

Course Notes

Real Functions and Measures

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Chapter 1

Basics of Abstract Measure Theory

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1.1 Review of Topology

1.1.1 Basic Definitions

Def'n. 1.1.1 Let $X \neq \emptyset$ and $\tau \subseteq \mathcal{P}(X)$. We say that (X, τ) is a **topological space** if τ satisfies the following conditions:

1. $\emptyset \in \tau$ $X \in \tau$
2. $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3. $V_\alpha \in \tau$ for all $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

We call the elements of τ **open sets**.

Def'n. 1.1.2 $U \subseteq X$ is a **neighbourhood** of $x \in X$ if there is some $G \in \tau$ such that $x \in G \subset U$.

Def'n. 1.1.3 $F \subseteq X$ is **closed** if F^c is open.

Def'n. 1.1.4 The **closure** of a set $E \subset X$ is the smallest closed set containing E (denoted \bar{E}).

Def'n. 1.1.5 x is an **accumulation point** of H if all neighbourhoods of x contains infinitely points of H . Equivalently, x is a **limit point** of $H \setminus \{x\}$.

Def'n. 1.1.6 If $H \subseteq X$, we have a natural subspace topology $\tau|_H = \{G \cap H : G \in \tau\}$.

1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1. \mathbb{R} along with the open sets (denoted τ_e , the Euclidean topology).
2. The discrete topology, $\tau = \mathcal{P}(X)$ for any $X \neq \emptyset$. This is the “finest” topology.

3. The antidiscrete topology, $\tau = \{\emptyset, X\}$ for any $X \neq \emptyset$. This is the “coarsest” topology.
4. One can define the extended real line, $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

The same can be done with a single symbol as well. In either case, the extended real line is a compact set.

5. Any metric spaces induces a topology. Consider a set $X \neq \emptyset$ arbitrary, and let $d : X \times X \rightarrow \mathbb{R}$ such that

- (a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$.
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$

Then $G \in \tau$ if and only if for any $x \in G$, there exists r so that $B_r(x) \subset G$. There are many examples of metric spaces:

- (a) $X = \mathbb{R}$, $d(x, y) = |x - y|$
- (b) $X = \mathbb{R}$, $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
- (c) $X = \mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- (d) $X = \mathbb{R}^2$, $d(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$ for $p \geq 1$.
- (e) and similarly for $X = \mathbb{R}^n$
- (f) $X = C[0, 1]$, $d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$.
- (g) normed space: X is a vector space over \mathbb{R} , $\|\cdot\| : X \rightarrow \mathbb{R}$ such that
 - i. $\|x\| = 0$ if and only if $x = 0$
 - ii. $\|cx\| = |c| \|x\|$
 - iii. $\|x + y\| \leq \|x\| + \|y\|$

If $\|\cdot\|$ is a norm, then $d(x, y) = \|x - y\|$ is a metric.

6. The cofinite topology: $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$.

1.1.3 Other Definitions

Def’n. 1.1.7 $K \subset X$ is **compact** if every open cover of K contains a finite subcover.

Def’n. 1.1.8 A topological space is called **locally compact** if every point has a compact neighbourhood.

Prop. 1.1.9 $C[0, 1]$ with the sup norm is not locally compact.

PROOF I’ll do this later.

□

Def'n. 1.1.10 A topological space is called **Hausdorff** if for any $x \neq y$, there exists neighbourhoods $U \ni x$, $V \ni y$ so that $U \cap V = \emptyset$.

The anti-discrete topology is not Hausdorff.

1. On the discrete topology, K is compact if and only if K is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of X).
3. On (\mathbb{R}, τ_e) , K is compact if and only if K is closed and bounded.
4. On (X, d) metric space, K is compact if and only if K is complete and totally bounded.

Prop. 1.1.11 1. Let $K \subset X$ be compact, let $F \subset K$ closed. Then F is also compact.
2. Compact sets in a Hausdorff space are closed.

PROOF 1. Let $F \subset \bigcup V_\alpha$. Then $K \subset F^c \cup (\bigcup V_\alpha)$ is an open cover for K , so it has a finite subcover $F^c \cup V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$. But then since $F \cap F^c = \emptyset$, $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ is a finite subcover.
2. Let $K \subset X$ be compact, and prove that K^c is open. Thus let $x \in K^c$. For any $y \in K$, there exist U_y, V_y disjoint neighbourhoods of x and y respectively. Now consider the open cover $K \subset \bigcup_{y \in K} V_y$, and get our finite subcover $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. But then $U_{y_1} \cap \dots \cap U_{y_n} \cap K = \emptyset$ and is open since it is a finite intersection. \square

Def'n. 1.1.12 $\Gamma \subseteq \tau$ is a **base** for τ if every $U \in \tau$ can be written as a countable union of the elements of Γ . Γ is a **countable base** if Γ is countable.

Prop. 1.1.13 \mathbb{R} has a countable base of intervals.

PROOF Consider the collection $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$. To see this, for any open set U , one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left(\bigcup_{\{r: B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$ is obvious, so let $x \in U$ be arbitrary, and let s be maximal so that $B_s(x) \subseteq U$. Then choose $q \in \mathbb{Q}$ so that $|x - q| < s/3$ and $r \in \mathbb{Q}$ so that $0 < r < s/2$. Then by construction $B_r(q) \ni x$ and by the triangle inequality $B_{r/2}(q) \subseteq U$, so $x \in S$. Thus $U = S$ as desired. \square

Note that the exact same argument (with some work) can be generalized to show that \mathbb{R}^n has a countable base of open hyperrectangles.

Prop. 1.1.14 Every metric space which is a countable union of compact sets has a countable base.

PROOF See my PMATH 351 notes. \square

1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

Def'n. 1.1.15 Let $(x_n) \subset X$ be a sequence and let $x \in X$. Then x is the **limit** of (x_n) if for any neighbourhood U of x , there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.

Prop. 1.1.16 If $F \subset X$ is closed, then for all convergent sequences in F , the limit is also in F .

PROOF See Homework. □

Def'n. 1.1.17 Let $f : X \rightarrow Y$ be a function, and $x \in X$ an accumulation point of $D(f)$. The limit of f at x is $y \in Y$ if for any neighbourhood V of y there exists a neighbourhood U of x such that $f(U \cap D(f) \setminus \{x\}) \subseteq V$.

Def'n. 1.1.18 Let $f : X \rightarrow Y$ be a function, and let $x \in D(f)$. Then f is **continuous at x** if for any neighbourhood V of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of x .

Def'n. 1.1.19 $f : X \rightarrow Y$ is called **continuous** if it is continuous at every point.

Prop. 1.1.20 $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(G)$ is open for all G open.

