Course Notes

Real Functions and Measures

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Contents

Chapter 1

Basics of Abstract Measure Theory

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1.1 Review of Topology

1.1.1 Basic Definitions

Def'n. 1.1.1 *Let $X \neq \emptyset$ and $\tau \subseteq \mathcal{P}(X)$. We say that (X, τ) is a* ***topological space*** *if τ satisfies the following conditions:*

1. $\emptyset \in \tau$ $X \in \tau$
2. $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3. $V_\alpha \in \tau$ *for all* $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

We call the elements of τ ***open sets****.*

Def'n. 1.1.2 *$U \subseteq X$ is a* ***neighbourhood*** *of $x \in X$ if there is some $G \in \tau$ such that $x \in G \subset U$.*

Def'n. 1.1.3 *$F \subseteq X$ is* ***closed*** *if F^c is open.*

Def'n. 1.1.4 *The* ***closure*** *of a set $E \subset X$ is the smallest closed set containing E (denoted $\overline{E}$).*

Def'n. 1.1.5 *x is an* ***accumulation point*** *of H if all neighbourhoods of x contains infinitely points of H. Equivalently, x is a limit point of $H \setminus \{x\}$.*

Def'n. 1.1.6 *If $H \subseteq X$, we have a natural subspace topology $\tau|_H = \{G \cap H : G \in \tau\}$.*

1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1. $\mathbb{R}$ along with the open sets (denoted τ_e, the Euclidean topology).
2. The discrete topology, $\tau = \mathcal{P}(X)$ for any $X \neq \emptyset$. This is the "finest" topology.

3. The antidiscrete topology, $\tau = \{\emptyset, X\}$ for any $X \neq \emptyset$ This is the "coarsest" topology.
4. One can define the extended real line, $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

   The same can be done with a single symbol as well. In either case, the extended real line is a compact set.
5. Any metric spaces induces a topology. Consider a set $X \neq 0$ arbitrary, and let $d : X \times X \to \mathbb{R}$ such that
   (a) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$.
   (b) $d(x,y) = d(y,x)$ for all $x, y \in X$
   (c) $d(x,y) \leq d(x,z) + d(z,y)$ for any $x, y, z \in X$

   Then $G \in \tau$ if and only if for any $x \in G$, there exists r so that $B_r(x) \subset G$. There are many examples of metric spaces:
   (a) $X = \mathbb{R}$, $d(x,y) = |x-y|$
   (b) $X = \mathbb{R}$, $d(x,y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
   (c) $X = \mathbb{R}^2$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$
   (d) $X = \mathbb{R}^2$, $d(x,y) = (|x_1-y_1|^p + |x_2-y_2|^p)^{1/p}$ for $p \geq 1$.
   (e) and similarly for $X = \mathbb{R}^n$
   (f) $X = C[0,1]$, $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$.
   (g) normed space: X is a vector space over $\mathbb{R}$, $\|\cdot\| : X \to \mathbb{R}$ such that
      i. $\|x\| = 0$ if and only if $X = 0$
      ii. $\|cx\| = |c| \|x\|$
      iii. $\left\|x+y\right\| \leq \|x\| + \left\|y\right\|$

      If $\|\cdot\|$ is a norm, then $d(x,y) = \left\|x-y\right\|$ is a metric.
6. The cofinite topology: $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$.

1.1.3 Other Definitions

Def'n. 1.1.7 *$K \subset X$ is **compact** if every open cover of K contains a finite subcover.*

Def'n. 1.1.8 *A topological space is called **locally compact** if every point has a compact neighbourhood.*

Prop. 1.1.9 *$C[0,1]$ with the sup norm is not locally compact.*

Proof I'll do this later. □

Def'n. 1.1.10 *A topological space is called* ***Hausdorff*** *if for any $x \neq y$, there exists neighbourhoods $U \ni x$, $V \ni y$ so that $U \cap V = \emptyset$.*

The anti-discrete topology is not Hausdorff.

1. On the discrete topology, K is compact if and only if K is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of X).
3. On $(\mathbb{R}, \tau_e)$, K is compact if and only if K is closed and bounded.
4. On (X, d) metric space, K is compact if and only if K is complete and totally bounded.

