A macroprudential view on portfolio rebalancing and compression

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Abstract

We analyse the consequences of post-trade risk reduction services for systemic risk in derivatives markets. Our focus is on portfolio rebalancing, which is a mechanism of injecting new trades to reduce the overall counterparty exposure, and portfolio compression, which is a mechanism to reduce the outstanding notional amount by trades termination and replacement. We first provide a mathematical characterisation of (optimal) portfolio rebalancing. Then, we explore the effects of these services on the financial system from a network perspective by considering contagion arising from only partial repayments in networks of variation margin payments. We provide sufficient conditions for portfolio rebalancing to reduce systemic risk. We also investigate the effects under a scenario where financial institutions react to stress strategically and make delayed payments.

Key words: Systemic risk, liquidity, post-trade risk reduction, portfolio rebalancing, portfolio compression, multilateral netting.

JEL code: C62, D85, G01, G28, G33.

1 Introduction

Post-trade risk reduction (PTRR) services that aim to mitigate risk in derivative portfolios have recently attracted much attention for example with regards to the clearing exemption, see the consultation paper published by ESMA (2020a). Portfolio compression reduces the number of trades and/or the notional size of contracts, and portfolio rebalancing helps to decrease the exposure between counterparties, also reduces the overall exposure in the market. The European Securities and Markets Authority (ESMA) has released the report on PTRR services, see ESMA (2020b), in

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which it summarises that "PTRR transactions are successfully undertaken today and have reduced a considerable amount of risks in the market." In addition to risk reduction benefits, market participants have incentives to use these techniques because of regulatory reforms such as the Uncleared Margin Rule (UMR) and Leverage Ratio (LR) requirements. In the context of strict risk management practices in bilateral markets introduced by policy makers, PTRR services increase collateral availability by affording possibility for market participants to manage their collateral obligations efficiently, ESMA (2020b). The motivation of our work is to explore the effects that portfolio rebalancing and portfolio compression have on systemic risk. We will see that reducing risk on the individual level may not result in a reduction of systemic risk when we take network effects into consideration.

The main contributions of this paper are as follows: First, we provide what we believe to be the first formal mathematical characterisation of portfolio rebalancing and optimal portfolio rebalancing. We establish a formal relationship between portfolio rebalancing and portfolio compression. Second, we provide sufficient conditions for portfolio rebalancing to reduce systemic risk. Third, we analyse both PTRR services in a worst case scenario, that reflects banks' possible strategic response to liquidity stress.

Therefore, we extend the existing literature on the relationship between PTRR services and systemic risk in several dimensions. While Veraart (2020b) has considered portfolio compression that allows for reducing notional value on a closed chain of contracts, and its consequences for systemic risk, we mainly focus on portfolio rebalancing and by doing so consider a different mechanism that changes the underlying network structure.

We organise the paper as follows. In Section 2 we give a formal definition of what we refer to as *portfolio rebalancing* (Definition 2.4) and optimal portfolio rebalancing (Definition 2.7). We introduce a restrictive notion of compression for which we discuss the implications in Appendix B.1 and we show how it is related to an elementary type of portfolio rebalancing. Furthermore, we establish a formal relationship between optimal portfolio rebalancing and non-conservative portfolio compression by D'Errico & Roukny (2021) in Appendix B.2. Section 3 describes how we measure systemic risk in the financial networks using the clearing framework in Veraart (2020a) that was also used in Veraart (2020b) and extends the Eisenberg & Noe (2001) framework. We model payment flows in the form of variation margins and thus concentrate on liquidity. Section 4 contains the main theoretical results. In Section 4.1 we derive sufficient conditions for portfolio rebalancing to reduce systemic risk. In particular, we show in Theorem 4.1 that a rebalancing exercise will reduce systemic risk if participating banks do not default in the original network. We use examples in Appendix C to illustrate the potential harmfulness of portfolio rebalancing, compare it with portfolio compression and discuss implications for financial stability. Appendix D extends the contagion model with liquidation costs and continues our analysis to cover additional liquidity risk that arises from fire sales of illiquid assets. In Section 4.2 we examine the consequences of PTRR services in the case of sequential payments. We give a new interpretation for the Rogers & Veraart (2013) model with zero recovery rates by looking at the Full Payment Algorithm built in Bardoscia et al. (2019a). Then, we present results on potential influence of PTRR services on liquidity. In Section 5 we discuss policy implications of our theoretical results. In particular, we discuss in which sense portfolio compression and rebalancing could be used as macroprudential tools to address liquidity risk and dampen procyclicality and what the associated remaining risks are. Finally, Section 6 concludes.

1.1 Related literature

Many results in the literature on netting focus on centralised netting via the introduction of a central counterparty (CCP). For instance, Duffie & Zhu (2011) provide a baseline framework for multilateral netting via CCPs, in which the theoretical model indicates a trade-off between central clearing and bilateral clearing. The work has been extended in many different dimensions, see e.g. Cont & Kokholm (2014). However, there has not been much literature yet on portfolio rebalancing and portfolio compression as a form of netting in derivatives markets that happens through coordination.

We are not aware of any literature that examines the effects of portfolio rebalancing and possible implications for systemic risk. For portfolio compression, however, there is some literature available.

The literature on portfolio compression can be divided into two strands. The first mainly focuses on algorithms that generate compression outcomes. Several optimisation-based algorithms for multilateral netting are proposed in O'Kane (2017). The algorithm that aims at minimising the L_1-norm is considered to be one of the favourites since large reduction in gross exposure can be achieved. D'Errico & Roukny (2021) study different efficiency of portfolio compression according to different levels of preference. They also use transaction-level data set to analyse empirically the performance of portfolio compression on reducing market excess.

The second strand explores consequences of portfolio compression on systemic risk. Veraart (2020b) analyses the effects of portfolio compression on systemic risk by deriving sufficient conditions for compression to reduce systemic risk. Schuldenzucker & Seuken (2020) investigate the incentives for banks to conduct portfolio compression under the Rogers & Veraart (2013) framework and show that incentivised compression can be harmful for the financial system. Furthermore, Amini & Feinstein (2020) consider the problem of an optimal design for portfolio compression where systemic risk measures are used as an objective in an optimisation problem.

While these results analyse multilateral netting from network perspective, they remain focused on reducing notional amount by portfolio compression. In contrast, our setting accounts for portfolio rebalancing which increases notional amount by injecting market risk-neutral transactions but reduces the overall counterparty exposure in a way that no one is worse off. According to our characterisation (see Definition 2.4), on a net basis, any multilateral netting activity can be described as a rebalancing exercise in the weak sense. We investigate how the liquidity risk propagation is affected by modelling variation margin payments. We show that portfolio rebalancing that allows for rewiring of the entire financial network can reduce systemic risk under certain conditions. Therefore, in principle, both portfolio compression and rebalancing could contribute to systemic risk reduction. A key insight is that *notional size in the financial system cannot capture contagion effects*. To the best of our knowledge, this is the first paper that formulates portfolio rebalancing problem in the context of multilateral netting, and contains comprehensive analysis on the interactions between systemic risk and portfolio rebalancing.

Our work is also related to modelling liquidity stress in derivatives markets similar to Paddrik et al. (2020); Bardoscia et al. (2019b). As in this literature, we consider Variation Margin (VM) calls but in our paper we explore the impact of PTRR services on liquidity shortfalls in the financial system. Variation margins are exchanged on a daily basis to reflect the changes in valuations because of price movements. Counterparty with loss-making position provides margin to the counterparty with profit-making position. Because the aggregate variation margins net out to zero, it might suggest that nearly no extra margin is needed from the network point of view. However, the total demand for variation margins may not be zero in reality. On the one hand, this can be caused by frictional demands for variation margins. Duffie et al. (2015) propose in their model for collateral requirements an assumption of "velocity drag", which says the collateral that a bank posts as margin is not immediately available for its counterparties, therefore the liquidity is temporarily trapped. On the other hand, banks' different response behaviours under stress can also lead to the lags in availability of margins. Paddrik et al. (2020) argue that banks in fact react to liquidity stress in various ways depending on their liquidity buffers and risk management practices. In their contagion model built on the Eisenberg & Noe (2001) framework, they consider a form of stress response function corresponding to hard default to model the scenario where banks pay nothing if they meet a liquidity shortfall. It can be thought as incorporating initial margins into the Rogers & Veraart (2013) model with zero recovery rates. The model admits a greatest fixed point as the payment equilibrium. A closely related paper is Bardoscia et al. (2019b). In order to simulate payment shortfalls in derivatives markets, the authors apply a variation of the Full Payment Algorithm developed in Bardoscia et al. (2019a) that does not allow partial payments. In our paper, we consider a similar stress scenario, put forward an equivalence of the Full Payment Algorithm, and identify the effects of PTRR services.

2 Post-trade risk reduction services

2.1 The financial market

We consider a financial system consisting of N financial institutions with indices in $\mathcal{N} = \{1, 2, ..., N\}$. For simplicity we will call them banks even though a wide range of non-banks trade in derivatives markets and all our results apply to them as well. We denote by $C = (C_{ij}) \in [0, \infty)^{N \times N}$ the corresponding *notional matrix*, where the ij-th entry C_{ij} represents the notional amount of liabilities from bank i to bank j that arise from trading derivative contracts, and $C_{ii} = 0$ for all $i \in \mathcal{N}$. In this paper, we assume that the derivative contracts are traded over-the-counter (OTC) and fungible, for example single-name Credit Default Swap (CDS) contracts written on the same reference entity with the same maturity date and then C_{ij} represents the notional amount that bank i has promised to j if a credit event of the underlying reference entity occurs.[1] Therefore, the notional value of the liabilities between two counterparties i and j can be added and represented by C_{ij} and C_{ji}.

Definition 2.1. For every non-negative matrix X, we define the directed graph associated with X as the tuple $(\mathcal{N}^X, \mathcal{E}^X)$, where the nodes $\mathcal{N}^X = \{i \in \mathcal{N} \mid X_{ij} > 0 \text{ or } X_{ji} > 0\}$ represent the banks and the edges $\mathcal{E}^X = \{(i,j) \in \mathcal{N}^2 \mid X_{ij} > 0\}$ represent the liabilities between the counterparties. We denote the *bilaterally netted matrix with respect to* X by X^{bi}, where $X^{bi}_{ij} = \max(X_{ij} - X_{ji}, 0)$ for all $i, j \in \mathcal{N}$.

2.2 Portfolio rebalancing

There are different types of PTRR services. We first introduce portfolio rebalancing (also known as counterparty risk rebalancing). Portfolio rebalancing consists of injecting market risk neutral transactions such that for every participant net exposures remain unchanged while at the same time the counterparty exposures are reduced.[2] It has been developed to reduce imbalances across portfolios in uncleared and cleared markets, see CCP12 (2020).

Example 2.2 (Portfolio rebalancing)**.** We consider an example of portfolio rebalancing in a network with three banks. Figure 1 provides an illustration. In this example, three additional trades are inserted, whose notional positions are characterised by a matrix R. The original notional matrix C, the matrix R and the matrix $C+R$ that represents the notional positions after rebalancing are given by

$$C = \begin{pmatrix} 0 & 8 & 7 \\ 5 & 0 & 10 \\ 9 & 5 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}, \quad C+R = \begin{pmatrix} 0 & 8 & 11 \\ 9 & 0 & 10 \\ 9 & 9 & 0 \end{pmatrix}.$$

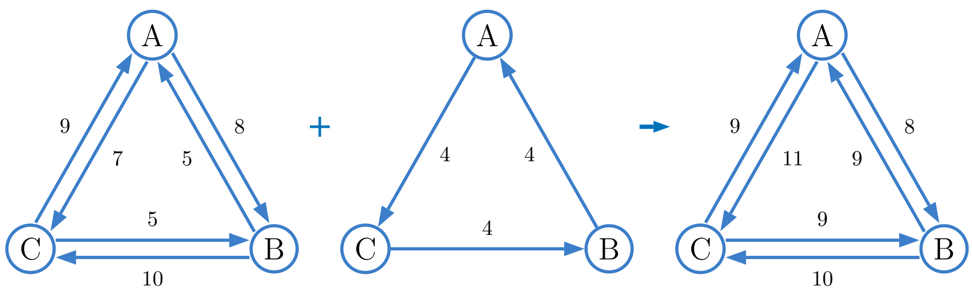
The corresponding bilaterally netted matrices C^{bi} and $(C+R)^{bi}$ are given by

$$C^{bi} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}, \quad (C+R)^{bi} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

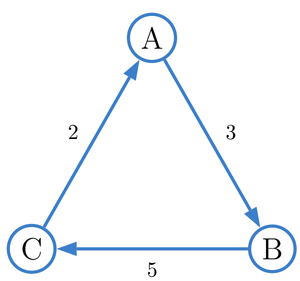
The net exposure of each bank is given by the corresponding column sum of C minus the corresponding row sum of C, i.e., the net exposures of the three banks in the network characterised by C are given by $(\mathbf{1}^\top C)^\top - C^\top \mathbf{1} = (-1, -2, 3)^\top$, where $\mathbf{1}$ is the column vector containing only 1s. These net exposures coincide with the net exposures $(\mathbf{1}^\top (C+R))^\top - (C+R)^\top \mathbf{1}$ in the rebalanced network.

[1] The derivative contracts are comparable in the sense that they have the same fundamental characteristics such as maturity and underlying. CDS contracts have been standardised in terms of coupons and maturity dates, therefore trade positions on theses contracts can be bucketed by the reference entity (single-name or index) and maturity.

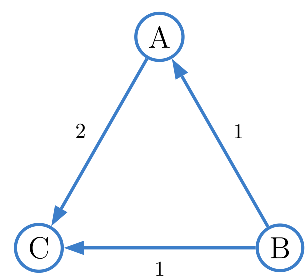
[2] Portfolio rebalancing is described in ESMA (2020a) as "new transactions are entered into to reduce counterparty risk by reducing the exposure between two counterparties and this is viewed by market participants as a way to also reduce systemic risk by decreasing the overall exposure in the market between counterparties".



(a) Starting from a notional matrix C and injecting rebalancing transactions characterised by matrix R results in the rebalanced notional matrix $C + R$.



(b) Initial transactions on a net basis as in C^{bi}.



(c) Final transactions on a net basis as in $(C + R)^{bi}$.

Figure 1: Example of portfolio rebalancing in a network of 3 institutions.

The gross notional did increase because of the non-negativity of R. The sum of "bilateral" net exposures computed by $\sum_i \sum_j |C_{ji} - C_{ij}|$, however, decreased from 20 to 8.

Hence, here we see that portfolio rebalancing keeps the net exposures unchanged and reduces the overall exposure in the network by reducing exposure between counterparties.

Now, we define portfolio rebalancing formally. We start by defining a rebalancing set, which characterises which banks take part in the rebalancing exercise, and a rebalancing matrix, which characterises the amount of trades that can be injected.

Definition 2.3 (Rebalancing set and rebalancing matrix)**.** We refer to the set $\mathcal{B} \neq \emptyset$, $\mathcal{B} \subseteq \mathcal{N}$, that represents all banks participating in the rebalancing, as *rebalancing set.* We refer to the matrix $R \in [0, \infty)^{N \times N}$ as *rebalancing matrix (with respect to $\mathcal{B}$)* if

$$\sum_{j \in \mathcal{B}} R_{ji} = \sum_{j \in \mathcal{B}} R_{ij} \quad \forall\, i \in \mathcal{B}, \tag{1}$$

$$R_{ij} = 0 \quad \forall\, (i, j) \notin \mathcal{B} \times \mathcal{B}. \tag{2}$$

The conditions (1) and (2) guarantee that the net position of the matrix R is zero, i.e., $(\mathbf{1}^\top R)^\top - R\mathbf{1} = \mathbf{0}$, where $\mathbf{0}$ is the column vector containing only 0s. Therefore, the net positions of the matrix C and of the matrix $C + R$ coincide.

We will often choose rebalancing matrices that satisfy additionally $R_{ij} \cdot R_{ji} = 0\ \forall\, i, j \in \mathcal{B}$ to reflect the fact that adding transactions in both directions is not necessary for rebalancing. However, our results do not rely on this feature.

Definition 2.4 ($\mathcal{B}$-rebalancing exercise in the weak sense)**.** Let C be a notional matrix, let $\mathcal{B}$ be a rebalancing set and let R be a rebalancing matrix with respect to $\mathcal{B}$. We refer to the triple $(C, \mathcal{B}, R)$ as a *$\mathcal{B}$-rebalancing exercise in the weak sense.* We refer to $C^{\mathcal{B}}$ where $C^{\mathcal{B}} = C + R$, as *$\mathcal{B}$-rebalanced notional matrix.*

Any rebalancing exercise in the weak sense can be achieved by a sequence of elementary rebalancing exercises.

Definition 2.5 (Elementary rebalancing)**.** Let $(C, \mathcal{B}, R)$ be a *$\mathcal{B}$-rebalancing exercise in the weak sense.* If the graph associated with R contains exactly one cycle, then $(C, \mathcal{B}, R)$ represents *elementary rebalancing*, and we denote the capacity by τ, where $\tau = \max_{i,j \in \mathcal{B}} R_{ij} > 0$.

Proposition 2.6. Let $(C, \mathcal{B}, R)$ be a rebalancing exercise in the weak sense. Then, there exist K elementary rebalancing exercises $(C^{(k)}, \mathcal{B}^{(k)}, R^{(k)})$ in the weak sense, where $C^{(1)} = C$, $C^{(k)} = C^{(k-1)} + R^{(k-1)}$ for $k \in \{2, 3, \ldots, K\}$, and $\mathcal{B}^{(k)} \subseteq \mathcal{B}$ for all k, such that

$$R = \sum_{k=1}^{K} R^{(k)}, \text{ and } C^{\mathcal{B}} = C^{(K)} + R^{(K)}.$$

The proof of Proposition 2.6 and subsequent results can be found in Appendix A unless indicated otherwise.

The example provided in Figure 1 is an example of an elementary rebalancing exercise. In Figure 2, we demonstrate the example from TriOptima given in (ESMA, 2020b, Annex 1), which is an example of a rebalancing exercise that is not elementary.

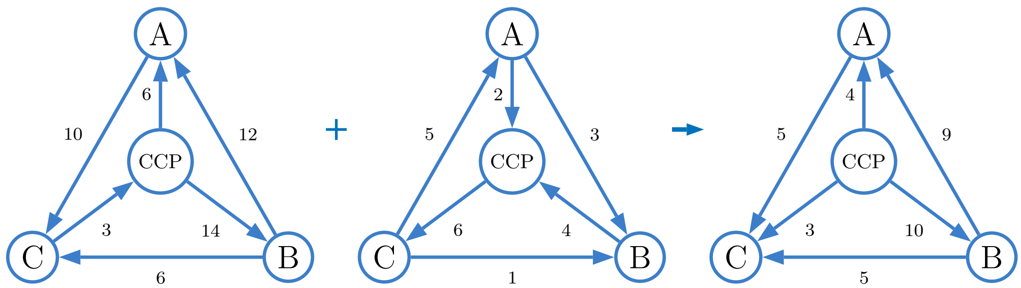


Figure 2: Portfolio rebalancing in which the CCP is treated as a counterparty as in (ESMA, 2020b, Annex 1). The portfolio positions in the final network are bilaterally netted, i.e., it represents $(C+R)^{bi}$.

In practice, portfolio rebalancing is done to reduce the bilateral exposures either for each pair of counterparties or across the system. One possible objective for rebalancing would be to inject trades such that the bilaterally netted position for each pair in the rebalancing set is less or equal than the bilaterally netted position in the original network, i.e.,

$$\left|C_{ij}^{\mathcal{B}} - C_{ji}^{\mathcal{B}}\right| \leq |C_{ij} - C_{ji}|, \quad \forall\, i,j \in \mathcal{B}. \tag{3}$$

Therefore, condition (3) guarantees that net exposures cannot increase after rebalancing for every pair of participants. This is consistent with the qualitative requirement formulated in (ESMA, 2020a, Section 8.2.5) that "no participant to the PTRR service should be worse off for the transactions included in the PTRR exercise than if the PTRR exercise had not taken place." Furthermore, condition (3) implies that

$$\sum_{j\in\mathcal{B}} (C^{\mathcal{B}})_{ij}^{bi} \leq \sum_{j\in\mathcal{B}} C_{ij}^{bi}, \quad \forall\, i \in \mathcal{B}, \tag{4}$$

i.e., the overall gross position in the bilaterally netted network does not increase with rebalancing. Condition (4) could also be used as an objective for portfolio rebalancing, but it is less strict than condition (3). In this paper, we are primarily interested in rebalancing exercises (in the weak sense) that satisfy condition (3) or condition (4) and we call them *$\mathcal{B}$-rebalancing exercises*. It would be possible to add more restrictions on the individual positions. We will see, however, in Section 4.1, that several results on the implications of rebalancing for systemic risk already hold for rebalancing exercises in the weak sense.

Next, we introduce the notion of *optimal rebalancing*. We formulate an optimisation problem in which we determine a rebalancing matrix such that the notional amounts in the bilaterally netted rebalanced network are minimised.

Definition 2.7 (Optimal rebalancing)**.** Let C be a notional matrix and let $\mathcal{B}$ be a rebalancing set. We refer to the optimisation problem

$$\min_{R_{ij}} \quad \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}(C+R)^{bi}_{ij} \tag{5}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{B}} R_{ji} = \sum_{j\in\mathcal{B}} R_{ij} \qquad \forall\, i \in \mathcal{B}, \tag{6}$$

$$R_{ij} \geq 0 \qquad \forall\, i,j \in \mathcal{B}, \tag{7}$$

as *optimal rebalancing problem.*

Let $\hat{R}^* \in [0,\infty)^{|\mathcal{B}|\times|\mathcal{B}|}$ be a solution to the optimal rebalancing problem and define a matrix $R^* \in [0,\infty)^{N\times N}$ as $R^*_{\mathcal{B}\mathcal{B}} = \hat{R}^*$ and $R^*_{ij} = 0$ for all $(i,j) \notin \mathcal{B}\times\mathcal{B}$. Then we refer to the triple $(C,\mathcal{B},R^*)$ as an *optimal $\mathcal{B}$-rebalancing exercise.*

The next theorem establishes that an optimal rebalancing exercise always exists.

Theorem 2.8. Let C be a notional matrix and let $\mathcal{B}$ be a rebalancing set. Then, there exists an optimal $\mathcal{B}$-rebalancing exercise.

In Appendix B, we will show how the optimal rebalancing problem is related to the non-conservative compression problem introduced in D'Errico & Roukny (2021), which corresponds to L_1-compression introduced in O'Kane (2017) . We will also see that a solution to the optimal rebalancing problem in Definition 2.7 is often not unique. All the results related to optimal rebalancing in this paper will not be affected by multiple solutions, unless stated otherwise.

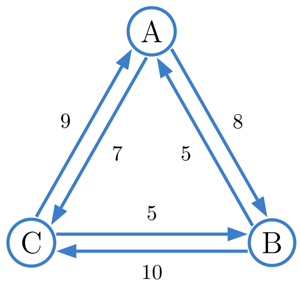
2.3 Portfolio compression

There is another PTRR service called portfolio compression. Compression techniques are offered by some third-party service providers, such as CME Group's TriOptima, and the compression algorithms in reality are proprietary.[3] The aim of portfolio compression is to reduce gross notional exposures while keeping net exposures unchanged.[4] Figure 3 shows a simplified example illustrating the idea of compression. We see that compressing multilaterally is advantageous in terms of the reduction of outstanding notional amount. Portfolio compression is used in both non-centrally

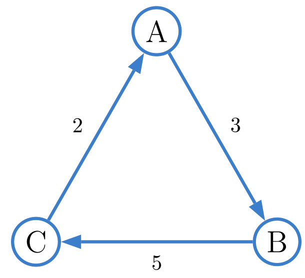
[3]Provisions in Article 17 of Commission delegated regulation (EU) 2017/567 states the conditions that portfolio compression service providers should accomplish, including agreements on legal effects and elements in compression proposal, see https://eur-lex.europa.eu/eli/reg_del/2017/567/oj. Essentially, portfolio compression consists of three steps. First, counterparties that decide to participate submit their portfolios information and specify risk tolerance. Second, the compression service provider runs their compression algorithm and publishes a compression proposal. Third, once all participants receive the proposal they may or may not agree to the proposal, and the compression is complete only if all participants accept it.

[4]ESMA (2020b) describes the main purpose of portfolio compression as "In principle to reduce notional amount outstanding and the number of transactions."

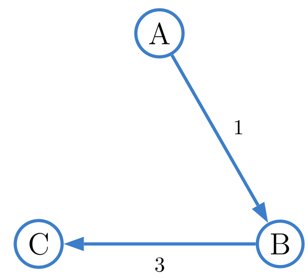
cleared and in centrally cleared markets.[5] In this paper, however, we do not consider compression by CCPs and we only focus on portfolio compression in OTC (not centrally cleared) markets.



(a) Non-compressed.



(b) Bilateral compression.



(c) Multilateral compression.

Figure 3: Example of portfolio compression.

Now we consider the formal definition of portfolio compression that is undertaken on closed chains of claims.[6]

Definition 2.9. Let C be a notional matrix and let C^{bi} be the bilaterally netted notional matrix with respect to C according to Definition 2.1. A *compression network cycle* $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges})$ is a directed cycle in the graph associated with C^{bi}. We let $\mu^{\max} = \min_{(i,j)\in\mathcal{C}_{edges}} C^{bi}_{ij} > 0$ be the maximal capacity of the compression network cycle. For any $0 < \mu \le \mu^{\max}$, we define the matrix $C^{\mathcal{C},\mu}$ with

$$C^{\mathcal{C},\mu}_{ij} = \begin{cases} C_{ij} - \mu, & \text{if } (i,j) \in \mathcal{C}_{edges}, \\ C_{ij}, & \text{otherwise,} \end{cases}$$

as the μ-compressed notional matrix using cycle $\mathcal{C}$.