PROOF Exercise. □

Thm. 1.1.21 Let $f : X \rightarrow Y$ be continuous and $K \subset X$ be compact. Then $f(K)$ is compact.

PROOF Recall that continuous functions pull back open sets. Let $f(K) \subset \bigcup U_\alpha$ be an open cover. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover for K , and has a finite subcover $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. But then $f(f^{-1}(U_{\alpha_1})) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$ is a subcover of $f(K)$. □

1.2 Measure Theory

1.2.1 σ -algebras

Def'n. 1.2.1 Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a **σ -algebra** if

1. $X \in \mathcal{M}$
2. $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
3. If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$

The pair (X, \mathcal{M}) is called a **measurable space**. The elements of \mathcal{M} are called **measurable sets**.

Def'n. 1.2.2 Let (X, \mathcal{M}) be a measurable space, (Y, τ) be a topological space. Then $f : X \rightarrow Y$ is called **measurable** if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.

Here are some simple examples of σ -algebras.

Ex. 1.2.3 1. $\mathcal{M} = \{\emptyset, X\}$ is a σ -algebra.

2. $\mathcal{P}(X) = \mathcal{M}$ is a σ -algebra.

3. $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable}\}$. To see this, given $A_n \in \mathcal{M}$, if everything is countable, then $\bigcup A_n$ is countable. If some A_i is countable, then $(\bigcup A_n)^c = \bigcap A_n^c$ is countable, so $\bigcup A_n \in \mathcal{M}$.

We will later see some proper examples, like the σ -algebra of Lebesgue measurable sets.

We have the following properties of σ -algebras.

Prop. 1.2.4 1. $\emptyset \in \mathcal{M}$

2. $A_1, A_2, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{M}$
3. $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$
4. $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$
5. f is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$.

PROOF 1. $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.

2. We can extend this to a countable union by introduction $A_{n+i} = \emptyset$ for $i \in \mathbb{N}$.
3. By DeMorgan's identities, $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.
4. $A \setminus B = A \cap B^c \in \mathcal{M}$.
5. H^c is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$. □

Prop. 1.2.5 Let $f : X \rightarrow Y$ be measurable, let $g : Y \rightarrow Z$ be continuous, then $g \circ f : X \rightarrow Z$ is measurable.

PROOF Let $V \subset Z$ be open, so $g^{-1}(V) \subset Y$ is open, so $f^{-1}(g^{-1}(V)) \in \mathcal{M}$ which is $(g \circ f)^{-1}(V)$. □

Prop. 1.2.6 Let (X, \mathcal{M}) be a measurable space, Y be a topological space. Let $\phi : \mathbb{R}^2 \rightarrow Y$ be continuous. If $u, v : X \rightarrow \mathbb{R}$ are measurable, then $h(x) = \phi(u(x), v(x))$ is measurable.

PROOF Define $f : X \rightarrow \mathbb{R}^2$ by $f(x) = (u(x), v(x))$. We will see that f is measurable, so that $h = \phi \circ f$ is measurable since ϕ is continuous. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals, so $R = I_1 \times I_2$ is an open rectangle. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Let $G \subset \mathbb{R}^2$ be an open set, so there exist R_n open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that f is measurable. □

Cor. 1.2.7 1. If $u, v : X \rightarrow \mathbb{R}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.

2. $u + iv : X \rightarrow \mathbb{C}$ is measurable.
3. $f : X \rightarrow \mathbb{C}$ is measurable, $f = u + iv \Rightarrow u, v, |f|$ are measurable.

Prop. 1.2.8 Define

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then χ_E is measurable if and only if $E \in \mathcal{M}$.

PROOF Naturally, $\chi_E^{-1}(1) = E$ and $\chi_E^{-1}(0) = E^c$, so χ_E is measurable if and only if $E, E^c \in \mathcal{M}$. □

Thm. 1.2.9 Let $\mathcal{F} \subset \mathcal{P}(X)$, then there exists a smallest σ -algebra containing \mathcal{F} . This is denoted by $S(\mathcal{F})$, the σ -algebra generated by \mathcal{F} .

PROOF Let $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra, } \mathcal{F} \subset \mathcal{M}\}$. Certainly $\Omega \neq \emptyset$ since $\mathcal{P}(X) \in \Omega$. Let $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$. We will see that $S(\mathcal{F})$ is a σ -algebra.

- (i) Since $X \in \mathcal{M}$, it follows that $X \in \bigcap \mathcal{M}$.

(ii) If $A \in S(\mathcal{F})$, then $A \in \mathcal{M}$ for all \mathcal{M} . Thus $A^c \in \mathcal{M}$ for all \mathcal{M} and $A^c \in \bigcap \mathcal{M}$.

(iii) In the same way, if $A_n \in S(\mathcal{F})$ for all n , then $A_n \in \mathcal{M}$ for all n, \mathcal{M} . Thus $\bigcup A_n \in \mathcal{M}$ for all \mathcal{M} so $\bigcup A_n \in \bigcap \mathcal{M} = S(\mathcal{F})$.

By definition, $\mathcal{F} \subset \bigcap \mathcal{M}$. Finally, $S(\mathcal{F})$ is minimal, since if $\mathcal{F} \subset \mathcal{N}$ is a σ -algebra, then $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$, so we are done. \square

Def'n. 1.2.10 Let (X, τ) be a topological space. Then $\mathcal{B} = S(\tau)$ is called the **Borel σ -algebra**. Borel sets are the elements of $S(\tau)$. A function $f : X \rightarrow Y$ is Borel measurable if $f^{-1}(G) \in \mathcal{B}$ for all $G \subset Y$ open.

Prop. 1.2.11 1. If $F \subset X$ is closed, then $F \in \mathcal{B}$.

2. $G_n \subset X$ are open, then $\bigcap_{n=1}^{\infty} G_n \in \mathcal{B}$. These are called G_δ -sets.

3. $F_n \subset X$ are closed, then $\bigcup_{n=1}^{\infty} F_n \in \mathcal{B}$. These are called F_σ -sets.

PROOF These follow directly from the definition of a σ -algebra. \square

Ex. 1.2.12 $X = \mathbb{R}, \tau_e$, then $\mathcal{B} = S(\tau_e)$. Let $\Gamma_0 = \{(a, b) : a < b\}$ be a family of open intervals. We see that $S(\Gamma_0) = \mathcal{B}$. Since $\Gamma_0 \subset \tau$, $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$. Conversely, let $G \in \tau$, then we have open intervals $G = \bigcup_{n=1}^{\infty} I_n$ so that $G \in S(\Gamma_0)$. Thus $S(\tau) \subset S(\Gamma_0)$ and $S(\Gamma_0) = \mathcal{B}$.