Prop. 1.1.11 *1. Let $K \subset X$ be compact, let $F \subset K$ closed. Then F is also compact.*
2. Compact sets in a Hausdorff space are closed.

Proof 1. Let $F \subset \bigcup V_\alpha$. Then $K \subset F^c \cup (\bigcup V_\alpha)$ is an open cover for K, so it has a finite subcover $F^c \cup V_{\alpha_1} \cup \cdots V_{\alpha_n}$. But then since $F \cap F^c = \emptyset$, $F \subset V_{\alpha_1} \cup \cdots V_{\alpha_n}$ is a finite subcover.
2. Let $K \subset X$ be compact, and prove that K^c is open. Thus let $x \in K^c$. For any $y \in K$, there exist U_y, V_y disjoint neighbourhoods of x and y respectively. Now consider the open cover $K \subset \bigcup_{y \in K} V_y$, and get our finite subcover $K \subset V_{y_1} \cup \cdots \cup V_{y_n}$. But then $U_{y_1} \cap \cdots \cap U_{y_n} \cap K = \emptyset$ and is open since it is a finite intersection. □

Def'n. 1.1.12 *$\Gamma \subseteq \tau$ is a* ***base*** *for τ if every $U \in \tau$ can be written as a countable union of the elements of Γ. Γ is a* ***countable base*** *if Γ is countable.*

Prop. 1.1.13 *$\mathbb{R}$ has a countable base of intervals.*

Proof Consider the collection $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$. To see this, for any open set U, one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left(\bigcup_{\{r : B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$ is obvious, so let $x \in U$ be arbitrary, and let s be maximal so that $B_s(x) \subseteq U$. Then choose $q \in \mathbb{Q}$ so that $|x - q| < s/3$ and $r \in \mathbb{Q}$ so that $0 < r < s/2$. Then by construction $B_r(q) \ni x$ and by the triangle inequality $B_{r/2}(q) \subseteq U$, so $x \in S$. Thus $U = S$ as desired. □

Note that the exact same argument (with some work) can be generalized to show that $\mathbb{R}^n$ has a countable base of open hyperrectangles.

Prop. 1.1.14 *Every metric space which is a countable union of compact sets has a countable base.*

Proof See my PMATH 351 notes. □

1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

Def'n. 1.1.15 *Let $(x_n) \subset X$ be a sequence and let $x \in X$. Then x is the* ***limit*** *of (x_n) if for any neighbourhood U of X, there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.*

Prop. 1.1.16 *If $F \subset X$ is closed, then for all convergent sequences in F, the limit is also in F.*

Proof See Homework. □

Def'n. 1.1.17 *Let $f : X \to Y$ be a function, and $x \in X$ an accumulation point of $D(f)$. The limit of f at x is $y \in Y$ if for any neighbourhood V of y there exists a neighbourhood U of x such that $f(U \cap D(f) \setminus \{x\}) \subseteq V$.*

Def'n. 1.1.18 *Let $f : X \to Y$ be a function, and let $x \in D(f)$. Then f is* ***continuous at*** *x if for any neighbourhood V of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of x.*

Def'n. 1.1.19 *$f : X \to Y$ is called* ***continuous*** *if it is continuous at every point.*

Prop. 1.1.20 *$f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is open for all G open.*

Proof Exercise. □

Thm. 1.1.21 *Let $f : X \to Y$ be continuous and $K \subset X$ be compact. Then $f(K)$ is compact.*

Proof Recall that continuous functions pull back open sets. Let $f(K) \subset \bigcup U_\alpha$ be an open cover. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover for K, and has a finite subcover $U_{\alpha_1} \cup \cdots U_{\alpha_n}$. But then $f(f^{-1}(U_{\alpha_1})) \cup \cdots \cup f(f^{-1}(U_{\alpha_n}))$ is a subcover of $f(K)$. □

1.2 Measure Theory

1.2.1 σ−algebras

Def'n. 1.2.1 *Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ−**algebra** if*

1. *$X \in \mathcal{M}$*
2. *$A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$*
3. *If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$*

The pair $(X, \mathcal{M})$ is called a ***measurable space****. The elements of $\mathcal{M}$ are called* ***measurable sets****.*

Def'n. 1.2.2 *Let $(X, \mathcal{M})$ be a measurable space, (Y, τ) be a topological space. Then $f : X \to Y$ is called* ***measurable*** *if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.*

Here are some simple examples of σ−algebras.

