# Course Notes

## Real Functions and Measures

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# Contents

# Chapter 1

# Basics of Abstract Measure Theory

Prof contact: simonp@caesar.elte.hu Grading: HW each week for 25% Midterm 30% Final 45%

## 1.1 Review of Topology

### 1.1.1 Basic Definitions

**Def'n. 1.1.1** *Let $X \neq \emptyset$ and $\tau \subseteq \mathcal{P}(X)$. We say that $(X, \tau)$ is a **topological space** if $\tau$ satisfies the following conditions:*

1. $\emptyset \in \tau$ $X \in \tau$
2. $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3. $V_\alpha \in \tau$ *for all* $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

*We call the elements of $\tau$ **open sets**.*

**Def'n. 1.1.2** *$U \subseteq X$ is a **neighbourhood** of $x \in X$ if there is some $G \in \tau$ such that $x \in G \subset U$.*

**Def'n. 1.1.3** *$F \subseteq X$ is **closed** if $F^c$ is open.*

**Def'n. 1.1.4** *The **closure** of a set $E \subset X$ is the smallest closed set containing $E$ (denoted $\overline{E}$).*

**Def'n. 1.1.5** *$x$ is an **accumulation point** of $H$ if all neighbourhoods of $x$ contains infinitely points of $H$. Equivalently, $x$ is a limit point of $H \setminus \{x\}$.*

**Def'n. 1.1.6** *If $H \subseteq X$, we have a natural subspace topology $\tau|_H = \{G \cap H : G \in \tau\}$.*

### 1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1. $\mathbb{R}$ along with the open sets (denoted $\tau_e$, the Euclidean topology).
2. The discrete topology, $\tau = \mathcal{P}(X)$ for any $X \neq \emptyset$. This is the "finest" topology.

3. The antidiscrete topology, $\tau = \{\emptyset, X\}$ for any $X \neq \emptyset$ This is the "coarsest" topology.
4. One can define the extended real line, $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

   The same can be done with a single symbol as well. In either case, the extended real line is a compact set.
5. Any metric spaces induces a topology. Consider a set $X \neq 0$ arbitrary, and let $d : X \times X \to \mathbb{R}$ such that
   (a) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$.
   (b) $d(x,y) = d(y,x)$ for all $x, y \in X$
   (c) $d(x,y) \leq d(x,z) + d(z,y)$ for any $x, y, z \in X$

   Then $G \in \tau$ if and only if for any $x \in G$, there exists $r$ so that $B_r(x) \subset G$. There are many examples of metric spaces:
   (a) $X = \mathbb{R}$, $d(x,y) = |x-y|$
   (b) $X = \mathbb{R}$, $d(x,y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
   (c) $X = \mathbb{R}^2$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$
   (d) $X = \mathbb{R}^2$, $d(x,y) = (|x_1-y_1|^p + |x_2-y_2|^p)^{1/p}$ for $p \geq 1$.
   (e) and similarly for $X = \mathbb{R}^n$
   (f) $X = C[0,1]$, $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$.
   (g) normed space: $X$ is a vector space over $\mathbb{R}$, $\|\cdot\| : X \to \mathbb{R}$ such that
      i. $\|x\| = 0$ if and only if $X = 0$
      ii. $\|cx\| = |c| \|x\|$
      iii. $\left\|x+y\right\| \leq \|x\| + \left\|y\right\|$

      If $\|\cdot\|$ is a norm, then $d(x,y) = \left\|x-y\right\|$ is a metric.
6. The cofinite topology: $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$.

### 1.1.3 Other Definitions

**Def'n. 1.1.7** *$K \subset X$ is **compact** if every open cover of $K$ contains a finite subcover.*

**Def'n. 1.1.8** *A topological space is called **locally compact** if every point has a compact neighbourhood.*

**Prop. 1.1.9** $C[0,1]$ *with the sup norm is not locally compact.*

Proof I'll do this later. ∎

**Def'n. 1.1.10** *A topological space is called* ***Hausdorff*** *if for any $x \neq y$, there exists neighbourhoods $U \ni x$, $V \ni y$ so that $U \cap V = \emptyset$.*

The anti-discrete topology is not Hausdorff.

1. On the discrete topology, $K$ is compact if and only if $K$ is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of $X$).
3. On $(\mathbb{R}, \tau_e)$, $K$ is compact if and only if $K$ is closed and bounded.
4. On $(X, d)$ metric space, $K$ is compact if and only if $K$ is complete and totally bounded.

**Prop. 1.1.11**
1. *Let $K \subset X$ be compact, let $F \subset K$ closed. Then $F$ is also compact.*
2. *Compact sets in a Hausdorff space are closed.*

Proof
1. Let $F \subset \bigcup V_\alpha$. Then $K \subset F^c \cup (\bigcup V_\alpha)$ is an open cover for $K$, so it has a finite subcover $F^c \cup V_{\alpha_1} \cup \cdots V_{\alpha_n}$. But then since $F \cap F^c = \emptyset$, $F \subset V_{\alpha_1} \cup \cdots V_{\alpha_n}$ is a finite subcover.
2. Let $K \subset X$ be compact, and prove that $K^c$ is open. Thus let $x \in K^c$. For any $y \in K$, there exist $U_y, V_y$ disjoint neighbourhoods of $x$ and $y$ respectively. Now consider the open cover $K \subset \bigcup_{y \in K} V_y$, and get our finite subcover $K \subset V_{y_1} \cup \cdots \cup V_{y_n}$. But then $U_{y_1} \cap \cdots \cap U_{y_n} \cap K = \emptyset$ and is open since it is a finite intersection. ■

**Def'n. 1.1.12** $\Gamma \subseteq \tau$ *is a* ***base*** *for $\tau$ if every $U \in \tau$ can be written as a countable union of the elements of $\Gamma$. $\Gamma$ is a* ***countable base*** *if $\Gamma$ is countable.*

**Prop. 1.1.13** $\mathbb{R}$ *has a countable base of intervals.*

Proof Consider the collection $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$. To see this, for any open set $U$, one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left( \bigcup_{\{r : B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$ is obvious, so let $x \in U$ be arbitrary, and let $s$ be maximal so that $B_s(x) \subseteq U$. Then choose $q \in \mathbb{Q}$ so that $|x - q| < s/3$ and $r \in \mathbb{Q}$ so that $0 < r < s/2$. Then by construction $B_r(q) \ni x$ and by the triangle inequality $B_{r/2}(q) \subseteq U$, so $x \in S$. Thus $U = S$ as desired. ■

Note that the exact same argument (with some work) can be generalized to show that $\mathbb{R}^n$ has a countable base of open hyperrectangles.

### 1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

**Def'n. 1.1.14** *Let $(x_n) \subset X$ be a sequence and let $x \in X$. Then $x$ is the* ***limit*** *of $(x_n)$ if for any neighbourhood $U$ of $X$, there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.*

**Prop. 1.1.15** *If $F \subset X$ is closed, then for all convergent sequences in $F$, the limit is also in $F$.*

Proof See Homework. ■

**Def'n. 1.1.16** *Let $f : X \to Y$ be a function, and $x \in X$ an accumulation point of $D(f)$. The limit of $f$ at $x$ is $y \in Y$ if for any neighbourhood $V$ of $y$ there exists a neighbourhood $U$ of $x$ such that $f(U \cap D(f) \setminus \{x\}) \subseteq V$.*

**Def'n. 1.1.17** *Let $f : X \to Y$ be a function, and let $x \in D(f)$. Then $f$ is* ***continuous at*** *$x$ if for any neighbourhood $V$ of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of $x$.*

**Def'n. 1.1.18** *$f : X \to Y$ is called* ***continuous*** *if it is continuous at every point.*

**Prop. 1.1.19** *$f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is open for all $G$ open.*

Proof Exercise. ■

**Thm. 1.1.20** *Let $f : X \to Y$ be continuous and $K \subset X$ be compact. Then $f(K)$ is compact.*

Proof Recall that continuous functions pull back open sets. Let $f(K) \subset \bigcup U_\alpha$ be an open cover. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover for $K$, and has a finite subcover $U_{\alpha_1} \cup \cdots U_{\alpha_n}$. But then $f(f^{-1}(U_{\alpha_1})) \cup \cdots \cup f(f^{-1}(U_{\alpha_n}))$ is a subcover of $f(K)$. ■

## 1.2 Measure Theory

### 1.2.1 $\sigma-$algebras

**Def'n. 1.2.1** *Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a $\sigma-$**algebra** if*

1. *$X \in \mathcal{M}$*
2. *$A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$*
3. *If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup\limits_{n \in \mathbb{N}} A_n \in \mathcal{M}$*

*The pair $(X, \mathcal{M})$ is called a* ***measurable space****. The elements of $\mathcal{M}$ are called* ***measurable sets****.*

**Def'n. 1.2.2** *Let $(X, \mathcal{M})$ be a measurable space, $(Y, \tau)$ be a topological space. Then $f : X \to Y$ is called* ***measurable*** *if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.*

Here are some simple examples of $\sigma-$algebras.

**Ex. 1.2.3**
1. $\mathcal{M} = \{\emptyset, X\}$ is a $\sigma-$algebra.
2. $\mathcal{P}(X) = \mathcal{M}$ is a $\sigma-$algebra.
3. $\mathcal{M} = \{A \subset X : A$ or $A^c$ is countable.$\}$. To see this, given $A_n \in \mathcal{M}$, if everything is countable, then $\bigcup A_n$ is countable. If some $A_i$ is countable, then $(\bigcup A_n)^c = \bigcap A_n^c$ is countable, so $\bigcup A_n \in \mathcal{M}$.

We will later see some proper exaples, like the $\sigma-$algebra of Lebesgue measurable sets.

We have the following properties of $\sigma-$algebras.

**Prop. 1.2.4**
1. *$\emptyset \in \mathcal{M}$*
2. *$A_1, A_2, \ldots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{M}$*
3. *$A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ then $\bigcap_{n=1}^\infty A_n \in \mathcal{M}$*

4. *$A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$*
5. *$f$ is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$.*

Proof
1. $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.
2. We can extend this to a countable union by introduction $A_{n+i} = \emptyset$ for $i \in \mathbb{N}$.
3. By DeMorgan's identities, $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.
4. $A \setminus B = A \cap B^c \in \mathcal{M}$.
5. $H^c$ is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$. ∎

**Prop. 1.2.5** *Let $f : X \to Y$ be measurable, let $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is measurable.*

Proof Let $V \subset Z$ be open, so $g^{-1}(V) \subset Y$ is open, so $f^{-1}(g^{-1}(V)) \in \mathcal{M}$ which is $(g \circ f)^{-1}(V)$. ∎

**Prop. 1.2.6** *Let $(X, \mathcal{M})$ be a measurable space, $Y$ be a topological space. Let $\phi : \mathbb{R}^2 \to Y$ be continuous. If $u, v : X \to \mathbb{R}$ are measurable, then $h(x) = \phi(u(x), v(x))$ is measurable.*

Proof Define $f : X \to \mathbb{R}^2$ by $f(x) = (u(x), v(x))$ We will see that $f$ is measurable, so that $h = \phi \circ f$ is measurable since $\phi$ is continuous. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals, so $R = I_1 \times I_2$ is an open rectangle. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Let $G \subset \mathbb{R}^2$ be an open set, so there exist $R_n$ open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that $f$ is measurable. ∎

**Cor. 1.2.7**
1. *If $u, v : X \to \mathbb{R}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.*
2. *$u + iv : X \to \mathbb{C}$ is measurable.*
3. *$f : X \to \mathbb{C}$ is measurable, $f = u + iv \Rightarrow u, v, |f|$ are measurable.*

**Prop. 1.2.8** *Define*

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

*Then $\chi_E$ is measurable if and only if $E \in \mathcal{M}$.*

Proof Naturally, $\chi_E^{-1}(1) = E$ and $\chi_E^{-1}(0) = E^c$, so $\chi_E$ is measurable if and only if $E, E^c \in \mathcal{M}$. ∎

**Thm. 1.2.9** *Let $\mathcal{F} \subset \mathcal{P}(X)$, then there exists a smallest $\sigma$-algebra containing $\mathcal{F}$. This is denoted by $S(\mathcal{F})$, the $\sigma$-**algebra generated by** $\mathcal{F}$.*

Proof Let $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra}, \mathcal{F} \subset M\}$. Certainly $\Omega \neq \emptyset$ since $\mathcal{P}(X) \in \Omega$. Let $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$. We will see that $S(\mathcal{F})$ is a $\sigma$-algebra.

