Environmental Fluid Dynamics: Lecture 11

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Overview

1. Conservation of Momentum: Scale Analysis
2. Conservation of Momentum: Approximations and Simplifications
   - Geostrophic Approximation
   - Hydrostatic Approximation
   - Boussinesq Approximation

Conservation of Momentum:

Scale Analysis

Conservation of Momentum: Scale Analysis

- We showed that the vector equation of motion in a non-inertial (rotating) reference frame is

$$\underbrace{\frac{D\vec{U}}{Dt}}_{1} = \underbrace{-\frac{1}{\rho}\vec{\nabla}p}_{2} \underbrace{- 2\vec{\Omega} \times \vec{U}}_{3} + \underbrace{\vec{g}}_{4} + \underbrace{\nu\vec{\nabla}^2\vec{U}}_{5}$$

where

1. acceleration in rotating coordinate system
2. pressure gradient force
3. Coriolis
4. effective gravity
5. fric tion

Conservation of Momentum: Scale Analysis

- It is useful to understand the momentum balance equation at various dominant scales in the atmosphere.
- Scale analysis is a very useful method for establishing the importance of various processes in the atmosphere and terms in the governing equations.
- Based on the relative importance of these processes/terms, we can deduce the behavior of motion at such scales.

Conservation of Momentum: Scale Analysis

- Scale analysis makes use of order-of-magnitude reasoning.
- First, we assign a characteristic value for each of the governing variables.
- Next, we use these characteristic values to estimate the magnitude of each term.
- We then compare terms to determine their relative importance.
- Based on which terms are most important, we can describe the approximate flow behavior at a particular scale.

Conservation of Momentum: Scale Analysis (Synoptic)

- To simplify things, let's assume mid-latitude ($\phi = 45^\circ$)
- We can then define the Coriolis parameter as $f = 2\Omega \sin\phi = 2\Omega \cos\phi$
- We will also make use of the f-plane assumption (f=const).
- Frictional effects are ignored (check for yourself the viscous term is many orders of magnitude smaller than the dominant terms in each equation).
- Just to emphasize: the typical magnitude of change defines the scale, which is not always the same as the magnitude of the quantity itself.

Conservation of Momentum: Scale Analysis (Synoptic)

The typical scales in the horizontal momentum equation for the *synoptic scale* are:

- $V \sim 10 \text{ ms}^{-1}$
- $W \sim 0.1 \text{ ms}^{-1}$
- $L \sim 1000 \text{ km} = 10^6 \text{ m}$
- $H \sim 10 \text{ km} = 10^4 \text{ m}$
- $T \sim L/V \sim 10^5 \text{ s}$
- $f \sim 10^{-4} \text{ s}^{-1}$
- $\rho \sim 1 \text{ kg m}^{-3}$
- Δp in horizontal $\sim 10 \text{ mb} = 1000 \text{ Pa}$
- Δp over vertical length scale $H \sim 1000 \text{ mb} = 10^5 \text{ Pa}$

Conservation of Momentum: Scale Analysis (Synoptic)

Apply the typical scales to the u-component of velocity:

$$
\begin{array}{llllll}
\dfrac{\partial u}{\partial t} & +u\dfrac{\partial u}{\partial x} & +v\dfrac{\partial u}{\partial y} & +w\dfrac{\partial u}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & +fv \\[2em]
\dfrac{V}{T} & \dfrac{VV}{L} & \dfrac{VV}{L} & \dfrac{WV}{H} & \dfrac{\Delta p}{\rho L} & fV \\[2em]
\dfrac{10}{10^5} & \dfrac{10 \times 10}{10^6} & \dfrac{10 \times 10}{10^6} & \dfrac{0.1 \times 10}{10^4} & \dfrac{10^3}{1 \times 10^6} & 10^{-4} \times 10 \\[2em]
10^{-4} & 10^{-4} & 10^{-4} & 10^{-4} & 10^{-3} & 10^{-3}
\end{array}
$$

Conservation of Momentum: Scale Analysis (Synoptic)

Apply the typical scales to the w-component of velocity:

$$
\begin{array}{cccccc}
\dfrac{\partial w}{\partial t} & +u\dfrac{\partial w}{\partial x} & +v\dfrac{\partial w}{\partial y} & +w\dfrac{\partial w}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & -g \\[2em]
\dfrac{W}{T} & \dfrac{VW}{L} & \dfrac{VW}{L} & \dfrac{WW}{H} & \dfrac{\Delta p}{\rho H} & g \\[2em]
\dfrac{0.1}{10^5} & \dfrac{10 \times 0.1}{10^6} & \dfrac{10 \times 0.1}{10^6} & \dfrac{0.1 \times 0.1}{10^4} & \dfrac{10^5}{1 \times 10^4} & 10 \\[2em]
10^{-6} & 10^{-6} & 10^{-6} & 10^{-6} & 10 & 10
\end{array}
$$