Ex. 1.2.13 Let $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$. I claim that $S(\Gamma_\infty) = \mathcal{B}$. Certainly $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$. Then $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$. Similarly, $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n] \in S(\Gamma_\infty)$. Thus $(a, \infty) \cap (-\infty, b) = (a, b) \in S(\Gamma_0)$, and using the previous example, $\mathcal{B} = S(\Gamma_\infty)$.

Prop. 1.2.14 Let (X, \mathcal{M}) be a measurable space, and let $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the euclidean topology. If $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for any $\alpha \in \mathbb{R}$, then f is measurable.

PROOF Recall that f is measurable if its inverse image takes open sets to measurable sets.

We have $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty]))^c \in \mathcal{M}$. Similarly,

$$f^{-1}([-\infty, \alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty} [-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha, \beta)) = f^{-1}([-\infty, \beta) \cap (\alpha, \infty]) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for τ_e . Thus if $G \subset \overline{\mathbb{R}}$ is open, then there exists open intervals so that $G = \bigcup_{n=1}^{\infty} I_n$ and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired. \square

1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the Dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

Def'n. 1.2.15 Let $(a_n)_{n \in \mathbb{N}} \subset \overline{\mathbb{R}}$ be a sequence, and $b_k = \sup\{a_k, a_{k+1}, \dots\}$. Then $\beta = \inf_{k \in \mathbb{N}} b_k$ is called the \limsup of (a_n) . We can similarly define $c_k = \inf\{a_k, a_{k+1}, \dots\}$ and $\liminf = \sup_{k \in \mathbb{N}} c_k$.

Def'n. 1.2.16 Let $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of functions. Then $(\sup f_n) : X \rightarrow \overline{\mathbb{R}}$, $(\sup f_n)(x) = \sup f_n(x)$ for all $x \in X$. Similarly, $(\inf f_n) : X \rightarrow \overline{\mathbb{R}}$, $(\inf f_n)(x) = \inf f_n(x)$ for all $x \in X$. Then $(\liminf f_n)(x) = \liminf f_n(x)$. If $\lim f_n(x)$ exists for all x , then we say $(\lim f_n)(x) = \lim f_n(x)$.

Thm. 1.2.17 Let $f_n : X \rightarrow \overline{\mathbb{R}}$ be measurable. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ are measurable.

PROOF Let $g = \sup f_n$. It is enough to prove that $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$ for all α . Let $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$. Let $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$. We show that $H = \bigcup_{n=1}^{\infty} H_n$.

First let $x \in H$, so $\sup f_n(x) > \alpha$. Thus get N so that $f_N(x) > \alpha$, so $x \in H_N$ and x is in the union. The converse is obvious.

Thus g is measurable. In the exact same way, $\inf f_n$ is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k \geq i} f_k$$

is measurable. □

Cor. 1.2.18 If $\lim f_n$ exists, then it is measurable.

PROOF If $\lim f_n$ exists, then $\lim f_n = \limsup f_n$. □

Cor. 1.2.19 If f, g are measurable, then $\max\{f, g\}$, $\min\{f, g\}$ are measurable.

Cor. 1.2.20 Let f be a function. Then $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$ (the positive and negative parts of f) are measurable. Similarly, $|f| = f_+ + f_-$ is measurable.

1.2.3 Measures

Def'n. 1.2.21 Let (X, \mathcal{M}) be a measurable space. A function $\mu : \mathcal{M} \rightarrow [0, +\infty]$ is called a **(positive) measure** if it is countably additive and not constant $+\infty$. In other words,

$$1. \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \text{ if } A_i \cap A_j = \emptyset$$

$$2. \exists A \in \mathcal{M} \text{ so that } \mu(A) < \infty$$

(X, \mathcal{M}, μ) is called a **measure space**.

Prop. 1.2.22 1. $\mu(\emptyset) = 0$

2. If $A_i \cap A_j = \emptyset$ then $\mu\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$

3. $A \subset B$ implies $\mu(A) \leq \mu(B)$

4. $A_1 \subset A_2 \subset A_3 \cdots$ then $\lim_{n \rightarrow \infty} \mu A_n = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

5. $A_1 \supset A_2 \supset A_3 \cdots$ and $\mu(A_i) < \infty$ then $\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

PROOF 1. Let $A \in \mathcal{M}$ so that $\mu(A) < \infty$, and fix $A_1 = A$, $A_2 = A_3 = \cdots = \emptyset$. Then $\bigcup A_n = A$ so $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$ so $\mu(\emptyset) = 0$.

2. Obvious

3. Note that $B = A \cup (B \setminus A)$ is a disjoint union.

4. Define $B_1 := A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then $B_i \cap B_j = \emptyset$ and $\mu(A_n) = \mu\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mu(B_i)$. Similarly, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$. Therefore, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$.

5. Let $C_n = A_1 \setminus A_n$, $C_1 = \emptyset$. Then $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) + \mu(A_n) = \mu(A_1)$. Let $A = \bigcap_{n=1}^{\infty} A_n$ so $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$ and $(\bigcup C_n) \cup A = A_1$ is a disjoint union. But then $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$ so that

$$\mu(A_1) - \mu(A) = \mu\left(\bigcup C_n\right) = \lim_{n \rightarrow \infty} \mu(C_n) = \mu(A_n) - \lim_{n \rightarrow \infty} \mu(A_n)$$

Since $\mu(A_1)$ is finite, we have $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$. \square

Ex. 1.2.23 Here are a few examples of measures that exist on arbitrary sets.

1. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$, and

$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$

It is easy to verify it is countably additive.

2. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$. Fix $x_0 \in X$. Then

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

1.3 Towards Integration

1.3.1 Simple Functions

Def'n. 1.3.1 $s : X \rightarrow \mathbb{R}$ or \mathbb{C} is called a simple function if its range is finite.

Prop. 1.3.2 Let s be a simple function, so that $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $A_i = s^{-1}(\{\alpha_i\})$ and s is measurable if and only if $A_i \in \mathcal{M}$.

PROOF Obvious. □

The following theorem is used later to define the integral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function f .

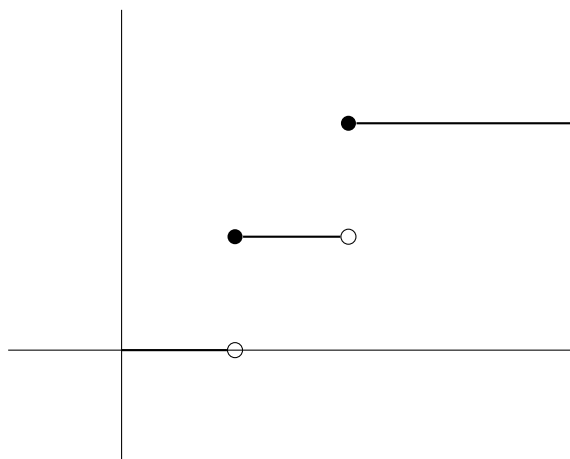
Thm. 1.3.3 Let $f : X \rightarrow [0, +\infty]$ be nonnegative measurable functions. Then there exists a sequence $s_n : X \rightarrow [0, +\infty]$ of simple measurable functions with

1. (s_n) is increasing and bounded above by f
2. $\lim s_n = f$ pointwise.

