_s) ds + \log(1-K_s) + V_0(p_{v_1}, v_1) \right\},$$

where $f(\cdot)$ is as given in (6). Let

$$L = \partial_t + \frac{1}{2}\left(\frac{(\mu_1 - \mu_2)p(1-p)}{\sigma}\right)^2 \partial_{pp} + \left[-(\lambda_1 + \lambda_2)p + \lambda_2\right]\partial_p$$

denote the generator of (t, p_t). Then, the associated HJB equations are

$$\begin{cases} \min\{-LV_0 - \rho, V_0 - V_1 + \log(1+K_b)\} = 0, \\ \min\{-LV_1 - f(p), V_1 - V_0 - \log(1-K_s)\} = 0, \end{cases} \tag{9}$$

with the terminal conditions

$$\begin{cases} V_0(p,T) = 0, \\ V_1(p,T) = \log(1-K_s). \end{cases} \tag{10}$$

Using the same technique as in Dai et al. [5], we can show that Problem (9)-(10) has a unique bounded strong solution (V_0, V_1), where $V_i \in W_q^{2,1}([\varepsilon, 1-\varepsilon] \times [0,T])$, for any $\varepsilon \in (0, 1/2)$, $q \in [1, +\infty)$. It should be pointed out that the differential operator L is degenerate at $p = 0, 1$ and the solution is only locally bounded in $W_q^{2,1}$.

Remark 3. In this paper, we restrict the state space of p to $(0,1)$ because both $p = 0$ and $p = 1$ are entrance boundaries (see Karlin and Taylor [9] and Dai et al. [5] for definition and discussions).

Now we define the buy region (BR), the sell region (SR), and the no-trading region (NT) as follows:

$$\begin{aligned} BR &= \{(p,t) \in (0,1) \times [0,T) : V_1(p,t) - V_0(p,t) = \log(1+K_b)\}, \\ SR &= \{(p,t) \in (0,1) \times [0,T) : V_1(p,t) - V_0(p,t) = \log(1-K_s)\}, \\ NT &= (0,1) \times [0,T) \setminus (BR \cup SR). \end{aligned}$$

To study the optimal strategy, we only need to characterize these regions.

3. Main results

In this section, we present the main theoretical results.

3.1. Characterization of the optimal trading strategy

Let

$$p_0 = \frac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2}, \quad a = \log\frac{1+K_b}{1-K_s}. \tag{11}$$

Theorem 1. *There exist two monotonically increasing boundaries $p_s^*(t)$, $p_b^*(t)$: $[0,T) \to [0,1]$ such that*

$$SR = \{(p,t) \in (0,1) \times [0,T) : p \leq p_s^*(t)\}, \tag{12}$$

$$BR = \{(p,t) \in (0,1) \times [0,T) : p \geq p_b^*(t)\}. \tag{13}$$

Moreover,

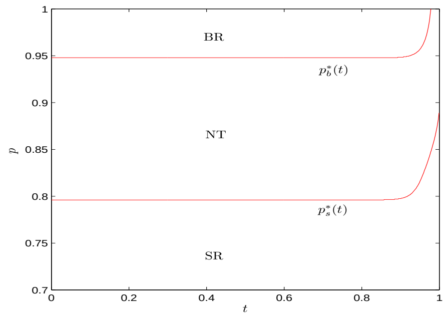
i) $p_b^*(t) \geq p_0 \geq p_s^*(t)$ for all $t \in [0,T)$;

ii) $\lim_{t \to T^-} p_s^*(t) = p_0$;

iii) there is a $\delta > \dfrac{a}{\mu_1 - \rho - \sigma^2/2}$ such that $p_b^*(t) = 1$ for $t \in (T - \delta, T)$;

iv) $p_s^*(t)$, $p_b^*(t) \in C^\infty$ if $p_s^*(t), p_b^*(t) \in (0,1)$.

FIGURE 1. Optimal buy and sell boundaries



Parameter values: $\lambda_1 = 0.36,\ \lambda_2 = 2.53,\ \mu_1 = 0.18,\ \mu_2 = -0.77,\ \sigma = 0.184,\ K_b = K_s = 0.001,\ \rho = 0.0679,\ T = 1$.

Proof. Denote $Z(p,t) \equiv V_1(p,t) - V_0(p,t)$. Similar to Lemma 2.2 in [5], we can show that $Z(p,t)$ satisfies the following double obstacle problem:

$$\min\left\{\max\left\{-LZ - f(p) + \rho, Z - \log(1+K_b)\right\}, Z - \log(1-K_s)\right\} = 0, \tag{14}$$

in $(0,1)\times[0,T)$, with the terminal condition $Z(p,T) = \log(1-K_s)$, and

$$\begin{cases} -LV_0 = \rho + (-LZ - f(p) + \rho)^- = \rho I_{\{Z<\log(1+K_b)\}} + f(p) I_{\{Z=\log(1+K_b)\}}, \\ V_0(p,T) = 0, \end{cases} \tag{15}$$

$$\begin{cases} -LV_1 = f(p) + (-LZ - f(p) + \rho)^+ = f(p) I_{\{Z>\log(1-K_s)\}} + \rho I_{\{Z=\log(1-K_s)\}}, \\ V_1(p,T) = \log(1-K_s). \end{cases} \tag{16}$$

Then we can use the same argument as in the proof of Theorem 2.5 in [5] to obtain the desired results. □

We call $p_s^*(t)$ ($p_b^*(t)$) the optimal sell (buy) boundary. To see better how Theorem 1 works, we provide a numerical result for illustration. In Figure 1, we plot the optimal sell and buy boundaries against time, where the parameter values used are $\lambda_1 = 0.36,\ \lambda_2 = 2.53,\ \mu_1 = 0.18,\ \mu_2 = -0.77,\ \sigma = 0.184,\ K_b = K_s = 0.001,\ \rho = 0.0679$, and $T = 1$. It can be seen that $p_s^*(t)$ and $p_b^*(t)$ are almost flat except when t is close to T where they sharply increase with t. Moreover, the sell boundary $p_s^*(t)$ approaches the theoretical value

$$p_0 = \frac{\rho - \mu_2 + \sigma^2/2}{\mu_1 - \mu_2} = \frac{0.0679 + 0.77 + 0.184^2/2}{0.18 + 0.77} = 0.9,$$

as $t \to T = 1$. Between the two boundaries is the NT, above the buy boundary is the BR, and below the sell boundary is the SR. Also, we observe that there is a δ such that $p_b^*(t) = 1$ for $t \in [T-\delta, T]$, which indicates that it is never optimal to buy stock when t is very close to T. Using Theorem 1, the lower bound of δ is estimated as

$$\frac{a}{\mu_1 - \rho - \sigma^2/2} = \frac{\log(1.001/0.999)}{0.18 - 0.0657 - 0.184^2/2} = 0.021,$$

which is consistent with the numerical result.