Conservation of Momentum: Scale Analysis (Synoptic)

- For horizontal motions, the **pressure gradient force** and **Coriolis** are the two dominant scales.
- For vertical motions, the **pressure gradient force** and **gravity** are the two dominant scales.
- The flow is quasi-two-dimensional (because $w << u$)
- The flow is roughly hydrostatic
- The flow is quasi-geostrophic

Conservation of Momentum: Scale Analysis (Mesoscale)

The typical scales in the horizontal momentum equation for the *mesoscale* are:

- $\nu \sim 10^-5 \text{ m}^2 \text{ s}^{-1}$
- $V \sim 10 \text{ ms}^{-1}$
- $W \sim \mathbf{1 \text{ ms}^{-1}}$ $(\uparrow)$
- $L \sim \mathbf{100 \text{ km} = 10^5 \text{ m}}$ $(\downarrow)$
- $H \sim 10 \text{ km} = 10^4 \text{ m}$
- $T \sim \boldsymbol{L/V = 10^4 \text{ s}}$ $(\downarrow)$
- $f \sim 10^{-4} \text{ s}^{-1}$
- $\rho \sim 1 \text{ kg m}^{-3}$
- Δp in horizontal $\sim \mathbf{1 \text{ mb} = 100 \text{ Pa}}$ $(\downarrow)$
- Δp over vertical length scale $H \sim 1000 \text{ mb} = 10^5 \text{ Pa}$

Conservation of Momentum: Scale Analysis (Mesoscale)

Apply the typical scales to the u-component of velocity:

$$
\begin{array}{cccccc}
\dfrac{\partial u}{\partial t} & +u\dfrac{\partial u}{\partial x} & +v\dfrac{\partial u}{\partial y} & +w\dfrac{\partial u}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & +fv \\[2em]
\dfrac{V}{T} & \dfrac{VV}{L} & \dfrac{VV}{L} & \dfrac{WV}{H} & \dfrac{\Delta p}{\rho L} & fV \\[2em]
\dfrac{10}{10^4} & \dfrac{10 \times 10}{10^5} & \dfrac{10 \times 10}{10^5} & \dfrac{1 \times 10}{10^4} & \dfrac{10^2}{1 \times 10^5} & 10^{-4} \times 10 \\[2em]
10^{-3} & 10^{-3} & 10^{-3} & 10^{-3} & 10^{-3} & 10^{-3}
\end{array}
$$

Conservation of Momentum: Scale Analysis (Mesoscale)

Apply the typical scales to the w-component of velocity:

$$
\begin{array}{cccccc}
\dfrac{\partial w}{\partial t} & +u\dfrac{\partial w}{\partial x} & +v\dfrac{\partial w}{\partial y} & +w\dfrac{\partial w}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & -g \\[2em]
\dfrac{W}{T} & \dfrac{VW}{L} & \dfrac{VW}{L} & \dfrac{WW}{H} & \dfrac{\Delta p}{\rho H} & g \\[2em]
\dfrac{1}{10^4} & \dfrac{10 \times 1}{10^5} & \dfrac{10 \times 1}{10^5} & \dfrac{1 \times 1}{10^4} & \dfrac{10^5}{1 \times 10^4} & 10 \\[2em]
10^{-4} & 10^{-4} & 10^{-4} & 10^{-4} & 10 & 10
\end{array}
$$

Conservation of Momentum: Scale Analysis (Mesoscale)

- For horizontal motions, all terms in the equation are of the same magnitude (none of them can be neglected) and We no longer have geostrophy!
- For vertical motions, the **pressure gradient force** and **gravity** are the two dominant scales.
- The flow is quasi-two-dimensional (because $w << u$)
- The flow is roughly hydrostatic
- The flow is non-geostrophic

Conservation of Momentum: Scale Analysis (Storm Scale)

The typical scales in the horizontal momentum equation for the *mesoscale* are:

- $V \sim 10 \text{ ms}^{-1}$
- $W \sim \mathbf{10 \text{ ms}^{-1}}$ (↑)
- $L \sim \mathbf{10 \text{ km} = 10^4 \text{ m}}$ (↓)
- $H \sim 10 \text{ km} = 10^4 \text{ m}$
- $T \sim \boldsymbol{L/V} \mathbf{= 10^3 \text{ s}}$ (↓)
- $f \sim 10^{-4} \text{ s}^{-1}$
- $\rho \sim 1 \text{ kg m}^{-3}$
- Δp in horizontal $\sim 1 \text{ mb} = 100 \text{ Pa}$
- Δp over vertical length scale $H \sim 1000 \text{ mb} = 10^5 \text{ Pa}$

Conservation of Momentum: Scale Analysis (Storm Scale)

Apply the typical scales to the u-component of velocity:

$$\frac{\partial u}{\partial t} \quad +u\frac{\partial u}{\partial x} \quad +v\frac{\partial u}{\partial y} \quad +w\frac{\partial u}{\partial z} \quad = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad +fv$$

$$\frac{V}{T} \quad \frac{VV}{L} \quad \frac{VV}{L} \quad \frac{WV}{H} \quad \frac{\Delta p}{\rho L} \quad fV$$

$$\frac{10}{10^3} \quad \frac{10\times 10}{10^4} \quad \frac{10\times 10}{10^4} \quad \frac{10\times 10}{10^4} \quad \frac{10^2}{1\times 10^4} \quad 10^{-4}\times 10$$

$$10^{-2} \quad 10^{-2} \quad 10^{-2} \quad 10^{-2} \quad 10^{-2} \quad 10^{-3}$$

Conservation of Momentum: Scale Analysis (Storm Scale)

Apply the typical scales to the w-component of velocity - which are now based on the Bousinessq equations of motion.

- $\nabla \cdot \vec{u} \approx 0$
- $\rho =$ constant everywhere, except when paired with gravity
- viscosity, thermal diffusivity, and specific heat are assumed constant

The Boussinesq approximation is used because it is the residual between the PGF and buoyancy force terms. Therefore we want to estimate the terms in terms of the deviations/perturbations from the hydrostatically balanced base state.

Conservation of Momentum: Scale Analysis (Storm Scale)

Apply the typical scales to the w-component of velocity:

$$
\begin{array}{cccccc}
\dfrac{\partial w}{\partial t} & +u\dfrac{\partial w}{\partial x} & +v\dfrac{\partial w}{\partial y} & +w\dfrac{\partial w}{\partial z} & = -\dfrac{1}{\overline{\rho}}\dfrac{\partial p'}{\partial z} & +\dfrac{\theta'}{\overline{\theta}}g \\[2em]
\dfrac{W}{T} & \dfrac{VW}{L} & \dfrac{VW}{L} & \dfrac{WW}{H} & \dfrac{\Delta p}{\overline{\rho}H} & \dfrac{\Delta\theta}{\theta_0}g \\[2em]
\dfrac{10}{10^3} & \dfrac{10\times 10}{10^4} & \dfrac{10\times 10}{10^4} & \dfrac{10\times 10}{10^4} & \dfrac{10^2}{0.5\times 10^4} & \dfrac{1\times 10}{300} \\[2em]
10^{-2} & 10^{-2} & 10^{-2} & 10^{-2} & 2\times 10^{-2} & 3\times 10^{-2}
\end{array}
$$

Conservation of Momentum: Scale Analysis (Storm Scale)

- For horizontal motions, all terms in the equation are of the same magnitude except Coriolis, which can be neglected.
- For vertical motions, all terms in the equation are of the same magnitude (none of them can be neglected)!
- The flow is three-dimensional ($u \sim w$)
- The flow is nonhydrostatic
- The flow is ageostrophic

Conservation of Momentum:

Approximations and Simplifications

Geostrophic Approximation

- We first examined large (synoptic) scales, focusing on the horizontal components

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + \nu \vec{\nabla}^2 u + g_x$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + \nu \vec{\nabla}^2 v + g_y$$

- We defined the Coriolis parameter ($f = 2\Omega \sin\phi$) and assumed mid-latitude ($\phi = 45°$; a.k.a. the f-plane assumption since f is constant)
- Assumed steady-state ($\partial u/\partial t = 0$) and neglected non-linear terms($u\ \partial u/\partial x + ... = 0$)
- Assumed $U \gg W$ and aligned gravity with $\hat{\boldsymbol{k}}$
- Neglected frictional effects

