Finite-Difference Methods in Electromagnetics
Lectures 1 and 2

Stanislav I. Maslovski

University of Aveiro

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Maxwell's Equations

Maxwell's equations in the time domain

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \text{(Faraday's law of induction)} \tag{1}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \qquad \text{(Amper's law + displacement current)} \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{(Gauss's law for electric field)} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's law for magnetic field)} \tag{4}$$

ρ	electric charge density (vector)	[C/m^3]
$\mathbf{J}$	electric current density (vector)	[A/m^2]
$\mathbf{E}$	electric field (vector)	[V/m]
$\mathbf{D}$	electric displacement (vector)	[C/m^2]
$\mathbf{H}$	magnetic field (vector)	[A/m]
$\mathbf{B}$	magnetic induction (vector)	[Wb/m^2]

Material Relations

Vectors $\mathbf{D}$, $\mathbf{B}$ are related to $\mathbf{E}$, $\mathbf{H}$ and the material polarization/magnetization

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad \text{(di)(para)(ferro)(..)electrics} \qquad (5)$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \qquad \text{(dia)(para)(ferro)(..)magnetics} \qquad (6)$$

ε_0	vacuum permittivity	[$\approx 8.8541878 \times 10^{-12}$ F/m]
μ_0	vacuum permeability	[$\approx 1.2566371 \times 10^{-6}$ H/m]
$\mathbf{P}$	electric polarization (vector)	[C/m^2]
$\mathbf{M}$	magnetic polarization (vector)	[Wb/m^2]

Examples of Material Relations

Medium	Polarization	Magnetization
vacuum	$\mathbf{P} = 0$	$\mathbf{M} = 0$
isolators	$\mathbf{P} = \varepsilon_0(\overline{\overline{\varepsilon}}_\mathrm{r} - 1) \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
conductors	$\mathbf{J} = \overline{\overline{\sigma}} \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
magnetics	$\mathbf{P} = 0$	$\mathbf{M} = \mu_0(\overline{\overline{\mu}}_\mathrm{r} - 1) \cdot \mathbf{H}$
e^- plasma	$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau}\frac{\partial}{\partial t} + \omega_\mathrm{p}^2\right)\mathbf{P} = \varepsilon_0\omega_\mathrm{p}^2\mathbf{E}$	$\mathbf{M} \approx 0$
ferromagnetics	$\mathbf{P} = \varepsilon_0(\overline{\overline{\varepsilon}}_\mathrm{r} - 1) \cdot \mathbf{E}$	$\frac{\partial}{\partial t}\mathbf{M} = -\gamma\mathbf{M} \times \mathbf{H} - \lambda\mathbf{M} \times (\mathbf{M} \times \mathbf{H})$

($\overline{\overline{\sigma}}$ - conductvity, ω_p - plasma freq., τ^{-1} - collision freq., γ - gyromagnetic ratio, λ - dumping par.)

Boundary Conditions

To solve for the EM fields in a given domain one has to define the *boundary conditions*

On good conductors:	$\mathbf{E}_\mathrm{t} = 0$	PEC BC ($\approx$ Dirichelet)
As dual concept:	$\mathbf{H}_\mathrm{t} = 0$	PMC BC ($\approx$ Neumann)
Generalization:	$\mathbf{E}_\mathrm{t} = \overline{\overline{Z}}_\mathrm{s} \cdot (\hat{\mathbf{n}} \times \mathbf{H}_\mathrm{t})$	impedance BC

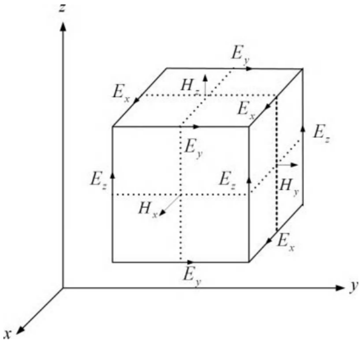
If a domain is split into subdomains:

$$\mathbf{E}_\mathrm{t}\big|_1 - \mathbf{E}_\mathrm{t}\big|_2 = 0 \qquad \text{(continuous at the boundary)}$$

$$\mathbf{H}_\mathrm{t}\big|_1 - \mathbf{H}_\mathrm{t}\big|_2 = \hat{\mathbf{n}} \times \mathbf{J}_\mathrm{s} \qquad \text{(continuous when } \mathbf{J}_\mathrm{s} = 0 \text{ at the boundry)}$$