Intuitively, compression is removing economically redundant and offsetting contracts along cycles, therefore reducing gross notional volumes. According to Definition 2.9, gross notional exposures are reduced by μ times the length of compression network cycle $\mathcal{C}$, and the net exposures are maintained.[7]

[5]The European Association of CCP Clearing Houses (EACH) explains compression by CCPs as "a very common practice within CCPs and happens automatically, as participants need to specifically opt out to avoid compression. [...] Compression cycles should be performed independently by CCPs - which perform portfolio compression as part of their netting activities - or other providers for the bilateral space", EACH (2020). Notwithstanding the similarity in multilateral netting, we are aware that in the response to the ESMA PTRR consultation paper (Question 2), EBF & ISDA (2020) distinguish multilateral netting from compression by CCPs.

[6]We use similar notations as in Veraart (2020b), but the maximal capacity $\mu^{\max}$ and the compression network cycle are determined by the bilaterally netted notional matrix. The type of compression we consider in this paper is related to the *conservative compression* in D'Errico & Roukny (2021), we discuss this in more detail in Appendix B. We also present the *non-conservative compression* considered in D'Errico & Roukny (2021) in Appendix B. Hereafter, we mean "conservative" compression when we discuss portfolio compression, unless specified otherwise.

[7]The compressed notional matrix $C^{\mathcal{C},\mu}$ is not necessarily an "optimal" result such as the outcome of the optimisation problem described in Appendix B (see Definition B.2), because we can also do bilateral compression even if we take $\mu = \mu^{\max}$. Since we are considering variation margin payments which are bilaterally netted, it makes no difference whether we compress bilaterally or not in Definition 2.9. However, it should be noted that partial termination of contracts may be unfavourable with operational costs taken into consideration.

Our definition of compression builds on the idea of O'Kane (2017) that proposes a loop compression algorithm by finding and eliminating all closed loops on net exposures matrix. We define portfolio compression in the sense that *counterparties do not want to increase their credit exposures to each other*. This is stated in a patent of TriOptima, see Brouwer (2013), in which there is an example using exposure matrix for bilaterally netted positions. In our analysis, we focus on compression along one closed loop only, but we provide a more general optimisation-based definition in Appendix B.

Remark 2.10. From Definition 2.9, on a net basis, we can think of portfolio compression as a special case of an elementary rebalancing exercise. One of the main differences is that the former reduces notional amount while the latter increases notional amount.

2.4 General comparison between portfolio rebalancing and portfolio compression

Portfolio rebalancing and portfolio compression are fundamentally different services as can be seen from Table 1 which compares some of their key characteristics. In particular, portfolio rebalancing is a mechanism that can only add new trades whereas portfolio compression can both terminate trades and replace old trades with new trades that often have less notional, ESMA (2020b).

	Portfolio Compression	Portfolio Rebalancing
Main objective	Reduce notional amount.	Reduce counterparty risk.
Mechanism	Early termination and/or replacement.	Inject offsetting trades with equal amounts of buy and sell exposures.
Notional size	Decreases.	Increases.

Table 1: Comparison of the two PTRR services based on the summary in (ESMA, 2020b, p. 14).

Portfolio compression shows strengths mainly in reducing complexities by (partially) terminating trades and replacing those with new trades, which reduces aggregate notional size and operational risk in particular.

For portfolio rebalancing, there are concerns that a build-up of transactions would add operational burdens.[8] However, several suggestions have been made to mitigate these operational burdens. First, transactions can be compressed bilaterally. Although portfolio compression is performed by a third party in general which is not a counterparty to any involved compression participants, "the process of portfolio compression, and primarily on a bilateral basis, is something that counterparties to a derivative contract can and currently do themselves", ESMA (2020b).[9] Second,

[8]In the view of the ESMA, "the benefits reducing market risks in the non-cleared netting sets outweigh, inter alia, the increased potential operational burden on market participants and regulators, the increase in gross risk in the non-cleared netting sets (in case of portfolio rebalancing)", ESMA (2020b).

[9]The ISDA 2017 Portfolio Compression Agreement is a template for bilateral counterparties to compress their existing trades where both parties can act as Provider, see https://www.isda.org/2017/11/28/

it is suggested that in order to alleviate the operational pressure and potential risk caused by build-up of trades, rebalancing transactions are compressed at a regular basis through the rebalancing exercise.[10]

Despite key differences between portfolio rebalancing and portfolio compression, we will see that there are still significant similarities when considering these services from the perspective of systemic risk. A netted payment arising from a rebalanced portfolio can coincide with a netted payment arising from a compressed portfolio. In this paper, we will consider compression or re-balancing exercise conducted under the International Swaps and Derivatives Association (ISDA) Master Agreement. This implies that we could consider netted variation margin payments that arise from the different types of outstanding derivatives positions without changing our analysis significantly. We will look into the mechanism of quantifying systemic risk in more detail next.

3 Assessing systemic risk

We will now describe how we measure systemic risk in financial networks and how we quantify the effect of PTRR services on systemic risk.

We assume that due to the outstanding liabilities from derivative positions modelled by the notional matrix C, payments obligations become due in the form of variation margin calls[11] if the market conditions change. For instance, consider the case of American International Group, Inc. (AIG) in the CDS market during the financial crisis. The protection seller faced great pressure on the margin calls from the protection buyers after a sudden shock to the credit markets. In the following, we will only consider the liabilities matrix in Definition 3.1 to account for the changes in the notional amount of positions.

Definition 3.1. Let C be a notional matrix and let $\mathcal{B}$ be a rebalancing set.

- We define the associated *payment liabilities matrix* L by $L = \psi \cdot C^{bi}$, where $\psi \in (0, \infty)$ and C^{bi} is the bilaterally netted notional matrix with respect to C.

isda-2017-portfolio-compression-agreement/. In addition, in the response to ESMA's consultation paper on MI-FID II and MIFIR, see ESMA (2014), it is mentioned that "compression is sometimes performed without reduction in the notional value of the portfolio but for the purpose of simplification. Portfolio compression can be used to aggregate contracts into fewer contracts without reduction of the notional amount. The purpose of this exercise could be to standardise the coupons and coupons period, to make them eligible for clearing or to facilitate the management of the contract." Even though such process without notional reduction was not considered to be in the scope of compression, its economic benefits cannot be overlooked.

[10] "ESMA notes that the market seems to self-regulate this concern as it is in the interest of the participants (particularly rebalancing conducted with more complex products that contain expiry features) to constantly compress such trades [...]. It is also in the interest of the providers to deliver efficient PTRR services and to continuously compress the rebalancing trades", ESMA (2020b).

[11] "Variation margin protects the transacting parties from the current exposure that has already been incurred by one of the parties from changes in the mark-to-market value of the contract after the transaction has been executed. The amount of variation margin reflects the size of this current exposure. It depends on the mark-to-market value of the derivatives at any point in time, and can therefore change over time", BCBS & IOSCO (2020). Hence, it is reasonable to assume that variation margins are proportional to exposures and we use liabilities matrix to model margin requirements.

- We define the associated $\mathcal{B}$*-rebalanced payment liabilities matrix* $L^{\mathcal{B}}$ by $L^{\mathcal{B}} = \psi \cdot (C^{\mathcal{B}})^{bi}$, where $(C^{\mathcal{B}})^{bi}$ is the bilaterally netted notional matrix with respect to the $\mathcal{B}$-rebalanced notional matrix $C^{\mathcal{B}}$.

- We refer to $\bar{L} \in [0,\infty)^N$, where $\bar{L}_i = \sum_{j=1}^N L_{ij}$, as the *total payment liabilities*. We refer to $\bar{A} \in [0,\infty)^N$, where $\bar{A}_i = \sum_{j=1}^N L_{ji}$, as *total interbank assets*.

- We refer to $\bar{L}^{\mathcal{B}} \in [0,\infty)^N$, where $\bar{L}_i^{\mathcal{B}} = \sum_{j=1}^N L_{ij}^{\mathcal{B}}$, as the *total* $\mathcal{B}$*-rebalanced payment liabilities.*

- We consider $A^e \in [0,\infty)^N$, where $A_i^e \geq 0$ represents the *external assets* that bank i holds from outside the network.

- We refer to the pair (L, A^e) as *financial network* and to the pair $(L^{\mathcal{B}}, A^e)$ as $\mathcal{B}$*-rebalanced financial network.*

In the following we will measure systemic risk by considering a quantity referred to as re-evaluated equity, introduced in Veraart (2020a) and considered in the context of portfolio compression in Veraart (2020b), which intuitively represents the difference between actual assets received and nominal payments due and it is a generalisation of the clearing mechanism introduced in Eisenberg & Noe (2001) and Rogers & Veraart (2013). The actual assets that each bank have, depend on the payments that all banks in the network are able to make. Therefore, the re-evaluated equity is characterised as a fixed point. We use the same notation as in Veraart (2020a,b) but apply it in the context of portfolio rebalancing.

Definition 3.2 (Re-evaluated equity)**.** 1. A valuation function $\mathbb{V} : \mathbb{R} \to [0,1]$ is defined by

$$\mathbb{V}(y) = \begin{cases} 1, & \text{if } y \geq 1 + k, \\ r(y), & \text{if } y < 1 + k, \end{cases} \tag{8}$$

where $k \geq 0$ and $r : (-\infty, 1+k) \to [0,1]$ is a non-decreasing and right-continuous function.

2. Consider a financial network (L, A^e) and a valuation function $\mathbb{V}$. Let $\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$ and $\mathcal{E} = [-\bar{L}, A^e + \bar{A} - \bar{L}]$. We define a function $\Phi = \Phi(\cdot; \mathbb{V}) : \mathcal{E} \to \mathcal{E}$, where

$$\Phi_i(E) = \Phi_i(E; \mathbb{V}) = A_i^e + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \quad \forall i \in \mathcal{N}. \tag{9}$$

We refer to a vector $E \in \mathcal{E}$ satisfying

$$E = \Phi(E) \tag{10}$$

as *re-evaluated equity in the original network.*

3. Consider a $\mathcal{B}$-rebalanced financial network $(L^{\mathcal{B}}, A^e)$ and a valuation function $\mathbb{V}$. Let $\mathcal{M}^{\mathcal{B}} = \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{B}} > 0\}$ and $\mathcal{E}^{\mathcal{B}} = [-\bar{L}^{\mathcal{B}}, A^e + \bar{A}^{\mathcal{B}} - \bar{L}^{\mathcal{B}}]$, where $\bar{A}_i^{\mathcal{B}} = \sum_{j=1}^N L_{ji}^{\mathcal{B}}$ for all $i \in \mathcal{N}$. We

define a function $\Phi^{\mathcal{B}} = \Phi^{\mathcal{B}}(\cdot; \mathbb{V}) : \mathcal{E}^{\mathcal{B}} \to \mathcal{E}^{\mathcal{B}}$, where

$$\Phi_i^{\mathcal{B}}(E) = \Phi_i^{\mathcal{B}}(E; \mathbb{V}) = A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}} \mathbb{V}\left(\frac{E_j + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \bar{L}_i^{\mathcal{B}}, \quad \forall\, i \in \mathcal{N}. \tag{11}$$

We refer to a vector $E \in \mathcal{E}^{\mathcal{B}}$ satisfying

$$E = \Phi^{\mathcal{B}}(E) \tag{12}$$

as *re-evaluated equity in the $\mathcal{B}$-rebalanced network.*

As discussed in Veraart (2020a,b) several financial contagion models can be recovered from the definition of the re-evaluated equity by considering special valuation functions $\mathbb{V}$. In particular, it is shown in Veraart (2020a) that clearing vector in the Eisenberg & Noe (2001) model can be recovered from the re-evaluated equity in distress contagion model if $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, and vice versa, where $k = 0$ and

$$\mathbb{V}^{\mathrm{EN}}(y) = 1 \wedge y^+.$$

Furthermore, if we set $k = 0$ and

$$\mathbb{V}^{\mathrm{RV}}(y) = \begin{cases} 1, & \text{if } y \geq 1, \\ \beta y^+, & \text{if } y < 1, \end{cases}$$

then this corresponds to the Rogers & Veraart (2013) model with default cost parameters $\alpha = \beta \in [0, 1]$. For $k \geq 0$, the zero recovery rate valuation function is defined by Veraart (2020b) as

$$\mathbb{V}^{\mathrm{zero}}(y) = \mathbb{1}_{\{y \geq 1+k\}}.$$

Note that the valuation function $\mathbb{V}$ is non-decreasing and right-continuous. Hence, by Tarksi's fixed point theorem (see (Tarski, 1955, Theorem 1)), the re-evaluated equity in (10) and (12) exist as a fixed point solution of Φ and $\Phi^{\mathcal{B}}$ given by (9) and (11) respectively, see also Veraart (2020a). Unless stated otherwise, we will consider the greatest fixed point of Φ and $\Phi^{\mathcal{B}}$, which gives us greatest re-evaluated equity in the original and an $\mathcal{B}$-rebalanced network respectively.

We use (L, A^e) to denote a financial network with liabilities matrix L and external assets A^e. We will use a valuation function $\mathbb{V}$ to assess systemic risk associated with the financial networks, and we will also refer to the triple $(L, A^e; \mathbb{V})$ as a *financial network.*

We define default, reduction and harmfulness of systemic risk as in Veraart (2020b) but we consider rebalancing here.

Definition 3.3. Let E^* be the greatest re-evaluated equity in the original financial network $(L, A^e; \mathbb{V})$ with total liabilities $\bar{L}$. Let $E^{\mathcal{B};*}$ be the greatest re-evaluated equity in the $\mathcal{B}$-rebalanced financial network $(L^{\mathcal{B}}, A^e; \mathbb{V})$ with total liabilities $\bar{L}^{\mathcal{B}}$. Then, the default set in the original network is given by

$$\mathcal{D}(L, A^e; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^* < 0\}, \tag{13}$$

and the default set in the $\mathcal{B}$-rebalanced network is given by

$$\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^{\mathcal{B};*} < 0\}. \tag{14}$$

Furthermore, we say that $\mathcal{B}$-rebalancing *reduces systemic risk* if $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) \subseteq \mathcal{D}(L, A^e; \mathbb{V})$. We say that $\mathcal{B}$-rebalancing *strongly reduces systemic risk* if $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) \subsetneq \mathcal{D}(L, A^e; \mathbb{V})$. We say that $\mathcal{B}$-rebalancing is *harmful* if $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) \setminus \mathcal{D}(L, A^e; \mathbb{V}) \neq \emptyset$.

We analyse the effects of PTRR services from an ex-post point of view. We assume that portfolio compression or rebalancing occurs *prior to* the shock. We consider both the original financial network and a rebalanced (or compressed) financial network and we apply the same shock to both networks and we compare the outcomes.

Hence, we see that the concept of a strong reduction of systemic risk can be interpreted as a Pareto improvement. In particular, if rebalancing strongly reduces systemic risk, this means that it does not cause any additional defaults and at least one bank that defaults in the original financial network no longer defaults in the $\mathcal{B}$-rebalanced network. A $\mathcal{B}$-rebalancing exercise reduces systemic risk if all banks that default in the $\mathcal{B}$-rebalanced network also default in the original network. The rebalancing is said to be *harmful* if there is at least one bank defaults in the rebalanced network, which would not have defaulted in the original network.

Note that Veraart (2020b) define the default set in the financial network $(L, A^e; \mathbb{V})$ as

$$\mathcal{D}(L, A^e; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^* < k\bar{L}_i\},$$

where $k \geq 0$ coincides with that used in the valuation function $\mathbb{V}$ defined by (8), while we set $k = 0$ in the default sets in Definition 3.3. In principle, as discussed in (Veraart, 2020b, Remark A.1), a PTRR service that reduces systemic risk has the benefit of moving participants further away from the default boundary by allowing $k > 0$. In this paper, banks in default are treated as illiquid banks because we are modelling *illiquidity* rather than insolvency. The liabilities are particularly in the form of daily variation margins due, and the assets are short-term assets which do not account for bank's full balance sheet. In view of this, hereafter, *we assume that* $k = 0$ *in the valuation function and default set.*

In Definition 3.1, we use a parameter ψ to account for credit exposures, which by our assumption are proportional to the notional exposures. The parameter can be interpreted as quantifying the degree of market volatility, so a larger ψ stands for a more volatile financial market with higher magnitude of variation margin calls.

Proposition 3.4. Let C be a notional matrix. Let $\psi_1, \psi_2 \in (0, \infty)$ be two parameters and let $(L^{(r)}, A^e; \mathbb{V})$ be the financial network where $L^{(r)} = \psi_r \cdot C^{bi}$ for $r \in \{1, 2\}$. Let E^1 and E^2 be the greatest re-evaluated equities in the financial networks $(L^{(1)}, A^e; \mathbb{V})$ and $(L^{(2)}, A^e; \mathbb{V})$ respectively.

Suppose that

$$\psi_1 > \psi_2. \tag{15}$$

Then,

$$\frac{E^2}{\psi_2} \geq \frac{E^1}{\psi_1}. \tag{16}$$

In particular,

1. if $E_i^1 \geq 0$ for some $i \in \mathcal{N}$, then $E_i^2 \geq 0$,
2. if $E_i^2 < 0$ for some $i \in \mathcal{N}$, then $\left|\frac{E_i^2}{E_i^1}\right| \leq \frac{\psi_2}{\psi_1}$,
3. $\mathcal{D}(L^{(2)}, A^e; \mathbb{V}) \subseteq \mathcal{D}(L^{(1)}, A^e; \mathbb{V})$.

Proposition 3.4 shows some ordering results with regards to the re-evaluated equities in financial networks with different levels of volatility. If bank i does not default in the market with high volatility, then it cannot default in the market with low volatility provided the same external assets. As a consequence, if bank i is in default in the mild market that is not so volatile corresponding to the smaller parameter ψ_2, then it would also default in the market with $\psi_1 > \psi_2$. Furthermore, the liquidity shortfall in the less severe scenario has an upper bound, which is given by $|E_i^2| \leq \frac{\psi_2}{\psi_1}|E_i^1|$.

4 Effects of PTRR services on systemic risk

In this section, we first discuss the main theoretical results on when portfolio rebalancing reduces systemic risk using the framework described in the previous section. Then, we investigate the consequences of PTRR services under a worst case response of bank's to liquidity stress.

4.1 Systemic risk reduction for portfolio rebalancing

4.1.1 The role of defaults among banks conducting rebalancing

In the following we will derive two conditions which are both individually sufficient for portfolio rebalancing to reduce systemic risk. They are consistent with the conditions derived in Veraart (2020b) for the case of compression.

The first condition assumes that no bank participating in rebalancing defaults in the original network.

Theorem 4.1. Consider a $\mathcal{B}$-rebalancing exercise in the weak sense and suppose that there exists no default in $\mathcal{B}$ in the original network. Then, for every bank, the re-evaluated equities with and without rebalancing coincide. In particular, this rebalancing exercise reduces systemic risk.

The theorem says that if there is no bank in the rebalancing set that default in the original financial network, then rebalancing cannot be harmful. As a result, high-quality banks with low probability of default can be encouraged to participate in order to reduce counterparty risk and also reduce systemic risk.

An interesting implication is that the result applies to any rebalancing exercise in the weak sense, even if the exercise increases counterparty exposure for every participant strictly. For example, think about the "opposite" portfolio compression, that is increasing notional amount along a cycle with the same value. Although systemic risk reduction in this case hardly make sense, we get the insight that adding offsetting transactions might not contribute to risk contagion as long as the participants are capable of resisting the shock.

The second condition assumes that no participating bank defaults in the rebalanced network.

Proposition 4.2. Consider a $\mathcal{B}$-rebalancing exercise in the weak sense and suppose that there exists no default in $\mathcal{B}$ in the rebalanced network. Then, for every bank, the re-evaluated equities with and without rebalancing coincide. In particular, this rebalancing exercise reduces systemic risk.

If there is no default in the rebalancing set in the network after rebalancing, then banks outside the rebalancing set would not be hurt by the rebalancing exercise. In other words, no defaulting banks in the rebalancing set in the rebalanced network guarantees that banks not participate in rebalancing would not default if they do not default in the network before rebalancing. Therefore, the situation after rebalancing cannot be worse and systemic risk is reduced if the condition in Proposition 4.2 is satisfied.

This result also shows that if the $\mathcal{B}$-rebalancing is harmful, then in the rebalancing set there should be at least one defaulting bank not only in the original network but also in the rebalanced network.

Remark 4.3. Examples in Appendix C imply that conditions in Theorem 4.1 and Proposition 4.2 for systemic risk reduction are neither necessary nor redundant.

Remark 4.4 (Generalisation of the results to a a market with fire sales)**.** So far we have assumed that banks hold external assets A^e which are liquid, such as cash, but we may add a fire sales contagion mechanism in the spirit of Cifuentes et al. (2005). We can extend the results in Veraart (2020b) and this section to account for liquidation costs and give sufficient conditions for asset price in the compressed or rebalanced network to be equal to asset price in the original network. We introduce the mathematical framework for contagion model with fire sales and provide theoretical results as an extension of Section 4.1 in Appendix D.

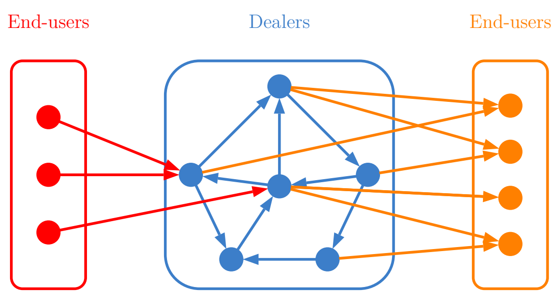
In fact, the effects of PTRR services on the asset price are consistent with the results for the re-evaluated equity without price impact. For both types of PTRR services, no defaults in participants in the original or compressed (resp. rebalanced) network is a sufficient condition for the equilibrium asset prices in the compressed (resp. rebalanced) and the original financial network to coincide. Hence, the results for the first contagion channel carry over to our analysis with the second contagion channel, namely fire sales. This means that portfolio compression and rebalancing have a positive influence on realising liquidity and constraining illiquidity spirals as long as participating banks do not get into trouble. The new assumption is stronger than those in Theorem 4.1 or Proposition 4.2, because it requires that participating banks also withstand fire sales risk in order to achieve systemic risk reduction. Hence, it is important that PTRR services participants (e.g. large dealers) retain enough liquidity buffers.

4.1.2 The role of optimal rebalancing and zero recovery rates

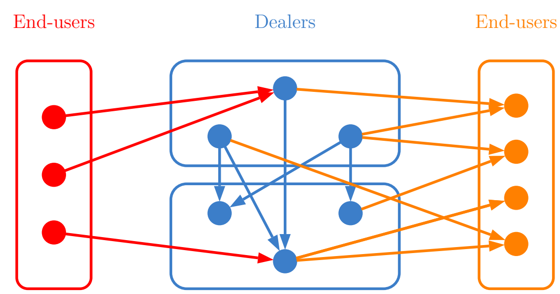
We first distinguish between fundamental default and contagious default in Definition A.2. We refer to a bank as in *fundamental default* if its external assets plus total interbank assets are not enough to cover its total payment liabilities, i.e., it cannot pay its liabilities in full even if we assume it receives all payments from its counterparties. A defaulting bank that is not in fundamental default is said to be in *contagious default*.

The following proposition provides a result on the type of default in a financial network in which all banks take part in optimal rebalancing.

Proposition 4.5. Consider an optimal $\mathcal{B}$-rebalancing exercise and suppose $\mathcal{B} = \mathcal{N}$. Then, all the defaulting banks in the rebalanced financial network are in fundamental default. Moreover, this rebalancing exercise reduces systemic risk.



(a) VM flows in the original network.



(b) VM flows in the optimally rebalanced network.

Figure 4: Stylised networks of VM flows in the CDS market. The direction of arrows represents selling protection.

Proposition 4.5 shows potential contribution of rebalancing, especially optimal rebalancing, to establishing resilience of the system. We highlight the outstanding feature using an example of CDS market. Figure 4 provides a stylised comparison between VM payment networks with and without (optimal) rebalancing. The direction of the arrows indicates net VM owed between counterparties because of mark-to-market devaluations. In Figure 4a, we can recognise feedback loops in the dealers section that may amplify default cascades, whereas they disappear in Figure 4b owing to rebalancing that breaks up the contagion channel.

As shown in (Veraart, 2020b, Proposition 4.11), zero recovery rates are sufficient for portfolio compression to reduce systemic risk. However, this conclusion does not hold for $\mathcal{B}$-rebalancing exercise in general. There we need the additional assumptions that the rebalancing is optimal and there are no defaults among the banks in $\mathcal{B}$ in the original network that have a negative net credit exposure.

Proposition 4.6. Let $(L, A^e; \mathbb{V}^{\text{zero}})$ be a financial network with zero recovery rates and let $\mathcal{B}$ be a rebalancing set.