(i) Since $X \in \mathcal{M}$, it follows that $X \in \cap \mathcal{M}$.
(ii) If $A \in S(\mathcal{F})$, then $A \in \mathcal{M}$ for all $\mathcal{M}$. Thus $A^c \in \mathcal{M}$ for all $\mathcal{M}$ and $A^c \in \cap \mathcal{M}$.
(iii) In the same way, of $A_n \in S(\mathcal{F}$ for all $n$, then $A_n \in \mathcal{M}$ for all $n, \mathcal{M}$. Thus $\bigcup A_n \in \mathcal{M}$ for all $\mathcal{M}$ so $\bigcup A_n \in \mathcal{M} \in \bigcap \mathcal{M} = S(\mathcal{F})$.

By definition, $\mathcal{F} \subset \bigcap \mathcal{M}$. Finally, $S(\mathcal{F})$ is minimal, since if $\mathcal{F} \subset \mathcal{N}$ is a $\sigma-$algebra, then $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$, so we are done. ■

**Def'n. 1.2.10** *Let $(X, \tau)$ be a topological space. Then $\mathcal{B} = S(\tau)$ is called the **Borel $\sigma$−algebra**. Borel sets are the elements of $S(\tau)$. A function $f : X \to Y$ is Borel measurable if $f^{-1}(G) \in \mathcal{B}$ for all $G \subset Y$ open.*

**Prop. 1.2.11**
1. *If $F \subset X$ is closed, then $F \in \mathcal{B}$.*
2. *$G_n \subset X$ are open, then $\bigcap_{n=1}^{\infty} G_n \in B$. These are called $G_\delta$−sets.*
3. *$F_n \subset X$ are closed, then $\bigcup_{n=1}^{\infty} F_n \in B$. These are called $F_\sigma$−sets.*

Proof These follow directly from the definition of a $\sigma-$algebra. ■

**Ex. 1.2.12** $X = \mathbb{R}, \tau_e$, then $\mathcal{B} = S(\tau_e)$. Let $\Gamma_0 = \{(a, b) : a < b\}$ be a family of open intervals. We see that $S(\Gamma_0) = \mathcal{B}$. Since $\Gamma_0 \subset \tau$, $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$. Conversely, let $G \in \tau$, then we have open intervals $G = \bigcup_{n=1}^{\infty} I_n$ so that $G \in S(\Gamma_0)$. Thus $S(\tau) \subset S(\Gamma_0)$ and $S(\Gamma_0) = \beta$.

**Ex. 1.2.13** Let $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$. I claim that $S(\Gamma_\infty) = \mathcal{B}$. Certainly $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$. Then $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$.Similarly, $(-\infty, a) = \bigcup_{n=1}^{\infty}(-\infty, a-1/n] \in S(\Gamma_\infty)$.Thus $(a, \infty) \cap (-\infty, b) = (a, b) \in S(\gamma_0)$, and using the previous example, $\mathcal{B} = S(\Gamma_\infty)$.

**Prop. 1.2.14** *Let $(X, \mathcal{M})$ be a measurable space, and let $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the eucildean topology. If $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for any $\alpha \in \mathbb{R}$, then $f$ is measurable.*

Proof Recall that $f$ is measurable if its inverse image takes open sets to measurable sets.
We have $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty])^c \in \mathcal{M}$. Similarly,

$$f^{-1}([-\infty, \alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty}[-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha, \beta) = f^{-1}([-\infty, \beta) \cap (\alpha, \infty]) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for $\tau_e$. Thus if $G \subset \overline{\mathbb{R}}$ is open, then there exists open intervals so that $G = \bigcup_{n=1}^{\infty} I_n$ and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired. ■

### 1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

**Def'n. 1.2.15** *Let $(a_n)_{n\in\mathbb{N}} \subset \overline{R}$ be a sequence, and $b_k = \sup\{a_k, a_{k+1}, \ldots\}$. Then $\beta = \inf_{k\in\mathbb{N}} b_k$ is called the* $\limsup$ *of $(a_n)$. We can similarly define $c_k = \inf\{a_k, a_{k+1}, \ldots\}$ and $\liminf = \sup_{k\in\mathbb{N}} c_k$.*

**Def'n. 1.2.16** *Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of functions. Then $(\sup f_n) : X \to \overline{\mathbb{R}}$, $(\sup f_n)(x) = \sup f_n(x)$ for all $x \in X$. Similarly, $(\inf f_n) : X \to \overline{\mathbb{R}}$, $(\inf f_n)(x) = \inf f_n(x)$ for all $x \in X$. Then $(\liminf f_n)(x) = \liminf f_n(x)$. If $\lim f_n(x)$ exists for all $x$, then we say $(\lim f_n)(x) = \lim f_n(x)$.*

**Thm. 1.2.17** *Let $f_n : X \to \overline{R}$ be measurable. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ are measurable.*

Proof Let $g = \sup f_n$. It is enough to prove that $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$ for all $\alpha$. Let $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$. Let $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$. We show that $H = \bigcup_{n=1}^{\infty} H_n$.

First let $x \in H$, so $\sup f_n(x) > \alpha$. Thus get $N$ so that $f_N(x) > \alpha$, so $x \in H_N$ and $x$ is in the union. The converse is obvious.

Thus $g$ is measureable. In the exact same way, $\inf f_n$ is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k\geq i} f_k$$

is measurable. ■

**Cor. 1.2.18** *If $\lim f_n$ exists, then it is measurable.*

Proof If $\lim f_n$ exists, then $\lim f_n = \limsup f_n$. ■

**Cor. 1.2.19** *If $f, g$ are measurable, then $\max\{f, g\}$, $\min\{f, g\}$ are measurable.*

**Cor. 1.2.20** *Let $f$ be a function. Then $f_+ = \max\{f, 0)\}$ and $f_- = -\min\{f, 0\}$ (the positive and negative parts of $f$) are measurable. Similarly, $|f| = f_+ + f_i$ is measurable.*

### 1.2.3 Measures

**Def'n. 1.2.21** *Let $(X, \mathcal{M})$ be a measurable space. A function $\mu : \mathcal{M} \to [0, +\infty]$ is called a* ***(positive) measure*** *if it is countably additive and not constant $+\infty$. In other words,*

1. $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ *if* $A_i \cap A_j = \emptyset$
2. $\exists A \in \mathcal{M}$ *so that* $\mu(A) < \infty$

*$(X, \mathcal{M}, \mu)$ is called a* ***measure space****.*

**Prop. 1.2.22**
1. $\mu(\emptyset) = 0$
2. *If* $A_i \cap A_j = \emptyset$ *then* $\mu\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$
3. $A \subset B$ *implies* $\mu(A) \leq \mu(B)$
4. $A_1 \subset A_2 \subset A_3 \cdots$ *then* $\lim_{n\to\infty} \mu A_n = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

5. *$A_1 \supset A_2 \supset A_3 \cdots$ and $\mu(A_i) < \infty$ then* $|lim_{n\to\infty}\mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

Proof 1. Let $A \in \mathcal{M}$ so that $\mu(A) < \infty$, and fix $A_1 = A$, $A_2 = A_3 = \cdots = \emptyset$. Then $\bigcup A_n = A$ so $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$ so $\mu(\emptyset) = 0$.

2. Obvious

3. Note that $B = A \cup (B \setminus A)$ is a disjoint union.

4. Define $B_1 := A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then $B_i \cap B_j = \emptyset$ and $\mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$. Similary, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ Therefore, $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$.

5. Let $C_n = A_1 \setminus A_n$, $C_1 = \emptyset$. Then $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) + \mu(A_n) = \mu(A_1)$. Let $A = \bigcap_{n=1}^{\infty} A_n$ so $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$ and $(\bigcup C_n) \cup A = A_1$ is a disjoint union. But then $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$ so that
$$\mu(A_1) - \mu(A) = \mu(\bigcup C_n) = \lim_{n\to\infty} \mu(C_n) = \mu(A_n) - \lim \mu(A_n)$$
Since $\mu(A_1)$ is finite, we have $\mu(A) = \lim \mu(A_n)$. ■

**Ex. 1.2.23** Here are a few examples of measures that exist on arbitrary sets.

1. $X$ arbitrary, $\mathcal{M} = \mathcal{P}(X)$, and
$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$
It is easy to verify it is countably additive.

2. $X$ arbitrary, $\mathcal{M} = \mathcal{P}(X)$. Fix $x_0 \in X$. Then
$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

## 1.3 Towards Integration

### 1.3.1 Simple Functions

**Def'n. 1.3.1** *$s : X \to \mathbb{R}$ or $\mathbb{C}$ is called a* ***simple function*** *if its range is finite.*

**Prop. 1.3.2** *Let $s$ be a simple function, so that $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i = s^{-1}(\{\alpha_i\})$ and $s$ is* ***measurable*** *if and only if $A_i \in \mathcal{M}$.*

Proof Obvious. ■

The following theorem is used later to define the intergral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function $f$.

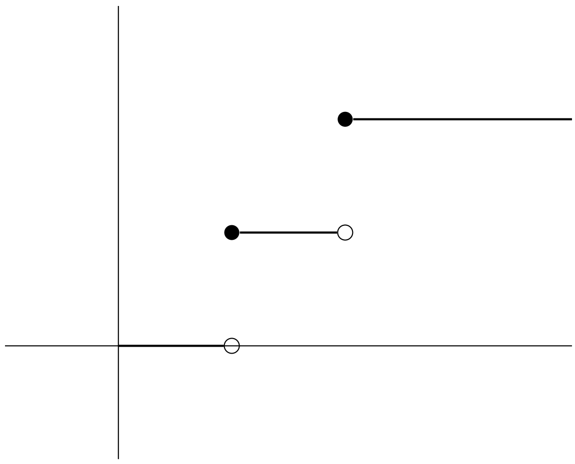
**Thm. 1.3.3** *Let $f : X \to [0, +\infty]$ be nonnegative measurable functions. Then there exists a sequence $s_n : X \to [0, +\infty]$ of simple measurable functions with*

1. *$(s_n)$ is increasing and bounded above by $f$*
2. *$\lim s_n = f$ pointwise.*

Proof Let $n \in \mathbb{N}$, $t \geq 0$, and define $k_n(t) = [2^n \cdot t]$ (i.e. $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn $\phi_1$ below:



Then $t - 2^{-n} \leq \phi_n(t) \leq t$, $\lim \phi_n(t) = t$ uniformly, and $\phi_n \leq \phi_{n+1}$, so the sequence of functions is monotone. Define $s_n = \phi_n \circ f$, so for any $x \in X$, $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$. Note that $s_n$ is simple since it has finite range (from $\phi_n$), and $s_n \leq s_{n+1}$ because $\phi_n \leq \phi_{n+1}$, and $s_n \leq f$ since $\phi_n(t) \leq t$. Furthermore, $\phi_n$ is measurable since its level sets are intervals, so $s_n = \phi_n \circ f$ is measurable. ■

### 1.3.2 Integration of Positive Functions

**Def'n. 1.3.4** *Let $s : X \to [0, +\infty)$ be a measurable simple function $s = \sum_{n=1}^{N} \alpha_i X_{A_i}$. Let $E \in \mathcal{M}$. Then define the* ***integral*** *of $s$ over $E$ with respect to $\mu$ as*

$$\int_E s \, \mathrm{d}\mu = \sum_{n=1}^{N} \alpha_i \mu(A_i \cap E)$$

*where we define $0 \cdot \infty = 0$.*

**Def'n. 1.3.5** *Let $f : X \to [0,+\infty]$ be a measurable function. Let $E \in \mathcal{M}$. Then the* ***(Lebesgue) integral*** *of $f$ over $E$ with respect to $\mu$ is*

$$\int_E f \,\mathrm{d}\mu = \sup\left\{\int_E s\,\mathrm{d}\mu : 0 \le s \le f;\ s \text{ is simple measurable}\right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only.

**Prop. 1.3.6** *Let $f,g : X \to [0,+\infty]$ be measurable functions. Let $E,A,B \in \mathcal{M}$.*

1. *If $f \le g$ then $\int_E f\,\mathrm{d}\mu \le \int_E g\,\mathrm{d}\mu$*
2. *If $A \subset B$, then $\int_A f\,\mathrm{d}\mu \le \int_B f\,\mathrm{d}\mu$*
3. *$\int_E c\cdot f\,\mathrm{d}\mu = c\cdot\int_E f\,\mathrm{d}\mu$ for all $c \ge 0$*
4. *If $f(x) = 0$ for all $x \in E$, then $\int_E f\,\mathrm{d}\mu = 0$*
5. *If $\mu(E) = 0$, then $\int_E f\,\mathrm{d}\mu = 0$*
6. *$\int_E f\,\mathrm{d}\mu = \int_X f\cdot\chi_E\,\mathrm{d}\mu$.*

Proof

1. This follows directly since

$$\left\{\int_E s\,\mathrm{d}\mu : 0 \le s \le f\right\} \subset \left\{\int_E s\,\mathrm{d}\mu : 0 \le s \le g\right\}$$

2. Let $0 \le s \le f$ be simple measurable. Then

$$\int_A s\,\mathrm{d}\mu = \sum \alpha_i\mu(A\cap A_i) \le \sum \alpha_i\mu(B\cap A_i) = \int_B s\,\mathrm{d}\mu$$

Take the supremum for all $0 \le s \le f$, then the result follows.

