Stock Market Bubbles and Anti-Bubbles *

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Abstract

Using a simple model of equity valuation, we define stock market bubbles and anti-bubbles as periods in which the dynamics of valuation is temporarily explosive. We identify a mechanism for the creation and destruction of bubbles and anti-bubbles that depends on the interaction between valuation and expected change in corporate profitability. Topically, we find that valuation dynamics are explosive in 2017, suggesting the possible formation of an equity bubble in the US.

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1 Introduction

Price bubbles have beguiled investors across time and in just about every asset class. Despite this long and distinguished history speculative manias have defied simple categorization. Stock markets in particular are a natural setting to study bubbles since stocks have a fundamental foundation based on the present value of future dividends. Furthermore, because of the stock market's essential capital allocation role in a competitive economy, it is important to understand extreme stock market events.

While the concept of bubbles has been around for a long time, their existence is controversial. In a standard textbook dividend discount model, a transversality condition, which sets to zero the discounted infinite-horizon price, "rules out 'bubbles' in which prices grow so fast that people will buy now just to resell at higher prices later, even if there are no dividends."[1] And while bubbles, which reflect high prices, receive the lion's share of focus, there are also anti-bubbles, which reflect low prices that fall so fast that people will sell now just to rebuy at even lower prices later.

LeRoy and Porter (1981), as well as Shiller (1981), show that stock price indices are too volatile to be consistent with (expected) dividends discounted at a constant rate. Shiller (1981) points out that time-varying discount rates can lead to excess volatility. If expected returns vary through time and are persistent, then news flow affecting expected returns can have a large impact on asset prices. Sparked by this insight, it is now well known that a dividend discount model with time varying discount rates is able to capture some of the salient volatility, predictability, and correlation characteristics of the stock market.[2] In this context, Cochrane (2011), page 1053, reframes the notion of a price bubble in terms of discount rates, stating that "all a 'price bubble' can possibly mean now is that the equivalent discount rate is 'too low' relative to some theory." In this paper we adopt this discount rate orientation.

In our discount rate model, the logarithm of price to fundamental value, rather than simply the ratio of price to fundamental value, emerges as a natural measure of valuation. This logarithmic transformation, which reflects nothing more than the assumption of the lognormality of prices,[3] has the substantial benefit of placing high prices and low prices (relative to fundamental value) on the same footing. The implication is that bubbles and

[1] See Cochrane (2005), page 25.

[2] See Summers (1986), Fama and French (1988), Poterba and Summers (1988), Fama and French (1989) for the original, pioneering work.

[3] See MacKenzie (2008), Chapter 2, and Samuelson (1965).

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anti-bubbles are two sides of the same "price deviation from fundamental value" coin.[4] Our basic discount rate approach is similar to the methodology in Fama and French (2002). However, their focus is on the average level of discount rates, whereas our focus is on the dynamics of discount rates.

Cochrane's dictum that a bubble (or anti-bubble) can only be defined in the context of a particular theory highlights an important point about identifying bubbles (or anti-bubbles) and is worth emphasizing in the context of fundamental value. We assume that (i) fundamental value is a fixed multiple of trailing ten year real earnings, (ii) growth is mean reverting, and (iii) growth shocks are not persistent. This means that our bubble and anti-bubble identifications are sensitive to these assumptions.

As shown in Tarlie (2013), under certain reasonable assumptions, e.g. mean reverting return on equity, the time varying discount rate models cited above can be expressed in terms of mean reverting valuation. Typically, mean reversion is modeled as a simple AR(1) process with a constant mean reversion speed.[5] Our main technical innovation is to allow for dynamics of this mean reversion speed.

Using data for the US stock market, we find empirical evidence of meaningful variation in the mean reversion speed of valuation. This is significant because the mean reversion speed controls the sensitivity of expected returns to valuation. Furthermore, from a bubble and anti-bubble perspective, there are periods in which the mean reversion speed actually becomes negative. In these situations, the dynamics of valuation, rather than exhibiting mean reversion, temporarily exhibits mean aversion. This means the dynamics are briefly explosive. Furthermore, during these periods the value effect is actually reversed and, for example, high valuations imply positive returns from changes in valuation.

These periods of temporarily explosive dynamics lead us to identify bubbles and anti-bubbles as periods of temporarily explosive dynamics. We find for the US stock market that such periods are rare. From 1881-2017, we count only five episodes of explosive dynamics. In this paper we focus on four of these periods: the anti-bubbles of the late 1910s and early 1980s, and the bubbles of the late 1920s and late 1990s.

The fifth episode begins in 2017 and is characterized by high valuation – an expensive market – and explosive dynamics. But we do not focus on this period because at the time of this writing the events are still unfolding. While our model is not able to predict when this, or any other, bubble will burst, it does provide insight into the mechanism for both

[4]While there is no universally agreed upon definition of situations in which prices are substantially below fundamental value, Shiller (2003) and Sornette and Cauwels (2015) refer to these as negative bubbles. In this paper we choose to use the terms bubble/anti-bubble rather than positive bubble/negative bubble, which saves us from having to constantly qualify a bubble as positive or negative.

[5]See Campbell (1991), Priestley (2001).

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the creation and destruction of bubbles. The mechanism, which is tied to the interaction between valuation and expected changes in corporate profitability, suggests that the bursting is likely to cause significant return volatility.

To gain insight into the drivers of the time variation in the mean reversion speed, we express the mean reversion speed in terms of a small number of terms nonlinear in valuation, and a term bilinear in the interaction between valuation and expected change in profitability. This model provides insight into a mechanism for the creation and destruction of bubbles and anti-bubbles: bubbles are prone to form when high valuations combine with positive expected changes in profitability. And in keeping with our basic notion that bubbles and anti-bubbles are two sides of the same coin, anti-bubbles are prone to form when low valuations combine with negative expected changes in profitability. The collapse of the bubble, or the reflation of the anti-bubble, is triggered by the flipping of the sign of the expected change in profitability.

The sign flipping of the expected change in profitability and its impact on valuation dynamics is consistent with Kindleberger's view that bubble bursting is related to "...some change of view that leads a market participant with a large position to sell. Prices fall. Expectations are reversed."[6] The changing view that reverses expectations is a changing view on profitability.

From an economic perspective, the collapse of bubbles and the role of the interplay between profitability and valuation – the bilinear term mentioned above – can be understood by thinking of capital supply. In bubbles, profitability is high: prices are high in good times. But in our model, the net investment rate is proportional to profitability. So when profitability is high, net investment is also high.[7] However, this increase in supply ultimately drives down profitability, i.e. profitability is mean reverting. This increase in supply of capital provides the underlying economic mechanism for the bubble bursting catalyst in our model: the expected change in profitability going from positive to negative. In keeping with our theme of symmetry between bubbles and anti-bubbles, for anti-bubbles the capital supply mechanism applies in reverse.

The collapse of bubbles, or the reflation of anti-bubbles, provides a mechanism for time varying volatility. The bursting of a bubble, triggered by expected change in profitability flipping from strongly positive to strongly negative, produces large changes in valuation, and therefore returns. This time varying volatility is distinct, and in addition to, any GARCH type effects. In fact, in our estimation methodology we incorporate GARCH terms in the

[6] See Kindleberger and Aliber (2005), page 104.

[7] This mechanism is similar to the Q theory of investment. In the Q theory, investment is high when price exceeds replacement cost. In our model, investment is high when profitability is high. But to the extent that price to replacement cost is proportional to profitability, Q theory and our model for capital supply have similar implications.

valuation dynamics.

In the next section we discuss this paper in the context of recent literature. In Section 3 we introduce the basic model, and in Section 4 we discuss the empirical estimates of the model using data for the US stock market from 1871-2017. In Section 5 we discuss return implications, in Section 6 we analyze a mechanism for the creation and destruction of bubbles and anti-bubbles implied by the empirical results, and in Section 7 we take a closer look at a bubble and anti-bubble to understand the process more deeply. Section 8 concludes.

2 Relation to recent literature

Phillips, Shi, and Yu (2015) point out that econometric identification of multiple bubbles is substantially more difficult than identification of single bubbles, and is an important, open question. The question is important because a reliable, low error rate methodology is particularly useful as a financial surveillance tool. The question is open because the power of unit root tests diminishes as the number of bubbles rises; see Evans (1991). Phillips, Shi, and Yu (2015), building on Phillips, Shi, and Yu (2014), provides a method based on flexible window unit root tests. This flexible window method is based on modifications of (Augmented) Dickey-Fuller unit root tests. The conceptual idea is that bubbles are framed as transitions of the system from stationary, through a unit root, to a temporarily explosive state, characterized by the AR(1) regression coefficient exceeding unity. By contrast, explosive dynamics in our approach arises when the mean reversion speed, modeled as an AR(1) process, becomes temporarily negative. In our approach, we frame the explosive behavior characteristic of bubbles in terms of a nonlinear dynamical system. This framing of the dynamics in terms of a nonlinear dynamical system is well-documented in the general literature (Ozaki (1992)), and is an example of the relationship between time variation in the mean reversion speed and nonlinearities. A primary benefit of framing the dynamics in this way is that multiple bubbles are naturally identified as periods of negative mean reversion speed. A secondary benefit of this framing is that it allows us to visualize the dynamics.

Another generalization of the unit root approach is to model the time variation in the coefficients using a Markov switching approach; (see, for example, Hall, Psaradakis, and Sola (1999), and Gürkaynak (2008) for a more extensive review). While the Markov switching approach and our approach are conceptually similar in that both methods contemplate time varying coefficients, in the Markov switching approach the coefficients typically only take on a discrete set of values, whereas in our approach the time varying coefficient (which follows an AR(1) process) can take on a continuous set of values.

Our characterization of bubbles and anti-bubbles differs from the log periodic power

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law model, another approach to identifying bubbles. In the log periodic power law model, bubbles are associated with singularities.[8] By contrast, in our model bubbles are associated with instabilities. In bubbles defined by singularities, dynamics at the time of the singularity are undefined, and there is a clear separation between the dynamics before the singularity – the formation process – and after – the destruction process. Furthermore, identification is effectively limited to single bubbles. By contrast, for bubbles defined by instabilities dynamics are well defined throughout the process, the process of creation is naturally related to the process of destruction, and multiple bubbles are handled naturally.

An advantage of the unit root and log periodic power law approaches is that they apply to any price series. While our model in this paper is specific to equity markets, it has the advantage of providing insight into a mechanism for the creation and destruction of bubbles and anti-bubbles. Nevertheless, our approach of treating the mean reversion speed as an unobserved state variable may be useful in identifying bubbles in other assets. The most obvious cases are ones in which there is a measure of fundamental value so that valuation is stationary. A recent example is gold, where Lucey and O'Connor (2013) use gold lease rates as a measure of fundamental value but find mixed evidence for bubbles using Markov Switching Augmented Dickey-Fuller tests for multiple bubbles. More generally, our approach might also be useful in situations where price is trend stationary.

3 The model

We start with the idea that, on average, market value of equity equals fundamental value. We proxy fundamental value as a constant multiple of normalized real earnings. Changes in fundamental value, which we can also think of as economic (equity) capital, are determined by retained earnings, the difference between earnings and total dividends over the relevant time period. With these assumptions we derive an expression for return that depends on expected change in valuation (log price to fundamental value), expected growth in fundamental value, and income.

The first step is to define, using the standard definition, the geometric return $r(t, t+dt)$ between times t and $t+dt$:

$$r(t, t+dt) = \ln \frac{P(t+dt) + D(t, t+dt)}{P(t)}, \tag{1}$$

In this expression, $P(t)$ is price at time t and $D(t, t+dt)$ is the total dividends paid between times t and $t+dt$. The next step is to introduce the notion of fundamental value, which we

[8]See, for example, Johansen, Ledoit, and Sornette (2000) and Sornette (2003).

denote by $F(t)$. We assume that changes in fundamental value, which we can also think of as economic (equity) capital, are determined by earnings and dividends via

$$F(t+dt) = F(t) + (E(t,t+dt) - D(t,t+dt)), \tag{2}$$

where $E(t,t+dt)$ is the flow of earnings between times t and $t+dt$. Note that both $E(t,t+dt)$ and $D(t,t+dt)$ are proportional to dt. This equation expresses the idea that changes in fundamental value are determined by retained earnings, i.e. earnings minus dividends. In this formulation, dividends include all (net) cash flows that do not flow through the income statement. This definition obviously includes cash payouts, but also includes capital raises (which come with a negative sign as they are cash flows from shareholders), share repurchases, and option exercise, among other possibilities. Dividing both sides of Eq. (2) by $F(t+dt)$ we can write

$$1 = \frac{F(t)}{F(t+dt)} + (\rho(t,t+dt) - \delta(t,t+dt)) \tag{3}$$

where

$$\rho(t,t+dt) = \frac{E(t,t+dt)}{F(t+dt)} \tag{4}$$

and

$$\delta(t,t+dt) = \frac{D(t,t+dt)}{F(t+dt)}. \tag{5}$$

We refer to the quantity $\rho(t,t+dt)$ as profitability as it is a measure of the rate at which the capital base generates earnings.

