

Finite-Difference Methods in Electromagnetics

Lectures 1 and 2

Stanislav I. Maslovski

University of Aveiro

May 21–22 and 28–29, 2020

Maxwell's Equations

Maxwell's equations in the time domain

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (\text{Faraday's law of induction}) \quad (1)$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad (\text{Amper's law + displacement current}) \quad (2)$$

$$\nabla \cdot \mathbf{D} = \rho \quad (\text{Gauss's law for electric field}) \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's law for magnetic field}) \quad (4)$$

ρ	electric charge density (vector)	$[\text{C}/\text{m}^3]$
\mathbf{J}	electric current density (vector)	$[\text{A}/\text{m}^2]$
\mathbf{E}	electric field (vector)	$[\text{V}/\text{m}]$
\mathbf{D}	electric displacement (vector)	$[\text{C}/\text{m}^2]$
\mathbf{H}	magnetic field (vector)	$[\text{A}/\text{m}]$
\mathbf{B}	magnetic induction (vector)	$[\text{Wb}/\text{m}^2]$

Material Relations

Vectors **D**, **B** are related to **E**, **H** and the material polarization/magnetization

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (\text{di})(\text{para})(\text{ferro})(\text{..})\text{electrics} \quad (5)$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \quad (\text{dia})(\text{para})(\text{ferro})(\text{..})\text{magnetics} \quad (6)$$

ε_0 vacuum permittivity $[\approx 8.8541878 \times 10^{-12} \text{ F/m}]$

μ_0 vacuum permeability $[\approx 1.2566371 \times 10^{-6} \text{ H/m}]$

P electric polarization (vector) $[\text{C/m}^2]$

M magnetic polarization (vector) $[\text{Wb/m}^2]$

Examples of Material Relations

Medium	Polarization	Magnetization
vacuum	$\mathbf{P} = 0$	$\mathbf{M} = 0$
isolators	$\mathbf{P} = \varepsilon_0(\bar{\bar{\varepsilon}}_r - 1) \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
conductors	$\mathbf{J} = \bar{\bar{\sigma}} \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
magnetics	$\mathbf{P} = 0$	$\mathbf{M} = \mu_0(\bar{\bar{\mu}}_r - 1) \cdot \mathbf{H}$
e^- plasma	$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau} \frac{\partial}{\partial t} + \omega_p^2 \right) \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E}$	$\mathbf{M} \approx 0$
ferromagnetics	$\mathbf{P} = \varepsilon_0(\bar{\bar{\varepsilon}}_r - 1) \cdot \mathbf{E}$	$\frac{\partial}{\partial t} \mathbf{M} = -\gamma \mathbf{M} \times \mathbf{H} - \lambda \mathbf{M} \times (\mathbf{M} \times \mathbf{H})$

($\bar{\bar{\sigma}}$ - conductivity, ω_p - plasma freq., τ^{-1} - collision freq., γ - gyromagnetic ratio, λ - dumping par.)

Boundary Conditions

To solve for the EM fields in a given domain one has to define the *boundary conditions*

On good conductors:	$\mathbf{E}_t = 0$	PEC BC (\approx Dirichlet)
As dual concept:	$\mathbf{H}_t = 0$	PMC BC (\approx Neumann)
Generalization:	$\mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\hat{\mathbf{n}} \times \mathbf{H}_t)$	impedance BC

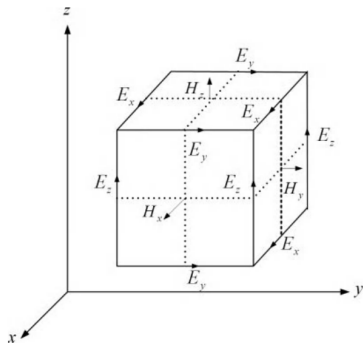
If a domain is split into subdomains:

$$\mathbf{E}_t|_1 - \mathbf{E}_t|_2 = 0 \quad (\text{continuous at the boundary})$$

$$\mathbf{H}_t|_1 - \mathbf{H}_t|_2 = \hat{\mathbf{n}} \times \mathbf{J}_s \quad (\text{continuous when } \mathbf{J}_s = 0 \text{ at the boundary})$$

($\hat{\mathbf{n}}$ - unit vector normal to the boundary, \mathbf{J}_s - surface current density [A/m])

Discretizing the Maxwell Equations (in space)



Yee's discretization scheme:

- ▶ Split space in cubic/rectangular cells
- ▶ Define H -points at the face centers
- ▶ Define E -points at the edge centers

$$(i, j, k) \leftrightarrow (i\Delta x, j\Delta y, k\Delta z)$$

$$(\nabla \times \mathbf{E})_x|_{i,j,k} \approx \frac{E_z|_{i,j+0.5,k} - E_z|_{i,j-0.5,k}}{\Delta y} - \frac{E_y|_{i,j,k+0.5} - E_y|_{i,j,k-0.5}}{\Delta z}$$

$$(\nabla \times \mathbf{H})_y|_{i,j,k+0.5} \approx \frac{H_x|_{i,j,k+1} - H_x|_{i,j,k}}{\Delta z} - \frac{H_z|_{i+0.5,j,k+0.5} - H_z|_{i-0.5,j,k+0.5}}{\Delta x}$$

etc.

Discretization in Time

Similarly, define D -points at times $t_n = n\Delta t$, and B -points at $t_{n+0.5} = t_n + 0.5\Delta t$:

$$\begin{aligned}\left.\frac{\partial \mathbf{D}}{\partial t}\right|^{n+0.5} &= \frac{\mathbf{D}|^{n+1} - \mathbf{D}|^n}{\Delta t} \\ \left.\frac{\partial \mathbf{B}}{\partial t}\right|^n &= \frac{\mathbf{B}|^{n+0.5} - \mathbf{B}|^{n-0.5}}{\Delta t}\end{aligned}$$

The material relations allow us to relate \mathbf{D} and \mathbf{B} to \mathbf{E} and \mathbf{H} .

E.g., in a non-dispersive isotropic magnetodielectric:

$$\begin{aligned}\left.\frac{\partial \mathbf{D}}{\partial t}\right|^{n+0.5} &= \varepsilon_0 \varepsilon_r \frac{\mathbf{E}|^{n+1} - \mathbf{E}|^n}{\Delta t} \\ \left.\frac{\partial \mathbf{B}}{\partial t}\right|^n &= \mu_0 \mu_r \frac{\mathbf{H}|^{n+0.5} - \mathbf{H}|^{n-0.5}}{\Delta t}\end{aligned}$$

Discretization in 2D case (TE waves)

Let us consider a 2D problem in xy -plane.

Let \mathbf{E} and \mathbf{J}^e be \perp to the plane and \mathbf{H} in the plane (the TE waves).

Discretized Maxwell's equations [Eqs. (1) and (2)]:

$$\frac{E_z|_{i,j+0.5}^n - E_z|_{i,j-0.5}^n}{\Delta y} + \mu_0\mu_r \frac{H_x|_{i,j}^{n+0.5} - H_x|_{i,j}^{n-0.5}}{\Delta t} = 0$$

$$-\frac{E_z|_{i,j+0.5}^n - E_z|_{i-1,j+0.5}^n}{\Delta x} + \mu_0\mu_r \frac{H_y|_{i-0.5,j+0.5}^{n+0.5} - H_y|_{i-0.5,j+0.5}^{n-0.5}}{\Delta t} = 0$$

$$\frac{H_y|_{i+0.5,j+0.5}^{n+0.5} - H_y|_{i-0.5,j+0.5}^{n+0.5}}{\Delta x} - \frac{H_x|_{i,j+1}^{n+0.5} - H_x|_{i,j}^{n+0.5}}{\Delta y} - \varepsilon_0\varepsilon_r \frac{E_z|_{i,j+0.5}^{n+1} - E_z|_{i,j+0.5}^n}{\Delta t} = J_z^e|_{i,j+0.5}^{n+0.5}$$

Yee's Leapfrog Update Scheme

After some algebra, we get the update equations:

$$H_x|_{i,j}^{n+0.5} = H_x|_{i,j}^{n-0.5} - \frac{\Delta t}{\mu_0 \mu_r} \frac{E_z|_{i,j+0.5}^n - E_z|_{i,j-0.5}^n}{\Delta y}$$

$$H_y|_{i-0.5,j+0.5}^{n+0.5} = H_y|_{i-0.5,j+0.5}^{n-0.5} + \frac{\Delta t}{\mu_0 \mu_r} \frac{E_z|_{i,j+0.5}^n - E_z|_{i-1,j+0.5}^n}{\Delta x}$$

$$E_z|_{i,j+0.5}^{n+1} = E_z|_{i,j+0.5}^n + \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left(\frac{H_y|_{i+0.5,j+0.5}^{n+0.5} - H_y|_{i-0.5,j+0.5}^{n+0.5}}{\Delta x} - \frac{H_x|_{i,j+1}^{n+0.5} - H_x|_{i,j}^{n+0.5}}{\Delta y} - J_z^e|_{i,j+0.5}^{n+0.5} \right)$$