The behavior of the thresholds $p_s^*(\cdot)$ and $p_b^*(\cdot)$ when t approaches T is due to our technical requirement of liquidating all the positions at T. Interested in long-term investment, we will approximate these thresholds, as in [5], by constants $p_s^* = \lim_{T-t\to\infty} p_s^*(t)$ and $p_b^* = \lim_{T-t\to\infty} p_b^*(t)$. Assuming that the initial position is flat and the initial conditional probability $p(0) \in (p_s^*, p_b^*)$, our trading strategy can be described as follows: as p_t goes up to hit p_b^*, we take a long position, that is, investing all the wealth in the stock. We will close out the position only when p_t goes down and hits p_s^*. According to (2)-(3), we have

$$dp_r = g(p_r)dr + \frac{(\mu_1-\mu_2)p_r(1-p_r)}{\sigma^2} d\log S_r, \tag{17}$$

where

$$g(p) = -(\lambda_1+\lambda_2)\,p + \lambda_2 - \frac{(\mu_1-\mu_2)p_t(1-p_t)\left((\mu_1-\mu_2)p+\mu_2-\sigma^2/2\right)}{\sigma^2}.$$

Relation (17) implies that p_t, the conditional probability in the bull market, increases (decreases) as the stock price goes up (down). Hence, our optimal trading strategy buys while the stock price is going up and sells when the stock price declines. In other words, it is trend-following in nature.

We have seen from Proposition 1 that both the buy and sell boundaries are increasing with time and that the buy (sell) boundary boundary is bounded from below (above) by p_0. Note that $p=0$ and $p=1$ are entrance boundaries that cannot be reached from the interior of the state space (see Remark 2 in Dai et al. [5]). A natural question is whether or not the sell (buy) boundary can coincide with $p=0$ ($p=1$). The following theorem provides an affirmative answer and sufficient conditions.

Theorem 2. *Let p_0 and a be given as in (11).*
i) If

$$p_0 < \min\left\{\frac{1}{3}, \frac{\lambda_2}{6(\lambda_1+\lambda_2)}\right\} \tag{18}$$

and

$$\frac{p_0}{\frac{\lambda_2}{12(\mu_1-\mu_2)p_0} - \frac{\lambda_1+\lambda_2}{2(\mu_1-\mu_2)}} \le a \le \frac{p_0}{\frac{9(\mu_1-\mu_2)}{\sigma^2} + \frac{2+6\lambda_1}{\mu_1-\mu_2}}, \tag{19}$$

then

$$p_s^*(t) \equiv 0,\ \ \forall\, t \le T - \frac{1}{p_0} - \frac{12p_0}{\lambda_2}.$$

ii) If $\lambda_1 > \lambda_2$ and

$$p_0 \ge 1 - \min\left[\frac{1}{3}, \frac{\lambda_1-\lambda_2}{6(\lambda_1+\lambda_2)}, \frac{\sigma^2(\lambda_1+\lambda_2)}{18(\mu_1-\mu_2)^2}\right], \quad a \ge \frac{\sigma^2(1-p_0)}{\mu_1-\mu_2}, \tag{20}$$

then

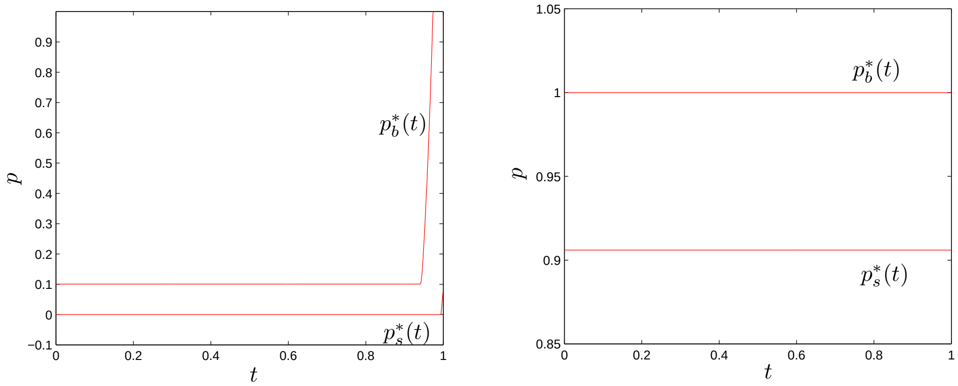
$$p_b^*(t) \equiv 1, \quad \forall\, t < T.$$

The proof of Theorem 2 relies on a technical partial different equation approach and is postponed to Appendix.

Figure 2 below illustrates situations where the parameter values do satisfy the conditions in Theorem 2. In Figure 2(a), the sell boundary coincides with the entrance boundary $p=0$ before $t=0.98$. Hence, one should never sell stock except when t is very close to 1. In Figure 2(b), the buy boundary remains at the entrance boundary $p=1$, which means that one should never buy any stock.

Now we present a sufficient condition to ensure that both the sell boundary and the buy boundary are attainable when t is not close to the terminal time T.

FIGURE 2. Scenarios of $p_s^*(t) = 0$, $p_b^*(t) \equiv 1$



(a) (b)

Parameter values. Case (a): $\lambda_1 = 0.2,\ \lambda_2 = 30,\ \mu_1 = 0.15,\ \mu_2 = 0.1,\ \sigma = 0.2,\ K_b = K_s = 0.0006,\ \rho = 0.085,\ T = 1$; Case (b): $\lambda_1 = 20,\ \lambda_2 = 1,\ \mu_1 = 0.2,\ \mu_2 = 0,\ \sigma = 0.45,\ K_b = K_s = 0.05,\ \rho = 0.08,\ T = 1$.

THEOREM 3. *Let p_0 and a be as given in (11). If $p_0 < \frac{1}{3}$ and*

$$a \leq \min\left\{ \frac{p_0}{\frac{9(\mu_1-\mu_2)}{\sigma^2} + \frac{2+6\lambda_1}{\mu_1-\mu_2}},\ \frac{p_0}{\frac{8(\mu_1-\mu_2)}{\sigma^2} + \frac{16\lambda_2}{(\mu_1-\mu_2)p_0}} \right\}, \tag{21}$$

then

$$p_s^*(t) > 0,\ \ p_b^*(t) < 1,\ \ \forall\, t \leq T - \frac{1}{p_0}.$$

Again we postpone the technical proof to Appendix.