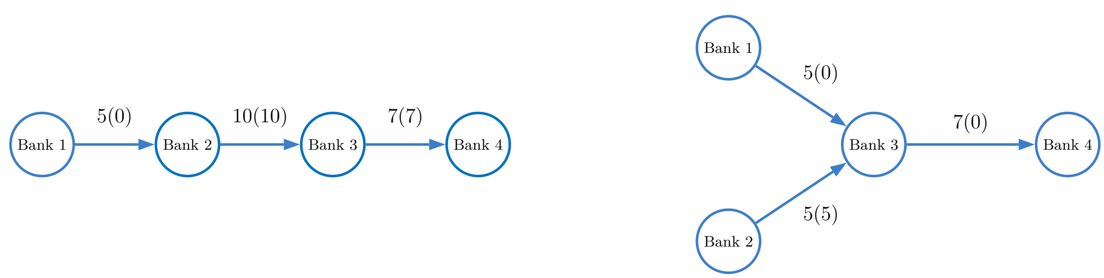
1. Let the $\mathcal{B}$-rebalancing exercise be optimal. Suppose

$$\left\{ i \in \mathcal{B} \,\middle|\, \sum_{j \in \mathcal{B}} (L_{ji} - L_{ij}) < 0 \right\} \cap \mathcal{D}(L, A^e; \mathbb{V}^{\text{zero}}) = \emptyset. \tag{17}$$

   Then, for every bank, the re-evaluated equity with rebalancing is greater or equal than the re-evaluated equity without rebalancing. In particular, this rebalancing exercise reduces systemic risk.

2. There exists a harmful $\mathcal{B}$-rebalancing exercise for which condition (17) in part 1 is satisfied.

In the following we provide an example that provides more intuition on the zero-recovery rate case. We consider the Rogers & Veraart (2013) model, i.e., $\mathbb{V} = \mathbb{V}^{\text{RV}}$ with $\alpha = \beta = 0$, to measure systemic risk. Our example network consists of four financial institutions. We consider an optimal rebalancing and assume that the rebalancing set is given by $\mathcal{B} = \{1, 2, 3\}$. The clearing payments made in the original and in the $\mathcal{B}$-rebalanced network are shown in Figure 5. The numbers in brackets represent the liabilities actually paid. Furthermore, here $\{i \in \mathcal{B} \,|\, \sum_{j \in \mathcal{B}} (L_{ji} - L_{ij}) < 0\} \cap \mathcal{D}(L, A^e; \mathbb{V}^{\text{zero}}) = \{1\} \neq \emptyset$, meaning the additional condition required in Proposition 4.6 is not satisfied.



(a) Original network. (b) Rebalanced network.

Figure 5: $\mathbb{V} = \mathbb{V}^{\text{zero}}$. $A^e = (1, 10, 1, 0)^\top$. Rebalancing is harmful.

In the original network, bank 1 is the only bank in fundamental default, but it turns out that this does not trigger any further contagious defaults. Even though bank 2 receives 0 payments from bank 1, bank 2 can still repay its debt to bank 3 in full since it has sufficient external assets. Then also bank 3 can make all its required payments to bank 4.

In the $\mathcal{B}$-rebalanced network, bank 1 remains the only node in fundamental default, but this fundamental default now triggers the contagious default of bank 3. This is due to the fact that bank 1 is now facing bank 3 directly and repays 0 rather than 5 to bank 3. Even though bank 2 repays its debt in full to bank 3, bank 3 does not have enough external assets to survive the direct shock coming from bank 1 and defaults on its payments to bank 4.

We find that in this case bank 2 is incentivised to participate because it bears no loss from bank 1 after rebalancing, but bank 3 would not be willing to accept the outcome if it knew that bank 1 would not be able to satisfy its payment obligations.

There are two main reasons why a zero recovery rate is not sufficient to reduce systemic risk for portfolio rebalancing. First, rebalancing can change which counterparties the risk is shared with each other. For compression, original counterparty relationships are preserved by our assumption. Therefore, as we have seen losses from bank 1 are directly transmitted to bank 3 in the rebalanced network, whereas prior to rebalancing there was no direct counterparty relationship between bank 1 and bank 3. Second, optimal rebalancing aims to reduce counterparty exposure in a way that results in a network with less intermediation. In the original network, the external assets of bank 2 act as a buffer for bank 3 to mitigate the direct impact of losses coming from bank 1. This benefit of the intermediation chain, disappears after rebalancing.

4.2 The worst case scenario with strategic response to liquidity stress

In this section, we analyse the effects of PTRR services for the *worst scenario*, that is by considering the *least* fixed points under $\mathbb{V} = \mathbb{V}^{\text{zero}}$.

Definition 4.7. Let (L, A^e) be a financial network. Then, we refer to the least fixed point of Φ defined in (9) where $\mathbb{V} = \mathbb{V}^{\text{zero}}$ as the *least re-evaluated equity under zero recovery rate* in the financial network (L, A^e).

The main motivation for this analysis is that the least fixed point under a zero recovery rate assumption can be interpreted economically as a strategic response to stress by banks suffering a liquidity stress as described in Bardoscia et al. (2019a).

To see how this strategic response unfolds, we first introduce the Full Payment Algorithm (FPA) developed in Bardoscia et al. (2019a), through which VM flows are realised sequentially.[12] It proceeds as follows. Given a financial network (L, A^e), L is the network of payment obligations and A_i^e represents the liquid assets that bank i holds. The relative liabilities matrix Π is defined by

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{L}_i}, & \text{if } \bar{L}_i > 0, \\ 0, & \text{if } \bar{L}_i = 0. \end{cases}$$

We also define $\bar{l}(0) = \bar{L}$ and $e(0) = A^e$. Then, for every i, the Full Payment Algorithm iterates according to the following equations.

[12] The steps in the algorithm are shown in detail in Appendix A.4.

$$
l_i(t) = \begin{cases} \bar{l}_i(t), & \text{if } e_i(t) \geq \bar{l}_i(t), \\ 0, & \text{if } e_i(t) < \bar{l}_i(t), \end{cases} \tag{18a}
$$

$$
e_i(t+1) = e_i(t) + \sum_j \Pi_{ji} l_j(t) - l_i(t), \tag{18b}
$$

$$
\bar{l}_i(t+1) = \begin{cases} 0, & \text{if } e_i(t) \geq \bar{l}_i(t), \\ \bar{l}_i(t), & \text{if } e_i(t) < \bar{l}_i(t). \end{cases} \tag{18c}
$$

Institutions are assumed to cope with liquidity stress strategically. Banks with insufficient liquid assets wait for potential payments from their counterparties and only make payment in full once they receive sufficient liquidity buffers, see Equation (18a) and Equation (18c). The liquid assets that are available to meet margin calls are updated according to Equation (18b). The algorithm terminates if no more banks are able to make any payment, and the associated payment vector l is the *output of the FPA*.

Next, we derive a new characterisation of the FPA, and relate it to the modelling framework introduced in Section 3. Bardoscia et al. (2019a) formulate conditions under which the FPA and the Eisenberg & Noe (2001) model are equivalent regarding payment and payment shortfall. In fact, the equilibrium can be characterised by the Rogers & Veraart (2013) model with zero recovery rates.

Given a financial network (L, A^e), a *clearing vector* in the Rogers & Veraart (2013) model is a vector $L \in [0, \bar{L}]$ satisfying

$$
L = \Psi^{RV}(L),
$$

where the function Ψ^{RV} is given by

$$
\Psi^{RV}(L) = \begin{cases} \bar{L}_i, & \text{if } A_i^e + \sum_{j \in \mathcal{N}} \Pi_{ji} L_j \geq \bar{L}_i, \\ \left(\alpha A_i^e + \beta \sum_{j \in \mathcal{N}} \Pi_{ji} L_j\right)^+, & \text{otherwise}, \end{cases} \tag{19}
$$

where $\alpha, \beta \in [0, 1]$.

Proposition 4.8. The output of the FPA is the least clearing vector in the Rogers & Veraart (2013) model with $\alpha = \beta = 0$.

Proposition 4.9. Let (L, A^e) be a financial network and let L^* be the output of the FPA. Then, the least re-evaluated equity under zero recovery rate E^* in (L, A^e) can be written as

$$
E_i^* = A_i^e + \sum_{j:\bar{L}_j > 0} L_{ji} \frac{L_j^*}{\bar{L}_j} - \bar{L}_i, \quad \forall\, i \in \mathcal{N}. \tag{20}
$$

Proposition 4.8 provides a link between the FPA and the Rogers & Veraart (2013) model. The equity E_i^* defined in (20) is the liquid assets of bank i including what it receives minus its total liabilities. Proposition 4.9 relates the strategic clearing framework with our modelling framework under the worst scenario and it is saying that the equity as a result of the FPA which incorporates strategic responses becomes the quantity that we focus on for our worst scenario analysis.

As a consequence, the definition of systemic risk reduction and harmfulness, for example, will be adjusted accordingly under the setting of sequential payments in this section. In particular, $E_i^* < 0$ means that bank i is under liquidity stress because the payments it receives in addition to available liquid assets are not enough to pay its liabilities in full.

In spite of the new characterisation of the FPA using $\mathbb{V}^{\text{zero}}$, we have motivations to consider strategic behaviours in financial networks. In centrally cleared markets, CCPs can trap liquidity from margin calls. CCPs typically make intraday margin calls caused by market movements, but only pass through the variation margin gains on the following morning. In addition, as non-cash collateral is accepted, the current practice is passing through the cash for variation margin payout on a pro rata basis, and paying out the remaining margins the next morning, ESRB (2020b). Since many large financial institutions are clearing members of CCPs, this would impose additional costs to them because they also face margin calls in non-centrally cleared markets. In addition to precautionary measures such as the net positive demand for collateral considered in Duffie et al. (2015), financial institutions may take defensive actions to not fulfill their payment obligations in a timely basis, as discussed in Bardoscia et al. (2019b); Paddrik et al. (2020).

Now, we analyse the effects of PTRR services in terms of liquidity strains arising from variation margin calls while taking account of the bank's strategic behaviour.

First of all, as we see in Proposition 4.9, the equity determined by the clearing outcome of the FPA corresponds to the least fixed point under zero recovery rate. When the fixed point problem does not admit a unique solution, this might create potential failure in coordination and give rise to liquidity arrangement inefficiency in the financial network because of sequential payments. In other words, liquidity flows efficiently in the network if the fixed point problem is unique, i.e., the least re-evaluated equity under zero recovery rate becomes the greatest one. The following proposition demonstrates that this is the case for directed acyclical graphs.

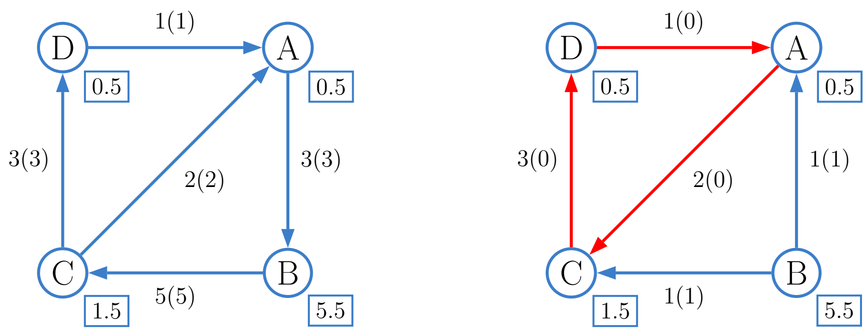
Proposition 4.10. If the graph associated with the financial network is a directed acyclic graph (DAG), then the clearing vector in the Rogers & Veraart (2013) model with $\alpha = \beta = 0$ is unique.

Next, we show in Theorem 4.11 that portfolio compression always reduces systemic risk when considering the worst case, i.e., the least re-evaluated equity with zero recovery rates. I.e., this means that the result derived in Veraart (2020b) that portfolio compression always reduces systemic risk under zero recovery rates when considering the greatest re-evaluated equity also holds when considering the least re-evaluated equity.

Furthermore, Theorem 4.11 also states that the statement of Theorem 4.1 does not hold when considering the worst case scenario, i.e., the least re-evaluated equity under zero recovery rate.

Theorem 4.11. 1. Consider a financial network and a compression network cycle. Then, for every bank, the least re-evaluated equity under zero recovery rate with compression is greater or equal than the least re-evaluated equity under zero recovery rate without compression. Therefore, this compression reduces systemic risk.

2. There exist a financial network and a harmful[13] rebalancing exercise for which there exists no default in the rebalancing set in the original network.



(a) Original network.

(b) Rebalanced network.

Figure 6: Example of a harmful portfolio rebalancing exercise under the worst case scenario.

Example 4.12 (Harmful portfolio rebalancing under stress)**.** We consider an example of portfolio rebalancing shown in Figure 6. In this example, banks A, B and C participate in the rebalancing exercise. Liquid assets are given in boxes. The liabilities are next to the arrows and the numbers in brackets stand for payments that are actually made, which are the outcomes of the FPA.

In the original network illustrated by Figure 6a all banks are able to make full payments including those rebalancing banks. In the rebalanced network, however, banks A, C and D are in default. We use red arrows in Figure 6b to represent liabilities that are not settled. In the following, we discuss our observations on the consequences.

First, note that if we clear the rebalanced network using the Rogers & Veraart (2013) model with zero recovery rates and choose the greatest clearing vector, then there is no default in the financial network. The default cascade arises because the cycle A-C-D is formed after the rebalancing exercise and the liquidity is trapped there. This is in line with Proposition 4.10 that the inefficiency is caused by cycles of claims.

Second, the example suggests that there is more uncertainty from rebalancing exercises under the worst case scenario that incorporates strategic behaviours. The outcomes in both networks reflect that the clearing method is very sensitive to the change of network structure and the liquid assets play a very important role in this circumstance.

Third, we have shown in Theorem 4.1 that rebalancing banks not defaulting in the original network is sufficient for systemic risk reduction, while this condition no longer holds under the

[13] Here, harmfulness is defined by considering default sets derived from the least re-evaluated equity rather than the greatest one.

worst case scenario. Therefore, we think from a financial stability point of view that it is beneficial to have policies to avoid this worst scenario and to ensure that the final payments converge to the greatest possible outcome.

5 Policy implications

In the response to the ESMA's consultation paper, the European Systemic Risk Board (ESRB) is satisfied with the overall performance of PTRRs and it generalises the function of those techniques as "By reducing the number of contracts in bilateral OTC derivatives portfolio and the size of the aggregate gross notional exposures, as well as by shortening the intermediation chains, these services reduce the operational complexity of the risk intermediation network by making it more resilient to the possible default of single nodes, reducing interconnectedness and increasing the transparency of bilateral exposures", ESRB (2020a).

In this section, we provide some insights based on the theoretical results. In particular, since we have shown that PTRR services do not need to reduce systemic risk, but they can reduce it under certain circumstances, we would like to discuss advantages and disadvantages of these services that would need to be considered in a wider cost-benefit analysis. First, we discuss how PTRR services could potentially mitigate the inherent procyclicality of margin requirements, but that this still does come with risks. Then, we discuss financial stability in centrally cleared markets.

5.1 Liquidity and procyclicality

ESRB (2017) summarises the important role of collateral in risk mitigation and sets out systemic risk stemming from procyclical collateral requirements. It is stated that procyclicality from margins is inherent because of valuation mechanism in the upswing and downswing of the asset price cycle. During the upswing, asset prices rise so the valuations of collateral increase, while the downswing starts as asset prices fall and more collateral is needed. Moreover, the procyclicality can interact with leverage within the financial system.[14]

Post-trade risk reduction services may be used as macroprudential tools to mitigate procyclicality of margins and address systemic risk in the following way. Both portfolio rebalancing and portfolio compression reduce the total amounts of variation margins that needs to be exchanged between counterparties when markets move.[15] In this sense, they already mitigate the effect of procyclicality by reducing the magnitudes of variation margins becoming due.

Nevertheless, as discussed before, risks from PTRR services remain since we have shown that there exist situations where they can be harmful, i.e., cause new defaults. So the main question is when these harmful situations could arise. The results in Section 4.1, especially Theorem 4.1,

[14]To be more specific, as explained by Geanakoplos (2010), in time of upswing, parties use the collateral freed up by higher valuation to increase either borrowing (financial leverage) or contingent commitments from derivatives (synthetic leverage), which raises asset prices and results in build up of excessive leverage. As asset prices depreciate, market participants have to post more margins which could lead to downward spiral and price-mediated contagion.

[15]Since variation margins are netted, also portfolio rebalancing results in lower variation margin payments.

suggest that portfolio compression and rebalancing are qualified risk mitigation tools in the markets if only firms with low default risk engage in these services. During periods of low volatility this would usually include a large number of institutions. Hence, in situations of low probability of default in the financial system, PTRR services are unlikely to cause liquidity risk propagation and they could be binding in the upswing to contain build-up of leverage.[16]

The analysis in Section 4.2 highlights that strategic response of banks under liquidity stress could result in the system reaching its equilibrium in the worst case state, i.e., the least re-evaluated equity with zero recovery rates. Still, even then Theorem 4.11 shows that portfolio compression always reduces systemic risk when comparing the worst case outcome of the compressed system to the non-compressed system. For rebalancing, however, this is not the case, and rebalancing can be harmful in the worst case scenario even if the non-rebalanced network in the worst case scenario did not have any defaults to start with.

Overall, from an economic point of view it is therefore beneficial to have mechanisms in place that avoid the situation that the system converges on the least fixed point. For this it is important that market participants have sufficient liquidity buffers. In line with time-varying macroprudential margin buffer proposed in ESRB (2017) and cash collateral buffers proposed in ESRB (2020b), one approach could be requiring a proportion of saved collateral to become margin buffers, which does not interfere with the risk management frameworks of market participants. In addition to benefits in counter-cyclical manner, the additional costs of the above proposals may create incentives to encourage central clearing for the non-centrally cleared transactions.

5.2 PTRR services and central clearing

We know that mandatory central clearing of OTC derivatives is one of the most important financial reforms after the 2008 financial crisis to mitigate systemic risk. In the meantime, it is still the fact that not all OTC derivatives transactions are centrally cleared and firms trade OTC portfolios actively. Although regulatory reforms promote central clearing where possible, it can depend on the mandatory clearing obligation, standardisation of the product, and interests of market participants on a voluntary basis, so it is inevitable that financial institutions will have both centrally-cleared and non-centrally cleared portfolio.

According to the results and discussions in previous sections, we think that PTRR services could achieve some benefits of CCPs in risk mitigation for non-centrally cleared products and they may be complementary to central clearing in uncleared space. It is noted in ESMA (2020b) that "the G20 commitment to improve the OTC derivatives market include the reporting obligation, the trading obligation, the clearing obligation and the requirement to increase use of collateral and

[16]ESRB (2017) considers a number of potential macroprudential tools with different objectives depending on the targeted phase of the asset price cycle. For instance, in order to limit the build-up of leverage during upswings, fixed or time-varying numerical floors on initial margins are proposed. The policy option of initial margin floors to prevent initial margins from falling too much is also identified in ESRB (2020b). The floors for initial margins could contribute to removing excessive leverage when asset prices increase. Although they are intended primarily for the upswings, they may have indirect influence on the downswings because the "overcollaterisation" can cause less deleveraging during bad times.

risk mitigation techniques." Therefore, we may leave some uncleared trades OTC and use targeting approaches such as margin requirements and PTRR services to mitigate risk, which are in line with the G20 commitment. Meanwhile, however, the EACH has realised additional risks that PTRR services pose, which include inappropriate supervision because "PTRR services could be used to manage particular exposures towards a counterparty", EACH (2020). One possible and desirable proposal would be to undertake PTRR services via CCPs, and utilise them complementary to CCPs' activities.[17] This not only ensures efficiency and lessens operational risk for individual firms by taking advantage of PTRR services, but also creates more transparent markets and fosters a safer financial system as a whole.

PTRR services share some valuable features of CCPs such as multilateral netting benefits that offer firms an efficient way to manage risk in the bilateral markets effectively. Building on the results for portfolio rebalancing, we provide an implication on multilateral netting by CCPs. Similar to Remark 2.10, we can see that multilateral netting activities can be characterised as rebalancing exercises in the weak sense with regards to bilaterally netted positions. Consider multilateral netting by a CCP. We use 0 to label the CCP when there is a single CCP in the financial market. Let $C^{uncleared} \in [0,\infty)^{(N+1)\times(N+1)}$ be the notional matrix representing transactions to be centrally cleared for all banks in $\mathcal{N}$, where $C^{uncleared}_{0i} = C^{uncleared}_{i0} = 0$ for all $i \in \mathcal{N}$. We define matrix R^{CCP} where

$$
\begin{aligned}
R^{CCP}_{0i} &= \left(\sum_{j\in\mathcal{N}} C^{uncleared}_{ji} - \sum_{j\in\mathcal{N}} C^{uncleared}_{ij}\right)^{+}, \quad \forall\, i \in \mathcal{N}, \\
R^{CCP}_{i0} &= \left(\sum_{j\in\mathcal{N}} C^{uncleared}_{ji} - \sum_{j\in\mathcal{N}} C^{uncleared}_{ij}\right)^{-}, \quad \forall\, i \in \mathcal{N}, \\
R^{CCP}_{ij} &= \left(C^{uncleared}\right)^{bi}_{ji}, \quad \forall\, i,j \in \mathcal{N}.
\end{aligned}
\tag{21}
$$

R^{CCP} is a rebalancing matrix by Definition 2.3. Finally, we let the rebalanced notional matrix be $C^{cleared} = C^{uncleared} + R^{CCP}$ and we can check that the graph associated with $\left(C^{cleared}\right)^{bi}$ is star-shaped and the CCP is in the centre. We obtain the centrally cleared market from the non-centrally market in terms of bilaterally netted positions. Taking account of multiple CCPs or novating to central clearing partially is also eligible by constructing rebalancing matrix analogous to (21). In the following, we generalise a result for central clearing which is a direct corollary to Theorem 4.1, so we omit the proof. Since CCPs only pass through variation margin gains, we assume for simplicity that they hold zero external assets.

Corollary 5.1. Consider the following financial markets consisting of banks in $\mathcal{N}$ and CCPs:

[17] CCP12 (2020) also notes that "Service providers who help to compress options activity concentrate on a number of metrics to improve the profitability of options trading. These include reducing gross notional, as well as lowering the gross amount of initial margin that must be posted under the UMRs. These are significant innovations from the industry, but it remains the case that these activities would be more efficient in a cleared model, where all risks could be multilaterally netted."

1. Let $C^{(1)}$ be a notional matrix for transactions in financial market without CCPs (non-centrally cleared market).

2. Let $C^{(2)}$ be a notional matrix for transactions in financial market with a single CCP labelled by index 0 (non-centrally cleared and centrally cleared markets).

3. Let $C^{(3)}$ be a notional matrix for transactions in financial market with multiple CCPs labelled by indices in $\mathcal{H}$ (non-centrally cleared and centrally cleared markets).

For $r \in \{2, 3\}$, every CCP in financial market $C^{(r)}$ has zero net position, i.e.,

$$\sum_{i \in \mathcal{N}} \left(C^{(2)}_{i0} - C^{(2)}_{0i} \right) = \sum_{i \in \mathcal{N} \cup \mathcal{H}} \left(C^{(3)}_{ih} - C^{(3)}_{hi} \right) = 0, \quad \forall\, h \in \mathcal{H}. \tag{22}$$

Moreover, we assume that

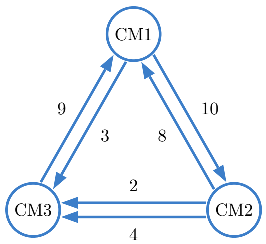
$$\sum_{j \in \mathcal{N}} \left(C^{(1)}_{ji} - C^{(1)}_{ij} \right) = \sum_{j \in \mathcal{N} \cup \{0\}} \left(C^{(2)}_{ji} - C^{(2)}_{ij} \right) = \sum_{j \in \mathcal{N} \cup \mathcal{H}} \left(C^{(3)}_{ji} - C^{(3)}_{ij} \right), \quad \forall\, i \in \mathcal{N}, \tag{23}$$

so every bank has same net positions in financial market $C^{(r)}$ for $r \in \{1, 2, 3\}$. Let $(L^{(r)}, A^e, \mathbb{V})$ be the financial network for the corresponding financial market represented by $C^{(r)}$, where $r \in \{1, 2, 3\}$. Suppose all financial institutions (including CCPs) do not default in one of the financial networks $(L^{(r)}, A^e, \mathbb{V})$ for some $r \in \{1, 2, 3\}$. Then, all financial institutions do not default in any of the remaining financial networks.

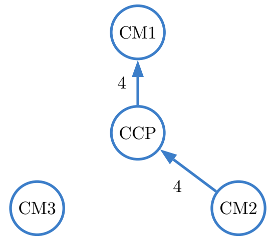
Unlike PTRR services that aim at specified classes of derivative products, Corollary 5.1 allows us to consider all derivative contracts in non-centrally cleared and centrally cleared markets. Condition (22) is saying that CCPs always have net positions equal to zero. In addition, the total net positions in every financial market are the same because of condition (23). Both conditions allow counterparties that have transactions in non-centrally cleared markets to decide to what extent they novate their OTC trades to transactions with CCPs without changing their net exposures. In addition, in the non-centrally cleared or centrally cleared markets, banks are able to choose whether to take part in PTRR exercises. Examples consisting of envisaged portfolios that made up of clearing members and CCPs are demonstrated in Figure 7.

Corollary 5.1 suggests that as long as the financial institutions are not likely to default, for example in periods of favourable market conditions, to what extent market participants use central clearing and how many CCPs they clear with does not make a big difference in affecting their liquidity conditions in facing margin calls because every institution can avoid failure in funding liquidity. However, note that this corollary applies under the consideration of the greatest re-evaluated equities. In particular, Theorem 4.11 implies that different clearing scenarios can have different outcomes under the worst scenario. Although CCPs break up cycles by establishing star-shaped networks, they may trap liquidity to a great extent during stressed periods, ESRB (2020b).

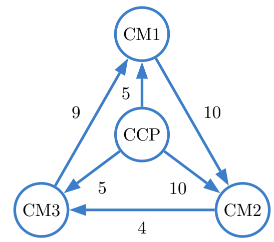
Central clearing plays an important role in keeping the financial sector safe and transparent. Since most of the rules for the clearing mandate have been conducted, regulators' attention has



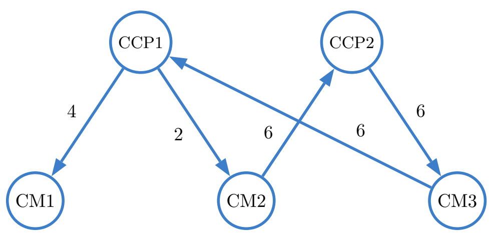
(a) Transactions without central clearing.