Stability Condition (Courant condition)

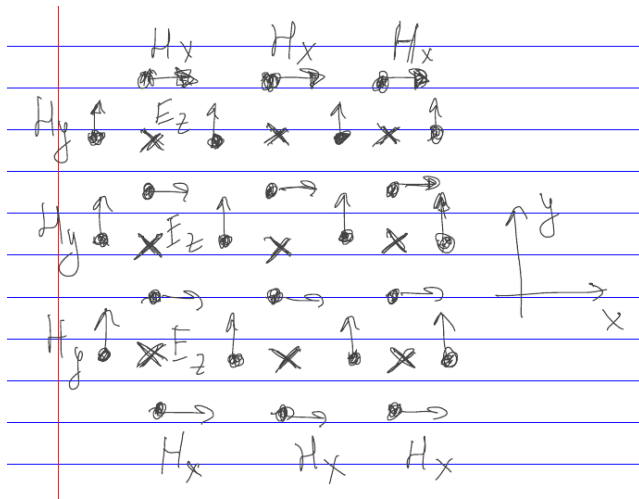
The Yee's algorithm is stable when the following condition is satisfied:

$$v\Delta t \leq \frac{1}{\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}}$$

Here, $v = c/\sqrt{\epsilon_r\mu_r}$ is the velocity of light in the material.

Grid Termination and the Boundary Conditions

The H -points at the edges of this 2D grid cannot be updated by Yee's scheme.



- ▶ At the non-updated H -points $H_{x,y} = 0$, effectively as for a PMC.
- ▶ Reflection occurs!
- ▶ To model open space, grid needs proper termination.
- ▶ Two main types of grid termination exist:
 - ▶ Absorbing Boundary Conditions (ABC)
 - ▶ Perfectly Matched Layer (PML)

Absorbing Boundary Conditions

Consider a TE plane wave propagating in free space:

$$E_z = E_0 \cos(k_x x + k_y y - \omega t)$$

From Maxwell's equations, $k_x^2 + k_y^2 = \varepsilon_0 \mu_0 \omega^2 \equiv k_0^2$ (dispersion relation), and

$$H_x = \frac{k_y}{\eta_0 k_0} E_z, \quad H_y = -\frac{k_x}{\eta_0 k_0} E_z, \quad \text{where } \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \quad (*)$$

If at the grid boundaries we enforce the conditions (*) for the plane waves *approaching* the boundary, such waves will not be reflected.

However, there is a problem: In FDTD we do not have information about k_x, k_y (and even k_0)!

When the wave is normally incident on x or y boundary,

$$k_y/k_0 = \pm 1 \text{ or } k_x/k_0 = \pm 1.$$

Another problem is that the grid points for E_z and $H_{x,y}$ *do not coincide*.

Practical Work

1. Modify the YEE-2D.m program in a way that it plots the fields in the complete domain $\text{size}_x \times \text{size}_y$ cells.
2. Make a modification such that the plotting (and pausing of the calculations) starts from the time step $t = 30$. For that, use the show variable (which is currently unused) and modify the code within the main loop.
3. Instead of exciting the structure by electric field at the boundary $x = x_{\max}$, excite it with the y -component of magnetic field at the opposite boundary $x = x_{\min}$. Note that in a plane EM wave, the electric and magnetic field magnitudes are related by the free space impedance η_0 (variable eta0): $E = \eta_0 H$, $\eta_0 \approx 377 \Omega$, therefore adjust the amplitude of the magnetic field source respectively. Observe the wave propagation, explain what you see.
4. Set the m variable in the code to 1.42. Observe the wave propagation, explain what you see. Change the value to 1.41, what happens? As a bonus question, explain why what you saw happened at $m > 1.41$ (and not at $m > 1$)?