($\hat{\mathbf{n}}$ - unit vector normal to the boundary, $\mathbf{J}_\mathrm{s}$ - surface current density [A/m])

Discretizing the Maxwell Equations (in space)



Yee's discretization scheme:

- Split space in cubic/rectangular cells
- Define H-points at the face centers
- Define E-points at the edge centers

$$(i, j, k) \leftrightarrow (i\Delta x, j\Delta y, k\Delta z)$$

$$(\nabla \times \mathbf{E})_x\Big|_{i,j,k} \approx \frac{E_z|_{i,j+0.5,k} - E_z|_{i,j-0.5,k}}{\Delta y} - \frac{E_y|_{i,j,k+0.5} - E_y|_{i,j,k-0.5}}{\Delta z}$$

$$(\nabla \times \mathbf{H})_y\Big|_{i,j,k+0.5} \approx \frac{H_x|_{i,j,k+1} - H_x|_{i,j,k}}{\Delta z} - \frac{H_z|_{i+0.5,j,k+0.5} - H_z|_{i-0.5,j,k+0.5}}{\Delta x}$$

etc.

Discretization in Time

Similarly, define D-points at times $t_n = n\Delta t$, and B-points at $t_{n+0.5} = t_n + 0.5\Delta t$:

$$\left.\frac{\partial \mathbf{D}}{\partial t}\right|^{n+0.5} = \frac{\mathbf{D}|^{n+1} - \mathbf{D}|^{n}}{\Delta t}$$

$$\left.\frac{\partial \mathbf{B}}{\partial t}\right|^{n} = \frac{\mathbf{B}|^{n+0.5} - \mathbf{B}|^{n-0.5}}{\Delta t}$$

The material relations allow us to relate $\mathbf{D}$ and $\mathbf{B}$ to $\mathbf{E}$ and $\mathbf{H}$.

E.g., in a non-dispersive isotropic magnetodielectric:

$$\left.\frac{\partial \mathbf{D}}{\partial t}\right|^{n+0.5} = \varepsilon_0 \varepsilon_{\mathrm{r}} \frac{\mathbf{E}|^{n+1} - \mathbf{E}|^{n}}{\Delta t}$$

$$\left.\frac{\partial \mathbf{B}}{\partial t}\right|^{n} = \mu_0 \mu_{\mathrm{r}} \frac{\mathbf{H}|^{n+0.5} - \mathbf{H}|^{n-0.5}}{\Delta t}$$

Discretization in 2D case (TE waves)

Let us consider a 2D problem in xy-plane.

Let $\mathbf{E}$ and $\mathbf{J}^{\mathrm{e}}$ be $\perp$ to the plane and $\mathbf{H}$ in the plane (the TE waves).

Discretized Maxwell's equations [Eqs. (1) and (2)]:

$$\frac{E_z|_{i,j+0.5}^{n} - E_z^{n}|_{i,j-0.5}}{\Delta y} + \mu_0\mu_{\mathrm{r}}\frac{H_x|_{i,j}^{n+0.5} - H_x|_{i,j}^{n-0.5}}{\Delta t} = 0$$

$$-\frac{E_z|_{i,j+0.5}^{n} - E_z|_{i-1,j+0.5}^{n}}{\Delta x} + \mu_0\mu_{\mathrm{r}}\frac{H_y|_{i-0.5,j+0.5}^{n+0.5} - H_y|_{i-0.5,j+0.5}^{n-0.5}}{\Delta t} = 0$$

$$\frac{H_y|_{i+0.5,j+0.5}^{n+0.5} - H_y|_{i-0.5,j+0.5}^{n+0.5}}{\Delta x} - \frac{H_x|_{i,j+1}^{n+0.5} - H_x|_{i,j}^{n+0.5}}{\Delta y} - \varepsilon_0\varepsilon_{\mathrm{r}}\frac{E_z|_{i,j+0.5}^{n+1} - E_z|_{i,j+0.5}^{n}}{\Delta t} = J_z^{\mathrm{e}}|_{i,j+0.5}^{n+0.5}$$