3. Let $S$ be simple and measurable, so $s = \sum \alpha_i\chi_{A_i}$. Then

$$\int_E c\cdot s\,\mathrm{d}\mu = \sum_{i=1}^{n} \alpha_I\cdot c\cdot\mu(E\cap A_i) = c\cdot\sum \alpha_i\mu(E\cap A_i) = c\int_E s\,\mathrm{d}\mu$$

Thus

$$\begin{aligned}\int_E c\cdot f\,\mathrm{d}\mu &= \sup\left\{\int_E s\,\mathrm{d}\mu : 0 \le s \le cf\right\}\\ &= \sup\left\{\int_E c\cdot t\,\mathrm{d}\mu : 0 \le t \le f\right\}\\ &= c\cdot\sup\left\{\int_E t\,\mathrm{d}\mu : 0 \le t \le f\right\}\\ &= c\cdot\int_E f\,\mathrm{d}\mu\end{aligned}$$

4. If $0 \le s \le f$, then $s = \sum \alpha_i\chi_{A_i}$. If $x \in A_i\cap E$, then $s(x) = \alpha_i$ and $\alpha_i = 0$. Then $\alpha_i\mu(A_i\cap E) = 0$ for all $i$: either $A_i\cap E = \emptyset$, or $A_i\cap E$ is not empty, and $\alpha_i = 0$. This is true for any $0 \le s \le f$, and taking supremums yields the result.

5. If $\mu(E) = 0$ then $\mu(A_i \cap E) = 0$, and $\int_E s \, \mathrm{d}\mu = \sum \alpha_i \mu(A_i \cap E) = 0$ and taking supremums, the result holds.

6. Exercise. First prove if $0 \leq s \leq f \cdot \chi_E$, then $\int_X s \, \mathrm{d}\mu = \int_E s \, \mathrm{d}\mu$. Then prove

$$\left\{ \int_E s \, \mathrm{d}\mu : 0 \leq s \leq f \cdot \chi_E \right\} = \left\{ \int_E s \, \mathrm{d}\mu : 0 \leq s \leq f \right\} \qquad \blacksquare$$

**Prop. 1.3.7** *Let $s$ be a simple and measurable. Then $\phi(E) = \int_E s \, \mathrm{d}\mu$ is a measure.*

Proof $\phi(\emptyset) = 0$, so $\phi$ is not constant $+\infty$. Let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union. Then

$$\begin{aligned} \phi(E) &= \sum_{i=1}^{m} \alpha_i \mu(A_i \cap E) \\ &= \sum_{i=1}^{m} \alpha_i \mu\left( A_i \cap \left( \bigcup_{n=1}^{\infty} E_n \right) \right) = \sum_{i=1}^{m} \alpha_i \mu\left( \bigcup_{n=1}^{\infty} (A_i \cap E_n) \right) \\ &= \sum_{i=1}^{m} \alpha_i \sum_{n=1}^{\infty} \mu(A_i \cap E_n) = \sum_{n=1}^{\infty} \sum_{i=1}^{m} \alpha_i \mu(A_i \cap E_n) \\ &= \sum_{n=1}^{\infty} \int_{E_n} s \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \phi(E_n) \end{aligned} \qquad \blacksquare$$

**Prop. 1.3.8** *Let $s, t$ be nonnegative, measurable simple functions. Then*

$$\int_X (s + t) \, \mathrm{d}\mu = \int_X s \, \mathrm{d}\mu + \int_X t \, \mathrm{d}\mu$$

Proof Write

$$s = \sum_{i=1}^{m} \alpha_i \chi_{A_i}, \quad t = \sum_{j=1}^{n} \beta_j \chi_{\beta_j}$$

and let $E_{ij} = A_i \cap B_j$, so $X = \bigcup_{i,j} E_{ij}$ is a disjoint union. We now have

$$\int_{E_{ij}} (s + t) \, \mathrm{d}\mu = (\alpha_i + \beta_j) \mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s \, \mathrm{d}\mu + \int_{E_{ij}} t \, \mathrm{d}\mu$$

Let $\phi(E)=\int_E(s+t)\,\mathrm{d}\mu$, which is a measure as above. Thus

$$\begin{aligned}\int_X(s+t)\,\mathrm{d}\mu=\phi(X)=\phi\left(\bigcup_{i,j}E_{ij}\right)\\ &=\sum_{i,j}\phi(E_{ij})=\sum_{i,j}\int_{E_{ij}}(s+t)\,\mathrm{d}\mu\\ &=\sum_{i,j}\left(\int_{E_{ij}}s\,\mathrm{d}\mu+\int_{E_{ij}}t\,\mathrm{d}\mu\right)\\ &=\sum_{i,j}\varphi(E_{ij})+\sum_{i,j}\theta(E_{ij})\\ &=\int_X s\,\mathrm{d}\mu+\int_X t\,\mathrm{d}\mu\end{aligned}$$

where $\varphi(E)=\int_E s\,\mathrm{d}\mu$, $\theta(X)=\int_E t\,\mathrm{d}\mu$. ■

To extend this result to general functions, we need some stronger machinery. In particular, we have the first of many convergence theorems.

## 1.4 Lebesgue's Monotone Convergence Theorem

**Thm. 1.4.1 (Lebesgue's Monotone Convergence)** *Let $f_n : X \to [0,+\infty]$ be measurable, such that*

*(i)* $0\leq f_1\leq f_2\leq\cdots$

*(ii)* $f(x):=\lim_{n\to\infty}f_n(x)$ *for all* $x\in X$

*Then $f$ is measurable, and $\int_X f\,\mathrm{d}\mu=\lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu$.*

Proof It was already proven that $f$ is measurable. We have $\int_X f_n\,\mathrm{d}\mu\leq\int_X f_{n+1}\,\mathrm{d}\mu$ for all $n$, so $\alpha:=\lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu$ exists. We also have $f_n\leq f$, so $\int_X f_n\,\mathrm{d}\mu\leq\int_X f\,\mathrm{d}\mu$ and $\alpha\leq\int_X f_n\,\mathrm{d}\mu$. Thus we wish to show $\alpha\geq\int_X f\,\mathrm{d}\mu$. It suffices to prove that $\alpha\geq\int_X s\,\mathrm{d}\mu$ for any simple $s\leq f$. Furthermore, if $c\in(0,1)$, it suffices to show that $\alpha\geq\int_X c\cdot s\,\mathrm{d}\mu$.

Define $E_n=\{x\in X : f_n(x)\geq c\cdot s(x)\}$. We have $E_1\subset E_2\subset\cdots$ so that $\bigcup\limits_{n=1}^{\infty}E_n=X$. Then

$$\int_X f_n\,\mathrm{d}\mu\geq\int_{E_n}f_n\,\mathrm{d}\mu\geq\int_{E_n}c\cdot s\,\mathrm{d}\mu$$

Let $\phi(E)=\int_E s\,\mathrm{d}\mu$, so $\int_{E_n}s\,\mathrm{d}\mu=\phi(E_n)$. Thus $\lim_{n\to\infty}\phi(E_n)=\phi(X)=\int_X s\,\mathrm{d}\mu$. Thus

$$\alpha\geq c\cdot\lim_{n\to\infty}\phi(E_n)=c\cdot\int_X s\,\mathrm{d}\mu=\int_X c\cdot s\,\mathrm{d}\mu$$

as desired. ■

Note that in the proof, choosing $c\in(0,1)$ was important since it lets us define the $E_n$ so that $\bigcup_{n=1}^{\infty}E_n=X$ - this would not work by defining $E_n=\{x\in X : f_n(x)\geq s(x)\}$.

### 1.4.1 Applications of Monotone Convergence

Now, as advertised, we can prove linearity of the integral for general measurable functions using the Monotone Convergence Theorem.

**Thm. 1.4.2** *Let $f, g : X \to [0, +\infty]$ measurable, then $\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu$.*

Proof We proved that there exists increasing sequences of simple functions $s_n, t_n$ such that $\lim s_n(x) = f(x)$, $\lim t_n(x) = g(x)$ monotonically. Then $s_n(x) + t_n(x) \to f(x) + g(x)$ monotonically so that

$$\begin{aligned}
\int_X (f+g)\,\mathrm{d}\mu &= \int_X \lim_{n\to\infty} (s_n + t_n)\,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \int_X (s_n + t_n)\,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \left( \int_X s_n\,\mathrm{d}\mu + \int_X t_n\,\mathrm{d}\mu \right) \\
&= \int_X \lim_{n\to\infty} s_n\,\mathrm{d}\mu + \int_X \lim_{n\to\infty} t_n\,\mathrm{d}\mu \\
&= \int_X f\,\mathrm{d}\mu + \int_X g(\mathrm{d}\mu)
\end{aligned}$$

relying on linearity of the integral over simple functions. ■

**Cor. 1.4.3** *If $f_n : X \to [0, +\infty]$ is a sequence of measurable functions, then*

$$\sum_{n=1}^{\infty} \int_X f_n\,\mathrm{d}\mu = \int_X \sum_{n=1}^{\infty} f_n\,\mathrm{d}\mu$$

Proof Define $g_n = \sum_{i=1}^{n} f_i$ is monotonically increasing. ■

**Ex. 1.4.4** Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(X)$, $\mu(E)$ is the counting measure. Let $a : X \to [0, \infty)$ be a function. This is a sequence. Every function is measurable. Let $s_n(i) = a(i)$ for $i \leq n$ and 0 otherwise, which is a simple function, and $s_n \leq s_{n+1}$. Then $\lim_{n\to\infty} s_n(i) = a(i)$ so $s_n \to a$ pointwise, so by LMC $\int_X s_n\,\mathrm{d}\mu = \int_X a\,\mathrm{d}\mu$. Also,

$$\int_X s_n\,\mathrm{d}\mu = \sum_{i=1}^{n} a(i)\mu(\{i\}) = \sum_{i=1}^{n} a(i)$$

so $\int_X a\,\mathrm{d}\mu = \sum\limits_{n=1}^{\infty} a(n)$.

For the following Lemma, we will see an application of it during the section on Complex measures (to follow).

**Lemma 1.4.5 (Fatou)** *Let $f_n : X \to [0, \infty)$ be a sequence of measurable functions. Then*

$$\int_X \liminf f_n\,\mathrm{d}\mu \leq \liminf \int_X f_n\,\mathrm{d}\mu$$

Proof Let $g_k = \inf\{f_k, f_{k+1}, \dots\}$ so $\liminf f_n = \lim_{n\to\infty} g_n$ and $g_n$ is increasing. Note that $g_k \le f_k$ for any $k$, so $\int_X g_k \,\mathrm{d}\mu \le \int_X f_k \,\mathrm{d}\mu$. Thus

$$\begin{aligned}\int_X \liminf f_n \,\mathrm{d}\mu &= \int_X \lim_{n\to\infty} g_n \,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu \\ &= \liminf \int_X g_n \,\mathrm{d}\mu \\ &\le \liminf \int_X f_n \,\mathrm{d}\mu\end{aligned}$$

Note that the inequality can hold strictly (exercise). ■

Our next application is a strengthening of the previous result for simple functions.

**Thm. 1.4.6** *Let $f : X \to [0,\infty]$ be measurable. Let $\phi(E) = \int_E f \,\mathrm{d}\mu$, $E \in \mathcal{M}$. Then $\phi$ is a measure and $\int_X g \,\mathrm{d}\phi = \int_X g \cdot f \,\mathrm{d}\mu$.*

Proof Certainly $\phi(\emptyset) = 0$, so $\phi \ne +\infty$. Thus let $E = \bigcup_{i=1}^\infty E_i$ be a disjoint union and $\chi_E f = \sum_{i=1}^\infty \chi_{E_i} f$. Thus we have

$$\begin{aligned}\phi(E) &= \int_E f \,\mathrm{d}\mu = \int_X \chi_E f \,\mathrm{d}\mu \\ &= \int_X \sum_{i=1}^\infty \chi_{E_i} f \,\mathrm{d}\mu = \sum_{i=1}^\infty \int_X \chi_{E_i} f \,\mathrm{d}\mu \\ &= \sum_{i=1}^\infty \int_{E_i} \mathrm{d}\mu = \sum_{i=1}^\infty \phi(E_i)\end{aligned}$$

where we interchange the summation using the previous corollary. Now, we prove that $\int_X g \,\mathrm{d}\phi = \int_X g f \,\mathrm{d}\mu$.