Practical Work

5. Set $m = 0.9$. Implement the impedance boundary condition $E_z = \eta_0 H_y$ at the domain edge $x = x_{\min}$ and $E_z = -\eta_0 H_y$ at $x = x_{\max}$. Note that the same conditions can be written as $H_y = \pm \frac{1}{\eta_0} E_z$ and that they must appear in the code before setting the source.
6. Shift the magnetic source location from $x = x_{\min}$ to approximately the middle of the domain at $x = (x_{\min} + x_{\max})/2$. Observe the wave propagation, explain.
7. Restrict the excitation to the line of nodes satisfying (approximately) $x = (x_{\min} + x_{\max})/2$, $y_{\min} + 35\Delta y \leq y \leq y_{\max} - 35\Delta y$. Observe the wave propagation, explain what you see.
8. Change the value of m to 1.01. Observe the wave propagation, explain.

Finite Differences in the Frequency Domain

Maxwell's Equations in the Frequency Domain

Maxwell's equations in the frequency domain

Fourier transform in $t \Rightarrow \frac{\partial}{\partial t} \leftrightarrow j\omega$, where $j = \sqrt{-1}$

$$\nabla \times \mathbf{E} + j\omega \mathbf{B} = 0 \quad (\text{Faraday's law of induction})$$

$$\nabla \times \mathbf{H} - j\omega \mathbf{D} = \mathbf{J} \quad (\text{Amper's law + displacement current})$$

$$\nabla \cdot \mathbf{D} = \rho \quad (\text{Gauss's law for electric field})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's law for magnetic field})$$

In these equations, \mathbf{E} , \mathbf{H} , \mathbf{B} , and \mathbf{D} are complex amplitudes of the fields oscillating with the frequency $f = \frac{\omega}{2\pi}$. They do not depend on time!

Material Relations in the Frequency Domain

Material relations:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

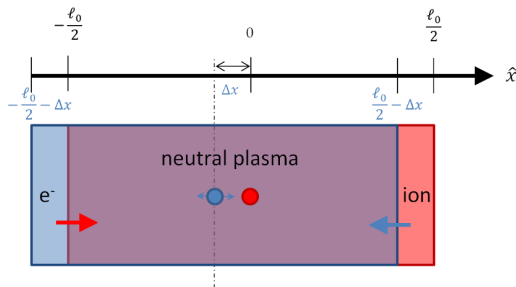
$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$$

Medium	Polarization	Magnetization
vacuum	$\mathbf{P} = 0$	$\mathbf{M} = 0$
pure dielectrics	$\mathbf{P} = \varepsilon_0 (\bar{\bar{\varepsilon}}_r(\omega) - 1) \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
pure magnetics	$\mathbf{P} \approx 0$	$\mathbf{M} = \mu_0 (\bar{\bar{\mu}}_r(\omega) - 1) \cdot \mathbf{H}$
conductors	$\mathbf{J} = \bar{\bar{\sigma}}(\omega) \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
e^- plasma	$(-\omega^2 + j\omega\tau^{-1} + \omega_p^2) \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E}$	$\mathbf{M} \approx 0$
ferromagnetics	$\mathbf{P} = \varepsilon_0 (\bar{\bar{\varepsilon}}_r(\omega) - 1) \cdot \mathbf{E}$	$j\omega \mathbf{M} = -\gamma \mathbf{M} \times \mathbf{H} - \lambda \mathbf{M} \times (\mathbf{M} \times \mathbf{H})$

No time dependence \Rightarrow great simplification!

Frequency-Dispersive Electric Susceptibility

Example: e^- plasma



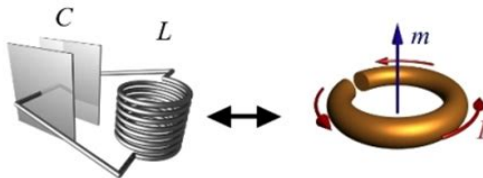
$$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau} \frac{\partial}{\partial t} + \omega_p^2 \right) \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E} \quad \Leftrightarrow \quad (-\omega^2 + j\omega\tau^{-1} + \omega_p^2) \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E}$$

$$\mathbf{P} = \frac{\varepsilon_0 \omega_p^2}{\omega_p^2 - \omega^2 + j\omega\tau^{-1}} \mathbf{E} \equiv \chi_{ee}(\omega) \mathbf{E}$$

$\chi_{ee}(\omega)$ - effective electric susceptibility

Frequency-Dispersive Magnetic Susceptibility

Example: artificial medium formed by split ring resonators (SRR)