The conditions in Theorem 3 is not sharp. However, condition (21) always holds if the transaction costs are sufficiently small. We also emphasize that the conditions presented in Theorem 3 are sufficient but not necessary. In fact, our numerical tests reveal that for reasonable parameter values, the sell and buy boundaries are strictly between (0,1) when t is not close to the terminal time T.

3.2. A verification theorem We now present a verification theorem, indicating that the solutions V_0 and V_1 of problem (9)-(10) are equal to the value functions and sequences of optimal stopping times can be constructed by using (p_s^*, p_b^*).

THEOREM 4. *(Verification Theorem) Let $(w_0(p,t), w_1(p,t))$ be the unique solution to problem (9)-(10) and $p_b^*(t)$ and $p_s^*(t)$ be the associated free boundaries, where $w_i \in W_q^{2,1}([\varepsilon, 1-\varepsilon] \times [0,T]),\ i = 0,1$, for any $\varepsilon \in (0,1/2)$, $q \in [1,+\infty)$. Then, $w_0(p,t)$ and $w_1(p,t)$ are equal to the value functions $V_0(p,t)$ and $V_1(p,t)$, respectively.*

Moreover, let

$$\Lambda_0^* = (\tau_1^*, v_1^*, \tau_2^*, v_2^*, \cdots),$$

where the stopping times $\tau_1^* = T \wedge \inf\{r \geq t:\ p_r \geq p_b^*(r)\}$, $v_n^* = T \wedge \inf\{r \geq \tau_n^*:\ p_r \leq p_s^*(r)\}$, and $\tau_{n+1}^* = T \wedge \inf\{r > v_n^*:\ p_r \geq p_b^*(r)\}$ for $n \geq 1$, and let

$$\Lambda_1^* = (v_1^*, \tau_2^*, v_2^*, \tau_3^*, \cdots),$$

where the stopping times $v_1^* = T \wedge \inf\{r \geq t:\ p_r \leq p_s^*(r)\}$, $\tau_n^* = T \wedge \inf\{r > v_{n-1}^*:\ p_r \geq p_b^*(r)\}$, and $v_n^* = T \wedge \inf\{r \geq \tau_n^*:\ p_r \leq p_s^*(r)\}$ for $n \geq 2$. Then Λ_0^* and Λ_1^* are optimal.

Note that in Theorem 4, we removed the technical condition $v_n^* \to T$ used in [5]. In addition, the solution to problem (9)-(10) is not smooth enough to use the Itô lemma. We will employ an approximation approach to overcome this difficulty. Note that one cannot directly utilize the results of Lamberton and Zervos [11] which are for a stationary problem.

Before proving Theorem 4, we introduce two lemmas. The first indicates that the optimal trading strategy incurs only a finite number of trades almost surely.

Lemma 2. *Let v_n^*, τ_n^* be as given in Theorem 4. Define*

$$\mathcal{N} = \inf\{n : v_n^* = T \text{ or } \tau_{n+1}^* = T\} \quad \text{and} \quad \inf \emptyset = +\infty.$$

Then there exists a constant C such that

$$\mathbb{E}(\mathcal{N}) \le C.$$

In particular, $\mathcal{N}(\omega)$ is finite almost surely. In other words, for fixed path, $v_n^* = \tau_n^* = T$ when n is large enough.

Proof. Recalling $p_b^*(r) \ge p_0 \ge p_s^*(r)$, p_s^*, $p_b^* \in C^\infty$ (see Theorem 1), and

$$V_1(r, p_b^*(r)) - V_0(r, p_b^*(r)) = \log(1+K_b) > \log(1-K_s) = V_1(r, p_s^*(r)) - V_0(r, p_s^*(r)),$$

we deduce that $p_b^*(r) > p_s^*(r)$ and there is a $\delta > 0$ such that

$$p_b^*(r) - p_s^*(r) > 4\delta.$$

Denote

$$P_r^1 = p_t + \int_t^r \Big[-(\lambda_1 + \lambda_2) p_u + \lambda_2 \Big] du - p_s^*(r), \quad P_r^2 = \int_t^r \frac{(\mu_1 - \mu_2) p_u (1 - p_u)}{\sigma} d\widehat{B}_u,$$

where P^1 is an absolutely continuous stochastic process and P^2 is a martingale. Apparently

$$P_r^1 + P_r^2 = p_r - p_s^*(r). \tag{22}$$

Since stochastic process p_r has continuous paths, the definitions of p_s^*, p_b^* imply that

$$\begin{aligned} &(P^1_{\tau_n^*} - P^1_{v^*_{n-1}}) + (P^2_{\tau_n^*} - P^2_{v^*_{n-1}}) = (P^1_{\tau_n^*} + P^2_{\tau_n^*}) - (P^1_{v^*_{n-1}} + P^2_{v^*_{n-1}}) \\ &= (p_{\tau_n^*} - p_s^*(\tau_n^*)) - (p_{v^*_{n-1}} - p_s^*(v^*_{n-1})) \\ &= p_b^*(\tau_n^*) - p_s^*(\tau_n^*) > 4\delta. \end{aligned}$$

Hence, we deduce

$$\text{either} \quad P^1_{\tau_n^*} - P^1_{v^*_{n-1}} > 2\delta \quad \text{or} \quad P^2_{\tau_n^*} - P^2_{v^*_{n-1}} > 2\delta. \tag{23}$$

On the other hand, P^1 is clearly bounded since $p_r, p_s^*(r) \in [0,1]$. Owing to (22), we infer that P^2 is bounded as well. Hence, we can choose a positive integer M such that

$$|P^2| \le M\delta.$$

If $P^2_{\tau_n^*} - P^2_{v^*_{n-1}} > 2\delta$, then the continuity of P^2 implies that the martingale P^2 should cross upward at least one of the intervals $[i\delta, (i+1)\delta]$ $(i = -M, -M+1, \cdot, \cdot, \cdot, M-1)$ during $[v^*_{n-1}, \tau_n^*]$.

