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Inefficient Bubbles and Efficient Drawdowns in Financial Markets

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Abstract

At odds with the common "rational expectations" framework for bubbles, economists like Hyman Minsky, Charles Kindleberger and Robert Shiller have documented that irrational behavior, ambiguous information or certain limits to arbitrage are essential drivers for bubble phenomena and financial crises. Following this understanding that asset price bubbles are generated by market failures, we present a framework for explosive semimartingales that is based on the antagonistic combination of (i) an excessive, unstable pre-crash process and (ii) a drawdown starting at some random time. This unifying framework allows one to accommodate and compare many discrete and continuous time bubble models in the literature that feature such market inefficiencies. Moreover, it significantly extends the range of feasible asset price processes during times of financial speculation and frenzy and provides a strong theoretical background for future model design in financial and risk management problem settings. This conception of bubbles also allows us to elucidate the status of rational expectation bubbles, which, by design, suffer from the paradox that a rational market should not allow for misvaluation. While the discrete time case has been extensively discussed in the literature and is most criticized for its failure to comply with rational expectations equilibria, we argue that this carries over to the finite time "strict local martingale"-approach to bubbles. Our framework will simplify and foster interdisciplinary exchange at the intersection of economics and mathematical finance and encourage further research.

Keywords — Financial Bubbles, Financial Crashes, Explosive processes, Bubble decomposition, Strict local martingale approach, Infinite horizon bubbles
JEL: G01, G10, C60

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1 Introduction

The term *asset price bubble* typically describes a situation where the market price of an asset exceeds its fair price, referred to as the asset's *fundamental value*. Asset price bubble models can essentially be grouped into two main paradigms – those where bubbles appear in an efficient market and those that are based on a violation of market efficiency.

The first group has become known as *rational expectation bubbles*, which tries to square the idea of asset over-valuation (and thus an inefficiency) with efficient markets. This inherent discrepancy prohibits the existence of bubbles in rational market equilibria and leads to joint hypothesis issues and internal inconsistencies in bubble detection; see section 2.1 below. While these problems have been discussed in the literature for discrete time interpretation of the rational expectations paradigm, we argue, using results from the literature, that they also permeate the more recent continuous time modeling approach. Despite these issues, presumably because such models can readily be described in a simple, unified framework, rational expectation bubbles are still used extensively in both discrete and continuous time models, ranging from economics and econometrics to mathematical finance.

Meanwhile, we have seen a flurry of models that are based on the understanding that bubbles are caused by a break of market efficiency in the sense of *perfect information rational expectations*. Various mechanisms have been proposed – such as asymmetric information Allen and Gorton (1993), Allen et al. (1993), heterogeneous beliefs Harrison and Kreps (1978), Scheinkman and Xiong (2003) and noise trading such as positive feedback activity De Long et al. (1990b), Shleifer (2000), Sornette (2003) in combination with limits to arbitrage Abreu and Brunnermeier (2003), De Long et al. (1990a), Shleifer and Summers (1990), Shleifer and Vishny (1997). In one way or another, such mechanisms create unsustainable behavior in asset prices and the risk of a crash. To capture the essence of such models, we propose to dissect the stock price into three so-called *bubble characteristics* –

1. a pre-drawdown process $\tilde{S}$,
2. the random time of the drawdown τ_J and (1.1)
3. the shape of the drawdown X.

These three bubble characteristics allow one to describe the full stock price $(S_t)_{t\in[0,\infty)}$ as

$$S_t = \tilde{S}_t \mathbb{1}_{\{t<\tau_J\}} + X_t \mathbb{1}_{\{\tau_J \leqslant t\}}, \tag{1.2}$$

following $\tilde{S}$ up to the crash at τ_J and X afterwards. Figure 1 shows such a decomposition for the Dow Jones stock price bubble and crash of 1987. To complement our framework, we need a notion of *market efficiency*. We include such a notion through an abstract condition, which distinguishes efficient from inefficient markets and allows us to classify the bubble characteristics (1.1). The pre-drawdown process $\tilde{S}$ is understood to show exuberant, excessive behavior, thus we require that

$$\tilde{S} \text{ violates market efficiency,} \tag{1.3}$$

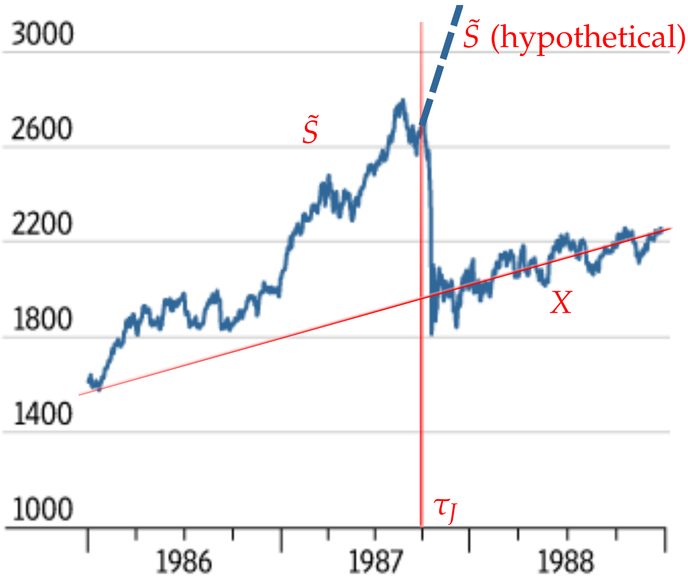

Figure 1: The Dow Jones stock price bubble and its bubble characteristics $(\tilde{S}, \tau_J, X)$. Source: WSJ Markets Data Group.

in a suitable way made precise below. The drawdown (τ_J, X) is understood as a mechanism to restore market efficiency, where two main bubble types can be extracted from the literature, gradually departing from rational expectation bubbles.

1.1 Type-I bubbles.

We can model the risk and shape of a drawdown (τ_J, X) being caused by an inefficiency, while extraordinary (pre-drawdown) returns ensure that this risk is "appropriately" priced in the market: in this case,

$$\text{the full price process } S \text{ is an efficient market,} \tag{1.4}$$

while both the pre-drawdown process $\tilde{S}$ and the drawdown process X – seen in isolation – are inefficient. We call this a type-I bubble.

This understanding of a bubble has been put forward in section (B) of Blanchard (1979) and continued in Johansen et al. (1999, 2000). Note that, while originally this type of bubble has been discussed within the rational expectations framework, the exact valuation of fundamentals plays a minor role here. The essence of this understanding of a bubble can be captured by a quote from Blanchard (1979), that "during the duration of the bubble, [the bubble process] is growing faster (...) because asset holders have to be compensated for the probability of a crash." What drives a type-I bubble is the mere existence of the risk of a crash (captured in our framework by τ_J and X), compensated by a pre-crash process that earns abnormal returns ($\tilde{S}$ violating market efficiency) such that the full price process S is an efficient market.

This conception of a bubble resolves an important methodological issue of (classic) rational expectation bubbles: a type-I inefficient bubble is verified by the occurrence of a crash and does not rely on an infinite payoff structure. However, in such models the precise mechanism of how *rational market participants are aware of the crash risk and price it in the market* remains unclear. While a rational, omniscient investor would stay invested in such a market, the underlying stock price S needs to be postulated a priori.

1.2 Type-II bubbles.

In order to avoid this issue of postulating market efficiency for S to accommodate rational investors, we can either look at markets that are not populated by rational investors with symmetric information or add limits to arbitrage. In the absence of rational investors going against exuberant development, $\tilde{S}$ is driven by some failure of market efficiency. To arrive at a full-fledged bubble model, we assume that the very cause for this inefficiency disappears/gets resolved at τ_J and price levels collapse to "appropriate" levels: in this case, within the decomposition (1.2),

$$\text{the drawdown process } X \text{ is an efficient market,} \tag{1.5}$$

while the full process S is inefficient. We call this a type-II bubble.

This representation captures the view for instance espoused by Fama (1989) and Pástor and Veronesi (2006) *(inefficiency: (ex-post) overestimation of future payoffs)*, Scheinkman and Xiong (2003) *(inefficiency: heterogeneous beliefs and overconfidence)* or Abreu and Brunnermeier (2003) *(inefficiency: noise trading and synchronization risk)*. While the underlying mechanisms causing a break of market efficiency are quite diverse,[1] the nature of the crash in each case is captured by decomposition (1.2) and the time of the crash represents the time where rationality is restored. Such models avoid the above problem of assuming an efficient market process S, as noted above for type-I bubbles. However, for type-II bubbles the efficient process X (and thus a model for S) can in general only be determined under additional assumptions on the assets payoff structure. Note that in case one is interested only in why market efficiency can fail (that is, only in the process $\tilde{S}$) and not in the specific role of a drawdown, then the distinction between type-I and type-II bubbles is not relevant. This includes many of the early papers, e.g., Allen et al. (1993), De Long et al. (1990a,b).

1.3 Bubbles as overvaluation.

One may like to think of bubbles as overvaluation, and for both bubble types irrationality in the pre-drawdown process $\tilde{S}$ leads to a valuation that is *too high* in the following sense.

1. In a type-I bubble, the asset is overvalued for those who mistake $\tilde{S}$ for the true dynamics of the price process. Projecting the pre-drawdown stock dynamics to final payoffs (ignoring the risk of a drawdown)[2], one overvalues the true payoff of the asset. An omniscient ob-

[1] The reader may be surprised to see Fama (1989) in the list of references to underpin inefficient market bubbles, as the author is essentially arguing that markets are efficient before, during and after the crash. However, a (hypothetical) omniscient observer with a proper valuation of future payoffs correctly sees a failure of market efficiency, which is restored only after the crash. For details see section 4.3.3.

[2] In other words, this means operating under the belief that $\tau_J \equiv \infty$ or $X \equiv \tilde{S}$, almost surely.

server, assessing the correct underlying S, does not see an overvaluation, only the risk of a drawdown compensated by high returns. A rational, omniscient investor stays invested.

2. In a type-II bubble, the asset is overvalued, even if the true dynamics S are known – X represents the correct valuation of the final payoff. The omniscient observer sees a bubble and, consequently, the risk of a drawdown as possible revaluation. A rational, omniscient investor would exit the market eventually, depending on the likelihood of possible crash times.

Below we provide a simple framework for such bubble characteristics $(\tilde{S}, \tau_J, X)$ and market efficiency conditions that allows one to accommodate, compare and ultimately justify several approaches in the literature. Many of these models have not yet been integrated in a wider framework and have been perceived as quite diverse attempts to bubble modeling, thus preventing efficient dialogue between research disciplines. Our framework is proposed as a way to strengthen the understanding of bubbles as a failure of market efficiency and serve as a basis for simple models that – contrary to rational expectation models – are based on *the* essential property of a bubble: the possibility of a drawdown fueled by market failure.

In section 2 we discuss alternative approaches in the literature – with a focus on rational expectation bubbles and their shortcomings.[3] In particular, we collect results from the literature that characterize rational expectation bubbles in both discrete and continuous time. In section 3 we provide a general framework for the bubble characteristics (1.1) and the two types of inefficient market bubbles discussed above. Sections 4 and 5 discuss applications and examples of inefficient market bubbles in continuous and discrete time, respectively. We end each section with a comparison to the rational expectations framework, highlighting the main differences and possible similarities. Exploiting the simple structure of our framework, we close with a discussion of issues in bubble detection and bubble modeling and some further research questions in section 6. The appendix summarizes some auxiliary definitions and compiles our technical results.

2 Review of bubble literature

Besides the literature mentioned in the introduction (modeling a break of market efficiency), the lion's share of bubble models emerges from the rational expectations framework. Thus, our discussion in this section will mainly focuses on this strand of the literature.

2.1 Rational expectation bubbles

This approach is based on the theoretical definition of an asset's fundamental value in rational expectation markets.

[3] As the main purpose of the present paper is to introduce a novel framework, the discussion of rational expectation bubbles is not devised as a comprehensive literature review and limited to general ideas and seminal publications.

2.1.1 Definitions

Setting (Rational expectation bubbles). For simplicity, we assume appropriately discounted prices and dividend streams and thus 0 interest rate.[4] Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, $\tau : \Omega \to [0, \infty]$ the lifetime of a non-negative asset $(S_t)_{t\in[0,\tau]}$ with cumulative dividend stream $(D_t)_{t\in[0,\tau]}$ and a final payoff S_τ. Then, following Jarrow et al. (2007), one can define the wealth process $(V_t)_{t\in[0,\tau]}$ of the asset by

$$V_t = S_t + D_t, \quad t \in [0, \tau]. \tag{2.1}$$

Finally, we assume that both S and D are semimartingales and that *rational expectations* holds in the form of *absence of arbitrage opportunities*. To describe this in our general continuous setting, we use the condition of *No Free Lunch with Vanishing Risk (NFLVR)* and the notion of *self-financing, admissible* trading strategies; see Delbaen and Schachermayer (1998) for details.[5] A general version of the first fundamental theorem of asset pricing in Delbaen and Schachermayer (1998) then implies that there exists a nonempty set $\mathcal{Q}$ of $\mathbb{P}$-equivalent measures such that V is a $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathcal{Q}$.

All definitions above include discrete time models by using $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ and the index set $[0, \tau] \cap \mathbb{N}$.

Fundamental values and bubble definition. If we make the additional assumption that the market generated by S is complete (that is, $\mathcal{Q}$ is a singleton), then the fundamental value of an asset can simply be defined as

$$S_t^* = E_{\mathbb{Q}}\left[(D_\tau - D_t) + S_\tau \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_t\right], \quad t \in [0, \tau], \tag{2.2}$$

the expectation of future payoff streams under the unique market implied probability $\mathbb{Q}$. In general, however, the market generated by S is incomplete and the straightforward definition (2.2) does not extend immediately. We can distinguish two notions of fundamental value and use the naming from Herdegen and Schweizer (2016).

Definition 2.1 (Strong rational expectation bubble). Assume the setting above. Then the fundamental value of an asset is defined as

$$S_t^* = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}} E_{\mathbb{Q}}\left[(D_\tau - D_t) + S_\tau \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_t\right], \quad t \in [0, \tau]. \tag{2.3}$$

As above, we say that S has a rational expectation bubble at t if $S_t^* < S_t$ and call $B = S - S^*$ the bubble component of S.

Definition 2.2 (Rational expectation $\mathbb{Q}$-bubble). Assume the setting above. For some $\mathbb{Q} \in \mathcal{Q}$, we define a $\mathbb{Q}$-fundamental value as

$$S_t^{\mathbb{Q}} = E_{\mathbb{Q}}\left[(D_\tau - D_t) + S_\tau \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_t\right], \quad t \in [0, \tau]. \tag{2.4}$$

[4] This assumption requires the a priori choice of a bank account, which is bounded from below and can be used as a numeraire. Thus, it is inflicted with a certain loss of generality; see, e.g., the discussion in Herdegen (2017).

[5] Below we always refer to self-financing, admissible strategies when using the terms *trading* and *replication*.

We say that S has a rational expectation $\mathbb{Q}$-bubble at t if $S_t^{\mathbb{Q}} < S_t^* = S_t$ and call $B^{\mathbb{Q}} = S - S^{\mathbb{Q}}$ the $\mathbb{Q}$-bubble component of S.

Remark 2.3.

(a) In the case of a complete market, the definitions collapse to the fundamental value in (2.2). It is clear that, if S has a strong bubble, then S has a $\mathbb{Q}$-bubble for every $\mathbb{Q} \in \mathcal{Q}$.

(b) The fundamental value in (2.3) has the the *superreplication property* S. By Kramkov (1996)'s optional decomposition, it is equal to the cheapest starting capital that allows one to replicate all payoffs of S on $[t, \tau]$ and, as such, has a solid economic interpretation as *fundamental* value. For a general discussion of fundamental values; see, e.g., section 6.1 in Herdegen and Schweizer (2016)

(c) Santos and Woodford (1997) were among the first to recognize that the fundamental value does not immediately extend from complete markets and discuss both definitions in incomplete markets; in fact, many of the early papers on rational expectation bubbles ignore this issue. Subsequently Loewenstein and Willard (2000), Heston et al. (2007), Loewenstein and Willard (2013) and Herdegen and Schweizer (2016) use the superreplication price (2.3) and the corresponding bubble definition,[6] while Jarrow et al. (2010) and Biagini et al. (2014) use a dynamic version of (2.4).

Example 2.4.

(a) **Special case: discrete time bubbles.** The classical framework is based on an infinite time horizon $\tau \equiv \infty$ and uses a discrete set of trading dates; see, e.g., Flood and Garber (1980), Blanchard (1979) and Diba and Grossman (1988a). Camerer (1989) provides an excellent overview of the early literature, Giglio et al. (2016) a recent overview. To arrive at the classical framework in the setting above, we add the structural assumption that for some $R \in [0, \infty)$ there exists a "discounting" measure $\mathbb{Q}_R \in \mathcal{Q}$ with the property that, for all $n, \tau \in \mathbb{N}$,

$$E_{\mathbb{Q}_R}\left[S_{n+\tau}|\mathcal{F}_n\right] = \mathbb{E}\left[\frac{S_{n+\tau}}{(1+R)^{\tau}}\Big|\mathcal{F}_n\right] \quad \text{and} \quad E_{\mathbb{Q}_R}\left[D_{n+\tau} - D_{n+\tau-1}|\mathcal{F}_n\right] = \mathbb{E}\left[\frac{D_{n+\tau} - D_{n+\tau-1}}{(1+R)^{\tau}}\Big|\mathcal{F}_n\right]. \tag{2.5}$$

The measure $\mathbb{Q}_R \in \mathcal{Q}$ discounts future prices and payoffs and ensures a constant required rate of return R. Using equation (2.5) and the $\mathbb{Q}_R$-martingale property of the value process $V = S + D$ yields the standard one-step No-Arbitrage condition, cf. chapter 7 of Campbell et al. (1997),

$$(1+R)S_n = \mathbb{E}\left[S_{n+1} + D_{n+1} - D_n|\mathcal{F}_n\right], \quad n \in \mathbb{N}. \tag{2.6}$$

Assuming the market is complete, R defines a unique martingale measure $\mathbb{Q}_R$ and the fundamental value (2.2) takes the well-known form of future discounted dividend payments, that

[6] Let us note that Herdegen and Schweizer (2016) use a dynamic version of (2.3), which allows for the concept of *bubble birth*.

is,

$$S_n^* = E_{\mathbb{Q}_R}\left[(D_\infty - D_n)|\mathcal{F}_n\right] = \mathbb{E}\left[\sum_{k=1}^{\infty} \frac{D_{n+k} - D_{n+k-1}}{(1+R)^k}\middle|\mathcal{F}_n\right], \quad n \in \mathbb{N} \cup \{\infty\}. \tag{2.7}$$

The assumption of completeness, however, is often implicit and unjustified; see example 4.4 in Santos and Woodford (1997).