- *Let $g = \chi_E$ be a characteristic function.* Then

$$\int_X \chi_E \,\mathrm{d}\phi = \phi(E) = \int_E f \,\mathrm{d}\mu = \int_X \chi_E f \,\mathrm{d}\mu$$

- *Let $g = \sum_{i=1}^n \alpha_i \chi_{A_i}$ be a simple function.* Then

$$\begin{aligned}\int_X \sum_{i=1}^n \alpha_i \chi_{A_i} \,\mathrm{d}\phi &= \sum_{i=1}^n \alpha_i \int_X \chi_{A_i} \,\mathrm{d}\phi \\ &= \sum_{i=1}^n \alpha_i \int_X \chi_{A_i} f \,\mathrm{d}\mu \\ &= \int_X \sum_{i=1}^n \alpha_i \chi_{A_i} f \,\mathrm{d}\mu\end{aligned}$$

  applying the result for characteristic functions.

- *Let $g$ be an arbitrary measurable function.* For the final step, we apply monotone convergence. Let $(s_n) \to g$ be an increasing sequence of simple functions, and note that $s_n f \to gf$. Thus
$$\begin{aligned}\int_X g\,\mathrm{d}\phi &= \int_X \lim s_n\,\mathrm{d}\phi = \lim \int_X s_n\,\mathrm{d}\phi\\ &= \lim \int_X s_n f\,\mathrm{d}\mu = \int_X \lim(s_n f)\,\mathrm{d}\mu\\ &= \int_X g\cdot f\,\mathrm{d}\mu\end{aligned}$$
as desired.

Thus we have our result. ■

## 1.5 Integration of Complex Valued Functions

**Def'n. 1.5.1** *A function $f : X \to \mathbb{C}$ is called **Lebesgue integrable** if $\int_X |f|\,\mathrm{d}\mu < \infty$. The collection of such functions is $L^1(\mu)$.*

### 1.5.1 Basic Properties

**Def'n. 1.5.2** *Let $f \in L^1(\mu)$. Then $f = u + iv$ and denote $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Let $E \in \mathcal{M}$; then the integral of $f$ over $E$ with respect to $\mu$ is*
$$\int_E f\,\mathrm{d}\mu = \int_E u^+\,\mathrm{d}\mu - \int_E u^-\,\mathrm{d}\mu + i\left(\int_E v^+\,\mathrm{d}\mu - \int_E v^-\,\mathrm{d}\mu\right)$$
*with the integral as defined for positive functions.*

**Thm. 1.5.3** *Let $f, g \in L^1(\mu)$, $\alpha, \beta \in \mathbb{C}$, so $\alpha f + \beta g = L^1(\mu)$ and*
$$\int_X (\alpha f + \beta g)\,\mathrm{d}\mu = \alpha \int_X f\,\mathrm{d}\mu + \beta \int_X g\,\mathrm{d}\mu$$

Proof Note that $\alpha f + \beta g$ is measurable, so $\int_X |\alpha f + \beta g|\,\mathrm{d}\mu \le |\alpha| \int_X |f|\,\mathrm{d}\mu + |\beta| \int_X |g|\,\mathrm{d}\mu < \infty$. For real measurable functions, $\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu$ directly by expanding the definition and using additivity over positive functions. We thus show $\int_X \alpha f\,\mathrm{d}\mu = \alpha \int_X f\,\mathrm{d}\mu$. If $\alpha \ge 0$, then
$$\begin{aligned}\int_X \alpha f\,\mathrm{d}\mu = \int_X \alpha(u+iv) &= \int_X (\alpha u^+ - \alpha u^- + i\alpha v^+ - i\alpha v^-)\,\mathrm{d}\mu\\ &= \int_X ((\alpha u)^+ - (\alpha u)^- + (i\alpha v)^+ - (i\alpha v)^-)\,\mathrm{d}\mu\\ &= \int_X (\alpha u)^+\,\mathrm{d}\mu - \int_X (\alpha u)^-\,\mathrm{d}\mu + \int_X i(\alpha v)^+\,\mathrm{d}\mu - \int_X i(\alpha v)^-\,\mathrm{d}\mu\\ &= \alpha \int_X u^+\,\mathrm{d}\mu - \alpha \int_X u^-\,\mathrm{d}\mu + \alpha \int_X iv^+\,\mathrm{d}\mu - \alpha \int_X iv^-\,\mathrm{d}\mu\\ &= \alpha \int_X f\,\mathrm{d}\mu\end{aligned}$$

and similarly for $\alpha = -1$, $\alpha = i$. ■

As usual, we have a triangle-type inequality for complex integration.

**Thm. 1.5.4** *Let $f \in L^1(\mu)$. Then $\left|\int_X f \,\mathrm{d}\mu\right| \leq \int_X |f| \,\mathrm{d}\mu$.*

Proof Let $z = \int_X f \,\mathrm{d}\mu$. Let $\alpha = \frac{|z|}{z}$ if $z \neq 0$, and $\alpha = 1$ otherwise. Then $\alpha \int_X f \,\mathrm{d}\mu = |z|$. Let $\mathrm{Re}(\alpha \cdot f) \leq |\alpha \cdot f| \leq |f|$ since $|\alpha| = 1$ so that

$$\begin{aligned}\left|\int_X f \,\mathrm{d}\mu\right| &= \alpha \cdot \int_X f \,\mathrm{d}\mu \\ &= \int_X \alpha f \,\mathrm{d}\mu \\ &= \int_X \mathrm{Re}(\alpha f) \,\mathrm{d}\mu + i \int_X \mathrm{Im}(\alpha f) \,\mathrm{d}\mu \\ &= \int_X \mathrm{Re}(\alpha f) \,\mathrm{d}\mu \\ &\leq \int_X |f| \,\mathrm{d}\mu\end{aligned}$$

since $\alpha \int_X f \,\mathrm{d}\mu \in \mathbb{R}$, we have $\int_X \mathrm{Im}(\alpha f) \,\mathrm{d}\mu = 0$. ■

### 1.5.2 Lebesgue's Dominated Convergence Theorem

The next theorem is a natural generalization of the Monotone Convergence Theorem to complex measurable functions (which do not have a standard partial ordering). Note this follows from the Monotone Convergence Theorem since we apply Fatou's Lemma. The Monotone Convergence Theorem is a special case of this theorem.

**Thm. 1.5.5 (Lebesgue's Dominated Convergence)** *Let $f_n : X \to \mathbb{C}$ be measurable functions such that $f = \lim f_n$. Assume that there is some $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all $n$. Then $f \in L^1(\mu)$ and $\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$.*

Proof We certainly know that $f$ is measurable, and $|f| \leq g$, so $f \in L^1(\mu)$. As well, the triangle inequality show that $|f - f_n| \leq 2g$ for any $n$. We will see that $0 \leq \liminf \int_X |f - f_n| \,\mathrm{d}\mu \leq \limsup \int_X |f - f_n| \,\mathrm{d}\mu \leq 0$. Assuming that this holds, then $\lim_{n\to\infty} \int_X |f - f_n| \,\mathrm{d}\mu = 0$ and

$$0 \leq \lim_{n\to\infty} \left|\int_X f \,\mathrm{d}\mu - \int_X f_n \,\mathrm{d}\mu\right| \leq \lim_{n\to\infty} \int_X |f - f_n| \,\mathrm{d}\mu = 0$$

and the desired result follows directly.

The first two inequalities are obvious: we must show that $\limsup \int_X |f_n| \, d\mu \le 0$. Firstly, we have

$$\begin{aligned}
\int_X 2g \, d\mu &= \int_X \left(2g - \lim_{n\to\infty} |f - f_n|\right) d\mu \\
&= \int_X \liminf (2g - |f - f_n|) \, d\mu \\
&\le \liminf \int_X (2g - |f - f_n|) \, d\mu && \text{By Fatou's Lemma} \\
&= \int_X 2g + \liminf \left(- \int_X |f - f_n| \, d\mu\right) \\
&= \int_X 2g - \limsup \int_X |f - f_n| \, d\mu
\end{aligned}$$

and since $\int_X 2g \, d\mu$ is finite, we subtract and $\limsup \int_X |f - f_n| \, d\mu \le 0$. ■

**Ex. 1.5.6** Consider $\lim_{n\to\infty} \int_0^n e^{-nx} \, dx$. Define

$$f_n(x) = \begin{cases} e^{-nx} & \text{if } x \le n \\ 0 & \text{if } x > n \end{cases}$$

Note that $f_n(x) \le g(x) = e^{-x}$ and $\int_0^\infty e^{-x} \, dx < \infty$. Thus

$$\begin{aligned}
\lim_{n\to\infty} \int_0^n e^{-nx} \, dx &= \int_{[0,\infty)} \lim_{n\to\infty} f_n(x) \, dx \\
&= \int_{[0,\infty)]} \chi_{\{0\}} \, dx \\
&= 0
\end{aligned}$$

**Rmk. 1.5.7** For the Riemann integral, we have $\int \lim f_n = \lim \int f_n$ as long as the convergence of $f_n$ is uniform.

# Chapter 2

# Construction of Regular Measures

## 2.1 The Vector Space $L^1(\mu)$

### 2.1.1 Almost Everywhere

Let $(X, \mathcal{M}, \mu)$ be a measure space.

**Def'n. 2.1.1** *Let $E \in \mathcal{M}$. We say that property $P$ holds almost everywhere in $E$ if there exists $N \in \mathcal{M}$ such that $\mu(N) = 0$, $N \subset E$, and $P$ holds in $E \setminus N$.*

**Ex. 2.1.2** Two functions $f, g : X \to \mathbb{C}$ are equal almost everywhere if $\exists N \subset X$ such that $\mu(N)$ and $f(x) = g(x)$ on $X \setminus N$.

**Prop. 2.1.3** *Let $E \subset X$ be such that $A_1, A_2, B_1, B_2 \in \mathcal{M}$ for which $\int_X f \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu$. Then $A_1 \subset E \subset B_1$, $A_2 \subset E \subset B_2$, and $\mu(B_1 \setminus A_1) = 0$ and $\mu(B_2 \setminus A_2) = 0$. Then $\mu(A_1) = \mu(A_2)$.*

Proof Note that $A_1 \setminus A_2 \subset E \setminus A_2 \subset B_2 \setminus A_2$. As well, $\mu(A_1 \setminus A_2) \le \mu(B_2 \setminus A_2) = 0$. Then

$$\mu(A_1) = \mu(A_1 \cap A_2^c) + \mu(A_1 \cap A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2)$$
$$\mu(A_2) = \mu(A_2 \cap A_1^c) + \mu(A_2 \cap A_1) = \mu(A_2 \setminus A_1) + \mu(A_2 \cap A_1) = \mu(A_1 \cap A_2) \qquad \blacksquare$$

**Prop. 2.1.4** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let*

$$\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, A \subset E \subset B, \mu(B \setminus A) = 0\}$$

*Then $\mathcal{M}^*$ is a $\sigma$–algebra, and $\mu^* : \mathcal{M}^* \to [0, +\infty]$ defined by $\mu^*(E) = \mu(A)$.*

Proof We show that $\mathcal{M}^*$ is a $\sigma$–algebra, and $\mu$ is countably additive.

1. $X \in \mathcal{M}$ so $X \in \mathcal{M}^*$.
2. If $E \in \mathcal{M}^*$, get $A \subset E \subset B$ so $B^c \subset E^c \subset A^c$, $A^c, B^c \in \mathcal{M}$. As well, $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \setminus A) = 0$, so $E^c \in \mathcal{M}^*$.
3. If $E_i \in \mathcal{M}^*$ is a countable collection, then get $A_i \subset E_i \subset B_i$. Fix $A = \bigcup A_i$ and $B = \bigcup B_i$. Then $B \setminus A = \bigcup (B_i \setminus A) \subset U(B_i \subset A_i)$ so $\mu(B \setminus A) = 0$ and $A \subset \bigcup E_i \subset B$ so $\bigcup E_i \in \mathcal{M}^*$.
4. Let $E_i$ be disjoint, $E = \bigcup E_i$, and $E_i \in \mathcal{M}^*$. Get $A_i \subset E_i \subset B_i$. Then $\mu^*(\bigcup E_i) = \mu(\bigcup A_i) = \sum \mu(A_i) = \sum \mu(E_i)$. $\blacksquare$

**Def'n. 2.1.5** *We call the space $(X, \mathcal{M}^*, \mu^*)$ the **completion** of $(X, \mathcal{M}, \mu)$.*

In particular, every subset of a set with measure 0 is measurable.