Yee's Leapfrog Update Scheme

After some algebra, we get the update equations:

$$H_x\big|_{i,j}^{n+0.5} = H_x\big|_{i,j}^{n-0.5} - \frac{\Delta t}{\mu_0 \mu_\mathrm{r}} \frac{E_z\big|_{i,j+0.5}^{n} - E_z^{n}\big|_{i,j-0.5}}{\Delta y}$$

$$H_y\big|_{i-0.5,j+0.5}^{n+0.5} = H_y\big|_{i-0.5,j+0.5}^{n-0.5} + \frac{\Delta t}{\mu_0 \mu_\mathrm{r}} \frac{E_z\big|_{i,j+0.5}^{n} - E_z\big|_{i-1,j+0.5}^{n}}{\Delta x}$$

$$E_z\big|_{i,j+0.5}^{n+1} = E_z\big|_{i,j+0.5}^{n} + \frac{\Delta t}{\varepsilon_0 \varepsilon_\mathrm{r}} \left(\frac{H_y\big|_{i+0.5,j+0.5}^{n+0.5} - H_y\big|_{i-0.5,j+0.5}^{n+0.5}}{\Delta x} - \frac{H_x\big|_{i,j+1}^{n+0.5} - H_x\big|_{i,j}^{n+0.5}}{\Delta y} - J_z^\mathrm{e}\big|_{i,j+0.5}^{n+0.5} \right)$$

Stability Condition (Courant condition)

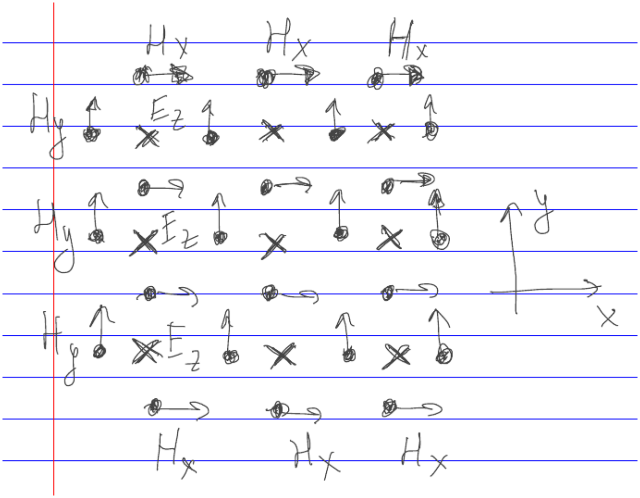
The Yee's algoritm is stable when the following condition is satisfied:

$$v\Delta t \leq \frac{1}{\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}}$$

Here, $v = c/\sqrt{\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}}$ is the velocity of light in the material.

Grid Termination and the Boundary Conditions

The H-points at the egdes of this 2D grid cannot be updated by Yee's scheme.



- At the non-updated H-points $H_{x,y} = 0$, effectively as for a PMC.
- Reflection occurs!
- To model open space, grid needs proper termination.
- Two main types of grid termination exist:
  - Absorbing Boundary Conditions (ABC)
  - Perfectly Matched Layer (PML)

Absorbing Boundary Conditions

Consider a TE plane wave propagating in free space:

$$E_z = E_0 \cos(k_x x + k_y y - \omega t)$$

From Maxwell's equations, $k_x^2 + k_y^2 = \varepsilon_0 \mu_0 \omega^2 \equiv k_0^2$ (dispersion relation), and

$$H_x = \frac{k_y}{\eta_0 k_0} E_z, \quad H_y = -\frac{k_x}{\eta_0 k_0} E_z, \quad \text{where } \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \qquad (*)$$

If at the grid boundaries we enforce the conditions (*) for the plane waves *approaching* the boundary, such waves will not be reflected.

However, there is a problem: In FDTD we do not have information about k_x, k_y (and even k_0)!