(b) **Special case: strict local martingale price process.** In the seminal paper of Loewenstein and Willard (2000), the authors have realized that the definition of a fundamental value in equation (2.3) allows for bubbles even when restricting to a finite time horizon $\tau \equiv T \in (0, \infty)$. This is possible only for continuous time models and thus a priori excluded in the classical bubble literature. Assume the setting above, zero dividend payments and, for simplicity, that the market is complete. Then we have a single payoff S_T and the fundamental value is given by $S_t^* = E_{\mathbb{Q}}\left[S_T|\mathcal{F}_t\right]$. The essential notion is that of a strict local martingale.

Definition 2.5 (Strict local martingale). A local martingale S is a strict local martingale if it fails to be a martingale.

As such, strict local martingales are fair games locally but fail to satisfy the martingale property. Thus, if S is a strict local martingale, we have

$$E_{\mathbb{Q}}\left[S_T|\mathcal{F}_0\right] < S_0 \tag{2.8}$$

and a strong bubble according to definition 2.1.

To understand the underlying mechanism, we can invoke Kramkov (1996)'s optional decomposition theorem, which tells us that there exists a trading strategy with initial capital $E_{\mathbb{Q}}\left[S_T|\mathcal{F}_0\right]$, replicating S_T almost surely. In addition, a trivial buy-and-hold strategy on $[0, T]$ with initial capital S_0 is always a replicating strategy; see figure 2 for a visualization. Note that the existence of two such strategies does not interfere with No-Arbitrage (NFLVR), as exploiting the difference would require a net short position in the asset, which, as a strict local martingale, is necessarily unbounded. Unbounded portfolio losses, however, are excluded by the admissibility condition.

2.1.2 Modeling rational expectation bubbles

One may be tempted to formulate the following for *rational expectation markets*.

> *Given an asset S with present market price S_0 that entitles the holder to a series of diverse payoff streams during its lifetime, then, if the market is rational in a basic sense, the fundamental price of this asset should be the market-implied present value of exactly those payoff streams: S_0.*

A deviation of the fundamental value from an asset's market price in a rational expectation market seems thus inconsistent with the notion of a market equilibrium populated by rational traders. Making precise this inconsistency in a rational equilibrium model of traders, a celebrated result by Tirole (1982) for strong rational expectation bubbles is often used in the literature to (theoretically)

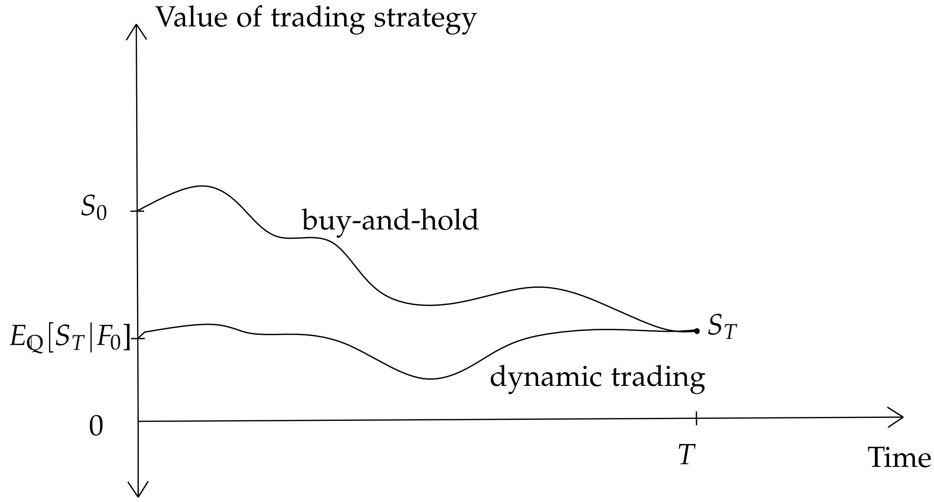

Figure 2: Sample paths of two self-financing, admissible strategies with final payoff $H = S_T$ in a strict local martingale model.

dismiss the existence of discrete time bubbles. This reasoning can be extended to bubbles in continuous time, as we argue below. These arguments do not apply to the case of Q-bubbles that fail to be strong bubbles, which we discuss in a separate paragraph.

Strong rational expectation bubbles. The fundamental value (2.3) in definition 2.1 is the superreplication price associated to S. By Kramkov (1996)'s optional decomposition, S_t^* is equal to the cheapest starting capital that allows one to replicate all payoffs of S on $[t, \tau] \cap [0, \infty)$ and, as such, has a solid economic interpretation as *fundamental* value (see, e.g., the discussion in section 6.1 in Herdegen and Schweizer (2016)). This defining property of a strong rational expectation bubble implies that, given $S_t^* < S_t$, no investor preferring more to less would accept a price S_t for payoffs that may be replicated with strictly smaller initial capital S_t^*. In discrete time complete markets, this rationale was made precise in Tirole (1982) (see the discussion therein and especially propositions 3 and 6).

Proposition 2.6. *Assume the setting above with a discrete set of trading dates, $\tau \equiv \infty$ and market completeness. Then in the following situations a strong rational expectation bubble (definition 2.1) cannot emerge as the equilibrium*[7] *price process of an economy populated by rational, utility-maximizing investors.*

1. *We have a finite number of short-run maximizing investors.*
2. *We have any (finite or infinite) number of long-run maximizing investors.*

In theorem 3.1 of Santos and Woodford (1997), this was extended to a slightly more general equilibrium framework and incomplete markets. For continuous time models in finite time $\tau \equiv$

[7] For definitions of equilibria with short-run maximizing investors (myopic rational expectations equilibrium) and long-run maximizing investors (fully dynamic rational expectations equilibrium) we refer to Tirole (1982).

$T \in (0, \infty)$ (which includes strict local martingales as in example 2.4(b)), we can utilize and reformulate theorem 3.2 of Jarrow and Larsson (2012).

Proposition 2.7. *Assume the setting above and let $\tau \equiv T \in (0, \infty)$. Then a strong rational expectation bubble (definition 2.1) cannot emerge as the equilibrium*[8] *price process of an economy populated by rational, utility-maximizing investors.*

While these negative results depend on the specific notion of *rational expectations equilibrium* employed and some leave room for bubbles in special types of models,[9] they show that strong rational expectation bubbles are necessarily driven by market inefficiencies that transcend equilibria in markets populated by rational, utility-maximizing investors.

In this sense, propositions (2.6) and (2.7) confirm the intuitive reasoning above that there is no such thing as a rational (expectation) bubble. To arrive at a price process that follows a strong rational expectations bubble, the behavior of agents need to depart *in one way or another* from the standard assumption – that is, there needs to be at least one agent that is not rational and/or not utility-maximizing. However, in this case we agree with Tirole (1982) that "(...) research should be devoted to the explanation of actual price bubbles by non-rational behavior" and one should strive to explicitly model such behavior. Eventually, this insight leads to the models discussed in section 1 and, thus, is the cornerstone of our general framework introduced in section 3.

Rational expectation $\mathbb{Q}$-bubbles. In the above setting, assume for some $t \in [0, \tau)$ and $\mathbb{Q} \in \mathcal{Q}$ we have that

$$S_t^{\mathbb{Q}} < S_t^* = S_t. \tag{2.9}$$

Then S has a $\mathbb{Q}$-bubble but not a strong bubble. As the superreplication price S^* equals the asset price, in this situation the buy-and-hold strategy is the cheapest way of replicating the payoffs of S and the arguments for strong bubbles above do not apply.

However, there is a caveat: strict inequality in $S_t^{\mathbb{Q}} < S_t^*$ and the superreplication property imply that any investor seeking to replicate the payoffs of S on $[t, \tau] \cap [0, \infty)$ with initial capital $S_t^{\mathbb{Q}}$ will fail to do so with positive probability. In other words, no market participant is able to generate the payoff stream of S with initial capital $S_t^{\mathbb{Q}}$. The argument that the asset is *overpriced* at S_t does not hold in this situation and the relation of processes satisfying (2.9) to asset price bubbles (understood as mispricing) remains questionable.

2.1.3 Detecting rational expectation bubbles

If we choose to sideline these theoretical concerns, failing to explicitly model the generating mechanism of a bubble leads to challenges in the empirical detection of rational expectation bubbles.

[8] A market equilibrium in this setting is defined in section 2.7 of Jarrow and Larsson (2012).

[9] See, for example, the overlapping generations model of Tirole (1985) or examples 4.1-4.4 in Santos and Woodford (1997).

Discrete infinite time. As mentioned in example 2.4(a), classical rational expectation bubbles rely on an infinite time horizon $\tau \equiv \infty$. This leads to several requirements on rational expectation bubbles that do not appear to be essential characteristics of asset price bubbles, but rather structural requirements of the infinite horizon framework.

1. As noted, assets with bounded lifetime cannot have a bubble. This implies that a test for bubbles is necessarily inconclusive before the asset has reached the end of its lifetime.
2. If one puts aside the latter point by *assuming* an infinite payoff structure, a bubble process B may be any martingale.

These points imply that the infinite horizon framework does not produce testable implications and additional hypothesis on the price and dividend structure are necessary for bubble detection. For a more detailed and extensive discussion of such *joint hypothesis* issues see, e.g., Camerer (1989), Gürkaynak (2008) or Jarrow (2016).

As an example, a popular test for rational expectation bubbles is based on equation (2.6); see, e.g., Diba and Grossman (1988b), Phillips et al. (2015) or Homm and Breitung (2012). One assumes that one-period dividend payments $(d_t)_{t\in\mathbb{N}}$ follow a linear autoregressive process of the form

$$d_{t+1} = c + \delta d_t + \epsilon_t, \quad t \in \mathbb{N}, \tag{2.10}$$

for $\delta < 1$, $c \in \mathbb{R}$ and a sequence of normally distributed random variables $(\epsilon_t)_{t\in\mathbb{N}}$. Then equation (2.7) implies that the stock price follows a linear autoregressive process as well. Consequently, only if there is a bubble component B one will be able to detect non-stationary behavior $\delta > 1$ in a linear model of the form (2.10) for the stock price. Fortunately, such tests for non-stationarity, based on *unit-root tests* have been applied and developed for several decades and are very well understood (Dickey-Fuller, Phillips-Perron, etc.). Unfortunately, however reliably one can detect $\delta > 1$ in a linear specification (2.10), the existence of a bubble is based on a *joint hypothesis* on required returns, dividend streams and the stock price. In other words, non-stationary behavior can be attributed to a misspecified model of future dividends, time-varying/stochastic expected returns *or* a bubble component. In section 5.3.1 we provide a detailed comparison of this approach with our framework and review some of the issues raised above in greater detail.

Continuous finite time. Example 2.4(b) discusses the fact that, for a finite deterministic lifetime $\tau \equiv T \in (0, \infty)$, S has a bubble if and only if it is a strict local martingale. There have not been many attempts to apply the strict local martingale approach for the purpose of bubble detection. One that is worth mentioning is based on the strict local martingale property of homogeneous diffusions with explosive volatility; see Jarrow et al. (2011a), Jarrow et al. (2011b) and chapter 4 of Protter (2013). In this specific framework, bubble detection is possible if and only if the volatility increases sufficiently *before* a crash happens. However, Sornette et al. (2018) analyzed over 40 bubbles in a model free approach and conclude that more often than not volatility is low before the crash. Thus, empirically, explosive volatility does not seem to be a reliable bubble indicator.

A class of diffusion processes with a crash (see section 4.4 for details) has been shown to be a strict local martingale if and only if the process is explosive and its relative jump size increases to one as a function of pre-drawdown price levels. While the first condition may well be tested, the second one seems unobservable even ex-post.

2.2 Other approaches

Below we describe two approaches that use a notion of bubbles based solely on stylized (empirical) facts of the resulting price process. A break of market efficiency is not explicitly modeled and thus an essential feature of an asset price bubble left out, whence these approaches are not ideal or fully satisfying – however, they avoid using the fundamental value in a bubble definition.

2.2.1 Growth and decline

Shiryaev et al. (2014) and Shiryaev and Zhitlukhin (2012) consider a geometric Brownian motion with a drift that changes at some random time τ. The asset price in their model follows the stochastic differential equation

$$\frac{dS_t}{S_t} = \left(\mu_1 \mathbb{1}_{\{t<\tau\}} + \mu_2 \mathbb{1}_{\{\tau \leqslant t\}}\right) dt + \sigma dW_t, \quad t \in [0,T]. \tag{2.11}$$

The authors define a bubble as a process that satisfies (2.11) with $\mu_1 > 0 > \mu_2$. This translates into the simple statement that an asset experiences a bubble if initially it has positive expected returns followed by negative expected returns. In section 4.2.3 below we see that this approach can be included in our framework.

2.2.2 Drift burst hypothesis

In a recent article, Christensen et al. (2016) postulate the so-called *drift burst hypothesis* to model short-lived *flash crashes* in high-frequency tick data. In particular, on a finite time horizon $[0,T]$, in a diffusion model

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(t)dW_t, \quad t \in [0,T], \tag{2.12}$$

they look at explosive drift $\mu(t) = (T-t)^{-\alpha}$ accompanied by explosive diffusion $\sigma(t) = (T-t)^{-\beta}$ for $\alpha \in (1/2, 1)$ and $\beta \in (0, 1/2)$. Now, if $\alpha - \beta < \frac{1}{2}$, a Girsanov measure change, shows that the process S satisfies No-Arbitrage; see lemma A.3. Explosive drift has already been considered in Johansen et al. (1999) and Johansen et al. (2000) to model asset price bubbles, where the authors use a crash with accelerating hazard rate to allow for the explosive drift $\mu(t) = (T-t)^{-\alpha}$. While there are interesting parallels between the two approaches, Christensen et al. (2016) try to adhere to the No-Arbitrage condition (by imposing $\alpha - \beta < \frac{1}{2}$), whereas adding a crash explicitly allows for arbitrage in the process given by (2.12). We will see that this difference is an essential feature of our definition of a bubble in the following section.

3 General definition of a bubble

As we have seen in propositions 2.6 and 2.7 above, rational expectation bubbles are essentially situations where *absence of arbitrage* is fulfilled, while the existence of a rational equilibrium in an economy with utility-maximizing investors is violated. This inconsistency is not ruling out rational bubbles, if one accepts the notion that bubbles represent *market failure*. However, not explicitly modeling the underlying mechanism of such market failure seems not satisfactory and leads to issues in detecting such bubbles, as discussed in section 2.1.3. To accommodate models from the literature that explicitly model market failures to describe bubbles, cf. the literature cited in section 1, we employ a nonlinear decomposition that complies with basic stylized facts of a bubble and allows one to transcend the rational expectations framework. In particular, we introduce a general definition of the bubble characteristics (1.1) to decompose the stock price as

$$S_t = \tilde{S}_t \mathbb{1}_{\{t<\tau_J\}} + X_t \mathbb{1}_{\{\tau_J \leqslant t\}}. \tag{3.1}$$

The use of the pre-crash process $\tilde{S}$ allows (or forces) one to model market failure as the driving mechanism of a bubble.

3.1 Notation

We use the following notation throughout the paper. For two real numbers $a, b \in \mathbb{R}$ let $a \wedge b = \min\{a, b\}$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random times $\tau_1, \tau_2 \colon \Omega \to [0, \infty]$ we use the *stochastic interval* notation

$$[[\tau_1, \tau_2)) = \{(\omega, t) \in \Omega \times [0, \infty) \colon \tau_1(\omega) \leqslant t < \tau_2(\omega)\}. \tag{3.2}$$

Similarly we use $[[\tau_1, \tau_2]]$ and $((\tau_1, \tau_2]]$. For a stochastic process $(X_t)_{t\in[0,\infty)}$ we denote its left-continuous version by $(X_{t-})_{t\in[0,\infty)}$, that is, the process with the property that $X_{t-} = \lim_{s\nearrow t} X_s$ for all $t \in [0, \infty)$. For a stochastic processes $(X_t)_{t\in[0,\infty)}, (Y_t)_{t\in[0,\infty)}$ and a random time $\tau \colon \Omega \to [0, \infty]$ we denote by $(X_t^{\tau,Y})_{t\in[0,\infty)}$ the stopped process with the property that

$$X_t^{\tau,Y} = X_t \mathbb{1}_{\{t<\tau\}} + Y_t \mathbb{1}_{\{\tau \leqslant t\}}. \tag{3.3}$$

In the special case of $Y = X_\tau$ we write $X_t^\tau = X^{\tau, X_\tau}$. For a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ and an $\mathbb{F}$-stopping time $\tau : \Omega \to [0, \infty)$ we define the (itself right-continuous) filtration $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$ consisting of the σ-algebras $\mathcal{F}_{t\wedge\tau-}$ given by

$$\mathcal{F}_{t\wedge\tau-} = \sigma\left(\{A \cap \{s < \tau\} \colon 0 \leqslant s \leqslant t, A \in \mathcal{F}_s\} \cup \mathcal{F}_0\right). \tag{3.4}$$

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ satisfies the *usual hypotheses* if $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. By replacing a filtration with its *usual augmentation*, we assume without loss of generality that the usual hypotheses hold. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the property that $\mathbb{F}$ satisfies the usual hypotheses is called *stochastic basis*. For a stochastic process $(X_t)_{t\in[0,\infty)}$ we denote by $\mathbb{F}^X$ the smallest filtration satisfying the usual hypothesis such that X is $\mathbb{F}^X$-adapted. For two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,\infty)}$ we write $\mathbb{F} \subseteq \mathbb{G}$ if $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0,\infty)$. We assume familiarity with the concept of semimartingales and (local) martingales; see, e.g., chapter I of Jacod and Shiryaev (2003).[10] We will call X a $(\mathbb{P},\mathbb{F})$-(semi)martingale or simply $\mathbb{F}$-(semi)martingale if X is a (semi)martingale on a stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$.

3.2 Market setting

Let $(\Omega,\mathcal{G},\mathbb{G},\mathbb{P})$ be a stochastic basis, let $(D_t)_{t\in[0,\infty)}$ be a nonnegative $\mathbb{G}$-semimartingale representing dividend payments and let $(B_t)_{t\in[0,\infty)}$ be a $\mathbb{G}$-semimartingale representing a money market account with the property that

$$\mathbb{P}\left[\inf_{t\in[0,\infty)} B_t > 0\right] = 1. \tag{3.5}$$

We assume that $D \equiv 0$ and units are denominated in B such that $B \equiv 1$, which corresponds to no dividends and zero interest rate.[11] For simplicity, below we consider only nonnegative stochastic processes, although most definitions and results extend to general stochastic processes (that is, negative prices) with little adaption.

The definitions in the following section can be adapted to discrete time (using piecewise constant processes and filtrations) and a finite time horizon $T \in (0,\infty)$ (replacing the intervals $[0,\infty)$ and $[0,\infty]$ with $[0,T]$ and $[0,T] \cup \{\infty\}$, respectively.)

3.3 Definitions

To describe an asset's market price, we use semimartingales as the natural class for which the Itô integral is well-defined and thus provide a concept of *trading*. To account for possible explosive behavior in bubble prices, we allow a priori for explosive semimartingales. For the following (somewhat non-standard) definition, cf. section 2 of Cheridito et al. (2005).