Rearranging Eq. (3), we can write (assuming $g(t,t+dt) \neq 1$)

$$\frac{F(t+dt)}{F(t)} = \frac{1}{1 - g(t,t+dt)} = 1 + g(t,t+dt) + \ldots \tag{6}$$

where

$$g(t,t+dt) = \rho(t,t+dt) - \delta(t,t+dt) \tag{7}$$

is the growth of fundamental value between t and $t+dt$. The second equality in Eq. (6) represents a first order Taylor series in $g(t,t+dt)$. Since $g(t,t+dt)$ is proportional to dt, the dots in Eq. (6) represent terms of order dt^2 and higher.

In this model, we can think of $E(t, t + dt) - D(t, t + dt)$ as net investment. Thus, the rate of net investment is high when profitability is high, and vice versa. But high net investment leads to increased capital supply, which ultimately drives down returns, implying mean reversion in profitability.

Because Eq. (2) does not include a random shock, and because we assume that ρ and δ are smooth, fundamental value also changes smoothly. These assumptions are consistent with a relatively stable production function and a model of growth that is relatively benign in that growth is mean reverting.

It is convenient to define valuation $q(t)$ as

$$q(t) = \ln \frac{P(t)}{F(t)}. \tag{8}$$

Our assumption that, on average, (log) price equals (log) fundamental value means that $\mathrm{E}[q(t)] = 0$.[9] If we assume that $F(t) = bE_{10}(t)$ (see, for example, Tarlie (2013)), where $E_{10}(t)$ is trailing 10 year real earnings, then $\mathrm{E}[q(t)] = 0$ implies $\ln b = \mathrm{E}[\ln P(t)/E_{10}(t)]$. Furthermore, the logarithmic transformation from $P(t)/F(t)$ to $\ln P(t)/F(t)$ enables symmetry between bubbles and anti-bubbles. Even though both $P(t)$ and $F(t)$ are always positive, the logarithmic transformation allows $q(t)$ to vary symmetrically around zero.

Using the definitions of valuation q, growth g, and dividends divided by fundamental value δ, and expanding to leading order in dt, we can write Eq. (1) for the geometric return as

$$r(t, t + dt) = dq(t, t + dt) + g(t, t + dt) + \delta(t, t + dt)e^{-q(t)} + \cdots \tag{9}$$

where $dq(t, t + dt) = q(t + dt) - q(t)$ and the dots denote terms of order dt^2 and higher. (Appendix B contains the details of the derivation.) This equation has an intuitive interpretation. The return $r(t, t + dt)$ between times t and $t + dt$ is defined by three sources of return: (i) change in valuation $(dq(t, t + dt))$, (ii) growth in fundamental value $(\rho(t, t + dt) - \delta(t, t + dt))$, and (iii) income $(\delta(t, t + dt)e^{-q(t)})$.

Rearranging Eq. (9) by combining the $\delta(t, t + dt)$ terms, we can write

$$r(t, t + dt) = dq(t, t + dt) + \rho(t, t + dt) + \delta(t, t + dt)\left(e^{-q(t)} - 1\right) + \cdots . \tag{10}$$

This expression forms the basis for the empirical analysis in the paper. Our goal is to estimate each of the three terms on the right hand side. A virtue of this approach is that

[9] Technically, when we say on average, price equals fundamental value we mean that on average, log price equals log fundamental value.

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we obtain, not only an estimate of expected return, but also its decomposition in terms of expected change in valuation, expected growth, and expected income. Our main innovation is to model valuation as a mean reverting process but with a time varying mean reversion speed. This feature drives our main empirical results.

4 Empirics

In this section we present empirical estimates of the three terms on the right hand side of Eq. (10). Our choice of models for valuation $q(t)$ and profitability $\rho(t)$ is motivated by their empirical autocorrelations. As shown in Tarlie (2013) and as is discussed below, the autocorrelation and partial autocorrelation results suggest that $q(t)$ follows an AR(1) process and $\rho(t)$ an AR(2) process. We therefore model $q(t)$ as an AR(1) process. But rather than treat the mean reversion speed as a constant, we model a time varying mean reversion speed. For profitability, $\rho(t)$, we do not find empirically that it is necessary to allow for time varying coefficients so we model $\rho(t)$ as a simple AR(2) process with constant coefficients.

While our specifications for the dynamics of q and ρ do not explicitly link the dynamics of these two quantities – an apparently restrictive assumption – we show in Section 6.2 that the time variation in the mean reversion speed on valuation involves the interaction between valuation and the expected change in profitability. In the following subsection we explain our data and in the following subsections we discuss our empirical findings.

4.1 Data

We use quarterly sampled price, dividend, earnings, and Consumer Price Index data from Robert Shiller (Shiller (2017)), and we measure time in quarterly units so that $dt = 1$ corresponds to 3 months and all quantities are measured in real terms using the Consumer Price Index as a deflator. The dividend data in the Shiller spreadsheet ends in September, 2017 and the earnings data ends in June, 2017. For the remainder of 2017, we supplement the dividend data using Cash Dividends Per Share and the earnings data using estimates of As Reported Earnings Per Share from the S&P 500 Earnings and Estimate Report, which we downloaded from the Standard & Poor's website.

A central quantity in our model is fundamental value, $F(t)$. Following the discussion in Section 3 (see also Tarlie (2013)) we estimate fundamental value as $F(t) = bE_{10}(t)$, where b is a constant and E_{10} is trailing ten year real earnings (see Campbell and Shiller (1988)).[10]

[10]Following the computation in Shiller (2017), for quarterly sampling we have $E_{10}(t) = (1/40)\sum_{i=0}^{39} E(t-i)$, where $E(t)$ is trailing annual real earnings.

These assumptions imply that $\ln b = \mathrm{E}[\ln P(t)/E_{10}(t)]$, which for the US stock market from January 1881-December 2017 gives $b = 15.5$. The motivation for basing our estimate of $F(t)$ on earnings rather than dividends is that earnings are more sensitive to the business cycle, total dividends are not observable, and earnings are not directly dependent on corporate payout policy or the form of payout.

Our assumption that price equals fundamental value means that the unconditional expected change in valuation is zero, i.e. $\mathrm{E}[dq(t, t+dt)] = 0$.[11] If we take the unconditional expectation of both sides of Eq. (10) we can write

$$\bar{r} = \bar{\rho} + \mathrm{E}\left[\mathrm{E}_t[\delta(t, t+dt)]\left(e^{-q(t)} - 1\right)\right], \tag{11}$$

where $\bar{r}$ and $\bar{\rho}$ are the unconditional expectations of expected return and profitability. This condition implies a relationship between average expected returns, profitability, and the ratio of dividends to fundamental value. The last term on the right hand side is at least an order of magnitude smaller than the other two terms because of the $e^{-q(t)} - 1$ term, which is a measure of the deviation of price from fundamental value. This means that $\bar{r} \approx \bar{\rho}$, i.e. on average, the cost of capital equals the return on capital. A subtlety of our dataset is that for the full sample from 1881-2017 we have $\mathrm{E}[E(t-4, t)/F(t)] = 0.0704$ but average annualized returns of 0.0627, a relatively large difference of about 0.77% per year. For the time period from 1926-2017 we have $\mathrm{E}[E(t-4, t)/F(t)] = 0.0719$ and average annualized returns of 0.0684, a smaller difference of only 0.35% per year. Given that prior to 1926 the dataset is from Cowles and Associates rather than Standard & Poor's, we attribute the large deviation between average returns and profitability for the 1881-2017 to be due to earnings quality issues. To impose the condition that $\bar{r} \approx \bar{\rho}$ we therefore adjust earnings by a multiplicative factor $0.0627/0.0704 = 0.89$.

In the left panel of Fig. 1 we show the time series of $q(t) = \ln P(t)/F(t)$ using $F(t) = 15.5E_{10}(t)$. The dotted lines in this figure represent one standard deviation. From a valuation perspective, five periods stand out in this chart: (i) the early 1920s, (ii) late 1920s, early 1930s, (iii) early 1980s, and (iv) late 1990s, early 2000s, and (v) 2017. In all of these cases valuations reached extreme levels, with two cases (early 1920s, early 1980s) representing particularly cheap markets, and three cases (late 1920s, late 1990s, 2017) representing particularly expensive markets. From a profitability perspective, shown in the right panel of Fig. 1, two periods stand out: (i) the early 1920s, and (ii) 2008. We will see in Section 6.2 that expected change in profitability plays an important role in shaping the dynamics of valuation.

[11] This condition is implied by our assumption that, on average, (log) price equals (log) fundamental value.

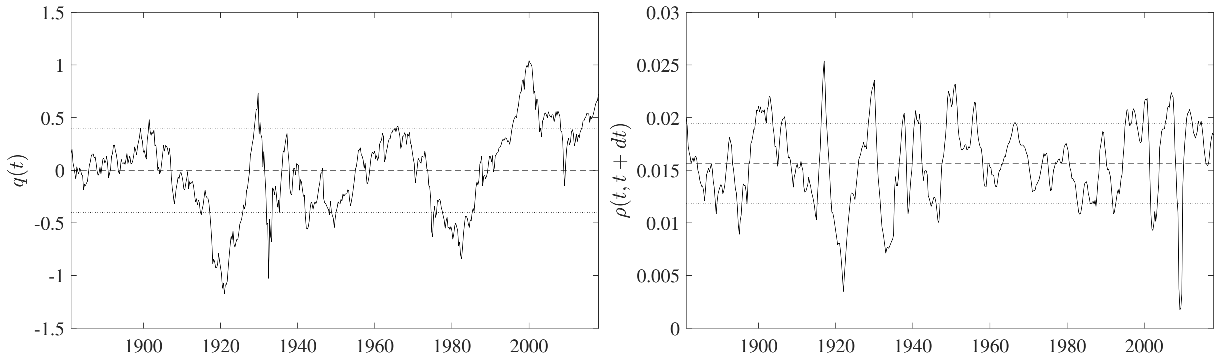

Figure 1. Plot of the time series of $q(t) = \ln P(t)/F(t)$ and $\rho(t) = E(t)/F(t)$ for $F(t) = 15.5E_{10}(t)$ using quarterly sampled US data from 1871-2017. The dashed horizontal lines represent the unconditional means, and the dotted lines represent one standard deviation. The mean of the valuation ($q(t)$) series is zero (by construction) and the standard deviation is 0.40. The mean of the profitability ($\rho(t)$) series is 0.0157 per quarter and the standard deviation is 0.0038.

One of the main themes of our paper is that bubbles and anti-bubbles are two sides of the same coin. This notion follows from the more general assumption that the distribution of $q(t) = \ln P(t)/F(t)$ is symmetric around zero. Visual inspection of the time series $q(t)$ in Fig. 1 is suggestive of this symmetry, and a Monte Carlo simulation of a simple AR(1) model $q(t+1) = Fq(t) + u(t+1)$, with F = 0.977 and u iid Normal with mean zero and $\text{std}(u) = 0.089$ (the sample estimates), indicates that the sample skewness of 0.24 for our 1881-2017 data is in the 24th percentile of observations. Thus, the hypothesis of symmetry cannot be rejected with a fairly high degree of confidence.[12] The assumption of symmetry plays an important role in our discussion of nonlinearities in Section 6.2.