When the wave is normally incident on x or y boundary,
$k_y/k_0 = \pm 1$ or $k_x/k_0 = \pm 1$.

Another problem is that the grid points for E_z and $H_{x,y}$ *do not coincide*.

Practical Work

1. Modify the `YEE-2D.m` program in a way that it plots the fields in the complete domain `sizex` $\times$ `sizey` cells.
2. Make a modification such that the plotting (and pausing of the calculations) starts from the time step $t = 30$. For that, use the `show` variable (which is currently unused) and modify the code within the main loop.
3. Instead of exciting the structure by electric field at the boundary $x = x_{\mathrm{max}}$, excite it with the y-component of magnetic field at the opposite boundary $x = x_{\mathrm{min}}$. Note that in a plane EM wave, the electric and magnetic field magnitudes are related by the free space impedance η_0 (variable `eta0`): $E = \eta_0 H$, $\eta_0 \approx 377\ \Omega$, therefore adjust the amplitude of the magnetic field source respectively. Observe the wave propagation, explain what you see.
4. Set the `m` variable in the code to 1.42. Observe the wave propagation, explain what you see. Change the value to 1.41, what happens? As a bonus question, explain why what you saw happened at $\mathtt{m} > 1.41$ (and not at $\mathtt{m} > 1$)?

Practical Work

5. Set $\mathtt{m} = 0.9$. Implement the impedance boundary condition $E_z = \eta_0 H_y$ at the domain edge $x = x_{\min}$ and $E_z = -\eta_0 H_y$ at $x = x_{\max}$. Note that the same conditions can be written as $H_y = \pm\frac{1}{\eta_0}E_z$ and that they must appear in the code before setting the source.
6. Shift the magnetic source location from $x = x_{\min}$ to approximately the middle of the domain at $x = (x_{\min} + x_{\max})/2$. Observe the wave propagation, explain.
7. Restrict the excitaion to the line of nodes satisfying (approximately) $x = (x_{\min} + x_{\max})/2$, $y_{\min} + 35\Delta y \leq y \leq y_{\max} - 35\Delta y$. Observe the wave propagation, explain what you see.
8. Change the value of $\mathtt{m}$ to 1.01. Observe the wave propagation, explain.

Finite Differences in the Frequency Domain

Maxwell's Equations in the Frequency Domain

Maxwell's equations in the frequency domain

Fourier transform in $t \Rightarrow \frac{\partial}{\partial t} \leftrightarrow j\omega$, where $j = \sqrt{-1}$

$$\nabla \times \mathbf{E} + j\omega\mathbf{B} = 0 \qquad \text{(Faraday's law of induction)}$$

$$\nabla \times \mathbf{H} - j\omega\mathbf{D} = \mathbf{J} \qquad \text{(Amper's law + displacement current)}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{(Gauss's law for electric field)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's law for magnetic field)}$$

In these equations, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{D}$ are complex amplitudes of the fields oscillating with the frequency $f = \frac{\omega}{2\pi}$. They do not depend on time!

Material Relations in the Frequency Domain

Material relations:

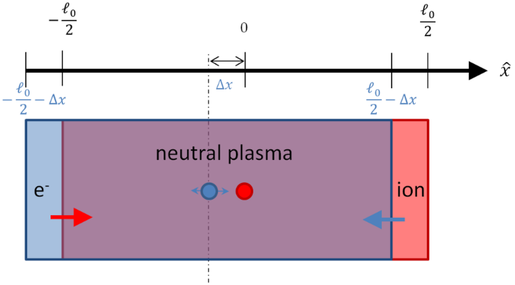
$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$$