Hence, by virtue of (23), we deduce that

$$\mathcal{N} \le \sum_{i=-M}^{M-1} \mathcal{U}_{[i\delta, (i+1)\delta]}(P^2) + \mathcal{U}_{2\delta}(P^1), \tag{24}$$

where $\mathcal{U}_{[i\delta,(i+1)\delta]}(P^2)$ denotes the number of crossing upward the interval $[i\delta,(i+1)\delta]$ for P^2 during $[0,T]$, and $\mathcal{U}_{2\delta}(P^1)$ denotes the number of crossing upward a 2δ-length interval for P^1 during $[0,T]$. In view of the inequality for crossing upward, we infer

$$\mathbb{E}(\mathcal{U}_{[i\delta,(i+1)\delta]}(P^2)) \leq \frac{1}{\delta}\Big(\mathbb{E}(|P^2|)+|i\delta|\Big) \leq \frac{1}{\delta}\mathbb{E}(|P^2|)+M \leq \frac{C}{4M}, \tag{25}$$

where C is a constant large enough. Since $p_r \in [0,1]$ and p_s^* is increasing, it is easy to see

$$\mathcal{U}_{2\delta}(P^1) \leq \frac{C}{2}. \tag{26}$$

The combination of (24), (25), and (26) yields the desired result. □

Our next lemma indicates that the solution to problem (9)-(10) has the same bounds as the value function (see Lemma 1).

Lemma 3. *Let $(w_0(p,t), w_1(p,t))$ be the solution to problem (9)-(10). Then*

$$\rho(T-t) \leq w_0(p,t) \leq \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t)$$

and

$$\log(1-K_s)+\rho(T-t) \leq w_1(p,t) \leq \log(1-K_s)+\left(\mu_1-\frac{\sigma^2}{2}\right)(T-t).$$

Proof. Clearly

$$-L(w_0-\rho(T-t)) = -Lw_0-\rho \geq 0,$$

from which we immediately infer by the maximum principle $w_0 \geq \rho(T-t)$. Owing to $w_1 - w_0 - \log(1-K_s) \geq 0$, we have $w_1 \geq \log(1-K_s)+\rho(T-t)$.

To prove the right hand side inequalities, we utilize (15) and (16) to get

$$\begin{aligned}
-Lw_0 &\leq \max\{\rho, f(p)\} \leq \mu_1 - \frac{\sigma^2}{2},\\
-Lw_1 &\leq \max\{\rho, f(p)\} \leq \mu_1 - \frac{\sigma^2}{2}.
\end{aligned}$$

Again by the maximum principle, the desired result follows. □

Now we are ready to prove the verification theorem.

Proof of Theorem 4. First, we show that for any stopping times $\theta_2 \geq \theta_1 \geq t$,

$$E_t w_1(p_{\theta_1},\theta_1) \geq E_t\left[\int_{\theta_1}^{\theta_2} f(p_r)dr + w_1(p_{\theta_2},\theta_2)\right] = E_t\left[\log\frac{S_{\theta_2}}{S_{\theta_1}} + w_1(p_{\theta_2},\theta_2)\right] \text{ a.s.} \tag{27}$$

Since w_1 is only *locally* bounded in $W_q^{2,1}((0,1)\times[0,T])$, we cannot directly use the Itô formula. To overcome the difficulty, we introduce the following stopping times:

$$\beta_m = \inf\{r \geq \theta_1 : p_r \in (0,1/m)\cup(1-1/m,1)\}\wedge\theta_2,\ m=1,2,\cdots.$$

Note that $p=0$ and $p=1$ cannot be reached from the interior of $(0,1)$ (see Remark 2 in [5]). We then infer that $\beta_m \to \theta_2$ as $m\to\infty$.

Due to $w_1 \in W_q^{2,1}([1/m,1-1/m]\times[0,T])$, applying the Itô formula to $w_1(p_r,r)$ in $[\theta_1,\beta_m]$ yields (c.f. Krylov [10])

$$w_1(p_{\theta_1},\theta_1) = w_1(p_{\beta_m},\beta_m) - \int_{\theta_1}^{\beta_m} Lw_1(p_r,r)dr - \int_{\theta_1}^{\beta_m}\partial_p w_1(p_r,r)\frac{(\mu_1-\mu_2)p_r(1-p_r)}{\sigma}d\widehat{B}_r \quad \text{a.e.}$$

By the Sobolev embedding theory, $\partial_p w_1 \in C([1/m, 1-1/m]\times[0,T])$, which implies that the last term in the above equation is a martingale. Taking conditional expectation in the above equation, we deduce that

$$E_t w_1(p_{\theta_1},\theta_1) = E_t\left[w_1(p_{\beta_m},\beta_m) - \int_{\theta_1}^{\beta_m} Lw_1(p_r,r)dr\right]. \tag{28}$$

Since w_0, $w_1 \in W^{2,1}_{q,\mathrm{loc}}$, we can rewrite

$$\begin{aligned} Lw_1 &= -f(p)I_{\{w_1>w_0+\log(1-K_s)\}} + L(w_0+\log(1-K_s))I_{\{w_1=w_0+\log(1-K_s)\}} \\ &= -f(p)I_{\{w_1>w_0+\log(1-K_s)\}} - \rho I_{\{w_1=w_0+\log(1-K_s)\}}. \end{aligned}$$

Hence

$$E\left[\int_0^T |\, Lw_1(p_r,r)\,|\, dr\right] < \infty. \tag{29}$$

Sending $m\to\infty$ in (28) and using (29) and Lemma 3, we have by the dominated convergence theorem

$$E_t w_1(p_{\theta_1},\theta_1) = E_t\left[-\int_{\theta_1}^{\theta_2} Lw_1(p_r,r)dr + w_1(p_{\theta_2},\theta_2)\right] \text{ a.s.}$$

Using $-Lw_1 - f(p)\geq 0$, we then obtain (27). In a similar way we can show

$$E_t w_0(p_{\theta_1},\theta_1) \geq E_t[\,\rho(\theta_2-\theta_1) + w_0(p_{\theta_2},\theta_2)\,] \text{ a.s.} \tag{30}$$

We next show, for any Λ_1 and $k=1,2,\ldots,$

$$\begin{aligned} E_t w_0(p_{v_k},v_k) \geq E_t\Bigg[&\rho(\tau_{k+1}-v_k) + \log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} \\ &+ w_0(p_{v_{k+1}},v_{k+1}) + (\log(1-K_s)-\log(1+K_b))I_{\{\tau_{k+1}<T\}}\Bigg]. \end{aligned} \tag{31}$$