Definition 3.1 (Semimartingales with possible explosion ($\mathcal{S}_E$)). Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a stochastic basis, let $(S_t)_{t\in[0,\infty)}$ be an RCLL, $\mathbb{F}$-adapted stochastic process taking values in $[0,\infty]$, let $(\tau_n)_{n\in\mathbb{N}}$ be the $\mathbb{F}$-stopping times[12] with the property that

$$\tau_n = \inf\{t \in [0,\infty) | S_t \geqslant n\}, \tag{3.6}$$

[10] While the concepts are introduced to cover rather general processes, the essential mechanism can be understood entirely without previous knowledge of continuous time stochastic processes. The reason we choose semimartingales is twofold. For one, they allow for the notion of trading in the sense of an Itô integral. Of course, more general concepts of stochastic integration (and thus trading) exist. Secondly, semimartingales cover most asset price models in the literature, especially those in bubble modeling.

[11] By applying notions of market efficiency to the value process $V = S + D$ instead of a stock price S, the assumption of zero dividends is made without loss of generality. The a priori choice of a numeraire B, however, is afflicted with a certain loss of generality as represented by (3.5); see the discussion in Herdegen (2017).

[12] See, e.g., theorem 1.27 on page 7 of Jacod and Shiryaev (2003).

and assume that S^{τ_n} is an $\mathbb{F}$-semimartingale for all $n \in \mathbb{N}$. Then S is called an $\mathbb{F}$-*semimartingale with possible explosion* and the predictable $\mathbb{F}$-stopping time[13] τ given by

$$\tau = \lim_{n\to\infty} \tau_n = \inf\{t \in [0,\infty) | S_{t-} = \infty\} \tag{3.7}$$

is called *explosion time* of S. We call S *explosive* $\mathbb{F}$-*semimartingale* if $\mathbb{P}[\tau < \infty] > 0$ and denote by $\mathcal{S}_E(\mathbb{F})$ the set of all possibly explosive $\mathbb{F}$-semimartingales for a given stochastic basis.

Definition 3.2 (Efficient market condition). Let the setting in section 3.2 be fulfilled and let $\mathcal{S}_E(\mathbb{G})$ be the set of possibly explosive $\mathbb{G}$-semimartingales as introduced in definition 3.1. Then a binary operator $\mathcal{E} : \mathcal{S}_E(\mathbb{G}) \to \{0,1\}$ is called *efficient market condition* with the following terminology

$$\mathcal{E}(S) = 1 \iff S \text{ is an } \mathcal{E}\text{-efficient market (given the information in } \mathbb{G}). \tag{3.8}$$

Definition 3.3 (Asset price drawdown). Let the setting in section 3.2 be fulfilled, let $\tilde{S} \in \mathcal{S}_E(\mathbb{G})$ with (possibly infinite) explosion time τ, let $\tau_J : \Omega \to [0,\infty]$ be a random time with $\tau_J < \tau$ on $\{\tau < \infty\}$, let $(X_t)_{t\in[0,\infty)}$ be a $\mathbb{G}$-semimartingale and let $(S_t)_{t\in[0,\infty)}$ be the stopped process

$$S_t = \tilde{S}_t^{\tau_J,X} = \tilde{S}_t \mathbb{1}_{\{t<\tau_J\}} + X_t \mathbb{1}_{\{\tau_J \leqslant t\}}, \quad t \in [0,\infty). \tag{3.9}$$

Then (τ_J, X) is called *asset price drawdown for* $\tilde{S}$ if it holds that

$$X\mathbb{1}_{\{\cdot<\tau_J\}} \leqslant \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} \quad \text{and} \quad \Delta S_{\tau_J} = X_{\tau_J} - \tilde{S}_{\tau_J-} \leqslant 0. \tag{3.10}$$

Definition 3.4 (Bubble). Let the setting in section 3.2 be fulfilled, S be a $\mathbb{G}$-semimartingale and $\mathcal{E}$ be an efficient market condition. Then S has an $\mathcal{E}$-*bubble* if and only if there exists a possibly explosive semimartingale $\tilde{S} \in \mathcal{S}_E(\mathbb{G})$ and an asset price drawdown (τ_J, X) for $\tilde{S}$ such that

1. $S = \tilde{S}^{\tau_J,X} = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} + X\mathbb{1}_{\{\tau_J \leqslant \cdot\}}$.
2. $\forall Y \in \mathcal{S}_E\left(\mathbb{F}^{\tilde{S}}\right) \cap \mathcal{S}_E(\mathbb{G})\colon Y\mathbb{1}_{\{\cdot<\tau_J\}} = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} \Longrightarrow \mathcal{E}(Y) = 0$.

3a. $\mathcal{E}(S) = 1$ (type-I bubble).

3b. $\mathcal{E}(S) = 0$ and $\mathcal{E}(X) = 1$ (type-II bubble).

If 1,2,3a hold, we call S an inefficient bubble of type-I; if 1,2,3b hold, we call S an inefficient bubble of type-II. The triplet $(\tilde{S}, \tau_J, X)$, defining the structure of the bubble, is referred to as *bubble characteristics*. As opposed to the full market information $\mathbb{G}$, introduced and fixed in section 3.2, the filtration $\mathbb{F}^{\tilde{S}}$ denotes to the information flow in a drawdown-free market (with the property that either $\mathbb{F}^{\tilde{S}} \subsetneq \mathbb{G}$ or $\mathbb{F}^{\tilde{S}} = \mathbb{G}$, depending on the model and structure of the drawdown).

[13] See, e.g., proposition 1.2(d) on page 51 of Ethier and Kurtz (1986).

3.4 Comments on technical conditions

(a) The definition 3.3 of an asset price drawdown (τ_J, X) for $\tilde{S}$ requires that

1. $X \leqslant \tilde{S}$ on $[[0, \tau_J))$, and
2. the sample paths of $S = \tilde{S}^{\tau_J, X}$ are non-increasing at the time of the drawdown ($\Delta S_{\tau_J} \leqslant 0$).

For type-I bubbles, one can always choose $X \equiv 0$ on $[[0, \tau_J))$ such that the first condition is fulfilled. The second condition ensures that adding a downward correction to $\tilde{S}$ leads to an efficient market S. For type-II bubbles, on the other hand, X (and not S) is constrained by market efficiency, and the first condition becomes necessary to ensure that the asset is initially overvalued at S, with a downward correction to efficient levels X. In both cases, we can interpret the inefficiency of $\tilde{S}$ as *overvaluation*.

(b) Instead of the notion of a fundamental price – the cornerstone of rational expectation bubbles – above definitions merely depend on an understanding of market efficiency. Such an efficient market condition $\mathcal{E}$ can take various forms; see section 4.1.3 below for examples. To arrive at a meaningful definition of a bubble, it is essential to use a meaningful notion of market efficiency, for example

$$\mathcal{E}(S) = 1 \Longleftrightarrow \text{trading in } (B, S) \text{ using } \mathbb{G}\text{-predictable trading strategies}$$
$$\text{does not allow for } \textit{riskless profits},$$

or the more restrictive

$$\mathcal{E}(S) = 1 \Longleftrightarrow \text{trading in } (B, S) \text{ using } \mathbb{G}\text{-predictable trading strategies}$$
$$\text{does not allow for } \textit{profits on average.}$$

(c) Condition 2. in definition 3.4 ensures that the pre-drawdown process $\tilde{S}$ violates market efficiency, in the sense that every decomposition $S = Y^{\tau_J, X}$ with an $\mathbb{F}^{\tilde{S}}$-adapted market Y necessarily has $\mathcal{E}(Y) = 0$. In this case we can confidently label $\tilde{S}$ – or, more precisely, the part of $\tilde{S}$ that is relevant in the decomposition of S – an inefficient market. To put differently, this ensures that the notion of a bubble is well-defined and does not suffer from identifiability issues. Note that in the case of a type-I bubble this requires $\mathbb{F}^{\tilde{S}} \subsetneq \mathbb{G}$, otherwise S itself would be an $\mathbb{F}^{\tilde{S}}$-adapted, efficient market that is indistinguishable from $\tilde{S}$ on $[[0, \tau_J))$. In most examples below it holds that

$$Y \in \mathcal{S}_E\left(\mathbb{F}^{\tilde{S}}\right) \cap \mathcal{S}_E(\mathbb{G}) \text{ and } Y\mathbb{1}_{\{\cdot < \tau_J\}} = \tilde{S}\mathbb{1}_{\{\cdot < \tau_J\}} \Longrightarrow Y = \tilde{S}, \tag{3.11}$$

and condition 2. can be succinctly stated as $\mathcal{E}(\tilde{S}) = 0$.

3.5 Discussion

(a) We understand bubbles as times of irrational exuberance and unsustainable growth – this part is captured by the break in market efficiency of the pre-drawdown process $\tilde{S}$, the apparent price development for an *observer ahead of the crash*. It seems that extraordinary profits are present in the market. However, the consequence of such price developments is the existence of a drawdown – this is captured by the stopped process $S = \tilde{S}^{\tau,X}$, which is the true dynamics perceived by an *omniscient observer*. As such, we formalize the minimal criteria a bubble process should satisfy, based on the following observations.

(i) Minsky (1972), who bases his *Financial Instability Hypothesis* on the observation that the "fundamental instability of a capitalist economy is a tendency to explode – to enter into a boom or euphoric state" and that these "sustained economic growth and business cycle booms (...) generate conditions conducive to disaster for the entire economic system",

(ii) Kindleberger (1996), who, within a more general *Anatomy of a Typical Crisis*, calls a bubble "an upward movement of prices that then implodes" as opposed to "(...) the technical language of some economists, [where] a bubble is any deviation from fundamentals",

(iii) Brunnermeier and Oehmke (2013), who describe a bubble through "(i) a *run-up phase*, in which bubble imbalances form, and (ii) a *crisis phase*, during which the risk that has built up in the background materializes (...)" and "stress that the run-up and crisis phases cannot be seen in isolation - they are two sides of the same coin."

As already discussed in the introduction, there are two types of inefficient bubbles, which differ in their understanding of how the drawdown counteracts exuberance.

(b) **Type-I bubbles:** The motivation behind a type-I bubble is as follows: we view a bubble as a stock price development so exceptional that, in an efficient market, it can only be explained by the possibility of a crash. Or, looking at the other side of the coin: a bubble is equivalent to the risk premium of a crash, which leads to exorbitant (inefficient) stock price development. The essential assumption 3a in the definition of a type-I bubble is efficiency of the full process

$$S = \tilde{S}^{\tau_J,X} = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} + X\mathbb{1}_{\{\tau_J \leqslant \cdot\}}. \tag{3.12}$$

This type of bubble rests on the view that, while instabilities and the risk of a drawdown are inherently present in the market, as a whole it does a good job in assessing and pricing this risk.

(c) **Type-II bubbles:** Alternatively, one may argue that, while a bubble materializes as a temporary departure from market efficiency, once this inefficiency gets resolved, prices return to efficient levels. In the full price process,

$$S = \tilde{S}^{\tau_J,X} = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} + X\mathbb{1}_{\{\tau_J \leqslant \cdot\}}, \tag{3.13}$$

X represents those efficient levels. During a type-I bubble, market failure creates an inherent instability that is (efficiently) priced in the market, whereas for type-II bubbles market failure leads to exuberant prices and a drawdown happens when this inefficiency gets resolved and prices return to efficient levels. The difference $\tilde{S} - X$ may be called the *bubble component*, while X (as an efficient process) may be called the *fundamental component*. While this yields a similar additive decomposition as for rational bubbles, note that for inefficient bubbles it is not the case that all components $\tilde{S}$, X *and* $\tilde{S} - X$ are efficient markets.

(d) Processes with explosive dynamics are generally not considered in asset price theory as obviously no frenzy would ever cause a stock to be valued at ∞ at any point in time. However, adding (the possibility of) a downturn enlarges the picture and allows for more excessive processes accompanied by a looming crash. This theoretical reasoning has a straightforward analogy in practice, where ignoring[14] the possibility of a drawdown seems to create large riskless returns during a bubble that pass most measures of risk.[15] We believe that acknowledging (or forcing to acknowledge) the risk of a drawdown in times of large, accelerating growth can lead to more robust models for financial applications and risk management.

4 Continuous time modelling

While the definitions above can be applied to both continuous and discrete time models, to increase readability we want to treat them separately as there is hardly any overlap of these model types in the literature.

To present a classification of bubbles in sections 4.2 and 4.3 below, at various points we make use of propositions. Their proofs are largely trivialities, merely checking the conditions of our main definition 3.4 above, but included for the convenience of the reader.

4.1 Examples of bubble characteristics

In this section we provide examples of the essential building blocks of continuous time inefficient market bubbles, that is, the bubble characteristics $(\tilde{S}, \tau_J, X)$ and a market efficiency condition $\mathcal{E}$. These building blocks will be used in sections 4.2 and 4.3 to present a variety of bubble models from the literature.

4.1.1 Examples of semimartingales with explosion $\tilde{S}$

The following examples are intended to serve as illustrations of (possibly) explosive pre-drawdown processes $\tilde{S}$ in the characteristic description of a bubble $S \longleftrightarrow (\tilde{S}, \tau_J, X)$. In continuous time modeling, Itô processes with explosion play an important role.

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, let $(W_t)_{t\in[0,\infty)}$ be a real valued $\mathbb{F}$-adapted[16] Brownian motion, let $(S_t)_{t\in[0,\infty)}$ be a nonnegative, continuous, $\mathbb{F}$-adapted process and $\tau_n, \tau : \Omega \to [0, \infty]$ be $\mathbb{F}$-predictable stopping times given by

$$\tau_n = \inf\{t \in (0, \infty] | S_t \geqslant n\} \text{ and } \tau = \lim_{n\to\infty} \tau_n \tag{4.1}$$

[14] Carelessly or deliberately - exposure to bubbles can take various forms.

[15] See, e.g., Borio and Drehmann (2011) for an account on this issue in the setting of financial stability and monetary policy.

[16] Although this will be the case most of the time, we do not a priori assume that $\mathbb{F}$ is generated by W.

with the property that there exists a constant $S_0 \in (0,\infty)$ and $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-predictable processes $\alpha\colon [[0,\tau)) \to \mathbb{R}$ and $\beta\colon [[0,\tau)) \to (0,\infty)$ such that, for all $n \in \mathbb{N}$, S is a unique[17] strong solution of the stochastic integral equation

$$S_{t\wedge\tau_n} = S_0 + \int_0^{t\wedge\tau_n} \alpha_s ds + \int_0^{t\wedge\tau_n} \beta_s dW_s, \quad t \in [0,\infty). \tag{4.2}$$

Then S is called a possibly explosive *Itô-process*. If $\tau \equiv \infty$, S is called non-explosive Itô-process or simply Itô-process.

It is clear that an explosive Itô-process S is an explosive semimartingale with explosion time τ.

Geometric Brownian motion. Let $\tau \equiv \infty$ and for $\mu \in \mathbb{R}$, $\sigma \in (0,\infty)$ let $\alpha\colon [[0,\infty)) \times (0,\infty) \to \mathbb{R}$ and $\beta\colon [[0,\infty)) \times (0,\infty) \to (0,\infty)$ be given by

$$\alpha_t = \mu S_t, \quad \beta_t = \sigma S_t. \tag{4.3}$$

Then S is a geometric Brownian motion given by $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma}{2}\right) t + \sigma W_t\right)$ for $t \in [0,\infty)$ and thus a non-explosive Itô-process.

Lipschitz continuous coefficients. Let $\tilde{\alpha} : [0,\infty) \times \Omega \times [0,\infty) \to \mathbb{R}$ and $\tilde{\beta} : [0,\infty) \times \Omega \times [0,\infty) \to [0,\infty)$ be real-valued functions that satisfy $\tilde{\alpha}(\cdot,\cdot,0) \geqslant 0$ and $\tilde{\beta}(\cdot,\cdot,0) = 0$, and are *locally random Lipschitz continuous* uniformly for $t \in [0,\infty)$, that is, for $f \in \{\tilde{\alpha}, \tilde{\beta}\}$ and for every $n \in \mathbb{N}$ there exists a finite random variable $K_n : \Omega \to (0,\infty)$ such that for all $x, y \in [0,n]$ and all $\omega \in \Omega$ it holds that

$$\sup_{t\in[0,\infty)} |f(t,\omega,x) - f(t,\omega,y)| \leqslant K_n(\omega)|x-y|. \tag{4.4}$$

This ensures (see, e.g., chapter 5 in Protter (1990)) that there exists a strong solution of equation (4.2) with

$$\alpha_t(\omega) = \tilde{\alpha}(t,\omega,S_t(\omega)), \quad \beta_t(\omega) = \tilde{\beta}(t,\omega,S_t(\omega)) \tag{4.5}$$

up to a predictable stopping time τ, the exit time of the domain $[0,\infty)$ at its upper boundary ∞. The boundary conditions on $\tilde{\alpha}$ and $\tilde{\beta}$ at the lower boundary 0 ensure that the process is non-negative. For homogeneous functions $\tilde{\alpha}(t,\omega,S_t(\omega)) = \tilde{\alpha}(S_t(\omega))$ and $\tilde{\beta}(t,\omega,S_t(\omega)) = \tilde{\beta}(S_t(\omega))$ one can proceed using *Feller's test for explosion* (see, e.g., corollary 4.4 in Cherny and Engelbert (2005)) to determine whether the process is explosive. For a specific example see section 4.2.2 below.

Stochastic exponential. Assume there exists an $\mathbb{F}$-predictable stopping time $\tau : \Omega \to [0,\infty)$ and predictable processes $\mu : [[0,\tau)) \to \mathbb{R}$, $\sigma : [[0,\tau)) \to [0,\infty)$ with $\mathbb{P}\left[\int_0^\tau \sigma_t^2 dt < \infty\right] = 1$. Let $\alpha\colon [[0,\tau)) \times (0,\infty) \to \mathbb{R}$ and $\beta\colon [[0,\tau)) \times (0,\infty) \to [0,\infty)$ be given by

$$\alpha_t(\omega) = \mu_t(\omega) S_t(\omega), \quad \beta_t(\omega) = \sigma_t(\omega) S_t(\omega) \quad \text{for } (\omega,t) \in [[0,\tau)). \tag{4.6}$$

[17] Uniqueness here is considered as *pathwise uniqueness*, in the sense that if there exist two solutions X^1 and X^2, then $\mathbb{P}\left[\{\omega | \forall t\colon (w,t) \in [[0,\tau)) \Rightarrow X_t^1(\omega) = X_t^2(\omega)\}\right] = 1$

Then theorem 5.3 in Protter (1977) ensures that equation (4.2) has a unique strong solution on $[[0,\tau))$. Moreover, for this special form of α and β, one can write S as stochastic exponential of the process $\int_0^{\cdot} \mu_s ds + \int_0^{\cdot} \sigma_s dW_s$, that is,

$$S_t = \exp\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s\right) \quad \text{on } [[0,\tau)). \tag{4.7}$$

The process S is explosive if and only if $\mathbb{P}\left[\int_0^\tau \mu_t dt = \infty\right] > 0$ and strictly positive if and only if $\mathbb{P}\left[\int_0^\tau \mu_t dt = -\infty\right] = 0$. For a specific example see section 4.2.1 below.