### 2.1.2 $L^1(\mu)$ as a normed space

**Prop. 2.1.6** *1. Let $f : X \to [0,+\infty)$ be measurable, $E \in \mathcal{M}$. If $\int_E f \, \mathrm{d}\mu = 0$, then $f = 0$ almost everywhere in $E$.*

*2. Let $f \in L^1(\mu)$. If $\int_E f \, \mathrm{d}\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ almost everywhere in $X$.*

PROOF 1. Let $A_n = \{x \in E : f(x) > 1/n\}$, so that

$$\frac{1}{n}\mu(A_n) \le \int_{A_n} \mathrm{d}\mu \le \int_E f \, \mathrm{d}\mu = 0 \Longrightarrow \mu(A_n) = 0$$

for all $n$. But then

$$N = \{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n \to \mu(N) \le \sum \mu(A_n) = 0$$

2. Write $f = u + iv$ so that

$$\int_E f \, \mathrm{d}\mu = \int_E u^+ \, \mathrm{d}\mu - \int_E u^- \, \mathrm{d}\mu + i \int_E v^+ \, \mathrm{d}\mu - i \int_E v^- \, \mathrm{d}\mu$$

We show that $u^+ = 0$ almost everywhere (the other terms are identical). Let $E = \{x \in X : u(x) \ge 0\}$, so $\int_E f \, \mathrm{d}\mu = 0$, so its real part is zero and $\int_E u^+ \, \mathrm{d}\mu = 0$. Thus $u^+ = 0$ almost everywhere in $E$. The result follows. ■

**Def'n. 2.1.7** *A **normed space** over $\mathbb{R}$ is a vector space $V$ over $\mathbb{R}$ with a map $\|\cdot\| : V \to \mathbb{R}$ such that*

*(i) $x \in V \Rightarrow \|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$.*

*(ii) $\|\lambda x\| \le |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in V$*

*(iii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in V$.*

Now $L^1(\mu) = \{f : X \to \mathbb{C}$ measurable and $\int_X |f| \mathrm{d}\mu < \infty\}$. We certainly have that $L^1(\mu)$ is a vector space. We wish to define $\|f\| = \int_X |f| \mathrm{d}\mu$. The only problem is that

$$\int_X |f| \mathrm{d}\mu = 0 \Longrightarrow f = 0 \text{ almost everywhere}$$

To deal with this problem, we quotient our space by the equivalence relation $f \sim g$ if and only if $f = g$ almost everywhere. With this in mind, define $V = L^1(\mu)/\sim$ denote the set of equivalence classes. We need to define $+, \cdot, \|\cdot\|$ on $V$. Let $[f]$ denote the class of $f$. Then

$$\begin{aligned} [f] + [g] &= [f + g] \\ c[f] &= [cf] \\ \|[f]\| &= \int_X |f| \mathrm{d}\mu \end{aligned}$$

Let's verify that this is well defined: if $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$. Indeed, this is true since the sums are equal except perhaps on a union of measure zero sets, so equality holds almost everywhere. The second definition is obviously well defined. Finally, by a homework assignment, $\|[f]\|$ is also well defined. Now, let's verify the properties of the norm.

(i) Certainly $\|[f]\| \geq 0$, and $\|[f]\| = 0$ implies $f = 0$ almost everywhere, so $[f] = [0] = 0$.

(ii) We have $\|\lambda \cdot [f]\| = \int_X |\lambda f| \,\mathrm{d}\mu = |\lambda| \int_X |f| \,\mathrm{d}\mu = |\lambda| \|[f]\|$

(iii) We have $\|[f] + [g]\| = \int_X |f + g| \,\mathrm{d}\mu \leq \int_X |f| + \int_X |g| = \|[f]\| + \|[g]\|$

In $L^1(\mu)$, two functions are the same if they are equal almost everywhere. However, this can be a challenge: if $f \in L^1(\mu)$ and $x_0 \in X$, then $f(x_0)$ is not well defined. For example, it is challenging to give meaning to boundary conditions of functions.

### 2.1.3 Construction of the Lebesgue measure

We begin from the Riemann integral $\int_a^b f(x)\,\mathrm{d}x$ for a continuous function $f$. Define $\operatorname{supp} f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. For continuous functions with compact (bounded) support, define $\Lambda f = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$ is the Riemann integral, which is a functional. In particular,

$$\text{measure}((a,b)) = \text{length}((a,b)) = \sup\{\Lambda f : f \text{ is continuous, compact support, } 0 \leq f \leq 1, \operatorname{supp} f \subset (a,b)\}$$

We will extend this to a $\sigma$−algebra containing the Borel sets. In order to define these, for open sets, $\mu(G) = \sup\{\Lambda f : 0 \leq f \leq 1, \operatorname{supp} f \subset G\}$, where $\Lambda$ is the Riemann integral. For an arbitrary set, $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$. However, this "measure" is not countably additive: the $\sigma$−algebra $\mathcal{P}(X)$ is too large (Vitali's construction). Instead, we will define $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$, which means that $E \cap K$ is regular for any $K$ compact, and regular means that the outer measure and inner measure are equal. The outer measure is $\sup\{\mu(K) : K \subset E \text{ compact}\} = \mu(E)$.

## 2.2 The Riesz Representation Theorem

In this section, we assume that $(X, \tau)$ be a locally compact, Hausdorff topological space.

**Def'n. 2.2.1** *We denote the space of continuous functions with compact support by* $C_c(X) = \{f : X \to \mathbb{C} \mid f \in C(X), \operatorname{supp} f \text{ is compact}\}$.

**Def'n. 2.2.2** *Let* $\Lambda : C_c(X) \to \mathbb{C}$ *be a* ***linear functional****, i.e.* $\Lambda(cf + g) = c\Lambda f + \Lambda g$. $\Lambda$ *is called a* ***positive*** *linear functional if* $f \geq 0 \Rightarrow \Lambda f \geq 0$.

By positivity, if $f \leq g$, then $g - f \geq 0$ so $\Lambda g - \Lambda g = \Lambda(g - f) \geq 0$ and $\Lambda f \leq \Lambda g$.

**Def'n. 2.2.3** *We say that* $K \prec f$ *if* $K$ *is compact and* $f \in C_c(X)$, $0 \leq f \leq 1$ *implies that* $x \in K \Rightarrow f(x) = 1$. *We say that* $f \prec G$ *if* $G$ *is open,* $f \in C_c(X)$, $0 \leq f \leq 1$, *and* $\operatorname{supp} f \subset G$.

**Lemma 2.2.4 (Urysohn)** *Let* $G \in \tau$, $K \subset G$ *compact. Then there exists* $f \in C_c(X)$ *such that* $K \prec f \prec G$.

Proof Will do later. ∎

**Lemma 2.2.5 (Partition of Unity)** *Let* $G_1, G_2, \ldots, G_n \in \tau$, *an let* $K \subset G_1 \cup \cdots \cup G_n$ *be compact. Then there are functions* $h_i \in C_c(X)$ *such that* $h_i \prec G_i$ *and* $K \prec \sum h_i$.

Proof Also will do later. ■

How can we create a positive linear functional on $C_c(X)$? If $\mu$ is a measure, and functions on $C_c(X)$ are measurable, then $\Lambda f = \int_X f \,\mathrm{d}\mu$ is a positive linear functional. The representation theorem says that there are no other examples.

**Thm. 2.2.6 (Riesz Representation)** *Let $(X,\tau)$ be as above. If $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique measure space $(X,\mathcal{M},\mu)$ such that $\Lambda f = \int_X f \,\mathrm{d}\mu$ for any $f \in C_c(X)$, $\mathcal{M} \supset \tau$, and*

*(i) $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$ for all $E \in \mathcal{M}$.*
*(ii) $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$.*
*(iii) $\mu(K) < \infty$ for any $K$ compact.*
*(iv) $\mathcal{M}$ is complete.*

First, let's get some definitions out of the way. Fix the notation as above.

**Def'n. 2.2.7** *Fix a Borel measure $\mu$. The **Lebesgue outer measure** is defined $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$.*

**Def'n. 2.2.8** *We say that $E \subset X$ is **regular** if $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$. Similarly, $E \subset X$ is **locally regular** for every compact $K$, $K \cap E$ is regular.*

**Claim 0: Definition of $\mu$ and $\mathcal{M}$; completeness of $\mathcal{M}$.**

Proof For an open set $G \in \tau$, let $\mu(G) = \sup\{\Lambda f : f \prec G\}$. Then $\mu(\emptyset) = 0$ and $G_1 \subset G_2$ implies that $\mu(G_1) \le \mu(G_2)$. Then extend $\mu$ to arbitrary $E \subset X$ as an outer measure. Now let $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$. Note that $\mathcal{M}$ contains compact sets, since they are regular. This is direct from the definition, since $\mu(F) \le \mu(K)$ for any compact $F \subseteq K$ and the supremum occurs exactly at $K$.

As well, $\mathcal{M}$ is complete: let $E \in \mathcal{M}$, $\mu(E) = 0$ and $A \subset E$. We want to show that $A \in \mathcal{M}$. Let $K$ be an arbitrary compact set; then $\mu(K \cap A) = 0$. Now if $F \subset K \cap A$ is compact, $\mu(F) = 0$. Thus $\sup\{\mu(F) : F \subset K \cap A \text{ compact}\} = 0$, so $K \cap A$ is regular and $A$ is locally regular and an element of $\mathcal{M}$. ■

**Claim 1: $\mu$ is $\sigma$−subadditive. In other words, if $E_1, E_2, \ldots$ are arbitrary subsets of $X$, then**

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu(E_i)$$

Proof If $\mu(E_j) = \infty$ for some $j$, then we are done. Thus assume $\mu(E_j) < \infty$ for all $j$. Let $\epsilon > 0$, $\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ be arbitrary. We will show that $\gamma \le \sum_{i=1}^{\infty} \mu(E_i)$.

Let $G_j \supset E_j$ be open, such that $\mu(G_j) \le \mu(E_j) + \frac{\epsilon}{2^j}$. Then

$$\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \le \mu\left(\bigcup_{j=1}^{\infty} G_j\right)$$

so there exists some $f \prec \bigcup_{j=1}^\infty G_j$ so $\gamma < \Lambda f$ by the definition of $\mu$ on open sets. Let $K = \operatorname{supp} f$ so that

$$K \subset \bigcup_{j=1}^\infty G_j \implies K \subset \bigcup_{j=1}^n G_j$$

since $\{G_j\}$ are an open cover for $K$ and $K$ is compact. Get a partition of unity $h_j \prec G_j$ for each $j = 1, \dots, n$ which satisfies $\sum_{j=1}^n h_j(x) = 1$ for any $x \in K$. Then $f \cdot h_j \prec G_j$ and $f = f \cdot \sum_{j=1}^n h_j$ so that

$$\begin{aligned}\gamma < \Lambda f = \Lambda\left(\sum_{j=1}^n f h_j\right) &= \sum_{j=1}^n \Lambda(f h_j) \\ \leq \sum_{j=1}^n \mu(G_j) &\leq \sum_{j=1}^n \left(\mu(E_j) + \frac{\epsilon}{2^j}\right) \\ \leq \sum_{j=1}^\infty \left(\mu(E_j)\right) + \epsilon\end{aligned}$$

which holds for all $\epsilon > 0$ if and only if $\gamma \leq \sum_{j=1}^\infty \mu(E_j)$. This holds for any $\gamma \leq \mu\left(\bigcup_{j=1}^\infty E_j\right)$ and the result follows. ■

**Claim 2: If $K \prec f \prec G$, then $\mu(K) \leq \Lambda f \leq \mu(G)$. Thus if $K$ is compact, $\mu(K) < \infty$.**

Proof It is direct from the definition of $\mu$ that $\Lambda f \leq \mu(G)$. Thus let $\gamma < \mu(K)$ and $\alpha \in (0,1)$. Let $V_\alpha := \{x \in X : f(x) > \alpha\}$ and $K \subset V_\alpha$ since $f \equiv 1$ on $K$. Since $f$ is continuous, $V_\alpha = f^{-1}((\alpha, \infty))$ is the preimage of an open set and thus open.

Now $\gamma < \mu(K) \leq \mu(V_\alpha)$, so we have some $h \prec V_\alpha$ such that $\gamma < \Lambda h$. Then $\alpha \cdot h \leq f$ since in $V_\alpha$, $\alpha \cdot h \leq \alpha < f$ and in $V_\alpha^c$, $\alpha \cdot h = 0 \leq f$. Now $\alpha \cdot \Lambda h = \Lambda(\alpha h) \leq \Lambda f$ so $\gamma < \Lambda f / \alpha$. This is true for all $\alpha \in (0,1)$ and $\gamma \leq \Lambda f$. Since this holds for all $\gamma < \mu(K)$, we have $\mu(K) \leq \Lambda f$ as required.