4.2 Change in valuation

To estimate the dynamic process of the change in valuation, the first term in the expression for return, we assume that $q(t)$ is stationary and follows an AR(1) process. This assumption is discussed in Tarlie (2013), and is supported by the partial autocorrelations, which for the first four lags are 0.98, -0.025, -0.15, and -0.076 (with approximate confidence bounds of ± 0.086). This pattern of partial autocorrelations – a statistically significant value for the

[12]More formally, if $H_0 : S = 0$ is the null hypothesis that skewness S is zero, and $H_a : S \neq 0$ is the alternative hypothesis, the sample skewness of 0.24 is well within the typical 5% confidence values required to reject the null. Specifically, for 10,000 Monte Carlo simulations of length 547 (the number of quarters in the valuation sample), 5% of the skewness estimates are below -0.65, and 5% of the skewness estimates are above 0.56. The sample estimate of 0.24 is therefore well within the confidence band typically required not to reject the null.

first lag, followed by values close to zero for subsequent lags – is a classic AR(1) signature.

In this paper we allow for time variation in the speed of mean reversion. We therefore write the change in $q(t)$ as

$$dq(t, t+1) = -\gamma_q(t)q(t) + w_q(t+1) \tag{12}$$

where $dq(t, t+1) = q(t+1) - q(t)$. We do not include a constant term in the dynamic equation for $q(t)$ because, by assuming that on average price equals fundamental value, the time average of $q(t)$ is zero.

The mean reversion speed $\gamma_q(t)$ plays a central role in our analysis because it determines the impact of valuation on return, i.e. Eq. (12) inserted into Eq. (10) implies $r(t, t+1) = -\gamma_q(t)q(t) + \ldots$, where the dots represent the remaining terms. When $\gamma_q(t)$ is large and positive, valuation mean reverts rapidly and the impact of valuation on return is high, leading to larger than average return predictability. By contrast, when $\gamma_q(t)$ is close to zero, valuation mean reverts slowly and the impact of valuation on return is low, leading to lower than average return predictability.

The time varying coefficient $\gamma_q(t)$ is designed to capture possible non-linearities in the evolution of $q(t)$ (Ozaki (1992)). However, not accounting for heteroskedasticity in the error term can generate spurious dynamics in the time-varying coefficients (see Cogley and Sargent (2005) for a discussion). Therefore, for the conditional volatility $w_q(t+1)$ we assume a GARCH(1,1) process, which we express as

$$w_q(t+1) \sim \mathcal{N}(0, h_q(t+1)) \tag{13}$$

$$h_q(t+1) = \alpha_{q0} + \alpha_{q1}w^2(t) + \alpha_{q2}h_q(t), \tag{14}$$

where $\mathcal{N}$ denotes a Normal distribution. For $\gamma_q(t)$, the time varying mean reversion speed, we assume AR(1) dynamics so that

$$\gamma_q(t+1) = a_{\gamma q} + F_{\gamma q}\gamma_q(t) + v_q(t+1) \tag{15}$$

where $a_{\gamma q}$ and $F_{\gamma q}$ are constants and $v_q(t+1)$ is a temporally uncorrelated Gaussian random variable with mean zero and variance $Q_{\gamma q}$ that is also uncorrelated with $w_q(t+1)$. The quantities $a_{\gamma g}$ and $F_{\gamma q}$ define the average $\bar{\gamma}_q = a_{\gamma q}/(1 - F_{\gamma q})$. Imposing finite variance on $q(t)$ requires

$$\frac{8a_{\gamma q}(1 - F_{\gamma q})}{Q_{\gamma q}} > 1. \tag{16}$$

We refer to this quantity, appropriate for quarterly sampling, as the stability statistic. The basic intuition is that the dynamics of valuation are bounded if the average value of the mean reversion speed is positive and large enough; see Appendix A for details, especially Eq. (33).

While valuation $q(t)$ is observable, the mean reversion speed $\gamma_q(t)$ is not. However, if we treat Eq. (15) for $\gamma_q(t)$ as the equation for an unobserved state variable, and Eq. (12) for dq as an observation equation, then these two expressions form a standard state space system (Hamilton (1994)). To account for the time varying volatility, we use the framework outlined in Harvey, Ruiz, and Sentana (1992) and Kim, Nelson, et al. (1999) for estimating $\gamma_q(t)$. Using the data from the US stock market as described in the introduction to this section, Table 1 presents our estimates of the model and Fig. 2 plots the time series of $\gamma_q(t)$.

Table 1
Estimation of the $q(t)$ process

Parameter estimates of the dynamics of $q(t)$ with time varying mean reversion speed $\gamma_q(t)$ and GARCH(1,1) noise w_q. Specifically, we have $dq(t, t+1) = -\gamma_q(t)q(t) + w_q(t+1)$. The specification for $\gamma_q(t)$ is also an AR(1) process, with $\gamma_q(t+1) = a_{\gamma q} + F_{\gamma q}\gamma_q(t) + v_q(t+1)$. We model the noise $w_q(t+1)$ as a GARCH(1,1) process, i.e. $w_q(t+1) \sim (0, h_q(t+1))$ with $h_q(t+1) = \alpha_{q0} + \alpha_{q1}w^2(t) + \alpha_{q2}h_q(t)$.

Parameter	Value	Std error
$a_{\gamma q}$	0.0016	0.0036
$F_{\gamma q}$	0.904	0.087
$Q_{\gamma q}$	0.0010	0.001
α_{q0}	0.0008	0.0003
α_{q1}	0.22	0.06
α_{q2}	0.67	0.07
Stability statistic	1.2	
Loglikelihood	611.5	

Even though the stability statistic is greater than one, indicating that the process is asymptotically stable, we see in Fig. 2 that there are a number of periods in which $-\gamma_q(t)$ is above zero, i.e. the mean reversion speed is negative. During these periods the dynamics of valuation is (temporarily) explosive. Evident in this graph are five episodes of temporarily explosive dynamics, including the year 2017, the time of this writing. We discuss this in more detail below.

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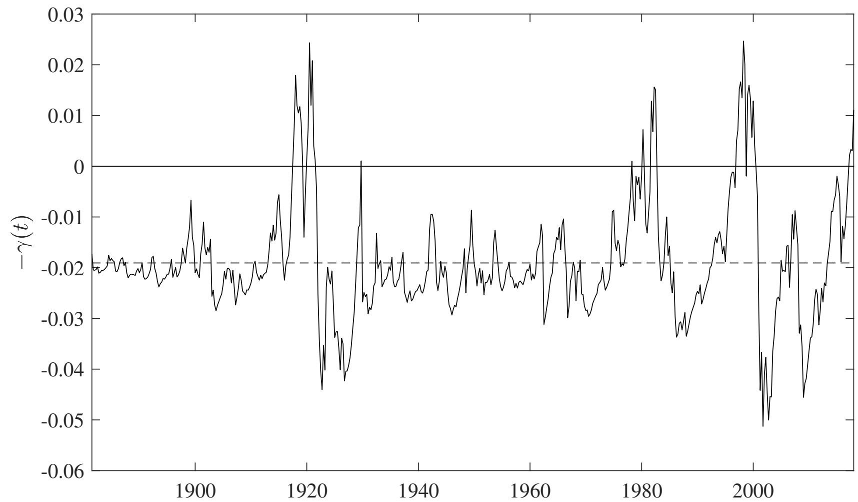

Figure 2. Plot of the time series of $-\gamma_q(t)$ generated from the Kalman filter using quarterly sampling of US data from 1871-2017. In quarterly units, the time series average of $-\gamma_q(t)$ is $\bar{\gamma}_q = -0.021$ (dashed line), the standard deviation is 0.014, and the half life of γ_q is $\ln 0.5/\ln|F_{\gamma q}| = 6.9$ quarters. The solid horizontal line is at zero; for $-\gamma_q(t) > 0$ the dynamics of $q(t)$ is temporarily explosive.

4.3 Profitability

We model profitability $\rho(t, t+dt)$ as an AR(2) process. This assumption follows the empirical evidence that the dynamics of profitability reflects the economic cycle, and is also supported by the partial autocorrelations. The partial autocorrelations for the first four lags are 0.96, -0.62, -0.095, and 0.031 (with approximate confidence bounds of ± 0.086). This pattern of partial autocorrelations is the classic signature of an AR(2) process, i.e. statistically significant values for the first two lags, with values for the subsequent lags close to zero.

While we might expect the coefficients in the AR(2) model to vary over time, empirically we find little meaningful time variation if we assume that the coefficients follow a two dimensional vector AR(1) process. More specifically, we allow the coefficients $\beta_{\rho 1}$ and $\beta_{\rho 2}$ in Eq. (17) to vary in time according to a two dimensional vector AR(1) process, and we find little time variation except for a narrow period around the Global Financial Crisis. For simplicity, we therefore take the coefficients to be constant so that we write

$$\rho(t, t+1) = \alpha_\rho + \beta_{\rho 1}\rho(t-1, t) + \beta_{\rho 2}\rho(t-2, t-1) + v_\rho(t+1), \tag{17}$$

where α_ρ, $\beta_{\rho 1}$, and $\beta_{\rho 2}$ are constants and $v_\rho(t+1)$ is white noise. Table 2 contains the results of a standard OLS regression.

We can gain intuition about the dynamics of $\rho(t) \equiv \rho(t-1, t)$ by recognizing that the

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Table 2
Estimation of the $\rho(t)$ process

Parameter estimates for the AR(2) specification of $\rho(t, t+1)$. In this specification, we have $\rho(t, t+1) = \alpha_\rho + \beta_{\rho 1}\rho(t-1, t) + \beta_{\rho 2}\rho(t-2, t-1) + v_\rho(t+1)$. Estimates are in quarterly units. The parameters γ_ρ, k_ρ, $\bar{\rho}$, and ω_ρ are derived from α_ρ, $\beta_{\rho 1}$, and $\beta_{\rho 2}$ using Eqs. (22).

Parameter	Value	Std error
α_ρ	0.0011	0.0038
$\beta_{\rho 1}$	1.55	0.034
$\beta_{\rho 2}$	-0.62	0.034
R^2	0.95	
γ_ρ	0.19	
k_ρ	0.067	
$\bar{\rho}$	0.0156	
ω_ρ	0.18	

AR(2) process defining $\rho(t)$ is that of an underdamped harmonic oscillator. To see this, define three constants $\bar{\rho}$, γ_ρ, and k_ρ, via

$$\gamma_\rho = 1 + \beta_{\rho 2}, \quad k_\rho = 1 - \beta_{\rho 1} - \beta_{\rho 2}, \quad \bar{\rho} = \frac{\alpha_\rho}{1 - \beta_{\rho 1} - \beta_{\rho 2}} \tag{18}$$

With these definitions, Eq. (17) for the evolution of $\rho(t)$ can be written as

$$d^2\rho(t) = -2\gamma_\rho d\rho(t) - k_\rho(\rho(t) - \bar{\rho}) + v_\rho(t+1), \tag{19}$$

where $d^2\rho(t) = \rho(t+1) + \rho(t-1) - 2\rho(t)$ and $d\rho(t) = \rho(t) - \rho(t-1)$. This expression defines the dynamics of a damped harmonic oscillator with unit mass, average value $\bar{\rho}$, damping coefficient γ_ρ, and spring force k. The angular oscillation frequency is given by $\omega_\rho = \sqrt{k - \gamma_\rho^2}$. We estimate this angular frequency to be 0.18, which implies a total periodicity of $2\pi/\omega_\rho \approx 36$ quarters. This nine year period means that trough-to-peak is about 4.5 years, roughly consistent with a five year economic expansion.

The harmonic oscillator interpretation of the dynamics of $\rho(t)$ is useful for two reasons. First, the parameters γ_ρ and k_ρ have economic interpretations (Tarlie (2013)). The parameter γ_ρ represents the strength of the frictional forces that slow the equilibration of $\rho(t)$, with $\gamma_\rho = 0$ representing a frictionless system. The parameter k_ρ represents the strength of the competitive economic forces that restore equilibrium, with $k_\rho = \infty$ representing an instantaneously competitive economy, i.e. perfect competition. Second, as we will see in

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Section 6.2, the expected change in $\rho(t)$, which we can interpret as the velocity of the oscillator, interacts with valuation in an important way. Together, valuation $q(t)$ and expected change in $\rho(t)$ play a significant role in influencing the dynamics of the mean reversion speed $\gamma_q(t)$.