In fact, using (27) and (30) and noticing that

$$w_0 \geq w_1 - \log(1+K_b) \text{ and } w_1 \geq w_0 + \log(1-K_s),$$

we have

$$\begin{aligned} &E_t w_0(p_{v_k},v_k) \\ &\quad\geq E_t[\rho(\tau_{k+1}-v_k) + w_0(p_{\tau_{k+1}},\tau_{k+1})] \\ &\quad\geq E_t[\rho(\tau_{k+1}-v_k) + \big(w_1(p_{\tau_{k+1}},\tau_{k+1}) - \log(1+K_b)\big)\, I_{\{\tau_{k+1}<T\}}] \\ &\quad\geq E_t[\rho(\tau_{k+1}-v_k) + \left(\log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_1(p_{v_{k+1}},v_{k+1}) - \log(1+K_b)\right) I_{\{\tau_{k+1}<T\}}] \\ &\quad\geq E_t[\rho(\tau_{k+1}-v_k) + \left(\log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{v_{k+1}},v_{k+1}) + \log(1-K_s) - \log(1+K_b)\right) I_{\{\tau_{k+1}<T\}}] \\ &\quad= E_t[\rho(\tau_{k+1}-v_k) + \log\frac{S_{v_{k+1}}}{S_{\tau_{k+1}}} + w_0(p_{v_{k+1}},v_{k+1}) + (\log(1-K_s)-\log(1+K_b))\, I_{\{\tau_{k+1}<T\}}]. \end{aligned}$$

Note that the above inequalities also work when starting at t in lieu of v_1, i.e.,

$$w_0(p_t,t) \geq E_t\left[\rho(\tau_1-t) + \log\frac{S_{v_1}}{S_{\tau_1}} + w_0(p_{v_1},v_1) + (\log(1-K_s)-\log(1+K_b))I_{\{\tau_1<T\}}\right].$$

Use this inequality and iterate (31) with $k=1,2,\ldots$, and note $w_0\geq 0$ to obtain

$$w_0(p,t)\geq V_0(p,t).$$

Similarly, we can show that

$$w_1(p_t,t)\geq E_t\left[\log\frac{S_{v_1}}{S_t}+w_1(p_{v_1},v_1)\right]\geq E_t\left[\log\frac{S_{v_1}}{S_t}+w_0(p_{v_1},v_1)+\log(1-K_s)\right].$$

Use this and iterate (31) with $k=1,2,\ldots$ as above to obtain

$$w_1(p,t)\geq V_1(p,t).$$

By Lemma 2, we immediately obtain $v_k^*,\tau_k^*\to T$ as $k\to\infty$. It can be seen that the equalities hold when $\tau_k=\tau_k^*$ and $v_k=v_k^*$. This completes the proof. □

We conclude this section by showing that for the optimal trading strategy, the limsup in the reward function defined in Section 2 is, in fact, a limit. Hence, the definition of the reward function makes sense in practice.

THEOREM 5. *The limit of* $\mathbb{E}[\Theta(m)]$ *as* m *tends infinity exists, where*

$$\Theta(m)=\sum_{n=1}^{m}\left[\log\frac{S_{v_n^*}}{S_{\tau_n^*}}+\rho(\tau_{n+1}^*-v_n^*)+\log\left(\frac{1-K_s}{1+K_b}\right)I_{\{\tau_n^*<T\}}\right].$$

Proof. Lemma 2 implies that for fixed path, $\tau_n=v_n=T$ for n large enough. So the sum is finite a.s., and $\lim_{m\to\infty}\Theta(m)$ exists a.s.

Next, we estimate the bound of $\Theta(m)$. Similar to the proof of Lemma 1, we can obtain

$$\Theta(m)\leq\left(\mu_1-\frac{\sigma^2}{2}\right)(T-t).$$

Using the same argument as in the proof of Lemma 1, we have

$$\sum_{n=1}^{m}\left[\log\frac{S_{v_n^*}}{S_{\tau_n^*}}+\rho(\tau_{n+1}^*-v_n^*)\right]\geq\left(\mu_2-\frac{\sigma^2}{2}\right)(T-t).$$

Moreover, it is clear that

$$\sum_{n=1}^{m}\log\left(\frac{1-K_s}{1+K_b}\right)I_{\{\tau_n^*<T\}}\geq\log\left(\frac{1-K_s}{1+K_b}\right)\mathcal{N}\ \text{ for any } m.$$

Lemma 2 implies that

$$\mathbb{E}\left[\left(\mu_2-\frac{\sigma^2}{2}\right)(T-t)+\log\left(\frac{1-K_s}{1+K_b}\right)\mathcal{N}\right]$$

exists. The convergence of $\mathbb{E}[\Theta(m)]$ follows from the Lebesgue dominated convergence theorem. □

4. Simulation and market tests In this section, we carry out numerical simulations and backtesting to examine the effectiveness of our trading strategy. To estimate p_t, the conditional probability in a bull market, we use a discrete version of the stochastic differential equation (17), for $t=0,1,\ldots,N$ with $dt=1/252$,

$$p_{t+1}=\min\left(\max\left(p_t+g(p_t)dt+\frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma^2}\log(S_{t+1}/S_t),0\right),1\right),\tag{32}$$

where the price process S_t is determined by the simulated paths or the historical market data. The min and max are added to ensure the discrete approximation p_t of the conditional probability in the bull market stays in the interval $[0,1]$.

4.1. Simulations For simulation we use the parameters given in Table 1. These numbers were used in [5]. The time horizon is 40 years.

λ_1	λ_2	μ_1	μ_2	σ	K	ρ
0.36	2.53	0.18	-0.77	0.184	0.001	0.0679

Table 1. Parameter values

We solve the HJB equations and derive $p_s^* = 0.796$ and $p_b^* = 0.948$. We run the 5000 round simulations for 10 times. Starting with \$1, the mean of the total/annualized return and the standard deviation are given in Table 2. The trend following strategy clearly outperforms the buy and hold in terms of return. Moreover, the trend following strategy has a monthly Sharp ratio of 0.22 while the return of the buy and hold strategy is lower than the riskfree rate $\rho = 0.0679$.

	Trend Following	Buy and Hold	No. of Trades
Mean	75.76(11.4%)	5.62(4.4%)	41.16
Stdev	2.48	0.39	0.29

Table 2. Statistics of ten 5000-path simulations

Comparing to the simulation results in [5] we only observe a slight improvement in terms of the ratio of mean return of the trend following strategy to that of the buy and hold strategy. However, the improvement is not significant enough to distinguish statistically from the results in [5] despite theoretically the present paper is more solid than [5]. Together with sensitivity tests on thresholds conducted in [5], this reveals that using the conditional probability in the bull market as trade signals is rather robust against the change of thresholds. It is analogous to the scenario when technical analysis is used: the effects of using 200-day moving average and 150-day moving average as trade signals are likely comparable.