Now, let $K$ be compact so that $\mu(K) \leq \Lambda f$ for all $K \prec f$. Let $\epsilon > 0$ and get $G \in \tau$, $G \supset K$ such that $\mu(G) \leq \mu(K) + \epsilon$. Then by Urysohn's lemma, get some $K \prec f \prec G$ so that $\mu(K) \leq \Lambda f \leq \mu(G)$, so $\Lambda f \leq \mu(K) + \epsilon$ and the result holds. Now suppose $K$ is compact, so $\mu(K) = \inf\{\mu(G) : K \subset G \in \tau\}$. By Urysohn's Lemma, get $f$ with $K \prec f \prec G$, and by (1.), $\mu(K) \leq \Lambda f \leq \mu(G)$ so that $\mu(K) = \inf\{\Lambda f : K \prec f\}$. As a corollary, we have that $\mu(K) < \infty$ (since $\Lambda$ is a positive linear functional). Besides demonstrating one of our properties, this provides a convenient way of computing the measure of compact sets. ■

**Claim 3: If $G \in \tau$, then $G$ is regular.**

Proof We first show that if $0 \leq f \leq 1$, then $\Lambda f \leq \mu(\operatorname{supp} f)$. Let $G \supset \operatorname{supp} f$ be open, so $f \prec G$ and $\mu(G) \geq \Lambda f$. Then $\mu(\operatorname{supp} f) = \inf\{\mu(G) : E \subset G \in \tau\} \geq \Lambda f$.

Now we want to show $\mu(G) = \sup\{\mu(K) : K \subset G \text{ compact}\}$. It suffices to show that $\sup\{\mu(K) : K \subset G \text{ compact}\} \geq \mu(G)$, so let $\gamma < \mu(G)$ and we want $K$ compact so that $\mu(K) > \gamma$. Let $f \prec G$ be such that $\Lambda f > \gamma$. Then $\mu(\operatorname{supp} f) > \gamma$ by the previous claim is compact, as desired. ■

**Claim 4: Suppose $E_1, E_2, \dots$ are disjoint regular. Then**

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i)$$

**If we assume additionally that $\mu(\cup E_i) < \infty$, then $\bigcup_{i=1}^{\infty} E_i$ is regular.**

Proof We first prove this for two compact sets. Thus let $K_1, K_2$ be disjoint compact sets. Then $K_2^c$ is open and $K_2^c \supset K_1$. By Urysohn's lemma, get $f \in C_c(X)$ so that $K_1 \prec f \prec K_2^c$ and $x \in K_1$ implies $f(x) = 1$, and $x \in K_2$ implies $f(x) = 0$.

Since $K_1 \cup K_2$ is compact, for all $\epsilon > 0$, get $g \prec K_1 \cup K_2$ such that $\mu(K_1 \cup K_2) + \epsilon > \Lambda g$ (by Claim 2). Furthermore, $K_1 \prec f \cdot g$ and $K_2 \prec (1-f) \cdot g$. Thus $\mu(K_1) + \mu(K_2) \leq \Lambda(f \cdot g) + \Lambda((1-f) \cdot g = \Lambda g < \mu(K_1 \cup K_2) + \epsilon$ which is true for any $\epsilon > 0$. Thus $\mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$ and equality holds, as required.

Now, by Claim 1, it remains to show that $\mu(\cup E_i) \geq \sum \mu(E_i)$. If $\mu(\cup E_i) = +\infty$, we are done, so assume $\mu(\cup E_i) < +\infty$. Since the $E_i$ are regular, there is a compact set $H_i \subset E_i$ so that $\mu(H_i) > \mu(E_i) - \frac{\epsilon}{2^i}$ for each $i \in \mathbb{N}$. Let $K_n = \bigcup_{i=1}^{n} H_i$. Now

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \epsilon$$

Taking the limit as $n$ goes to infinity gives $\mu(\cup E_i) \geq \sum \mu(E_i) - \epsilon$ for any $\epsilon > 0$, so we are done.

Let's now see the second part. For any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{N} \mu(E_i) + \epsilon \leq \mu(K_N) + 2\epsilon$$

with $K_N$ compact defined in the same way as above. Since $\epsilon > 0$ was arbitrary, the result follows directly. ∎

**Claim 5: $E$ is regular and $\mu(E) < \infty$ if and only if for any $\epsilon > 0$, there exists $K$ compact, $G$ open so that $K \subset E \subset G$ and $\mu(G \setminus K) < \epsilon$.**

Proof There exists by regularity (and the definition of the outer measure) $K \subset E \subset G$ so that

$$\mu(E) - \frac{\epsilon}{2} \leq \mu(K) \leq \mu(G) \leq \mu(E) + \epsilon/2$$

As well, $\mu(G) = \mu(K \cup (G \setminus K)) = \mu(K) + \mu(G \setminus K)$ and $\mu(G \setminus K) = \mu(G) - \mu(K) < \epsilon$.

Conversely, let $K \subset E \subset G$ and $\mu(G \setminus K) < \epsilon$. Then

$$\mu(E) \leq \mu(G) = \mu(K) + \mu(G \setminus K) < \mu(K) + \epsilon$$

so $\mu(E) < \infty$ and $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$, so $E$ is regular. ∎

**Claim 6:**

1. **Let $A, B$ be regular with $\mu(A), \mu(B) < \infty$. Then $A \setminus B$, $A \cup B$, $A \cap B$ are regular and have finite measure.**
2. **If $\mu(E) < \infty$, then $E$ is regular if and only if $E$ is locally regular.**
3. **If $E_i$ are regular, then $\bigcup_{i=1}^{\infty} E_i$ is regular.**

Proof Recall that for any $\epsilon > 0$, there exists $K_1 \subset A \subset G_1$ and $K_2 \subset B \subset G_2$ such that $\mu(G_1 \setminus K_1) < \epsilon$ and $\mu(G_2 \setminus K_2) < \epsilon$.

1. Note that $A\setminus B \subset G_1\setminus K_2 \subset (G_1\setminus K_1)\cup(K_1\setminus G_2)\cup(G_2\setminus K_2)$, where $K_1\setminus G_2$ is compact. Thus $\mu(A\setminus B) \leq \epsilon+\mu(K_1\setminus G_1)+\epsilon < \infty$ and $\mu(A\setminus B)-2\epsilon \leq \mu(K_1\setminus G_2)$ so $A\setminus B$ is regular. Finally since $A\cup B = (A\setminus B)\cup B$, $A\cup B$ is regular and $\mu(A\cup B) < \infty$. Thus $A\cap B = (A\cup B)\setminus((A\setminus B)\cup(B\setminus A))$ is regular and has measure less than infinity.

2. Let $\mu(E) < \infty$, and first suppose $E$ is regular. Let $K$ be a compact set. Then $\mu(K) < \infty$ and $K$ is regular, so $E\cap K$ is regular (by 1.) so $E$ is locally regular.

   Conversely, suppose $E$ is locally regular. Let $\epsilon > 0$ and $G \supset E$ be open so that $\mu(G) < \mu(E)+1 < \infty$. As well, $G$ is regular, so there exists $K$ with $\mu(G) < \mu(K)+\epsilon/2$. Now,

$$\begin{aligned}\mu(E) &= \mu((E\setminus K)\cup(E\cap K)) \leq \mu(E\setminus K)+\mu(E\cap K)\\ &\leq \mu(G\setminus K)+\mu(E\cap K)\\ &< \frac{\epsilon}{2}+\mu(E\cap K)\end{aligned}$$

   so $\mu(E\cap K) > \mu(E)-\epsilon/2$. Then since $E$ is locally regular, $E\cap K$ is regular and get a compact set $L\subset E\cap K$ such that $\mu(L) > \mu(E\cap K)-\epsilon/2 > \mu(E)-\epsilon$. This holds for any $\epsilon > 0$, so $E$ is regular.

3. Set $F_1 = E_1$, $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$ so $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and the $F_i$ as disjoint. By Claim 4, $\cup F_i$ is regular and $F_i$ are regular (TODO: finiteness requirement?) ■

**Claim 7: $\mathcal{M}$ is a $\sigma$−algebra, $M\subset\tau$, and $\mu$ is countably additive on $\mathcal{M}$.**

Proof We demonstrate the requirements:

- Let $A\in\mathcal{M}$: we see that $A^c\in\mathcal{M}$. If $K$ is an arbitrary compact set, then $A^c\cap K = K\setminus(A\cap K)$ is regular by Claim 7 since $K$ is regular (and thus locally regular), and $A\cap K$ is regular since $A$ is locally regular.
- Now let $A_n\in\mathcal{M}$; we will show that $A := \bigcup_{n=1}^{\infty} A_n \in\mathcal{M}$. Indeed, if $K$ is an arbitrary compact set, then

$$A\cap K = \bigcup_{n=1}^{\infty}(A_n\cap K)$$

  is regular by Claim 6.
- We now show $\mathcal{M}\supset\tau$. It suffices by closure under complements to show that all closed sets are in $\mathcal{M}$. If $A$ is closed, then $A\cap K$ is compact and thus regular, so $A\in\mathcal{M}$.
- Finally, let $E_i\in\mathcal{M}$ be locally regular and disjoint; it suffices to show that

$$\mu\left(\bigcup_{i=1}^{\infty}E_i\right) \geq \sum_{i=1}^{\infty}\mu(E_i)$$

  If $\mu(E_i) = +\infty$, we are done, so assume $\mu(E_i) < \infty$ for all $i$. But then by Claim 6.2, the $E_i$ are regular, so the result holds by Claim 6.3. ■

**Claim 8: $\Lambda f = \int_X f\,\mathrm{d}\mu$ for all $f\in C_c(X)$.**

Proof We are finally almost done: we just need to show that $\mu$, as defined, actually represents $\Lambda$. Let's start by simplifying $f$ as much as possible.

- *It suffices to do this for real valued functions.* If $f = u + iv$, then $\Lambda f = \Lambda u + i\Lambda v = \int_X u \,\mathrm{d}\mu + i\int_X v \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu$.
- *It suffices to show $\Lambda f \leq \int_X f \,\mathrm{d}\mu$.* If this holds for all $f$, then $\Lambda(-f) \leq \int_X -f \,\mathrm{d}\mu$ so that $-\Lambda f \leq -\int_X f \,\mathrm{d}\mu$ and $\Lambda f \geq \int_X f \,\mathrm{d}\mu$ and equality holds.
- *It is enough to prove that $\Lambda f \leq \int_X f \,\mathrm{d}\mu$ for $f \geq 0$.* Assuming so, let $f$ be arbitrary and let $K = \operatorname{supp} f$ be compact, and $a = \min f$, $b = \max f$. The general idea of the proof is to translate $f$ by the value $|a|$ so that it is positive. However, we cannot do this directly since $f + |a|$ is not compactly supported; however, we can use Urysohn's Lemma to translate it on its support. Now, let $\epsilon > 0$ be arbitrary. Fix $K = \operatorname{supp} f$ and get $G \supset K$ so that $\mu(G) \leq \mu(K) + \epsilon$. By Urysohn's lemma, there exists $h \in C_c(X)$ so that $K \prec h \prec G$. Thus $|a| \cdot h(x) = |a|$ for all $x \in K$, so $F := f + |a|h \geq 0$ since $f \geq -|a|$. Now by assumption,

$$\Lambda F \leq \int_X F \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu + |a| \int_X h \,\mathrm{d}\mu$$

so that

$$\begin{aligned}\Lambda f &= \Lambda F - |a|\Lambda h \\ &\leq \int_X f \,\mathrm{d}\mu + |a|\int_X h \,\mathrm{d}\mu - |a|\Lambda h \\ &\leq \int_X f \,\mathrm{d}\mu + |a|\left(\int_X h \,\mathrm{d}\mu - \Lambda h\right)\end{aligned}$$

We now want to show $|\int_X h \,\mathrm{d}\mu - \Lambda h| < \epsilon$, and the result will follow. By Claim 2, $\mu(K) \leq \Lambda h \leq \mu(G)$. As well, $\int_X h \,\mathrm{d}\mu \leq \mu(G)$ since $h \prec G$ implies $h \leq \chi_G$. Thus since $h \geq 0$, by assumption

$$\mu(K) \leq \Lambda h \leq \int_X h \,\mathrm{d}\mu \leq \mu(G)$$

and the result follows since $\mu(G) - \mu(K) < \epsilon$. Thus $\Lambda f \leq \int f + |a|\epsilon$ for all $\epsilon > 0$, so $\Lambda f \leq \int f$ as desired.

It now remains to show $\Lambda f \leq \int_X f \,\mathrm{d}\mu$ for $f \geq 0$. Since $f = Mf/M$ where $M = \max f$, we can assume $0 \leq f \leq 1$. Fix $K = \operatorname{supp} f$, let $\epsilon > 0$ be arbitrary. Let $0 = c_0 < c_1 < c_2 < \cdots < c_n = 1$ with $c_k - c_{k-1} < \epsilon$ for all $k$ and $\mu(f^{-1}(c_k)) = 0$ for all $k = 1, \ldots, n-1$. The existence of such a set follows from Assignment 6. Let $K_j = K \cap f^{-1}([c_{j-1}, c_j])$ for $j = 1, 2, \ldots, n$ and $L_j = K \cap f^{-1}([c_{j-1}, c_j))$ for $j = 1, 2, \ldots, n-1$.