4.4 Dividends

While we observe both valuation $q(t)$ and profitability $\rho(t)$, the challenge for the income term in the expression for return is that we do not observe total dividends, i.e. cash flows that do not flow through the income statement and are therefore not accounted for by earnings. While we observe cash dividends, there are other cash flows, including share repurchases, capital raises, and option issuance, that are generally not observed, especially over the long history of interest to us. However, if we decompose the change in valuation $dq(t, t + dt)$ and profitability $\rho(t, t+dt)$ into expected and unexpected components, then we can cast our basic expression for return, Eq. (10), as

$$r(t, t + 1) - \mathrm{E}_t[dq(t, t + 1)] - \mathrm{E}_t[\rho(t, t + 1)] = \delta(t, t + 1) \left(e^{-q(t)} - 1\right) + w_r(t + 1). \tag{20}$$

where the noise term is given by

$$w_r(t + 1) = \{dq(t, t + 1) - \mathrm{E}_t[dq(t, t + 1)]\} + \{\rho(t, t + 1) - \mathrm{E}_t[\rho(t, t + 1)]\} + \epsilon_r(t + 1), \tag{21}$$

and $\mathrm{E}_t[\cdot]$ represents the expectation operator conditional on time t information. In the expression for w_r, the quantity ϵ_r represents the higher order terms in Eq. (10). Because a component of w_r is the residual $w_q(t + 1) = dq(t, t + 1) - \mathrm{E}_t[dq(t, t + 1)]$ given in Eq. (13), we model w_r as a GARCH(1,1) process in exactly the same way as for w_q. We further assume that $\delta(t, t + 1)$ follows AR(1) dynamics, i.e.

$$\delta(t, t + 1) = a_\delta + F_\delta\, \delta(t - 1, t) + v_\delta(t + 1) \tag{22}$$

where a_δ and F_δ are constants and v_δ is a temporally uncorrelated Gaussian random variable with mean zero and variance Q_δ, which is also contemporaneously uncorrelated with w_r. This model implies that the impact on growth due to dividend payout is mean reverting if $|F_\delta| < 1$.

The system of equations (20) and (22) is of the standard state space form with Eq. (20) the observation equation and Eq. (22) the state equation. The left hand side of Eq. (20) is generated from observations of quarterly returns $r(t, t + 1)$, the output of the Kalman

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filter estimate of $\mathrm{E}_t[dq(t,t+1)]$, and the fitted value $\mathrm{E}_t[\rho(t,t+1)]$ of the AR(2) process for $\rho(t,t+1)$. Table 3 presents the values of the estimates, and Fig. 3 displays the time series of $\delta(t,t+1)$. To illustrate the difference between $\delta(t,t+1)$ and $\rho(t,t+1)$ with respect to both magnitude and variability, we include as the dotted line in Fig. 3 the plot of $\rho(t,t+1)$ from the right hand panel of Fig. 1.

There are two time periods in Fig. 3 in which $\delta(t-1,t)$ is negative. As dividends in our model are net cash flows to shareholders not accounted for by earnings, a straightforward interpretation of the negative values is that they are associated with aggregate equity capital raises. While plausible, we caution against reading too much into this portrayal because the right hand side of Eq. (20) is essentially a residual, and as such is sensitive to anything missed by the models for q and ρ.

Table 3
Estimation of the $\delta(t)$ process

Parameter Estimates of Eq. (25), i.e. $\delta(t,t+1) = a_\delta + F_\delta \delta(t-1,t) + v_\delta(t+1)$. The parameters α_{r0}, α_{r1}, and α_{r2} are the parameters of the GARCH(1,1) process for the noise in the observation equation (20) defining $\delta(t,t+1)$.

Parameter	Value	Std error
a_δ	0.0031	0.0042
F_δ	0.58	0.46
Q_δ	0.0002	0.0004
α_{r0}	0.0010	0.0003
α_{r1}	0.20	0.05
α_{r2}	0.66	0.08
Loglikelihood	616.1	

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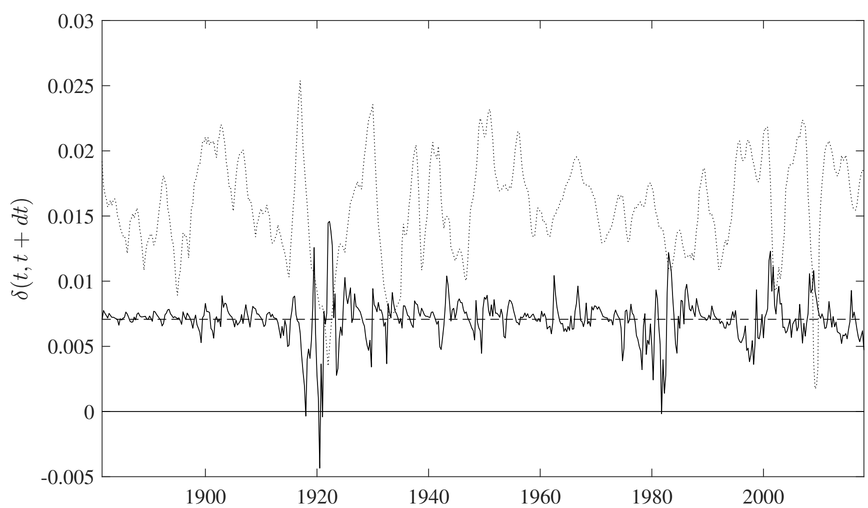

Figure 3. Plot of the time series of $\delta(t, t+1)$, in quarterly units, generated from the Kalman filter using quarterly sampling of US data from 1871-2017. The time average of $\delta(t, t+1)$ is 0.0072 and the standard deviation is 0.0017. Included for comparison is a plot of $\rho(t, t+1)$ (dotted line), illustrating the difference in magnitude and variation between $\rho(t, t+1)$ and $\delta(t, t+1)$.

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5 Return implications

The key driver of our results is the time varying mean reversion speed of valuation, $\gamma_q(t)$. The importance of this quantity derives from the fact that it determines the loading of valuation on return, i.e. $r(t,t+1) = -\gamma_q(t)q(t) + \cdots$, where the dots denote the contribution from expected growth and income.

5.1 *Expected returns*

The expression for the time varying expected return follows by taking conditional expectations of both sides of Eq. (10) so that

$$\mathrm{E}_t[r(t,t+dt)] = \mathrm{E}_t[dq(t,t+dt)] + \mathrm{E}_t[\rho(t,t+dt)] + \mathrm{E}_t[\delta(t,t+dt)]\left(e^{-q(t)} - 1\right). \tag{23}$$

We estimate the right hand side of this expression by combining the estimates, presented in Section 4, of (i) expected change in valuation $\mathrm{E}_t[dq(t,t+1)]$, (ii) expected profitability $\mathrm{E}_t[\rho(t,t+1)]$, and (iii) expected income $\mathrm{E}_t[\delta(t,t+1)]$. The expected change in valuation and expected income are direct outputs of the respective Kalman filters, and expected profitability is the fitted value of the AR(2) model defining profitability.

In Fig. 4 we plot (solid line) the time varying expected return, in annualized units, for the case of time varying $\gamma_q(t)$. For comparison, we plot (dotted line) the expected return for the case where $\gamma_q(t)$ is equal to the constant $\bar{\gamma}_q$, which is the time series average of $\gamma_q(t)$.

There are a number of time periods, indicated by the arrows in Fig. 4, in which expected returns for the model with time varying mean reversion speed differ substantially from the model with constant mean reversion speed. Two of these time periods, the late 1910s and the early 1980s are anti-bubbles, i.e. periods of explosive dynamics and low valuation. Three of these time periods, 1929, the late 1990s, and 2017, are bubbles, i.e. periods of explosive dynamics and high valuation.

Most of the time expected returns for static and dynamic mean reversion speeds are a relatively close match: these are the cases when $\gamma_q(t)$ is positive and valuation is mean reverting. When $\gamma_q(t)$ is larger than average, valuation has a larger than average impact on expected return. And, as is evident from Fig. 2, there are times when $\gamma_q(t)$ is more than one standard deviation above the mean and valuation has a particularly strong influence on expected return.

But, there are also times when $\gamma(t)$ is small and the impact of valuation on expected return is diminished: small, positive $\gamma_q(t)$ means that expected returns are effectively disconnected from valuation $q(t)$. Furthermore, in rare instances $\gamma_q(t)$ falls below zero. In

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these cases, rather than exhibiting mean reverting behavior, valuation $q(t)$ is mean averting: high valuation is followed by higher valuation or low valuation is followed by lower valuation. This is effectively momentum in the valuation component of expected return. The longer $\gamma_q(t)$ stays negative, the more valuation $q(t)$ mean averts and the larger the deviation of price from fundamental value. The result is either a bubble or an anti-bubble.

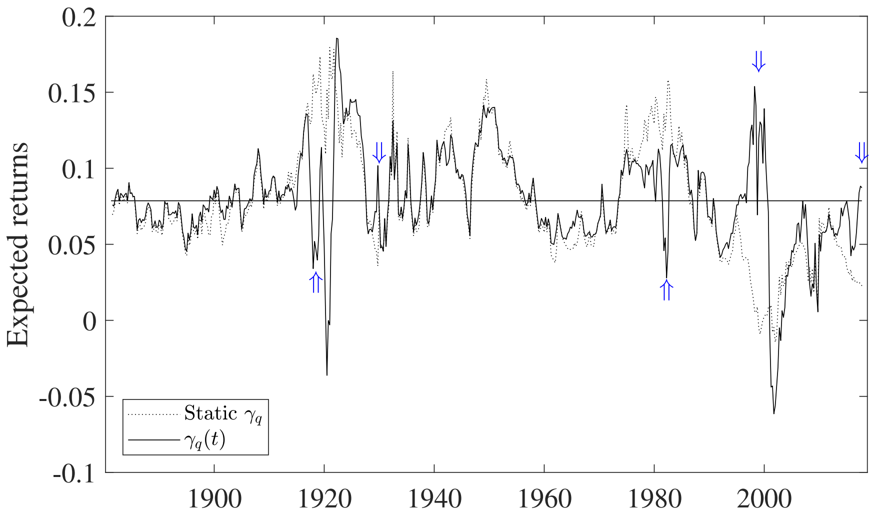

Figure 4. The solid line is a plot of the expected return $\mathrm{E}_t[r(t,t+1)]$, in annualized units. The dotted line is a plot of the discount rate assuming a constant mean reversion speed equal to $\bar{\gamma}_q$. The arrows indicate five periods in which the mean reversion speed $\gamma_q(t)$ is negative and the valuation effect is reversed: (i) late 1910s, (ii) 1929, (iii) early 1980s, (iv) late 1990s, and (v) 2017.

But importantly, as long as $\gamma_q(t)$ satisfies the stability condition given in Eq. (16), the explosive behavior is temporary because $\gamma_q(t)$ is itself a mean reverting process with a positive mean and volatility that is not excessive.

From an expected return perspective, negative $\gamma_q(t)$ means that expensive valuation contributes positively to expected returns, and cheap valuation contributes negatively. During these bubble or anti-bubble periods the valuation effect is actually reversed.

5.2 Term structure of expected returns

To better visualize the anatomy of bubbles and anti-bubbles it is useful to examine the term structure of expected returns during these extreme market events. The assumptions we have made about the dynamics of valuation $q(t)$, profitability $\rho(t,t+1)$, and income to fundamental value $\delta(t,t+1)$ allow us to easily compute the term structure of discount rates by straightforward iteration of the dynamical equations (12) for valuation, (17) for

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profitability, and (22) for income. Computing the term structure of change in valuation also involves iterating Eq. (15) to generate the term structure of the mean reversion speed.

Most of the time, i.e. when $\gamma_q(t)$ is positive and close to the average, the term structure of expected returns is monotonic in horizon (see Tarlie (2013)). When valuations are low ($q(t) < 0$) instantaneous expected returns are higher than average and the term structure is downward sloping. Similarly, when valuations are high ($q(t) > 0$), instantaneous expected returns are lower than average and the term structure is upward sloping. However, during the bubble and anti-bubble periods and $\gamma_q(t)$ is negative, the term structure displays rich patterns.

In Fig. 5 we illustrate the term structure for the anti-bubbles of the early 1920s and early 1980s, and the bubbles of 1929 and late 1990s. In these plots, the point $\tau = 0$ on the horizontal axes represents the date of interest, e.g. 031920 for the top-left panel, and determines the initial conditions for the $q(\tau)$, $\rho(\tau, \tau + 1)$, and $\delta(\tau, \tau + 1)$ iterations.