PROOF Let $n \in \mathbb{N}$, $t \geq 0$, and $k_n(t) = [2^n \cdot t]$ (i.e. $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn ϕ_1 below:



Then $t - 2^{-n} \leq \phi_n(t) \leq t$, $\lim \phi_n(t) = t$, and $\phi_n \leq \phi_{n+1}$. Define $s_n = \phi_n \circ f$, so for any $x \in X$, $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$. Note that s_n is simple since it has finite range (from ϕ_n), and $s_n \leq s_{n+1}$ because $\phi_n \leq \phi_{n+1}$, and $s_n \leq f$ since $\phi_n(t) \leq t$. Furthermore, ϕ_n is measurable since its level sets are intervals, so $\phi_n \circ f$ is measurable. □

1.3.2 Integration of Positive Functions

Let (X, \mathcal{M}, μ) be a measure space.

Def'n. 1.3.4 Let $S : X \rightarrow [0, +\infty)$ be a measurable simple function $s = \sum_{n=1}^n \alpha_i X_{A_i}$. Let $E \in \mathcal{M}$. Then define the **integral of s over E with respect to μ** as

$$\int_E s \, d\mu = \sum_{n=1}^n \alpha_i \mu(A_i \cap E)$$

where we define $0 \cdot \infty = 0$.

Def'n. 1.3.5 Let $f : X \rightarrow [0, +\infty]$ be a measurable function. Let $E \in \mathcal{M}$. Then the **(Lebesgue) integral of f over E with respect to μ** is

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f; s \text{ is simple measurable} \right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only.

Prop. 1.3.6 Let $f, g : X \rightarrow [0, +\infty]$ be measurable functions. Let $E, A, B \in \mathcal{M}$.

1. If $f \leq g$ then $\int_E f \, d\mu \leq \int_E g \, d\mu$
2. If $A \subset B$, then $\int_A f \, d\mu \leq \int_B f \, d\mu$
3. $\int_E c \cdot f \, d\mu = c \cdot \int_E f \, d\mu$ for all $c \geq 0$
4. If $f(x) = 0$ for all $x \in E$, then $\int_E f \, d\mu = 0$
5. If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$
6. $\int_E f \, d\mu = \int_X f \cdot \chi_E \, d\mu$.

PROOF 1. Note that

$$\left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\} \subset \left\{ \int_E s \, d\mu : 0 \leq s \leq g \right\}$$

$0 \leq s \leq f$ be simple measurable. Then

$$\int_A s \, d\mu = \sum \alpha_i \mu(A \cap A_i) \leq \sum \alpha_i \mu(B \cap A_i) = \int_B s \, d\mu$$

Take the supremum for all $0 \leq s \leq f$, then the result follows.

3. Let S be simple and measurable, so $s = \sum \alpha_i \chi_{A_i}$. Then

$$\int_E c \cdot s \, d\mu = \sum_{i=1}^n \alpha_i \cdot c \cdot \mu(E \cap A_i) = c \cdot \sum \alpha_i \mu(E \cap A_i) = c \int_E s \, d\mu$$

Thus

$$\begin{aligned}\int_E c \cdot f \, d\mu &= \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq cf \right\} \\ &= \sup \left\{ \int_E c \cdot t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \sup \left\{ \int_E t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \int_E f \, d\mu\end{aligned}$$

4. If $0 \leq s \leq f$, then $s = \sum \alpha_i \chi_{A_i}$. If $x \in A_i \cap E$, then $s(x) = \alpha_i$ and $\alpha_i = 0$. Then $\alpha_i \mu(A_i \cap E) = 0$ for all i : either $A_i \cap E = \emptyset$, or $A_i \cap E$ is not empty, and $\alpha_i = 0$. This is true for any $0 \leq s \leq f$, and taking supremums yields the result.
5. If $\mu(E) = 0$ then $\mu(A_i \cap E) = 0$, and $\int_E s \, d\mu = \sum \alpha_i \mu(A_i \cap E) = 0$ and taking supremums, the result holds.
6. Exercise. First prove if $0 \leq s \leq f \cdot \chi_E$, then $\int_X s \, d\mu = \int_E s \, d\mu$. Then prove $\left\{ \int_E s \, d\mu : 0 \leq s \leq f \cdot \chi_E \right\} = \left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\}$. \square

Prop. 1.3.7 Let s be a simple and measurable. Then $\phi(E) = \int_E s \, d\mu$ is a measure.

PROOF $\phi(\emptyset) = 0$, so ϕ is not constant $+\infty$. Let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union. Then

$$\begin{aligned}\phi(E) &= \sum_{i=1}^m \alpha_i \mu(A_i \cap E) \\ &= \sum_{i=1}^m \alpha_i \mu \left(A_i \cap \left(\bigcup_{n=1}^{\infty} E_n \right) \right) = \sum_{i=1}^m \alpha_i \mu \left(\bigcup_{n=1}^{\infty} (A_i \cap E_n) \right) \\ &= \sum_{i=1}^m \alpha_i \sum_{n=1}^{\infty} \mu(A_i \cap E_n) = \sum_{n=1}^{\infty} \sum_{i=1}^m \alpha_i \mu(A_i \cap E_n) \\ &= \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu = \sum_{n=1}^{\infty} \phi(E_n)\end{aligned}$$

\square

Prop. 1.3.8 Let s, t be nonnegative, measurable simple functions. Then

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

PROOF Write

$$s = \sum_{i=1}^m \alpha_i \chi_{A_i}, \quad t = \sum_{j=1}^n \beta_j \chi_{B_j}$$

and let $E_{ij} = A_i \cap B_j$, so $X = \bigcup_{i,j} E_{ij}$ is a disjoint union. We now have

$$\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j) \mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu$$

Let $\mu(E) = \int_E (s+t) d\mu$, which is a measure as above. Thus

$$\begin{aligned} \int_X (s+t) d\mu &= \phi(X) = \phi\left(\bigcup_{i,j} E_{ij}\right) \\ &= \sum_{i,j} \phi(E_{ij}) = \sum_{i,j} \int_{E_{ij}} (s+t) d\mu \\ &= \sum_{i,j} \left(\int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu \right) \\ &= \sum_{i,j} \varphi(E_{ij}) + \sum_{i,j} \theta(E_{ij}) \\ &= \int_X s d\mu + \int_X t d\mu \end{aligned}$$

where $\varphi(E) = \int_E s d\mu$, $\theta(X) = \int_X t d\mu$. □

1.3.3 Lebesgue's Monotone Convergence Theorem

Thm. 1.3.9 (Lebesgue's Monotone Convergence) *Let $f_n : X \rightarrow [0, +\infty]$ be measurable, such that*

(i) $0 \leq f_1 \leq f_2 \leq \dots$

(ii) $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in X$

Then f is measurable, and $\int_X f d\mu = \lim \int_X f_n d\mu$.