Geostrophic Approximation

- The resulting expression is a balance between Coriolis and the pressure gradient force in the horizontal direction
- This balance is called the *geostrophic approximation*

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

- The geostrophic balance is a diagnostic expression (no time derivatives) that gives the relationship between the pressure field and horizontal wind in large-scale systems

Geostrophic Approximation

- We can now define a horizontal wind that satisfies the geostrophic approximation - called the *geostrophic wind*

$$
\begin{aligned}
-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} &\Rightarrow v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x} \\
fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} &\Rightarrow u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}
\end{aligned}
$$

Combining yields

$$
\vec{V}_g \equiv u_g\hat{\boldsymbol{\imath}} + v_g\hat{\boldsymbol{\jmath}}
$$

$$
\boxed{\vec{V}_g \equiv \hat{\boldsymbol{k}} \times \frac{1}{f\rho}\vec{\nabla}p}
$$

Geostrophic Approximation

$$\vec{V}_g \equiv \hat{\boldsymbol{k}} \times \frac{1}{f\rho}\vec{\nabla}p$$

- Thus, if we know the distribution of the pressure field then we know the geostrophic wind
- Remember, this is only valid for quasi-steady, large-scale motions away from the equator (at equator, $\phi = 0 \rightarrow \sin\phi = 0 \rightarrow f = 0$) and above the PBL (w/in PBL, friction and vertical momentum exchange are important)
- For synoptic scales, the geostrophic wind is within 10-15% of the actual horizontal wind

Geostrophic Approximation

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

- These expressions imply that $\vec{V}_g \cdot \vec{\nabla} p = 0$ (i.e., $\vec{V}_g \perp \vec{\nabla} p$)

$$(u_g \hat{\boldsymbol{\imath}} + v_g \hat{\boldsymbol{\jmath}}) \cdot (\frac{\partial p}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial y}\hat{\boldsymbol{\jmath}}) = u_g \frac{\partial p}{\partial x} + v_g \frac{\partial p}{\partial y}$$

Recall that

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \qquad v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

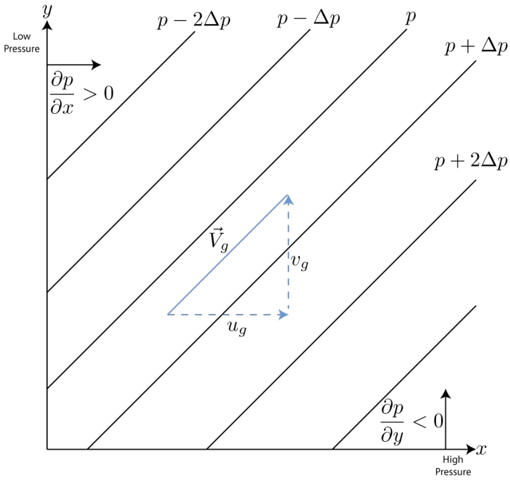
Substitution yields

$$-\frac{1}{f\rho}\frac{\partial p}{\partial y}\frac{\partial p}{\partial x} + \frac{1}{f\rho}\frac{\partial p}{\partial x}\frac{\partial p}{\partial y} = 0$$

Thus, $\vec{V}_g \parallel$ to lines of constant pressure

Geostrophic Approximation

- Since, $\vec{V}_g \parallel$ to lines of constant pressure, that means isobars on a weather map are nearly streamlines



Geostrophic Approximation

- If we want to develop a prognostic equation using the geostrophic approximation, we must retain the acceleration term

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

Recall that

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \qquad v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

Substitution yields

$$\frac{Du}{Dt} = f(v - v_g)$$
$$\frac{Dv}{Dt} = -f(u - u_g)$$

- In other words, accelerations are proportional to the departure of the wind from the geostrophic wind

Geostrophic Approximation

$$\frac{Du}{Dt} = f(v - v_g) \qquad \frac{Dv}{Dt} = -f(u - u_g)$$

- If you recall from scale analysis, the acceleration terms are an order of magnitude smaller than the Coriolis and pressure terms
- While geostrophic balance is a useful diagnosis tool, its application to weather forecasting is fraught with difficulty
- Why? The acceleration (very important to get right) is equal to the small difference between two large terms - which means it is susceptible to measurement errors in velocity or pressure gradient