Medium	Polarization	Magnetization
vacuum	$\mathbf{P} = 0$	$\mathbf{M} = 0$
pure dielectrics	$\mathbf{P} = \varepsilon_0 (\overline{\overline{\varepsilon}}_\mathrm{r}(\omega) - 1) \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
pure magnetics	$\mathbf{P} \approx 0$	$\mathbf{M} = \mu_0 (\overline{\overline{\mu}}_\mathrm{r}(\omega) - 1) \cdot \mathbf{H}$
conductors	$\mathbf{J} = \overline{\overline{\sigma}}(\omega) \cdot \mathbf{E}$	$\mathbf{M} \approx 0$
e^- plasma	$\left(-\omega^2 + j\omega\tau^{-1} + \omega_\mathrm{p}^2\right) \mathbf{P} = \varepsilon_0 \omega_\mathrm{p}^2 \mathbf{E}$	$\mathbf{M} \approx 0$
ferromagnetics	$\mathbf{P} = \varepsilon_0 (\overline{\overline{\varepsilon}}_\mathrm{r}(\omega) - 1) \cdot \mathbf{E}$	$j\omega\mathbf{M} = -\gamma \mathbf{M} \times \mathbf{H} - \lambda \mathbf{M} \times (\mathbf{M} \times \mathbf{H})$

No time dependence $\Rightarrow$ great simplification!

Frequency-Dispersive Electric Susceptibility

Example: e^- plasma



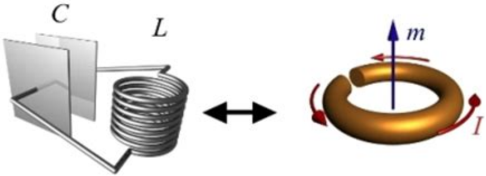
$$\left(\frac{\partial^2}{\partial t^2}+\frac{1}{\tau}\frac{\partial}{\partial t}+\omega_{\mathrm{p}}^2\right)\mathbf{P}=\varepsilon_0\omega_{\mathrm{p}}^2\mathbf{E} \quad \Leftrightarrow \quad \left(-\omega^2+j\omega\tau^{-1}+\omega_{\mathrm{p}}^2\right)\mathbf{P}=\varepsilon_0\omega_{\mathrm{p}}^2\mathbf{E}$$

$$\mathbf{P}=\frac{\varepsilon_0\omega_{\mathrm{p}}^2}{\omega_{\mathrm{p}}^2-\omega^2+j\omega\tau^{-1}}\mathbf{E}\equiv\chi_{\mathrm{ee}}(\omega)\mathbf{E}$$

$\chi_{\mathrm{ee}}(\omega)$ - effective electric susceptibility

Frequency-Dispersive Magnetic Susceptibility

Example: artificial medium formed by split ring resonators (SRR)



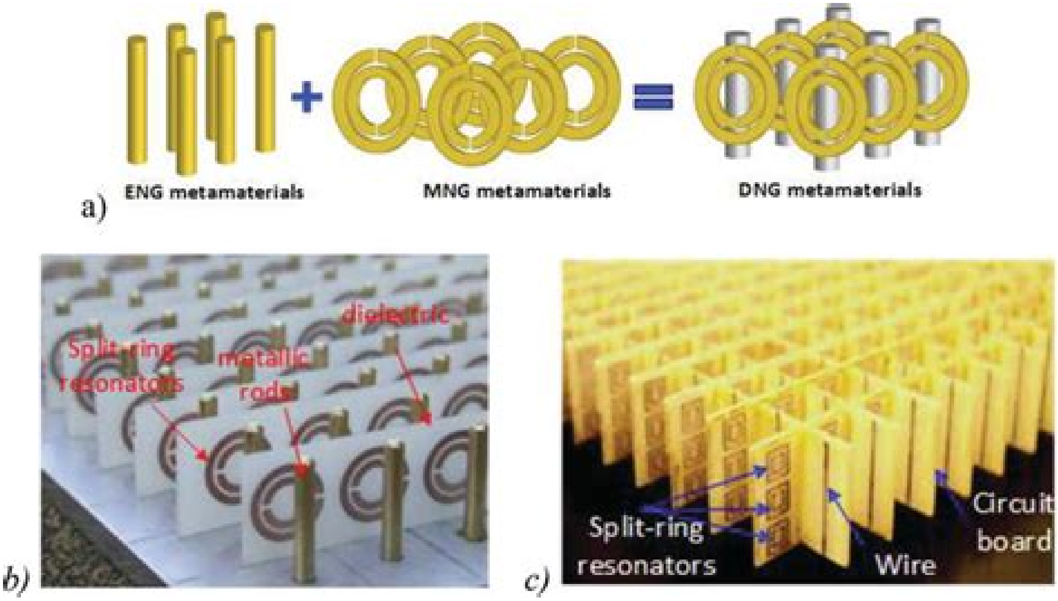
$$\mathcal{E} = -j\omega\Phi = -j\omega\mu_0 HS$$

