Exercise # 3. Numerical Solution of the Poisson Problem.

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1 Method

I will describe relevant steps of developing this implementation and respective theory. Based on the homework text I implemented each step in Matlab.

1. **Input files from specified mesh:**

   1. The user selects a mesh refinement level (0 to 4);

   2. All 3 files are loaded: `topol`, `bound` and `coord`.

2. **create pattern for the stiffness matrix:**

   1. I created a range vector 1 to `Ne`, then place it 3 times as a column in a matrix and reshaped the matrix to obtain the column vector $row = [1, 1, 1, 2, 2, 2, 3, \ldots]^T$.

   2. The `col` vector is simply obtained by reshaping the `topol` as a column vector.

   3. Then we can compute the adjacency matrix as `A = sparse(row,col,1)` and the pattern for the stiffness matrix as `H = A' * A`, finally we clean the matrix as we only want the sparse pattern, `H = H * 0`.

3. **Stiffness matrix:** I created the function `[H, delta] = computeStiff(H, topol, coord)` to encapsulate all the following computations. It has the topology and coordinates matrices as inputs, as well as the H matrix with its pattern already defined before. Its outputs are the final H matrix and the `delta` vector with the surface measures of each element. Using a for loop for each element:

   1. Get coordinates of the 3 nodes that define the element, compute the surface measure for that element and save it in the `delta` vector.

   $$\Delta = \frac{1}{2} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

   2. Compute the b and c coefficients of the basis functions (a is not needed):

   $$b_i = y_j - y_m \qquad c_i = x_m - x_j,$$

   others are obtained using anticlockwise indices permutation.

   3. Compute `Hloc` for an element as

   $$H_{loc} = \frac{1}{4\Delta}\{b^T b + c^T c\}$$

   where $b = [b_i, b_j, b_m]$ and $c = [c_i, c_j, c_m]$

   4. Assemble the stiffness matrix H from the local matrices using algorithm 3.3.

4. **Right hand size:** Using a for loop to iterate in the list of nodes:

    1. Get the node coordinates;

    2. Find elements that have this node as a vertex and get their surface measures from the `delta` vector;

    3. Compute the right hand size vector as $f_i \approxeq (-4 + 2x_i^2 + 2y_i^2)\frac{\sum_e \Delta_e}{3}$

5. **Boundary Conditions:** As explained in the homework text we can simply change the diagonal value `H(i,i)`. So I changed the diagonal value of H at the node `i` of the boundary, `H(i,i)=Rmax`.

6. **Solve the Linear System:** Following recommendations I used tolerance as 1×10^{-8} and Matlab's PCG method, Jacobi preconditioner as `M = sparse(diag(diag(H)))`, Choledsky preconditioner as `L = ichol(H)`. Then we can call the PCG method to solve the linear system.

7. **Error computation:** Using a for loop to visit each node:

    1. Get the coordinates of the node;

    2. Compute the analytical solution as $u(x, y) = x^2 + y^2 - x^2y^2 - 1$;

    3. Sum surface measures of each element that have this node as a vertex;

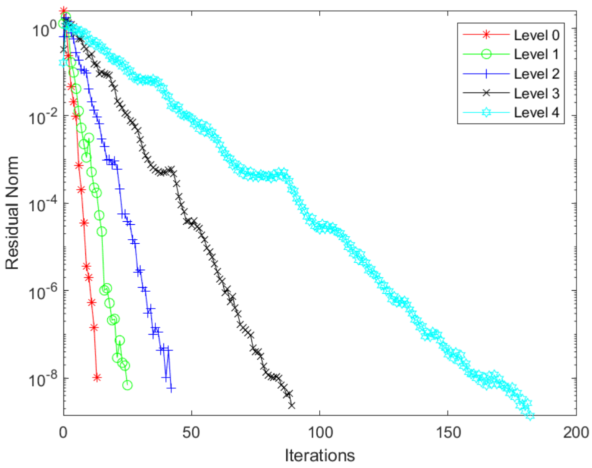
    4. Compute local error as $(u_i - u(x_i, y_i))^2 \frac{\sum_e \Delta_e}{3}$;

    5. Sum all the local errors and ϵ will be its square root.

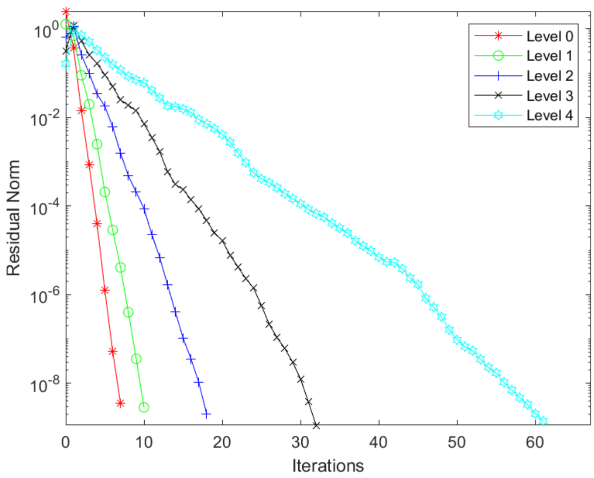
2 Results

2.1 Convergence Plots

As required the following images show the convergence plots for all refinement levels, one for each preconditioner.



(a) Jacobi



(b) Choledsky

Figure 1: Convergence plots - Residual norms vs Iterations

There are clear differences in convergence regarding the preconditioner used. The Choledsky preconditioner significantly improves convergence by reducing the number of iterations needed in up to 3x.

There is also a clear slowdown of convergence when increasing the level of refinement.

I also recorded the computational time needed to solve each PCG method but found no relevant differences.

2.2 Error

Level	**Error** ϵ	**Error Ratio**
0	6.912×10^{-2}	N/A
1	1.631×10^{-2}	0.236
2	3.984×10^{-3}	0.244
3	9.883×10^{-4}	0.248
4	2.465×10^{-4}	0.249

Table 1: FEM Convergence table

Each level of refinment drecreases l by a factor of 2, $l_{i+1} = \frac{1}{2} l_i$. The error is proportional to l^2 so $\epsilon_{i+1} = \frac{1}{4} \epsilon_i$. This corresponds to our experimental values and ratio.