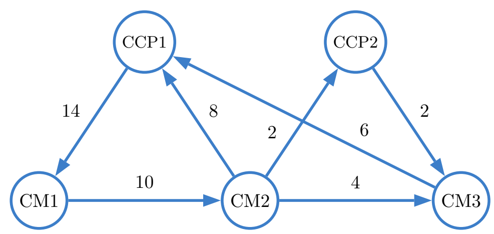
(b) Central clearing with a single CCP.



(c) Partial central clearing with a single CCP.



(d) Central clearing with multiple CCPs.



(e) Partial central clearing with multiple CCPs.

Figure 7: Panels (b) through (e) illustrate various financial markets for which central clearing is implemented.

shifted to the analysis of the impact of the reforms and policy revision.[18] The implication of Corollary 5.1, to some extent, could be that from a stability perspective we should put more focus on CCPs' risk management regimes under extreme market conditions such as a centralised default management process.

6 Conclusion

In this paper, we have provided a mathematical characterisation for (optimal) rebalancing and have derived conditions for portfolio rebalancing to reduce systemic risk. Namely, if there are no defaults in the rebalancing set in the original network or if there are no defaults in the rebalancing set in the rebalanced network, then rebalancing reduces systemic risk. We have also derived sufficient conditions for optimal rebalancing to reduce systemic risk under zero recovery rates.

We have provided examples showing that portfolio rebalancing changes the network structure, so loss propagation is different in the rebalanced network and that there exist situations in which

[18]For instance, the post-implementation evaluation of the effects of the G20 financial regulatory reforms reported by Financial Stability Board (2018). See also Duffie (2018) for a survey on post-crisis financial reforms. Murphy (2020) uses mandatory central clearing as an example to underline issues that post-crisis regulation review should emphasise, in which it is stated that one reason for the emergence of debates on clearing policy is "a clear link was never established between the ultimate policy making goal – financial stability – and the means chosen to achieve it in OTC derivatives markets – mandatory central clearing."

portfolio rebalancing can be harmful. In particular, portfolio rebalancing which mitigates counterparty credit risk on the individual level does not reflect systemic risk reduction in general. Because the primary goal of PTRR services for systemic risk would not be to increase resilience of individual market participants, it is essential to have a system-wide perspective to monitor the overall influence of PTRR services on financial stability.

Our results in Section 4.1 suggest that participation of banks in PTRR services that are not likely to be illiquid will not cause risk propagation through the financial network, therefore those banks can be encouraged to use PTRR services.

In Section 4.2 we have seen the benefits of PTRR services, in particular portfolio compression, on reducing liquidity strains during periods of extreme stress, i.e., in our worst cases analysis. Recently, in March 2020, the COVID-19 crisis caused a significant liquidity stress in financial markets with large variation margins becoming due[19] exactly at a time where liquidity was already under strain (EBF & ISDA, 2020). Our analysis implies that appropriate PTRR services could reduce parts of these pressures. Hence, PTRR services can bring risk mitigation benefits and they may contribute to anti-procyclicality as well. As a result, PTRR services could have a macroprudential role to play in mitigating aspects of systemic risk in the OTC markets.

Nevertheless, we have also seen that PTRR services can be harmful under some circumstances, and hence they are not a risk-free tool that is guaranteed to only reduce systemic risk. Therefore one will need to assess their advantages and disadvantages carefully, and we think that any such analysis should take contagion effects into consideration.

Our findings on PTRR services also inspires thinking on central clearing from a novel perspective. We may ask the questions: Does interconnectedness increase from the clearing members via CCP's loss mutualisation mechanism during times of market stress? Is portfolio compression or rebalancing a more efficient means to limit knock-on effects and mitigate moral hazard given heterogeneity? As put forward in Cont (2017), risk in the financial system does not disappear and central clearing transforms counterparty credit risk into liquidity risk. Understanding the challenges on risk management of CCPs such as robustness of CCP design will be one of the keys for improvement. Since PTRR services complement the clearing obligation, we hope our paper could stimulate discussions on the interactions between CCPs and PTRR services.

A Proofs

A.1 Additional definitions

Definition A.1 (Initial equity in the original and rebalanced financial network)**.** Let $(L, A^e; \mathbb{V})$ and $(L^{\mathcal{B}}, A^e; \mathbb{V})$ be the original and $\mathcal{B}$-rebalanced financial network respectively. For all $i \in \mathcal{N}$ we

[19] During the most recent severe liquidity stress, i.e., the COVID-19 crisis in March 2020, although change in net VM flows was moderate, "gross VM paid and received increased from the order of three or four times to the order of ten to twelve times. This points to significantly increased liquidity requirements exactly when funding markets might be less liquid", EBF & ISDA (2020)

define the initial equity in the original and $\mathcal{B}$-rebalanced network by

$$\begin{aligned} E_i^{(0)} &= A_i^e + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i, \\ E_i^{\mathcal{B}(0)} &= A_i^e + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{B}} - \bar{L}_i^{\mathcal{B}}. \end{aligned} \tag{24}$$

The initial equity is the equity that bank should have when all banks pay their liabilities in full, i.e., no default occurs in the financial network.

Definition A.2 (Fundamental and contagious default sets)**.** Let $(L, A^e; \mathbb{V})$ and $(L^{\mathcal{B}}, A^e; \mathbb{V})$ be the original and $\mathcal{B}$-rebalanced financial network respectively. Let $E^{(0)}$ and $E^{\mathcal{B}(0)}$ be initial equities given by (24). Then, $\mathcal{F} = \{i \in \mathcal{N} \,|\, E_i^{(0)} < 0\}$ and $\mathcal{D}(L, A^e; \mathbb{V}) \setminus \mathcal{F}$ are defined as fundamental default set and contagious default set in the financial network $(L, A^e; \mathbb{V})$ respectively. Similarly, we refer to $\mathcal{F}^{\mathcal{B}} = \{i \in \mathcal{N} \,|\, E_i^{\mathcal{B}(0)} < 0\}$ and $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) \setminus \mathcal{F}^{\mathcal{B}}$ as fundamental default set and contagious default set in the $\mathcal{B}$-rebalanced financial network $(L^{\mathcal{B}}, A^e; \mathbb{V})$.

A.2 Proofs of results in Sections 2 and 3

Proof of Proposition 2.6. From the Definition of a rebalancing matrix, it follows immediately that any rebalancing matrix R can be interpreted as representing a flow in network theory, i.e., R_{ij} is a flow along edge (i, j). In particular, this flow is a circulation because the in-flow and out-flow are the same at each node, see Definition 2.3. Therefore, we are able to apply the flow decomposition theorem, see (Ahuja et al., 1993, Theorem 3.5), to this circulation. By (Ahuja et al., 1993, Property 3.6), R can be represented as a sum of cycles, i.e., there exist matrices $R^{(k)}$ for $k \in \{1, 2, \dots, K\}$, where the graph associated with each $R^{(k)}$ consists of exactly one cycle, such that

$$R = \sum_{k=1}^{K} R^{(k)}.$$

We let $\mathcal{B}^{(k)} = \{i \in \mathcal{B} \,|\, \exists\, j \text{ such that } R_{ij}^{(k)} > 0\} \subseteq \mathcal{B}$ for each k. Moreover, $R^{(k)}$ is a rebalancing matrix for every k because $R_{ij}^{(k)} \cdot R_{ji}^{(k)} = 0$ for all $i, j \in \mathcal{B}^{(k)}$ follows from the property of R. Therefore, $(C^{(k)}, \mathcal{B}^{(k)}, R^{(k)})$ is an elementary rebalancing exercise in the weak sense for each k. By the recursive definition of notional matrix, we obtain that

$$C^{(K)} = C + R^{(1)} + \cdots + R^{(K-1)}.$$

Hence,

$$C^{(K)} + R^{(K)} = C + \sum_{k=1}^{K} R^{(k)} = C + R = C^{\mathcal{B}}.$$

□

Proof of Theorem 2.8. As argued in Appendix B, the non-conservative $\mathcal{B}$-compression problem in

Definition B.6 is a linear programming problem and it admits a solution, since the feasible region is non-empty (since the matrix $(C_{\mathcal{B}\mathcal{B}})$ satisfies both constraints (57) and (58)) and since the objective function is bounded from below by 0. Therefore, the existence of a solution to the optimal rebalancing problem follows from the existence of the solution of the non-conservative compression problem by using the explicit construction of such a solution established in the proof of part 1. of Lemma B.8. □

Proof of Proposition 3.4. Let L be the liabilities matrix corresponding to $\psi = 1$, i.e., $L = C^{bi}$. For $r \in \{1, 2\}$, let $\bar{A}_i^{(r)}$ be the interbank assets of bank i in $(L^{(r)}, A^e; \mathbb{V})$. Let $\bar{L}$ be the total liabilities for the financial network $(L, A^e; \mathbb{V})$. It follows that for all $r \in \{1, 2\}$, $\bar{L}^{(r)} = \psi_r \bar{L}$ and $L_{ij}^{(r)} = \psi_r L_{ij}$ for all $i, j \in \mathcal{N}$. Moreover, for all $i \in \mathcal{N}$, $\bar{L}_i^{(r)} > 0$ if and only if $\bar{L}_i > 0$. We set $\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$. By definition of the function Φ defined in (9), we have for all $i \in \mathcal{N}$,

$$
\begin{aligned}
E_i^1 &= A_i^e + \sum_{j \in \mathcal{M}} L_{ji}^{(1)} \mathbb{V}\left(\frac{E_j^1 + \bar{L}_j^{(1)}}{\bar{L}_j^{(1)}}\right) - \bar{L}_i^{(1)}, \\
E_i^2 &= A_i^e + \sum_{j \in \mathcal{M}} L_{ji}^{(2)} \mathbb{V}\left(\frac{E_j^2 + \bar{L}_j^{(2)}}{\bar{L}_j^{(2)}}\right) - \bar{L}_i^{(2)}.
\end{aligned}
\tag{25}
$$

For $r \in \{1, 2\}$, we define the function $\Phi^r : [-\bar{L}^{(r)}, A^e + \bar{A}^{(r)} - \bar{L}^{(r)}] \to [-\bar{L}^{(r)}, A^e + \bar{A}^{(r)} - \bar{L}^{(r)}]$, where

$$
\Phi_i^r(E) = A_i^e + \psi_r \left[\sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \psi_r \bar{L}_j}{\psi_r \bar{L}_j}\right) - \bar{L}_i\right], \quad \forall i \in \mathcal{N}. \tag{26}
$$

Then, by (25), for all $i \in \mathcal{N}$,

$$
\begin{aligned}
E_i^1 &= A_i^e + \psi_1 \left[\sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^1 + \psi_1 \bar{L}_j}{\psi_1 \bar{L}_j}\right) - \bar{L}_i\right], \\
E_i^2 &= A_i^e + \psi_2 \left[\sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^2 + \psi_2 \bar{L}_j}{\psi_2 \bar{L}_j}\right) - \bar{L}_i\right].
\end{aligned}
\tag{27}
$$

Furthermore, E^1 is the greatest fixed point of Φ^1 and E^2 is the greatest fixed point of Φ^2.

We define a third function $\tilde{\Phi} : [-\bar{L}^{(1)}, A^e/(\psi_2/\psi_1) + \bar{A}^{(1)} - \bar{L}^{(1)}] \to [-\bar{L}^{(1)}, A^e/(\psi_2/\psi_1) + \bar{A}^{(1)} - \bar{L}^{(1)}]$, where

$$
\tilde{\Phi}_i(E) = \frac{A_i^e}{(\psi_2/\psi_1)} + \psi_1 \left[\sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \psi_1 \bar{L}_j}{\psi_1 \bar{L}_j}\right) - \bar{L}_i\right], \quad \forall i \in \mathcal{N}. \tag{28}
$$

Let $\tilde{E}$ be the greatest fixed point of $\tilde{\Phi}$. From (28), we obtain that

$$\Phi_i^2\left(\frac{\psi_2}{\psi_1}\tilde{E}\right) = A_i^e + \psi_2\left[\sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{\tilde{E}_j + \psi_1\bar{L}_j}{\psi_1\bar{L}_j}\right) - \bar{L}_i\right] = \frac{\psi_2}{\psi_1}\tilde{E}_i, \quad \forall\, i \in \mathcal{N}. \tag{29}$$

Hence, $\frac{\psi_2}{\psi_1}\tilde{E}$ is a fixed point of Φ^2. Because E^2 is the greatest fixed point of Φ^2, we get

$$E^2 \geq \frac{\psi_2}{\psi_1}\tilde{E}. \tag{30}$$

It remains to show that $\tilde{E} \geq E^1$, where $\tilde{E}$ and E^1 are the greatest fixed points of $\tilde{\Phi}$ and Φ^1 respectively. Let $\tilde{E}^{(0)} = A^e/(\psi_2/\psi_1) + \bar{A}^{(1)} - \bar{L}^{(1)}$ and define recursively $\tilde{E}^{(n+1)} = \tilde{\Phi}\left(\tilde{E}^{(n)}\right)$ for $n \in \mathbb{N}_0$. Similarly, we let $E^{(0)} = A^e + \bar{A}^{(1)} - \bar{L}^{(1)}$ and define $E^{(n+1)} = \Phi^1\left(E^{(n)}\right)$ for $n \in \mathbb{N}_0$. $\tilde{\Phi}$ and Φ^1 are non-decreasing functions. By (Veraart, 2020a, Theorem 2.6), sequences $(\tilde{E}^{(n)})$ and $(E^{(n)})$ are non-increasing. Furthermore,

$$\lim_{n\to\infty}\tilde{E}_i^{(n)} = \tilde{E}_i, \quad \lim_{n\to\infty}E_i^{(n)} = E_i^1, \quad \forall\, i \in \mathcal{N}. \tag{31}$$

We prove by induction that for all $n \in \mathbb{N}_0$,

$$\tilde{E}_i^{(n)} \geq E_i^{(n)}, \quad \forall\, i \in \mathcal{N}. \tag{32}$$

Let $n = 0$. Then, we have $\tilde{E}_i^{(0)} \geq E_i^{(0)}$ for all $i \in \mathcal{N}$ because $\psi_2/\psi_1 < 1$ by assumption (15). Now suppose the induction hypothesis (32) holds for a fixed $n \in \mathbb{N}_0$, and we show that it also holds for $n+1$. By (28) and (26), we obtain

$$\begin{aligned}
\tilde{E}_i^{(n+1)} &= \tilde{\Phi}_i\left(\tilde{E}^{(n)}\right) = \frac{A_i^e}{(\psi_2/\psi_1)} + \psi_1\left[\sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{\tilde{E}_j^{(n)} + \psi_1\bar{L}_j}{\psi_1\bar{L}_j}\right) - \bar{L}_i\right], \\
E_i^{(n+1)} &= \Phi_i^1\left(E^{(n)}\right) = A_i^e + \psi_1\left[\sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \psi_1\bar{L}_j}{\psi_1\bar{L}_j}\right) - \bar{L}_i\right].
\end{aligned}$$

Because the valuation function $\mathbb{V}$ is non-decreasing, the induction hypothesis (32) implies that $\tilde{E}_i^{(n+1)} \geq E_i^{(n+1)}, \forall\, i \in \mathcal{N}$. This finishes the induction proof since $\tilde{E} \geq E^1$ follows by (31). Combining this with (30) we have

$$E^2 \geq \frac{\psi_2}{\psi_1}\tilde{E} \geq \frac{\psi_2}{\psi_1}E^1,$$

as required by (16).

The remaining statements are direct consequences of (16) and we omit the proof. □

A.3 Proofs of results in Section 4.1

Throughout the proofs of results in Section 4.1, we will use the quantities summarised in Definition A.3.

Definition A.3 (Market elements)**.** Let $\mathbb{V}$ be a valuation function with $k = 0$.

- Let $(L, A^e; \mathbb{V})$ be the original financial network with total liabilities $\bar{L}$.
- Let $\mathcal{B}$ be the rebalancing set and let $(L^{\mathcal{B}}, A^e; \mathbb{V})$ be the $\mathcal{B}$-rebalanced financial network. Let $\bar{L}^{\mathcal{B}}$ be the total liabilities in the $\mathcal{B}$-rebalanced financial network.
- Let E^* be the greatest re-evaluated equity in the original network. Let $\mathcal{D}(L, A^e; \mathbb{V})$ be the default set defined by (13).
- Let $E^{\mathcal{B};*}$ be the greatest re-evaluated equity in the $\mathcal{B}$-rebalanced network. Let $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$ be the default set defined by (14).

We will use the following lemmas to prove the the main results in Section 4.1.

Lemma A.4. Let $(C, \mathcal{B}, R)$ be a $\mathcal{B}$-rebalancing exercise in the weak sense. Let (L, A^e) be a financial network and let $(L^{\mathcal{B}}, A^e)$ be the $\mathcal{B}$-rebalanced financial network given in Definition 3.1. Then,

1. $\bar{L}_i = \bar{L}_i^{\mathcal{B}}$ for all $i \in \mathcal{N} \setminus \mathcal{B}$.
2. For all $i \in \mathcal{N}$,
$$\sum_{j\in\mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}}) = \sum_{j\in\mathcal{B}} (L_{ji} - L_{ij}), \tag{33}$$
and
$$\bar{L}_i - \bar{L}_i^{\mathcal{B}} = \sum_{j\in\mathcal{B}} L_{ji} - \sum_{j\in\mathcal{B}} L_{ji}^{\mathcal{B}}. \tag{34}$$

Proof of Lemma A.4. 1. This is a direct consequence of Definition 2.3 and Definition 2.4.

2. Let C and $C^{\mathcal{B}}$ be the notional matrix and $\mathcal{B}$-rebalanced notional matrix respectively. Then, (33) holds for all $i \in \mathcal{N} \setminus \mathcal{B}$. By Definition 2.1 and Definition 3.1, for all $i \in \mathcal{B}$,

$$\begin{aligned}
\sum_{j\in\mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}}) &= \psi \sum_{j\in\mathcal{B}} \left((C^{\mathcal{B}})_{ji}^{bi} - (C^{\mathcal{B}})_{ij}^{bi} \right) = \psi \left(\sum_{C_{ji}^{\mathcal{B}} \ge C_{ij}^{\mathcal{B}}} (C_{ji}^{\mathcal{B}} - C_{ij}^{\mathcal{B}}) - \sum_{C_{ji}^{\mathcal{B}} < C_{ij}^{\mathcal{B}}} (C_{ij}^{\mathcal{B}} - C_{ji}^{\mathcal{B}}) \right) \\
&= \psi \sum_{j\in\mathcal{B}} (C_{ji}^{\mathcal{B}} - C_{ij}^{\mathcal{B}}) = \psi \left(\sum_{j\in\mathcal{B}} (C_{ji} - C_{ij}) + \sum_{j\in\mathcal{B}} (R_{ji} - R_{ij}) \right) \\
&= \psi \sum_{j\in\mathcal{B}} (C_{ji} - C_{ij}) = \psi \left(\sum_{C_{ji} \ge C_{ij}} (C_{ji} - C_{ij}) - \sum_{C_{ji} < C_{ij}} (C_{ij} - C_{ji}) \right) \\
&= \psi \sum_{j\in\mathcal{B}} (C_{ji}^{bi} - C_{ij}^{bi}) = \sum_{j\in\mathcal{B}} (L_{ji} - L_{ij}),
\end{aligned}$$

where the third line uses $\sum_{j\in\mathcal{B}} R_{ji} = \sum_{j\in\mathcal{B}} R_{ij}$ from Definition 2.3. In addition, (34) follows immediately from (33).

□

Lemma A.5. Let $(C, \mathcal{B}, R)$ be a rebalancing exercise in the weak sense. Set

$$\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}, \qquad \mathcal{M}^{\mathcal{B}} = \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{B}} > 0\}.$$

Let $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{B}}$. Then,

1. $j \in \mathcal{B}$,
2. $L_{ji} = 0 \; \forall \, i \in \mathcal{N} \setminus \mathcal{B}$,
3. if additionally the rebalancing exercise is elementary, then $L_{ji} = 0 \; \forall \, j \in \mathcal{B} \setminus \{p(i)\}$.

Proof of Lemma A.5. Let $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{B}}$.

1. By the definitions of $\mathcal{M}$ and $\mathcal{M}^{\mathcal{B}}$ we have $\bar{L}_j > 0$ and $\bar{L}_j^{\mathcal{B}} = 0$. Therefore, $\bar{L}_j \neq \bar{L}_j^{\mathcal{B}}$ which implies that $j \in \mathcal{B}$ by Lemma A.4.
2. Now suppose there exists an $i \in \mathcal{N} \setminus \mathcal{B}$ such that $L_{ji} > 0$. Then, $\bar{L}_j^{\mathcal{B}} = \sum_{k\in\mathcal{N}} L_{jk}^{\mathcal{B}} \geq L_{ji} > 0$ which contradicts the assumption that $j \notin \mathcal{M}^{\mathcal{B}}$.
3. Now let $i \in \mathcal{B}$, let $j \in \mathcal{B} \setminus \{p(i)\}$ and suppose that $L_{ji} > 0$. Then, $L_{ji}^{\mathcal{B}} = L_{ji} > 0$ and $\bar{L}_j^{\mathcal{B}} = \sum_{k\in\mathcal{N}} L_{jk}^{\mathcal{B}} \geq L_{ji}^{\mathcal{B}} > 0$, which is a contradiction to $\bar{L}_j^{\mathcal{B}} = 0$.

□

Lemma A.6. Consider the market elements in Definition A.3. For all $i \in \mathcal{N}$, let $E_i^{(0)}$ and $E_i^{\mathcal{B}(0)}$ be the initial equity in the original network and $\mathcal{B}$-rebalanced network defined in (24) respectively. For $n \in \mathbb{N}$, we define two sequences recursively

$$\begin{aligned} E^{(n)} &= \Phi\left(E^{(n-1)}\right), \\ E^{\mathcal{B}(n)} &= \Phi^{\mathcal{B}}\left(E^{\mathcal{B}(n-1)}\right), \end{aligned} \tag{35}$$

where the functions Φ and $\Phi^{\mathcal{B}}$ are defined in (9) and (11) respectively. Then,

1. $E_i^{(0)} = E_i^{\mathcal{B}(0)}, \quad \forall \, i \in \mathcal{N}$.
2. The sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{B}(n)}\right)$ are non-increasing, i.e., for all $i \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$, it holds that
$$E_i^{(n)} \geq E_i^{(n+1)}, \qquad E_i^{\mathcal{B}(n)} \geq E_i^{\mathcal{B}(n+1)}.$$

3. For all $i \in \mathcal{N}$, the sequences $\left(E_i^{(n)}\right)$ and $\left(E_i^{\mathcal{B}(n)}\right)$ converge to the greatest fixed points of Φ and $\Phi^{\mathcal{B}}$ respectively, i.e.,

$$\lim_{n\to\infty} E_i^{(n)} = E_i^*, \qquad \lim_{n\to\infty} E_i^{\mathcal{B}(n)} = E_i^{\mathcal{B};*}.$$

Proof of Lemma A.6. 1. We can rewrite the initial equities as

$$\begin{aligned} E_i^{(0)} &= A_i^e + \sum_{j\in\mathcal{N}\setminus\mathcal{B}} (L_{ji} - L_{ij}) + \sum_{j\in\mathcal{B}} (L_{ji} - L_{ij}), \\ E_i^{\mathcal{B}(0)} &= A_i^e + \sum_{j\in\mathcal{N}\setminus\mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}}) + \sum_{j\in\mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}}) = A_i^e + \sum_{j\in\mathcal{N}\setminus\mathcal{B}} (L_{ji} - L_{ij}) + \sum_{j\in\mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}}). \end{aligned}$$

Therefore, it is sufficient to show that $\sum_{j\in\mathcal{B}} (L_{ji} - L_{ij}) = \sum_{j\in\mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}})$ for all $i \in \mathcal{N}$, which holds by Lemma A.4.

2. We know from (Veraart, 2020a, Lemma A.1) that the functions Φ and $\Phi^{\mathcal{B}}$ are non-decreasing, so the statement follows from (Veraart, 2020a, Theorem 2.6).

3. The two sequences defined by (35) converge to the corresponding greatest re-evaluated equity in financial network with and without rebalancing also follows from (Veraart, 2020a, Theorem 2.6).

$\square$

Proof of Theorem 4.1. We consider the fixed point iteration as in the proof of (Veraart, 2020b, Proposition 4.8). Consider the market elements in Definition A.3. Let

$$\begin{aligned} \mathcal{M} &= \{i \in \mathcal{N} \,|\, \bar{L}_i > 0\}, \\ \mathcal{M}^{\mathcal{B}} &= \{i \in \mathcal{N} \,|\, \bar{L}_i^{\mathcal{B}} > 0\}, \end{aligned} \tag{36}$$

which are the set of banks that have positive liabilities in the original network $(L, A^e; \mathbb{V})$ and $\mathcal{B}$-rebalanced network $(L^{\mathcal{B}}, A^e; \mathbb{V})$ respectively. We consider sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{B}(n)}\right)$ defined by (35).

We will show by induction that if $\{i \in \mathcal{B} \,|\, E_i^* < 0\} = \emptyset$, then

$$E_i^{\mathcal{B}(n)} = E_i^{(n)}, \quad \forall\, i \in \mathcal{N} \tag{37}$$

holds for all $n \in \mathbb{N}_0$. The above statement then leads to

$$E_i^{\mathcal{B};*} = \lim_{n\to\infty} E_i^{\mathcal{B}(n)} = \lim_{n\to\infty} E_i^{(n)} = E_i^*, \quad \forall\, i \in \mathcal{N}$$

by Lemma A.6.

