Course Notes

Introduction to Probability

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Contents

Chapter 1

Introduction

1.1 Basic Principles

1.1.1 Probability Spaces

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1.2 Ω

Ω is a set, called the sample space, and $\omega \in \Omega$ are called outcomes and $A \subset \Omega$ are called events.

Ex. 1.1.1 A horserace with 3 horses, a, b, c, has $\Omega = \{(a,b,c),(a,c,b),\dots,(c,b,a)\}$. Then $|\Omega| = 6$ and $A = \{a \text{ wins the race}\} = \{(a,b,c),(a,c,b)\}$.

Ex. 1.1.2 Roll two fair dice, a white die and a yellow die. Then $\Omega = \{(1,1),(1,2),\dots,(6,6)\}$ and $|\Omega| = 36$.

Ex. 1.1.3 Continue flipping a coin until there is a head. Then

$$\Omega = \{(H),(T,H),(T,T,H),\dots\}$$

Then define

$$A = \{\text{there are an even number of rolls}\} = \{(T,H),(T,T,T,H),\dots\}$$

Ex. 1.1.4 Consider $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 100\}$. Then $A = \{\text{you score 50 points}\} = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

Def'n. 1.1.5 *If $A \cap B = \emptyset$, we say that A and B are* ***mutually exclusive*** *events. If $A \subset B$, we say that A* ***implies*** *B.*

Write $A^c = \Omega \setminus A$. Recall distributivity, the deMorgan relations, etc.

1.1.3 $\mathcal{F}$

$\mathcal{F}$ is a collection of subsets of Ω, which denote the events that we consider.

- If Ω is countable, then typically $\mathcal{F}$ is just the collection of all subsets of Ω.
- If Ω is a domain in $\mathbb{R}^n$, then it is a strict subset of $\mathbb{R}^n$.

In any case, $\mathcal{F}$ has to be closed under the following operations:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

in other words, that $\mathcal{F}$ is a σ−algebra.

1.1.4 $\mathbb{P}$

Finally, $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is a function that satisfies 3 axioms:

1. For any $A \in \mathcal{F}$, then $\mathbb{P}(A) \geq 0$
2. $\mathbb{P}(\Omega) = 1$
3. (σ−additivity) Let $A_1, A_2, A_3, \ldots$ be a sequence of mutually exclusive events. Then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

1.1.5 Consequences

- $\mathbb{P}(A^c) + \mathbb{P}(A) = \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1$.
- If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ since $\mathbb{P}(B) = \mathbb{P}((A^c \cap B) \cup (A \cap B)) = \mathbb{P}(A^c \cap B) + \mathbb{P}(A \cap B) = \mathbb{P}(A^c \cap B) + \mathbb{P}(A)$
- For any A, B, we have

  $$\mathbb{P}(A \cup B) = \mathbb{P}((A^c \cap B) \cup (A \cap B) \cup (A \cap B^c)) = \mathbb{P}(A^c \cap B) + \mathbb{P}(A \cap B) + \mathbb{P}(B^c \cap A) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

  Similarly,

  $$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

  which generlizes arbitrarily:

  $$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_r})$$

PROOF We have already proved the base case for $n = 2$, so assume the formula holds for a union of n events. Then

$$\mathbb{P}(A_1 \cup \cdots A_n \cup A_{n+1}) = \mathbb{P}(A_1 \cup \cdots \cup A_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}((A_1 \cup \cdots \cup A_n) \cap A_{n+1})$$

We can distribute the first and third terms using the induction hypothesis, and the result follows. □

Def'n. 1.1.6 *We say $D_1, D_2, \ldots$ is a **decreasing** sequence of events of $D_{k+1} \subset D_k$. We say $D_1, D_2, \ldots$ is a **increasing** sequence of events of $D_{k+1} \supset D_k$.*

Let $\lim_{n\to\infty} D_n = \bigcap_{n=1}^{\infty} {}_n$ and $\lim_{n\to\infty} I_n = \bigcup_{n=1}^{\infty} I_n$.

