Exercise # 1. Numerical methods for ODES.

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Methods

In this exercise I used the following methods to solve Ordinary Differential Equations:

Simpson's Method

4-stage Runge-Kutta method (RK4)

Backwards Differentiation Formulas (BDF)

Crank Nicolson (CN)

Answers

Question 1

$$y(t) = e^{-5t} \tag{1}$$

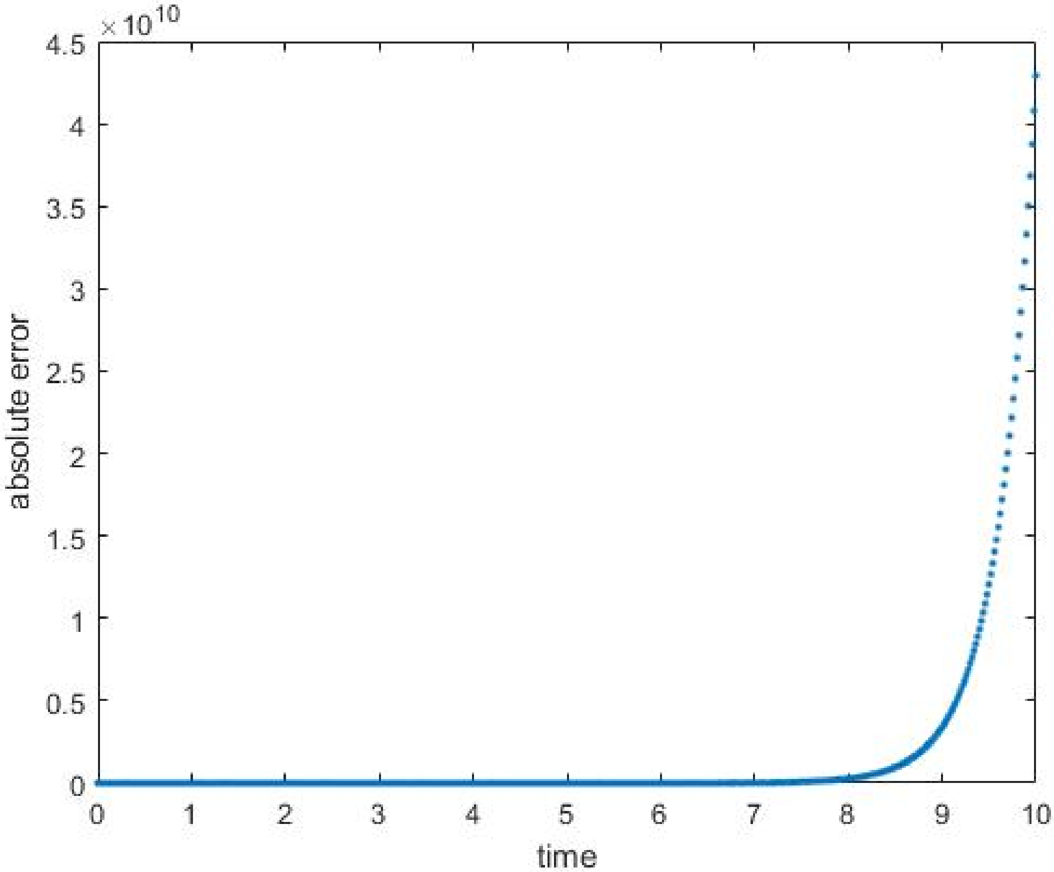

Figure 1: Absolute error in function of time using Forward Euler method to compute y(1)

The final error was 4.2916×10^{10}.