Geostrophic Approximation

- A useful comparison of the magnitude of the acceleration with the Coriolis force is through the ratio of their respective characteristic scales

$$\frac{U^2/L}{f_o L}$$

- This non-dimensional number is called the *Rossby NUmber*

$$\mathrm{Ro} \equiv \frac{U}{f_o L}$$

- Named after Swedish meteorologist Carl-Gustav Rossby
- As Ro becomes smaller, rotational effects are more important and the geostrophic approximation is more valid
- Ro breaks down near the equator, where $f \to 0$

Hydrostatic Approximation

- Next, we examined large (synoptic) scales and applied scale analysis to the vertical component

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u \cos\phi + \nu \vec{\nabla}^2 w + g_z$$

- We assumed mid-latitude, steady-state, and neglected non-linear terms and frictional effects
- Assumed $U \gg W$ and aligned gravity with $\hat{\boldsymbol{k}}$

Hydrostatic Approximation

- The resulting expression is a balance between the vertical pressure gradient and gravity
- This balance is called the *hydrostatic approximation*

$$\frac{\partial p}{\partial z} = -\rho g$$

- This expression can be related to the hypsometric equation (see Lecture 5)
- In other words, vertical accelerations are small compared to gravity at large scales and so the pressure at any point is equal to the weight of the overlying air

Boussinesq Approximation

- This is really 3 assumptions motivated byt he fact that compressibility effects are often small for atmospheric flows
  1. Assume flow is incompressible: $\vec{\nabla} \cdot \vec{U} = 0$
  2. Material properties (viscosity, thermal diffusivity, and specific heat) are assumed constant
  3. Density is allowed to vary in gravity term but not in inertia term
- Let's look at assumption 3

Boussinesq Approximation

- In reality, ρ may vary in $(x,\ y,\ z,\ t)$
- We want to approximate it based on the smallness of its deviation from the mean
- Define a base-state pressure based on constant density ρ_0 (i.e., reference atmosphere has constant density)

$$\rho = \rho_0 + \rho'$$

Then

$$p = \bar{p} + p'$$

where $\bar{p}$ is the solutions of $\partial\bar{p}/\partial z = -\rho_0 g$

Boussinesq Approximation

- The equations of motion become

$$(\rho_0 + \rho')\frac{D\vec{U}}{Dt} = -\vec{\nabla}p' \underbrace{-\vec{\nabla}\bar{p} - \rho_0 g\hat{\boldsymbol{k}}}_{\text{These cancel}} - \rho' g\hat{\boldsymbol{k}} + \mu\vec{\nabla}^2\vec{U}$$

divide by ρ_0

$$\left(1 + \frac{\rho'}{\rho_0}\right)\frac{D\vec{U}}{Dt} = -\frac{1}{\rho_0}\vec{\nabla}p' - \frac{\rho'}{\rho_0}g\hat{\boldsymbol{k}} + \nu\vec{\nabla}\vec{U}$$

Assume $\rho'/\rho_0 \ll 1$, so finally

$$\frac{D\vec{U}}{Dt} = -\frac{1}{\rho_0}\vec{\nabla}p' - \frac{\rho'}{\rho_0}g\hat{\boldsymbol{k}} + \nu\vec{\nabla}\vec{U}$$

Boussinesq Approximation

- Now let's consider the term $\rho^{'}/\rho$. The equation of state is given by $p = \rho R T$, which is decomposed as

$$\frac{\overline{p}}{R} + \frac{p^{'}}{R} = \overline{\rho}\,\overline{T} + \overline{\rho}T^{'} + \rho^{'}\overline{T} + \rho^{'}T^{'}$$

Next, we apply Reynolds averaging

$$\frac{\overline{\overline{p}}}{R} + \frac{\overline{p^{'}}}{R} = \overline{\overline{\rho}\,\overline{T}} + \overline{\overline{\rho}T^{'}} + \overline{\rho^{'}\overline{T}} + \overline{\rho^{'}T^{'}} \longrightarrow \frac{\overline{p}}{R} = \overline{\rho}\,\overline{T} + \overline{\rho^{'}T^{'}} \rightarrow \frac{\overline{p}}{R} = \overline{\rho}\,\overline{T}$$

- Thus, the equation of state holds in the mean. This is reasonable because the equation of state was originally formulated from measurements made with crude, slow-response sensors. These sensors were essentially measuring mean quantities.