We know from Lemma A.6 that the two sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{B}(n)}\right)$ are non-increasing. The assumption $\{i \in \mathcal{B} \,|\, E_i^* < 0\} = \emptyset$ implies that $E_i^{(n)} \geq \lim_{k\to\infty} E_i^{(k)} = E_i^* \geq 0$ for all $i \in \mathcal{B}$ and for all $n \in \mathbb{N}_0$. Hence,

$$\{i \in \mathcal{B} \,|\, E_i^{(n)} < 0\} = \emptyset, \quad \forall\, n \in \mathbb{N}_0. \tag{38}$$

Moreover, (38) taken jointly with (37) also implies that for all $n \in \mathbb{N}_0$,

$$E_i^{\mathcal{B}(n)} = E_i^{(n)} \geq 0, \quad \forall\, i \in \mathcal{B},$$

or equivalently,

$$\{i \in \mathcal{B} \,|\, E_i^{\mathcal{B}(n)} < 0\} = \emptyset, \quad \forall\, n \in \mathbb{N}_0. \tag{39}$$

Now we start to prove (37) by induction. Let $n = 0$, then we know that $E_i^{\mathcal{B}(0)} = E_i^{(0)}$ is true for all $i \in \mathcal{N}$ by Lemma A.6. We assume that (37) holds for a fixed $n \in \mathbb{N}_0$, and we show that it is also true for $n+1$. By definition, the expressions for $E_i^{(n+1)}$ and $E_i^{\mathcal{B}(n+1)}$ are

$$E_i^{(n+1)} = \Phi_i\left(E^{(n)}\right) = A_i^e + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i,$$

$$E_i^{\mathcal{B}(n+1)} = \Phi_i^{\mathcal{B}}\left(E^{\mathcal{B}(n)}\right) = A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}} \mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \bar{L}_i^{\mathcal{B}}.$$

For all $j \in \mathcal{B}$, we have

$$\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) \overset{(39)}{=} 1 \overset{(38)}{=} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right). \tag{40}$$

In addition, for all $j \in \mathcal{N} \setminus \mathcal{B}$, it holds that $\bar{L}_j = \bar{L}_j^{\mathcal{B}}$, and combining this with the induction hypothesis (37), gives

$$\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) = \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right). \tag{41}$$

Let $i \in \mathcal{N} \setminus \mathcal{B}$. Then,

$$\begin{aligned}
E_i^{\mathcal{B}(n+1)} &= A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}}} \underbrace{L_{ji}^{\mathcal{B}}}_{= L_{ji}} \mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \underbrace{\bar{L}_i^{\mathcal{B}}}_{= \bar{L}_i \text{ by Lemma A.4}} \\
&= A_i^e + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \underbrace{\sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{B}}} L_{ji} \mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)}_{=0 \text{ by Lemma A.5, 2.)}} - \bar{L}_i \\
&= A_i^e + \sum_{j \in \mathcal{M} \cap \mathcal{B}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)}_{= 1 = \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) \text{ by (40)}} + \sum_{j \in \mathcal{M} \setminus \mathcal{B}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)}_{= \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) \text{ by (41)}} - \bar{L}_i \\
&= A_i^e + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= E_i^{(n+1)}.
\end{aligned}$$

Let $i \in \mathcal{B}$. Then,

$$\begin{aligned}
E_i^{\mathcal{B}(n+1)} &= A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}} \mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \bar{L}_i^{\mathcal{B}} \\
&= A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}} \cap \mathcal{B}} L_{ji}^{\mathcal{B}} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)}_{= 1} + \sum_{j \in \mathcal{M}^{\mathcal{B}} \setminus \mathcal{B}} \underbrace{L_{ji}^{\mathcal{B}}}_{= L_{ji}} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)}_{= \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right)} \\
&\quad - \bar{L}_i - (\bar{L}_i^{\mathcal{B}} - \bar{L}_i) \\
&= A_i^e + \sum_{j \in \mathcal{M} \setminus \mathcal{B}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \sum_{j \in \mathcal{M}^{\mathcal{B}} \cap \mathcal{B}} L_{ji}^{\mathcal{B}} - (\bar{L}_i^{\mathcal{B}} - \bar{L}_i) \quad \text{by } (\star) \\
&= A_i^e + \sum_{j \in \mathcal{M} \setminus \mathcal{B}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \sum_{j \in \mathcal{M} \cap \mathcal{B}} L_{ji} \quad \text{(By Lemma A.4)} \\
&= A_i^e + \sum_{j \in \mathcal{M} \cap \mathcal{B}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) + \sum_{j \in \mathcal{M} \setminus \mathcal{B}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= A_i^e + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= E_i^{(n+1)}.
\end{aligned}$$

where the third equality $(\star)$ uses the fact that $\mathcal{M} \setminus \mathcal{B} = \mathcal{M}^{\mathcal{B}} \setminus \mathcal{B}$. This finishes the induction proof and the systemic risk reduction follows immediately.

$\square$

Proof of Proposition 4.2. We consider the triple $(C, \mathcal{B}, R)$, where R is the rebalancing matrix with respect to $\mathcal{B}$. Then, $C^{\mathcal{B}} = C + R$.

We will now define a new rebalancing matrix $\tilde{R} \in [0, \infty)^{N \times N}$ that reverts the effects of R in the sense that

$$(C^{\mathcal{B}} + \tilde{R})^{bi} = C^{bi}.$$

By assumption $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) = \emptyset$, where $L^{\mathcal{B}} = \psi(C^{\mathcal{B}})^{bi}$ for a $\psi \in (0, \infty)$. Hence, we can apply Theorem 4.1 to the financial network $(L^{\mathcal{B}}, A^e)$ to conclude that the re-evaluated equities in the original financial network $(L^{\mathcal{B}}, A^e)$ and in the $\mathcal{B}$-rebalanced financial network $(\psi(C^{\mathcal{B}} + \tilde{R})^{bi}, A^e) = (L, A^e)$ coincide.

Hence, we only need to determine an appropriate $\tilde{R}$. We set $\tilde{R} = R^{\top}$. Then it follows directly from Definition 2.3, that $\tilde{R}$ is a rebalancing matrix with respect to the original rebalancing set $\mathcal{B}$. Furthermore, for all $i, j \in \mathcal{N}$

$$L_{ij} = \psi C_{ij}^{bi} = \psi \max\{C_{ij} - C_{ji}, 0\}$$

and

$$\begin{aligned}(C^{\mathcal{B}} + \tilde{R})_{ij}^{bi} &= (C + R + R^{\top})_{ij}^{bi} = \max\{C_{ij} + R_{ij} + R_{ji} - C_{ji} - R_{ji} - R_{ij}, 0\} = \max\{C_{ij} - C_{ji}, 0\} \\ &= C_{ij}^{bi}\end{aligned}$$

and hence indeed $(\psi(C^{\mathcal{B}} + \tilde{R})^{bi}, A^e) = (L, A^e)$.

$\square$

We will use Lemma A.7 (whose second and third statements are similar to (Veraart, 2020b, Lemma B.4)) to prove Proposition 4.5.

Lemma A.7. Consider the market elements in Definition A.3. Let $\mathcal{F}$ and $\mathcal{F}^{\mathcal{B}}$ be the fundamental default set in the original and $\mathcal{B}$-rebalanced network respectively according to Definition A.2. Then, 1.) $\mathcal{F}^{\mathcal{B}} = \mathcal{F}$, 2.) $\mathcal{F} \subseteq \mathcal{D}(L, A^e; \mathbb{V})$ and 3.) $\mathcal{F}^{\mathcal{B}} \subseteq \mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$.

Proof of Lemma A.7.

1. By Definition A.2 we have $\mathcal{F} = \{i \in \mathcal{N} \,|\, E_i^{(0)} < 0\}$ and $\mathcal{F}^{\mathcal{B}} = \{i \in \mathcal{N} \,|\, E_i^{\mathcal{B}(0)} < 0\}$. Since $E_i^{(0)} = E_i^{\mathcal{B}(0)}$ for every $i \in \mathcal{N}$ by Lemma A.6, we obtain $\mathcal{F}^{\mathcal{B}} = \mathcal{F}$.
2. We consider the sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{B}(n)}\right)$ defined by (35). For every $i \in \mathcal{F}$, Lemma A.6 implies that $\forall\, m \in \mathbb{N}$, $0 > E_i^{(0)} \geq E_i^{(m)} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$. Therefore, $i \in \mathcal{D}(L, A^e; \mathbb{V})$.
3. Now fix $i \in \mathcal{F}^{\mathcal{B}}$. Again by Lemma A.6 we get $\forall\, m \in \mathbb{N}$, $0 > E_i^{\mathcal{B}(0)} \geq E_i^{\mathcal{B}(m)} \geq \lim_{n\to\infty} E_i^{\mathcal{B}(n)} = E_i^{\mathcal{B};*}$, hence $i \in \mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$.

$\square$

Proof of Proposition 4.5. Here $\mathcal{B} = \mathcal{N}$. Consider the market elements in Definition A.3. We know from Lemma A.7 that $\mathcal{F}^{\mathcal{B}} \subseteq \mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$. We first prove $\mathcal{F}^{\mathcal{B}} = \mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$ by showing that $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) \subseteq \mathcal{F}^{\mathcal{B}}$.

Let $i \in \mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$. Then,

$$0 > E_i^{\mathcal{B};*} = A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}} \mathbb{V}\left(\frac{E_j^{\mathcal{B};*} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \bar{L}_i^{\mathcal{B}}.$$

Hence, $\bar{L}_i^{\mathcal{B}} > 0$. Since the graph associated with the $\mathcal{N}$-rebalanced network is bipartite by (D'Errico & Roukny, 2021, Lemma 1), this implies that $L_{ji}^{\mathcal{B}} = 0$ for all $j \in \mathcal{M}^{\mathcal{B}}$.[20] Hence, $\sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}} \mathbb{V}\left(\frac{E_j^{\mathcal{B};*} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) = 0 = \sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}}$ and hence

$$E_i^{\mathcal{B};*} = A_i^e - \bar{L}_i^{\mathcal{B}} = E_i^{\mathcal{B}(0)}.$$

This leads to $E_i^{\mathcal{B}(0)} < 0$, so $i \in \mathcal{F}^{\mathcal{B}}$. Hence, $\mathcal{F}^{\mathcal{B}} = \mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V})$. Furthermore, we have $\mathcal{F}^{\mathcal{B}} = \mathcal{F} \subseteq \mathcal{D}(L, A^e; \mathbb{V})$ by Lemma A.7, and hence $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}) \subseteq \mathcal{D}(L, A^e; \mathbb{V})$.

□

We will use the following lemma to prove Proposition 4.6 which corresponds to (Veraart, 2020b, Lemma B.5) therefore we omit the proof.

Lemma A.8. Let $E_i^{\mathcal{B}(n)}, E_i^{(n)}, \bar{L}_i^{\mathcal{B}}, \bar{L}_i \in \mathbb{R}, E_i^{\mathcal{B}(n)} \geq E_i^{(n)}, \bar{L}_i^{\mathcal{B}} \leq \bar{L}_i$ and $k \geq 0$. Then,

$$\mathbb{V}^{\text{zero}}\left(\frac{E_i^{\mathcal{B}(n)} + \bar{L}_i^{\mathcal{B}}}{\bar{L}_i^{\mathcal{B}}}\right) = \mathbb{1}_{\left\{E_i^{\mathcal{B}(n)} \geq k\bar{L}_i^{\mathcal{B}}\right\}} \geq \mathbb{1}_{\left\{E_i^{(n)} \geq k\bar{L}_i\right\}} = \mathbb{V}^{\text{zero}}\left(\frac{E_i^{(n)} + \bar{L}_i}{\bar{L}_i}\right).$$

Proof of Proposition 4.6. 2. We first prove part 2 by providing one example of a harmful rebalancing exercise that is not optimal, but satisfies (17).

Let C be a notional matrix and let $\mathcal{B}$ be a rebalancing set. Let $(L, A^e; \mathbb{V}^{\text{zero}})$ be the corresponding financial network (with $\psi = 1$) with zero recovery rates where

$$C = L = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^e = (5, 0, 0, 0)^\top.$$

We assume that $\mathcal{B} = \{1, 2, 3\}$ and the rebalancing exercise is an elementary $\mathcal{B}$-rebalancing

[20] To be more precise, we are able to apply (D'Errico & Roukny, 2021, Lemma 1) because we show in Proposition B.7 that the optimally rebalanced network solves the non-conservative $\mathcal{N}$-compression problem defined in Appendix B.

exercise with capacity $\tau = 5$, i.e.,

$$R = \begin{pmatrix} 0 & 0 & 5 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C^{\mathcal{B}} = C + R = \begin{pmatrix} 0 & 5 & 5 & 0 \\ 5 & 0 & 3 & 2 \\ 2 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L^{\mathcal{B}} = (C^{\mathcal{B}})^{bi} = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $L^{\mathcal{B}}$ is the liabilities matrix in the $\mathcal{B}$-rebalanced network $(L^{\mathcal{B}}, A^e; \mathbb{V}^{\text{zero}})$. In particular, we see that condition (17) is satisfied because $\left\{ i \in \mathcal{B} \,|\, \sum_{j \in \mathcal{B}} (L_{ji} - L_{ij}) < 0 \right\} = \{1\}$ and $\mathcal{D}(L, A^e; \mathbb{V}^{\text{zero}}) = \{3\}$. In particular, node 3 is the only fundamental default but it does not trigger any contagious defaults. However, we can check that $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}^{\text{zero}}) = \{2, 3\}$. Hence, in the rebalanced network the fundamental default of node 3 triggers the contagious default of node 2. So this rebalancing exercise is harmful.

1. Now we prove part 1. Consider the market elements in Definition A.3 and assume $\mathbb{V} = \mathbb{V}^{\text{zero}}$. Similar to the proof of Theorem 4.1, we take two sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{B}(n)}\right)$ defined by (35). For all $i \in \mathcal{B}$, we define $\Lambda_i^{\mathcal{B}} = \sum_{j \in \mathcal{B}} (L_{ji} - L_{ij})$. By Lemma A.4, $\Lambda_i^{\mathcal{B}} = \sum_{j \in \mathcal{B}} (L_{ji}^{\mathcal{B}} - L_{ij}^{\mathcal{B}})$. Since the graph associated with $L_{\mathcal{B},\mathcal{B}}^{\mathcal{B}}$ is bipartite according to (D'Errico & Roukny, 2021, Lemma 1)[21], we get for all $i \in \mathcal{B}$,

$$\begin{cases} L_{ji}^{\mathcal{B}} = 0, & \forall\, j \in \mathcal{B}, \text{ if } \Lambda_i^{\mathcal{B}} < 0, \\ L_{ij}^{\mathcal{B}} = 0, & \forall\, j \in \mathcal{B}, \text{ if } \Lambda_i^{\mathcal{B}} \geq 0. \end{cases} \tag{42}$$

We prove by induction that $\{i \in \mathcal{B} \,|\, \Lambda_i^{\mathcal{B}} < 0\} \cap \mathcal{D}(L, A^e; \mathbb{V}^{\text{zero}}) = \emptyset$ implies

$$E_i^{\mathcal{B}(n)} \geq E_i^{(n)}, \quad \forall\, i \in \mathcal{N} \tag{43}$$

holds for all $n \in \mathbb{N}_0$. Once this has been shown, by Lemma A.6, it follows that

$$E_i^{\mathcal{B};*} = \lim_{n \to \infty} E_i^{\mathcal{B}(n)} \geq \lim_{n \to \infty} E_i^{(n)} = E_i^*, \quad \forall\, i \in \mathcal{N}.$$

Let $n = 0$ and we know $E_i^{\mathcal{B}(0)} = E_i^{(0)}$ for all $i \in \mathcal{N}$ by Lemma A.6. Now suppose (43) is true for a fixed $n \in \mathbb{N}_0$, and we show that it also holds for $n+1$. Substituting $\mathbb{V}$ with $\mathbb{V}^{\text{zero}}$ in (9) and (11) we get

$$E_i^{(n+1)} = \Phi_i\left(E^{(n)}\right) = A_i^e + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i,$$

$$E_i^{\mathcal{B}(n+1)} = \Phi_i^{\mathcal{B}}\left(E^{\mathcal{B}(n)}\right) = A_i^e + \sum_{j \in \mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{\mathcal{B}(n)} + \bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}} \right) - \bar{L}_i^{\mathcal{B}}.$$

[21] See also footnote 20.

By Lemma A.8, induction hypothesis (43) implies

$$\mathbb{V}^{\text{zero}}\left(\frac{E_i^{\mathcal{B}(n)}+\bar{L}_i^{\mathcal{B}}}{\bar{L}_i^{\mathcal{B}}}\right)=\mathbb{1}_{\left\{E_i^{\mathcal{B}(n)}\geq k\bar{L}_i^{\mathcal{B}}\right\}}\geq\mathbb{1}_{\left\{E_i^{(n)}\geq k\bar{L}_i\right\}}=\mathbb{V}^{\text{zero}}\left(\frac{E_i^{(n)}+\bar{L}_i}{\bar{L}_i}\right).$$

Therefore,

$$\begin{aligned}E_i^{\mathcal{B}(n+1)}&=A_i^e+\sum_{j\in\mathcal{M}^{\mathcal{B}}}L_{ji}^{\mathcal{B}}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{B}(n)}+\bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)-\bar{L}_i^{\mathcal{B}}\\&=A_i^e+\sum_{j\in\mathcal{N}}L_{ji}^{\mathcal{B}}\mathbb{1}_{\left\{E_j^{\mathcal{B}(n)}\geq k\bar{L}_j^{\mathcal{B}}\right\}}-\bar{L}_i^{\mathcal{B}}\\&\geq A_i^e+\sum_{j\in\mathcal{N}}L_{ji}^{\mathcal{B}}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\bar{L}_i^{\mathcal{B}}=:(*)\end{aligned}$$

Let $i\in\mathcal{N}\setminus\mathcal{B}$. Then,

$$\begin{aligned}(*)&=A_i^e+\sum_{j\in\mathcal{N}}L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\bar{L}_i\\&=A_i^e+\sum_{j\in\mathcal{M}}L_{ji}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)-\bar{L}_i\\&=E_i^{(n+1)}.\end{aligned}$$

Let $i\in\mathcal{B}$ with $\Lambda_i^{\mathcal{B}}<0$. Then,

$$\begin{aligned}(*)&=A_i^e+\sum_{j\in\mathcal{M}^{\mathcal{B}}\setminus\mathcal{B}}L_{ji}^{\mathcal{B}}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\bar{L}_i^{\mathcal{B}}\quad\text{(by (42))}\\&=A_i^e+\sum_{j\in\mathcal{M}\setminus\mathcal{B}}L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\bar{L}_i^{\mathcal{B}}\quad(\text{since }\mathcal{M}\setminus\mathcal{B}=\mathcal{M}^{\mathcal{B}}\setminus\mathcal{B})\\&=A_i^e+\sum_{j\in\mathcal{M}}L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\sum_{j\in\mathcal{B}}L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\bar{L}_i^{\mathcal{B}}\\&\geq A_i^e+\sum_{j\in\mathcal{M}}L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}-\underbrace{\left(\sum_{j\in\mathcal{B}}L_{ji}+\bar{L}_i^{\mathcal{B}}\right)}_{=\bar{L}_i\text{ by Lemma A.4 and (42)}}\\&=A_i^e+\sum_{j\in\mathcal{M}}L_{ji}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)-\bar{L}_i=E_i^{(n+1)}.\end{aligned}$$

Let $i\in\mathcal{B}$ with $\Lambda_i^{\mathcal{B}}\geq 0$. Then, for all $j\in\mathcal{M}^{\mathcal{B}}\cap\mathcal{B}$ such that $L_{ji}^{\mathcal{B}}>0$ we have $\Lambda_j^{\mathcal{B}}<0$ by (42). Moreover, the assumption $\{i\in\mathcal{B}\,|\,\Lambda_i^{\mathcal{B}}<0\}\cap\mathcal{D}(L,A^e;\mathbb{V}^{\text{zero}})=\emptyset$ implies that such j do not default and hence, $\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}}=1$ for all $j\in\mathcal{M}^{\mathcal{B}}\cap\mathcal{B}$ for which $L_{ji}^{\mathcal{B}}>0$. Therefore, we obtain

$$\begin{aligned}
(*) &= A_i^e + \sum_{j\in\mathcal{M}^{\mathcal{B}}\cap\mathcal{B}} L_{ji}^{\mathcal{B}}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}} + \sum_{j\in\mathcal{M}^{\mathcal{B}}\setminus\mathcal{B}} L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}} - \bar{L}_i^{\mathcal{B}} \\
&= A_i^e + \sum_{j\in\mathcal{M}^{\mathcal{B}}\cap\mathcal{B}} L_{ji}^{\mathcal{B}} + \sum_{j\in\mathcal{M}\setminus\mathcal{B}} L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}} - \bar{L}_i^{\mathcal{B}} \quad (\text{since } \mathcal{M}\setminus\mathcal{B} = \mathcal{M}^{\mathcal{B}}\setminus\mathcal{B}) \\
&= A_i^e + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}} - \sum_{j\in\mathcal{M}\cap\mathcal{B}} L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}} + \sum_{j\in\mathcal{M}^{\mathcal{B}}\cap\mathcal{B}} L_{ji}^{\mathcal{B}} - \bar{L}_i^{\mathcal{B}} \\
&\geq A_i^e + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{1}_{\left\{E_j^{(n)}\geq k\bar{L}_j\right\}} - \sum_{j\in\mathcal{M}\cap\mathcal{B}} L_{ji} + \sum_{j\in\mathcal{M}^{\mathcal{B}}\cap\mathcal{B}} L_{ji}^{\mathcal{B}} - \bar{L}_i^{\mathcal{B}} \\
&= A_i^e + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \quad (\text{by Lemma A.4}) \\
&= E_i^{(n+1)}.
\end{aligned}$$

This finishes the induction proof and the proof is complete now as $\mathcal{D}(L^{\mathcal{B}}, A^e; \mathbb{V}^{\text{zero}}) \subseteq \mathcal{D}(L, A^e; \mathbb{V}^{\text{zero}})$ follows immediately.

□

A.4 Proofs and further details on the results in Section 4.2

For a given financial network (L, A^e), Algorithm 1 computes a vector $\tilde{l}_*$ that corresponds to the payments made by all banks in the financial network. At time t, $e(t)$ consists of available liquid assets including received payments, and $\mathcal{A}(t)$ comprises banks that are able to pay in full. The assumption that banks either make full payment or pay nothing is incorporated into step 7. The main difference between the FPA and hard default in Paddrik et al. (2020) is that in the former setting we assume banks cannot anticipate future payments. This is modelled in the sense that there is a sequence of payments and banks can only pay in full if they have received sufficient liquidity. Hence, unlike the Eisenberg & Noe (2001) framework in finding equilibrium payment vector, there is no coordination among banks in the FPA to determine the payments.

Next, we consider an algorithm that returns the least clearing vector in the model by Rogers & Veraart (2013). We consider only the case where the defaulting nodes make zero payments which corresponds to setting $\alpha = \beta = 0$ in the model by Rogers & Veraart (2013) and hence the algorithm presented there simplifies to Algorithm 2. We refer to Algorithm 2 as the Least Clearing Vector Algorithm (LA). The algorithm starts by assuming that initially there is no "solvent" bank that would be able to make any payment. $\mathcal{S}^{(0)}$ is the set of banks that would be able to pay liabilities in full even if all other banks did not meet their obligations. Similar to the construction in (Rogers & Veraart, 2013, Theorem 3.7), as the algorithm terminates, the output is the least clearing vector.

We use Lemma A.9 to prove Proposition 4.8.

Lemma A.9. Consider the FPA (Algorithm 1) and the LA (Algorithm 2) described in Figure 8.

Algorithm 1 Full Payment Algorithm (FPA) in Bardoscia et al. (2019a)

1: Set $e(0) := A^e$, $l(0) := \mathbf{0}$, and $\mathcal{A}(0) := \emptyset$. Set $t = 1$.
2: For all $i \in \mathcal{N}$, set

$$e_i(t) = e_i(t-1) + \sum_{j \in \mathcal{N}} l_j(t-1)\Pi_{ji} - l_i(t-1). \tag{44}$$

3: Determine

$$\mathcal{A}(t) = \{i \in \mathcal{N} \mid e_i(t) \geq \bar{L}_i\} \setminus \bigcup_{s=0}^{t-1} \mathcal{A}(s). \tag{45}$$

4: **if** $\mathcal{A}(t) \equiv \emptyset$ **then**
5:     **return** $\tilde{l}_* = \sum_{s=0}^{t-1} l(s)$.
6: **else**
7:     set $l_i(t) = \bar{L}_i$ for all $i \in \mathcal{A}(t)$, and $l_i(t) = 0$ otherwise.
8: **end if**
9: Set $t = t + 1$ and go back to step 2.

Algorithm 2 Least Clearing Vector Algorithm (LA) for Rogers & Veraart (2013) model with $\alpha = \beta = 0$

1: Set $t = 0$, $l^{(0)} := \mathbf{0}$, and $\mathcal{D}^{(-1)} := \mathcal{N}$.
2: For all $i \in \mathcal{N}$, determine

$$v_i^{(t)} := A_i^e + \sum_{j \in \mathcal{N}} l_j^{(t)} \Pi_{ji} - \bar{L}_i. \tag{46}$$

3: Define

$$\mathcal{D}^{(t)} := \{i \in \mathcal{N} \mid v_i^{(t)} < 0\} \text{ and } \mathcal{S}^{(t)} := \{i \in \mathcal{N} \mid v_i^{(t)} \geq 0\}. \tag{47}$$

4: **if** $\mathcal{D}^{(t)} \equiv \mathcal{D}^{(t-1)}$ **then**
5:     **return** $l_* = l^{(t-1)}$.
6: **else**
7:     set $l_i^{(t+1)} = \bar{L}_i$ for all $i \in \mathcal{S}^{(t)}$, and $l_i^{(t+1)} = 0$ for all $i \in \mathcal{D}^{(t)}$.
8: **end if**
9: Set $t = t + 1$ and go back to step 2.

