Exercicio 1.

$$p(\theta) = \frac{\sin\theta}{2} \quad , \quad 0 \leq \theta \leq \pi$$

Confirmar que $p(\theta)$ está normalizado

$$\int_0^{\pi} p(\theta)\, d\theta = \int_0^{\pi} \frac{\sin\theta}{2}\, d\theta = \frac{1}{2}\left(-\cos\theta\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(1 - (\cos\pi)\right) = 1$$

$$F(x) = \int_0^{x} p(\theta)\, d\theta$$

$$= \int_0^{x} \frac{\sin\theta}{2}\, d\theta = \frac{1}{2}\left(-\cos\theta\right)\Big|_0^{x}$$

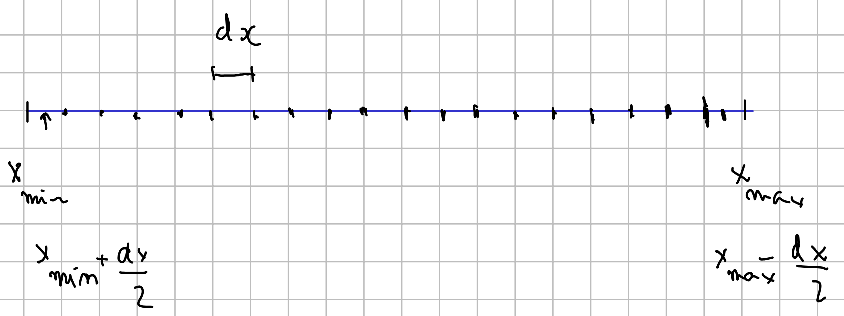
$$= \frac{1}{2}\left(1 - \cos x\right)$$

$$u = F(x) \Leftrightarrow u = \frac{1}{2}\left(1 - \cos x\right)$$

$$1 - \cos x = 2u$$

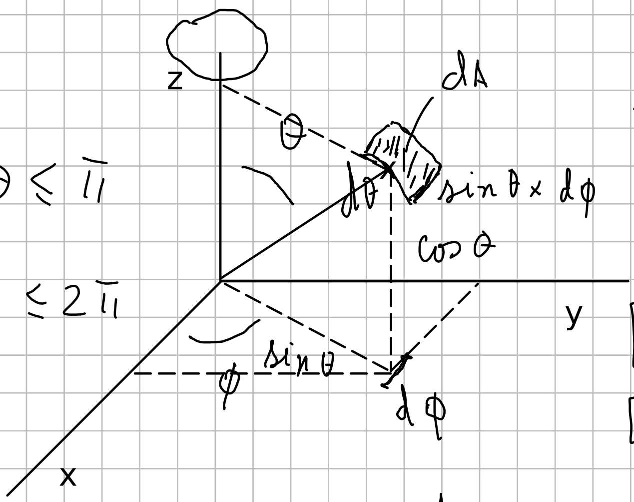
$$\cos x = 1 - 2u$$

$$x = \arccos(1 - 2u)$$



b)

Distribuição uniforme dos pontos
sobre a superficie esférica



$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

$$p(\theta, \phi)\, d\theta\, d\phi = \frac{dA}{A_{esfera}}$$

$[\theta, \theta + d\theta]$

$[\phi, \phi + d\phi]$

(x,y,z)

$z = \cos\theta$

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$r = 1 \qquad A_{esfera} = 4\pi$

$$dA = d\theta \times \sin\theta \times d\phi$$

$$p(\theta, \phi)\, \cancel{d\theta}\, \cancel{d\phi} = \frac{\sin\theta}{4\pi}\, \cancel{d\theta}\, \cancel{d\phi}$$

$$p(\theta, \phi) = \frac{\sin\theta}{2} \quad \frac{1}{2\pi}$$

$$= p(\theta) \quad p(\phi)$$

Exercicio 2.

$$p_{k,\lambda}(x) = \lambda^k \frac{x^{k-1}}{\Gamma(k)} e^{-\lambda x}, \quad 0 \leq x \leq \infty$$

$$\int_0^\infty p_{k,\lambda}(x)\, dx = \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-\lambda x} dx$$

$y = \lambda x$

$dy = \lambda\, dx$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \left(\frac{y}{\lambda}\right)^{k-1} e^{-y} \frac{dy}{\lambda}$$

$$= \frac{1}{\Gamma(k)} \int_0^\infty y^{k-1} e^{-y} dy = 1$$

a) $Y = X_1 + X_2$, $P_{X_1}(x) = \lambda e^{-\lambda x}$

$P_{X_2}(x) = \lambda e^{-\lambda x}$

$P_Y(y) = ?$ $\quad P_{X_1}(x)\, dx = \text{prob}(x \leq X_1 \leq x + dx)$

$$\boxed{P_Y(y) = \int_0^y P_{X_1}(x)\, dx \; P_{X_2}(y-x)}$$

$$P_Y(y) = \int_0^y dx_1 \int_0^y dx_2 \, P_{X_1}(x_1)\, P_{X_2}(x_2)\, \delta(y - (x_1 + x_2))$$

$$P_Y(y) = \int_0^y \lambda e^{-\lambda x} dx \; \lambda e^{-\lambda(y-x)}$$

$$P_{2,\lambda}(y) = \lambda^2 e^{-\lambda y} \int_0^y dx = \lambda^2 y e^{-\lambda y}$$

$$Y = X_1 + X_2$$

$$Y = X_1 + \underbrace{X_2 + X_3}$$

$$P_Y(y) = \int_0^y P_{X_1}(x_1)\, P_{2,\lambda}(y - x_1)\, dx_1$$

$$= \int_0^y \lambda e^{-\lambda x_1} \lambda^2 e^{-\lambda(y-x_1)} (y - x_1)\, dx_1$$

$$= \lambda^3 e^{-\lambda y} \int_0^y (y - x_1)\, dx_1$$

$$= \lambda^3 e^{-\lambda y} \left. \frac{(y-x_1)^2}{-2} \right|_0^y$$

$$P_Y(y) = \lambda^3 \frac{y^2}{2} e^{-\lambda y}$$

$$P_{3,\lambda}(y) \qquad Y = X_1 + \overbrace{X_2 + X_3 + X_4}$$

$$P_{4,\lambda} = \int_0^y P_{1,\lambda}(x)\, P_{3,\lambda}(y-x)\, dx$$

Se $\quad Y = \sum_{i=1}^{K} X_i = X_1 + X_2 + \cdots + X_K$

$$P_Y(y) = P_{K,\lambda}(y) = \frac{\lambda^K}{(K-1)!} y^{K-1} e^{-\lambda y}$$

Gerar números com densidade de probabilidade exponencial

$$F(x) = \int_0^x \lambda e^{-\lambda y}\, dy$$

$$= \lambda \left. \frac{e^{-\lambda y}}{-\lambda} \right|_0^x = 1 - e^{-\lambda x}$$

$$u = F(x) = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - u$$

$$-\lambda x = \ln(1-u)$$

$$x = -1/\lambda \, \ln(1-u)$$