For each $K_j$ and any $\epsilon > 0$, there exists $\tau \ni G_j \supset K_j$ such that $\mu(G_j) \leq \mu(K_j) + \frac{\epsilon}{2^j}$. By Urysohn's lemma, get $h_j$ so that $K_j \prec h_j \prec G_j$. Then $f \leq \sum_{j=1}^n c_j h_j$: if $x \in K^c$, $f = 0$. Otherwise, if $x \in K$, then $x \in K_j$ for some $j$. Since $h_j = 1$ and $f(x) \leq c_j$ on $K_j$, we have $f(x) \leq c_j = c_j h_j(x) \leq \sum_{i=1}^n c_i h_i$.

Now, there is just a lot of algebra.

$$\begin{aligned}
\Lambda f \leq \Lambda\left(\sum c_j h_j\right) &= \sum_{i=1}^{n} c_j \Lambda h_j && \text{(linearity and positivity)} \\
&\leq \sum_{j=1}^{n} c_j \mu(G_j) && (h_j \prec G_j) \\
&\leq \sum_{j=1}^{n} c_j \mu(K_j) + \sum_{j=1}^{n} c_j \frac{\epsilon}{2^j} && \text{(choice of } K_j\text{)} \\
&\leq \sum_{j=1}^{n} (c_{j-1} + c_j - c_{j-1}) \mu(L_j) + \epsilon && (L_j \subset K_j, |c_j| \leq 1) \\
&\leq \sum_{j=1}^{n} c_{j-1} \mu(L_j) + \epsilon \cdot \mu(K) + \epsilon && (L_j \text{ disjoint}, c_j - c_{j-1} < \epsilon)
\end{aligned}$$

Now define $g$ so $g(x) = c_{j-1}$ if $x \in L_j$, and $g \equiv 0$ outside $K$. Then $g$ is a simple function, so that the summation above is precisely the integral of $g$. Furthermore, $g \leq f$ so $\int_X g \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu$ and

$$\begin{aligned}
\Lambda f &\leq \int_X g \, \mathrm{d}\mu + \epsilon + \epsilon \mu(K) \\
&\leq \int_X f \, \mathrm{d}\mu + \epsilon(1 + \mu(K))
\end{aligned}$$

and, since $\mu(K) < \infty$, because this holds for any $\epsilon > 0$, we are done! ■

## 2.3 Regularity Properties of Borel Measures

At the beginning of the Riesz Representation Theorem, we introduced a variety of conditions which we will summarize here independetly.

**Def'n. 2.3.1** *A measure defined on the family of Borel sets is called a* ***Borel measure****.*

**Def'n. 2.3.2** *Let $\mu : \mathcal{B} \to [0, +\infty]$ be a Borel measure.*

1. *$E$ is called* ***outer regular*** *if $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$.*
2. *$E$ is called* ***inner regular*** *if $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$*
3. *$\mu$ is called* ***regular*** *if every $E \in \mathcal{B}$ is inner and outer regular.*

The next condition is a finiteness condition: naturally, we like spaces that aren't too big.

**Def'n. 2.3.3** *A set $E \subset X$ is called $\sigma-$**compact** if $E = \bigcup_{n=1}^{\infty} E_n$, for $E_n$ compact.*

The sets in the next definition are standard in real analysis.

**Def'n. 2.3.4** *A $G_\delta$ set is one of the form $\bigcap\limits_{n=1}^{\infty} A_n$ with $A_n$ open, and a $F_\sigma$ set is one of the form $\bigcup\limits_{n=1}^{\infty} B_n$ for $B_n$ closed.*

Measure spaces $(X, \mathcal{M}, \mu)$ which satisfy these properties are particularly nice. To be precise, by "nice", we have the following theorem:

**Thm. 2.3.5** *Let $X$ be a locally compact, $\sigma-$compact Hausdorff space. Let $\mathcal{M} \supset \mathcal{B}$ be a $\sigma-$algebra, $\mu : \mathcal{M} \to [0, +\infty]$ be a measure such that*
*(i) $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$ (outer regularity)*
*(ii) $\mu(E) = \sup\{\mu(K) : K \subset E$ compact$\}$, $\mu(E) < \infty$ (inner regularity for finite measure sets)*
*(iii) $\mu(K) < \infty$ for $K$ compact (finite on compact sets)*
*Then*
1. *For all $E \in \mathcal{M}$ and $\epsilon > 0$, there exists $F$ closed and $G$ open so that $F \subset E \subset G$ and $\mu(G \setminus F) < \epsilon$.*
2. *$\mu$ is regular*
3. *For all $E \in \mathcal{M}$, there exists a $F_\sigma$ set $A$ and a $G_\delta$ set $B$ so $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.*

Thankfully, the proof is not too hard.

Proof Since $X$ is $\sigma-$compact, write $X = \bigcup\limits_{n=1}^{\infty} K_n$, $K_n$ compact.

1. By (iii), we have $\mu(K_n \cap E) < \infty$. Thus by (i), get $G_n$ open so that $G_n \supset K_n \cap E$ with $\mu(G_n \setminus (K_n \cap E)) < \frac{\epsilon}{2^{n+1}}$. Let $G = \bigcup\limits_{n=1}^{\infty} G_n$ be open, so that
$$G \setminus E \subset \bigcup_{n=1}^{\infty} G_n \setminus (K_n \cap E)$$
and
$$\mu(G \setminus E) \leq \sum_{n=1}^{\infty} \mu(G_n \setminus (K_n \cap E)) < \frac{\epsilon}{2}$$
Repeat this for $E^c$: get an open set $H$ such that $\mu(H \setminus E^c) < \frac{\epsilon}{2}$. Then $F = H^c \subset E$ satisfies $\mu(E \setminus F) = \mu(F^c \setminus E^c) = \mu(H \setminus E^c) < \frac{\epsilon}{2}$. Thus $\mu(G \setminus F) \leq \mu(G \setminus E) + \mu(E \setminus F) < \epsilon$.
2. $E$ is outer regular by (i). If $\mu(E) < \infty$, then $E$ is inner regular by (ii), so $E$ is regular; thus suppose $\mu(E) = \infty$.
Let $F \subset E$ be given by 1, so that $\mu(F) = +\infty$ (or $\mu(E)$ would be finite). Note that $H_n := \bigcup_{k=1}^{n} (F \cap K_k)$ is a compact set, so that $H_n \subset F$. Then $\bigcup_{n=1}^{\infty} H_n = F$, and $\mu(H_n) \to \mu(F) = \infty$. Thus $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$.
3. Apply 1 wih $\epsilon = 1/j$ for $j \in \mathbb{N}$. Then there exists $F_j \subset E \subset G_j$ so $\mu(G_j \setminus F_j) < \frac{1}{j}$. Define
$$A = \bigcup_{j=1}^{\infty} F_j, \qquad B = \bigcap_{j=1}^{\infty} G_j$$
Then $A \subset E \subset B$ and $\mu(B \setminus A) \leq \mu(G_j \setminus F_j) < \frac{1}{j}$ for any $j \in \mathbb{N}$, so $\mu(B \setminus A) = 0$. ■

As a corollary to this, if we assume that $X$ is a locally compact and $\sigma-$compact space and $\Lambda$ is a positive linear functional on $C_c(X)$, then the measure $\mu$ representing $\Lambda$ is a regular measure. More generally, if we assume that *every* open set is $\sigma-$compact, we have the following theorem:

**Thm. 2.3.6** *Let $X$ be locally compact and Hausdorff, and assume that every open set is $\sigma-$compact. Let $\lambda : \mathcal{B} \to [0,\infty]$ be a Borel measure such that $\lambda(K) < \infty$ for any compact set $K$. Then $\lambda$ is regular.*

Proof Let $\Lambda f = \int_X f \, d\lambda$. Then $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional. By the Riesz representation theorem, there exists $\mu : \mathcal{M} \to [0,\infty]$ such that $\int_X f \, d\mu = \Lambda f = \int_X f \, d\lambda$. We see that $\lambda = \mu$ on $\mathcal{B}$, so that $\lambda$ is regular since $\mu$ is.

We first prove this for open sets. Let $G \in \tau$; then there exists compact $K_n$ so $G = \bigcup_{n=1}^{\infty} K_n$. By Urysohn's lemma, there exists $f_i$ such that $K_i \prec f_i \prec G$. Let $g_n = \max\{f_1, f_2, \ldots, f_n\}$, so $g_n \in C_c(X)$, and $g_n \to \chi_G$ pointwise. But then applying Lebesgue's Monotone Convergence theorem (and the fact that $\lambda = \mu$ on $C_c(X)$),

$$\begin{aligned}
\lambda(G) &= \int_X \chi_G \, d\lambda = \int_X \lim_{n\to\infty} g_n \, d\lambda = \lim_{n\to\infty} \int_X g_n \, d\lambda \\
&= \lim_{n\to\infty} \int_X g_n \, d\mu = \int_X \lim_{n\to\infty} g_n \, d\mu = \int_X \chi_G \, d\mu \\
&= \mu(G)
\end{aligned}$$

Now for any $E \in \mathcal{B}$, get $F$ closed, $G$ open so that $F \subset E \subset G$ and $\mu(G \setminus F) < \epsilon$. Since $G \setminus F$ is open, $\lambda(G \setminus F) = \mu(G \setminus F) < \epsilon$ so $\lambda(G) \leq \lambda(E) + \epsilon$. Thus $|\mu(E) - \lambda(E)| < \epsilon$ for all $\epsilon > 0$ so $\lambda(E) = \mu(E)$. ■

## 2.4 Construction of the Lebesgue Measure

We have the Riesz Representation Theorem in a locally compact Hausdorff space.

**Def'n. 2.4.1** *Let $E \subset \mathbb{R}^k$, $x \in \mathbb{R}^k$. Then $E + x = \{y + x : y \in E\}$ is the* ***translate*** *of $E$.*

**Def'n. 2.4.2** *We define a $k-$cell in $\mathbb{R}^k$ by $W = I_1 \times I_2 \times \cdots \times I_k$ where $I_j$ is an interval. We also define* $\operatorname{vol}(W) = (b_1 - a_1)(b_2 - a_1)\cdots(b_k - a_k)$ *where $a_j, b_j$ are the endpoints of the $I_j$.*

We know that $\operatorname{vol}(W + x) = \operatorname{vol}(W)$ for any $k-$cell $W$ and $x \in \mathbb{R}$.

**Thm. 2.4.3** *There exists a $\sigma-$algebra $\mathcal{M}$ in $\mathbb{R}^k$ and a complete measure $m : \mathcal{M} \to [0,+\infty]$ satisfying*

1. *$m(W) = \operatorname{vol}(W)$ for any $k-$cell $W$.*
2. *$\mathcal{M} \supset \mathcal{B}$ and $E \in \mathcal{M}$ if and only if there exists $A \in F_\sigma, B \in G_\delta$ such that $A \subset E \subset B$ and $m(B \setminus A) = 0$.*
3. *$m$ is translation invarient: $m(E + x) = m(E)$.*
4. *If $\mu$ is a translation invariant Borel measure, and $\mu(K) < \infty$ for all $K$ compact, then there exists $c \in \mathbb{R}$ so that $\mu(E) = c \cdot m(E)$.*
5. *If $T : \mathbb{R}^k \to \mathbb{R}^k$ is linear, then there exists $\Delta(T) \in \mathbb{R}$ such that $m(T(E)) = \Delta(T) \cdot m(E)$.*

Proof For $f \in C_c(\mathbb{R}^k)$, let $\Lambda f = \int_{R^k} f(x) \, dx$ (the Riemann Integral). Then $\Lambda : C_c(\mathbb{R}^k) \to \mathbb{C}$ is a positive linear functional, so by the Riesz representation theorem, there exists a unique measure $m$ and $\mathcal{M} \supset \mathcal{B}$ so for all $f \in C_c(\mathbb{R}^k)$, $\Lambda f = \int_{\mathbb{R}^k} f \, dm$. Let's prove that this measure has the appropriate properties:

1. By the definition of $m$, for an open $k-$cell $W$, $m(W) = \sup\{\Lambda f : f \prec W\} = \operatorname{vol}(W)$ (by definition of the Riemann integral). If $W$ is an arbitrary $k-$cell, then there exist open $k-$cells $W_n$ such that $W = \bigcap_{n=1}^{\infty} W_n$. Then $\operatorname{vol}(W_n) \to \operatorname{vol}(W)$, so $m(W_n) \to m(W)$, and $\operatorname{vol}(W_n) = m(W_n)$. Thus $\operatorname{vol}(W) = m(W)$.