Prop. 1.1.7 *σ–additivity implies that for any increasing sequence,*

$$\Pr\left(\lim_{n\to\infty} I_n\right) = \lim_{n\to\infty} \Pr(I_n)$$

and similarly for any decreasing sequence

$$\Pr\left(\lim_{n\to\infty} D_n\right) = \lim_{n\to\infty} \Pr(D_n)$$

PROOF Note that (2) implies (1): if D_k is a decreasing sequence, then $I_k = D_k^c$ is an increasing sequence and

$$\left(\lim_{n\to\infty} D_n\right)^c = \left(\bigcap_{n=1}^{\infty} D_n\right)^c = \bigcup_{n=1}^{\infty} I_n = \lim_{n\to\infty} I_n$$

and taking probabilities,

$$\Pr\left(\lim_{n\to\infty} D_n\right) = 1 - \Pr\left(\lim_{n\to\infty} I_n\right) = 1 - \lim_{n\to\infty} \Pr(I_n) = \lim_{n\to\infty} \Pr(D_n)$$

To prove that σ–additivity implies (1), let $I_1, I_2, \ldots$ be increasing. Let $A_1 = I_1$ and for $k \geq 2$ let $A_k = I_k \setminus I_{k-1}$. Then $A_1, A_2, \ldots$ are mutually exclusive and for any $k \geq 1$,

$$\bigcup_{k=1}^{K} A_k = I_k$$

Thus

$$\bigcup_{k=1}^{\infty} A_k = \lim_{n\to\infty} I_n$$

Now note that $\Pr(I_K) = \sum_{k=1}^{K} \Pr(A_k)$ while

$$\begin{aligned}\Pr\left(\lim_{n\to\infty} I_n\right) &= \Pr\left(\bigcup_{k=1}^{\infty} A_k\right) \\ &= \sum_{k=1}^{\infty} \Pr(A_k) \\ &= \lim_{K\to\infty} \sum_{k=1}^{K} \Pr(A_k) \\ &= \lim_{K\to\infty} \Pr(I_K)\end{aligned}$$

□

1.2 Styles of Problems

1.2.1 Finite Uniform Probabilities

We assume that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ and $\mathbb{P}(\{\omega_i\}) = \mathbb{P}(\{\omega_j\})$. Then $\mathbb{P}(\{\omega_i\}) = \frac{1}{N}$ and $\mathbb{P}(A) = |A|/N$.

Ex. 1.2.1 In an urn there are 6 blue balls and 5 red balls. Draw 3 balls out of this 11. What is the change that among the 3 there are exactly 2 blue balls and 1 red ball?

Let us pretend that the balls are labelled, 1 through 11, and set Ω to be all the ordered triples of disjoint elements. Then $A = \{\text{exactly 2 blue and 1 red}\}$, and note that $A = A^1 \cup A^2 \cup A^3$ where A^i has a red in position i and blue in the other two positions. Now, $|A^i| = 5 \cdot 6 \cdot 5$, so $|A| = 3 \cdot 6 \cdot 5 \cdot 6$ and

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{3 \cdot 6 \cdot 5 \cdot 6}{11 \cdot 10 \cdot 9}$$

We now suppose that $\Omega = \{\Lambda \subset \{1, \ldots, 11\} \mid |\Lambda| = 3\}$, so $|\Omega| = \binom{11}{3}$. Now

$$A = \{\Lambda_1 \cup \Lambda_1 | \Lambda_1 \subset \{1, \ldots, 6\}, |\Lambda_1| = 2, \Lambda_2 \subset \{7, \ldots, 11\}, |\Lambda_2| = 1\}$$

So $|A| = \binom{6}{2} \cdot 5$.

Ex. 1.2.2 Consider a group of N people. What is the chance that there is at least one pair amoung them who have the same birthday?

Define $\Omega = \{(i_1, i_2, \ldots, i_N) \mid i_j \in \{1, \ldots, 365\}\}$. We want $A = \{\text{there is at least one common birthday}\}$. We can write

$$A^c = \{(i_1, \ldots, i_n) \in \Omega \mid i_j \neq i_k \forall j \neq k\}$$

Then $|A^c| = 365 \cdot 364 \cdots (365 - N + 1)$ and

$$P_N = \mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{365 \cdot 364 \cdots (365 - N + 1)}{365^N}$$

Ex. 1.2.3 Suppose we have N people at a party. The following day, everyone leaves one after another, and chooses a single phone from a pile. What is the chance that nobody chooses her own phone?