The above simulation results are based on the average outcomes of large numbers of simulated paths. We now investigate the performance of our strategy with individual sample paths. Table 3 collects simulation results on 10 single paths using buy-sell thresholds $p_s^* = 0.795$ and $p_b^* = 0.948$ with the same data given in Table 1. We can see that the simulation is very sensitive to individual paths. Nevertheless, on large number of trials our strategy clearly outperforms the buy and hold strategy statistically.

Note that this observation is consistent with the measurement of an effective investment strategy in marketplace. For example, O'Neil's CANSLIM works during a period of time does not mean it works on each stock when applied. How it works is measured based on the overall average when applied to a group of stocks fitting the prescribed selection criteria.

Trend Following	Buy and Hold	No. of Trades
67.080	3.2892	36.000
24.804	2.2498	42.000
22.509	0.40591	42.000
1887.8	257.75	33.000
26.059	0.16373	48.000
60.267	1.5325	43.000
34.832	5.7747	42.000
8.6456	0.077789	46.000
128.51	30.293	37.000
224.80	29.807	40.000

Table 3. Ten single-path simulations

4.2. Market tests We now turn to the question whether the trend following trading strategy presented works in real markets. In view of the path sensitivity discussed in the end of the last section we conduct our tests using a broad based stock index which reflects the aggregation of the behaviors of a large number of stocks. While ex-post tests are employed in [5], we conduct the ex-ante tests for the SP500 index – a broad based index that has a set of accessible historical data reasonably long for our tests. Our goal is evaluating whether our theoretically optimal trend following strategy provides useful guidance in real market.

The historical data for SP500 is available since 1962. We assume that any trading action will take place at the close of the market and, therefore, will use the SP500 daily closing price for our test. We define an up trend to be rally at least 20% and a down trend decline at least 20%. For any giving period of the SP500 historical data, say 5 or 10 years, one can find several up and down trends. We can use the statistics of the duration and total appreciation/depreciation of these trends to empirically calibrate the parameters $\mu_i, \lambda_i, i=1,2$ and σ. However, after quickly scanning several such periods of data we find that the empirical estimate of these parameters is quite different in different time periods. The change of the parameters, of course, is not unanticipated. Many social, economic and technological factors contribute to such a change and make it difficult to precisely predict. However, these exogenous impacts on the parameters happen over time. Thus, we make the following working assumptions: (a) the parameters gradually change over a long time horizon (say 10 years) yet they are relatively stable in a short time horizon (say 1 year) and (b) recent data is more relevant compared to the data in the distant past. Base on these assumptions we determine the parameters by beginning with the statistical estimate of the 10 year data from 1962 to 1972 as follows: μ_1 and λ_1 are estimated as the average of annualized return and reciprocal of the length of the up trends and μ_2 and λ_2 are the average of annualized return and reciprocal of the length of the down trends. We conduct the trend following strategy using these parameters and the corresponding thresholds in the following year and then update the parameters and the corresponding thresholds at the beginning of a new year using the new data that become available if a new up or down trend is completed. To reflect assumption (b), we update the parameters using the so called exponential average method in which the update of the parameters is determined by the old parameters and new parameters with formula

$$\text{update} = (1-2/N)\text{old} + (2/N)\text{new},$$

where we chose $N=6$ based on the number of up and down trends between 1962–1972. The exponential average allows us to overweight the recent information while avoiding unwanted abrupt changes due to dropping old information. Then we use the yearly updated parameters to calculate the corresponding thresholds. Finally, we use these parameters and thresholds to test the SP500 index from 1972-2011. The equity curve of the trend following strategy is compared to the buy and hold strategy in the same period of time in Figure 3. The upper, middle and the lower curves represent the equity curves of the trend following strategy, the buy and hold strategy including dividend, and the SP500 index without dividend adjustment, respectively.

As we can see, the trend following strategy not only outperforms the buy and hold strategy in total return, but also has a smoother equity curve, which means a higher Sharpe ratio; see Table 4.

Index(time frame)	TF	TF Sharpe	BH	BH Sharpe	10 year bonds
SP500 (1972-2011)	11.03%	0.217	9.8%	0.128	6.79%

Table 4. Testing results for trend following trading strategies

The test result for SP500 here is, if not better, at least comparable to the ex-pose test in [5] showing that trends indeed exist in the price movement of SP500. It is worthwhile pointing out that in [5], there is a mistake that the dividends are not treated as reinvestment. As a correction, the returns of the buy and hold strategy and the trend following strategy in [5] (Table 10) should be respectively 54.6 and 70.9, instead of 33.5 and 64.98, for SP500 (1962-2008).

We note that although an index such as the SP500 reflects the aggregation of the behavior of many individual stocks, trading it with the trend following strategy could still experience an

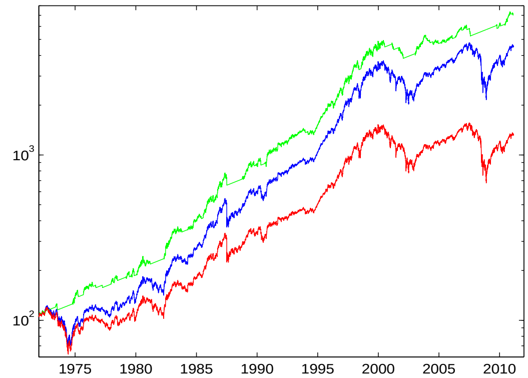


FIGURE 3. Trend following trading of SP500 1972–2011 compared with buy and hold

instability as observed in the end of last section. Using the trend following strategy simultaneously on a large number of stocks should smooth out the fluctuation of the performance and achieve better stability. In spite that such tests belong to the area of developing proprietary trading strategies and do not fall in the scope of this paper, the testing methods used here are relevant and useful.

5. Conclusion We have considered a finite horizon investment problem in a bull-bear switching market, where the drift of the stock price switches between two parameters corresponding to an uptrend (bull market) and a downtrend (bear market) according to an unobservable Markov chain. The goal is to maximize the expected log-utility of the terminal wealth. We restricted attention to allowing flat and long positions only and used a sequence of stopping times to indicate the time of entering and exiting long positions. We have shown that the optimal trading strategy is trend following, characterized by the conditional probability in the uptrend crossing the buy and sell boundaries.