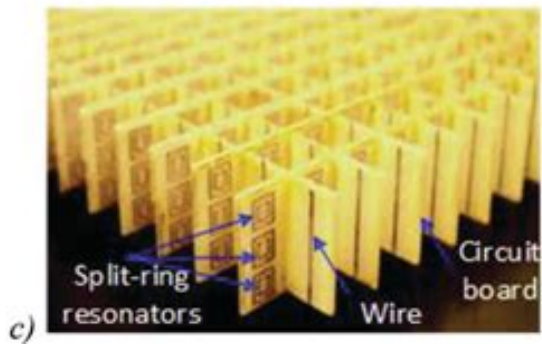
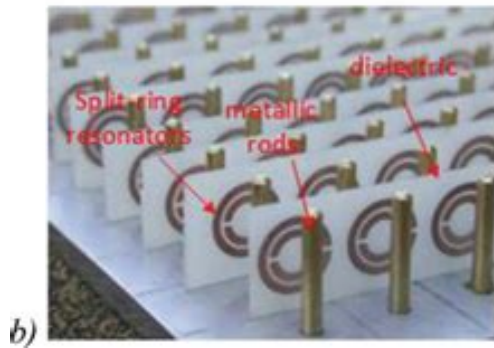
$$\mathcal{E} = -j\omega\Phi = -j\omega\mu_0 HS$$

$$I = \frac{\mathcal{E}}{j\omega L + \frac{1}{j\omega C} + R}$$

$$M = \mu_0 n I S = -\frac{j\omega n \mu_0^2 S^2}{j\omega L + \frac{1}{j\omega C} + R} H = \frac{\omega^2 n \mu_0^2 S^2 / L}{\omega_0^2 - \omega^2 + j\omega\tau^{-1}} \equiv \chi_{\text{mm}}(\omega) H$$

$$\omega_0 = 1/\sqrt{LC}, \quad \tau = L/R$$

Metamaterials



Electric Field Equation for Dispersive Magnetodielectrics

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = (\varepsilon_0 + \bar{\bar{\chi}}_{ee}(\omega)) \cdot \mathbf{E} \equiv \bar{\bar{\varepsilon}}(\omega) \cdot \mathbf{E}$$

$\bar{\bar{\varepsilon}}(\omega)$ - frequency-dispersive permittivity (tensor)

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = (\varepsilon_0 + \bar{\bar{\chi}}_{mm}(\omega)) \cdot \mathbf{H} \equiv \bar{\bar{\mu}}(\omega) \cdot \mathbf{H}$$

$\bar{\bar{\mu}}(\omega)$ - frequency-dispersive permeability (tensor)

Rewritten Maxwell's equations:

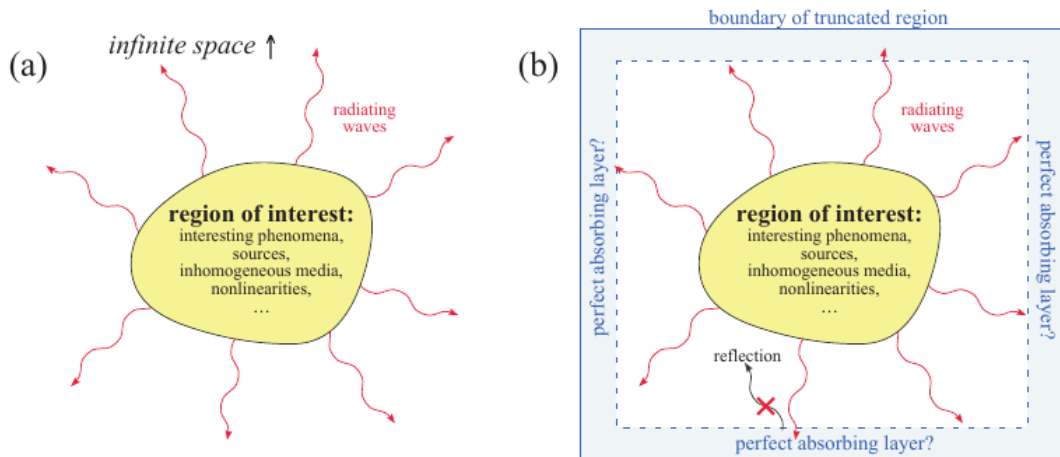
$$\mathbf{H} = -\frac{1}{j\omega} \bar{\bar{\mu}}(\omega)^{-1} \cdot (\nabla \times \mathbf{E})$$

$$\mathbf{E} - \frac{1}{j\omega} \bar{\bar{\varepsilon}}(\omega)^{-1} \cdot (\nabla \times \mathbf{H}) = -\frac{\bar{\bar{\varepsilon}}^{-1}(\omega) \cdot \mathbf{J}}{j\omega}$$