Define $\Omega = \{(i_1, \ldots, i_N) \mid \text{permutations of } \{1, \ldots, N\}\}$, so $\omega = (i_1, \ldots, i_k)$ means person k chooses phone i_k. Then $|\Omega| = N!$. Fix $B = \{\text{nobody picks her/his phone}\}$. Define $A_1 = \{\text{person 1 picks his phone}\}$, so $|A_1| = (N-1)!$, and similarly for A_2, etc. Then $B = A_1^c \cap A_2^c \ldots \cap A_N^c = (A_1 \cup \ldots \cup A_N)^c$, and $\mathbb{P}(A_i) = \frac{1}{N}$. Now in general,

$$\Pr(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{(N-k)!}{N!}$$

for i_k distinct. Thus we now have

$$\begin{aligned}\Pr(B) &= 1 - \Pr(A_1 \cup A_2 \cup \ldots \cup A_n)\\ &= 1 - \sum_{r=1}^{N} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 \cdots < i_r \leq N} \Pr(A_{i_1} \cap \cdots \cap A_{i_r})\\ &= \sum_{r=1}^{n} (-1)^{r+1} \binom{N}{r} \frac{(N-r)!}{N!}\\ &= \sum_{r=1}^{N} (-1)^{r+1} \frac{1}{r!}\end{aligned}$$

so that

$$\Pr(B) = 1 + \sum_{r=1}^{N} (-1)^r \frac{1}{r!} = \sum_{r=0}^{N} (-1)^r \frac{1}{r!}$$

Thus $\lim_{N\to\infty} \Pr(B) = \frac{1}{e}$.

Ex. 1.2.4 (Round table seating) Consider a round table with 20 seats, and 10 married couples sit. What is the change that no couples sit together?

Define $\Omega = \{\text{permutations of } \{1, \ldots, 20\}/\sim\}$ where $(i_1, \ldots, i_{20}) \sum (i_{20}, i_1, \ldots, i_{19})$. Then $|\Omega| = 19!$. Define $B = \{\text{no couples together} = A_1^c \cap A_2^c \cap \cdots \cap A_{10}^c\}$, where

$$A_k = \{\text{the 8th woman sits next to her spouse}\}$$

so that

$$\Pr(B) = 1 - \Pr(A_1 \cup \cdots \cup A_{10})$$

Note that

$$\Pr(A_i) = \frac{18!2}{19!} = \frac{2}{19}$$

by "joining" the couple together, arranging them around the table, and permuting the couple internally. Thus generalizes to

$$\Pr(A_{i_1} \cap \cdots \cap \Pr(a_{i_r}) = \frac{2^r(19-r)!}{19!}$$

Then by inclusion-exclusion,

$$\Pr(B) = 1 - \binom{10}{1} \cdot \frac{18!2}{19!} + \binom{10}{2} \frac{17!2^2}{19!} - \binom{10}{3} \frac{16!2^3}{19!} \cdots + \binom{10}{10} \frac{9!2^{10}}{19!} \approx 0.339$$

Ex. 1.2.5 (Poker hand probabilities) A poker hand is a straight if the 5 cards are of increasing value and not all of the same suit, starting with $A, 2, 3, 4, \ldots, 10$.

Define $\Omega = \{5 \text{ element subsets of the 52 cards}\}$. Then $|\Omega| = \binom{52}{5}$. Thus

$$\Pr(\text{straight}) = \frac{10 \cdot (4^5 - 4)}{\binom{52}{5}}$$

$$\Pr(\text{full house}) = \frac{13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}}$$

Ex. 1.2.6 (Bridge hand probabilities) In bridge, each of the 4 players get 13 cards. Let $\Omega = \{13 \text{ cards that North gets}\}$.

$$\Pr(\text{North receives all spaces}) = \frac{1}{\binom{52}{13}}$$

Pr(North does not receive all 4 suits of any value) = Pr(There is some value such that all suits are at N)

Let $V_k = \{\text{North gets all four suits of value } k\}$. Then

$$\Pr(V_1) = \frac{\binom{48}{9}}{\binom{52}{13}}$$

$$\Pr(V_1 \cap V_2) = \frac{\binom{44}{5}}{\binom{52}{13}}$$

$$\Pr(V_1 \cap V_2 \cap V_4) = \frac{\binom{40}{1}}{\binom{52}{13}}$$

Thus

$$1 - \Pr(V_1 \cup V_2 \cup \cdots \cup V_{13}) = 1 - \frac{\binom{48}{9}}{\binom{52}{13}} \cdot 13 + \binom{13}{2}\frac{\binom{44}{5}}{\binom{52}{13}} - \binom{13}{3}\frac{40}{\binom{52}{5}}$$

What is the change that each player receives one ace? There are

$$\frac{52!}{13!13!13!13!}$$

possible hands. There are 4! ways to arrange the aces, which gives

$$\Pr(E) = \frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}}$$