Figure 8: Clearing algorithms.

Fix an iteration $t \in \mathbb{N}_0$. Then,

$$\bigcup_{s=0}^{t+1} \mathcal{A}(s) = \mathcal{S}^{(t)}. \tag{48}$$

In particular, the banks that make payments in the FPA up to time $t+1$ are identical to the banks that make payments in the LA up to time $t+1$.

Proof of Lemma A.9. We prove the result by induction. Let $t = 0$. By plugging the initial values of Algorithm 1 into (44) and (45) we obtain that $\bigcup_{s=0}^{1} \mathcal{A}(s) = \mathcal{A}(0) \cup (\mathcal{A}(1) \setminus \mathcal{A}(0)) = \mathcal{A}(1) = \{i \in \mathcal{N} \,|\, e_i(1) \geq \bar{L}_i\} = \{i \in \mathcal{N} \,|\, A_i^e \geq \bar{L}_i\}$. Furthermore, $\mathcal{S}^{(0)} = \{i \in \mathcal{N} \,|\, A_i^e - \bar{L}_i \geq 0\}$ in Algorithm 2 and hence $\bigcup_{s=0}^{1} \mathcal{A}(s) = \mathcal{S}^{(0)}$.

Now suppose (48) holds for a fixed t. We show that it also holds for $t+1$, i.e., we show that $\bigcup_{s=0}^{t+2} \mathcal{A}(s) = \mathcal{S}^{(t+1)}$.

We rewrite

$$\bigcup_{s=0}^{t+2} \mathcal{A}(s) = \bigcup_{s=0}^{t+1} \mathcal{A}(s) \cup \mathcal{A}(t+2) \text{ and } \mathcal{S}^{(t+1)} = \mathcal{S}^{(t)} \cup \left(\mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}\right),$$

and therefore it is sufficient to prove that $\mathcal{A}(t+2) = \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$.

First, note that according to the definition of Algorithm 1, for every $i \in \mathcal{N}$, we can write $e_i(t+1)$ as

$$\begin{aligned}
e_i(t+1) &= e_i(t) + \sum_{j \in \mathcal{N}} l_j(t)\Pi_{ji} - l_i(t) \\
&= e_i(t-1) + \sum_{j \in \mathcal{N}} l_j(t-1)\Pi_{ji} - l_i(t-1) + \sum_{j \in \mathcal{N}} l_j(t)\Pi_{ji} - l_i(t) \\
&= \cdots \\
&= A_i^e + \sum_{j \in \mathcal{N}} \Pi_{ji} \sum_{s=0}^{t} l_j(s) - \sum_{s=0}^{t} l_i(s) \\
&\overset{(\star)}{=} A_i^e + \sum_{j \in \bigcup_{s=0}^{t} \mathcal{A}(s)} \Pi_{ji} \sum_{s=0}^{t} l_j(s) - \sum_{s=0}^{t} l_i(s) \\
&\overset{(\star\star)}{=} A_i^e + \sum_{j \in \bigcup_{s=0}^{t} \mathcal{A}(s)} \bar{L}_j \Pi_{ji} - \sum_{s=0}^{t} l_i(s),
\end{aligned} \tag{49}$$

where $(\star)$ follows from the fact that $\sum_{s=0}^{t} l_j(s) > 0$ implies $j \in \bigcup_{s=0}^{t} \mathcal{A}(s)$ (see step 3-7 in Algorithm 1). Furthermore, $(\star\star)$ holds, since for these $j \in \bigcup_{s=0}^{t} \mathcal{A}(s)$ it holds that $\sum_{s=0}^{t} l_j(s) = \bar{L}_j$.

In addition, we can rewrite (46) at $t+1$ as

$$v_i^{(t+1)} = A_i^e + \sum_{j \in \mathcal{N}} l_j^{(t+1)}\Pi_{ji} - \bar{L}_i = A_i^e + \sum_{j \in \mathcal{S}^{(t)}} \bar{L}_j \Pi_{ji} - \bar{L}_i. \tag{50}$$

First, we show that $\mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)} \subseteq \mathcal{A}(t+2)$.

Let $i \in \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$, then since by the induction hypothesis $\mathcal{S}^{(t)} = \bigcup_{s=0}^{t+1} \mathcal{A}(s)$, it holds that $i \notin \bigcup_{s=0}^{t+1} \mathcal{A}(s)$. From the definition of $l_i(s)$, $s \in \{0, \ldots, t+1\}$ this implies that $\sum_{s=0}^{t+1} l_i(s) = 0$. Combining these results gives,

$$
\begin{aligned}
0 \leq v_i^{(t+1)} &= A_i^e + \sum_{j \in \mathcal{S}^{(t)}} \bar{L}_j \Pi_{ji} - \bar{L}_i + \underbrace{0}_{= \sum_{s=0}^{t+1} l_i(s)} = A_i^e + \sum_{j \in \bigcup_{s=0}^{t+1} \mathcal{A}(s)} \bar{L}_j \Pi_{ji} + \sum_{s=0}^{t+1} l_i(s) - \bar{L}_i \\
&= e_i(t+2) - \bar{L}_i
\end{aligned}
$$

and hence $i \in \mathcal{A}(t+2)$.

Second, we show that $\mathcal{A}(t+2) \subseteq \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$.

Let $i \in \mathcal{A}(t+2)$. Hence,

$$
\begin{aligned}
\bar{L}_i \leq e_i(t+2) &= A_i^e + \sum_{j \in \bigcup_{s=0}^{t+1} \mathcal{A}(s)} \bar{L}_j \Pi_{ji} - \sum_{s=0}^{t+1} l_i(s) \overset{\text{ind.hyp.}}{=} A_i^e + \sum_{j \in \mathcal{S}(t)} \bar{L}_j \Pi_{ji} - \underbrace{\sum_{s=0}^{t+1} l_i(s)}_{\overset{(\diamond)}{=} 0} \\
&= A_i^e + \sum_{j \in \mathcal{S}(t)} \bar{L}_j \Pi_{ji},
\end{aligned}
\tag{51}
$$

where $(\diamond)$ holds, since the definition of $\mathcal{A}(t+2)$ implies that $i \notin \bigcup_{s=0}^{t+1} \mathcal{A}(s) \overset{\text{ind.hyp}}{=} \mathcal{S}^{(t)}$, which implies that $\sum_{s=0}^{t+1} l_i(s) = 0$. Combining this with (51) implies that $0 \leq A_i^e + \sum_{j \in \mathcal{S}(t)} \bar{L}_j \Pi_{ji} - \bar{L}_i = v_i^{(t+1)}$ and therefore $i \in \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$. $\square$

Now we can prove the result that the outcomes of Algorithms 1 and 2 in Figure 8 coincide.

Proof of Proposition 4.8. Suppose for a fixed $t > 0$ at one iteration it holds that, $\mathcal{A}(t) = \emptyset$. Then $\tilde{l}_\star = \sum_{s=0}^{t-1} l(s)$, where $\tilde{l}_{\star,i} = \bar{L}_i$ if $i \in \bigcup_{s=0}^{t-1} \mathcal{A}(s)$ and 0 otherwise.

Since by Lemma A.9, $\bigcup_{s=0}^{t-1} \mathcal{A}(s) = \mathcal{S}^{(t-2)}$, this means that $\mathcal{A}(t) = \emptyset$ is equivalent to $\mathcal{D}^{(t)} = \mathcal{D}^{(t-1)}$ in the LA. Furthermore, the LA returns $l_\star = l^{(t-1)}$, where $l_{\star,i} = \bar{L}_i$ if $i \in \mathcal{S}(t-2)$ and 0 otherwise.

Therefore, both algorithms terminate when the same banks are selected and all their payments are identical.

Moreover, by Rogers & Veraart (2013), the LA generates a sequence of vectors increasing to the least clearing vector, so the statement follows immediately. $\square$

Proof of Proposition 4.9. Note that $\mathbb{V}^{\text{zero}} = \mathbb{V}^{\text{RV}}$ by letting $\alpha = \beta = 0$. This result is the analogue to the result for the greatest clearing vector presented in (Veraart, 2020a, Theorem 2.9) and can be proved in the same way. Therefore, we omit the proof. $\square$

Proof of Proposition 4.10. Given financial network (L, A^e), let $\underset{\sim}{l}$ and $\tilde{l}$ be the least and greatest clearing vector respectively.

It is sufficient to show that $\underset{\sim}{l}_i = \tilde{l}_i$ for all $i \in \mathcal{N}$ such that $\bar{L}_i > 0$. We prove by contradiction. By definition, $\underset{\sim}{l}_i \leq \tilde{l}_i$, and they take values from 0 or $\bar{L}_i > 0$. Suppose $\underset{\sim}{l} \neq \tilde{l}$. It follows that there exists j such that $\underset{\sim}{l}_j = 0$ and $\tilde{l}_j = \bar{L}_j$. Let B_i be the set of direct "borrowers" of i, i.e., there exists a path of length 1 from j to i for every $j \in B_i$. Since j's payments are different in two clearing vectors, payments that j receives cannot be the same, implying that there exists $k \in B_j$ such that $\underset{\sim}{l}_k = 0$ and $\tilde{l}_k = \bar{L}_k$. We continue and end up with m for which $B_m = \emptyset$ because the graph is a DAG and the reachability relation forms a partial order $\leq$ on the nodes. Two nodes a and b are ordered as $a \leq b$ when there exists a path from a to b. Hence, we can always find such m. In particular, we have $\underset{\sim}{l}_m = 0$ and $\tilde{l}_m = \bar{L}_m$, which is a contradiction as the clearing payment for m is unique since it never receives any payments from any other nodes in the system.

$\square$

Proof of Theorem 4.11. Part 2. has been shown by Example 4.12, so we only prove part 1. here. Let $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges})$ be a compression network cycle. Let l^* and $l^{\mathcal{C};*}$ be the outputs of the FPA for the original network (L, A^e) and μ-compressed network $(L^{\mathcal{C}}, A^e)$ respectively. Set

$$\mathcal{A} = \{i \in \mathcal{N} \,|\, l_i^* > 0\},$$
$$\mathcal{A}^{\mathcal{C}} = \{i \in \mathcal{N} \,|\, l_i^{\mathcal{C};*} > 0\}.$$

By Proposition 4.9, for all $i \in \mathcal{N}$, the least re-evaluated equity under zero recovery rate in (L, A^e) is

$$E_i^* = A_i^e + \sum_{j:\bar{L}_j>0} L_{ji} \frac{l_j^*}{\bar{L}_j} - \bar{L}_i = A_i^e + \sum_{j \in \mathcal{A}} L_{ji} - \bar{L}_i.$$

Similarly, the least re-evaluated equity under zero recovery rate in $(L^{\mathcal{C}}, A^e)$ is

$$E_i^{\mathcal{C};*} = A_i^e + \sum_{j \in \mathcal{A}^{\mathcal{C}}} L_{ji}^{\mathcal{C}} - \bar{L}_i^{\mathcal{C}}, \quad \forall\, i \in \mathcal{N}.$$

In order to prove $E_i^* \leq E_i^{\mathcal{C};*}$ for all $i \in \mathcal{N}$, it is sufficient to show that $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{C}}$, i.e., $l_i^* > 0$ implies $l_i^{\mathcal{C};*} > 0$.

Let $\tilde{l}^*$ be the output of the FPA for $(L^{\mathcal{C}}, \tilde{A}^e)$, where

$$\tilde{A}_i^e = \begin{cases} A_i^e - \mu, & \text{if } i \in \mathcal{C}_{nodes}, \\ A_i^e, & \text{otherwise.} \end{cases}$$

We first show that $l^* = \tilde{l}^*$ by considering three types of banks. A bank is *fundamentally liquid* if its external assets are sufficient to cover its total liabilities. A bank that is not fundamentally liquid but is able to make full payment provided it receives payments from its counterparties is *conditionally liquid*, otherwise it is said to be *fundamentally illiquid*. For bank i that is either fundamentally liquid or fundamentally illiquid, $l_i^* = \tilde{l}_i^*$. Now suppose that bank i is conditionally liquid. Let $\mathcal{A}(t)$ and $\tilde{\mathcal{A}}(t)$ be the sets defined by (45) for financial networks (L, A^e) and $(L^{\mathcal{C}}, \tilde{A}^e)$ respectively. Since $A_i^e - \bar{L}_i = \tilde{A}_i^e - \bar{L}_i^{\mathcal{C}}$, it follows that at each iteration t, $\mathcal{A}(t) = \tilde{\mathcal{A}}(t)$. Hence, $l_i^* = \tilde{l}_i^*$.

Furthermore, it is obvious that $l^{\mathcal{C};*} \geq \tilde{l}^*$ because $A^e \geq \tilde{A}^e$. Therefore, $l^{\mathcal{C};*} \geq l^*$.

□

B Additional results on PTRRs

B.1 Definitions of portfolio compression

D'Errico & Roukny (2021) formulate a linear programming characterisation for conservative compression, which is given as follows.

Definition B.1 (Conservative compression problem in D'Errico & Roukny (2021))**.** Let $C \in [0,\infty)^{N\times N}$ be a notional matrix. The optimisation problem is

$$\begin{aligned} \min \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}^c \\ \text{s.t.} \quad & \sum_{j=1}^{N}(C_{ji}^c - C_{ij}^c) = \sum_{j=1}^{N}(C_{ji} - C_{ij}) && \forall\, i \in \mathcal{N}, \\ & 0 \leq C_{ij}^c \leq C_{ij} && \forall\, i \in \mathcal{N},\, j \in \mathcal{N}. \end{aligned}$$

We provide a new optimisation-based characterisation for conservative compression, that builds on the idea of O'Kane (2017).

Definition B.2 (Conservative compression optimisation problem)**.** Let $C \in [0,\infty)^{N\times N}$ be a notional matrix. The *conservative compression optimisation problem* is given by

$$\min \quad \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}^c \tag{52}$$

$$\text{s.t.} \quad \sum_{j=1}^{N}(C_{ji}^c - C_{ij}^c) = \sum_{j=1}^{N}(C_{ji} - C_{ij}) \qquad \forall\, i \in \mathcal{N}, \tag{53}$$

$$0 \leq C_{ij}^c \leq \max\left(C_{ij} - C_{ji}, 0\right) \qquad \forall\, i \in \mathcal{N},\, j \in \mathcal{N}. \tag{54}$$

Conservative compression by Definition B.2 takes place on loops of claims defined by bilaterally netted notional matrix C^{bi}. In fact, as explained in O'Kane (2017), bilateral netting is implicit in the fungible market, so we may implement compression algorithms based on net exposures between each pair of counterparties.

Definition B.2 implies that we can reduce gross notional exposures through multilateral netting (other than bilateral netting) only if there exists at least one cycle in the graph associated with

C^{bi}. Note that constraint (54) is more stringent than the conservative compression tolerance in Definition B.1.

Next, we present the non-conservative compression problem introduced in D'Errico & Roukny (2021) in the spirit of O'Kane (2017) under the name L_1 compression.[22]

Definition B.3 (Non-conservative compression problem in D'Errico & Roukny (2021))**.** Let $C \in [0,\infty)^{N\times N}$ be a notional matrix. The optimisation problem is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}^{nc} \\
\text{s.t.} \quad & \sum_{j=1}^{N}(C_{ji}^{nc} - C_{ij}^{nc}) = \sum_{j=1}^{N}(C_{ji} - C_{ij}) && \forall\, i \in \mathcal{N}, \\
& C_{ij}^{nc} \geq 0 && \forall\, i \in \mathcal{N},\, j \in \mathcal{N}.
\end{aligned}$$

As mentioned in Section 2.2, the optimisation problem in Definition B.3 often has multiple solutions.

Proposition B.4. Let $C \in [0,\infty)^{N\times N}$ be a notional matrix and $\mathcal{N}$ the corresponding set of banks with $|\mathcal{N}| = N$ where $N \geq 4$. Let $\Lambda_i^{net} = \sum_{j=1}^{N}(C_{ji} - C_{ij})$ be the net notional exposure of bank i. Let $\mathcal{N}^+ = \{s \in \mathcal{N} \mid \Lambda_s^{net} > 0\}$ and $\mathcal{N}^- = \{s \in \mathcal{N} \mid \Lambda_s^{net} < 0\}$. Then, there are multiple solutions to the optimisation problem in Definition B.3 if and only if $|\mathcal{N}^+| \geq 2$ and $|\mathcal{N}^-| \geq 2$.

Proof of Proposition B.4. The "only if" direction is obvious because the graph associated with the compressed notional matrix is bipartite by (D'Errico & Roukny, 2021, Lemma 1), so we prove the "if" direction only. Without loss of generality, suppose we have banks in $\mathcal{N} = \{1, 2, 3, 4\}$. Let $\mathcal{N}^+ = \{3, 4\}$ and $\mathcal{N}^- = \{1, 2\}$, and let the compressed notional matrix be

$$C^{nc} = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a, b > 0$. Let $\delta^{max} = \min\{a, b\}$ and $0 < \delta \leq \delta^{max}$. Then, the reconstructed notional matrix

$$C^{nc'} = \begin{pmatrix} 0 & 0 & a-\delta & \delta \\ 0 & 0 & \delta & b-\delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[22] Mathematical programs that characterise the solution of different types of compression have been summarised in O'Kane (2017) and D'Errico & Roukny (2021). By (D'Errico & Roukny, 2021, Lemma 1), the non-conservatively compressed network is bipartite, in which banks with positive net exposures will only have incoming edges and banks with negative net exposures will only have outgoing edges. Therefore, non-conservative compression can also be characterised as a transportation problem.

maintains the net notional exposure for all banks and keeps the gross notional exposure unchanged, thus is another solution for the optimisation problem. Hence, there are infinitely many solutions. □

We notice that non-conservative portfolio compression achieves both netting and counterparty risk rebalancing, so it can be seen as a *hybrid* between conservative compression and (optimal) rebalancing on a net basis.

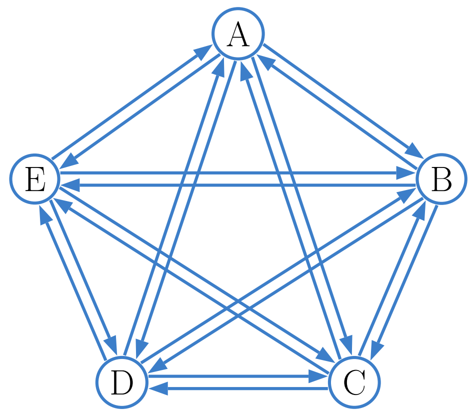
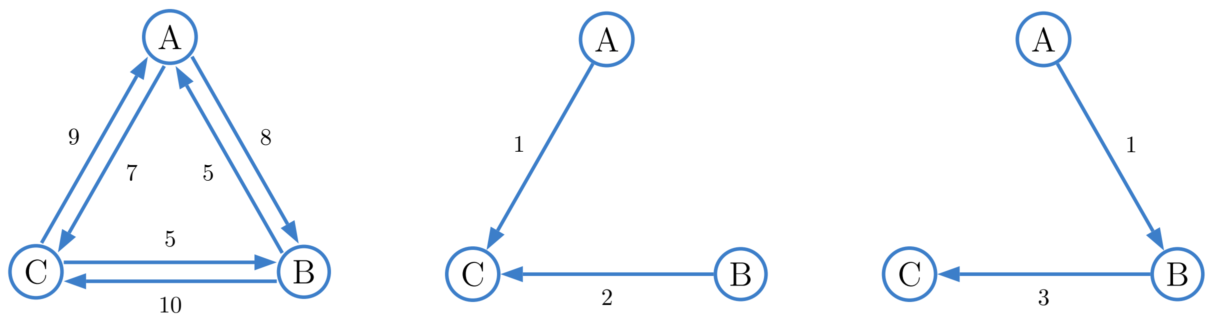


Figure 9: Illustration of the dealers market.

In our paper, we consider Definition 2.9 and Definition B.2 for (conservative) portfolio compression instead of Definition B.1. First, we assume that portfolio compression takes place in inter-dealer markets. Major dealers trade with each other and have contracts in both directions, therefore the trading relationship between each pair of dealers always exists, which means that dealers are buyers and sellers to each other (see Figure 9). Second, counterparty relationships (or net credit exposures) can change over time by a variety of reasons, such as new transactions, terminations etc.. We use Figure 10 to demonstrate the difference between Definition B.1 and our Definition B.2 of conservative compression. There exists a cycle in the graph associated with the bilaterally netted network, so by Definition B.2 the netted positions on the cycle are reduced by 1 and the outcome is shown in Figure 10c. According to Definition B.1, notional amount can be reduced as long as it is not larger than the original amount. Therefore, we see in Figure 10b that B has obligation 1 (which is less than 7) towards A while A has no obligation to B. Indeed, this network is also the outcome of non-conservative compression problem in Definition B.3. Figure 10b and 10c reflect that our definition of conservative compression does not allow switching bilateral relationships (i.e., net buyer or net seller).

We provide a sufficient condition under which conservative and non-conservative compression in the sense of D'Errico & Roukny (2021) are equivalent.



(a) Non-compressed. (b) Definition B.1. (c) Definition B.2.

Figure 10: Outcomes of conservative compression according to different optimisation problems. Numbers represent notional amount of contracts.

Proposition B.5. Consider a notional matrix C representing the dealers market, so

$$\min_{i,j\in\mathcal{N}} C_{ij} > 0.$$

Let $\Lambda_i^{net} = \sum_{j=1}^{N}(C_{ji} - C_{ij})$ be the net notional exposure of bank i. Suppose

$$\max_{i\in\mathcal{N}} |\Lambda_i^{net}| \leq \min_{i,j\in\mathcal{N}} C_{ij}. \tag{55}$$

Then, the conservatively and non-conservatively compressed networks according to Definition B.1 and Definition B.3 respectively have the same gross notional exposure.

Proof of Proposition B.5. It suffices to show that the non-conservatively compressed notional matrix C^{nc} is a feasible solution to the conservative compression optimisation problem in Definition B.1. Because the constraints on net notional exposure are the same, it remains to show that $C_{ij}^{nc} \leq C_{ij}$ for all $i, j \in \mathcal{N}$. By (55), we obtain

$$C_{ij}^{nc} \leq |\Lambda_i^{net}| \leq \max_{i\in\mathcal{N}} |\Lambda_i^{net}| \leq \min_{i,j\in\mathcal{N}} C_{ij} \leq C_{ij}.$$

□

Condition (55) implies that the net-gross ratio satisfies

$$\frac{\sum_{i\in\mathcal{N}} |\Lambda_i^{net}|}{\sum_{i,j\in\mathcal{N}} C_{ij}} \leq \frac{\sum_{i\in\mathcal{N}} \max_{i\in\mathcal{N}} |\Lambda_i^{net}|}{\sum_{i,j\in\mathcal{N}} \min_{i,j\in\mathcal{N}} C_{ij}} \leq \frac{N \min_{i,j\in\mathcal{N}} C_{ij}}{N^2 \min_{i,j\in\mathcal{N}} C_{ij}} = \frac{1}{N}.$$

Therefore, a compression exercise consisting of 6 banks will have net-gross ratio less than 16.7%. It gives us an intuition that in the financial networks with relatively low net-gross ratio, conservative and non-conservative compression in the D'Errico & Roukny (2021) setting have comparable performance in removing market excess, as the example in Figure 10 illustrates.

D'Errico & Roukny (2021) also characterise a hybrid compression, which is composed of non-conservative compression in the inter-dealer segment and conservative compression in the dealer-customer segment. According to different compression tolerances, they apply linear programming framework to their data set and compute compression efficiency which is defined as the ratio of exposures that can be removed and the total excess (i.e., maximum removable exposures without additional constraints). From the statistics of compression efficiency, the authors find similar performance of hybrid and conservative compression. Both types of compression show very close results on the minimum, maximum and mean value. They interpret this as "results from the conservative compression show that, even under rather stringent constraints, the vast majority of market excess can be eliminated." Based on the intuition from Proposition B.5, the finding could potentially just be the consequence of a relatively low net-gross ratio in their data set.

B.2 Optimal portfolio rebalancing and non-conservative portfolio compression

Both O'Kane (2017) and D'Errico & Roukny (2021) consider a type of portfolio compression that can be characterised as a linear programming problem with the objective to reduce the gross exposures in the network while keeping the net exposures of each counterparty unchanged. We define non-conservative portfolio compression as follows.

Definition B.6 (Non-conservative $\mathcal{B}$-compression). Let C be a notional matrix and let $\mathcal{B} \subseteq \mathcal{N}$. We refer to the optimisation problem

$$\min \quad \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}_{ij} \tag{56}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{B}}(\tilde{C}_{ji} - \tilde{C}_{ij}) = \sum_{j\in\mathcal{B}}(C_{ji} - C_{ij}) \qquad \forall\, i \in \mathcal{B}, \tag{57}$$

$$\tilde{C}_{ij} \geq 0 \qquad \forall\, i \in \mathcal{B},\, j \in \mathcal{B}, \tag{58}$$

as *non-conservative $\mathcal{B}$-compression problem.*

Our definition of the non-conservative $\mathcal{B}$-compression reduces to the non-conservative compression problem of D'Errico & Roukny (2021) for $\mathcal{B} = \mathcal{N}$. The only additional constraints on the new positions are that they are non-negative. This implies in particular that an optimal solution to this optimisation problem can results in situations where new counterparty relationships are formed between counterparties that have zero bilateral exposure prior to compression, or the original counterparty relationships are swapped after the exercise. In this sense, one can consider this as an *non-conservative* exercise.