Regarding future research, it would be interesting to see how the approach works in models with more than two states, e.g., (bull, bear, sideways markets). In addition, substantial empirical tests on much broader selections of stocks will be useful to reveal when the trend following method works and when it fails in the marketplace.

Appendix. Proofs of Theorems 2 and 3.

Proof of Theorem 2. i) First we prove

$$Z(p,t) \equiv \log(1+K_b), \;\; \forall\, p \geq 3p_0,\, 0 \leq t \leq T - 1/p_0. \tag{33}$$

Let us construct a function:

$$Z_1 = \begin{cases} -a[(p-2p_0)(T-t)-1]^2 + \log(1+K_b), & 2p_0 \leq p \leq \min\{2p_0 + \frac{1}{T-t}, 1\}; \\ \log(1+K_b), & \min\{2p_0 + \frac{1}{T-t}, 1\} < p \leq 1. \end{cases}$$

We claim that Z_1 is a subsolution of (14) in $(2p_0, 1) \times (T - 1/p_0, T)$. Indeed,

$$Z_1\left(2p_0 + \frac{1}{T-t}, t\right) = \log(1+K_b), \;\; \partial_p Z_1\left(2p_0 + \frac{1}{T-t}, t\right) = 0, \;\; \forall\, 2p_0 + \frac{1}{T-t} \leq 1.$$

So, $Z_1 \in W_q^{2,1}((2p_0,1)\times(0,T))$. Moreover, for $2p_0 \le p \le \min\{2p_0+\frac{1}{T-t},1\}$ and $T-1/p_0 \le t \le T$, we have

$$
\begin{aligned}
-LZ_1 &= -2a(p-2p_0)[(p-2p_0)(T-t)-1] + \frac{a(\mu_1-\mu_2)^2 p^2(1-p)^2(T-t)^2}{\sigma^2} \\
&\quad -2a\lambda_1 p(T-t)[(p-2p_0)(T-t)-1] + 2a\lambda_2(T-t)[(p-2p_0)(T-t)-1](1-p) \\
&\le 2a + \frac{a(\mu_1-\mu_2)^2[\,(p-2p_0)^2+4p_0(p-2p_0)+4p_0^2)\,](1-p)^2(T-t)^2}{\sigma^2} \\
&\quad +2a\lambda_1[(p-2p_0)+2p_0](T-t)[1-(p-2p_0)(T-t)],
\end{aligned}
$$

where the inequality is due to $0 \le p-2p_0 \le 1$ and $-1 \le (p-2p_0)(T-t)-1 \le \frac{1}{T-t}(T-t)-1 \le 0$. Noticing that $0 \le 1-p \le 1,\ 0 \le (p-2p_0)(T-t) \le 1$, and $0 \le p_0(T-t) \le 1$, we then deduce

$$
\begin{aligned}
-LZ_1 &\le 2a + \frac{a(\mu_1-\mu_2)^2(1+4+4)}{\sigma^2} + 2a\lambda_1(1+2) \\
&= \left[2 + \frac{9(\mu_1-\mu_2)^2}{\sigma^2} + 6\lambda_1\right] a \\
&\le (\mu_1-\mu_2)p_0,
\end{aligned}
$$

where the last inequality is due to the right hand side condition in (19). It is clear that for any $\min\{2p_0+\frac{1}{T-t},1\} \le p \le 1$, we have

$$
-LZ_1 = -L(\log(1+K_b)) = 0 \le (\mu_1-\mu_2)p_0.
$$

On the other hand, in the domain $\mathcal{M} \triangleq \{(p,t)\in[2p_0,1)\times[T-1/p_0,T] : Z(p,t) < \log(1+K_b)\}$, one has

$$
-LZ \ge f(p)-\rho \ge f(2p_0)-\rho = (\mu_1-\mu_2)p_0 \ge -LZ_1.
$$

Apparently,

$$
Z_1(2p_0,t) = \log(1-K_s) \le Z(2p_0,t), \quad Z_1(p,T) = \log(1-K_s) \le Z(p,T).
$$

Using the maximum principle in the domain $\mathcal{M}$, we infer $Z \ge Z_1$ in $[2p_0,1)\times[T-1/p_0,T]$. In particular,

$$
Z(3p_0, T-1/p_0) \ge Z_1(3p_0, T-1/p_0) = \log(1+K_b).
$$

It is not hard to show that $Z(p,t)$ is decreasing with respect to t and increasing with respect to p. We then obtain (33).

Consider another function:

$$
\underline{Z} = \log(1-K_s) + \frac{a}{6p_0}\left[p-3p_0+\frac{\lambda_2}{2}\left(T-\frac{1}{p_0}-t\right)\right] \text{ in } \mathcal{N},
$$

where $\mathcal{N} \triangleq (0,3p_0)\times(T-1/p_0-12p_0/\lambda_2, T-1/p_0)$. We now show that $\underline{Z}$ is a subsolution of (14) in $\mathcal{N}$. It is easy to verify

$$
\partial_t \underline{Z} < 0, \quad \partial_p \underline{Z} > 0, \quad \underline{Z}(3p_0, T-1/p_0-12p_0/\lambda_2) = \log(1+K_b), \quad \underline{Z} < \log(1+K_b) \text{ in } \mathcal{N}.
$$

Moreover,

$$
-L\underline{Z} = \frac{a}{6p_0}\left[(\lambda_1+\lambda_2)p - \frac{\lambda_2}{2}\right] \le \frac{a}{6p_0}\left[3(\lambda_1+\lambda_2)p_0 - \frac{\lambda_2}{2}\right].
$$

In the domain $\{(p,t)\in\mathcal{N}: Z(p,t)<\log(1+K_b)\}$,

$$-LZ\geq f(p)-\rho\geq-(\mu_1-\mu_2)p_0\geq\frac{a}{6p_0}\left[3(\lambda_1+\lambda_2)p_0-\frac{\lambda_2}{2}\right]\geq-L\underline{Z},$$

where the third inequality is due to (18) and the left hand side condition in (19). It is clear that

$$\begin{aligned}&\underline{Z}(p,T-1/p_0)\leq\underline{Z}(3p_0,T-1/p_0)=\log(1-K_s)\leq Z(p,T-1/p_0),\ \forall\, p\in(0,3p_0],\\&\underline{Z}(3p_0,t)\leq\log(1+K_b)=Z(3p_0,t),\ \forall\, t\in(T-1/p_0-12p_0/\lambda_2,T-1/p_0).\end{aligned}$$