Ex. 1.2.3

1. $\mathcal{M} = \{\emptyset, X\}$ is a σ−algebra.
2. $\mathcal{P}(X) = \mathcal{M}$ is a σ−algebra.
3. $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable.}\}$. To see this, given $A_n \in \mathcal{M}$, if everything is countable, then $\bigcup A_n$ is countable. If some A_i is countable, then $(\bigcup A_n)^c = \bigcap A_n^c$ is countable, so $\bigcup A_n \in \mathcal{M}$.

We will later see some proper exaples, like the σ−algebra of Lebesgue measurable sets.

We have the following properties of σ−algebras.

Prop. 1.2.4 *1. $\emptyset \in \mathcal{M}$*
2. *$A_1, A_2, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{M}$*
3. *$A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$*
4. *$A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$*
5. *f is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$.*

Proof 1. $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.
2. We can extend this to a countable union by introduction $A_{n+i} = \emptyset$ for $i \in \mathbb{N}$.
3. By DeMorgan's identities, $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.
4. $A \setminus B = A \cap B^c \in \mathcal{M}$.
5. H^c is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$. □

Prop. 1.2.5 *Let $f : X \to Y$ be measurable, let $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is measurable.*

Proof Let $V \subset Z$ be open, so $g^{-1}(V) \subset Y$ is open, so $f^{-1}(g^{-1}(V)) \in \mathcal{M}$ which is $(g \circ f)^{-1}(V)$. □

Prop. 1.2.6 *Let $(X, \mathcal{M})$ be a measurable space, Y be a topological space. Let $\phi : \mathbb{R}^2 \to Y$ be continuous. If $u, v : X \to \mathbb{R}$ are measurable, then $h(x) = \phi(u(x), v(x))$ is measurable.*

Proof Define $f : X \to \mathbb{R}^2$ by $f(x) = (u(x), v(x))$ We will see that f is measurable, so that $h = \phi \circ f$ is measurable since ϕ is continuous. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals, so $R = I_1 \times I_2$ is an open rectangle. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Let $G \subset \mathbb{R}^2$ be an open set, so there exist R_n open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that f is measurable. □

Cor. 1.2.7 *1. If $u, v : X \to \mathbb{R}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.*
2. *$u + iv : X \to \mathbb{C}$ is measurable.*
3. *$f : X \to \mathbb{C}$ is measurable, $f = u + iv \Rightarrow u, v, |f|$ are measurable.*

Prop. 1.2.8 *Define*

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

Then χ_E is measurable if and only if $E \in \mathcal{M}$.

Proof Naturally, $\chi_E^{-1}(1) = E$ and $\chi_E^{-1}(0) = E^c$, so χ_E is measurable if and only if $E, E^c \in \mathcal{M}$. □

Thm. 1.2.9 *Let $\mathcal{F} \subset \mathcal{P}(X)$, then there exists a smallest $\sigma-$algebra containing $\mathcal{F}$. This is denoted by $S(\mathcal{F})$, the $\sigma-$**algebra generated by** $\mathcal{F}$.*

Proof Let $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{–algebra}, \mathcal{F} \subset M\}$. Certainly $\Omega \neq \emptyset$ since $\mathcal{P}(X) \in \Omega$. Let $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$. We will see that $S(\mathcal{F})$ is a $\sigma-$algebra.
(i) Since $X \in \mathcal{M}$, it follows that $X \in \cap \mathcal{M}$.

(ii) If $A \in S(\mathcal{F})$, then $A \in \mathcal{M}$ for all $\mathcal{M}$. Thus $A^c \in \mathcal{M}$ for all $\mathcal{M}$ and $A^c \in \cap\mathcal{M}$.
(iii) In the same way, of $A_n \in S(\mathcal{F}$ for all n, then $A_n \in \mathcal{M}$ for all $n, \mathcal{M}$. Thus $\bigcup A_n \in \mathcal{M}$ for all $\mathcal{M}$ so $\bigcup A_n \in \mathcal{M} \in \bigcap \mathcal{M} = S(\mathcal{F})$.