PROOF It was already proven that f is measurable. We have $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$ for all n , so $\alpha := \lim_{n \rightarrow \infty} \int_X f_n d\mu$ exists. We also have $f_n \leq f$, so $\int f_n \leq \int f$ and $\alpha \leq \int_X f_n d\mu$. Thus we wish to show $\alpha \geq \int_X f d\mu$. It suffices to prove that $\alpha \geq \int_X s d\mu$ for any simple $s \leq f$. Let $c \in (0, 1)$; it suffices to show that $\alpha \geq \int_X c \cdot s d\mu$. Define $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\}$. We have $E_1 \subset E_2 \subset \dots$ so that $\bigcup E_n = X$. Then

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} c \cdot s d\mu$$

Let $\phi(E) = \int_E s d\mu$, so $\int_{E_n} s d\mu = \phi(E_n) \rightarrow \phi(\bigcup E_n) = \phi(X) = \int_X s d\mu$. Thus

$$\alpha \geq c \cdot \lim_{n \rightarrow \infty} \phi(E_n) = c \cdot \int_X s d\mu = \int_X cs d\mu$$

as desired. □

Ex. 1.3.10 Consider the function consisting of a triangle with base $2/n$ and height n . Then $\int_0^1 f_n = 1$ as a Riemannian integral. However, $\lim_{n \rightarrow \infty} f_n(x) = 0$ for any x , so $\int_0^1 f = 0 \neq 1 = \lim \int_0^1 f_n$.

Thm. 1.3.11 Let $f, g : X \rightarrow [0, +\infty]$ measurable, then $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$.

PROOF We proved that there exists increasing sequences of simple functions s_n, t_n such that $\lim s_n(x) = f(x)$, $\lim t_n(x) = g(x)$. Then $s_n(x) + t_n(x) \rightarrow f(x) + g(x)$ monotonically. But then

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X \lim_{n \rightarrow \infty} (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \int_X (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \left(\int_X s_n d\mu + \int_X t_n d\mu \right) \\ &= \int_X \lim_{n \rightarrow \infty} s_n d\mu + \int_X \lim_{n \rightarrow \infty} t_n d\mu \\ &= \int_X f d\mu + \int_X g d\mu \end{aligned} \quad \square$$

Cor. 1.3.12 If $f_n : X \rightarrow [0, +\infty]$ is a sequence of measurable functions, then

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=1}^{\infty} f_n d\mu$$

Ex. 1.3.13 Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(X)$, $\mu(E)$ is the counting measure. Let $a : X \rightarrow [0, \infty)$ be a function. This is a sequence. Every function is measurable. Let $s_n(i) = a(i)$ for $i \leq n$ and 0 otherwise, which is a simple function, and $s_n \leq s_{n+1}$. Then $\lim_{n \rightarrow \infty} s_n(i) = a(i)$ so $s_n \rightarrow a$ pointwise, so by LMC $\int_X s_n d\mu = \int_X a d\mu$. Also,

$$\int_X s_n d\mu = \sum_{i=1}^n a(i) \mu(\{i\}) = \sum_{i=1}^n a(i)$$

$$\text{so } \int_X a d\mu = \sum_{n=1}^{\infty} a(n).$$

Lemma 1.3.14 (Fatou) Let $f_n : X \rightarrow [0, \infty)$ be a sequence of measurable functions. Then

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu$$

PROOF Let $g_k = \inf\{f_k, f_{k+1}, \dots\}$ so $\liminf f_n = \lim_{n \rightarrow \infty} g_n$ and g_n is increasing. Note that $g_k \leq f_k$ for any k , so $\int_X g_k d\mu \leq \int_X f_k d\mu$. Thus

$$\begin{aligned} \int_X \liminf f_n d\mu &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \lim \int_X g_n d\mu \\ &= \liminf \int_X g_n d\mu \\ &\leq \liminf \int_X f_n d\mu \end{aligned} \quad \square$$

Ex. 1.3.15 It is possible for the inequality to be strict. Define $f_{2n} = \chi_{[0,1]}$ and $f_{2n+1} = \chi_{[1,2]}$. Thus $\liminf f_n(x) = 0$ so $\int_{[0,2]} \liminf f_n d\mu = 0$ but $\inf_{[0,2]} \int_{[0,2]} f_n d\mu = 1$

Thm. 1.3.16 Let $f : X \rightarrow [0, \infty]$ be measurable. Let $\phi(E) = \int_E f d\mu$, $E \in \mathcal{M}$. Then ϕ is a measure and $\int_X g d\phi = \int_X g \cdot f d\mu$.

PROOF Certainly $\phi(\emptyset) = 0$, so $\phi \neq +\infty$. Thus let $E = \bigcup_{i=1}^{\infty} E_i$ be a disjoint union. Then $\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f$. Thus we have

$$\begin{aligned} \phi(E) &= \int_E f d\mu \\ &= \int_X \chi_E f d\mu \\ &= \int_X \sum_{i=1}^{\infty} \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_X \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \phi(E_i) \end{aligned}$$

Now, we prove that $\int_X g d\mu = \int_X g f d\mu$.

First, we do this for $g = \chi_E$. Then $\int_X \chi_E d\mu = \phi(E)$ on the left, and $\int_X \chi_E f d\mu = \int_E f d\mu = \phi(E)$ and equality holds.

Now, let $g = \sum_{i=1}^n \alpha_i \chi_{A_i}$ be a simple function. Then $\int_X \sum \alpha_i \chi_{A_i} d\phi = \sum \alpha_i \int_X \chi_{A_i} d\phi$ on the left and $\int_X \sum \alpha_i \chi_{A_i} f d\mu = \sum \alpha_i \int_X \chi_{A_i} f d\mu$.

Finally, let g be an arbitrary measurable function, and let $(s_n) \rightarrow g$ be an increasing sequence of simple functions. Note that $s_n f \rightarrow g f$. Thus

$$\begin{aligned} \int_X g d\phi &= \int_X \lim s_n d\phi = \lim \int_X s_n d\phi \\ &= \lim \int_X s_n f d\mu = \int_X \lim (s_n f) d\mu \\ &= \int_X g \cdot f d\mu \end{aligned}$$

as desired. □

1.4 Integration of Complex Valued Functions

Def'n. 1.4.1 A function $f : X \rightarrow \mathbb{C}$ is called *Lebesgue integrable* if $\int_X |f| d\mu < \infty$. The collection of such functions is $L^1(\mu)$.