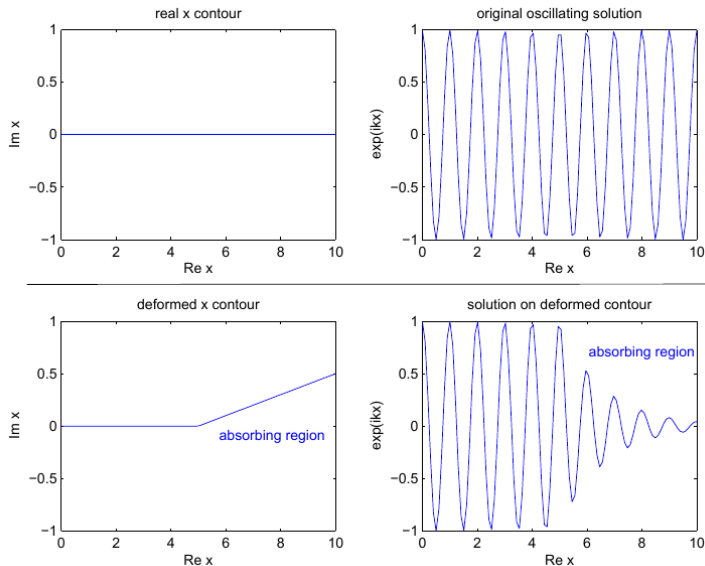
Electric field equation:

$$\mathbf{E} - \frac{1}{\omega^2} \bar{\bar{\varepsilon}}(\omega)^{-1} \cdot (\nabla \times (\bar{\bar{\mu}}(\omega)^{-1} \cdot \nabla \times \mathbf{E})) = -\frac{\bar{\bar{\varepsilon}}(\omega)^{-1} \cdot \mathbf{J}}{j\omega}$$

Perfectly Matched Layers in the Frequency Domain



Perfectly Matched Layers in the Frequency Domain



$$e^{ikx} = e^{ik(\text{Re}(x)+i\text{Im}(x))}, \quad i = \sqrt{-1}, \quad \text{decays when } \text{Im}(x) > 0$$

Coordinate Scaling

Introduce $x_{\text{old}} = x_{\text{new}} + if(x_{\text{new}})$, then

$$\frac{\partial}{\partial x_{\text{old}}} = \frac{1}{1 + if'(x_{\text{new}})} \frac{\partial}{\partial x_{\text{new}}} \equiv \frac{1}{s_x} \frac{\partial}{\partial x_{\text{new}}}$$

When done for all x, y, z :

$$\nabla = \frac{\mathbf{x}_0}{s_x} \frac{\partial}{\partial x} + \frac{\mathbf{y}_0}{s_y} \frac{\partial}{\partial y} + \frac{\mathbf{z}_0}{s_z} \frac{\partial}{\partial z}$$

Substituting into the electric field equation results (for diagonal $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$) in

$$\begin{aligned} \epsilon_{xx}^{\text{new}} &= \frac{s_y s_z}{s_x} \epsilon_{xx}^{\text{old}}, & \epsilon_{yy}^{\text{new}} &= \frac{s_x s_z}{s_y} \epsilon_{yy}^{\text{old}}, & \epsilon_{zz}^{\text{new}} &= \frac{s_x s_y}{s_z} \epsilon_{zz}^{\text{old}} \\ \mu_{xx}^{\text{new}} &= \frac{s_y s_z}{s_x} \mu_{xx}^{\text{old}}, & \mu_{yy}^{\text{new}} &= \frac{s_x s_z}{s_y} \mu_{yy}^{\text{old}}, & \mu_{zz}^{\text{new}} &= \frac{s_x s_y}{s_z} \mu_{zz}^{\text{old}} \end{aligned}$$