By definition, $\mathcal{F} \subset \bigcap \mathcal{M}$. Finally, $S(\mathcal{F})$ is minimal, since if $\mathcal{F} \subset \mathcal{N}$ is a $\sigma-$algebra, then $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$, so we are done. □

Def'n. 1.2.10 *Let (X,τ) be a topological space. Then $\mathcal{B} = S(\tau)$ is called the **Borel $\sigma-$algebra**. Borel sets are the elements of $S(\tau)$. A function $f : X \to Y$ is Borel measurable if $f^{-1}(G) \in \mathcal{B}$ for all $G \subset Y$ open.*

Prop. 1.2.11 *1. If $F \subset X$ is closed, then $F \in \mathcal{B}$.*
2. $G_n \subset X$ are open, then $\bigcap_{n=1}^{\infty} G_n \in B$. These are called $G_\delta-$sets.
3. $F_n \subset X$ are closed, then $\bigcup_{n=1}^{\infty} F_n \in B$. These are called $F_\sigma-$sets.

Proof These follow directly from the definition of a $\sigma-$algebra. □

Ex. 1.2.12 $X = \mathbb{R}, \tau_e$, then $\mathcal{B} = S(\tau_e)$. Let $\Gamma_0 = \{(a,b) : a < b\}$ be a family of open intervals. We see that $S(\Gamma_0) = \mathcal{B}$. Since $\Gamma_0 \subset \tau$, $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$. Conversely, let $G \in \tau$, then we have open intervals $G = \bigcup_{n=1}^{\infty} I_n$ so that $G \in S(\Gamma_0)$. Thus $S(\tau) \subset S(\Gamma_0)$ and $S(\Gamma_0) = \beta$.

Ex. 1.2.13 Let $\Gamma_\infty = \{(a,\infty) : a \in \mathbb{R}\}$. I claim that $S(\Gamma_\infty) = \mathcal{B}$. Certainly $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$. Then $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$.Similarly, $(-\infty, a) = \bigcup_{n=1}^{\infty}(-\infty, a-1/n] \in S(\Gamma_\infty)$.Thus $(a,\infty) \cap (-\infty, b) = (a,b) \in S(\gamma_0)$, and using the previous example, $\mathcal{B} = S(\Gamma_\infty)$.

Prop. 1.2.14 *Let $(X,\mathcal{M})$ be a measurable space, and let $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the eucildean topology. If $f^{-1}((\alpha,\infty]) \in \mathcal{M}$ for any $\alpha \in \mathbb{R}$, then f is measurable.*

Proof Recall that f is measurable if its inverse image takes open sets to measurable sets.
We have $f^{-1}([-\infty,\alpha]) = (f^{-1}((\alpha,\infty])^c \in \mathcal{M}$. Similarly,

$$f^{-1}([-\infty,\alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty}[-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha,\beta) = f^{-1}([-\infty,\beta) \cap (\alpha,\infty]) = f^{-1}([-\infty,\beta)) \cap f^{-1}((\alpha,\infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for τ_e. Thus if $G \subset \overline{\mathbb{R}}$ is open, then there exists open intervals so that $G = \bigcup_{n=1}^{\infty} I_n$ and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired. □

1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

Def'n. 1.2.15 *Let* $(a_n)_{n\in\mathbb{N}} \subset \overline{R}$ *be a sequence, and* $b_k = \sup\{a_k, a_{k+1}, \ldots\}$. *Then* $\beta = \inf_{k\in\mathbb{N}} b_k$ *is called the* $\limsup$ *of* (a_n). *We can similarly define* $c_k = \inf\{a_k, a_{k+1}, \ldots\}$ *and* $\liminf = \sup_{k\in\mathbb{N}} c_k$.

Def'n. 1.2.16 *Let* $f_n : X \to \overline{\mathbb{R}}$ *be a sequence of functions. Then* $(\sup f_n) : X \to \overline{\mathbb{R}}$, $(\sup f_n)(x) = \sup f_n(x)$ *for all* $x \in X$. *Similarly,* $(\inf f_n) : X \to \overline{\mathbb{R}}$, $(\inf f_n)(x) = \inf f_n(x)$ *for all* $x \in X$. *Then* $(\liminf f_n)(x) = \liminf f_n(x)$. *If* $\lim f_n(x)$ *exists for all* x, *then we say* $(\lim f_n)(x) = \lim f_n(x)$.