4.1.2 Examples of drawdowns (τ_J, X)

Continuous time crash. In many continuous models the drawdown X takes the form of a single jump ("crash") happening at a totally inaccessible[18] random time τ_J with a crash size depending on the information available at that time. We can adapt a standard approach in the literature to construct such a random time; see e.g. section 6.5 in Bielecki and Rutkowski (2002). Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, let $\tilde{S}$ be an $\mathbb{F}$-semimartingale with explosion time τ, let $(\Lambda_t)_{t\in[0,\infty)}$ be an $\mathbb{R} \cup \{\infty\}$-valued, continuous, increasing, $(\mathcal{F}_{t\wedge\tau-})_{t\in[0,\infty)}$-adapted process with the property that $\Lambda_\tau = \infty$. Then, on a larger probability space $(\Omega, \mathcal{G}, \mathbb{P})$, we can construct a random time $\tau_J : \Omega \to [0,\infty)$ and a filtration $\mathbb{G} \supseteq \mathbb{F}$ with the property that $\tilde{S}$ is a $\mathbb{G}$-semimartingale and

$$\mathbb{1}_{\{\tau_J \leqslant t\}} - \Lambda_{t\wedge\tau_J} \text{ is a } \mathbb{G}\text{-local martingale;} \tag{4.8}$$

see lemma A.1 for details. To model the crash size, let $\kappa : \Omega \to [0,1]$ be a $\mathcal{G}_{\tau_J-}$-measurable random variable and the drawdown X be given by

$$X_t = \tilde{S}_{\tau_J-}(1-\kappa)\mathbb{1}_{\{\tau_J \leqslant t\}}, \quad t \in [0,\infty). \tag{4.9}$$

Then the stopped process is the single jump process

$$\tilde{S}_t^{\tau_J, X} = \tilde{S}_t \mathbb{1}_{\{t<\tau_J\}} + \tilde{S}_{\tau_J-}(1-\kappa)\mathbb{1}_{\{\tau_J \leqslant t\}}, \quad t \in [0,\infty), \tag{4.10}$$

and is, by construction, a $\mathbb{G}$-semimartingale. Moreover, (τ_J, X) is a drawdown as in definition 3.3.

Note that the semimartingale property follows from this construction only due to the continuity of the hazard process Λ. See example 3.12 in Herdegen and Herrmann (2016) for a single jump process with a discontinuous hazard process that fails to be a semimartingale.

Bifurcation. Depending on the dynamics of the underlying markets and its fundamentals, it can be reasonable to model the asset price drawdown as a crash that only happens with a certain probability and include an alternative scenario, where the potential positive influence of exuberant behavior turns out to be sustainable.

[18] A totally inaccessible random time almost surely avoids predictable stopping times; see, e.g., definition 2.20 in Jacod and Shiryaev (2003). A predictable stopping time can be approximated by a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times $\tau_n < \tau$, i.e., it can be "foreseen".

Assume the setting of the last paragraph and in addition assume that $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ is rich enough to support a Bernoulli variable Y on $\mathcal{G}_{\tau_J -}$ with probability of success $\alpha \in [0,1]$. Define the drawdown X to be

$$X_t = \begin{cases} \tilde{S}_{\tau_J -} \mathbb{1}_{\{\tau_J \leqslant t\}}, & \text{if } Y = 1 \\ \tilde{S}_{\tau_J -} (1-\kappa) \mathbb{1}_{\{\tau_J \leqslant t\}}, & \text{if } Y = 0 \end{cases} \quad \text{for } t \in [0, \infty). \tag{4.11}$$

Again, the $\mathbb{G}$-semimartingale property of the stopped process $\tilde{S}^{\tau_J, X}$ follows from construction.

Diffusive drawdown. While a single jump (crash) at τ_J can be a reasonable model assumption if there are liquidity problems during times of market distress, in general a drawdown materializes in some time interval $[\tau_J, \cdot]$, which may be modeled by a diffusive drawdown. For this, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, W be a real-valued $\mathbb{F}$-Brownian motion and $\tilde{S}$ be a possibly explosive Itô process as in definition 4.1 with explosion time τ and $\alpha \geqslant 0$. Let τ_J be a random time constructed by a $(\mathcal{F}_{t \wedge \tau -})_{t \in [0,\infty)}$-adapted hazard process as given by equation (4.8). For two constants $\mu \in (-\infty, 0)$ and $\sigma \in (0, \infty)$ define the drawdown X to be

$$X_t = \tilde{S}_t \mathbb{1}_{\{t < \tau_J\}} + \tilde{S}_{\tau_J} \exp\left(\left(\mu - \frac{\sigma^2}{2} \right)(t - \tau_J) + \sigma \left(W_t - W_{\tau_J} \right) \right) \mathbb{1}_{\{\tau_J \leqslant t\}}, \quad t \in [0, \infty). \tag{4.12}$$

Let $\mathbb{G}$ be the filtration generated by $\mathbb{F}$ and the stopped process $S = \tilde{S}^{\tau_J, X}$ and satisfying the usual hypothesis. Observe that W remains a Brownian motion with respect to $\mathbb{G}$ and $W_t - W_{\tau_J}$ is a restarted Brownian motion independent of $\mathcal{G}_{\tau_J}$. In particular, $\tilde{S}$ and S are a $\mathbb{G}$-semimartingale with continuous sample paths.

4.1.3 Examples of efficient market conditions $\mathcal{E}$

Below we present measures of market efficiency in increasing strength that are common in the literature. Before stating the definitions, let us give a descriptive introduction.

1. **Exclude markets that allow for riskless gains:** The weakest condition is classical No-Arbitrage, which is known as *No Free Lunch with Vanishing Risk* developed for possibly unbounded processes in continuous time. For a full characterization in a general continuous-time equilibrium model set-up see Jarrow and Larsson (2012). It is similar to the (slightly weaker) *law of one price*, which basically means that the same good cannot trade for different prices. Shefrin (2008) calls this *riskless arbitrage*.
2. **Exclude markets that allow for large gains with little risk:** Starting with Cochrane and Saa-Requejo (2000) in an intention to get reasonable bounds for relative pricing approaches without resorting to strong assumptions on collective risk preferences, there has been an effort to rule out not only riskless arbitrage opportunities but also arbitrage opportunities that are afflicted with little risk, so called *Good Deals*. A common measure of riskiness employed is the Sharpe-ratio; see Cochrane and Saa-Requejo (2000), Černý (2003) or Björk and Slinko (2006). *Riskiness*, however, can be captured by a variety of utility/performance

measures different from the Sharpe ratio; see Klöppel and Schweizer (2007). We employ the notion of good-deals not for pricing purposes, but to classify markets with a payoff structure that is "too good to be true". Shefrin (2008) calls this type of arbitrage *risky arbitrage*, although he does not specify a particular measure to contrast risk and return.

3. **Exclude markets with stochastic or time-dependent required returns:** In the economics literature an additional, arguably strong assumption on risk preferences is sometimes employed to analyze markets and simplify the description of asset prices. It is assumed that investors have, on average, constant risk-aversion such that the required return on the asset is constant over time; see, e.g., Blanchard (1979), Johansen et al. (1999), Sornette and Andersen (2002), Phillips et al. (2011) or chapter 7 of Campbell et al. (1997) for a textbook treatment. Note that, while this condition is derived from *rational expectations* or *No-Arbitrage* by a priori fixing a risk-neutral measure, it is considerably stronger than the No-Arbitrage condition in point 1. above and one has to be careful when using it, especially in negations.[19] A more general version of this condition would be to a priori fix some level of risk premia. In fact, most absolute asset pricing models essentially use this condition, and the pricing measure is informed by certain (macro-)economic variables.

Definition 4.2 (No Free Lunch with Vanishing Risk (NFLVR)). Let the setting in section 3.2 be fulfilled and let $S \in \mathcal{S}_E(\mathbb{G})$ be a non-negative semimartingale with possible explosion. Then S is called a (NFLVR)-efficient market if it does not explode and no riskless profits can be made through trading with self-financing, $\mathbb{G}$-predictable strategies with bounded losses. In continuous time, this definition involves a limiting description for *riskless*; see Delbaen and Schachermayer (1998) for details.

Remark 4.3. The fundamental theorem of asset pricing in Delbaen and Schachermayer (1998) implies that a (non-negative) $\mathbb{G}$-semimartingale S satisfies (NFLVR) if and only if there exists at least one measure $\mathbb{Q} \approx \mathbb{P}$ such that S is a $(\mathbb{Q}, \mathbb{G})$-local martingale.

For the next definition, we restrict to a finite time horizon to simplify presentation and avoid additional assumptions.

Definition 4.4 (No-Good-Deal (NGD) #1). Let the setting in section 3.2 be fulfilled, let $S \in \mathcal{S}_E(\mathbb{G})$ be a non-negative semimartingale with possible explosion, let $T \in [0, \infty)$, $K \in (0, \infty]$, let $\mathcal{Q}_T$ be the set of local martingale measures for S equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then S is called a *static* $(\text{NGD})_{K,f}$-efficient market if there exists a measure $\mathbb{Q} \in \mathcal{Q}_T$ such that for the density process $Z_t^{\mathbb{Q}} = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{G}_t\right]$, $t \in [0, T]$, it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\left[f\left(\frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}}\right) \middle| \mathcal{G}_t \right] < K. \tag{4.13}$$

[19] A market with stochastically varying risk premia can hardly be called inefficient, although it violates constant required rate of return. This lies at the heart of Fama (1991)'s joint hypothesis problem and LeRoy (1976)'s (incomplete) critique of the efficient market hypothesis.

Remark 4.5. Depending on the choice of the function f, the bound in equation (4.13) allows one to derive a bound on the expected utilities of $\mathbb{Q}$-valued payoffs, for details and specific choices for f see Klöppel and Schweizer (2007) and Albrecher (2009). In the setting of a jump diffusion in the filtration generated by a Brownian motion and a pure jump process, an instantaneous (or *dynamic*) version of *No-Good-Deal* with a quadratic f has been introduced in Björk and Slinko (2006). The explicit description of the density of an equivalent martingale measure in such models allows one to derive instantaneous conditions; see appendix A.2.

Definition 4.6 (Constant required return (CR)). Let the setting in section 3.2 be fulfilled, let $S \in \mathcal{S}_E(\mathbb{G})$ be a non-negative semimartingale with possible explosion and let $r \in [0, \infty)$. Then S is called a $(\mathrm{CR})_r$-efficient market if S satisfies (NFLVR) and $(e^{-rt}S_t)_{t\in[0,\infty)}$ is a $(\mathbb{P}, \mathbb{G})$-local martingale.

Example 4.7. We close this section viewing the above definitions from two different angles, a general and a specific one. Let $f(x) = x^2 - 1$, such that (4.13) is a bound on the *variance* of the densities.

1. In terms of the set of $\mathbb{P}$-equivalent local martingale measures $\mathcal{Q}$ for a strictly positive $\mathbb{G}$-semimartingale S on $[0, T]$, $T \in [0, \infty)$.

   a) (NFLVR) $\Longleftrightarrow$ $\mathcal{Q} \neq \varnothing$ (a pricing measure exists).
   b) $(\mathrm{NGD})_{K,f}$ $\Longleftrightarrow$ there exists a measure $\mathbb{Q} \in \mathcal{Q}$ such that its density process satisfies (4.13) (and thus gives rise to a market without "K-Good Deals").
   c) $(\mathrm{CR})_r$ $\Longleftrightarrow$ there exists a measure $\mathbb{Q} \in \mathcal{Q}$ such that its density process satisfies[20]

   $$\frac{d[S, Z^{\mathbb{Q}}]_t}{S_t Z_t^{\mathbb{Q}}} = -r\,dt, \quad t \in [0, T]. \tag{4.14}$$

   If $e^{-r\cdot}S$ and S are true $\mathbb{P}$- and $\mathbb{Q}$-martingales, respectively, this condition simplifies to $\mathbb{E}_{\mathbb{Q}}[S_T|\mathcal{F}_t] = e^{-r(T-t)}\mathbb{E}[S_T|\mathcal{F}_t]$, for $t \in [0, T]$.

   Note that if $\mathbb{P} \in \mathcal{Q}$, then S fulfills $(\mathrm{NGD})_{\epsilon,f}$ for any $\epsilon > 0$ and $(\mathrm{CR})_0$.

2. In terms of a one-period model starting at S_0 with two states of the world $\Omega = \{\omega_1, \omega_2\}$ with equal probability $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. At time $T = 1$ the price either moves to $S_1(\omega_1) = S_0(1 + u)$ or to $S_1(\omega_2) = S_0(1 + d)$ for $d < u \in [-1, \infty)$. Then it holds that

   a) (NFLVR) $\Longleftrightarrow$ $(d < 0 < u)$ or $(d = u = 0)$.
   b) $(\mathrm{NGD})_{K,f}$ $\Longleftrightarrow$ (NFLVR) and $\left(2 + \frac{8ud}{(u-d)^2} < K\right)$.
   c) $(\mathrm{CR})_r$ $\Longleftrightarrow$ (NFLVR) and $r = \log\left(1 + \frac{u+d}{2}\right)$.

[20] Here we use the quadratic variation $[X, Y]$, defined for any two semimartingales X, Y; see, e.g., section II.6 in Protter (1990).

Condition (2a) describes that riskless profit is possible only if the market moves up or down almost surely, (2b) is derived from explicitly calculating the variance of the random variable $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$ for this simple model. Finally, condition (2c) follows from calculating the continuously discounted growth rate for the discrete process.

4.2 Examples of type-I bubbles

Following definition 3.4, a type-I bubble is given by its characteristic triplet $(\tilde{S}, \tau_J, X)$, where $\tilde{S}$ fails market efficiency, while the full stock price S is an efficient market. We do not focus on the generating mechanism of inefficiency (for which we refer to the original papers), but classify resulting processes as inefficient market bubbles for various market efficiency conditions. A stronger market efficiency condition $\mathcal{E}$ leads to a weaker notion of a bubble.

4.2.1 JLS model

The setting considered here is a simple version the classical model of Johansen et al. (1999),Johansen et al. (2000), which is based on an analysis of a hierarchical system of traders. Let $[0, T]$ be a finite time horizon for some $T \in [0, \infty)$, let $\alpha \in [0, \infty)$, $\mu_0, \sigma_0 \in (0, \infty)$ be constants, let $\mu : [0, T) \to [0, \infty)$ be given by

$$\mu(t) = \frac{\mu_0}{(T-t)^\alpha}. \tag{4.15}$$

For a constant $\kappa \in (0, 1)$, consider a random time τ_J given by a (deterministic) hazard process

$$\Lambda_t = \int_0^t \frac{\mu(s)}{\kappa} ds, \quad t \in [0, T] \tag{4.16}$$

as in (4.8). The probability that there is no crash on $[0, T]$ is $1 - e^{-\Lambda_T}$ and nonzero if and only if $\alpha \in [0, 1)$. Moreover, let $(W_t)_{t \in [0,T]}$ be a Brownian motion in its natural filtration $\mathbb{F}^W$ and let the pre-drawdown process $\tilde{S}$ be an Itô process as in section 4.1.1 with (possible) explosion time T that satisfies the stochastic differential equation

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu(t)dt + \sigma_0 dW_t. \tag{4.17}$$

It is easy to see that $\tilde{S}$ explodes to ∞ at T if and only if $\alpha \in [1, \infty)$. Finally, to complement the drawdown (τ_J, X), let X be a crash as defined in (4.9) with a relative jump size κ.

Then the stopped process $S = \tilde{S}^{X, \tau_J}$ satisfies a stochastic differential equation of the form

$$\frac{dS_t}{S_{t-}} = \mu(t)\mathbb{1}_{\{t<\tau_J\}}dt + \sigma_0 \mathbb{1}_{\{t<\tau_J\}}dW_t - \kappa dJ_t, \quad t \in [0, T] \tag{4.18}$$

where $(J_t)_{t \in [0,T]}$ is the single jump process $J = \mathbb{1}_{\{\tau_J \leqslant \cdot\}}$. Our simple formulation implies in particular that the process is constant after τ_J. Let $\mathbb{G}$ be the filtration generated by $\mathbb{F}^W$ and assume that the market setting in section 3.2 is fulfilled.

Proposition 4.8. *Let the setting above be fulfilled, let $K \in (\mu_0/\sigma_0, \infty]$, S be given as in* (4.18) *with bubble characteristics $(\tilde{S}, \tau_J, X)$ and (NGD) refer to the dynamic No-Good-Deal condition from section A.2.3. Then it holds that*

(a) $\alpha \geqslant \frac{1}{2} \iff S$ *has a (NFLVR)-bubble.*

(b) $\alpha > 0 \iff S$ *has a* $(NGD)_K$*-bubble.*

(c) $\alpha = 0 \implies S$ *has a* $(NGD)_{\frac{\mu_0}{\sigma_0}}$*-bubble.*

Proof of proposition 4.8. First we note that $\mathbb{F}^{\tilde{S}} = \mathbb{F}^W$ and (3.11) holds. Lemma A.3 implies that $\tilde{S}$ violates (NFLVR) if and only if $\alpha \in [\frac{1}{2}, \infty)$. By definition A.4, $\tilde{S}$ violates $(\mathrm{NGD})_K$ if for some $t \in [0, T]$ we have

$$\frac{\mu_0}{\sigma_0 (T-t)^\alpha} \geqslant K. \tag{4.19}$$

Thus, $\tilde{S}$ violates $(\mathrm{NGD})_K$ if and only if $\alpha > 0$ and $(\mathrm{NGD})_{\frac{\mu_0}{\sigma_0}}$ if $\alpha = 0$. On the other hand, the full price process S in (4.18) is a $\mathbb{P}$-martingale by definition and satisfies (NFLVR) and $(\mathrm{NGD})_K$ for all $K > 0$. ☐

Remark 4.9.

(a) While the behavior of the random time τ_J is derived from analyzing a hierarchical network of traders, the pre-drawdown process $\tilde{S}$ in equation (4.17) is chosen to ensure market efficiency. It is not a priori clear why $\tilde{S}$ should follow this behavior with instantaneous return μ. As noted in the introduction, it needs to be *postulated* to ensure rational investors do not exit the market.

(b) In particular, $\tilde{S}$ is chosen such that S is a $\mathbb{P}$-martingale, which is a considerably stronger assumption than dictated by market efficiency (NFLVR) or (NGD). One possibility is to extend the JLS model as in Herdegen and Herrmann (2018), where the authors assume that the crash only accounts for some part of the drift in the pre-drawdown process. Another possibility is to focus on the behavior close to explosion time T, in which case mild additional assumptions suffice; see the discussion in section A.3 in the appendix.