Discretizing Maxwell's Equations in the Frequency Domain

To approximate $\nabla \times$ we use the same principle as in the time domain (Yee's scheme):

$$(i, j, k) \leftrightarrow (i\Delta x, j\Delta y, k\Delta z)$$

$$(\nabla \times \mathbf{E})_x|_{i,j+0.5,k} \approx \frac{E_z|_{i,j+1,k} - E_z|_{i,j,k}}{\Delta y} - \frac{E_y|_{i,j+0.5,k+0.5} - E_y|_{i,j+0.5,k-0.5}}{\Delta z}$$

$$(\nabla \times \mathbf{E})_y|_{i+0.5,j,k} \approx \frac{E_x|_{i+0.5,j,k+0.5} - E_x|_{i+0.5,j,k-0.5}}{\Delta z} - \frac{E_z|_{i+1,j,k} - E_z|_{i,j,k}}{\Delta x}$$

$$(\nabla \times \mathbf{H})_z|_{i,j,k} \approx \frac{H_y|_{i+0.5,j,k} - H_y|_{i-0.5,j,k}}{\Delta x} - \frac{H_x|_{i,j+0.5,k} - H_x|_{i,j-0.5,k}}{\Delta y}$$

(etc.)

Discretization in 2D case for TE waves

Let us consider a 2D problem in xy -plane.

Let \mathbf{E} and \mathbf{J}^e be \perp to the plane and \mathbf{H} in the plane (the TE waves).

Discretized Maxwell's equations in the frequency domain:

$$H_x|_{i,j+0.5} = -\frac{\mu_{xx}^{-1}(\omega, x_i, y_{j+0.5})}{j\omega} \frac{E_z|_{i,j+1} - E_z|_{i,j}}{\Delta y}$$

$$H_y|_{i+0.5,j} = \frac{\mu_{yy}^{-1}(\omega, x_{i+0.5}, y_j)}{j\omega} \frac{E_z|_{i+1,j} - E_z|_{i,j}}{\Delta x}$$

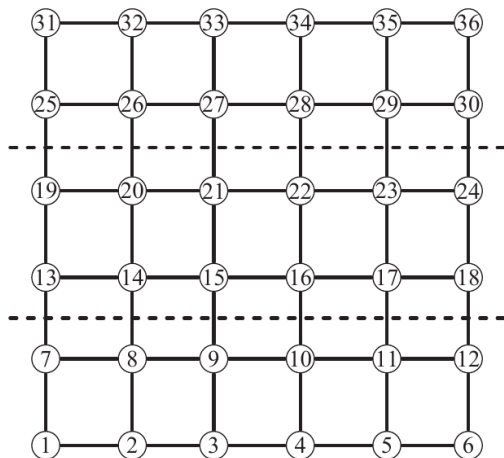
$$E_z|_{i,j} - \frac{\varepsilon_{zz}^{-1}(\omega, x_i, y_j)}{j\omega} \left(\frac{H_y|_{i+0.5,j} - H_y|_{i-0.5,j}}{\Delta x} - \frac{H_x|_{i,j+0.5} - H_x|_{i,j-0.5}}{\Delta y} \right) = -\frac{\varepsilon_{zz}^{-1}(\omega, x_i, y_j) J_z^e|_{i,j}}{j\omega}$$

No time dependence \Rightarrow no time index n !

The PML scale factors are included in $\bar{\bar{\varepsilon}}(\omega, x, y)$ and $\bar{\bar{\mu}}(\omega, x, y)$.

Matrix Formulation of the Problem

Grid of E -points



Example: 6×6 grid of E -points \Rightarrow
 $N = 6 \times 6 = 36$ unknowns in total

$$[A]_{N \times N} \cdot [E]_{N \times 1} = [S]_{N \times 1}$$

How to obtain the system matrix $[A]$?

Filling Columns of the System Matrix

To get the columns of the system matrix we can use this trick:

```
sparse matrix A <= all zeros
k = 1
for i from 1 to 6
  for j from 1 to 6
    arrays E_z, H_x, H_y <= all zeros
    element E_z(i,j) <= 1
    /* find H_x and H_y from E_z */
    array H_x <= <expressed through E_z>
    array H_y <= <expressed through E_z>
    /* express the LHS of EFE */
    array E_z <= <EFE via E_z, H_x, H_y>
    column k of A <= vec(E_z)
    k = k + 1
  end
end
```

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{N,1} \end{pmatrix}$$
$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1,2} \\ A_{2,2} \\ \vdots \\ A_{N,2} \end{pmatrix}$$
$$\vdots$$

Now switching to a practical example...