Thm. 1.2.17 *Let* $f_n : X \to \overline{R}$ *be measurable. Then* $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ *are measurable.*

Proof Let $g = \sup f_n$. It is enough to prove that $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$ for all α. Let $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$. Let $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$. We show that $H = \bigcup_{n=1}^{\infty} H_n$.

First let $x \in H$, so $\sup f_n(x) > \alpha$. Thus get N so that $f_N(x) > \alpha$, so $x \in H_N$ and x is in the union. The converse is obvious.

Thus g is measureable. In the exact same way, $\inf f_n$ is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k\geq i} f_k$$

is measurable. □

Cor. 1.2.18 *If* $\lim f_n$ *exists, then it is measurable.*

Proof If $\lim f_n$ exists, then $\lim f_n = \limsup f_n$. □

Cor. 1.2.19 *If* f, g *are measurable, then* $\max\{f, g\}$, $\min\{f, g\}$ *are measurable.*

Cor. 1.2.20 *Let* f *be a function. Then* $f_+ = \max\{f, 0)\}$ *and* $f_- = -\min\{f, 0\}$ *(the positive and negative parts of* f*) are measurable. Similarly,* $|f| = f_+ + f_i$ *is measurable.*

1.2.3 Measures

Def'n. 1.2.21 *Let* $(X, \mathcal{M})$ *be a measurable space. A function* $\mu : \mathcal{M} \to [0, +\infty]$ *is called a* ***(positive) measure*** *if it is countably additive and not constant* $+\infty$. *In other words,*

1. $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ *if* $A_i \cap A_j = \emptyset$
2. $\exists A \in \mathcal{M}$ *so that* $\mu(A) < \infty$

$(X, \mathcal{M}, \mu)$ *is called a* ***measure space***.

Prop. 1.2.22 *1. $\mu(\emptyset) = 0$*

2. If $A_i \cap A_j = \emptyset$ then $\mu\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$

3. $A \subset B$ implies $\mu(A) \leq \mu(B)$

4. $A_1 \subset A_2 \subset A_3 \cdots$ then $\lim_{n\to\infty} \mu A_n = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

5. $A_1 \supset A_2 \supset A_3 \cdots$ and $\mu(A_i) < \infty$ then $|lim_{n\to\infty}\mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

Proof 1. Let $A \in \mathcal{M}$ so that $\mu(A) < \infty$, and fix $A_1 = A$, $A_2 = A_3 = \cdots = \emptyset$. Then $\bigcup A_n = A$ so $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$ so $\mu(\emptyset) = 0$.

2. Obvious

3. Note that $B = A \cup (B \setminus A)$ is a disjoint union.

4. Define $B_1 := A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then $B_i \cap B_j = \emptyset$ and $\mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$. Similary, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ Therefore, $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$.

5. Let $C_n = A_1 \setminus A_n$, $C_1 = \emptyset$. Then $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) + \mu(A_n) = \mu(A_1)$. Let $A = \bigcap_{n=1}^{\infty} A_n$ so $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$ and $(\bigcup C_n) \cup A = A_1$ is a disjoint union. But then $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$ so that

$$\mu(A_1) - \mu(A) = \mu(\bigcup C_n) = \lim_{n\to\infty} \mu(C_n) = \mu(A_n) - \lim \mu(A_n)$$

Since $\mu(A_1)$ is finite, we have $\mu(A) = \lim \mu(A_n)$. □

Ex. 1.2.23 Here are a few examples of measures that exist on arbitrary sets.

1. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$, and

$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$

It is easy to verify it is countably additive.

2. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$. Fix $x_0 \in X$. Then

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

1.3 Towards Integration

1.3.1 Simple Functions

Def'n. 1.3.1 *$s : X \to \mathbb{R}$ or $\mathbb{C}$ is called a simple function if its range is finite.*

Prop. 1.3.2 *Let s be a simple function, so that $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i = s^{-1}(\{\alpha_i\})$ and s is measurable if and only if $A_i \in \mathcal{M}$.*