The non-conservative $\mathcal{B}$-compression problem is a linear programming problem and it admits a solution, since the feasible region is non-empty (since the matrix $C_{\mathcal{B}\mathcal{B}}$ satisfies both constraints (57) and (58)) and since the objective function is bounded from below by 0.

The next proposition establishes that an optimal rebalancing exercise always exists and how it is related to a solution of the non-conservative $\mathcal{B}$-compression problem.

Proposition B.7. Let C be a notional matrix and let $\mathcal{B}$ be a rebalancing set.

1. Let $\tilde{C}^*$ be a solution to the non-conservative $\mathcal{B}$-compression problem. Then, there exists a rebalancing matrix $\tilde{R}^*$ that depends on $\tilde{C}^*$ such that $C + \tilde{R}^*$ solves the optimal rebalancing problem.

2. Let R^* be a solution to the optimal rebalancing problem. Then, the matrix $(C + R^*)^{bi}$ solves the non-conservative $\mathcal{B}$-compression problem.

3. Let $\tilde{C}^*$ be a solution to the non-conservative $\mathcal{B}$-compression problem and let R^* be a solution to the optimal rebalancing problem. Then,

$$\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}^*_{ij} = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} (C + R^*)^{bi}_{ij},$$

   i.e., both optimisation problems attain the same minimum.

We will use the following Lemma to prove Proposition B.7.

Lemma B.8. 1. Let $\tilde{C}^*$ be a solution to the non-conservative $\mathcal{B}$-compression problem in Definition B.6. Then, define $\tilde{R}^* \in [0,\infty)^{N\times N}$ as follows.

(a) For all $(i,j) \in \mathcal{B}\times\mathcal{B}$ with $\tilde{C}^*_{ij} > 0$ and $C^{bi}_{ij} = 0$, we set

$$\tilde{R}^*_{ij} = C^{bi}_{ji} + \tilde{C}^*_{ij}, \qquad \tilde{R}^*_{ji} = 0 = \underbrace{C^{bi}_{ij}}_{=0} + \underbrace{\tilde{C}^*_{ji}}_{=0}.$$

(b) For all $(i,j) \in \mathcal{B}\times\mathcal{B}$ with $C^{bi}_{ij} \geq \tilde{C}^*_{ij} > 0$, we set

$$\tilde{R}^*_{ij} = 0 = \underbrace{C^{bi}_{ji}}_{=0} - \underbrace{\tilde{C}^*_{ji}}_{=0}, \qquad \tilde{R}^*_{ji} = C^{bi}_{ij} - \tilde{C}^*_{ij}.$$

(c) For all $(i,j) \in \mathcal{B}\times\mathcal{B}$ with $\tilde{C}^*_{ij} \geq C^{bi}_{ij} > 0$, we set

$$\tilde{R}^*_{ij} = \tilde{C}^*_{ij} - C^{bi}_{ij}, \qquad \tilde{R}^*_{ji} = 0 = \underbrace{\tilde{C}^*_{ji}}_{=0} - \underbrace{C^{bi}_{ji}}_{=0}.$$

(d) If $\tilde{C}^*_{ij} = \tilde{C}^*_{ji} = 0$ then we set $\tilde{R}^*_{ij} = C^{bi}_{ji}$ and $\tilde{R}^*_{ji} = C^{bi}_{ij}$.

(e) For all $(i,j) \in \mathcal{B}\times\mathcal{B}$ with $\tilde{C}^*_{ji} > 0$ the definitions above apply to the swapped index pair (j,i) with corresponding labels (a'), (b') and (c').

(f) For all $(i,j) \notin \mathcal{B}\times\mathcal{B}$ we set $\tilde{R}^*_{ij} = 0$.

Then, $(C, \mathcal{B}, \tilde{R}^*)$ is an optimal $\mathcal{B}$-rebalancing exercise and

$$\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}^*_{ij} = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} (C + \tilde{R}^*)^{bi}_{ij}. \tag{59}$$

2. Let $(C, \mathcal{B}, R^*)$ be an optimal $\mathcal{B}$-rebalancing exercise. Then, $\tilde{C} = (C + R^*)^{bi}$ solves the non-conservative $\mathcal{B}$-compression problem.

Note that (59) says that the minimum of the non-conservative compression problem is identical to the minimum of the optimal rebalancing problem. Furthermore, the Lemma establishes a link between the minimisers of these two optimisation problems.

Proof of Lemma B.8. 1. It follows directly from the definition of $\tilde{R}^*$ that $\tilde{R}^*_{ij} \geq 0 \; \forall \, i,j \in \mathcal{N}$. One can check that $\tilde{R}^*$ satisfies condition (1), that is

$$\sum_{j \in \mathcal{B}} \tilde{R}^*_{ji} = \sum_{j \in \mathcal{B}} \tilde{R}^*_{ij} \quad \forall \, i \in \mathcal{B}.$$

Indeed, for all $i \in \mathcal{B}$,

$$\begin{aligned}
\sum_{j \in \mathcal{B}} (\tilde{R}^*_{ji} - \tilde{R}^*_{ij}) &= \sum_{j:(a)} (-C^{bi}_{ji} - \tilde{C}^*_{ij}) + \sum_{j:(b)} (C^{bi}_{ij} - \tilde{C}^*_{ij}) + \sum_{j:(c)} (C^{bi}_{ij} - \tilde{C}^*_{ij}) + \sum_{j:(a')} (C^{bi}_{ij} + \tilde{C}^*_{ji}) \\
&\quad + \sum_{j:(b')} (-C^{bi}_{ji} + \tilde{C}^*_{ji}) + \sum_{j:(c')} (-C^{bi}_{ji} + \tilde{C}^*_{ji}) + \sum_{j:(d)} (C^{bi}_{ij} - C^{bi}_{ji}) \\
&= \underbrace{\sum_{j:(a)} -C^{bi}_{ji} + \sum_{j:(b)} C^{bi}_{ij} + \sum_{j:(c)} C^{bi}_{ij} + \sum_{j:(a')} C^{bi}_{ij} + \sum_{j:(b')} -C^{bi}_{ji} + \sum_{j:(c')} -C^{bi}_{ji} + \sum_{j:(d)} (C^{bi}_{ij} - C^{bi}_{ji})}_{=: \mathrm{I}} \\
&\quad + \underbrace{\sum_{j:(a)} -\tilde{C}^*_{ij} + \sum_{j:(b)} -\tilde{C}^*_{ij} + \sum_{j:(c)} -\tilde{C}^*_{ij} + \sum_{j:(a')} \tilde{C}^*_{ji} + \sum_{j:(b')} \tilde{C}^*_{ji} + \sum_{j:(c')} \tilde{C}^*_{ji}}_{=: \mathrm{II}} \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We have

$$\mathrm{I} = \sum_{j:(a)\cup(b)\cup(c)\cup(a')\cup(b')\cup(c')\cup(d)} (C_{ij} - C_{ji}) = \sum_{j \in \mathcal{B}} (C_{ij} - C_{ji})$$

and

$$\mathrm{II} = \sum_{j:(a)\cup(b)\cup(c)\cup(a')\cup(b')\cup(c')\cup(d)} (\tilde{C}^*_{ji} - \tilde{C}^*_{ji}) = \sum_{j \in \mathcal{B}} (\tilde{C}^*_{ji} - \tilde{C}^*_{ji}).$$

Hence, $\mathrm{I} + \mathrm{II} = 0$ follows from constraint (57) in Definition B.6.

Next, we show that $(C^{\mathcal{B}})^{bi}_{ij} = (C + \tilde{R}^*)^{bi}_{ij} = \tilde{C}^*_{ij} \; \forall \, i,j \in \mathcal{B}$.

(a) Here,

$$\begin{aligned}
C^{\mathcal{B}}_{ij} &= C_{ij} + \tilde{R}^*_{ij} = C_{ij} + C^{bi}_{ji} + \tilde{C}^*_{ij} = C_{ij} + C_{ji} - C_{ij} + \tilde{C}^*_{ij} = C_{ji} + \tilde{C}^*_{ij}, \\
C^{\mathcal{B}}_{ji} &= C_{ji} + \tilde{R}^*_{ji} = C_{ji}.
\end{aligned}$$

Therefore,

$$(C^{\mathcal{B}})^{bi}_{ij} = \max\{0, C^{\mathcal{B}}_{ij} - C^{\mathcal{B}}_{ji}\} = \tilde{C}^*_{ij},$$
$$(C^{\mathcal{B}})^{bi}_{ji} = \max\{0, C_{ji} - C_{ji} - \tilde{C}^*_{ij}\} = 0 = \tilde{C}^*_{ji}.$$

(b) Similarly, here

$$C^{\mathcal{B}}_{ij} = C_{ij} + \tilde{R}^*_{ij} = C_{ij},$$
$$C^{\mathcal{B}}_{ji} = C_{ji} + \tilde{R}^*_{ji} = C_{ij} - \tilde{C}^*_{ij}.$$

Therefore,

$$(C^{\mathcal{B}})^{bi}_{ij} = \max\{0, C_{ij} - (C_{ij} - \tilde{C}^*_{ij})\} = \tilde{C}^*_{ij},$$
$$(C^{\mathcal{B}})^{bi}_{ji} = \max\{0, C_{ij} - \tilde{C}^*_{ij} - C_{ij}\} = 0 = \tilde{C}^*_{ji}.$$

(c) Similarly, here

$$C^{\mathcal{B}}_{ij} = C_{ij} + \tilde{R}^*_{ij} = C_{ij} + \tilde{C}^*_{ij} - (C_{ij} - C_{ji}) = \tilde{C}^*_{ij} + C_{ji},$$
$$C^{\mathcal{B}}_{ji} = C_{ji} + \tilde{R}^*_{ji} = C_{ji}.$$

Therefore,

$$(C^{\mathcal{B}})^{bi}_{ij} = \max\{0, \tilde{C}^*_{ij} + C_{ji} - C_{ji}\} = \tilde{C}^*_{ij},$$
$$(C^{\mathcal{B}})^{bi}_{ji} = \max\{0, C_{ji} - (\tilde{C}^*_{ij} + C_{ji})\} = 0 = \tilde{C}^*_{ji}.$$

(d) Similarly, here

$$C^{\mathcal{B}}_{ij} = C_{ij} + \tilde{R}^*_{ij} = C_{ij} + C^{bi}_{ji},$$
$$C^{\mathcal{B}}_{ji} = C_{ji} + \tilde{R}^*_{ji} = C_{ji} + C^{bi}_{ij}.$$

Therefore,

$$(C^{\mathcal{B}})^{bi}_{ij} = \max\{0, (C_{ij} - C_{ji}) + (C^{bi}_{ji} - C^{bi}_{ij})\} = 0 = \tilde{C}^*_{ij},$$
$$(C^{\mathcal{B}})^{bi}_{ji} = \max\{0, (C_{ji} - C_{ij}) + (C^{bi}_{ij} - C^{bi}_{ji})\} = 0 = \tilde{C}^*_{ji}.$$

The results carry over to (a'), (b') and (c'). Hence, $(C^{\mathcal{B}})^{bi}_{ij} = (C + \tilde{R}^*)^{bi}_{ij} = \tilde{C}^*_{ij} \ \forall\, i, j \in \mathcal{B}$, which then implies that (59) holds.

We show that $\tilde{R}^*$ solves the optimal rebalancing problem. To do so we consider another optimisation problem first that we refer to as *bilaterally netted* optimisation problem. It is given by

$$\min \quad \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}(\tilde{C})^{bi}_{ij} \tag{60}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{B}}(\tilde{C}_{ji}-\tilde{C}_{ij}) = \sum_{j\in\mathcal{B}}(C_{ji}-C_{ij}) \qquad \forall\, i\in\mathcal{B}, \tag{61}$$

$$\tilde{C}_{ij} \geq 0 \qquad \forall\, i,j\in\mathcal{B}, \tag{62}$$

$$\tilde{C}_{ij} = \max\{\tilde{C}_{ij}-\tilde{C}_{ji},0\} \qquad \forall\, i,j\in\mathcal{B}. \tag{63}$$

Note that the set of feasible points for the bilaterally netted optimisation problem is given by

$$F_{BN} = \{\tilde{C}\in[0,\infty)^{|\mathcal{B}|\times|\mathcal{B}|} \mid \tilde{C} \text{ satisfies (63)}\} \cap F_{NC},$$

where

$$F_{NC} = \{\tilde{C}\in[0,\infty)^{|\mathcal{B}|\times|\mathcal{B}|} \mid \tilde{C} \text{ satisfies (57) and (58)}\}$$

denotes the set of feasible points for the non-conservative $\mathcal{B}$-compression problem.

As shown in (D'Errico & Roukny, 2021, Lemma 1), any solution $\tilde{C}^*$ of the non-conservative compression problem represents notional amount of banks that have either only outgoing or only incoming edges. This implies that $\tilde{C}^* = (\tilde{C}^*)^{bi}$. Therefore, any solution $\tilde{C}^*$ to the non-conservative $\mathcal{B}$-compression problem is a feasible point (since it satisfies (63) as well) for the bilaterally netted optimisation problem and hence in F_{BN}. This implies that the bilaterally netted optimisation problem admits a solution.

Furthermore, the objective function of the bilaterally netted optimisation problem satisfies

$$\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\tilde{C}^{bi}_{ij} = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\tilde{C}_{ij} \tag{64}$$

for all $\tilde{C}$ satisfying (63) and therefore coincides with the objective function of the non-conservative compression problem.

Furthermore, $F_{BN}\subseteq F_{NC}$ and hence combining this with (64) gives

$$\min_{\tilde{C}\in F_{BN}}\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\tilde{C}^{bi}_{ij} = \min_{\tilde{C}\in F_{BN}}\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\tilde{C}_{ij} \geq \min_{\tilde{C}\in F_{NC}}\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\tilde{C}_{ij}.$$

Since by (D'Errico & Roukny, 2021, Lemma 1) any solution of the non-conservative compression problem is feasible for the bilaterally netted optimisation problem this implies that the objective functions in these two optimisation problems attain the same minimum, i.e.,

$$\min_{\tilde{C}\in F_{BN}} \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}^{bi}_{ij} = \min_{\tilde{C}\in F_{NC}} \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}_{ij}.$$

This establishes that any solution to the non-conservative compression problem is a solution to the bilaterally netted optimisation problem and vice versa.

Next, we show how the optimal rebalancing problem is related to the bilaterally netted optimisation problem.

The optimal rebalancing problem can be rewritten as

$$\begin{aligned} \min_{R_{ij}} \quad & \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} (C+R)^{bi}_{ij} \\ \text{s.t.} \quad & \sum_{j\in\mathcal{B}} \left((C_{ji}+R_{ji})-(C_{ij}+R_{ij})\right) = \sum_{j\in\mathcal{B}} (C_{ji}-C_{ij}) && \forall\, i\in\mathcal{B}, \\ & R_{ij} \geq 0 && \forall\, i,j\in\mathcal{B}, \end{aligned}$$

which is equivalent to

$$\min_{\hat{C}_{ij}} \quad \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} (\hat{C})^{bi}_{ij} \tag{65}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{B}} \left(\hat{C}_{ji}-\hat{C}_{ij}\right) = \sum_{j\in\mathcal{B}} (C_{ji}-C_{ij}) \qquad \forall\, i\in\mathcal{B}, \tag{66}$$

$$\hat{C}_{ij} \geq C_{ij} \qquad \forall\, i,j\in\mathcal{B}. \tag{67}$$

which we refer to as the *rewritten optimal rebalancing problem*. Then, the feasible set of the rewritten optimal rebalancing problem is given by

$$F_{ROR} = \{\hat{C}\in[0,\infty)^{|\mathcal{B}|\times|\mathcal{B}|} \mid \hat{C} \text{ satisfies (67) } \}\cap F_{NC}.$$

Hence, $F_{ROR}\subseteq F_{NC}$ and therefore

$$\min_{\hat{C}\in F_{ROR}} \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \hat{C}^{bi}_{ij} \geq \min_{\tilde{C}\in F_{NC}} \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}_{ij}. \tag{68}$$

Since $\tilde{C}^*$ is a solution of the non-conservative $\mathcal{B}$-compression problem and since $(C^{\mathcal{B}})^{bi} = (C+\tilde{R}^*)^{bi} = \tilde{C}^*$ it follows that the minimum of the non-conservative portfolio compression optimisation problem satisfies

$$\min_{\tilde{C}\in F_{NC}} \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}_{ij} = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} \tilde{C}^*_{ij} = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}} (C+\tilde{R}^*)^{bi}_{ij}.$$

Combining this result and with the fact that $C + \tilde{R}^* \in F_{ROR}$ implies

$$\min_{\tilde{C} \in F_{NC}} \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} \tilde{C}_{ij} = \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} (C + \tilde{R}^*)^{bi}_{ij} \geq \min_{\hat{C} \in F_{ROR}} \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} \hat{C}^{bi}_{ij}. \tag{69}$$

Combining (68) and (69) implies

$$\min_{\hat{C} \in F_{ROR}} \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} \hat{C}^{bi}_{ij} \geq \min_{\tilde{C} \in F_{NC}} \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} \tilde{C}_{ij} = \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} (C + \tilde{R}^*)^{bi}_{ij} \geq \min_{\hat{C} \in F_{ROR}} \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} \hat{C}^{bi}_{ij}$$

and hence

$$\sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} (C + \tilde{R}^*)^{bi}_{ij} = \min_{\hat{C} \in F_{ROR}} \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}} \hat{C}^{bi}_{ij}.$$

Hence, we see that $C + \tilde{R}^*$ solves the rewritten optimal rebalancing problem and hence $\tilde{R}^*$ solves the optimal rebalancing problem.

2. Let R^* be a solution to the optimal rebalancing problem. Set $\tilde{C} = (C + R^*)^{bi}$. We first show that $\tilde{C} \in F_{BN}$. It is clear that all elements of $\tilde{C}$ are non-negative and satisfies condition (63). Next, we check that for all $i \in \mathcal{B}$

$$\sum_{j \in \mathcal{B}} (\tilde{C}_{ji} - \tilde{C}_{ij}) = \sum_{j \in \mathcal{B}} (C_{ji} - C_{ij}).$$

Let $i \in \mathcal{B}$. Then it holds that

$$
\begin{aligned}
&\sum_{j\in\mathcal{B}}(\tilde{C}_{ji}-\tilde{C}_{ij}) = \sum_{j\in\mathcal{B}}\Big((C+R^*)^{bi}_{ji}-(C+R^*)^{bi}_{ij}\Big)\\
&= \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}>0}\Big((C+R^*)^{bi}_{ji}-(C+R^*)^{bi}_{ij}\Big) + \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}\le 0}\Big((C+R^*)^{bi}_{ji}-(C+R^*)^{bi}_{ij}\Big)\\
&= \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}>0}(C+R^*)^{bi}_{ji} - \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}\le 0}(C+R^*)^{bi}_{ij}\\
&= \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}>0}(C_{ji}+R^*_{ji}-C_{ij}-R^*_{ij}) - \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}\le 0}(C_{ij}+R^*_{ij}-C_{ji}-R^*_{ji})\\
&= \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}>0}(C_{ji}+R^*_{ji}-C_{ij}-R^*_{ij}) + \sum_{j\in\mathcal{B},(C+R^*)^{bi}_{ji}\le 0}(C_{ji}+R^*_{ji}-C_{ij}-R^*_{ij})\\
&= \sum_{j\in\mathcal{B}}(C_{ji}+R^*_{ji}-C_{ij}-R^*_{ij})\\
&= \underbrace{\sum_{j\in\mathcal{B}}(R^*_{ji}-R^*_{ij})}_{=0} + \sum_{j\in\mathcal{B}}(C_{ji}-C_{ij})\\
&= \sum_{j\in\mathcal{B}}(C_{ji}-C_{ij}).
\end{aligned}
$$

Hence, $\tilde{C} \in F_{BN}$.

Furthermore, since R^* solves the optimal rebalancing problem and since according to (59) the minimum of the non-conservative compression problem and the minimum of the optimal rebalancing problem coincide, it holds that

$$
\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\tilde{C}_{ij} = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}(C+R^*)^{bi}_{ij} = \min_{\hat{C}\in F_{ROR}}\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\hat{C}^{bi}_{ij} = \min_{\hat{C}\in F_{NC}}\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}}\hat{C}_{ij}
$$

Since $\tilde{C} \in F_{BN}$, this implies that $\tilde{C}$ is an optimal solution to the non-conservative $\mathcal{B}$-compression problem.

□

Proof of Proposition B.7. 1. The statement follows from part 1. in Lemma B.8.

2. The statement follows from part 2. in Lemma B.8.

3. The statement follows from part 1. in Lemma B.8.

□

C Examples

In this section, we show possible outcomes after performing portfolio compression and rebalancing. We reach opposite conclusions on systemic risk reduction by comparing compression and rebalancing in two cases, and we understand the role of external assets.

Figure 11 shows the original network, compressed and rebalanced networks. Numbers next to the arrows are liabilities. The participants for compression are $\mathcal{C}_{nodes} = \{1, 2, 3\}$, for which we only consider $\mu = \mu^{\max} = 1$. Note that for the original network, the optimal rebalancing problem with respect to $\mathcal{B} = \{1, 2, 3\}$ has a unique solution. Additionally, by Proposition B.4, the optimal rebalancing problem with respect to $\mathcal{B} = \{1, 2, 3, 4\}$ has an infinite number of solutions. To see this note that liabilities matrices of the form

$$\begin{pmatrix} 0 & 1+\delta & 0 & 2-\delta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\delta \in [0, 1]$, can be used to represent a rebalancing result[23]. Figure 11d shows such a network.

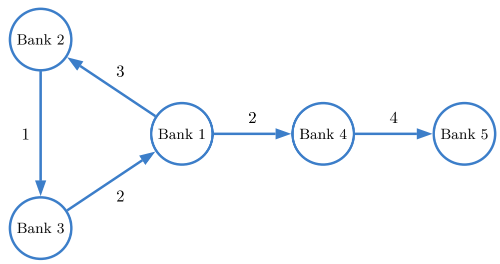
Table 2 summarises the equity of each bank and the total equity under the framework of Eisenberg & Noe (2001) for networks in Figure 11. We consider two cases.

Case 1. Let $A^e = A^{(1)}$ be the first case. We observe that portfolio rebalancing can be harmful for banks outside the rebalancing set. Larger proportions of liabilities from banks on rebalancing cycle to banks outside are allocated by rebalancing than compression. In the non-compressed and compressed networks, only bank 1 defaults, but banks $\{1, 4\}$ default in $\mathcal{B}$-rebalanced network when $\mathcal{B} = \{1, 2, 3\}$. Therefore, when rebalancing is performed and there are defaults on rebalancing cycle, more losses coming from banks that are creditors of the defaulting banks on rebalancing cycle would transmit to the banks outside rebalancing cycle compared with compression. Hence, being non-conservative in network rewiring may be a disadvantage in case of risk exacerbation.

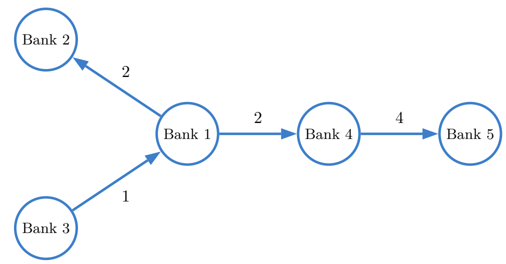
The equity results for the rebalanced network in Figure 11d depending on δ are listed in the last column of Table 2. Portfolio rebalancing with respect to banks $\{1, 2, 3, 4\}$ is harmful (i.e., bank 4 defaults in the rebalanced network) if and only if $\delta \in [0, 0.21)$, and it (weakly) reduces systemic risk if and only if $\delta \in [0.21, 1]$. This tells us that different choices for rebalancing outcomes can lead to different results.

Case 2. We keep the liabilities matrix L for the original network and set the external assets to be $A^e = A^{(2)}$. First we focus on situations where banks $\{1, 2, 3\}$ participate in PTRR services. In the original network shown in Figure 11a, bank 3 is in fundamental default and bank 1 is in contagious

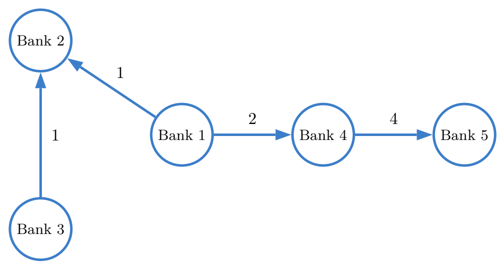
[23] We see that we can consider a cycle on a subset of matrix elements and add and subtract δ along this cycle. By doing this the corresponding net and gross positions remain unchanged. Such cycle moves have been considered in the context of network reconstruction from partial information, see Gandy & Veraart (2017).



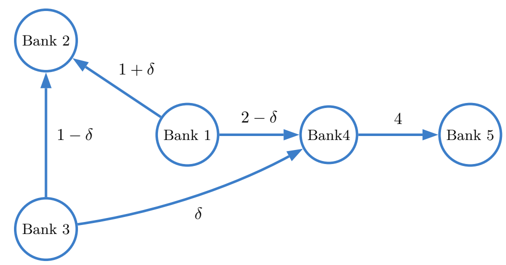
(a) Original network.



(b) Compressed network.



(c) Rebalanced network.



(d) Rebalanced network.