$$I = \frac{\mathcal{E}}{j\omega L + \frac{1}{j\omega C} + R}$$

$$M = \mu_0 n I S = -\frac{j\omega n \mu_0^2 S^2}{j\omega L + \frac{1}{j\omega C} + R} H = \frac{\omega^2 n \mu_0^2 S^2 / L}{\omega_0^2 - \omega^2 + j\omega\tau^{-1}} \equiv \chi_{\mathrm{mm}}(\omega) H$$

$$\omega_0 = 1/\sqrt{LC}, \quad \tau = L/R$$

Metamaterials



Electric Field Equation for Dispersive Magnetodielectrics

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = (\varepsilon_0 + \overline{\overline{\chi}}_{\mathrm{ee}}(\omega)) \cdot \mathbf{E} \equiv \overline{\overline{\varepsilon}}(\omega) \cdot \mathbf{E}$$

$\overline{\overline{\varepsilon}}(\omega)$ - frequency-dispersive permittivity (tensor)

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = (\varepsilon_0 + \overline{\overline{\chi}}_{\mathrm{mm}}(\omega)) \cdot \mathbf{H} \equiv \overline{\overline{\mu}}(\omega) \cdot \mathbf{H}$$

$\overline{\overline{\mu}}(\omega)$ - frequency-dispersive permeability (tensor)

Rewritten Maxwell's equations:

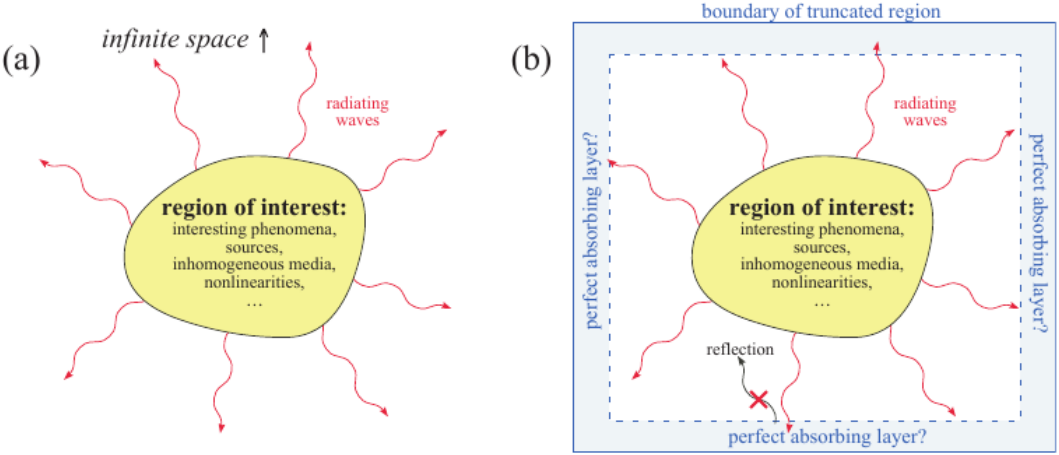
$$\mathbf{H} = -\frac{1}{j\omega}\,\overline{\overline{\mu}}(\omega)^{-1} \cdot (\nabla \times \mathbf{E})$$

$$\mathbf{E} - \frac{1}{j\omega}\,\overline{\overline{\varepsilon}}(\omega)^{-1} \cdot (\nabla \times \mathbf{H}) = -\frac{\overline{\overline{\varepsilon}}^{-1}(\omega) \cdot \mathbf{J}}{j\omega}$$