There are a number of notable features in these plots. First, for the anti-bubbles (the two left panels), we see that even though valuations are low (-0.97 for $t = 031920$ and -0.79 for $t = 031982$), expected return from change in valuation is actually negative. This is because $\gamma_q(t)$ is less than zero and valuations are expected to decline even further. For the bubble periods (the two right panels) we have the mirror image. Even though valuations are high (0.73 for $t = 091929$ and 1.04 for $t = 121999$), expected return from change in valuation is positive, i.e. the dashed line at $\tau = 0$ is above zero. This is because valuations are expected to increase even further. Note also the slow reversion back to long term average; for example, for 031999 the (expected) discount rate 20 years later is still about one percentage point below average.

While the instantaneous expected returns in bubbles is higher than average and in anti-bubbles is lower than average, the term structures during these periods are very steep. For example, in both 091929 and 121999 the annualized expected returns for $\tau = 0$ are around 10%, but for $\tau \approx 2$ years the expected returns implied by the term structure fall dramatically. This implies that these markets are highly unstable. In Section 6.2 we connect these periods of instability to nonlinear valuation dynamics controlled by expected changes in profitability.

Second, in the anti-bubble of the early 1920s expected growth plus expected income (i.e. carry) at $\tau = 0$ is below the long term average. Furthermore, for both bubbles the expected growth rate is above average. This suggests that valuation and growth are correlated. To examine this effect further we study, in the next section, the time series of the components of returns.

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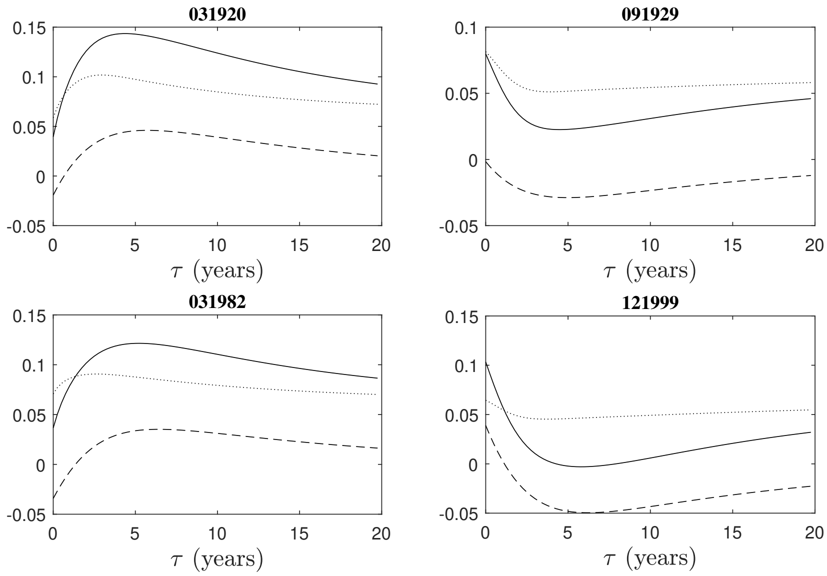

Figure 5. Term structures of discount rates for 031920, 091929, 031982, and 121999 in annual units. Total expected return is the solid line. The other lines represent a decomposition of total return into return from (i) expected change in valuation (dashed line), (ii) carry, i.e. expected growth plus expected income (dotted line).

5.3 Return decomposition

It is useful to decompose the expected return from Fig. 4 into: (i) expected change in valuation, and (ii) carry, which is expected growth plus expected income. These are shown, in annual units, in Fig. 6. We see clearly that changes in valuation make no contribution to long term returns; all returns come from a combination of growth and dividends, the sum of which we call carry. We also see that periods of high valuation tend to be associated with above average carry, mainly driven by high expected growth, and periods of low valuation with periods of below average carry.

From a bubble, anti-bubble perspective this means that bubbles tend to be associated with high levels of growth, and therefore profitability, and anti-bubbles with low levels of growth and profitability. More generally, the correlation between $q(t)$ and $\mathrm{E}_t[\rho(t, t+1)]$ is 0.45 with a Hansen-Hodrick (12 lag) corrected standard error of 0.11. This suggests that bubbles and anti-bubbles don't appear out of the blue. Rather, we will show in the next section that they arise out of the interaction between valuation and expected change in profitability.

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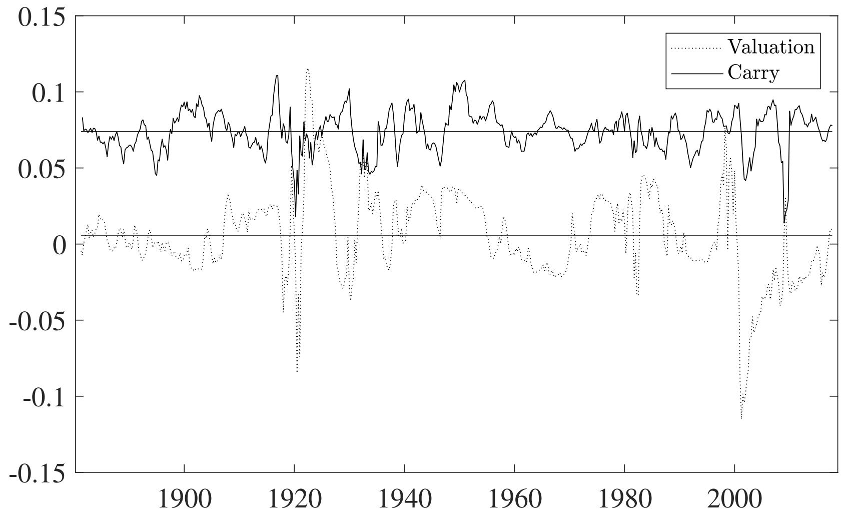

Figure 6. Decomposition of expected return into (i) expected change in valuation $\mathrm{E}_t[dq(t,t+1)]$ (solid line), and (ii) carry, i.e. expected growth $\mathrm{E}_t[\rho(t,t+1)-\delta(t,t+1)]$ plus expected income $\mathrm{E}_t[\delta(t,t+1)e^{-q(t)}]$ (dotted line). Series are shown in annual units. The time averages of the valuation and carry series are -0.0007, and 0.065, respectively.

6 A mechanism for the creation and destruction of bubbles and anti-bubbles

The importance of the mean reversion speed $\gamma_q(t)$ is that it controls the sensitivity of expected return to valuation. In the previous sections we illustrated how and when the dynamics of $\gamma_q(t)$ impact expected returns. In particular, negative values of $\gamma_q(t)$ lead to extreme valuations, i.e. bubbles and anti-bubbles. Our goal in this section is to develop some insight into the mechanism driving $\gamma_q(t)$.

We find that a small number of terms nonlinear in valuation explain a meaningful portion of the variation in $\gamma_q(t)$, and that a term bilinear in valuation and expected change in profitability controls the locally explosive dynamics of valuation. This bilinear term, which couples valuation and expected change in profitability, also breaks valuation symmetry. But because the time average of this bilinear term is zero, symmetry still holds on average. Furthermore, this term provides insight into a mechanism for both formation and collapse of bubbles and anti-bubbles.

The collapse of bubbles, or the reflation of anti-bubbles, provides a mechanism for time varying volatility. The flipping of the sign of the change in expected profitability can cause the forces driving valuation to change from explosive to strongly mean reverting. This can induce large changes in valuation, and therefore returns.

The bursting of a bubble, or the reflation of an anti-bubble, is catalyzed by the expected change in profitability flipping sign. We can think of the underlying economic mechanism for this sign flipping in terms of capital supply. In bubbles, profitability is high, which in our model means net investment is high. But high net investment means an increase in capital supply, ultimately leading to an oversupply and lower returns. This is the economic mechanism driving the expected change in profitability from positive to negative. For anti-bubbles, the capital supply mechanism applies in reverse.

6.1 The relationship between mean reversion speed and valuation

As a first step, in Fig. 7 we combine into a single scatter plot the time series of valuation $q(t)$ shown in Fig. 1 with the time series of mean reversion speed $\gamma_q(t)$ shown in Fig. 2. In this plot we see that extremes in valuation, both high and low, are associated with periods of negative mean reversion speeds. This graph also illustrates the symmetry between the bubble and anti-bubble periods, with the common feature of strongly negative mean reversion speeds for both high and low valuations strengthening the notion that bubbles and anti-bubbles are two sides of the same coin.

To relate this picture to the time series results presented in Section 4.2 we highlight the bubbles of 1929 and 1999, the anti-bubbles of the early 1920s and early 1980s, and the recent period of 2017. These periods are characterized by extreme valuations and mean reversion speeds $\gamma_q(t)$ less than zero ($-\gamma_q(t) > 0$ in the figure).

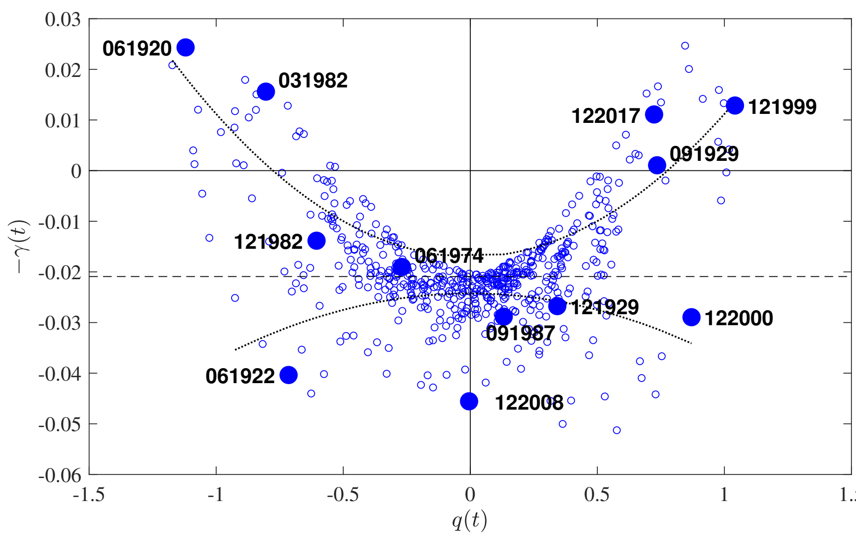

Figure 7. Scatter plot of $-\gamma_q(t)$ vs. $q(t)$ for the time series of $q(t)$ and $\gamma_q(t)$ illustrated in Figs. 1 and 2, respectively. The values of $\gamma_q(t)$ are in quarterly units, and the dashed horizontal line is the time median (0.021) of $\gamma_q(t)$.

To illustrate how dramatically $\gamma_q(t)$ can change, we also show the state (q, γ_q) at a date subsequent to the highlighted bubble and anti-bubble periods. Examining the figure, we see extreme changes in $\gamma_q(t)$ over relatively short periods of time. For example, the most rapid transition is from 091929 to 121929 where $-\gamma_q(t)$ goes from a value slightly above zero to, three months later, a value below the median of -0.021. For the anti-bubble of the early 1920s, $-\gamma_q(t)$ goes from a value close to 0.03 in 061920 to a value of about -0.04 in 061922. For the bubble of the late 1990s we see a very similar transition but from high valuation rather than low valuation. For example, $-\gamma_q(t)$ goes from a value of close to 0.02 in 121999 to a value close to -0.03 in 122000, just one year later.

These large, rapid moves in $\gamma_q(t)$ translate into large changes in expected returns (see Fig. 4), which in turn translate directly into heightened return volatility. This implies, for example, that the state $(q,\, \gamma_q)$ for 122017, which has approximately the same valuation as the peak of market in 1929 but a substantially more negative mean reversion speed, is likely to be followed by a state in the future with a larger than average value of $\gamma_q(t)$. Below, we make this notion more precise by presenting a mechanism for these transitions in $\gamma_q(t)$ that links time varying return volatility to the interaction between valuation and expected change in profitability.

For completeness, we also highlight a number of well-known market events, such as 1974, 1987, and 2008. While price changes were extreme during all three of these periods, they were unremarkable from a valuation perspective (valuations in all three cases were close to neutral) and the values of $\gamma_q(t)$ were either close to average (1974) or above average (1987 and 2008). These periods therefore do not exhibit the decoupling from valuation that is characteristic of bubbles and anti-bubbles; if anything they exhibit greater sensitivity to valuation.

The main takeaway from Fig. 7 is that $\gamma_q(t)$ depends in a nonlinear and symmetric way on $q(t)$. To illuminate both the nonlinearity and the symmetry, the two dashed lines (intended to guide the eye to the behavior we seek to explain) in Fig. 7 are quadratic fits for $-\gamma_q(t)$ greater than average (upper dotted curve) and $-\gamma_q(t)$ less than average (lower dotted curve). The symmetry of $\gamma_q(t)$ with respect to the sign of valuation $q(t)$ reflects a basic theme of our paper: the symmetry of bubbles and anti-bubbles. Furthermore, not necessarily obvious in this figure, but nevertheless important, is that the valuation symmetry is broken by profitability.