Figure 11: Financial networks with and without PTRR services. Compression banks are $\mathcal{C}_{nodes} = \{1,2,3\}$ in Figure 11b and rebalancing banks are $\mathcal{B} = \{1,2,3\}$ in Figure 11c. Figure 11d shows a configuration for $\mathcal{B}$-rebalanced network where $\mathcal{B} = \{1,2,3,4\}$ and $\delta \in [0,1]$.

	None	$\mathcal{C}_{nodes} = \{1,2,3\}$	$\mathcal{B} = \{1,2,3\}$	$\mathcal{B} = \{1,2,3,4\}$
Bank 1	0	0	0	0
Bank 2	2.4	2.5	2.67	$(8-\delta)/3$
Bank 3	1	1	1	1
Bank 4	0.2	0.1	0	$(\delta/3 - 0.07)^+$
Bank 5	4	4	3.93	$4 - (\delta/3 - 0.07)^-$
Total	7.6	7.6	7.6	7.6

(a) $A^{(1)} = (2, 1, 2, 2.6, 0)^\top$.

	None	$\mathcal{C}_{nodes} = \{1,2,3\}$	$\mathcal{B} = \{1,2,3\}$	$\mathcal{B} = \{1,2,3,4\}$
Bank 1	0	0	0.2	0.2
Bank 2	2.82	2.85	2.5	$2.5 + 0.5\delta$
Bank 3	0	0	0	0
Bank 4	0.01	0	0.13	$(0.13 - 0.5\delta)^+$
Bank 5	4	3.98	4	$4 - (0.13 - 0.5\delta)^-$
Total	6.83	6.83	6.83	6.83

(b) $A^{(2)} = (3.2, 1, 0.5, 2.13, 0)^\top$.

Table 2: Equity results corresponding to networks in Figure 11 with external assets A^e for $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, i.e., Eisenberg & Noe (2001) model. Gray represents confirmed defaults, and light gray represents banks that are vulnerable to defaults depending on δ.

default. Banks $\{1,3,4\}$ default in the compressed network, so compression is harmful in this case. Only bank 3 defaults in the $\{1,2,3\}$-rebalanced network, therefore $\{1,2,3\}$-rebalancing strongly reduces systemic risk. We see different consequences of compression and rebalancing since in Case 1 we have that compression is not harmful while rebalancing is harmful.

As with Case 1, the optimal rebalancing problem with respect to banks $\{1,2,3,4\}$ has an infinite number of solutions. Banks $\{1,3\}$ default in the network without rebalancing. Therefore, rebalancing strongly reduces systemic risk (corresponding to only bank 3 defaults in the rebalanced network) if and only if $\delta \in [0, 0.26)$, while it is harmful if and only if $\delta \in [0.26, 1]$ (corresponding to banks $\{3,4\}$ default in the rebalanced network).

We may wonder if it is always more beneficial for the financial system by allowing more banks to participate in rebalancing. Case 2 tells us that the answer is no, because we may have banks $\{3,4\}$ default when $\mathcal{B} = \{1,2,3,4\}$ while only bank 3 defaults when $\mathcal{B} = \{1,2,3\}$. The result depends on the network structure of the rebalancing outcome, which is controlled by δ in our example.

Remark C.1. By assigning different external assets without changing interbank liabilities for the original network, we have seen that compression is better than rebalancing in Case 1 and reached an opposite conclusion in Case 2. In fact, we can also highlight the role of external assets in PTRR services using the Rogers & Veraart (2013) model, then we may find some non-monotonicity on the comparisons.

D Contagion model with fire sales mechanism and results on price-mediated contagion

In this section, we add a fire sales contagion mechanism in the spirit of Cifuentes et al. (2005). We still consider a financial network $(L, A^e; \mathbb{V})$ as introduced in Section 3, but now we make a more specific assumption on the external assets of the banks, namely that they consist of cash and shares of an illiquid asset[24].

For each bank i, we assume that it holds cash with value $c_i \geq 0$, and $y_i \geq 0$ units of illiquid asset with fundamental value $P > 0$. Therefore, $A_i^e = c_i + y_i P$ are the nominal external assets.

Now we describe the mechanism of a price impact that occurs when banks are forced to sell their illiquid assets to satisfy their payment obligations. According to the above setting, the nominal cash balance for bank i is $c_i + \sum_{j \in \mathcal{N}} L_{ji} - \bar{L}_i$. If this value is negative, then bank i becomes illiquid and it has to sell some of its illiquid asset to compensate for the lack of liquidity. For an illiquid bank i, if it is not able to cover its liquidity shortfall by selling y_i units of illiquid asset, then bank i defaults and pays its interbank liabilities according to the proportionality rule. The forced liquidation has negative impact on the price of illiquid asset, which in turn might cause other banks' liquidity shortfall and further fire sales. We proceed as in Cifuentes et al. (2005) and consider an inverse

[24] We assume that there is only a single illiquid asset, but considering different asset classes and different demand functions would not change our analysis significantly.

demand function to model the interactions between liquidation costs and a clearing equilibrium in financial networks. The inverse demand function maps quantities to prices. It is defined as follows.

Definition D.1 (Inverse demand function)**.** We refer to function f with following two properties as an *inverse demand function*:

- $f(0) = P$.
- $f : [0, \sum_{i \in \mathcal{N}} y_i] \to [P_{min}, P]$ is continuous and non-increasing, where $P_{min} = f\left(\sum_{i \in \mathcal{N}} y_i\right)$.

This definition implies, that if the quantity being sold is 0, then there is no price impact, i.e., f returns the fundamental value P of the asset. It attains its smallest value P_{min} if all banks $i \in \mathcal{N}$ sell all their units of the illiquid asset. Moreover, if we set $f = P$, then there is no price impact and the model reduces to the framework introduced in Section 3. We will use $(L, c, y, f; \mathbb{V})$ to denote a financial network with liquidation costs, where $c = (c_1, ..., c_N)^\top$, $y = (y_1, ..., y_N)^\top$ and f are as just explained.

Remark D.2. When we consider clearing with fire sales, we set $k = 0$ in the valuation function and default set. We denote the default set in the financial network $(L, c, y, f; \mathbb{V})$ by $\mathcal{D}(L, c, y, f; \mathbb{V})$. The definition of systemic risk reduction in financial networks with liquidation costs can be adjusted accordingly.

We define the notions of an equilibrium asset price and a re-evaluated equity in the financial network $(L, c, y, f; \mathbb{V})$, which is an extended version of (Veraart, 2020b, Definition 3.6) that also accounts for a fire sales mechanism in the clearing framework. The incorporation of fire sales into other contagion mechanisms based on the Eisenberg & Noe (2001) model has been studied in previous literature, see e.g. Cifuentes et al. (2005); Weber & Weske (2017).

Definition D.3 (Equilibrium asset price and re-evaluated equity in distress contagion model with liquidation costs)**.** Consider a financial network $(L, c, y, f; \mathbb{V})$ and a given inverse demand function f. Let $\mathcal{M} = \{i \in \mathcal{N} \,|\, \bar{L}_i > 0\}$ and $\mathcal{E}^P = [P_{min}, P] \times [-\bar{L}, w]$, where $w_i = c_i + y_i P + \sum_{j \in \mathcal{N}} L_{ji} - \bar{L}_i$ for all $i \in \mathcal{N}$. Let $\mathbb{V} : \mathbb{R} \to [0, 1]$ be a valuation function defined in Definition 3.2. Let $\Phi^P = \Phi^P(\cdot; \mathbb{V}) : \mathcal{E}^P \to \mathcal{E}^P$ where

$$\begin{cases} \Phi_0^P(p, E) & = f\left(\sum_{i \in \mathcal{N}} \frac{\left[\bar{L}_i - c_i - \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right)\right]^+}{p} \wedge y_i\right), \\ \Phi_i^P(p, E) & = c_i + y_i p + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \quad \forall\, i \in \mathcal{N}. \end{cases} \tag{70}$$

Then, we refer to the vector $(p, E^P) \in \mathcal{E}^P$ satisfying

$$(p, E^P) = \Phi^P(p, E^P) \tag{71}$$

as *equilibrium asset price* p *and re-evaluated equity* E^P *in the financial network* $(L, c, y, f; \mathbb{V})$.

The existence of fix point solution to (71) is characterised in the following theorem. We use the greatest fixed point in Theorem D.4 as our choice for an equilibrium price and re-evaluated equity in the financial network $(L, c, y, f; \mathbb{V})$.

Theorem D.4. Consider a financial network $(L, c, y, f; \mathbb{V})$. Let Φ^P be the function defined in Definition D.3.

1. There exists a greatest fixed point $(p^*, E^{P;*})$ and a least fixed point (p_*, E_*^P), such that for all solutions (p, E^P) to the fixed point problem (71) we have

$$(p_*, E_*^P) \leq (p, E^P) \leq (p^*, E^{P;*}).$$

2. We set $(p^{(0)}, E^{(0);P}) = (P, w)$ and define recursively $(p^{(\kappa+1)}, E^{(\kappa+1);P}) = \Phi^P(p^{(\kappa)}, E^{(\kappa);P})$ for all $\kappa \in \mathbb{N}_0$. Then,

   (a) $(p^{(\kappa)}, E^{(\kappa);P})_{\kappa \in \mathbb{N}_0}$ is a monotonically non-increasing sequence, i.e.,

   $$(p^{(\kappa+1)}, E^{(\kappa+1);P}) \leq (p^{(\kappa)}, E^{(\kappa);P})$$

   for all $\kappa \in \mathbb{N}_0$.

   (b) The limit $\lim_{\kappa\to\infty}(p^{(\kappa)}, E^{(\kappa);P})$ exists and $(p^*, E^{P;*}) = \lim_{\kappa\to\infty}(p^{(\kappa)}, E^{(\kappa);P})$.

Proof of Theorem D.4. We first claim that the function Φ^P defined in Definition D.3 is right-continuous, non-decreasing and bounded. Since function $\mathbb{V}$ is right-continuous and non-decreasing and function f is continuous and non-increasing, it follows that Φ_0^P is right-continuous and non-decreasing in p and E. In addition, for $i \in \mathcal{N}$, it immediately follows from $\mathbb{V}$ being right-continuous and non-decreasing that $\Phi_i^P(p, E)$ is a right-continuous and non-decreasing function of p and E. We also note that $\Phi^P(P_{min}, -\bar{L}) \geq (P_{min}, -\bar{L})$ and $\Phi^P(P, w) \leq (P, w)$. Therefore, Φ^P is bounded from above by (P, w) and bounded from below by $(P_{min}, -\bar{L})$. This completes the proof of our claim. The remaining proof follows similarly as the proof of (Veraart, 2020a, Theorem 3.4) and we omit the details. $\square$

We first show that adding fire sales channel of systemic risk can only reduce the corresponding re-evaluated equity.

Proposition D.5. Consider a financial network without fire sales $(L, c, y, P; \mathbb{V})$ and a financial network $(L, c, y, f; \mathbb{V})$ with fire sales, where $A_i^e = c_i + y_i P$ for every $i \in \mathcal{N}$. Then, for every bank, the re-evaluated equity in $(L, c, y, f; \mathbb{V})$ cannot be larger than that in $(L, c, y, P; \mathbb{V})$.

Since Proposition D.5 can be proved by applying similar arguments as in the proof of Theorem D.6, we omit the proof.

Now we extend the results in Veraart (2020b) and Section 4.1 to account for liquidation costs and give sufficient conditions for asset price in the compressed or rebalanced network to be equal to asset price in the original network.

Theorem D.6. Let $(L, c, y, f; \mathbb{V})$ be a financial network for which we can undertake portfolio compression (resp. portfolio rebalancing). If participating banks do not default in the original or compressed (resp. rebalanced) network, then the asset prices in both financial networks coincide. In addition, the re-evaluated equities in both financial networks also coincide.

Proof of Theorem D.6. We only prove the statements for rebalancing. The case of compression is a direct corollary because it can be seen as a special case of rebalancing, see Remark 2.10. We prove the following two parts:

1. Rebalancing banks do not default in the original network implies corresponding asset prices and re-evaluated equities in both financial networks are the same.

2. Rebalancing banks do not default in the rebalanced network implies corresponding asset prices and re-evaluated equities in both financial networks are the same.

Part 1: We let $k = 0$ by Remark D.2. We set the initial asset price in financial networks with and without rebalancing to be $p^{\mathcal{B}(0)} = p^{(0)} = P$. Similar to Definition A.1, for all $i \in \mathcal{N}$, we define the initial equity in financial network $(L, c, y, f; \mathbb{V})$ and $(L^{\mathcal{B}}, c, y, f; \mathbb{V})$ by

$$E_i^{(0);P} = c_i + y_i p^{(0)} + \sum_{j \in \mathcal{N}} L_{ji} - \bar{L}_i,$$
$$E_i^{\mathcal{B}(0);P} = c_i + y_i p^{\mathcal{B}(0)} + \sum_{j \in \mathcal{N}} L^{\mathcal{B}}_{ji} - \bar{L}^{\mathcal{B}}_i.$$

Let $\mathcal{M}^{\mathcal{B}} = \{i \in \mathcal{N} \mid \bar{L}^{\mathcal{B}}_i > 0\}$. Let $\Phi^P : [P_{min}, P] \times [-\bar{L}, E^{(0);P}] \to [P_{min}, P] \times [-\bar{L}, E^{(0);P}]$ be the function defined in (70). Let $\Phi^{P,\mathcal{B}} : [P_{min}, P] \times [-\bar{L}^{\mathcal{B}}, E^{\mathcal{B}(0);P}] \to [P_{min}, P] \times [-\bar{L}^{\mathcal{B}}, E^{\mathcal{B}(0);P}]$ where

$$\begin{cases} \Phi_0^{P,\mathcal{B}}(p, E) = f\left(\sum_{i \in \mathcal{N}} \frac{\left[\bar{L}^{\mathcal{B}}_i - c_i - \sum_{j \in \mathcal{M}^{\mathcal{B}}} L^{\mathcal{B}}_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}^{\mathcal{B}}_j}{\bar{L}^{\mathcal{B}}_j} \right) \right]^+}{p} \wedge y_i \right), \\ \Phi_i^{P,\mathcal{B}}(p, E) = c_i + y_i p + \sum_{j \in \mathcal{M}^{\mathcal{B}}} L^{\mathcal{B}}_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}^{\mathcal{B}}_j}{\bar{L}^{\mathcal{B}}_j} \right) - \bar{L}^{\mathcal{B}}_i, \quad \forall i \in \mathcal{N}. \end{cases} \tag{72}$$

For $n \in \mathbb{N}$, we define recursively

$$(p^{(n)}, E^{(n);P}) = \Phi^P(p^{(n-1)}, E^{(n-1);P}),$$
$$(p^{\mathcal{B}(n)}, E^{\mathcal{B}(n);P}) = \Phi^{P,\mathcal{B}}(p^{\mathcal{B}(n-1)}, E^{\mathcal{B}(n-1);P}),$$

where Φ^P and $\Phi^{P,\mathcal{B}}$ are defined in (70) and (72) respectively. Then, according to Theorem D.4 we obtain that $(p^*, E^{P;*}) = \lim_{n\to\infty}(p^{(n)}, E^{(n);P})$ and $(p^{\mathcal{B};*}, E^{\mathcal{B};P;*}) = \lim_{n\to\infty}(p^{\mathcal{B}(n)}, E^{\mathcal{B}(n);P})$ exist. We will prove by induction that $\mathcal{D}(L, c, y, f; \mathbb{V}) \cap \mathcal{B} = \emptyset$ implies

$$p^{\mathcal{B}(n)} = p^{(n)}, \tag{73}$$

and

$$E_i^{\mathcal{B}(n);P} = E_i^{(n);P}, \quad \forall\, i \in \mathcal{N} \tag{74}$$

hold for all $n \in \mathbb{N}_0$. Once these have been shown, we get

$$(p^{\mathcal{B};*}, E^{\mathcal{B};P;*}) = \lim_{n\to\infty}(p^{\mathcal{B}(n)}, E^{\mathcal{B}(n);P}) = \lim_{n\to\infty}(p^{(n)}, E^{(n);P}) = (p^*, E^{P;*}).$$

We know that sequences $(p^{(n)}, E^{(n);P})$ and $(p^{\mathcal{B}(n)}, E^{\mathcal{B}(n);P})$ are non-increasing by Theorem D.4. Therefore, the assumption $\{i \in \mathcal{B} \,|\, E_i^{P;*} < 0\} = \emptyset$ implies that $E_i^{(n);P} \geq \lim_{k\to\infty} E_i^{(k);P} = E_i^{P;*} \geq 0$ for all $i \in \mathcal{B}$ and for all $n \in \mathbb{N}_0$, i.e.,

$$\{i \in \mathcal{B} \,|\, E_i^{(n);P} < 0\} = \emptyset, \quad \forall\, n \in \mathbb{N}_0. \tag{75}$$

Furthermore, combining (75) with induction hypothesis (74) we get

$$\{i \in \mathcal{B} \,|\, E_i^{\mathcal{B}(n);P} < 0\} = \emptyset, \quad \forall\, n \in \mathbb{N}_0. \tag{76}$$

Now we start to prove (73) and (74) by induction. Let $n = 0$, then we have $(p^{\mathcal{B}(0)}, E^{\mathcal{B}(0);P}) = (p^{(0)}, E^{(0);P})$ by Lemma A.6. Assume now (73) and (74) hold for a fixed $n \in \mathbb{N}_0$, and we show that they also hold for $n+1$. By definition of Φ^P and $\Phi^{P,\mathcal{B}}$, we know

$$p^{(n+1)} = f\left(\sum_{i\in\mathcal{N}} \frac{\left[\bar{L}_i - c_i - \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n);P}+\bar{L}_j}{\bar{L}_j}\right)\right]^+}{p^{(n)}} \wedge y_i\right),$$

$$p^{\mathcal{B}(n+1)} = f\left(\sum_{i\in\mathcal{N}} \frac{\left[\bar{L}_i^{\mathcal{B}} - c_i - \sum_{j\in\mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}}\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n);P}+\bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right)\right]^+}{p^{\mathcal{B}(n)}} \wedge y_i\right),$$

and

$$E_i^{(n+1);P} = c_i + y_i p^{(n)} + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n);P}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \quad \forall\, i \in \mathcal{N},$$

$$E_i^{\mathcal{B}(n+1);P} = c_i + y_i p^{\mathcal{B}(n)} + \sum_{j\in\mathcal{M}^{\mathcal{B}}} L_{ji}^{\mathcal{B}}\mathbb{V}\left(\frac{E_j^{\mathcal{B}(n);P}+\bar{L}_j^{\mathcal{B}}}{\bar{L}_j^{\mathcal{B}}}\right) - \bar{L}_i^{\mathcal{B}}, \quad \forall\, i \in \mathcal{N}.$$

Since $p^{\mathcal{B}(n)} = p^{(n)}$, we obtain that $E_i^{\mathcal{B}(n+1);P} = E_i^{(n+1);P}$ for all $i \in \mathcal{N}$ by substituting A_i^e with $c_i + y_i p^{\mathcal{B}(n)} = c_i + y_i p^{(n)}$ in the proof of Theorem 4.1. It is straightforward to check that the arguments in the proof of Theorem 4.1 can be adapted here because of (74), (75) and (76). Hence,

$$\begin{aligned} p^{\mathcal{B}(n+1)} &= f\left(\sum_{i\in\mathcal{N}} \frac{\left[y_i p^{\mathcal{B}(n)} - E_i^{\mathcal{B}(n+1);P}\right]^+}{p^{\mathcal{B}(n)}} \wedge y_i\right) \\ &= f\left(\sum_{i\in\mathcal{N}} \frac{\left[y_i p^{(n)} - E_i^{(n+1);P}\right]^+}{p^{(n)}} \wedge y_i\right) \\ &= p^{(n+1)}. \end{aligned}$$

This completes the proof of the first part.

Part 2: For the second part, the result follows by applying similar arguments as in the proof of Proposition 4.2.

□

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References

Ahuja, R. K., Magnanti, T. L. & Orlin, J. B. (1993). *Network flows : theory, algorithms, and applications*. Prentice Hall, Englewood Cliffs, N.J.

Amini, H. & Feinstein, Z. (2020). Optimal network compression. Available at SSRN: https://ssrn.com/abstract=3677587.

Bardoscia, M., Ferrara, G., Vause, N. & Yoganayagam, M. (2019a). Full payment algorithm. Available at SSRN: https://ssrn.com/abstract=3344580.

Bardoscia, M., Ferrara, G., Vause, N. & Yoganayagam, M. (2019b). Simulating liquidity stress in the derivatives market. Bank of England Working Paper No. 838.

BCBS & IOSCO (2020). Margin requirements for non-centrally cleared derivatives. Basel Committee on Banking Supervision, Board of the International Organization of Securities Commissions.

Brouwer, D. P. (2013). System and method of implementing massive early terminations of long term financial contracts. US Patent 8,606,688.

CCP12 (2020). Progress and Initiatives in OTC Derivatives – A CCP12 Report. Available at https://ccp12.org/wp-content/uploads/2020/02/Progress_and_Initiatives_in_OTC_Derivatives-A_CCP12_Report.pdf.

Cifuentes, R., Ferrucci, G. & Shin, H. S. (2005). Liquidity risk and contagion. *Journal of the European Economic Association* **3**, 556–566.

Cont, R. (2017). Central clearing and risk transformation. Norges Bank Working Paper 3/2017.

Cont, R. & Kokholm, T. (2014). Central clearing of OTC derivatives: Bilateral vs multilateral netting. *Statistics & Risk Modeling with Applications in Finance and Insurance* **31**, 3–22.

D'Errico, M. & Roukny, T. (2021). Compressing over-the-counter markets. *Operations Research* .

Duffie, D. (2018). Financial regulatory reform after the crisis: An assessment. *Management Science* **64**, 4835–4857.

Duffie, D., Scheicher, M. & Vuillemey, G. (2015). Central clearing and collateral demand. *Journal of Financial Economics* **116**, 237–256.

Duffie, D. & Zhu, H. (2011). Does a central clearing counterparty reduce counterparty risk? *The Review of Asset Pricing Studies* **1**, 74–95.

EACH (2020). EACH response to the ESMA Consultation Paper "Report on post trade risk reduction services with regards to the clearing obligation (EMIR Article 85(3a)". Available at https://www.eachccp.eu/wp-content/uploads/2020/06/EACH-response-to-the-ESMA-consultation-on-PTRR-services.pdf.

EBF & ISDA (2020). Response to the ESMA PTRR consultation. Available at https://www.isda.org/a/hVbTE/ISDA-PTRR-Response-final.pdf.

Eisenberg, L. & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science* **47**, 236–249.

ESMA (2014). Final Report, ESMA's Technical Advice to the Commission on MiFID II and MiFIR. Available at https://www.esma.europa.eu/sites/default/files/library/2015/11/2014-1569_final_report_-_esmas_technical_advice_to_the_commission_on_mifid_ii_and_mifir.pdf.

ESMA (2020a). Consultation Paper, Report on post trade risk reduction services with regards to the clearing obligation (EMIR Article 85(3a)). Available at https://www.esma.europa.eu/sites/default/files/library/esma70-151-2852_consultation_report_ptrr_services_-_article_853a_of_emir.pdf.

ESMA (2020b). Report to the European Commission, Report on post trade risk reduction services with regards to the clearing obligation (EMIR Article 85(3a)). Available at https://www.esma.europa.eu/sites/default/files/library/esma70-156-3351_report_on_ptrr_services_with_regards_to_the_clearing_obligation_0.pdf.

ESRB (2017). The macroprudential use of margins and haircuts. Available at https://www.esrb.europa.eu/pub/pdf/reports/170216_macroprudential_use_of_margins_and_haircuts.en.pdf.

ESRB (2020a). ESRB opinion on ESMA's report on post trade risk reduction services with regards to the clearing obligation (Article 85(3a) EMIR). Available at https://www.esrb.europa.eu/pub/pdf/other/esrb.opinion200625_on_ESMA_report_on_post_trade_risk_reduction_services~d9810f4f69.en.pdf.

ESRB (2020b). Mitigating the procyclicality of margins and haircuts in derivatives markets and securities financing transactions. Available at https://www.esrb.europa.eu/pub/pdf/reports/esrb.report_200109_mitigating_procyclicality_margins_haricuts~0f3e9f9e48.en.pdf.

Financial Stability Board (2018). Incentives to centrally clear over-the-counter (OTC) derivatives. Available at https://www.fsb.org/wp-content/uploads/R191118-1-1.pdf.

Gandy, A. & Veraart, L. A. M. (2017). A Bayesian methodology for systemic risk assessment in financial networks. *Management Science* **63**, 4428–4446.

Geanakoplos, J. (2010). The leverage cycle. *NBER macroeconomics annual* **24**, 1–66.

Murphy, D. (2020). Our Lessons Have Returned: Insights into Post-crisis Financial Regulation from Mandatory OTC Derivatives Clearing Policy. LSE Legal Studies Working Paper.

O'Kane, D. (2017). Optimising the multilateral netting of fungible OTC derivatives. *Quantitative Finance* **17**, 1523–1534.

Paddrik, M., Rajan, S. & Young, H. P. (2020). Contagion in derivatives markets. *Management Science* **66**, 3603–3616.

Rogers, L. C. & Veraart, L. A. M. (2013). Failure and rescue in an interbank network. *Management Science* **59**, 882–898.

Schuldenzucker, S. & Seuken, S. (2020). Portfolio compression in financial networks: Incentives and systemic risk. Available at SSRN: https://ssrn.com/abstract=3135960.

Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific journal of Mathematics* **5**, 285–309.

Veraart, L. A. M. (2020a). Distress and default contagion in financial networks. *Mathematical Finance* **30**, 705–737.

Veraart, L. A. M. (2020b). When does portfolio compression reduce systemic risk? Available at SSRN: https://ssrn.com/abstract=3688495.

Weber, S. & Weske, K. (2017). The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk* **2**, 1–38.