4.2.2 Andersen-Sornette model

The following model has been introduced in Sornette and Andersen (2002) and Andersen and Sornette (2004). Let W be a Brownian motion in its natural filtration $\mathbb{F}^W$, let $m \in (1, \infty)$, $\mu_0, \sigma_0 \in (0, \infty)$ be constants, let $\sigma\colon [0, \infty) \to [0, \infty)$ and $\mu\colon [0, \infty) \to [0, \infty)$ be functions given by $\mu(x) = \frac{m\sigma_0^2}{2} x^{2m-2} + \mu_0 x^{m-1}$ and $\sigma(x) = \sigma_0 x^{m-1}$ for $x \in [0, \infty)$ and let the pre-drawdown process $\tilde{S}$ be the Itô-process given by the stochastic differential equation

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu(\tilde{S}_t)dt + \sigma(\tilde{S}_t)dW_t, \quad t \in [0, \infty). \tag{4.20}$$

Feller's test for explosion shows immediately that $\tilde{S}$ is an explosive Itô-process with a predictable explosion time $\tau : \Omega \to [0, \infty)$. Alternatively it is rather straightforward (see Sornette and Andersen (2002)) to derive an explicit solution for $\tilde{S}$ with $\tau(\omega) = \inf\{t \in (0, \infty) : \mu_0 t + \sigma_0 W_t(\omega) = \mu_0 T_c\}$

for some $T_c \in (0,\infty)$ depending on the starting value $\tilde{S}_0 \in (0,\infty)$. For a constant relative jump size $\kappa \in (0,1)$, consider a drawdown (τ_J, X) with random time τ_J given by a hazard process

$$\Lambda_t = \int_0^t \frac{\mu(S_s)}{\kappa} ds, \quad t \in [0,\infty). \tag{4.21}$$

as in (4.8) and a crash X as in (4.9). Then we get that the stopped process $S = \tilde{S}^{\tau_J,\kappa}$ satisfies the stochastic differential equation

$$\frac{dS_t}{S_{t-}} = \mu(S_t)\mathbb{1}_{\{t<\tau_J\}}dt + \sigma(S_t)\mathbb{1}_{\{t<\tau_J\}}dW_t - \kappa dJ_t, \quad t \in [0,\infty), \tag{4.22}$$

where $(J_t)_{t\in[0,\infty)}$ is the single jump process $J = \mathbb{1}_{\{\tau_J \leqslant \cdot\}}$. Let $\mathbb{G}$ be the filtration generated by $\mathbb{F}^W$ and assume that the setting in section 3.2 is fulfilled.

Proposition 4.10. *Let the setting above be fulfilled and S be given as in* (4.22)*. Then S has a (NFLVR)-bubble.*

Proof of proposition 4.10. First we note that $\mathbb{F}^{\tilde{S}} \subseteq \mathbb{F}^W$ and (3.11) holds. As argued above, $\tilde{S}$ has an explosion time τ with $\mathbb{P}[\tau < T] > 0$ for all $T > 0$ and thus, in particular, $\tilde{S}$ violates (NFLVR). Lemma A.2 confirms that S is a $\mathbb{P}$-local martingale[21] and satisfies (NFLVR). The proof of proposition 4.10 is thus completed. □

Remark 4.11. Mirroring the situation discussed in remark 4.9 above, here μ and σ are chosen to model a simple form of inefficiency: nonlinear growth of the pre-drawdown process $\tilde{S}$. The subsequent definition of Λ as in equation (4.21), however, needs to be postulated to ensure that S is an efficient market. Moreover, its specific form is not necessary for S to satisfy (NFLVR) and other choices are possible; see the discussion in section 3 of Sornette and Andersen (2002) and the ideas in section A.3 below.

4.2.3 Growth and decline

In section 2.2.1 we have discussed an approach by Shiryaev et al. (2014), who model a bubble by a process S given by an Itô process of the form

$$\frac{dS_t}{S_t} = \left(\mu_1\mathbb{1}_{\{t<\tau_J\}} + \mu_2\mathbb{1}_{\{\tau_J \leqslant t\}}\right)dt + \sigma dW_t, \quad t \in [0,T], \tag{4.23}$$

for a random time of $\tau_J : \Omega \to (0,T)$, uniformly distributed on $(0,T)$ and independent of the $\mathbb{F}^W$-Brownian motion W. They say there is a bubble if $\mu_1 > 0 > \mu_2$. To capture this definition in our framework, we need to define its bubble characteristics $(\tilde{S}, \tau_J, X)$. Let $\tilde{S}$ be an Itô process as in definition 4.1 given by $\alpha_t = \mu_1\tilde{S}_t$ and $\beta_t = \sigma\tilde{S}_t$. Then $\tilde{S}$ is a geometric Brownian motion. Moreover, let (τ_J, X) be a diffusive drawdown as defined in section 4.1.2 with X associated with the parameters μ_2 and σ and a market filtration $\mathbb{G}$, being the smallest filtration satisfying the usual hypothesis that S is adapted to. The process S in (4.23) is then equal to the stopped process

[21] In fact, S is even a true martingale, as shown in Schatz and Sornette (2019).

$S = \tilde{S}^{\tau_J, X}$. While the decomposition of a bubble can be readily be applied here, its derivation is based merely on stylized facts and does not a priori allow for an interpretation using market efficiency. A (rather ad-hoc) possibility would be to choose a dynamic (NGD)$_K$-condition as in definition A.2, with a constant $K = \frac{\mu_1}{\sigma}$.[22] Then $\tilde{S}$ violates (NGD)$_K$ while S fulfills (NGD)$_K$ and, noting that (3.11) holds, S has a (NGD)$_K$-bubble of type-I in the sense of definition 3.4.

Remark 4.12. As for the examples of type-I bubbles in sections 4.2.1 and 4.2.2 above, the behavior of S needs to be postulated. If S is given as above, the return μ_1 may be seen as remuneration for negative returns to follow. However, while even an informed investor cannot observe τ_J and thus cannot pinpoint the exact time when the instantaneous returns turn negative ($\mu_2 < 0$), the unconditional expected return eventually turns negative on $[0, T]$, and it needs to be argued why market participants stay invested in such an environment, an argument that seems to be missing in Shiryaev et al. (2014).

4.3 Examples of type-II bubbles

Following definition 3.4, a type-II bubble is given by its characteristic triplet $(\tilde{S}, \tau_J, X)$, where $\tilde{S}$ fails market efficiency, while the drawdown X is an efficient market. Below we discuss examples of type-II bubbles featuring overconfidence, limits to arbitrage or overestimation of future payoffs to derive $\tilde{S}$ as the (partial) equilibrium price process emerging under these inefficiencies. As in section 4.2.1, it should be clear that extracting the pre-crash $\tilde{S}$ is a simplification of the underlying model and may omit important additional features, but allows us to concisely describe the resulting bubble process.

4.3.1 Heterogeneous beliefs on dividend payments

Following the seminal paper by Harrison and Kreps (1978), there has been an effort to describe a deviation from rational prices in a dynamic partial equilibrium framework using heterogeneous beliefs. This includes the models in Nutz and Scheinkman (2017), Scheinkman and Xiong (2003). As elaborated in chapter 1 of Harrison and Kreps (1978), instead of a homogeneous set of investors (the *representative investor*) we look at several investor classes with subjective beliefs about the distribution of future dividends. All investors are assumed rational and risk-neutral. To describe this model in our framework (and ensure that the correction is downward), we need to assume that at least one investor class is overconfident, that is, they overestimate future dividend payoffs. Then the bubble characteristics can be described as follows (here we omit, for simplicity, a description of the filtrations involved).

1. The pre-crash process $\tilde{S}$ is derived as (partial) equilibrium price if investors' beliefs are distinct and short-selling is prohibited (cf. proposition 2 in Harrison and Kreps (1978)).
2. One way to represent the fundamental value X in these models is to take an objective position and define X as the (risk-neutral) expectation of future dividends (cf. proposition 3

[22] To apply the (NGD) condition in this setting, we need to invoke theorem 7.13 in Liptser and Shiryaev (2000), which clarifies the structure of $\mathbb{P}$-equivalent measures on $\mathcal{G}_T$.

in Harrison and Kreps (1978)).[23] As described in section 5 of Scheinkman and Xiong (2003), a crash may be represented as a random time where investor beliefs collapse or fundamentals become observable, whence the new equilibrium price should settle at X.

3. The models work with risk neutral agents, such that the market efficiency condition is constant required returns as given by definition 4.6.

To sum up, the price process is given by

$$S = \tilde{S}^{\tau_J, X} = \tilde{S}\mathbb{1}_{\{\cdot < \tau_J\}} + X\mathbb{1}_{\{\tau_J \leqslant \cdot\}} \tag{4.24}$$

and has a type-II bubble in the sense of definition 3.4: the process $\tilde{S}$ exceeds risk-neutral valuation, while at time τ_J the bubble collapses and S follows the efficient process X.

4.3.2 Noise trading and synchronisation risk

The following model is from Abreu and Brunnermeier (2003) and seeks to describe bubbles that persist due to a synchronization problem among informed, rational traders, which is introduced through an unknown starting date of the mispricing. In particular, in the model it is assumed that the price grows deterministically with a growth rate $g \in (0, \infty)$. The fundamental uncertainty in the model is introduced by the random time $t_0 : \Omega \to [0, \infty)$, which is used to specify risk premia[24] given by $g\mathbb{1}_{[0,t_0)} + 0\mathbb{1}_{[t_0,\infty)}$. However, t_0 is unknown to the market participants and the mispricing can persist. Information about t_0 is sequentially revealed to informed investors over some time window $[t_0, t_0 + \eta]$, who then, maximizing their payoff, try to ride the bubble for a while but sell-out not too late. After a fraction $\kappa \in [0, 1]$ of informed traders wants to sell out, a crash happens. The model is completed with an exogenous bursting of the bubble at a time $t_0 + \overline{\tau}$ if a crash does not happen before. The bubble characteristics can be described as follows.

1. Let the pre-crash $\tilde{S}_t$ be an Itô process on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as in definition 4.1 given by $\alpha_t = g\tilde{S}_t$ and $\beta_t \equiv 0$. Then $\tilde{S}$ is the deterministically growing process $\tilde{S}_t = e^{gt}$.
2. Let the drawdown (τ_J, X) be given by a process X with $X_t = e^{gt}\mathbb{1}_{[0,t_0)} + e^{gt_0}\mathbb{1}_{[t_0,\infty)}$ and a random time $\tau_J^{\text{ex/end}}$ with
   (a) $\tau_J^{\text{ex}} = t_0 + \overline{\tau}$ if we have an exogenous crash as in section 5.1 of Abreu and Brunnermeier (2003) or
   (b) $\tau_J^{\text{end}} = t_0 + \eta\kappa + \tau^*$ for some uniquely specified $\tau^* \in (0, \overline{\tau} - \eta\kappa)$ if we have an endogenous crash as in section 5.2. of Abreu and Brunnermeier (2003).
3. The market efficiency condition $\mathcal{E}$ is a version of constant required return (definition 4.6), where the required return, by definition, is given by $g\mathbb{1}_{[0,t_0)} + 0\mathbb{1}_{[t_0,\infty)}$.

[23]Alternatively, one can use subjective fundamental values, that is, the risk-neutral expectation of future dividends under subjective beliefs of one of the investor classes, cf. footnote 7 on page 1186 of Scheinkman and Xiong (2003). The (overconfident) belief in being able to resell the asset to other market participants leads investors to pay more than their subjective fundamental value.

[24]Recall that, for simplicity, we assume discounted prices and thus $r = 0$.

Assume that $\mathbb{G}$ is generated by $\mathbb{F}$ and t_0. Then the stopped process $S = \tilde{S}^{\tau_J, X}$ has a type-II bubble in the sense of definition 3.4: the $\mathbb{G}$-semimartingales $\tilde{S}$ and S (and every $\mathbb{G}$-semimartingale that is equal to $\tilde{S}$ up until τ_J) violate efficiency and grow at the exuberant return g beyond t_0, while at time τ_J the bubble collapses and S follows the efficient $\mathbb{G}$-semimartingale X.

4.3.3 An "efficient" inefficient bubble

Here we deal with a mechanism that was initially discussed by Fama (1989) and appears in a similar form in Pástor and Veronesi (2006). Both argue that agents are rational during boom and crash and thus dismiss the claim that the *Dow Jones 1987 drop* and the *Nasdaq 1997-2003 epsiode*, respectively, were bubbles characterized by a deviation from fundamentals. Let us describe the essence of such models in a simplified example to show how this can be described as an inefficient market bubble.

Setting. Let $T \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ be a stochastic basis, let $(W_t)_{t \in [0,T]}$ be an $\mathbb{F}$-Brownian motion and $\mu \in \mathbb{R}$, $\sigma \in (0, \infty)$ be constants and assume dividend payments $(D_t)_{t \in [0,T]}$ given by $D_t = 0$ for $t < T$ and a final payoff

$$D_T = \exp\left(\mu T - \frac{\sigma^2}{2} T + \sigma W_T\right). \tag{4.25}$$

A standard way to model misperceptions about future payoffs would be to introduce a measure $\mathbb{Q}$ representing perceived distribution of the final payoff (or, as in Pástor and Veronesi (2006), of the excess profitability). For simplicity, we will instead just assume that rational agents belief the final payoff to be

$$\tilde{D}_T = d \exp\left(\mu T - \frac{\sigma^2}{2} T + \sigma W_T\right) \tag{4.26}$$

for some $d > 1$. Within this setting, we can now define the characteristic triplet of our bubble.

Characteristic triplet.

1. A drawdown process $(X_t)_{t \in [0,T]}$, given by the conditional expectation of the true (discounted) final payoff,
$$X_t = \mathbb{E}_{\mathbb{P}}\left[e^{-\mu(T-t)} D_T | \mathcal{F}_t\right], \quad t \in [0, T), \tag{4.27}$$
and $X_T = 0$.
2. A random time $\tau_J : \Omega \to (0, T)$, independent of W. This represents the time when agents realize that their beliefs about the final payoff were incorrect. We assume that the market filtration $\mathbb{G}$ is generated by W and τ_J.
3. A pre-crash process $(\tilde{S}_t)_{t \in [0,T]}$ derived under belief that the final payoff equals $\tilde{D}_T$, that is,
$$\tilde{S}_t = \mathbb{E}_{\mathbb{P}}\left[e^{-\mu(T-t)} \tilde{D}_T | \mathcal{F}_t\right], \quad t \in [0, T), \tag{4.28}$$
and $\tilde{S}_T = 0$. In this setting, (3.11) is fulfilled.

Finally, we assume investors (on average) require a return μ and let $\mathcal{E}$ be the market efficiency condition given by

$$\mathcal{E}(S) = 1 \Longleftrightarrow S_t + D_t \text{ satisfies } (\mathrm{CR})_\mu,$$

for a $\mathbb{G}$-semimartingale S. Here we have to adjust the market setting in section 3.2 and definition 4.6 for nonzero dividends.

Proposition 4.13. *Assume the setting above and let S be the stopped process*

$$S_t = \tilde{S}_t^{\tau_J, X} = \tilde{S}_t \mathbb{1}_{\{t<\tau_J\}} + X_t \mathbb{1}_{\{\tau_J \leqslant t\}}, \quad t \in [0, T]. \tag{4.29}$$

Then S has a type-II inefficient market bubble.

Proof of proposition 4.13. From 4.27 we can infer that X is given by

$$X_t = \exp\left(\mu t - \frac{\sigma^2}{2} t + \sigma W_t\right), \quad t \in [0, T). \tag{4.30}$$

Similarly, from 4.28,

$$\tilde{S}_t = d \exp\left(\mu t - \frac{\sigma^2}{2} t + \sigma W_t\right), \quad t \in [0, T). \tag{4.31}$$

Then, clearly, $e^{-\mu t}(X_t + D_t)_{t\in[0,T]}$ is a $\mathbb{G}$-martingale and X satisfies $(CR)_\mu$, while $e^{-\mu t}(\tilde{S}_t + D_t)_{t\in[0,T]}$ and $e^{-\mu t}(S_t + D_t)_{t\in[0,T]}$ fail to be $\mathbb{G}$-martingales. The conditions of definition 3.4 are fulfilled and S has an inefficient market bubble (of type-II). $\square$

Remark 4.14.

(a) As argued in Fama (1989), this effect of overvaluing future payoffs may be amplified by a change in risk premia. A sudden increase in risk aversion lowers today's *discounted* expected value even further.

(b) Given their beliefs, agents act rational all the time. Upon realizing overconfidence, the market efficiently adjusts to the value X of true expected payoffs. Thus, objectively, agents behave irrational prior to τ_J, which allows for the interpretation of bubbly behavior (overvaluation), verified by the crash. In this sense, there is an analogy between the "efficient" model here and models based on market failures: this analogy is covered by the decomposition $(\tilde{S}, \tau_J, X)$. The existence of a bubble is a question of perspective.

4.4 Comparison to rational expectation bubbles

We present a model class that allows for partly overlapping occurrence of rational expectation and type-I inefficient market bubbles. A range of bubble models in the literature (see, e.g., sections 4.2.1 to 4.2.2) are based on single jump processes of the form

$$dS_t = \alpha(t, S_t)\mathbb{1}_{\{t<\tau_J\}} dt + \beta(t, S_t)\mathbb{1}_{\{t<\tau_J\}} dW_t - \frac{\alpha(t, S_{t-})}{h(t, S_{t-})} dJ_t, \quad t \in [0, \infty), \tag{4.32}$$

where $(W_t)_{t\in[0,\infty)}$ is a Brownian motion, α and β are suitably regular coefficient functions, $(J_t)_{t\in[0,\infty)}$ is a $\{0,1\}$-valued single jump process, whose jump time $\tau_J = \inf\{t > 0\colon J_t = 1\}$ is associated with a hazard process

$$\Lambda_{t\wedge\tau_J} = \int_0^{t\wedge\tau_J} h(s, S_s)ds, \quad \text{for } t \in [0,\infty). \tag{4.33}$$

Within our framework, this amounts to using a possibly explosive Itô process as in definition 4.1 given by α and β and a crash (τ_J, X) as in section 4.1.2 where X as in (4.9) is given by

$$X_t \equiv \tilde{S}_{\tau_J-}(1-\kappa) \qquad \text{on } [[\tau_J,\infty)). \tag{4.34}$$

with a relative jump size

$$\kappa = \frac{\alpha(\tau_J, S_{\tau_J-})}{S_{\tau_J-}h(\tau_J, S_{\tau_J-})}. \tag{4.35}$$

Based on their examination of single jump processes with a deterministic hazard rate in Herdegen and Herrmann (2016), Herdegen and Herrmann (2018) consider (within a more general setting) a solution to the SDE (4.32) assuming a finite time horizon $T \in [0,\infty)$ and coefficients

$$\alpha(t, S_t) = \phi'(t)S_t,\ \beta(t, S_t) = \sigma_0 S_t \text{ and } h(t, S_t) = h(t) \tag{4.36}$$

for $\sigma_0 \in (0,\infty)$ and continuously differentiable functions $\phi, h : [0, T) \to (0,\infty)$. They show, in particular, that the process $(S_t)_{t\in[0,T]}$ is a strict local martingale if and only if

$$\int_0^T h(t)dt = \infty \quad \text{and} \quad \int_0^T \big(h(t) - \phi'(t)\big)\,dt < \infty. \tag{4.37}$$

Similarly, Schatz and Sornette (2019) consider the setting of a homogeneous diffusion with

$$\alpha(t, S_t) = \alpha(S_t),\ \beta(t, S_t) = \beta(S_t) \text{ and } h(t, S_t) = h(S_t) \tag{4.38}$$

for suitably regular functions α, β and h and show that S fails to be a uniformly integrable martingale if and only if

$$\lim_{x\to\infty} \frac{\alpha(x)}{h(x)x} = 1 \text{ or } \lim_{x\to\infty} \frac{h(x)x^2}{\beta^2(x)} = 0. \tag{4.39}$$

Comparison and discussion. Note that single jump processes as above generate an incomplete market, and the choice of fundamental value in the rational expectations literature is not unique, cf. the discussion in section 2.1. For simplicity, in the proposition below we content ourselves with diagnosing a bubble using either of the two fundamental values.