6.2 *Fitting the time varying mean reversion speed*

Motivated by these observations we fit $\gamma_q(t)$ to $q^2(t)$, $q^4(t)$, and the product of $q(t)$ and expected change in profitability, i.e.

$$\gamma_q(t) = \gamma_q(q, \mathrm{E}_t[\Delta\rho]) = a + b^0_{20} q^2(t) + b^0_{40} q^4(t) + b^0_{11} \mathrm{E}_t[\Delta\rho] q(t) + \epsilon_{\gamma_q}(t) \tag{24}$$

In this expression,

$$\mathrm{E}_t[\Delta\rho] = \mathrm{E}_t[\rho(t, t+1)] - \rho(t-1, t), \tag{25}$$

is the expected change in profitability. The "No lag" columns of Table 4 contain the results of this regression. Because b^0_{20} is negative, the $b^0_{20} q^2(t)$ term captures the negative curvature of $\gamma_q(t)$ evident in Fig. 7. The impact of this quadratic term is that as valuations deviate from neutral, either positively or negatively, $\gamma_q(t)$ declines in magnitude and with it, the sensitivity of expected return to valuation. But if $b^0_{20} q^2(t)$ were the only q-dependent term then, from a dynamical systems perspective $q(t)$ is asymptotically unbounded because for magnitudes of q large enough q is explosive. The role of the $b^0_{40} q^4(t)$ term is to asymptotically bound q.

Table 4
Fit of $\gamma_q(t)$

Parameter estimates of $\gamma_q(t) = a + \sum_{i=1}^{2} b^0_{2i} q^{2i}(t) + b^1_{20} q^2(t-4) + \epsilon_{\gamma_q}(t)$. The standard deviation of $q^2(t)$ is 0.23, and the standard deviation of $\mathrm{E}_t[\Delta\rho(t)]q(t)$ is 2.7×10^{-4}.

		No lag		With a 4 quarter lag	
Term	Parameter	Estimate	Std Error	Estimate	Std Error
constant	a	0.024	0.0011	0.023	0.0009
$q^2(t)$	b^0_{20}	-0.041	0.0177	-0.069	0.0170
$q^4(t)$	b^0_{40}	0.016	0.0147	0.019	0.0112
$\mathrm{E}_t[\Delta\rho]q(t)$	b^0_{11}	-10.3	2.6	-2.8	3.3
$q^2(t-4)$	b^1_{20}			0.034	0.0130
	R^2	0.33		0.49	

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6.3 Visualizing the dynamics

To visualize the effects of the terms nonlinear in valuation, and to also help understand the importance of the bilinear term in Eq. (26), we introduce the notion of a pseudopotential $V(q)$. By introducing the pseudopotential we do not mean to imply that valuation dynamics are fully described by a potential function. In fact, the scatter plot in Fig. 7 suggests that lags of the state variables are relevant in explaining the time variation of the mean reversion speed, a theme that we examine in more detail in the following section, Section 6.4. The aim of this section is to visualize the primary mechanism that drives both the creation and destruction of bubbles and anti-bubbles.

The pseudopotential is related to the force driving changes in valuation. If we write

$$\mathrm{E}_t[dq(t, t+dt)] = -\frac{dV(q)}{dq}, \tag{26}$$

then we see that the force driving valuation q is given by minus the derivative of the pseudopotential $V(q)$ with respect to valuation. With this framing we can visualize the dynamics of $q(t)$ by imagining a ball sliding down the sides of $V(q)$. Because $dq = -\gamma_q q dt$, we have that $V(q) = \int dq \mathrm{E}_t[\gamma_q q]$. For γ_q given in Eq. (24) we therefore get

$$V_z(q) = q^2 \left(\frac{a}{2} + \frac{b_{20}^0}{4} q^2(t) + \frac{b_{40}^0}{6} q^4(t) + \frac{b_{11}^0}{3} \mathrm{E}_t[\Delta\rho] q(t) \right). \tag{27}$$

We index $V_z(q)$ by $z(t) \equiv \mathrm{E}_t[\Delta\rho]/\sigma(\mathrm{E}_t[\Delta\rho])$, with $\sigma(\mathrm{E}_t[\Delta\rho])$ the standard deviation of $\mathrm{E}_t[\Delta\rho]$, because $\mathrm{E}_t[\Delta\rho]$ is a dynamic control parameter that controls the shape of the pseudopotential. This is illustrated in Fig. 8 where we plot $V_z(q)$ for five values of z.

We can think of the transition of z from positive to negative or negative to positive as driven by capital supply. In bubbles, profitability is high. But this means that net investment is also high, which leads to an oversupply of capital, ultimately driving down profitability. This is the economic mechanism driving z from positive to negative. Similarly, in anti-bubbles, profitability is low. This means that net investment is also low, which leads to an undersupply of capital, ultimately lifting profitability. This is the economic mechanism driving z from negative to positive.

The (middle) solid line in Fig. 8 corresponds to $z = 0$, i.e. expected change in profitability is zero. In this case we see that because b_{40}^0 is positive, the dynamics of q are bounded. Note also that because b_{20}^0 is negative, the effect of the $b_{20}^0 q^2$ term is to flatten the potential, i.e. decrease the magnitude of the restoring forces for intermediate levels of valuation.

Now consider the case of nonzero expected change in profitability, i.e. $z \neq 0$. The first thing to note is that the symmetry of $V_z(q)$ with respect to changing the sign of q is broken.

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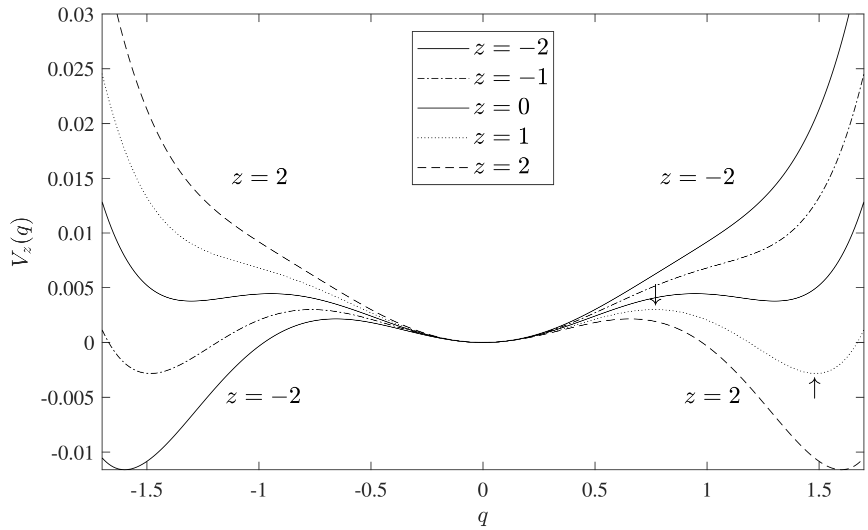

Figure 8. Pseudopotential $V_z(q)$ as a function of q for $V_z(q)$ given in Eq. (27). Each curve is for a fixed $z(t) = \mathrm{E}_t[\Delta\rho]/\sigma(\mathrm{E}_t[\Delta\rho])$, where $\sigma(\mathrm{E}_t[\Delta\rho]) \approx 7.0 \times 10^{-4}$ is the standard deviation of $\mathrm{E}_t[\Delta\rho]$. For $z = 1$, the down arrow marks the locally unstable fixed point at $q = 0.77$ and the up arrow marks the locally stable fixed point at $q = 1.48$.

For positive expected change in profitability ($z > 0$) and low valuation ($q < 0$), the strength of the forces pushing valuation back to neutral increase, and for high valuation the strength of the restoring forces decrease. A similar picture holds for negative expected change in profitability, but with signs reversed.

For concreteness, take $z = 1$, which corresponds to $\mathrm{E}_t[\Delta\rho] = \sigma(\mathrm{E}_t[\Delta\rho])$ and is the dotted line in Fig. 8. In this case, we see that for valuation $q < 0.77$ (indicated by the down arrow), the force $-dV/dq$ pushes q back to neutral valuation ($q = 0$). In this case, $q = 0$ is stable, but it is only locally stable because the point $q = 0.77$ is an unstable fixed point. For valuations larger than $q = 0.77$, q is pushed toward the stable fixed point at $q = 1.48$. Thus, for valuation between $q = 0.77$ and $q = 1.48$, q is pushed away from neutral ($q = 0$) and toward the locally stable fixed point at $q = 1.48$. A similar narrative holds for $z = -1$, but with signs reversed.

In this model, therefore, bubble and anti-bubble formation, and the explosive behavior in valuation that is characteristic of these extremes, is a consequence of (i) the weakening of the restoring forces by the $b^0_{20}q^2$ term, and (ii) the amplification of this weakening by the interaction between valuation and expected change in profitability by the $b^0_{11}\mathrm{E}_t[\Delta\rho]q(t)$ term. So, for example, when valuation is high, but not too high, and expected change in profitability is positive, e.g. the system is between the up and down arrows for $z = 1$ in Fig. 8, valuation increases. If valuation gets too extended, the $b^0_{40}q^4(t)$ term forces valuation

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back to the locally stable high valuation fixed point.

Over time, however, this situation of high valuations is not stable. The reason is that expected change in profitability ($\mathrm{E}_t[\Delta\rho]$) is the variable that controls the shape of potential $V_z(q)$. And expected change in profitability changes through time with a cyclical component that varies around zero and is tied to the business cycle.[13] Thus, when we look at the curves in Fig. 8 we should imagine the time variation as the control variable $z(t)$ moves through time.

With this picture in mind, the "bubble bursts" when expected change in profitability $\mathrm{E}_t[\Delta\rho]$ changes sign, e.g. z goes from positive to negative in Fig. 8. If the change in $\mathrm{E}_t[\Delta\rho]$ is large, e.g. z goes from, say, $z = 1$ to $z = -1$ or even $z = -2$, the effect is dramatic in that for large valuation and $z = 1$ or $z = 2$ the force pushing valuation back to neutral is extremely strong. And we see this large reversal of valuation and characterize it as the bursting of a bubble. More generally, the movements in $\mathrm{E}_t[\Delta\rho]$ and the associated reversals of valuation are a mechanism for time varying return volatility and clustering.

Of course, because of the symmetry we have imposed, the narrative for bubbles applies equally well to the situation of anti-bubbles, i.e. low valuations coupled, initially with expected change in profitability that is negative, that then over time flips to being positive.

It is useful to contrast the picture implied by the pseudopotentials in Fig. 8 with regime switching models. In the context of the pseudopotential, we can think of regime switching models as pseudopotentials with multiple locally stable states, e.g. for $z = 1$ the $q = 0$ and $q = 1.48$ local minima. However, if the dynamics is governed by a potential such as that given in Eq. (27) in which the control parameter $\mathrm{E}_t[\Delta\rho]$ is changing continuously over time such that the location of the locally stable states is also changing continuously, then the regime switching model, which is better suited to identifying static local minima, will have difficulty.

6.4 Fitting $\gamma_q(t)$ with lags

While the model given in Eq. (24) captures the negative curvature of $\gamma_q(t)$ as a function of $q(t)$, it does not effectively capture the positive curvature, highlighted by the lower dashed quadratic that is also evident in this figure. To capture this effect, we add a four quarter lag of q^2; regressions that include intermediate lags yield little additional information. The

[13]To see this more explicitly, use Eqs. (17) and (18) to write $\mathrm{E}_t[\Delta\rho] = k_\rho(\bar{\rho} - \rho(t)) + (1 - 2\gamma_\rho)\Delta\rho(t)$, where $k_\rho = 0.067$ and $1 - 2\gamma_\rho = 0.62$ (see Table 2). If $\rho(t) > \bar{\rho}$ then the first term contributes negatively to $\mathrm{E}_t[\Delta\rho]$. But if $\Delta\rho(t) = \rho(t) - \rho(t-1)$ is positive and large enough, then the second term can offset the negative contribution coming from the first term, resulting in a positive value for $\mathrm{E}_t[\Delta\rho]$. However, this cannot continue indefinitely. At some point $\rho(t)$ will exceed $\bar{\rho}$ by such a large amount that $\mathrm{E}_t[\Delta\rho]$ will change sign.