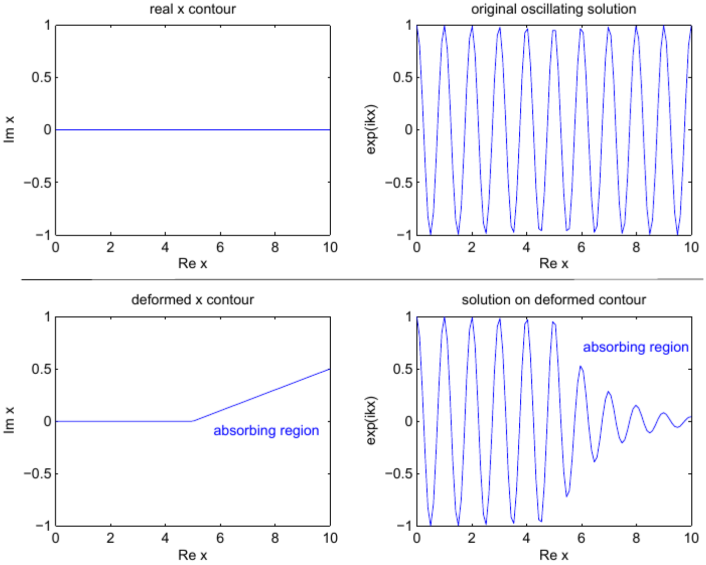
Electric field equation:

$$\mathbf{E} - \frac{1}{\omega^2}\,\overline{\overline{\varepsilon}}(\omega)^{-1} \cdot (\nabla \times (\overline{\overline{\mu}}(\omega)^{-1} \cdot \nabla \times \mathbf{E})) = -\frac{\overline{\overline{\varepsilon}}(\omega)^{-1} \cdot \mathbf{J}}{j\omega}$$

Perfectly Matched Layers in the Frequency Domain



Perfectly Matched Layers in the Frequency Domain



$e^{ikx} = e^{ik(\mathrm{Re}(x)+i\mathrm{Im}(x))}$, $i = \sqrt{-1}$, decays when $\mathrm{Im}(x) > 0$

Coordinate Scaling

Introduce $x_{\text{old}} = x_{\text{new}} + if(x_{\text{new}})$, then

$$\frac{\partial}{\partial x_{\text{old}}} = \frac{1}{1 + if'(x_{\text{new}})} \frac{\partial}{\partial x_{\text{new}}} \equiv \frac{1}{s_x} \frac{\partial}{\partial x_{\text{new}}}$$

When done for all x, y, z:

$$\nabla = \frac{\mathbf{x}_0}{s_x} \frac{\partial}{\partial x} + \frac{\mathbf{y}_0}{s_y} \frac{\partial}{\partial y} + \frac{\mathbf{z}_0}{s_z} \frac{\partial}{\partial z}$$

Substituting into the electric field equation results (for diagonal $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$) in

$$\varepsilon_{xx}^{\text{new}} = \frac{s_y s_z}{s_x} \varepsilon_{xx}^{\text{old}}, \quad \varepsilon_{yy}^{\text{new}} = \frac{s_x s_z}{s_y} \varepsilon_{yy}^{\text{old}}, \quad \varepsilon_{zz}^{\text{new}} = \frac{s_x s_y}{s_z} \varepsilon_{zz}^{\text{old}}$$

$$\mu_{xx}^{\text{new}} = \frac{s_y s_z}{s_x} \mu_{xx}^{\text{old}}, \quad \mu_{yy}^{\text{new}} = \frac{s_x s_z}{s_y} \mu_{yy}^{\text{old}}, \quad \mu_{zz}^{\text{new}} = \frac{s_x s_y}{s_z} \mu_{zz}^{\text{old}}$$