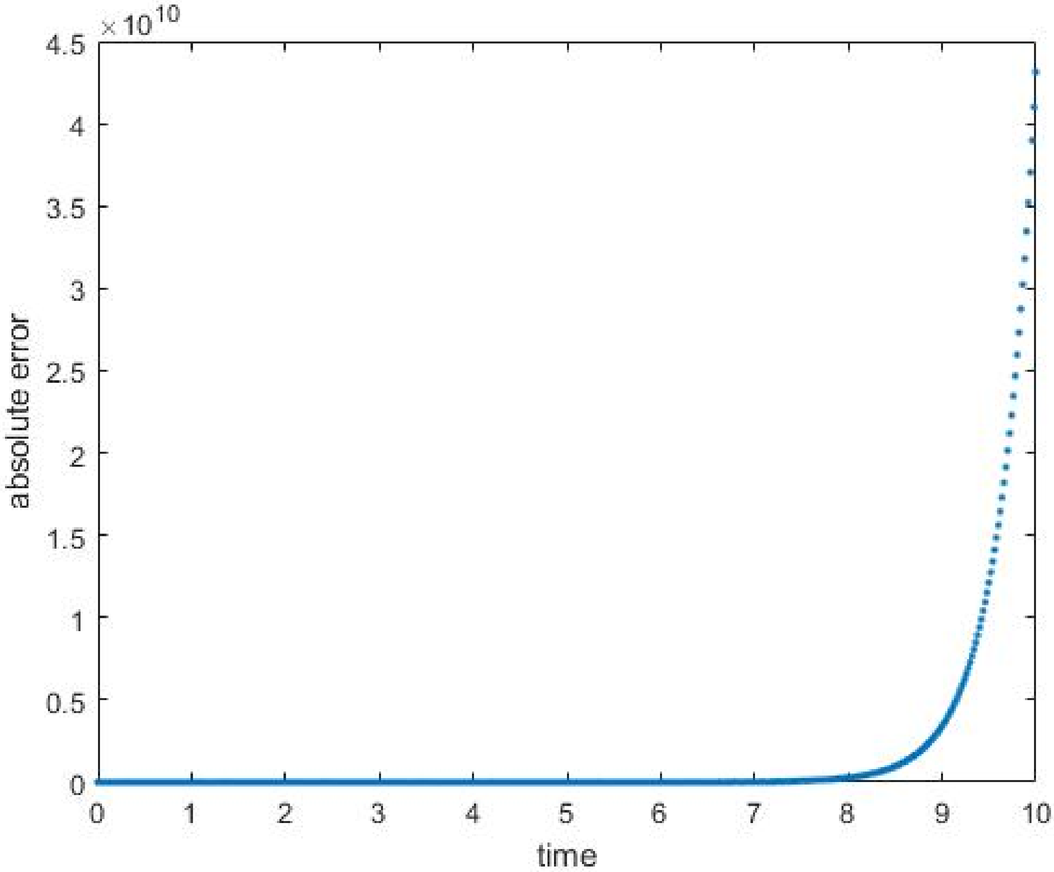

Figure 2: Absolute error in function of time using RK4 method to compute y(1)

The final error was 4.3146×10^{10}.

The Simpson's method has an empty stability region as proved by: ... We can notice the difference in the initial conditions in our results. The FE calculation for y(2) is better then the RK4 calculation given the best final error. This is, although, not relevant because the difference is of about 0.5×10^{-10}%.

Question 2

The exact solution can be found as:

$$y(t) = \frac{1}{10t + 1} \tag{2}$$

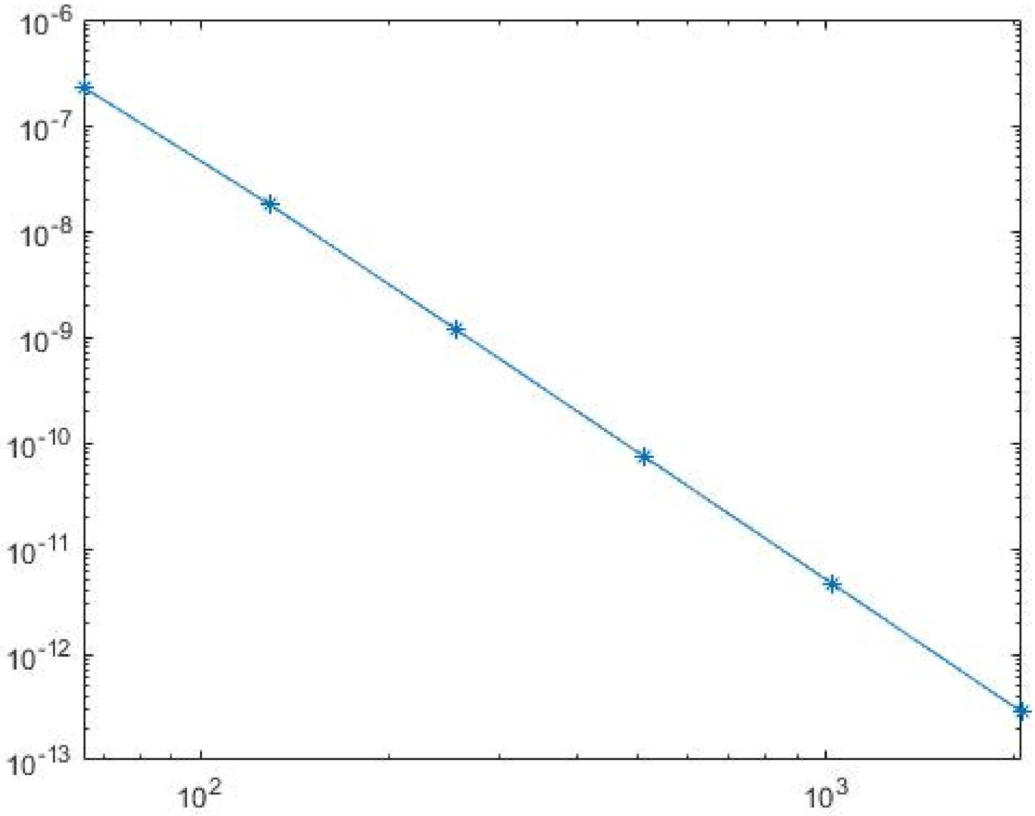

Figure 3: LogLog plot of the error as a function of the number of steps.

h	**error**
3.125000×10^{-2}	2.291844×10^{-7}
1.562500×10^{-2}	1.785763×10^{-8}
7.812500×10^{-3}	1.160234×10^{-9}
3.906250×10^{-3}	7.312862×10^{-11}
1.953125×10^{-3}	4.579586×10^{-12}
9.765625×10^{-4}	2.863750×10^{-13}

The error reduces with the increase of the number of steps (decrease of h) as expected in theory.

Question 3

Question 4

Stability for RK4

As explained in the reference book **??** on pages 19 and 20, we have the 4-th order Runge-Kutta method as:

$$y_{n+1} = (1 + \frac{1}{6}hk_1 + \frac{1}{3}hk_2 + \frac{1}{3}hk_3 + \frac{1}{6}hk_4)y_n \tag{3}$$

, where

$$k1 = f(y_n) \tag{4}$$

$$k2 = f(y_n + \frac{h}{2}k_1) \tag{5}$$

$$k3 = f(y_n + \frac{h}{2}k_2) \tag{6}$$

$$k4 = f(y_n + hk_3) \tag{7}$$

Using $\bar{h} = h\lambda$, one can simplify this equation to:

$$y_{n+1} = (1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3 + \frac{1}{24}\bar{h}^4)y_n \tag{8}$$

The relation of the current iteration value y_n with the initial value y_0 is:

$$y_{n+1} = (1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3 + \frac{1}{24}\bar{h}^4)^n y_o \tag{9}$$

This implies the absolute stability region satisfies the following inequality:

$$|1 + \bar{h} + \frac{1}{2}\bar{h}^2 + \frac{1}{6}\bar{h}^3 + \frac{1}{24}\bar{h}^4| < 1 \tag{10}$$

Assuming h as a real number we have the following stability region:

$$-2.78529 < \bar{h} < 0 \tag{11}$$

This can be extended to a system of ODEs by using the largest modulus eigenvalue as λ found using `lambda = -eigs(A,1,'lm')` to be $\lambda = -7.8388262 \times 10^4$.