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regression results including the lagged variable $q^2(t-4)$ are shown in the two rightmost columns of Table 4; the notation is such that the "1" superscript denotes the lagged variable.

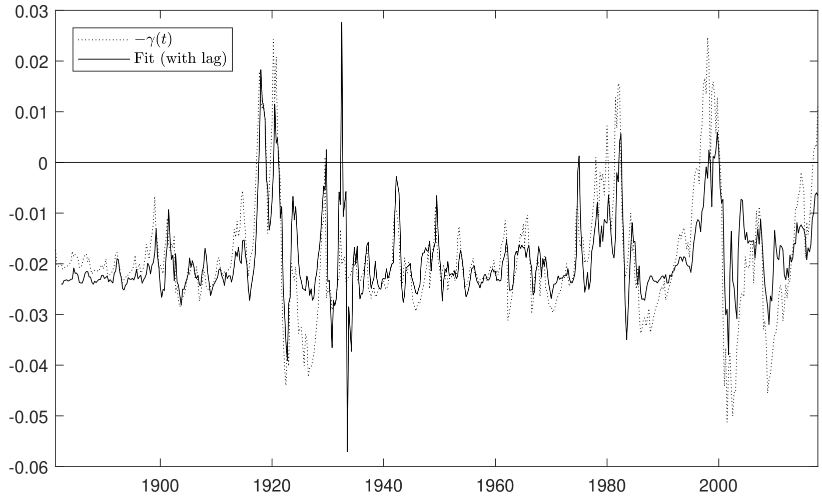

Figure 9. Fit of $\gamma_q(t)$ to a linear combination of $q^2(t)$, $q^4(t)$, $\mathrm{E}_t[\Delta\rho]q(t)$, and the four quarter lag $q^2(t-4)$. The R^2 of the fit is 0.49.

The explanatory power (adjusted R^2) of the regression including the four quarter lag term $q^2(t-4)$ is 0.49, higher than the explanatory power of the model without lags. Both the fit with no lags and the fit with the lags are able to capture the periods of temporarily explosive behavior associated with the negative mean reversion speeds. The quality of fit for the model with lags is illustrated in Fig. 9 where we plot the estimated $\gamma_q(t)$ and the time series of the fitted value for the regression including the lagged variables. The extra explanatory power of the model with lags relative to the model with no lags comes from the lagged model's ability to fit the positive values of $\gamma_q(t)$ that are larger than average. In Fig. 9 these are the points in the lower part of the figure.

This second regression including the lagged term implies a more subtle sensitivity to valuation. For example, if lagged valuation is either very high ($q(t-4) \gtrsim 1$) or very low ($q(t-4) \lesssim -1$), then the term $b^1_{20}q^2(t-4)$ makes a positive contribution to $\gamma_q(t)$. For situations in which valuation at time t is close to neutral ($q(t) \sim 0$) this results in $\gamma_q(t)$ significantly above average, precisely the results depicted in the lower part of Fig. 7. Also note in this regression that the coefficient $b^0_{20} = -0.069$ for the regression with lags is even more negative than the value of -0.041 for the regression with no lags. This means that for small valuations, the model with lags is less sensitive to current valuation than the model with no lags.

7 The anatomy of bubbles and anti-bubbles

In this section we examine in more detail the anti-bubble of the late 1910s, the bubble of the late 1990s, and the global financial crisis of 2008 and 2009. Our goal is to develop intuition about the dynamics of the mean reversion speed $\gamma_q(t)$ by taking a more focused look at these particular time periods. The pseudopotential defined in Section 6.3 provides a useful tool for visualization. The characteristics of the anti-bubble of the late 1910s and the bubble of the late 1990s are driven by the interaction of extreme valuation with expected change in profitability. By contrast, the global financial crisis is not characterized by extreme valuation, as valuation reached levels about one standard deviation expensive (see Fig. 1). However, expected change in profitability did reach extreme levels, and we dramatically illustrate the impact of this using the pseudopotential.

7.1 *The anti-bubble of the late 1910s*

To get a closer look at the dynamics of the anti-bubble of the late 1910s, the graphs in Fig. 10 plot the no lag fit of $-\gamma_q(t)$ and its components for the period from 1916 through 1922. The solid line in the figure is the fit of $\gamma_q(t)$ using Eq. (24); points in the upper part of the graph represent temporarily explosive dynamics and are characteristic of the anti-bubble.

The main idea is that expected change in profitability and valuation – the bilinear term – contributes to both the formation and destruction of the anti-bubble. In the first part of the period both $z(t)$ and $q(t)$ are negative so that $b^0_{11}\mathrm{E}_t[\Delta\rho]q(t)$ is positive, i.e. the dashed line in the figure from 1917 - 1919 is above zero, contributing to the negative mean reversion speed and temporarily explosive dynamics. The bottom left inset shows a snapshot of the pseudopotential for 091918. In this inset we see that the pseudopotential forces, driven by the bilinear term with $z = -1.2$, are pushing valuation further away from zero, even though valuation is already very negative.

On the other hand, in the second part of the period z flips sign and turns positive. The bottom right inset shows that as of 121922 we now have $z = 1.8$. But since valuation is still negative, now the contribution from the bilinear term goes the other way, and the pseudopotential forces are now pushing valuation back to zero.

The negative mean reversion speed leads to the formation of the anti-bubble. But how does the anti-bubble burst? Examining Fig. 10, we see that around 1922 the contribution from the bilinear term (the dotted line) leads to a reversal of the mean reversion speed, turning it positive. To visualize this effect, the inset figure on the lower right of Fig. 10 shows the pseudopotential for 121922. Valuation at this time is still very negative, but because expected change in profitability is now positive as manifested by $z = 2.1$, very

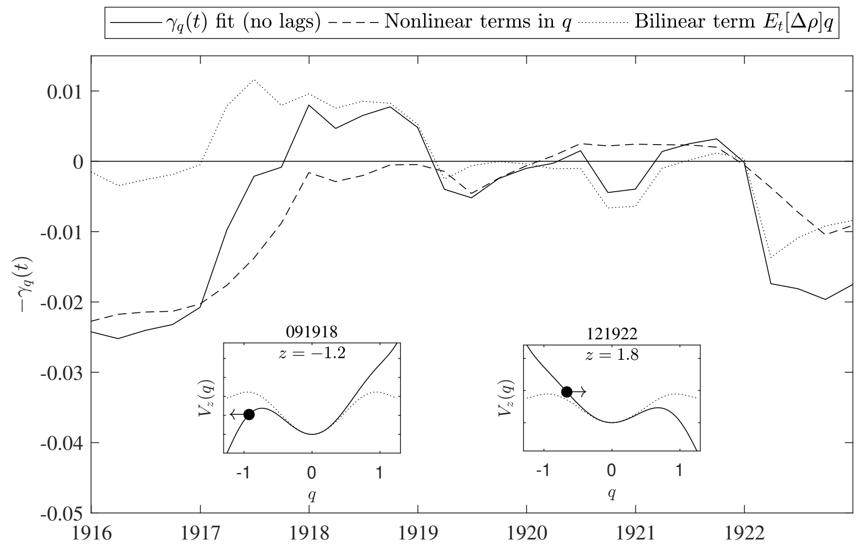

Figure 10. For the anti-bubble of the late 1910s, this figure shows a decomposition of the fit of $\gamma_q(t)$ using Eq. (24) into the terms nonlinear in q (q^2 and q^4) and the bilinear term $\mathrm{E}_t[\Delta\rho]q$. The solid line is the total, the dashed line the contribution from the constant plus nonlinear terms, and the dotted line the contribution from the bilinear term. The left inset shows the pseudopotential for 091918 with $z = -1.2$, and the right inset shows the pseudopotential for 121922 with $z = 1.8$. For reference, the dotted lines in the insets are the pseudopotentials with $z = 0$.

strong forces (as indicated by the steepness of the potential) are pushing $q(t)$ back to local equilibrium at $q = 0$.

7.2 The bubble of the late 1990s

The scenario for the bubble of the late 1990s is the mirror image of the anti-bubble of the late 1910s. In this case, valuations are high rather than low, and expected change in profitability is initially positive rather than negative. In this episode, using the fitted value of $\gamma_q(t)$ the dynamics of $q(t)$ are explosive from mid-1999 to the beginning of 2000. However, as can be seen in Fig. 2, $\gamma_q(t)$ estimated from the Kalman filter is negative over a substantially longer period, from 091996 through 091999 (except for 061998). The 091196 onset of the period of negative $\gamma_q(t)$ coincides with Alan Greenspan's irrational exuberance speech, which was delivered December 5, 1996. Furthermore, we see from the dotted line in Fig. 11 that the bilinear term $\mathrm{E}_t[\Delta\rho]q$, shown as the dotted line, is the primary driver of the fitted value, shown as the solid line, during this period.

The period of negative $\gamma_q(t)$ inflates the bubble. At the beginning of 2000, expected change in profitability is positive with $z = 0.9$ and valuation is high with $q \sim 1$. This combination results in the pseudopotential shown in the upper left inset in Fig. 11 for 121999. We see that since q is larger than the local maximum, pseudopotential forces push valuation away from equilibrium. This translates into the negative mean reversion speed evident as the solid line in the main part of Fig. 11. Valuation dynamics are explosive.

One year later, however, the pseudopotential has changed dramatically. While valuation hasn't changed much with $q = 0.73$, expected change in profitability is now significantly negative with $z = -2.2$. This means, as illustrated in the top right inset, that strong pseudopotential forces now push valuation back to the local equilibrium at $q = 0$. We can also see the effect of the bilinear term in the dotted line in Fig. 11. Clearly it is this term that drives the dynamics of $\gamma_q(t)$ in the crucial period from the end of 1999 to the end of 2000.

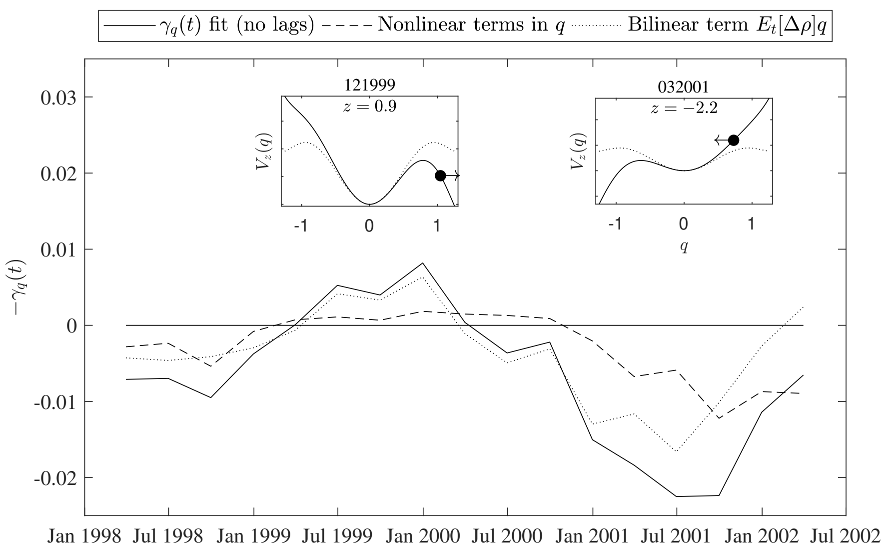

Figure 11. For the bubble of the late 1990s, this figure shows a decomposition of the fit of $\gamma_q(t)$ using Eq. (24) into the terms nonlinear in q (q^2 and q^4) and the bilinear term $\mathrm{E}_t[\Delta\rho]q$. The solid line is the total, the dashed line the contribution from the constant plus nonlinear terms, and the dotted line the contribution from the bilinear term. The bottom left inset shows the pseudopotential for 121999 with $z = 0.9$, and the bottom right inset shows the pseudopotential for 122000 with $z = -2.2$. For reference, the dotted lines in the insets are the pseudopotentials with $z = 0$.

7.3 The global financial crisis

Bubbles and anti-bubbles are characterized, at least in part, by extreme valuations. Because valuations during the global financial crisis of 2008-2009 are only moderately high, the conventional bubble label does not apply. However, the global financial crisis is characterized by extremely low profitability; see the right hand panel of Fig. 1.