Proposition 4.15. *Let the setting in section 3.2 be fulfilled, assume that the asset price S follows* (4.32) *with one of the following specifications under some $\mathbb{P}$-equivalent measure $\mathbb{Q}$ and assume that $\mathbb{G} = \mathbb{F}^S$.*

1. *The specification in section 4.2.1 with $\alpha \in [1,\infty)$ and asset lifetime $\tau = T$.*
2. *The specification in section 4.2.2 with asset lifetime $\tau = \tau_J$.*

Then S has both a rational expectation bubble[25] *and an inefficient market bubble (definition 3.4).*

[25] Either a strong rational expectation bubble (definition 2.1) or a rational expectation $\mathbb{Q}$-bubble (definition 2.2).

Proof of proposition 4.15. For 1, we can check that (4.37) is fulfilled and thus S a $\mathbb{Q}$-strict local martingale. As such, it holds that $E_{\mathbb{Q}}[S_T|\mathcal{F}_0] < S_0$ and we have two (and only two) possibilities:

(a) the superreplication price (2.3) is equal to S_0, in which case we have a rational expectation $\mathbb{Q}$-bubble as in definition 2.2, or

(b) the superreplication price (2.3) is strictly smaller than S_0, in which case we have a strong bubble as in definition 2.1.

In any case, S has a rational expectations bubble. Being a (strict) local martingale, we know that S satisfies (NFLVR). At the same time, we can define a pre-drawdown process $\tilde{S}$ given by

$$d\tilde{S}_t = \frac{\mu_0}{(T-t)^{\alpha}}dt + \sigma_0 dW_t, \quad t \in [0, T], \tag{4.40}$$

which satisfies (3.11), explodes $\mathbb{P}$-a.s. on $[0, T]$ and thus violates (NFLVR), whence S has a (NFLVR)-inefficient market bubble of type-I. This proves 1. Using the setting of section 4.2.2 and the results implied by (4.39), the second part follows analogously. $\square$

Remark 4.16. Although there is some overlap, following our discussion in section 2.1 we suggest this is coincidental: if one accepts that single jump processes of the form (4.32) are reasonable models for financial bubbles, then differentiating between *bubble* or *no bubble* solely based on the behavior of the relative size of the crash as suggested by (4.37) and (4.39) seems far-fetched. The analysis in section 4 in Herdegen and Herrmann (2018), specifically table 4.4, supports this statement, as for this type of single jump processes we see no fundamental differences in optimal investment strategies for strict local martingales and martingales.

5 Discrete time modelling

In this section we assume the setting in section 3.2, but restrict to discrete time, which simplifies the definitions in section 3.3 and allows us to state the following succinct definition of a bubble.

5.1 Discrete time inefficient market bubble

Definition 5.1. Let the setting in section 3 be fulfilled, $(S_n)_{n\in\mathbb{N}}$ be a $\mathbb{G}$-adapted discrete time stochastic process and $\mathcal{E}$ be an efficient market condition. Then S has an *$\mathcal{E}$-bubble* if and only if there exist $\mathbb{G}$-adapted processes $(\tilde{S}_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ and a random time $\tau_J : \Omega \to \mathbb{N} \cup \infty$ such that

1. $S = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} + X\mathbb{1}_{\{\tau_J \leqslant \cdot\}}$.
2. $X_n\mathbb{1}_{\{n<\tau_J\}} \leqslant \tilde{S}_n\mathbb{1}_{\{n<\tau_J\}}$, $n \in \mathbb{N}$, and $\Delta S_{\tau_J} = X_{\tau_J} - \tilde{S}_{\tau_J - 1} \leqslant 0$.
3. For every $\mathbb{F}^{\tilde{S}}$-adapted process Y it holds that $Y\mathbb{1}_{\{\cdot<\tau_J\}} = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} \Longrightarrow \mathcal{E}(Y) = 0$.

4a. $\mathcal{E}(S) = 1$ (type-I bubble).

4b. $\mathcal{E}(S) = 0$ and $\mathcal{E}(X) = 1$ (type-II bubble).

The pair (τ_J, X) is called *drawdown*. The bubbly stock price is fully described by its *bubble characteristics* $(\tilde{S}, \tau_J, X)$.

To see that this is equivalent to the general definition 3.4 when restricted to discrete time, we need to note the following.

1. Any discrete semimartingale with explosion as in definition 3.1 has explosion time $\tau = \infty$. A discrete process can reach infinity in finite time if and only if it jumps to ∞ at some random time τ while being finite at $\tau - 1$. Then S^{τ_n}, the sequence of stopped processes in definition 3.1, cannot be a sequence of non-explosive processes.
2. Any adapted discrete process is of finite variation and a (discrete time) semimartingale.

5.2 Examples

Below we provide examples of the essential building blocks of discrete time inefficient market bubbles (the bubble characteristics $(\tilde{S}, \tau_J, X)$ and a market efficiency condition $\mathcal{E}$) and an example of a bubble model from the literature.

In discrete time, as any adapted process $\tilde{S}$ is a valid pre-drawdown process and explosion cannot occur, we omit stating examples for pre-drawdown processes.

5.2.1 Drawdown as a discrete time crash

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a stochastic basis and $(p_n)_{n\in\mathbb{N}}$ be a sequence of $[0,1]$-valued $\mathbb{F}$-predictable random variables. Then we can define a random time $\tau_J : \Omega \to \mathbb{N}$ with the property that

$$\mathbb{P}\left[\tau_J = n | \mathcal{F}_{n-1}, \tau_J \geqslant n\right] = p_n, \quad n \in \mathbb{N}. \tag{5.1}$$

For an $\mathbb{F}$-adapted pre-drawdown process $(\tilde{S}_n)_{n\in\mathbb{N}}$ we may choose a process $(X_n)_{n\in\mathbb{N}}$ with $X_0 = 0$ and the property that, for $n \in \mathbb{N}$,

$$\begin{aligned} X_n &\in [0, \tilde{S}_{n-1}), && \text{if } \tau_J = n, \\ X_n &\equiv X_{n-1}, && \text{else.} \end{aligned} \tag{5.2}$$

If we let $\mathbb{G}$ be the filtration generated by $\mathbb{F}$, τ_J and X, the crashed process $S = \tilde{S}\mathbb{1}_{\{\cdot<\tau_J\}} + X\mathbb{1}_{\{\tau_J\leqslant\cdot\}}$ is a $\mathbb{G}$-semimartingale and (τ_J, X) is a drawdown as in definition 3.3.

5.2.2 Efficient market condition

For examples of market efficiency conditions we refer to section 4.1.3, which covers discrete time models without adaption. Very common in the literature is the condition of constant required returns introduced in definition 4.6. In discrete time, it can be stated using a one-step condition.

Definition 5.2 (Constant required return (CR)). Let the setting in section 3.2 be fulfilled in discrete time, let $(S_n)_{n\in\mathbb{N}}$ be an $\mathbb{G}$-adapted stochastic process that satisfies (NFLVR), and let $r \in \mathbb{R}$. Then S is a (CR)$_r$-efficient market if for $R = e^r - 1$ we have

$$\mathbb{E}\left[S_{n+1}|\mathcal{G}_n\right] = (1+R)S_n, \quad n \in \mathbb{N}. \tag{5.3}$$

With a slight abuse of notation, we use (CR)$_R$ to refer to constant required return with simple return R as given by the one-step condition (5.3). Note that we have assumed zero dividends and the wealth process is equal to the stock price. In the presence of a cumulative dividend process $(D_n)_{n\in\mathbb{N}}$, equation 5.3 has to be replaced with

$$\mathbb{E}\left[S_{n+1} + D_{n+1} - D_n|\mathcal{G}_n\right] = (1+R)S_n, \quad n \in \mathbb{N}. \tag{5.4}$$

5.2.3 The periodically collapsing bubble of Evans

As a classical example of a bubble model in the literature that has originally been used to show the shortcomings of early cointegration tests, the periodically collapsing bubble of Evans (1991) can be described in our framework and represents arguably the simplest model of a type-I bubble as in definition 5.1. Given parameters $R, \delta \in \mathbb{R}$, $\alpha \in (\delta/(1+R), \infty)$, $\pi \in (0,1)$ and a sequence $(u_n)_{n\in\mathbb{N}}$ of positive, iid random variables with $\mathbb{E}\left[u_1\right] = 1$, the bubble process $(S_n)_{n\in\mathbb{N}}$ is defined as

$$\begin{aligned}
S_{n+1} &= (1+R)S_n u_{n+1} && \text{for } S_n \leqslant \alpha \\
S_{n+1} &= \begin{cases} \left(\delta + \frac{1+R}{\pi}\left(S_n - \frac{\delta}{1+R}\right)\right) u_{n+1} & \text{with probability } \pi \\ \delta u_{n+1} & \text{else} \end{cases} && \text{for } S_n > \alpha
\end{aligned}$$

To identify the bubble characteristics, we define a pre-drawdown process $\tilde{S}$ given by

$$\begin{aligned}
\tilde{S}_{n+1} &= (1+R)\tilde{S}_n u_{n+1} && \text{for } \tilde{S}_n \leqslant \alpha \\
\tilde{S}_{n+1} &= \left(\delta + \frac{1+R}{\pi}\left(\tilde{S}_n - \frac{\delta}{1+R}\right)\right) u_{n+1} && \text{for } \tilde{S}_n > \alpha
\end{aligned}$$

and a drawdown (τ_J, X) as in definition 3.3 given by

$$\begin{aligned}
\mathbb{P}\left[\tau_J = n|\tilde{S}_n, \tau_J \geqslant n\right] &= \begin{cases} 1-\pi & \text{for } \tilde{S}_n > \alpha \\ 0 & \text{else} \end{cases} \\
X_n &= \delta u_n \quad \text{for } n \in \mathbb{N},
\end{aligned} \tag{5.5}$$

to get $S = \tilde{S}^{\tau_J, X}$. Note that (3.11) holds in this setting. Then, with a market efficiency condition of constant required returns given by (5.3) above,

$$\mathcal{E}(S) = 1 \leftrightarrow \text{S satisfies (CR)}_R, \tag{5.6}$$

we get that S has a $(CR)_R$-bubble of type-I according to definition 5.1: the pre-drawdown process $\tilde{S}$ grows at a rate $\frac{1+R}{\pi}$ while the full process S grows at rate $1+R$. To arrive at the periodically collapsing model with a series of crashes $\left(\tau_J^{(n)}\right)_{n\in\mathbb{N}}$, simply restart the process after a crash $\tau_J^{(i)}$ until the next crash at $\tau_J^{(i+1)}$.

Remark 5.3. Note that the model was originally described in the rational expectations framework, discussed in section 2.1, with an additive decomposition where a fundamental component F was added to the bubble S. This can be mimicked without changes in our framework. To see the difference of the two approaches, consider the case $\pi \equiv 1$: in the rational expectations framework, this is a valid bubble model. In our approach, however, the existence of a crash (and higher growth countering it) is an essential feature of a bubble and $\pi \equiv 1$ would not be an inefficient market bubble. Moreover, a bubble in our approach can naturally end after a crash, whereas within the rational expectations framework the bubble continues ad infinitum.

5.3 Comparison to rational expectation bubbles

As we have discussed in the introduction, (discrete time) rational expectation bubbles satisfy a one-step no-arbitrage condition, thus they cannot meaningfully be described in our framework, whose main characteristics is a break of market efficiency in the pre-drawdown process.

1. In section 5.3.1 we compare the two approaches and discuss the joint hypothesis issue mentioned in section 2.1.3 in a simplified setting.
2. In section 5.3.2 we discuss, using examples from the literature, how researchers seem to implicitly add natural elements of our framework to rational expectations bubbles.

5.3.1 Efficient vs. inefficient market bubbles

Consider the standard discrete time set-up from section 2.1 given by a stock price $(S_n)_{n\in\mathbb{N}}$, a cumulative dividend stream $(D_n)_{n\in\mathbb{N}}$ and the constant-return No-Arbitrage condition (5.3). Our goal is to illustrate the joint hypothesis issue discussed in section 2.1.3 in a simplified setting. For this, assume we want to model a technology stock that does not pay dividends in the near future $[0, T]$ for some $T \in \mathbb{N}$ and possibly experiences a bubble.[26]

Rational expectation bubble. Then, as in discussed in example 2.4(a), the standard description of a stock price $S = S^* + B$ with fundamental value S^* given by

$$S_n^* = \begin{cases} \sum_{i=T-n+1}^{\infty} \left(\frac{1}{1+R}\right)^i \mathbb{E}\left[D_{n+i} - D_{n+i-1} | \mathcal{F}_n\right], & n \in \{0, \dots, T\}, \\ \sum_{i=1}^{\infty} \left(\frac{1}{1+R}\right)^i \mathbb{E}\left[D_{n+i} - D_{n+i-1} | \mathcal{F}_n\right], & n \in \{T+1, \dots\}, \end{cases} \tag{5.7}$$

it holds that

$$\mathbb{E}\left[S_{n+1}^* | \mathcal{F}_n\right] = (1+R) S_n^*, \quad n \in \{0, \dots, T\}. \tag{5.8}$$

[26] Take, for instance, the case study of Zynga in Forró et al. (2012) or the recent development of the Bitcoin price.

Using the standard argument in the rational bubble literature, from equations (5.4) and (5.7) we can derive that the bubble component $B = S - S^*$ necessarily satisfies the martingale property on $[0, \infty)$,

$$\mathbb{E}\left[B_{n+1}|\mathcal{F}_n\right] = (1+R)B_n, \quad n \in \mathbb{N}. \tag{5.9}$$

We can conclude that the process with or without a bubble grows at the same rate and fails to cover the intuitive notion that *the expected return is larger if the stock price experiences a bubble*. Even if the true required return R is known (e.g., inferred from a consumption model for the pricing kernel), a bubble cannot be detected. This reasoning extends to the situation of multiple dividends, in the sense that the only way to derive testable hypothesis on bubble processes are assumptions on future dividends.

Inefficient market bubble (type-I). Assume instead there is the possibility of disclosure of bad news leading to a lower expected valuation of future dividends and a drop by a multiplicative fraction of $\kappa \in (0,1)$. At a date $n \in [0, T]$ this occurrence can be described by a random time τ_J with $\mathbb{P}\left[\tau_J = n+1|\tau_J > n\right] = p$. There is a probability $(1-p)^T$ that there is no downward correction on the whole time interval and high payoffs are realized. Then the full price process up to and including the crash can be written as

$$S_n^{\tau_J,\kappa} = S_n \mathbb{1}_{\{n<\tau_J\}} + (1-\kappa)S_{n-1}\mathbb{1}_{\{\tau_J=n\}}, \quad n \in \{0, \ldots, \tau_J\}, \tag{5.10}$$

where S represents the high payoffs in case of no downward correction and is assumed to grow at some constant (conditional) return R',

$$\mathbb{E}\left[S_{n+1}|\mathcal{F}_n, \text{ no drop at } n+1\right] = (1+R')S_n, \quad n \in \{0, \ldots, T\}. \tag{5.11}$$

The unconditional expected growth (that is, the expected return that a rational trader requires to hold the asset) is thus equal to

$$\mathbb{E}\left[S_{n+1}^{\tau_J,\kappa}\middle| \mathcal{F}_n, \tau_J > n\right] = \left[(1+R')(1-p) + p(1-\kappa)\right]S_n^{\tau_J,\kappa}, \quad n \in \{0, \ldots, \tau_J\}, \tag{5.12}$$

with a growth rate $1 + R = (1+R')(1-p) + p(1-\kappa) < 1 + R'$. The objective market efficiency condition is therefore No-Arbitrage with expected return R. The full model consists of

1. a *pre-drawdown process* S growing with expected return R',
2. *a crash* described by the random time τ_J and a fraction κ, and
3. an *objective price process* $S^{\tau_J,\kappa}$ growing with (true) expected return $R < R'$.

In other words, the pre-drawdown process may grow at a significantly higher rate R', remunerating for the possibility of a crash (that is, revaluation) and violating market efficiency, while the de facto price process $S^{\tau_J,\kappa}$ grows at R. Contrary to the rational expectations framework, if we know the true required return is equal to R', we can detect a bubble.

Remark 5.4.

(a) For the uninformed investor that ignores the possibility of a crash (i.e., trades in the belief that pre-drawdown process S is the true underlying), the high returns given by R' seem like a marvelous deal. However, even for rational, informed traders (think of hedge funds, banks or other large players) that correctly assess the probability of a crash, p and require a return R', it is fully rational to stay invested. In this conception of a bubble, the question in the rational bubble literature of why informed traders are not deflating a bubble is naturally answered: they don't have a reason to do so.[27]

(b) To conclude this section, note that, while the above simplified examples show the theoretical difference of the two frameworks, neither of the above descriptions allows one to detect a bubble before the crash if the required rate of return is unknown – all processes (including the bubbly pre-drawdown process) are assumed to have constant expected return and measuring that return (in the absence of dividends) will be inconclusive.[28] One possible remedy is provided by the observation that pre-drawdown trading behavior of noise traders is often characterized by positive feedback mechanisms, frenzy, overoptimism and thus increasing returns; see, e.g., De Long et al. (1990b), Hüsler et al. (2013), M.Leiss et al. (2015), Shleifer (2000), Sornette and Cauwels (2015) and section 4.2.1 to 4.2.2 for continuous time models of increasing (accelerating) returns.

5.3.2 Sidelining rational expectations in the literature

Below we show that some of the additional assumption and slight departures from rational expectations applied in the literature are, essentially, not too different from the basic ideas of our framework. We hope that this strengthens the case for our inefficient markets approach to bubbles introduced in section 3, which explicitly incorporates these ideas. For this, recall the bubble characteristics $(\tilde{S}, \tau_J, X)$ as introduced in definition 5.1, where $\tilde{S}$ is *a pre-drawdown process that violates market efficiency* and (τ_J, X) represents *the possibility of a drawdown*. These departures from the rational expectations paradigm seem to be further indications for its inadequacy in constituting a reasonable theoretical framework – explicitly including those stylized features can improve the modeling process significantly.