Boussinesq Approximation

- Now let's subtract the mean equation of state from the full linearized form

$$\frac{\overline{p}}{R}+\frac{p^{'}}{R}-\frac{\overline{p}}{R}=\overline{\rho}\,\overline{T}+\overline{\rho}T^{'}+\rho^{'}\overline{T}+\rho^{'}T^{'}-\overline{\rho}\,\overline{T}\longrightarrow\frac{p^{'}}{R}=\overline{\rho}T^{'}+\rho^{'}\overline{T}+\rho^{'}T^{'}.$$

- Finally divide by the mean equation of state

$$\frac{p^{'}}{R}\frac{R}{\overline{p}}=\frac{\overline{\rho}T^{'}}{\overline{\rho}\,\overline{\overline{T}}}+\frac{\rho^{'}\overline{\overline{T}}}{\overline{\rho}\,\overline{\overline{T}}}+\frac{\rho^{'}T^{'}}{\overline{\rho}\,\overline{\overline{T}}}\longrightarrow\frac{p^{'}}{\overline{p}}=\frac{\rho^{'}}{\overline{\rho}}+\frac{T^{'}}{\overline{\overline{T}}}+\frac{\rho^{'}}{\overline{\rho}}\frac{T^{'}}{\overline{\overline{T}}}.$$

- The last term is much smaller than the others is safely neglected, which leads to

$$\frac{p^{'}}{\overline{p}}=\frac{\rho^{'}}{\overline{\rho}}+\frac{T^{'}}{\overline{\overline{T}}}\ .$$

Boussinesq Approximation

$$\frac{p^{'}}{\overline{p}} = \frac{\rho^{'}}{\overline{\rho}} + \frac{T^{'}}{\overline{T}} \; .$$

- This is the linearized perturbation ideal gas law. Scale analysis shows that $p^{'}/\overline{p} \rightarrow 0.1 \text{ hPa}/1000 \text{ hPa} = 10^{-4}$, while $T^{'}/\overline{T} \rightarrow 1 \text{ K}/300 \text{ K} = 3.33^{-3}$. Thus, we can neglect the pressure term, yielding

$$\frac{\rho^{'}}{\overline{\rho}} \approx -\frac{T^{'}}{\overline{T}}$$

- Recalling that $T^{'} = T - \overline{T}$ and making use of the Poisson equation, one can write

$$\frac{T^{'}}{\overline{T}} = \frac{\theta\left(\dfrac{p}{p_0}\right)^{\frac{R}{c_p}} - \overline{\theta}\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}{\overline{\theta}\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}} = \frac{\theta\left(\dfrac{\overline{p}+p^{'}}{p_0}\right)^{\frac{R}{c_p}} - \overline{\theta}\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}{\overline{\theta}\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}$$

$$\approx \frac{\theta\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}} - \overline{\theta}\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}{\overline{\theta}\left(\dfrac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}} \approx \frac{\theta - \overline{\theta}}{\overline{\theta}}.$$

Boussinesq Approximation

- Thus, one can use the approximation that

$$\boxed{\frac{\rho^{'}}{\overline{\rho}} \approx -\frac{T^{'}}{\overline{T}} \approx -\frac{\theta^{'}}{\overline{\theta}}}$$

- We can use this approximation, we can write the equations of motion in terms of buoyancy

$$\boxed{\frac{D\vec{U}}{Dt} = -\frac{1}{\rho_0}\vec{\nabla}p' + \beta\theta' + \nu\vec{\nabla}\vec{U}}$$

where $\beta = g\hat{\boldsymbol{k}}/\overline{\theta}$ is the buoyancy parameter

Boussinesq Approximation

- We can rewrite this expression by noting that

$$\frac{1}{\rho_0}\vec{\nabla}p' = \frac{1}{\rho_0}\vec{\nabla}p - \frac{1}{\rho_0}\vec{\nabla}\bar{p}$$

where

$$\frac{1}{\rho_0}\vec{\nabla}\bar{p} = -g\hat{\boldsymbol{k}}$$

- Combined, we can write the equations of motion as

$$\boxed{\frac{D\vec{U}}{Dt} = \frac{1}{\rho_0}\vec{\nabla}p - g\hat{\boldsymbol{k}} + \beta\theta' + \nu\vec{\nabla}\vec{U}}$$

Boussinesq Approximation

$$\boxed{\frac{D\vec{U}}{Dt} = \frac{1}{\rho_0}\vec{\nabla}p - g\hat{\boldsymbol{k}} + \beta\theta' + \nu\vec{\nabla}\vec{U}}$$

- The Boussinesq approximation essentially describes the residual between the pressure gradient force and the buoyancy force
- This is why we estimate in terms of deviations/perturbation from the hydrostatically-balanced base state