So while q is not extreme, $z = \mathrm{E}_t[\Delta\rho]/\sigma(\Delta\rho)$ is. In Fig. 12 we show the fit of $\gamma_q(t)$ for the period 2008-2011. We see that the contribution from the nonlinear terms in q as shown in the dashed line is roughly constant and uninteresting. On the other hand, the contribution from the bilinear term $\mathrm{E}_t[\Delta\rho]$ as shown in the dotted line changes fairly dramatically.

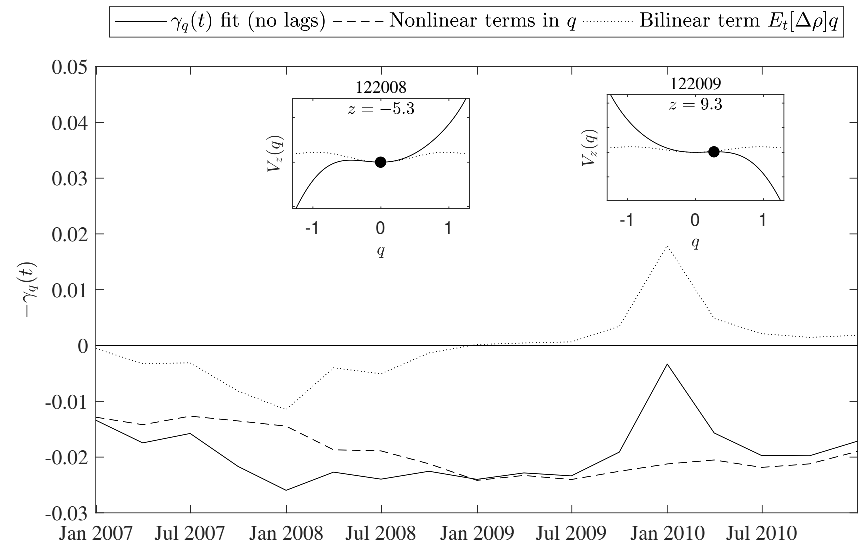

Figure 12. For the global financial crisis of 2008-2009, this figure shows a decomposition of the fit of $\gamma_q(t)$ using Eq. (24) into the terms nonlinear in q (q^2 and q^4) and the bilinear term $\mathrm{E}_t[\Delta\rho]q$. The solid line is the total, the dashed line the contribution from the constant plus nonlinear terms, and the dotted line the contribution from the bilinear term. The top left inset shows the pseudopotential for 122008 with $z = -5.3$, and the top right inset shows the pseudopotential for 122009 with $z = 9.3$. For reference, the dotted lines in the insets are the pseudopotentials with $z = 0$.

In terms of the effect on the pseudopotential, in 122008, in the depths of the financial crisis, we have $z = -5.3$, i.e. expected change in profitability is minus five times the typical value. This extreme value of z translates into the pseudopotential shown in the top left inset. The picture that emerges is that valuation ($q \approx 0$) is precariously situated on the edge of a cliff: if valuations fall too much ($q \lesssim -0.4$), dynamics become explosive and low valuations lead to even lower valuations, potentially resulting in an anti-bubble.

That this anti-bubble result did not play out can be understood by looking at the second inset, the one on the top right. This graphic shows the pseudopotential as of 122009. At this time z has completely flipped: rather than the 5.3 value of one year prior, now $z = 9.2$! At this point, rather than being precariously situated with anti-bubble risk, the market is faced with bubble risk.

This scenario highlights the sensitivity of market dynamics to expected change in profitability, and in particular, the role of profitability in the global financial crisis.

8 Conclusion

We present a model of equity discount rates based on a simple model of valuation. Our technical innovation is to allow for a time varying mean reversion in the dynamics of valuation. Empirically we find that for the US stock market there are periods where the mean reversion temporarily becomes negative. During these periods of negative mean reversion speeds valuation dynamics is explosive, leading to a definition of bubbles and anti-bubbles as periods in which valuation dynamics is temporarily explosive.

The time varying mean reversion speed captures nonlinearities that are important drivers of the dynamics. We find that the interaction between valuation and expected change in profitability is a key catalyst in generating bubbles and anti-bubbles. Bubbles are prone to form when a string of positive expected changes in profitability leads to and combines with expensive valuation. The bubble or anti-bubble is destroyed when expected change in profitability flips sign.

The time varying mean reversion speed has implications for returns. During bubble periods, valuation is temporarily mean averting. This means that expected returns from valuation change are positive, even though the market is expensive: valuations are high and expected to go higher. The reverse is true for anti-bubbles. Furthermore, the rich dynamics of valuation leads to rich term structures of expected returns during these bubble and anti-bubble periods.

From an econometric perspective, identifying multiple bubbles is a well known, open problem. We solve this problem by allowing the mean reversion speed to vary in time according to an AR(1) process, leading to a model that can be estimated using standard state space methods (i.e. Kalman filter). Multiple bubbles are then simply periods of negative mean reversion speed. Furthermore, analyzing the US stock market from 1881-2017 we only identify 5 periods of explosive dynamics, suggesting that this state space approach is a viable candidate for a low error rate, reliable bubble indicator that might be useful for financial surveillance.

Finally, our results have potential policy implications. Brunnermeier and Schnabel (2016) argue that a purely passive stance toward the build up of the bubbles is invariably costly. In our simple model, formation of bubbles and anti-bubbles is caused by the interplay between extreme valuation and views on changes in profitability (or growth). A necessary condition for bubbles and anti-bubbles are extremes of valuation, with expansion aided by

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the interplay between valuation and the expected change in growth. If the bubble or anti-bubble has formed, it pops or reflates when the sign of expected change in profitability changes. Given that bubbles and anti-bubbles are impacted by both valuation and expected change in profitability, policy makers should be mindful of the impact of their actions on both valuation and profitability.

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Appendix A Stability statistic

In this section we outline the computation of the stability statistic given in Eq. (16). Imposing finite variance on valuation $q(t)$ places restrictions on the dynamic properties of the time varying mean reversion speed $\gamma_q(t)$. The basic intuition is that if the average mean reversion is positive and large enough, then the (unconditional) variance of valuation is finite.

For this calculation it is useful to first work in continuous time. Our starting point is the dynamic equations for $q(t)$ and $\gamma_q(t)$:

$$dq(t, t+dt) = -\gamma_q(t)q(t) + dW_q(t+dt) \tag{28}$$

and

$$d\gamma_q(t, t+dt) = -\phi_{\gamma q}(\gamma_q(t) - \bar{\gamma}_q)dt + dV_{\gamma q}(t+dt), \tag{29}$$

where $\phi_{\gamma q} > 0$ is the constant, positive mean reversion speed of $\gamma_q(t)$. In these expressions we take $dW_q(t+dt)$ to have zero mean and to be wide sense stationary, but we do not need to make strong assumptions about its dynamics. We do, however, assume that $dW_q(t+dt)$ is uncorrelated with the temporally uncorrelated standard Brownian noise $dV_{\gamma q}(t+dt)$ that drives $\gamma_q(t)$. Integrating Eq. (28) from t to s we get

$$q(s) = q(t)e^{-\tilde{\gamma}_q(t,s)} + \int_t^s dW_q(s')e^{\tilde{\gamma}_q(t,s')-\tilde{\gamma}_q(t,s)} \tag{30}$$

where

$$\tilde{\gamma}_q(t,s) = \int_t^s ds'\gamma_q(s') \tag{31}$$

$$= \bar{\gamma}_q\,(s-t) + (\gamma_q(t) - \bar{\gamma}_q)\left(\frac{1 - e^{-\phi_{\gamma q}\,(s-t)}}{\phi_{\gamma q}}\right) + \int_t^s dV_{\gamma q}(s')\left(\frac{1 - e^{-\phi_{\gamma q}\,(s'-s)}}{\phi_{\gamma q}}\right). \tag{32}$$

Since dW_q has zero mean and is uncorrelated with $dV_{\gamma q}$, terms that are linear in dW_q vanish and the terms quadratic in dW_q are finite and do not depend on $\tilde{\gamma}_q$. Therefore, the finite variance condition[14] is satisfied if $\lim_{s\to\infty} E[e^{-2\tilde{\gamma}_q(t,s)}] < \infty$. This condition is satisfied as long as

$$\bar{\gamma}_q > \frac{1}{2}\frac{\sigma_V^2}{\phi_{\gamma q}^2}, \tag{33}$$

[14]Note that in this model finite variance also implies finite mean.

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where $\sigma_V^2 dt = E[dV^2]$. The stability condition reflects the intuition that if the average mean reversion speed $\bar{\gamma}_q$ is positive and large enough, then variance is finite. Furthermore, since the right hand side of Eq. (33) is positive definite, this V_{γ_q} condition implies that $\bar{\gamma}_q$ is also strictly positive, as expected. To see that the units of the left and right hand sides of this condition are the same, note that the units of σ_V^2 are time^{-3} and that the units of both $\bar{\gamma}_q$ and $\phi_{\gamma q}$ are time^{-1}.

To convert the stability condition in Eq. (33) into the discrete time expression given in Eq. (16), we convert the formulation of the dynamics given in Eq. (29) to the discrete form given in Eq. (15) using the following relations:

$$\bar{\gamma}_q = \frac{a_{\gamma q}}{1 - F_{\gamma q}}, \quad \phi_{\gamma q} = \frac{1 - F_{\gamma q}}{dt}, \quad Q_{\gamma q} = \sigma_V^2 dt. \tag{34}$$

Inserting these expressions into Eq. (33), and using the fact that $dt = 0.25$, we can write the discrete version of the stability condition appropriate for quarterly sampling as

$$\frac{8 a_{\gamma q}(1 - F_{\gamma q})}{Q_{\gamma q}} > 1. \tag{35}$$

Appendix B Return decomposition

In this appendix we outline the derivation of the return decomposition given in Eq. (9). The basic return formula for the geometric return $r(t, t + dt)$ given in Eq. (9) expresses, to leading order in dt, the geometric return as the sum of three terms: (i) change in valuation, (ii) growth, and (iii) income. The derivation exploits the fact that change in valuation $dq(t, t + dt) = q(t + dt) - q(t)$, growth $g(t, t + dt)$, and dividends $D(t, t + dt)$ and dividends divided by fundamental value $\delta(t, t + dt)$, are proportional to dt.

The first step in the derivation is to start with the definition of $r(t, t+dt)$ given in Eq. (1) and write

$$
\begin{aligned}
r(t, t + dt) &= \ln \frac{P(t + dt) + D(t, t + dt)}{P(t)} && (36)\\
&= \ln \frac{P(t + dt)}{P(t)} \left(1 + \frac{D(t, t + dt)}{P(t + dt)}\right) && (37)\\
&= \ln \frac{P(t + dt)}{P(t)} + \ln \left(1 + \frac{D(t, t + dt)}{P(t + dt)}\right) && (38)\\
&= \ln \frac{P(t + dt)}{P(t)} + \frac{D(t, t + dt)}{P(t + dt)} + \cdots && (39)
\end{aligned}
$$

where the dots denote terms of order dt^2 and higher. For the first term on the right hand

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side of Eq. (39) we can then write

$$\ln \frac{P(t+dt)}{P(t)} = \ln \frac{P(t+dt)}{B(t+dt)} \frac{B(t)}{P(t)} \frac{B(t+dt)}{B(t)} \tag{40}$$

$$= \ln e^{q(t+dt)} e^{-q(t)} \frac{1}{1-g(t,t+dt)} \tag{41}$$

$$= q(t+dt) - q(t) + g(t,t+dt) + \cdots \tag{42}$$

where the second line follows from the definitions of $g(t, t+dt)$ and $q(t)$ given in Eqs. (6) and (8), respectively, and the dots denote terms of order dt^2 and higher. For the second term on the right hand side of Eq. (39), we can write

$$\frac{D(t,t+dt)}{P(t+dt)} = \frac{D(t,t+dt)}{B(t+dt)} \frac{B(t+dt)}{P(t+dt)} \tag{43}$$

$$= \delta(t,t+dt) e^{-q(t+dt)} \tag{44}$$

$$= \delta(t,t+dt) e^{-q(t)} e^{-dq(t,t+dt)} \tag{45}$$

$$= \delta(t,t+dt) e^{-q(t)} + \cdots \tag{46}$$

where the dots denote terms of order dt^2 and higher. Putting Eqs. (42) and (46) together, we have to leading order in dt,

$$r(t,t+dt) \simeq q(t+dt) - q(t) + g(t,t+dt) + \delta(t,t+dt) e^{-q(t)}, \tag{47}$$

which is Eq. (9).

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