1.4.1 Basic Properties

Def'n. 1.4.2 Let $f \in L^1(\mu)$. Then $f = u + iv$ and denote $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Let $E \in \mathcal{M}$; then the integral of f over E with respect to μ is

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \left(\int_E v^+ d\mu - \int_E v^- d\mu \right)$$

Thm. 1.4.3 Let $f, g \in L^1(\mu)$, $\alpha, \beta \in \mathbb{C}$, so $\alpha f + \beta g \in L^1(\mu)$ and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$$

PROOF Note that $\alpha f + \beta g$ is measurable, so $\int_X |\alpha f + \beta g| d\mu \leq |\alpha| \int_X |f| d\mu + |\beta| \int_X |g| d\mu < \infty$. For real measurable functions, $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$ directly by expanding the definition and using additivity over positive functions. We thus show $\int_X \alpha f d\mu = \alpha \int_X f d\mu$. If $\alpha \geq 0$, then

$$\begin{aligned} \int_X \alpha f d\mu &= \int_X \alpha(u + iv) = \int_X (\alpha u^+ - \alpha u^- + i\alpha v^+ - i\alpha v^-) d\mu \\ &= \int_X ((\alpha u)^+ - (\alpha u)^- + (i\alpha v)^+ - (i\alpha v)^-) d\mu \\ &= \int_X (\alpha u)^+ d\mu - \int_X (\alpha u)^- d\mu + \int_X i(\alpha v)^+ d\mu - \int_X i(\alpha v)^- d\mu \\ &= \alpha \int_X u^+ d\mu - \alpha \int_X u^- d\mu + \alpha \int_X iv^+ d\mu - \alpha \int_X iv^- d\mu \\ &= \alpha \int_X f d\mu \end{aligned}$$

and similarly for $\alpha = -1$, $\alpha = i$. □

Thm. 1.4.4 Let $f \in L^1(\mu)$. Then $\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$.

PROOF Let $z = \int_X f d\mu$. Let $\alpha = \frac{|z|}{z}$ if $z \neq 0$, and $\alpha = 1$ otherwise. Then $\alpha \int_X f d\mu = |z|$. Let $u = \operatorname{Re}(\alpha \cdot f) \leq |\alpha \cdot f| \leq |f|$ since $|\alpha| = 1$. Thus

$$\begin{aligned} \left| \int_X f d\mu \right| &= \alpha \cdot \int_X f d\mu \\ &= \int_X \alpha f d\mu \\ &= \int_X \operatorname{Re}(\alpha f) d\mu \\ &\leq \int_X |f| d\mu \end{aligned}$$

□

1.4.2 More Dominated Convergence

Naturally, we want similar results as we have before. Indeed, we have the following theorem:

Thm. 1.4.5 (Lebesgue's Dominated Convergence) *Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions such that $f = \lim f_n$. Assume that there is some $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all n . Then $f \in L^1(\mu)$ and $\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$.*

PROOF We certainly know that f is measurable, and $|f| \leq g$, so $f \in L^1(\mu)$. As well, the triangle inequality show that $|f - f_n| \leq 2g$ for any n . We will see that $0 \leq \liminf \int_X |f - f_n| \, d\mu \leq \limsup \int_X |f - f_n| \, d\mu \leq 0$. Assuming that this holds, then $\lim \int_X |f - f_n| \, d\mu = 0$ and

$$0 \leq \lim \left| \int_X f \, d\mu - \int_X f_n \, d\mu \right| \leq \int_X |f - f_n| \, d\mu = 0$$

The first two inequalities are obvious: we must show that $\limsup \int_X |f - f_n| \, d\mu \leq 0$. Firstly, we have

$$\begin{aligned} \int_X 2g \, d\mu &= \int_X \left(2g - \lim_{n \rightarrow \infty} |f - f_n| \right) d\mu \\ &= \int_X \liminf (2g - |f - f_n|) \, d\mu \\ &\leq \lim \int_X (2g - |f - f_n|) \, d\mu && \text{By Fatou's Lemma} \\ &= \int_X 2g + \liminf \left(- \int_X |f - f_n| \, d\mu \right) \\ &= \int_X 2g - \limsup \int_X |f - f_n| \, d\mu \end{aligned}$$

and since $\int_X 2g \, d\mu$ is finite, we subtract and $\limsup \int_X |f - f_n| \, d\mu \leq 0$. \square

Ex. 1.4.6 Consider $\lim_{n \rightarrow \infty} \int_0^n e^{-nx} \, dx$. Define

$$f_n(x) = \begin{cases} e^{-nx} & \text{if } x \leq n \\ 0 & \text{if } x > n \end{cases}$$

Note that $f_n(x) \leq g(x) = e^{-x}$ and $\int_0^\infty e^{-x} \, dx < \infty$. Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^n e^{-nx} \, dx &= \int_{[0, \infty)} \lim_{n \rightarrow \infty} f_n(x) \, dx \\ &= \int_{[0, \infty)} \chi_{\{0\}} \, dx \\ &= 0 \end{aligned}$$

Rmk. 1.4.7 For the Riemann integral, we have $\int \lim f_n = \lim \int f_n$ as long as the convergence of f_n is uniform.

Chapter 2

The Lebesgue measure

2.1 The Vector Space $L^1(\mu)$

2.1.1 Almost Everywhere

Let (X, \mathcal{M}, μ) be a measure space.

Def'n. 2.1.1 Let $E \in \mathcal{M}$. We say that property P holds almost everywhere in E if there exists $N \in \mathcal{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds in $E \setminus N$.

Ex. 2.1.2 Two functions $f, g : X \rightarrow \mathbb{C}$ are equal almost everywhere if $\exists N \subset X$ such that $\mu(N) = 0$ and $f(x) = g(x)$ on $X \setminus N$.

Prop. 2.1.3 Let $E \subset X$ be such that $A_1, A_2, B_1, B_2 \in \mathcal{M}$ for which $\int_X f d\mu = \int_X g d\mu$. Then $A_1 \subset E \subset B_1$, $A_2 \subset E \subset B_2$, and $\mu(B_1 \setminus A_1) = 0$ and $\mu(B_2 \setminus A_2) = 0$. Then $\mu(A_1) = \mu(A_2)$.