So,

$$h_{max} = \frac{-2.78529}{-7.8388262 \times 10^4} = 3.5531978 \times 10^{-5} \tag{12}$$

$$0 < h < 3.5531978 \times 10^{-5} \tag{13}$$

I tested various values of h around this value. I found that the method produced `NaN` values for $h > 3.6 \times 10^{-5}$. The error for $h = h_{max}$ was $error_{h_{max}} = 0.2231543$. The experimental h_{max} I found to be in the region: $3.55596 \times 10^{-5} < h_{max} < 3.55675 \times 10^{-5}$ This was noticeable because the error increased from 0.22624 to 5.1645.

Results

Method	Number of steps	Error	CPU time (secs)
ODE45	9445	1.155269×10^{-5}	$8.791882s$
CN	100	4.467899×10^{-3}	$208.038764s$
CN	1000	4.441078×10^{-4}	$514.197773s$
CN	10000	4.438412×10^{-5}	$3086.958397s$
BDF3	100	4.482679×10^{-3}	$188.455409s$
BDF3	1000	4.442484×10^{-4}	$557.765280s$
BDF3	10000	4.438552×10^{-5}	$3399.768021s$

Question 5

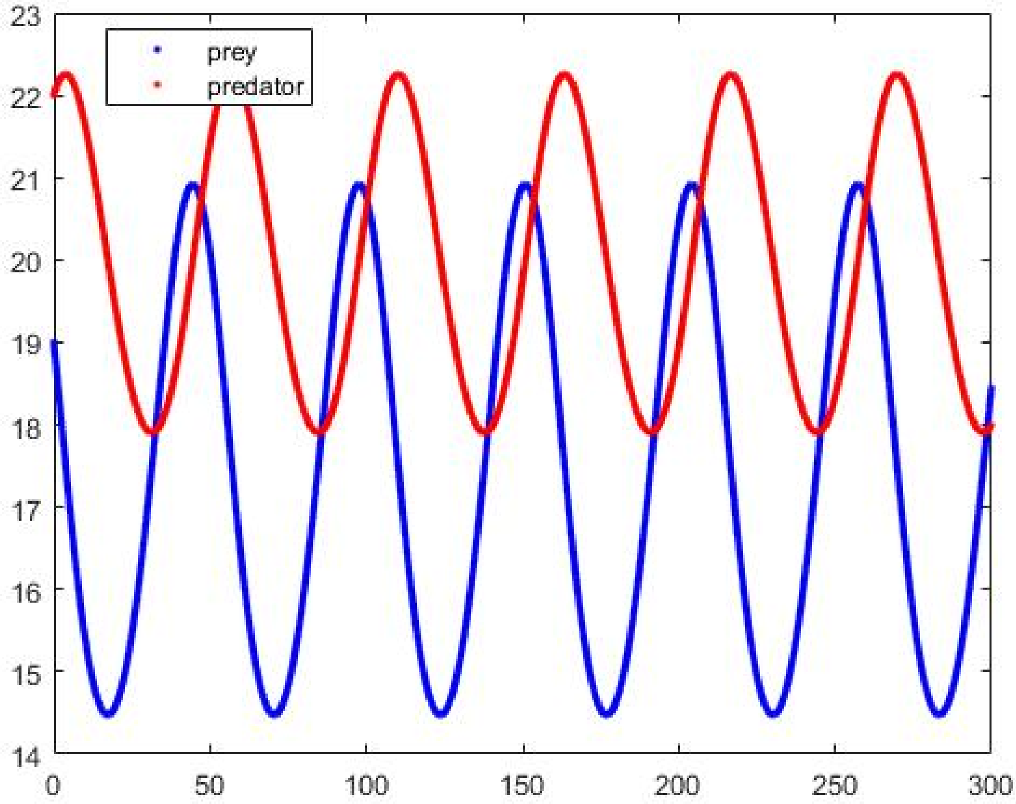

Figure 4: Evolution of the number of preys and predators.

Results

Outputs