Proof Obvious. □

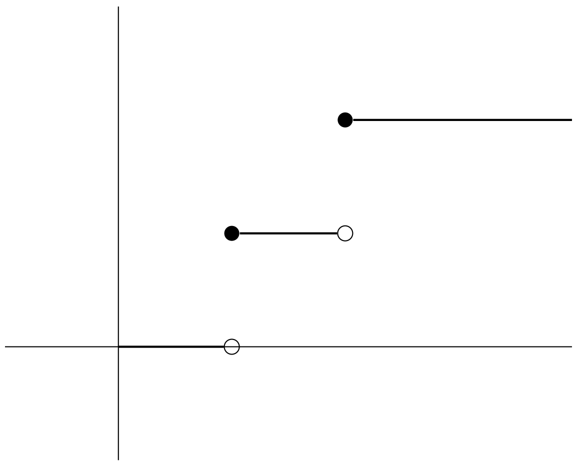
The following theorem is used later to define the intergral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function f.

Thm. 1.3.3 *Let $f : X \to [0, +\infty]$ be nonnegative measurable functions. Then there exists a sequence $s_n : X \to [0, +\infty]$ of simple measurable functions with*
1. *(s_n) is increasing and bounded above by f*
2. *$\lim s_n = f$ pointwise.*

Proof Let $n \in \mathbb{N}$, $t \geq 0$, and $k_n(t) = [2^n \cdot t]$ (i.e. $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn ϕ_1 below:



Then $t - 2^{-n} \leq \phi_n(t) \leq t$, $\lim \phi_n(t) = t$, and $\phi_n \leq \phi_{n+1}$. Define $s_n = \phi_n \circ f$, so for any $x \in X$, $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$. Note that s_n is simple since it has finite range (from ϕ_n), and $s_n \leq s_{n+1}$ because $\phi_n \leq \phi_{n+1}$, and $s_n \leq f$ since $\phi_n(t) \leq t$. Furthermore, ϕ_n is measurable since its level sets are intervals, so $\phi_n \circ f$ is measurable. □

1.3.2 Integration of Positive Functions

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Def'n. 1.3.4 *Let $S : X \to [0, +\infty)$ be a measurable simple function $s = \sum_{n=1}^{n} \alpha_i X_{A_i}$.Let $E \in \mathcal{M}$. Then define the **integral of** s over E be with respect to μ as*

$$\int_E s \, \mathrm{d}\mu = \sum_{n=1}^{n} \alpha_i \mu(A_i \cap E)$$

where we define $0 \cdot \infty = 0$.

Def'n. 1.3.5 *Let $f : X \to [0, +\infty]$ be a measurable function. Let $E \in \mathcal{M}$. Then the **(Lebesgue) integral** of f over E with respect to μ is*

$$\int_E f \, \mathrm{d}\mu = \sup\left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le f;\ s \textit{ is simple measurable}\right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only. Let $f, g : X \to [0, +\infty]$ be measurable functions. Let $E, A, B \in \mathcal{M}$.

1. If $f \le g$ then $\int_E f \, \mathrm{d}\mu$ and $\int_E g \, \mathrm{d}\mu$
2. If $A \subset B$, then $\int_A f \, \mathrm{d}\mu \le \int_b f \, \mathrm{d}\mu$
3. $\int_E c \cdot f \, \mathrm{d}\mu = c \cdot \int_E f \, \mathrm{d}\mu$ for all $c \ge 0$
4. If $f(x) = 0$ for all $x \in E$, then $\int_E f \, \mathrm{d}\mu = 0$
5. If $\mu(E) = 0$, then $\int_E f \, \mathrm{d}\mu = 0$
6. $\int_E f \, \mathrm{d}\mu = \int_X f \cdot \chi_E \, \mathrm{d}\mu$.

Proof 1. Note that

$$\left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le f\right\} \subset \left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le g\right\} Let$$

0$\le s \le f$ be simple measurable. Then

$$\int_A s \, \mathrm{d}\mu = \sum \alpha_i \mu(A \cap A_i) \le \sum \alpha_i \mu(B \cap A_i) = \int_B s \, \mathrm{d}mu$$

Take the supremum for all $0 \le s \le f$, then the result follows.