PROOF Note that $A_1 \setminus A_2 \subset E \setminus A_2 \subset B_2 \setminus A_2$. As well, $\mu(A_1 \setminus A_2) \leq \mu(B_2 \setminus A_2) = 0$. Then

$$\begin{aligned}\mu(A_1) &= \mu(A_1 \cap A_2^c) + \mu(A_1 \cap A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2) \\ \mu(A_2) &= \mu(A_2 \cap A_1^c) + \mu(A_2 \cap A_1) = \mu(A_2 \setminus A_1) + \mu(A_2 \cap A_1) = \mu(A_1 \cap A_2)\end{aligned}\quad \square$$

Prop. 2.1.4 Let (X, \mathcal{M}, μ) be a measure space. Let

$$\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, A \subset E \subset B, \mu(B \setminus A) = 0\}$$

Then \mathcal{M}^* is a σ -algebra, and $\mu^* : \mathcal{M}^* \rightarrow [0, +\infty]$ defined by $\mu^*(E) = \mu(A)$.

PROOF We show that \mathcal{M}^* is a σ -algebra, and μ is countably additive.

1. $X \in \mathcal{M}$ so $X \in \mathcal{M}^*$.
2. If $E \in \mathcal{M}^*$, get $A \subset E \subset B$ so $B^c \subset E^c \subset A^c$, $A^c, B^c \in \mathcal{M}$. As well, $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \setminus A) = 0$, so $E^c \in \mathcal{M}^*$.
3. If $E_i \in \mathcal{M}^*$ is a countable collection, then get $A_i \subset E_i \subset B_i$. Fix $A = \bigcup A_i$ and $B = \bigcup B_i$. Then $B \setminus A = \bigcup (B_i \setminus A) \subset \bigcup (B_i \setminus A_i)$ so $\mu(B \setminus A) = 0$ and $A \subset \bigcup E_i \subset B$ so $\bigcup E_i \in \mathcal{M}^*$.
4. Let E_i be disjoint, $E = \bigcup E_i$, and $E_i \in \mathcal{M}^*$. Get $A_i \subset E_i \subset B_i$. Then $\mu^*(\bigcup E_i) = \mu(\bigcup A_i) = \sum \mu(A_i) = \sum \mu(E_i)$. \square

Def'n. 2.1.5 We call the space $(X, \mathcal{M}^*, \mu^*)$ the **completion** of (X, \mathcal{M}, μ) .

In particular, every subset of a set with measure 0 is measurable.

2.1.2 $L^1(\mu)$ as a normed space

Prop. 2.1.6 1. Let $f : X \rightarrow [0, +\infty)$ be measurable, $E \in \mathcal{M}$. If $\int_E f \, d\mu = 0$, then $f = 0$ almost everywhere in E .

2. Let $f \in L^1(\mu)$. If $\int_E f \, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ almost everywhere in X .

PROOF 1. Let $A_n = \{x \in E : f(x) > 1/n\}$, so that

$$\frac{1}{n}\mu(A_n) \leq \int_{A_n} f \, d\mu \leq \int_E f \, d\mu = 0 \implies \mu(A_n) = 0$$

for all n . But then

$$N = \{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n \implies \mu(N) \leq \sum \mu(A_n) = 0$$

2. Write $f = u + iv$ so that

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \int_E v^+ \, d\mu - i \int_E v^- \, d\mu$$

We show that $u^+ = 0$ almost everywhere (the other terms are identical). Let $E = \{x \in X : u(x) \geq 0\}$, so $\int_E f \, d\mu = 0$, so its real part is zero and $\int_E u^+ \, d\mu = 0$. Thus $u^+ = 0$ almost everywhere in E . The result follows. \square

Def'n. 2.1.7 A **normed space** over \mathbb{R} is a vector space V over \mathbb{R} with a map $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

(i) $x \in V \implies \|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| \leq |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in V$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Now $L^1(\mu) = \{f : X \rightarrow \mathbb{C} \text{ measurable and } \int_X |f| \, d\mu < \infty\}$. We certainly have that $L^1(\mu)$ is a vector space. We wish to define $\|f\| = \int_X |f| \, d\mu$. The only problem is that

$$\int_X |f| \, d\mu = 0 \implies f = 0 \text{ almost everywhere}$$

To deal with this problem, we quotient our space by the equivalence relation $f \sim g$ if and only if $f = g$ almost everywhere. With this in mind, define $V = L^1(\mu)/\sim$ denote the set of equivalence classes. We need to define $+, \cdot, \|\cdot\|$ on V . Let $[f]$ denote the class of f . Then

$$[f] + [g] = [f + g]$$

$$c[f] = [cf]$$

$$\|[f]\| = \int_X |f| \, d\mu$$

Let's verify that this is well defined: if $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$. Indeed, this is true since the sums are equal except perhaps on a union of measure zero sets, so equality holds almost everywhere. The second definition is obviously well defined. Finally, by a homework assignment, $\|[f]\|$ is also well defined. Now, let's verify the properties of the norm.

- (i) Certainly $\|f\| \geq 0$, and $\|f\| = 0$ implies $f = 0$ almost everywhere, so $[f] = [0] = 0$.
- (ii) We have $\|\lambda \cdot f\| = \int_X |\lambda f| d\mu = |\lambda| \int_X |f| d\mu = |\lambda| \|f\|$
- (iii) We have $\|f + g\| = \int_X |f + g| d\mu \leq \int_X |f| + \int_X |g| = \|f\| + \|g\|$

In $L^1(\mu)$, two functions are the same if they are equal almost everywhere. However, this can be a challenge: if $f \in L^1(\mu)$ and $x_0 \in X$, then $f(x_0)$ is not well defined. For example, it is challenging to give meaning to boundary conditions of functions.

2.1.3 Construction of the Lebesgue measure

We begin from the Riemann integral $\int_a^b f(x) dx$ for a continuous function f . Define $\text{supp } f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. For continuous functions with compact (bounded) support, define $\Lambda f = \int_{\mathbb{R}} f(x) dx$ is the Riemann integral, which is a functional. In particular,

$$\text{measure}((a, b)) = \text{length}((a, b)) = \sup\{\Lambda f : f \text{ is continuous, compact support, } 0 \leq f \leq 1, \text{supp } f \subset (a, b)\}$$

We will extend this to a σ -algebra containing the Borel sets. In order to define these, for open sets, $\mu(G) = \sup\{\Lambda f : 0 \leq f \leq 1, \text{supp } f \subset G\}$, where Λ is the Riemann integral. For an arbitrary set, $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$. However, this “measure” is not countably additive: the σ -algebra $\mathcal{P}(X)$ is too large (Vitali’s construction). Instead, we will define $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$, which means that $E \cap K$ is regular for any K compact, and regular means that the outer measure and inner measure are equal. The outer measure is $\sup\{\mu(K) : K \subset E \text{ compact}\} = \mu(E)$.

2.2 The Riesz Representation Theorem

In this section, we assume that (X, τ) be a locally compact, Hausdorff topological space.

Def’n. 2.2.1 We denote the space of continuous functions with compact support by $C_c(X) = \{f : X \rightarrow \mathbb{C} \mid f \in C(X), \text{supp } f \text{ is compact}\}$.