A pre-drawdown process that violates market efficiency. Some authors study log-prices instead of prices, e.g., among others, Phillips et al. (2011) or Phillips et al. (2015). The main idea is to take a log-linear approximation of the constant-return No-Arbitrage condition

$$(1+R)S_n = \mathbb{E}\left[S_{n+1} + D_{n+1} - D_n(1+R)|\mathcal{F}_n\right], \quad n \in \mathbb{N}, \tag{5.13}$$

[27] This may be important to keep in mind in view of the growing amount of empirical bubble literature that present short sale bans/constraints as a driving mechanism of a bubble build-up. While short sale constraints will naturally have an influence on the shape of the bubble and are a driving force in many type-II bubble models (see, e.g., section 4.3.1), they seem not essential to generate the *exuberant returns & crash*-pattern that is characteristic for bubbles.

[28] Apart, of course, from a fundamental analysis, that can be highly uncertain if expected payoffs are in the distant future.

for a stock price $(S_n)_{n\in\mathbb{N}}$ and cumulative dividend payments $(D_n)_{n\in\mathbb{N}}$, as elaborated in Campbell and Shiller (1989) or section 7 of Campbell et al. (1997). Then it is argued that non-stationary behavior in an autoregressive specification for log-prices implies the existence of a bubble. However, this leads to behavior transcending the *rational expectations* framework, which can be seen as follows. Assume that the bubble component is large compared to single-period dividends $\tilde{D}_n = D_n - D_{n-1}(1+R)$, that is,

$$S_n \approx B_n \gg \tilde{D}_n = D_n - D_{n-1}(1+R), \quad n \in \mathbb{N} \tag{5.14}$$

and assume that, as in Phillips et al. (2011), the log stock price $s = \log(S)$ follows a linear autoregressive process

$$s_{n+1} = c + \delta s_n + \epsilon_n, \quad n \in \mathbb{N} \tag{5.15}$$

with $c, \delta \in (0, \infty)$. For simplicity of the argument, assume that $\epsilon_n \equiv 0$ for all n. Then, by recursion, we get for the price S that

$$S_{n+1} = e^{s_{n+1}} \geqslant S_0^{(\delta^{n+1})}, \quad n \in \mathbb{N}. \tag{5.16}$$

Now the standard approach in such bubble tests is to look for an explosive root ($\iff \delta > 1$) in equation (5.15). However, for $\delta > 1$, S cannot fulfill (5.13), violating the original model assumptions and one would detect a bubble precisely in situations where the No-Arbitrage condition (5.13) is *violated*. This test is thus close in spirit to point 2 in definition 3.4 : the pre-drawdown process shows exuberant behavior and violates market efficiency.

Introducing a drawdown through the backdoor. Therefore, to ensure that the one-step No-Arbitrage condition (5.13) can still be satisfied in case of so-called mildly explosive log-prices, Phillips et al. (2011), Lee and Phillips (2015) and Phillips et al. (2015) specify a stock market crash (or, rather, an *end date* of the bubble) by a random time of regime change. We refer to section 2.3 in Lee and Phillips (2015) for a detailed discussion. The rational expectation framework requires that the crash is completely exogenous – imposed by the modeler – and rational market participants are unaware of its possibility, otherwise it would reflect earlier in the price process. Imposing such a regime change with a crash to a *more reasonable* price level can be viewed as the acknowledgement of the fact that a bubble will not last forever (that is, until infinity), violating the prediction of the rational expectations framework. Thus, this model assumption is close in spirit to point 3 in our definition 3.4: there exists the risk of a crash and the price process including the crash constitutes a *reasonable* price process. However, in the present case, the (necessarily) exogenous description of a crash does not allow for the possibility that the pre-drawdown behavior of the stock price be related to the probabilistic structure of the crash.

6 Discussion

Let us close with a few lessons learned for the two essential tasks of **bubble detection** and **bubble modeling** within our framework. Bubble detection essentially refers to the binary decision *bubble* or *no bubble*, informed by pre-drawdown data; bubble modeling refers to an attempt at giving a full (probabilistic) description of an asset's price evolution in a bubbly episode, either pre- or post-drawdown.

Rational expectation models try to square the idea of overvaluation with market efficiency, which results in bubble models that are consistent with utility-maximizing investors only in a rather special situation: propositions 2.6 and 2.7 imply that a payoff at infinity and infinitely many market participants are necessary to allow for rational expectation bubbles.[29] As discussed in section 2.1.3, this type of models entirely relies on additional hypotheses, as, by construction, a payoff at infinity cannot be verified. One may well, of course, reverse the logic of proposition 2.6 and 2.7 and define a bubble as a process that **does not** emerge as the equilibrium price process of an economy populated by utility-maximizing investors. This means, however, that some kind of market failure drives the evolution of the stock price. To arrive at sensible models of bubbles, this implies that one should strive to explicitly model this inefficiency.

In an attempt to do so, we define inefficient market bubbles in section 3 based on a violation of efficiency, in accord with the possibility (risk) of a drawdown. Based on the bubble characteristics $(\tilde{S}, \tau_J, X)$, we introduce two types of bubbles (definition 3.4), which both are based on an inefficient pre-drawdown process $\tilde{S}$ and differ in their interpretation of the drawdown (τ_J, X). A type-I bubble can be understood as a first (partial) departure from rational expectations: while we acknowledge and model the existence of an instability (τ_J, X), the full stock price process $S = \tilde{S}^{\tau_J, X}$ is still an efficient market. More severely, in a type-II bubble the stock price S departs from efficient levels (represented by X) and returns to efficiency with a crash at τ_J. That said, in general, without additional information on the probabilistic structure of τ_J, we will not be able to distinguish type-I and type-II inefficient bubbles by looking at a single bubble episode.

Bubble detection, however, should take place on $[0, \tau_J)$ and thus merely concerns detecting inefficiency of $\tilde{S}$, irrespective of the type. This simpler task still exposes us to well-known joint hypothesis problems. Recall the discussion in section 4.3.3, suggesting that efficiency in *price levels* – and thus also the existence of a bubble – is a matter of perspective, as high valuation may a priori always be explained by high expected future payoffs (or profitability) and a drawdown as a result of changes in expectations and risk aversion. Instead, for bubble detection, we have to focus our attention on models that generate testable signals of inefficiency. Notable examples are situations characterised by heterogeneous beliefs and overconfidence (e.g., Scheinkman and Xiong (2003)), leading to increased trading volume, or momentum trading (e.g., Lin et al. (2018)), imitation and herding (e.g., Johansen et al. (1999)), leading to accelerating returns. These testable signals in *price evolution* and variables such as *trading volume* allow for bubble detection, verified by the occurrence of a crash. Such models have the potential to avoid *joint hypothesis problems* in testing for market efficiency, first comprehensively described by Fama (1991) and still open for discussion; see, e.g., Greenwood et al. (2017). As a straightforward example, if we find the dynamics of price or return to be explosive in finite time (e.g., the models in sections 4.2.1 and 4.2.2), then at some point we can expect a regime change. We know the exuberant/explosive/accelerating behavior necessarily has to end and can constitute a bubble without further assumptions (see example 6.2 below).

[29] As demonstrated in the overlapping generations model of Tirole (1985).

Having said that, the second task of **bubble modeling** is even more ambitious. If we aim for an accurate prediction of a drawdown or, more generally, increase robustness of financial problem formulations vis-à-vis asset price bubbles, the full structure of a bubble will become important. The decomposition $(\tilde{S}, \tau_J, X)$ is a good starting point, but will likely require choosing between type-I and type-II bubble as well as additional assumptions on the probabilistic structure of the drawdown.[30]

Further research. Apart from an immediate extension of the framework in section 3 to

1. a higher dimensional setting including several price processes and market instruments,
2. other types of market efficiency such as statistical arbitrage or macroeconomic specifications of risk premia, and
3. a more detailed description of the correction mechanism in type-II bubbles (in the present setting merely a jump from $\tilde{S}_{\tau_J -}$ to X_{τ_J}), which is not always as immediate as during the 1987 Dow Jones market crash (cf. figure 1),

let us give two examples of research questions induced by the decomposition of inefficient market bubbles that may be promising to investigate.

Example 6.1 (Explosive processes with a jump.). The breakdown of an asset price with a bubble within the framework of section 3 provides a solid theoretical justification for using explosive processes (in combination with a random time of a crash) in asset price modeling; see the processes in sections 4.2.1-4.2.2 above for applications in the literature. Such processes have received only little attention in the stochastic processes literature, with the notable exception of Herdegen and Herrmann (2014) and Herdegen and Herrmann (2018), who study single jump processes with deterministic, explosive jump intensity in combination with an independent Brownian motion. In particular, absolutely continuous changes of measure, which are essential to arbitrage theory and option pricing, have not been investigated yet in situations such as

1. section A.3, which is similar to Herdegen and Herrmann (2018) but allows for a more general relation of drift and jump intensity, and
2. section 4.2.2, where the jump intensity depends on the Brownian evolution.

Example 6.2 (Bubble detection.). Detecting a bubble with bubble characteristics $(\tilde{S}, \tau_J, X)$ in a market obeying a market efficiency condition $\mathcal{E}$ (cf. the definitions in section 3) in its build-up phase (that is, prior to τ_J) amounts to asserting whether the pre-drawdown process $\tilde{S}$ violates market efficiency. One may use the weak market efficiency condition $\mathcal{E} =$ (NFLVR) and a homogeneous diffusion of the form

$$d\tilde{S}_t = \mu_0 \tilde{S}^\alpha dt + \sigma_0 \tilde{S}^\beta dW_t, \quad t \in [0, \infty), \tag{6.1}$$

[30]E.g., an assumption on $\tilde{S}$ in section 4.2.1, cf. remark 4.9, an assumption on τ_J in section 4.2.2, cf. remark 4.11, a distributional assumption on τ_J in section 4.3.1 or an assumption on t_0 influencing τ_J in section 4.3.2.

for parameters $\mu_0, \sigma_0, \alpha, \beta \in (0, \infty)$ and a Brownian motion $(W_t)_{t\in[0,\infty)}$. Then a version of Feller's test for explosion (see, e.g., corollary 4.4 in Cherny and Engelbert (2005)) can be used in combination with a sufficient condition for $\tilde{S}$ to violate (NFLVR),

$$\exists t \in [0, \infty) \colon \mathbb{P}\left[\tilde{S}_t = \infty\right] > 0 \Longrightarrow \tilde{S} \text{ violates (NFLVR).} \tag{6.2}$$

Thus, the problem is reduced to estimating $(\hat{\mu}_0, \hat{\sigma}_0, \hat{\alpha}, \hat{\beta})$ from data – for which an adaption of classic results for non-explosive diffusions, cf. Aït-Sahalia (2002) and references therein, may be warranted. While it seems rather extreme to use what is essentially a relative pricing approach on a single asset, this is very robust and avoids joint hypothesis problems.

A Appendix

A.1 Technical results

Below we collect results from the literature that are used above; applied/extended to our setting of explosive processes.

Lemma A.1. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, let S be an $\mathbb{F}$-semimartingale with explosion as in definition 3.1 with explosion time τ and let $(\Lambda_t)_{t\in[0,\infty)}$ be a continuous, $\mathbb{F}$-adapted, increasing process with*

$$\begin{aligned} \Lambda &< \infty \ \ \text{on } [[0, \tau)), \\ \Lambda &\equiv \infty \ \ \text{on } [[\tau, \infty)). \end{aligned} \tag{A.1}$$

Then there exists a filtered probability space $(\hat{\Omega}, \mathcal{G}, \mathbb{G}, \hat{\mathbb{P}})$ and a random time τ_J with hazard process Λ such that the compensated jump process

$$\mathbb{1}_{\{\tau_J \leqslant t\}} - \Lambda_{t\wedge\tau_J}, \quad t \in [0, \infty), \tag{A.2}$$

is a $\mathbb{G}$-martingale and S is a $\mathbb{G}$-semimartingale with explosion.

Proof of lemma A.1. We follow a standard approach in the literature (see, e.g., section 6.5 in Bielecki and Rutkowski (2002)) and take $(\hat{\Omega}, \mathcal{G}, \hat{\mathbb{P}})$ to be the product extension of $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a uniformly $[0, 1]$-distributed random variable ξ, define

$$\tau_J = \inf\{t \in [0, \infty) | e^{-\Lambda_t} \leqslant \xi\} \tag{A.3}$$

and let $\mathbb{G}$ be the smallest right-continuous filtration generated by $\mathbb{F}$ and $\mathbb{1}_{\{\tau_J \leqslant \cdot\}}$. Then, by construction, τ_J is a $\mathbb{G}$-stopping time with $\mathbb{P}\left[\tau_J < \tau\right] = 1$ and the compensated jump process

$$\mathbb{1}_{\{\tau_J \leqslant t\}} - \Lambda_{t\wedge\tau_J}, \quad t \in [0, \infty), \tag{A.4}$$

is a $\mathbb{G}$-martingale. Moreover, by construction and continuity of Λ, any $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale (this is the so-called *martingale invariance property*) and hence any $\mathbb{F}$-semimartingale (with explosion) is a $\mathbb{G}$-semimartingale (with explosion). This completes the proof of lemma A.1.□

Next we show that a class of strictly positive semimartingales with explosion, whose stochastic logarithm is a quasi-left-continuous local submartingale with explosion, allows for a straightforward construction of a crash such that the crashed process is a local martingale. The result can in principle be extended to processes that may hit zero (and stay there), however then more care is needed in describing the stochastic exponential and logarithm; see Larsson and Ruf (2017) for a note on stochastic exponentials and logarithms on stochastic intervals.

Lemma A.2. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, let $(S_t)_{t\in[0,\infty)}$ be an $\mathbb{F}$-semimartingale with explosion as in definition 3.1 with explosion time τ such that $S > 0$ and $S_- > 0$, let $(Y_t)_{t\in[0,\infty)}$ be the stochastic logarithm*[31] *of S, assume that there exists a continuous, $\mathbb{F}$-adapted, increasing process $(A_t)_{t\in[0,\infty)}$ with $A \equiv \infty$ on $[[\tau, \infty))$ such that, for every $n \in \mathbb{N}$, there exists some decomposition $Y^{\tau_n} = Y_0 + M^n + A^{\tau_n}$ for a local martingale M^n and an $\mathbb{F}$-stopping time τ_n, and let $\kappa \in (0,1)$ be a constant. Then A/κ is the hazard process of a random time τ_J with the property that the stopped process*

$$S_t^{\tau_J,\kappa} = S_t \mathbb{1}_{\{t<\tau_J\}} + S_{\tau_J-}\,(1-\kappa)\,\mathbb{1}_{\{\tau_J \leqslant t\}}, \quad t \in [0,\infty) \tag{A.5}$$

is a $\mathbb{G}$-local martingale, where $\mathbb{G}$ is the filtration generated by $\mathbb{F}$ and $\mathbb{1}_{\{\tau_J \leqslant \cdot\}}$.

Proof of Lemma A.2. Continuity of A and the assumption on Y imply that, for every $n \in \mathbb{N}$, there exists a unique (up to indistinguishability) local martingale M^n and continuous finite variation process A^n such that $Y^{\tau_n} = Y_0 + M^n + A^n$. By definition, S fulfills the stochastic integral equation

$$S_t = \int_0^t S_{s-} dY_s = \int_0^t S_{s-}\left(dM_s^n + dA_s^n\right), \quad \text{on } [[0, \tau_n]]. \tag{A.6}$$

This implies that the crashed process $S^{\tau_J,\kappa}$ solves the integral equation

$$S_t^{\tau_J,\kappa} = \int_0^t \left(S_{s-}^{\tau_J,\kappa}\right) \mathbb{1}_{\{s<\tau_J\}} \left(d\left(M_s^n\right) + d\left(A_s^n\right) - \kappa d\mathbb{1}_{\{\tau_J \leqslant s\}}\right), \quad \text{on } [[0, \tau_n]]. \tag{A.7}$$

The construction in lemma A.1 ensures that for the filtration $\mathbb{G}$, generated by $\mathbb{F}$ and $\mathbb{1}_{\{\tau_J \leqslant \cdot\}}$, it holds that $\frac{1}{\kappa}A^{\tau_J} - \mathbb{1}_{\{\tau_J \leqslant \cdot\}}$ and M^n are $\mathbb{G}$-local martingales. Then, using $A = A^n$ on $[[0, \tau_n]]$ and equation (A.7), we know that, for every n, $(S^{\tau_J,\kappa})^{\tau_n}$ is a $\mathbb{G}$-local martingale. We can conclude, using the result of theorem 44(e) in Protter (1990), that $S^{\tau_J,\kappa}$ is a $\mathbb{G}$-local martingale. The proof of lemma A.2 is thus completed. □

With a straightforward application of Girsanov's theorem, we get a characterization of strictly positive Itô-processes that satisfy (NFLVR).

Lemma A.3. *Let the setting in section 3.2 be fulfilled on a time horizon $[0, T]$ for some $T \in [0,\infty)$, let S be a strictly positive Itô-process with explosion time $\tau : \Omega \to [0,\infty]$ as defined in definition 4.1, assume that the filtration $\mathbb{F} \subseteq \mathbb{G}$ is generated W and assume that every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale. Then S is a (NFLVR)-efficient market on $[0, T]$ if and only if the following properties are fulfilled.*

[31] The stochastic logarithm L of a strictly positive process X with explosion is a process that, for any $n \in \mathbb{N}$, is given as the solution of the integral equation $dL_t = 1/X_t dX_t$ on $[[0, \tau_n]]$. It is thus well-defined on $\cup_{n\in\mathbb{N}}[[0, \tau_n]] = [[0, \tau))$.

1. $\mathbb{P}\left[\tau > T\right] = 1$,
2. $\mathbb{P}\left[\int_0^T \left(\frac{\alpha_s}{\beta_s}\right)^2 ds < \infty\right] = 1$ *and*
3. $\mathbb{E}\left[\exp\left(-\int_0^T \frac{\alpha_s}{\beta_s} dW_s - \frac{1}{2}\int_0^T \left(\frac{\alpha_s}{\beta_s}\right)^2 ds\right)\right] = 1.$

Proof of Lemma A.3. Below we write g_s for a process g that may depend on (s, S_s). First assume that S satisfies *No Free Lunch with Vanishing Risk* on $[0, T]$. By the fundamental theorem of asset pricing in Delbaen and Schachermayer (1998), there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S is a $(\mathbb{Q}, \mathbb{G})$-local martingale on $[0, T]$. This implies that S does not explode $\mathbb{Q}$-a.s., hence $\mathbb{P}$-a.s. on $[0, T]$. Moreover, a result in Stricker (1977)[32] shows that S is a $(\mathbb{Q}, \mathbb{F})$-local martingale. Then, by the martingale representation theorem[33] there exists a predictable, square-integrable process $f : [0, T] \times \Omega \to \mathbb{R}$ such that the martingale density process $(Z_t)_{t\in[0,T]}$ given by $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right]$ has the form

$$Z_t = \exp\left(-\int_0^t f_s dW_s - \frac{1}{2}\int_0^t f_s^2 ds\right), \quad t \in [0, T]. \tag{A.8}$$

By the Girsanov-Meyer theorem[34] we know that Z defines a $\mathbb{Q}$-local martingale $(N_t)_{t\in[0,T]}$ through

$$N_t = \int_0^t \beta_s dW_s + \int_0^t \beta_s f_s ds, \quad t \in [0, T]. \tag{A.9}$$

As S is a $\mathbb{Q}$-local martingale, so is $S - N = \int_0^{\cdot} (\alpha_s - \beta_s f_s)\, ds$, which is the case if and only if we have $\frac{\alpha}{\beta} = f$. This implies that $\frac{\alpha}{\beta}$ is a square integrable process with probability 1 and that

$$\exp\left(-\int_0^T \frac{\alpha_s}{\beta_s} dW_s - \frac{1}{2}\int_0^T \left(\frac{\alpha_s}{\beta_s}\right)^2 ds\right) \tag{A.10}$$

is a well defined random variable with expected value 1.