3. Let S be simple and measurable, so $s = \sum \alpha_i \chi_{A_i}$. Then

$$\int_E c \cdot s \, \mathrm{d}\mu = \sum_{i=1}^{n} \alpha_I \cdot c \cdot \mu(E \cap A_i) = c \cdot \sum \alpha_i \mu(E \cap A_i) = c \int_E s \, \mathrm{d}\mu$$

Thus

$$\begin{aligned}\int_E c\cdot f\,\mathrm{d}\mu &= \sup\left\{\int_E s\,\mathrm{d}\mu : 0\leq s\leq cf\right\}\\ &= \sup\left\{\int_E c\cdot t\,\mathrm{d}\mu : 0\leq t\leq f\right\}\\ &= c\cdot\sup\left\{\vert\int_E t\,\mathrm{d}\mu : 0\leq t\leq f\right\}\\ &= c\cdot\int_E f\,\mathrm{d}\mu\end{aligned}$$

4. If $0\leq s\leq f$, then $s=\sum\alpha_i\chi_{A_i}$. If $x\in A_i\cap E$, then $s(x)=\alpha_i$ and $\alpha_i=0$. Then $\alpha_i\mu(A_i\cap E)=0$ for all i: either $A_i\cap E=\emptyset$, or $A_i\cap E$ is not empty, and $\alpha_i=0$. This is true for any $0\leq s\leq f$, and taking supremums yields the result.

5. If $\mu(E)=0$ then $\mu(A_i\cap E)=0$, and $\int_E s\,\mathrm{d}\mu=\sum\alpha_i\mu(A_i\cap E)=0$ and taking supremums, the result holds.

6. Exercise. First prove if $0\leq s\leq f\cdot\chi_E$, then $\int_X s\,\mathrm{d}\{\mu\}=\int_E s\,\mathrm{d}\mu$. Then prove $\left\{\int_E s\,\mathrm{d}\mu : 0\leq s\leq f\cdot\chi_E\right\}=\left\{\vert\int_E s\,\mathrm{d}\mu : 0\leq s\leq f\right\}$. □

Prop. 1.3.6 *Let s be a simple and measurable, and define $\phi(E)=\int_e s\,\mathrm{d}\mu$ is a measure.*

Proof $\phi(\emptyset)=0$, so ϕ is not constant $+\infty$. Let $E=\bigcup_{n=1}^\infty E_n$ be a disjoint union. Then

$$\begin{aligned}\phi(E) &= \sum_{i=1}^m \alpha_i\mu(A_i\cap E)\\ &= \sum_{i=1}^m \alpha_i\mu\left(A_i\cap\left(\bigcup_{n=1}^\infty E_n\right)\right)\\ &= \sum_{i=1}^m \alpha_i\mu\left(\bigcup_{n=1}^\infty (A_i\cap E_n)\right)\\ &= \sum_{i=1}^m \alpha_i\sum_{n=1}^\infty \mu(A_i\cap E_n)\\ &= \sum_{n=1}^\infty\sum_{i=1}^m \alpha_i\mu(A_i\cap E_n)\\ &= \sum_{n=1}^\infty\int_{E_n} s\,\mathrm{d}\mu\\ &= \sum_{n=1}^\infty \phi(E_n)\end{aligned}$$

□

Prop. 1.3.7 *Let s,t be nonnegative, measurable simple functions. Then*

$$\int_X (s+t)\,\mathrm{d}\mu = \int_X s\,\mathrm{d}\mu + \int_X t\,\mathrm{d}\mu$$

Proof Write

$$s = \sum_{i=1}^{m} \alpha_i X_{A_i}, \qquad t = \sum_{j=1}^{n} \beta_j X_{\beta_j}$$

and let $E_{ij} = A_i \cap B_j$, so $X = \bigcup_{i,j} E_{ij}$ is a disjoint union. We now have

$$\int_{E_{ij}} (s+t)\,\mathrm{d}\mu = (\alpha_i + \beta_j)\mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s\,\mathrm{d}\mu + \int_{E_{ij}} t\,\mathrm{d}\mu$$

Let $\mu(E) = \int_E (s+t)\,\mathrm{d}\mu$, which is a measure as above. Thus

$$\begin{aligned}
\int_X (s+t)\,\mathrm{d}\mu = \phi(X) &= \phi\left(\bigcup_{i,j} E_{ij}\right) \\
&= \sum_{i,j} \phi(E_{ij}) = \sum_{i,j} \int_{E_{ij}} (s+t)\,\mathrm{d}\mu \\
&= \sum_{i,j} \left(\int_{E_{ij}} s\,\mathrm{d}\mu + \int_{E_{ij}} t\,\mathrm{d}\mu \right) \\
&= \sum_{i,j} \varphi(E_{ij}) + \sum_{i,j} \theta(E_{ij}) \\
&= \int_X s\,\mathrm{d}\mu + \int_X t\,\mathrm{d}\mu
\end{aligned}$$

where $\varphi(E) = \int_E s\,\mathrm{d}\mu$, $\theta(X) = \int_E t\,\mathrm{d}\mu$. □