Def’n. 2.2.2 Let $\Lambda : C_c(X) \rightarrow \mathbb{C}$ be a **linear functional**, i.e. $\Lambda(cf + g) = c\Lambda f + \Lambda g$. Λ is called a **positive linear functional** if $f \geq 0 \Rightarrow \Lambda f \geq 0$.

Def’n. 2.2.3 We say that $K < f$ if K is compact and $f \in C_c(X)$, $0 \leq f \leq 1$ implies that $x \in K \Rightarrow f(x) = 1$. We say that $f < G$ if G is open, $f \in C_c(X)$, $0 \leq f \leq 1$, and $\text{supp } f \subset G$.

Lemma 2.2.4 (Urysohn) Let $G \in \tau$, $K \subset G$ compact. Then there exists $f \in C_c(X)$ such that $K < f < G$.

PROOF Will do later. It’s pretty fun - it’s a construction using the Dyadic rationals. □

Lemma 2.2.5 (Partition of Unity) Let $G_1, G_2, \dots, G_n \in \tau$, and let $K \subset G_1 \cup \dots \cup G_n$ be compact. Then there are functions $h_i \in C_c(X)$ such that $h_i < G_i$ and $K < \sum h_i$.

PROOF Also will do later. □

How can we create a positive linear functional on $C_c(X)$? If μ is a measure, and functions on $C_c(X)$ are measurable, then $\Lambda f = \int_X f d\mu$ is a positive linear functional. The representation theorem says that there are no other examples.

Thm. 2.2.6 (Riesz Representation) *Let (X, τ) be as above. If $\Lambda : C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional, then there exists a unique measure space (X, \mathcal{M}, μ) such that $\Lambda f = \int_X f d\mu$ for any $f \in C_c(X)$, $\mathcal{M} \supset \tau$, and*

- (i) $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$ for all $E \in \mathcal{M}$.
- (ii) $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$.
- (iii) $\mu(K) < \infty$ for any K compact.
- (iv) \mathcal{M} is complete.

First, let's get some definitions out of the way. Fix the notation as above.

Def'n. 2.2.7 *Fix a Borel measure μ . The **Lebesgue outer measure** is defined $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$.*

Def'n. 2.2.8 *We say that $E \subset X$ is **regular** if the inner and outer measures coincide. Similarly, $E \subset X$ is **locally regular** if there exists a compact K so that $K \cap E$ is regular.*

PROOF For an open set $G \in \tau$, let $\mu(G) = \sup\{\Lambda f : f < G\}$. Then $\mu(\emptyset) = 0$ and $G_1 \subset G_2$ implies that $\mu(G_1) \leq \mu(G_2)$. Then extend μ to arbitrary $E \subset X$ as an outer measure.

Now let $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$. Let's first see that \mathcal{M} satisfies the desired properties. We first see that \mathcal{M} is complete. Let $E \in \mathcal{M}$, $\mu(E) = 0$ and $A \subset E$. We want to show that $A \in \mathcal{M}$. Let K be compact, and consider $K \cap A$ so that $\mu(K \cap A) = 0$. Then if $F \subset K \cap A$ is compact, $\mu(F) = 0$ implies $\sup\{\mu(F) : F \subset K \cap A \text{ compact}\} = 0$.

Claim 1: μ is σ -subadditive.

PROOF If $\mu(E_j) = \infty$ for some j , then we are done. Thus assume $\mu(E_j) < \infty$ for all j . Let $\epsilon > 0$, $\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ be arbitrary. Let $G_j \supset E_j$ be open, such that $\mu(G_j) \leq \mu(E_j) + \frac{\epsilon}{2^j}$. Then

$$\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \mu\left(\bigcup_{j=1}^{\infty} G_j\right)$$

so there exists some $f < \bigcup_{j=1}^{\infty} G_j$ so $\gamma < \Lambda f$. Let $K = \text{supp } f$ and $K \subset \bigcup_{j=1}^{\infty} G_j$ and by compactness

there exists some n so $K \subset \bigcup_{j=1}^n G_j$. Apply our partition of unity and get some $h_j < G_j$ for each

$j = 1, \dots, n$ such that if $x \in K$, then $\sum_{j=1}^n h_j(x) = 1$. Then $f \cdot h_j < G_j$ so $f = f \cdot \sum_{j=1}^n h_j$.

Thus

$$\begin{aligned}\gamma < \Lambda f &= \Lambda \left(\sum_{j=1}^n f h_j \right) = \sum_{j=1}^n \Lambda(f h_j) \\ &\leq \sum_{j=1}^n \mu(G_j) \leq \sum_{j=1}^n \left(\mu(E_j) + \frac{\epsilon}{2^j} \right) \\ &\leq \sum_{j=1}^{\infty} \left(\mu(E_j) \right) + \epsilon\end{aligned}$$

which holds for all $\epsilon > 0$ if and only if

$$\gamma \leq \sum_{j=1}^{\infty} \mu(E_j)$$

for all $\gamma \leq \mu \left(\bigcup_{j=1}^{\infty} E_j \right)$ and the result follows. \square

Claim 2: If $K < f < G$, then $\mu(K) \leq \Lambda f \leq \mu(G)$. Thus if K is compact, $\mu(K) = \inf\{\Lambda f : K < f\}$, so $\mu(K) < \infty$.

PROOF It is obvious that $\Lambda f \leq \mu(G)$. Thus let $\gamma < \mu(K)$ and $\alpha \in (0, 1)$. Let $V_\alpha := \{x \in X : f(x) > \alpha\}$ and $K \subset V_\alpha$. Now $\gamma < \mu(K) \leq \mu(V_\alpha)$, so we have some $h < V_\alpha$ such that $\gamma < \Lambda h$. Then $\alpha \cdot h \leq f$ since in V_α , $\alpha \cdot h \leq \alpha < f$ and in V_α^c , $\alpha \cdot h = 0 \leq f$. Now $\alpha \cdot \Lambda h = \Lambda(\alpha h) \leq \Lambda f$ so $\gamma < \Lambda f / \alpha$. This is true for all $\alpha \in (0, 1)$ and $\gamma \leq \Lambda f$. Since this holds for all $\gamma < \mu(K)$, we have $\mu(K) \leq \Lambda f$ as required.

Now, let K be compact. Since $\mu(K) \leq \Lambda f$ for all $K < f$. Let $\epsilon > 0$, so we have $G \in \tau$, $G \supset K$ such that $\mu(G) \leq \mu(K) + \epsilon$. Then by Urysohn's lemma, get some f so that $\mu(K) \leq \Lambda f \leq \mu(G)$, so $\Lambda f \leq \mu(K) + \epsilon$ and the result holds. \square