Conversely, assume that conditions 1–3 hold. Then we can define a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ by its Radon-Nikodym derivative on $\mathcal{F}_T$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{\alpha_s}{\beta_s} dW_s - \frac{1}{2}\int_0^T \left(\frac{\alpha_s}{\beta_s}\right)^2 ds\right). \tag{A.11}$$

Again we apply the Girsanov-Meyer theorem to get that the process

$$\int_0^t \beta_s dW_s + \int_0^t \frac{\alpha_s}{\beta_s} \beta_s ds = S_t, \quad t \in [0, T] \tag{A.12}$$

is a $(\mathbb{Q}, \mathbb{F})$-local martingale and hence, by assumption, a $(\mathbb{Q}, \mathbb{G})$-local martingale, completing the proof of lemma A.3. □

[32] See theorem 3.6 in Föllmer and Protter (2011).
[33] See, e.g., corollary 4 in chapter IV of Protter (1990).
[34] See, e.g., theorem 20 in chapter III of Protter (1990)

A.2 Measure change and No-Good-Deal (NGD) for jump diffusions

Below we extend No-Good-Deal bounds introduced in section 4.1.3 above to a jump diffusion setting, where a useful instantaneous formulation can be derived. For simplicity we restrict ourselves to a finite time horizon $[0, T]$ for some $T \in (0, \infty)$.

A.2.1 Jump diffusion with integrable intensity

The following is a standard construction, details of which can found in, e.g., chapter III.5 of Jacod and Shiryaev (2003) or chapter 6.6 of Bielecki and Rutkowski (2002). Let the setting in section 3.2 be fulfilled, let $\mathbb{F} \subseteq \mathbb{G}$ be a filtration satisfying the usual hypothesis, let W be an $\mathbb{F}$-Brownian motion, let $(\lambda_t)_{t\in[0,T]}$ be an $\mathbb{F}$-progressively measurable, locally integrable process, $(N_t)_{t\in[0,T]}$ be a conditional Poisson counting process with intensity λ, further assume that the filtration $\mathbb{G}$ is generated by W and N, let μ, σ, κ be $\mathbb{G}$-predictable processes with $\kappa \geqslant -1$, suitably integrable, such that the equation

$$S_t = S_0 + \int_0^t S_{s-} \left(\mu_s ds + \sigma_s dW_s + \kappa_s dN_s\right), \tag{A.13}$$

has a unique strong solution $(S_t)_{t\in[0,T]}$ and let $(\gamma_t)_{t\in[0,T]}$ be an $\mathbb{G}$-predictable process with values in $(-1, \infty)$ satisfying

$$\int_0^T \frac{1}{\sigma_s^2}(\kappa_s(1+\gamma_s)\lambda_s)^2 ds < \infty \text{ and } \int_0^T (1+\gamma_s)\lambda_s < \infty, \quad \mathbb{P}\text{-a.s.} \tag{A.14}$$

Then the process $(Z_t)_{t\in[0,T]}$ given by

$$Z_t = 1 + \int_0^t Z_{s-} \left(-\frac{1}{\sigma_s}(\mu_s + \kappa_s(1+\gamma_s)\lambda_s)dW_s + \gamma_s\left(dN_s - \lambda_s ds\right)\right) \tag{A.15}$$

defines a measure $\mathbb{Q} \approx \mathbb{P}$ if $\mathbb{E}\left[Z_T\right] = 1$, in which case S is a $\mathbb{Q}$-local martingale. In fact, all measures $\mathbb{Q} \approx \mathbb{P}$ such that S is a local $\mathbb{Q}$-martingale admit a density of the form (A.15).

A.2.2 A single jump diffusion with possibly explosive intensity

The local integrability of the process λ above is essential to have a well-defined Poisson process N. If we restrict ourselves to the first jump of N, we can allow for a possibly non-integrable intensity and explosive behavior up to the jump. This has been rigorously elaborated in Herdegen and Herrmann (2018) in the setting of deterministic hazard rate. In particular, let the setting in section 3.2 be fulfilled, let $h : [0, T) \to [0, \infty)$ and $\kappa : [0, T) \to (0, 1]$ be differentiable functions, let $\mu_0 \in \mathbb{R}$, $\sigma_0 \in (0, \infty)$ be constants, let J be a single jump process with hazard rate h, let $\tau_J = \inf\{t \in [0, \infty) | J_t = 1\}$ be the random time of the jump, let W be a Brownian motion and assume that $\mathbb{G}$ is the filtration generated by W and J, let $(S_t)_{t\in[0,T]}$ be the solution of the integral equation

$$S_t = S_0 + \int_0^t S_{s-}\left((\mu_0 + \kappa(s)h(s)\mathbb{1}_{\{s \leqslant \tau_J\}})ds + \sigma_0 dW_s - \kappa(s)dJ_s\right), \tag{A.16}$$

and let $\gamma : [0,T) \to (-1,\infty)$ be a differentiable function that is uniformly bounded away from -1 such that

$$\int_0^T (\kappa(t)h(t)\gamma(t))^2 dt < \infty \text{ and } \int_0^T \mathbb{1}_{\{\mathbb{P}[T \leqslant J]>0\}} h(t)(1+\gamma(t))dt < \infty. \tag{A.17}$$

Then the process $(Z_t)_{t\in[0,T]}$ given by

$$Z_t = 1 + \int_0^t Z_{s-}\left(-\frac{1}{\sigma_0}(\mu_0 - \kappa(s)h(s)\gamma(s)\mathbb{1}_{\{s \leqslant \tau_J\}})dW_s + \gamma_s\left(dJ_s - h(s)\mathbb{1}_{\{s \leqslant \tau_J\}}ds\right)\right) \tag{A.18}$$

satisfies $\mathbb{E}[Z_T] = 1$ and defines a measure $\mathbb{Q} \approx \mathbb{P}$; with S being a $\mathbb{Q}$-local martingale. Note that μ and h might not be defined at T, which is negligible in the integral. This is a straightforward (though non-trivial) extension of equation (A.15) to the single jump setting. The first condition of (A.17) ensures that a Girsanov change of measure is well-defined, whereas the second condition ensures that, in case there is non-negligible probability that the jump does not happen on $[0,T]$, this persists under the new measure.

A.2.3 No-Good-Deal

From equations (A.15) and (A.18) one can derive straightforward bounds on the instantaneous Sharpe ratios of S and Sharpe ratios of derivatives and portfolios derived from S. See appendix A in Björk and Slinko (2006), where the seminal work in Hansen and Jagannathan (1991) is extended to the jump diffusion setting. In a slight abuse of notation (recall that we used an absolute bound on the price in equation (4.13), such a bound is called instantaneous or dynamic *No-Good-Deal bound*. Klöppel and Schweizer (2007) have generalized this to Lévy models.

Definition A.4 (No-Good-Deal (NGD)). Let the setting in A.2.1 be fulfilled and $K \in (0,\infty]$. Then S satisfies *dynamic* $(\text{NGD})_K$ if there exists a measure $\mathbb{Q} \approx \mathbb{P}$ with a density of the form (A.15) for a $\mathbb{G}$-predictable process γ such that for almost every $t \in [0,T]$ it holds that

$$\frac{1}{\sigma_t^2}(\mu_t + \kappa_t(1+\gamma_t)\lambda_t)^2 + \gamma_t^2\lambda_t < K. \tag{A.19}$$

For a single jump diffusion with possibly explosive intensity, as a direct extension of definition A.4, the No-Good-Deal condition accounts for the fact that the process follows a Geometric Brownian motion after the first jump.

Definition A.5 (No-Good-Deal (NGD) for a single jump diffusion). Let the setting in A.2.2 be fulfilled and and $K \in (0,\infty]$. Then S satisfies *dynamic* $(\text{NGD})_K$ if there exists a measure $\mathbb{Q} \approx \mathbb{P}$ with a density of the form (A.15) for a $\mathbb{G}$-predictable process γ such that for almost every $t \in [0,T]$ it holds that

$$\frac{1}{\sigma_0^2}(\mu_0 - \kappa(t)h(t)\gamma_t\mathbb{1}_{\{t \leqslant \tau_J\}})^2 + \gamma_t^2 h(t)\mathbb{1}_{\{t \leqslant \tau_J\}} < K. \tag{A.20}$$

As quantities given by the market implied measure $\mathbb{Q}$, the two quantities in the sum of (A.19) and (A.20) represent the diffusion risk and $\gamma^2 h$ represents the jump risk of the *instantaneous market price of risk*.

A.3 Beyond the bubble framework – further assumptions

As noted above in sections 4.2.1-4.2.2, the framework presented in this paper allows – necessarily – for a great deal of slack in the modeling process. We have criticized the rational expectations framework for its additional assumptions necessary in bubble detection (constant expected return and stationary dividend payments; see section 2.1.3) – here we want to discuss (considerably weaker) additional assumptions within our framework that implicitly have been used in the literature. For our discussion we need to use straightforward extensions of section A.2 that have NOT been rigorously presented in the literature yet.

General setting. For this purpose, we generalize the setting of the section above. Assume the setting in section 3.2 and let $T \in [0, \infty)$ be a finite time horizon, let $\sigma_0 \in (0, \infty)$ be a constant, let $\mu : [0, T) \to [0, \infty)$ be a continuous function with $\lim_{t \to T} \mu(t) = \infty$ and let $\tilde{S}$ be an Itô process as in section 4.1.1 with (possible) explosion time T that satisfies the stochastic differential equation

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu(t)dt + \sigma_0 dW_t. \tag{A.21}$$

Assume we want to model a (NGD)$_\infty$-bubble with a crash as in section 4.1.2 given by a constant relative crash size κ and a random time τ_J with hazard process Λ given by

$$\Lambda_t = \int_0^t h(s)ds, \quad t \in [0, T], \tag{A.22}$$

for a positive continuous function $h : [0, T) \to \mathbb{R}$. Then the crashed process S is given by

$$\frac{dS_t}{S_{t-}} = \mu(t)\mathbb{1}_{\{t<\tau_J\}}dt + \sigma_0 \mathbb{1}_{\{t<\tau_J\}}dW_t - \kappa dJ_t, \quad t \in [0, T], \tag{A.23}$$

where $(J_t)_{t\in[0,T]}$ is the single jump process $J = \mathbb{1}_{\{\tau_J \leqslant \cdot\}}$. In our setting, the pre-drawdown process $\tilde{S}$ violates (NGD)$_\infty$, as μ is unbounded. To get a bubble as in definition 3.4, we need to specify μ and h such that the crashed process S satisfies (NGD)$_\infty$.

No-Good-Deal bounds and additional assumptions. In section 4.2.1 we made the choice $\mu = \kappa h$, which implies that S is a martingale (under the probability measure $\mathbb{P}$). However, for the crashed process to satisfy (NGD)$_\infty$ we have a much weaker sufficient condition: that there exists a function $\gamma : [0, T) \to (-1, \infty)$, as in section A.2.1, bounded away from -1 and suitably integrable, such that the No-Good-Deal bound (A.19) up to the first jump is fulfilled. Let us look at this a little more closely. If we define $\phi : [0, T) \to \mathbb{R}$ by

$$\phi(t) = \frac{1}{\sigma_0}\left(\mu(t) - \kappa(1 + \gamma(t))h(t)\right), \quad t \in [0, T), \tag{A.24}$$

then S fulfills (NGD)$_\infty$ if

1. ϕ is square integrable
2. $(1 + \gamma)h$ is integrable if h is and

3. the No-Good-Deal bound

$$\left(\phi(t)^2 + \gamma(t)^2 h(t)\right) \mathbb{1}_{\{t \leqslant \tau_J\}} < \infty, \quad t \in [0, T]. \tag{A.25}$$

is fulfilled. We can write the expected instantaneous returns process $(R_t)_{t\in[0,T]}$ as

$$R(t) = \mu(t) - \kappa h(t) = \sigma_0\phi(t) + \kappa h(t)\gamma(t). \tag{A.26}$$

This formulation allows for the interpretation of ϕ as being the *market price of diffusion risk* and γ the *market price of jump risk*, cf., e.g., Björk and Slinko (2006). The conditions imposed by $(\mathrm{NGD})_\infty$ are rather weak, so one might want to add additional assumptions on risk-preferences of investors in the market (via assumptions on R, ϕ, and γ). For example, in increasing generality,

(a) Arguably quite restrictive, one can assume that investors are risk-neutral on average during the bubble and thus the required return satisfies $R \equiv 0$. This is implies immediately that $\mu = \kappa h$.

(b) One may impose a No-Good-Deal bound $K \in (0, \infty)$ on S, such that equation (A.25) is fulfilled with bound K.

(c) No additional assumption, merely that S has a $(\mathrm{NGD})_\infty$-bubble.

A specific model – JLS model. To analyze the implications of above assumptions (a)–(c) for a specific model choice, we consider the JLS model; see Sornette (2003) and references therein. In this model, a network of traders is analyzed to find that herding, positive feedback and imitative behavior in the market leads to systemic instability and thus a possible financial crash given by a hazard rate

$$h(t) = B_1(T - t)^{-\alpha} + B_2(T - t)^{-\alpha} \cos\left(\omega \ln(T - t) - \psi\right) \tag{A.27}$$

for parameters $\alpha, B_1, B_2, \omega, \psi \in (0, \infty)$ chosen such that h is non-negative. For a detailed derivation we refer to Sornette (2003) and Seyrich and Sornette (2016). This is a so-called risk-driven bubble, as the possibility of a crash (as opposed to the price process) is specified; see section 5 in Sornette (2003) for details on this terminology.

Using the assumptions above, we seek to derive conditions on the price S (or, equivalently, its return μ) to yield reasonable bubble models given by equation (A.23) based on a crash given by the hazard rate h.

(a) If we assume risk-neutral investors we get that $\mu = \kappa h$ and thus μ necessarily behaves like h in equation (A.27).

(b) If we assume a finite No-Good-Deal bound K, equation (A.25) implies that

$$\lim_{t\to T} \left|\mu(t)(T - t)^{\alpha} - \kappa(B_1 + B_2 \cos\left(\omega \ln(T - t) - \psi\right))\right| = 0 \tag{A.28}$$

Thus, with speed of convergence depending on α and K, μ eventually shows explosive behavior and log-periodic oscillations around the critical time T.

(c) If we only assume that S satisfies $(\text{NGD})_\infty$ then little can be said about the exact behavior of μ. As an example, we may have market prices of risk given by $\gamma(t) = 1/(B_1 + B_2 \cos(\omega \ln(T-t) - \psi))$ and $\phi \equiv 0$ and, from equation (A.26), $\mu(t) = (T-t)^{-\alpha}$. Thus, although market prices of risk ϕ and γ are both bounded, μ may show no log-periodic oscillations (while S still satisfies No-Arbitrage!). As another example, in the case $\alpha \in (0, \frac{1}{2})$ we could have $\gamma \equiv 0$ and $\phi = (c - \kappa h)/\sigma_0$ for some constant $c \in (0, \infty)$. Then ϕ is square integrable and an admissible market price of risk while $\mu = c - \kappa h + \kappa h \equiv \text{const}$ on $[0, T]$.

Discussion. In the papers that introduced the JLS model (see Johansen et al. (2000), Johansen and Sornette (1999) or Sornette (2003) for an overview) it is assumed that the No-Arbitrage condition holds *and* investors are risk-neutral, thus $\mu = \kappa h$ and the price mimics the explosive behavior and log-periodic oscillations of h. With this assumption, an analysis of the price process leads to a detailed knowledge about the hazard of a crash.

Responding to criticism from Ilinski (1999) regarding the assumption of *risk-neutrality*, in Johansen et al. (1999) it is claimed that changing risk preferences within a bounded expected return R or some bounded market price of crash risk γ[35] cannot prevent that explosive behavior and log-periodic signatures are apparent in the price. Feigenbaum (2001) extends the criticism presented in Ilinski (1999), pointing out that the expected required return need not be constant. In response to this, Sornette and Johansen (2001) and Zhou and Sornette (2006) introduce the general case of assuming only No-Arbitrage[36], however the discussion is reduced to the case $\gamma \equiv 0$; in this case it is claimed that log-periodic signatures of h persist. Long story short, the issue that additional assumptions are necessary to derive a relationship $\mu \leftrightarrow h$ is well known and has been discussed in the literature, but a comprehensive study has not been presented yet.

The above can be seen as a first step[37] towards such a comprehensive study, with a full description of possible discount factors and a clear framework that allows one to accurately determine the assumptions that are necessary to reach the desired conclusions. We have seen that, in general, No-Arbitrage (or the stronger $(\text{NGD})_\infty$) alone does not ensure that explosive behavior and log-periodic signature of h is observable in the price (see the examples in point (c) above) – in contradiction to some claims in the literature. In particular, it is possible that an extreme change in risk premia could lead to a situation where the increasing risk of a crash is not reflected in the price (resp. μ) until the crash happens. However, with reasonably weak additional assumptions (see point (b) – $(\text{NGD})_K$ for some $K \in (0, \infty)$ or, e.g., imposing a bound on required return R) one can infer that μ behaves like h around the critical time T and thus show explosive *and* log-periodic behavior.

To close the section, we want to note that similar conclusions, within the respective model framework, can be drawn for *return-driven* bubbles as defined in Sornette (2003), where an analysis

[35] Called ν and K, respectively, in Johansen et al. (1999).

[36] In the sense of (NFLVR) – the existence of an equivalent measure $\mathbb{Q}$ with associated density process $Z^{\mathbb{Q}}$. In Sornette and Johansen (2001) the equivalent terminology of a stochastic discount factor $M = Z^{\mathbb{Q}}$ is used.

[37] Note that we make the restrictive assumption that γ (and thus ϕ) are deterministic functions and thus, although covering a large variety, we can only apply our statements to a subclass of pricing kernels.

of (or assumptions on) market structures leads to a model for μ and one wants to infer information about the hazard of a crash h.

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