Discretizing Maxwell's Equations in the Frequency Domain

To approximate $\nabla\times$ we use the same principle as in the time domain (Yee's scheme):

$$(i,j,k) \leftrightarrow (i\Delta x, j\Delta y, k\Delta z)$$

$$(\nabla \times \mathbf{E})_x\big|_{i,j+0.5,k} \approx \frac{E_z|_{i,j+1,k} - E_z|_{i,j,k}}{\Delta y} - \frac{E_y|_{i,j+0.5,k+0.5} - E_y|_{i,j+0.5,k-0.5}}{\Delta z}$$

$$(\nabla \times \mathbf{E})_y\big|_{i+0.5,j,k} \approx \frac{E_x|_{i+0.5,j,k+0.5} - E_x|_{i+0.5,j,k-0.5}}{\Delta z} - \frac{E_z|_{i+1,j,k} - E_z|_{i,j,k}}{\Delta x}$$

$$(\nabla \times \mathbf{H})_z\big|_{i,j,k} \approx \frac{H_y|_{i+0.5,j,k} - H_y|_{i-0.5,j,k}}{\Delta x} - \frac{H_x|_{i,j+0.5,k} - H_x|_{i,j-0.5,k}}{\Delta y}$$

(etc.)

Discretization in 2D case for TE waves

Let us consider a 2D problem in xy-plane.

Let $\mathbf{E}$ and $\mathbf{J}^{\mathrm{e}}$ be $\perp$ to the plane and $\mathbf{H}$ in the plane (the TE waves).

Discretized Maxwell's equations in the frequency domain:

$$H_x|_{i,j+0.5} = -\frac{\mu_{xx}^{-1}(\omega, x_i, y_{j+0.5})}{j\omega} \frac{E_z|_{i,j+1} - E_z|_{i,j}}{\Delta y}$$

$$H_y|_{i+0.5,j} = \frac{\mu_{yy}^{-1}(\omega, x_{i+0.5}, y_j)}{j\omega} \frac{E_z|_{i+1,j} - E_z|_{i,j}}{\Delta x}$$

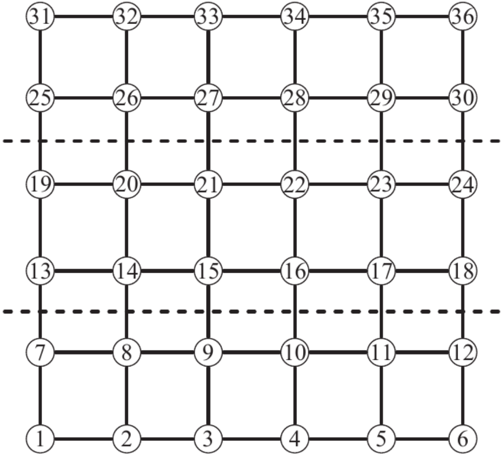
$$E_z|_{i,j} - \frac{\varepsilon_{zz}^{-1}(\omega, x_i, y_j)}{j\omega} \left(\frac{H_y|_{i+0.5,j} - H_y|_{i-0.5,j}}{\Delta x} - \frac{H_x|_{i,j+0.5} - H_x|_{i,j-0.5}}{\Delta y} \right) = -\frac{\varepsilon_{zz}^{-1}(\omega, x_i, y_j) J_z^{\mathrm{e}}|_{i,j}}{j\omega}$$

No time dependence $\Rightarrow$ no time index n!

The PML scale factors are included in $\overline{\overline{\varepsilon}}(\omega, x, y)$ and $\overline{\overline{\mu}}(\omega, x, y)$.

Matrix Formulation of the Problem

Grid of E-points



Example: 6×6 grid of E-points $\Rightarrow$
$N = 6 \times 6 = 36$ unknowns in total

$$[A]_{N\times N} \cdot [E]_{N\times 1} = [S]_{N\times 1}$$

How to obtain the system matrix $[A]$?

Filling Columns of the System Matrix

To get the columns of the system matrix we can use this trick:

```
sparse matrix A <= all zeros
k = 1
for i from 1 to 6
  for j from 1 to 6
    arrays E_z, H_x, H_y <= all zeros
    element E_z(i,j) <= 1
    /* find H_x and H_y from E_z */
    array H_x <= <expressed through E_z>
    array H_y <= <expressed through E_z>
    /* express the LHS of EFE */
    array E_z <= <EFE via E_z, H_x, H_y>
    column k of A <= vec(E_z)
    k = k + 1
  end
end
```

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{N,1} \end{pmatrix}$$

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1,2} \\ A_{2,2} \\ \vdots \\ A_{N,2} \end{pmatrix}$$

$$\vdots$$

Now switching to a practical example. . .