Again using the maximum principle, we deduce $\underline{Z}\leq Z$ in the domain $\mathcal{N}$. In particular,

$$\begin{aligned}Z(p,t)\ &\geq\ Z(p,T-1/p_0-12p_0/\lambda_2)\geq\underline{Z}(p,T-1/p_0-12p_0/\lambda_2)\\&>\ \underline{Z}(0,T-1/p_0-12p_0/\lambda_2)>\log(1-K_s),\ \forall\, p>0,\, t\leq T-1/p_0-12p_0/\lambda_2,\end{aligned}$$

which yields the desired result.

ii) From (20), we infer

$$p_0\geq 2/3,\qquad(\lambda_1+\lambda_2)(3p_0-2)-\lambda_2\geq\frac{\lambda_1+\lambda_2}{2};$$
$$\frac{4(\mu_1-\mu_2)^2(1-p_0)}{\sigma^2}\leq\frac{(\lambda_1+\lambda_2)}{2},\quad\frac{\sigma^2(\lambda_1+\lambda_2)}{18(\mu_1-\mu_2)}\geq(\mu_1-\mu_2)(1-p_0).$$

Construct the following function:

$$\overline{Z}(p,t)=\begin{cases}\log(1-K_s), & 0\leq p<3p_0-2,\\ \log(1-K_s)+\frac{\sigma^2\,[p-(3p_0-2)]^2}{9(\mu_1-\mu_2)(1-p_0)}, & 3p_0-2\leq p\leq 1.\end{cases}$$

It is easy to see that $\overline{Z}\geq\log(1-K_s)$ and $\overline{Z}\in W_q^{2,1}((0,1)\times[0,T))\cap C((0,1)\times[0,T])$, for any $q\geq 1$. For $0<p<3p_0-2$, we have

$$-L\overline{Z}=-L(\log(1-K_s))=0\geq f(3p_0-2)-\rho\geq f(p)-\rho.$$

For $3p_0-2\leq p\leq 2p_0-1$, we find

$$\begin{aligned}-L\overline{Z}\ &=\ \frac{\sigma^2}{9(\mu_1-\mu_2)(1-p_0)}\left\{\frac{-(\mu_1-\mu_2)^2p^2(1-p)^2}{\sigma^2}+2\,[\,(\lambda_1+\lambda_2)p-\lambda_2\,]\,[\,p-(3p_0-2)\,]\right\}\\&\geq\ -(\mu_1-\mu_2)(1-p_0)=f(2p_0-1)-\rho\geq f(p)-\rho.\end{aligned}$$

For $2p_0-1\leq p\leq 1$, we have

$$\begin{aligned}-L\overline{Z}\ &\geq\ \frac{\sigma^2}{9(\mu_1-\mu_2)(1-p_0)}\left[-\frac{(\mu_1-\mu_2)^2 4(1-p_0)^2}{\sigma^2}+(\lambda_1+\lambda_2)(1-p_0)\right]\\&\geq\ \frac{\sigma^2}{9(\mu_1-\mu_2)(1-p_0)}\,\frac{(\lambda_1+\lambda_2)(1-p_0)}{2}\\&\geq\ (\mu_1-\mu_2)(1-p_0)=f(1)-\rho\geq f(p)-\rho.\end{aligned}$$

Hence, $\overline{Z}$ must be a supersolution of (14). We then deduce that

$$Z(p,t)\leq\overline{Z}(p,t)<\overline{Z}(1,t)=\log(1-K_s)+\frac{\sigma^2(1-p_0)}{\mu_1-\mu_2}\leq\log(1-K_s)+a=\log(1+K_b),\ \forall\, p<1,$$

which implies that the buy region does not exist. So, $p_b^*(t)\equiv 1$ for all t. $\square$

Proof of Theorem 3. Consider an auxiliary function:

$$\overline{Z} = \begin{cases} \log(1-K_s) + a\left(\frac{4p}{p_0} - 1\right)^2, & \frac{p_0}{4} \le p \le \frac{p_0}{2}, \\ \log(1-K_s), & 0 \le p < \frac{p_0}{4}. \end{cases}$$

Clearly $\overline{Z} \in W_q^{2,1}((0,p_0/2)\times(0,T)) \cap C([0,p_0/2]\times[0,T])$ and

$$\overline{Z} \ge \log(1-K_s). \tag{34}$$

It is not hard to verify that for $p \in (p_0/4, p_0/2)$, we have

$$\begin{aligned} -L\overline{Z} &= \frac{a}{p_0^2}\left[\frac{-16(\mu_1-\mu_2)^2p^2(1-p)^2}{\sigma^2} + 8(\lambda_1+\lambda_2)p(4p-p_0) - 8\lambda_2(4p-p_0)\right] \\ &\ge -\left[\frac{4(\mu_1-\mu_2)^2}{\sigma^2} + \frac{8\lambda_2}{p_0}\right]a. \end{aligned}$$

Using (21), it follows

$$-L\overline{Z} \ge -(\mu_1-\mu_2)p_0/2 = f(p_0/2) - \rho \ge f(p) - \rho \tag{35}$$

for $p \in (p_0/4, p_0/2)$. In the case $p \in (0, p_0/4)$,

$$-L\overline{Z} = -L(\log(1-K_s)) = 0 \ge f(p) - \rho. \tag{36}$$

The combination of (34)-(36) yields

$$\min\left\{-L\overline{Z} - f(p) + \rho, \overline{Z} - \log(1-K_s)\right\} \ge 0$$

in $p \in (0, p_0/2)$, $t \in [0,T)$. Moreover, it is clear that

$$\overline{Z}(p,T) \ge \log(1-K_s) = Z(p,T), \qquad \overline{Z}(p_0/2,t) = \log(1+K_b) \ge Z(p_0/2,t).$$

Thus $\overline{Z}$ must be a supersolution of (14) in $[0,p_0/2]\times[0,T]$. By the maximum principle, we infer $\overline{Z} \ge Z$ in $[0,p_0/2]\times[0,T]$. Then, for $p < p_0/4$, we have

$$\log(1-K_s) \le Z \le \overline{Z} \equiv \log(1-K_s),$$

which implies $Z \equiv \log(1-K_s)$ for $p < p_0/4$. Note that we can obtain (33) in terms of $p_0 < 1/3$ and (21). The desired result then follows. □

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