1.3.3 Lebesgue's Monotone Convergence Theorem

Thm. 1.3.8 (Lebesgue's Monotone Convergence) *Let $f_n : X \to [0,+\infty]$ be measurable, such that*

(i) $0 \leq f_1 \leq f_2 \leq \cdots$

(ii) $f(x) := \lim_{n\to\infty} f_n(x)$ *for all* $x \in X$

Then f is measurable, and $\int_X f\,\mathrm{d}\mu = \lim \int_x f_n\,\mathrm{d}\mu$.

Proof It was already proven that f is measurable. We have $\int_X f_n\,\mathrm{d}\mu \leq \int_x f_{n+1}\,\mathrm{d}\mu$ for all n, so $\alpha := \lim_{n\to\infty} \int_X f_n\,\mathrm{d}\mu$ exists. We also have $f_n \leq f$, so $\int f_n \leq \int f$ and $\alpha \leq \int_X f_n\,\mathrm{d}\mu$. Thus we wish to show $\alpha \geq \int_x f\,\mathrm{d}\mu$. It suffices to prove that $\alpha \geq \int_x s\,\mathrm{d}\mu$ for any simple $s \leq f$. Let $c \in (0,1)$; it suffices to show that $\alpha \geq \int_x c \cdot s\,\mathrm{d}\mu$. Define $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\}$. We have $E_1 \subset E_2 \subset \cdots$ so that $\bigcup E_n = X$. Then

$$\int_X f_n\,\mathrm{d}\mu \geq \int_{E_n} f_n\,\mathrm{d}\mu \geq \int_{E_n} c \cdot s\,\mathrm{d}\mu$$

Let $\phi(E) = \int_E s \,\mathrm{d}\mu$, so $\int_{E_n} s \,\mathrm{d}\mu = \phi(E_n) \to \phi(\cup E_n) = \phi(X) = \int_X s \,\mathrm{d}\mu$. Thus

$$\alpha \geq c \cdot \lim_{n\to\infty} \phi(E_n) = c \cdot \int_X s \,\mathrm{d}\mu = \int_X cs \,\mathrm{d}\mu$$

as desired. □

Ex. 1.3.9 Consider the function consisting of a triangle with base $2/n$ and height n. Then $\int_0^1 f_n = 1$ as a Riemannian integral. However, $\lim f_n(x) = 0$ for any x, so $\int_0^1 f = 0 \neq 1 = \lim \int_0^1 f_n$.

Thm. 1.3.10 *Let $f, g : X \to [0, +\infty]$ measurable, then $\int_X (f + g) \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu$.*

Proof We proved that there exists increasing sequences of simple functions s_n, t_n such that $\lim s_n(x) = f(x)$, $\lim t_n(x) = g(x)$. Then $s_n(x) + t_n(x) \to f(x) + g(x)$ monotonically. But then

$$\begin{aligned}
\int_X (f+g) \,\mathrm{d}\mu &= \int_X \lim_{n\to\infty} (s_n + t_n) \,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \int_X (s_n + t_n) \,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \left(\int_X s_n \,\mathrm{d}\mu + \int_X t_n \,\mathrm{d}\mu \right) \\
&= \int_X \lim_{n\to\infty} s_n \,\mathrm{d}\mu + \int_X \lim_{n\to\infty} t_n \,\mathrm{d}\mu \\
&= \int_X f \,\mathrm{d}\mu + \int_X g(\mathrm{d}\mu)
\end{aligned}$$

□

Cor. 1.3.11 *If $f_n : X \to [0, +\infty]$ is a sequence of measurable functions, then*

$$\sum_{n=1}^{\infty} \int_X f_n \,\mathrm{d}\mu = \int_x \sum_{n=1}^{\infty} f_n \,\mathrm{d}\mu$$