*Let $\lambda$ be a Borel measure. If $\lambda(W) = m(W)$ for all $W$ $k$−cells, then $\lambda(E) = m(E)$ for all $E \in \mathcal{B}$.* For any $G$ open, $G = \bigcup_{n=1}^{\infty} W_n$ disjointly, so $\lambda(G) = m(W)$. Then since $\lambda$ and $m$ are regular, $\lambda(E) = \inf\{\lambda(G) : E \subset G \in \tau\} = \inf\{m(G) : E \subset G \in \tau\} = m(E)$ for all $E \in \mathcal{B}$.

We now see (iii). Dfine $\lambda(E) = m(E + X)$. If $W$ is a box, then $\lambda(W) = m(W + x) = \mathrm{vol}(W + x) = \mathrm{vol}(W) = m(W)$, so by the lemma, $\lambda(E) = m(E)$ for all $E \in \mathcal{B}$. Then regularity implies $\lambda(E) = m(E)$ for all measurable $E$.

We have (iv): let $c = \mu([0,1]^k) = c \cdot \mathrm{vol}([0,1]^k)$. Translation invariance of vol implies $\mu(W) = c \cdot \mathrm{vol}(W)$.

We have (v). If $\dim(\mathrm{Im}(T)) < k$, then $m(\mathrm{Im}(T)) = 0$ so $\Delta(T) = 0$. Otherwise, $T$ is a homeomorphism so $T(E) \in \mathcal{B}$ for all $E \in \mathcal{B}$. Let $\mu(E) = m(T(E))$. Then $\mu(E + x) = m(T(E) + T(x)) = m(T(E)) = \mu(E)$, so $\mu$ is translation invariant. Then by (iv), $\mu(E) = c \cdot m(E)$ and set $\Delta(T) = c$. ∎

**Thm. 2.4.4** *If $A \subset \mathbb{R}$ for which every set is Lebesgue measurable, then $m(A) = 0$.*

Proof Partition $\mathbb{R}$ into cosets by $\mathbb{Q}$; let $E$ be a set containing exactly one element of each class (axiom of choice). Now if $r \neq s$, $r, s \in \mathbb{Q}$, then $(E + r) \cap (E + s) = \emptyset$. But then $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r)$ disjointly. Given $A$, define $A_t = A \cap (E + t)$ for $t \in \mathbb{Q}$. Now let $K \subset A_t$, so $K \subset E + t$. Since $(K + r_1) \cap (K + r_2) = \emptyset$, define $H = \bigcup_{r \in \mathbb{Q} \cap [0,1]} (K + r)$ is a countable disjoint union. But then $\infty > m(H) = \sum_r m(K)$ so $m(K) = 0$ and $m(A_t) = 0$. But then

$$\bigcup_{t \in \mathbb{Q}} A_t = \cup(A \cap (E + t)) = A \cap \left( \bigcup_{t \in \mathbb{Q}} (E + t) \right) = A \cap \mathbb{R} = A$$

so $m(A) = 0$ as well. ∎

## 2.5 Measurability and Continuity

Let $X$ be a locally compact, Hausdorff topological space. Let $\mathcal{M}$ be a $\sigma$−algebra, $\mu$ be a measure satisfying the properties in the Riesz representation theorem. We then have

**Thm. 2.5.1 (Lusin)** *Let $f : X \to \mathbb{C}$ be a measurable function, with $\operatorname{supp} f \subset A$ and $\mu(A) < \infty$. Then for any $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$ and $\sup_X |G| \leq \sup_X |f|$.*

Proof It suffices to assume that we can do this for compact $K$. Assuming so, let $\mu(A) < \infty$ and get $K \subset A$ compact with $\mu(A \setminus K) < \epsilon/2$. Define $\hat{f}$ so that $\hat{f} = f$ on $K$ and $\hat{f} = 0$ otherwise, so $\sup \hat{f} \subset K$ and $f'$ is measurable. By assumption, get $g$ so that $\mu(\{x : g(x) \neq f'(x)\}) < \epsilon/2$. Then

$$\mu(\{x \in X : f(x) \neq g(x)\}) \leq \mu(A \setminus K) + \mu\Big(\{x \in K \cup A^c : \hat{f}(x) \neq g(x)\}\Big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since $f' = f$ on $(A \setminus K)^c$. Now, let's prove the statement for $A$ compact.

We first assume that $0 \leq f \leq 1$. For $t \geq 0$ and each $N \in \mathbb{N}$, define $k_n(t) = [2^n \cdot t]$, so $k_n(t) \in \mathbb{Z}$ and $k_n(t) \leq t \cdot 2^n < k_n(t) + 1$. Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n}, & 0 \leq t \leq n \\ n, & t > n \end{cases}$$

Let $s_n(x) = \phi_n(f(x))$ and $t_n = s_n - s_{n-1}$. Observe that $f = \sum_{n=1}^{\infty} t_n$; I claim that $2^n \cdot t_n \in \{0, 1\}$. To see this, first note that

$$k_{n-1}(t) \leq t \cdot 2^{n-1} < k_{n-1}(t) + 1 \Longrightarrow 2k_{n-1}(t) \leq t \cdot 2^n < 2k_{n-1}(t) + 2$$

so $2k_{n-1}(t)$ is the largest even number below $t \cdot 2^n$. Thus $k_n - 2k_{n-1} \in \{0, 1\}$ for all $t$. Since $0 \leq f \leq 1$, for all $n$ and $x$,

$$\begin{aligned} 2^n \cdot t_n(x) &= 2^n \cdot (\phi_n(f(x)) - \cdot \phi_{n-1}(f(x))) \\ &= 2^n \left(2^{-n} \cdot k_n(f(x)) - 2^{-(n-1)} \cdot k_{n-1}(f(x))\right) \\ &= k_n(f(x)) - 2k_{n-1}(f(x)) \in \{0, 1\} \end{aligned}$$

as required. Thus $2^n \cdot t_n$ is the characteristic function of some set $T_n \subset A$, so $\mu(T_n) < \infty$.

Let $V \supset A$ be open so that $\overline{V}$ is compact; this set exists since $X$ is locally compact and $A$ is compact (by assumption). *To construct it, for each $x \in A$, let $V_x \subset F_x$ be a compact neighbourhood of $x$. Since $\{V_x\}_{x \in A}$ is an open cover for $A$, there exists a subcover $\{V_{x_i}\}_{i=1}^n$. Then $A \subset \overline{\bigcup_{i=1}^n V_x} \subset \bigcup_{i=1}^n F_{x_i}$ is a closed subset of a compact set, and thus compact.* Since $\mu(T_n) < \infty$, get $K_n$ compact, $V_n$ open, so that $K_n \subset T_n \subset V_n$ with $\mu(V_n \setminus K_n) < \epsilon/2^n$. We can assume $V_n \subset V$ since we can always take $V_n \cap V$, which is open.

By Urysohn's lemma, there exists $h_n \in C_c(X)$ with $K_n \prec h_n \prec V_n$. Define $g = \sum_{n=1}^{\infty} 2^{-n} \cdot h_n$ is a uniform limit, so $g$ is continuous and $\operatorname{supp} g \subset \overline{V}$. If $x \in K_n$, then $h_n(x) = 1$ and $t_n(x) = 2^{-n}$, so $2^{-n} \cdot h_n(x) = t_n(x)$. If $x \notin V_n$, then $h_n(x) = 0$ so $t_n(x) = 0$ and $2^{-n} \cdot h_n(x) = t_n(x)$. Thus

$$S = \{x \in A : f(x) \neq g(x)\} \subset \bigcup_{n=1}^{\infty} (V_n \setminus K_n)$$

and $\mu(S) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus K_n) < \epsilon$.

If $-A \leq f \leq A$, then $0 \leq f + A \leq 2A$ and apply the above theorem to $(f + A)/(2A)$ and get some $\hat{g}$. Then $2A\hat{g} - A$ has the desired properties. Additionally, for any real valued function, let $B_n = \{x \in X : |f(x)| > n\}$. Then $\bigcap_{n=1}^{\infty} B_n = \emptyset$, $\mu(B_1) \leq \mu(\operatorname{supp} f) < \infty$, and $B_{n+1} \subset B_n$ for all $n$. Thus $\mu(B_n) \to \mu(\bigcap B_n) = 0$. Let $N$ be such that $\mu(B_N) < \epsilon/2$, so if $x \notin B_N$, $f(x) \leq N$, and define $\tilde{f}(x) = (1 - \chi_{B_n(x)})f(x)$. Then $\tilde{f}$ is bounded, and apply the above to get $g \in C_c(x)$ so that $\mu\left(\{x : \tilde{f}(x) \neq g(x)\}\right) < \epsilon/2$. But then

$$\mu(\{x : g(x) \neq f(x)\}) \leq \mu(\{x : f(x) \neq \tilde{f}(x)\}) + \mu(\{x : \tilde{f}(x) \neq g(x)\}) = \epsilon$$

All that is left is to do this for complex valued functions, satisfying the additional constraint. To be precise, let $f$ be complex valued and write $f = f_1 + if_2$. Then for $\epsilon > 0$, get $g_1, g_2 \in C_c(X)$

satisfying the requirements for $\epsilon/2$ and set $g = g_1 + ig_2$. We will prove that $\sup|G| \leq \sup|f|$. If $\sup|f| = \infty$ we are done, so let $R = \sup_X |f|$. Let

$$\phi(z) = \begin{cases} z & : |z| \leq R \\ \frac{R\cdot z}{|z|} & : |z| > R \end{cases}$$

then $\phi$ is continuous and $|\phi| \leq R$. We already have $g \in C_c(X)$ so that $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$. Let $\tilde{g} = \phi \circ g$, which is also continuous and $|\tilde{g}| \leq R$. Finally, if $\tilde{f} \neq \tilde{g}$, then certainly $f \neq g$, so

$$\mu\{\tilde{g} \neq f\} = \mu\{\phi \circ f \neq \phi \circ g\} \leq \mu\{g \neq f\} < \epsilon$$

and we are done. ■

For the following corollary, we require the same context for $(X, \mathcal{M}, \mu)$ as in Lusin's Theorem.

**Cor. 2.5.2** *Let $f : X \to \mathbb{C}$ be measurable,* $\operatorname{supp} f \subset A$*, and $\mu(A) < \infty$ and $|f| \leq 1$. Then there exists $g_n \in C_c(X)$ with $|g_n| \leq 1$ and* $\lim g_n(x) = f_n(x)$ *almost everywhere.*

Proof Apply Lusin's theorem with $\epsilon = 1/n$ for each $g_n$. ■

# Chapter 3

# Complex Measures

## 3.1 Hilbert Spaces

### 3.1.1 Basic Definitions

A complex vector space $H$ is called an inner product space if there is a pairing $\langle \cdot, \cdot \rangle : H \to \mathbb{C}$ satisfying, for any $x, y \in H$ and $\alpha \in \mathbb{C}$

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$
3. $\langle x, x \rangle \geq 0$
4. $\langle x, x \rangle = 0$ if and only if $x = 0$

One can define a norm by $\|x\| = \sqrt{\langle x, x \rangle}$. The fact that this is acutally a norm is a consequence of the following proposition:

**Prop. 3.1.1** *On an inner product space $H$, the following inequalities hold:*

1. ***Cauchy-Schwarz:*** $|\langle x, y \rangle| \leq \|x\| \|y\|$
2. ***Triangle:*** $\|x + y\| \leq \|x\| + \|y\|$

From the triangle inequality, we can define a metric $d(x, y) = \|x - y\|$. Thus we call $H$ a Hilbert space if it is also complete with respect to this metric.

There are some standard continuous functions on $H$:

**Prop. 3.1.2** *For a fixed $y \in H$, the mappings*

$$x \mapsto \langle x, y \rangle, \quad x \mapsto \langle y, x \rangle, \quad x \mapsto \|x\|$$

*are continuous functions.*

The first map is particularly important since, to some $x \in H$, we can associate the continuous linear functional $\phi_x$ defined by $\phi_x(y) = \langle x, y \rangle$. More importantly, in Hilbert spaces, the converse holds as well. The first part is the content of (the original) Riesz representation theorem:

**Thm. 3.1.3** *Let $H$ be a Hilbert space, and $H^*$ be the vector space of continuous linear functions on $H^*$.*

1. *For each $L \in H^*$, then there exists a unique $y \in H$ such that*
$$Lx = \langle x, y \rangle$$
*for all $x \in H$.*
2. *$H^*$ is a Hilbert space with inner product*
$$\langle \phi_u, \phi_v \rangle = \langle v, u \rangle$$
*and is isomorphic (as a Hilbert space) to $H$, where $\phi_x(y) = \langle x, y \rangle$.*

### 3.1.2 The space $L^2(\mu)$

In particular, consider the space $L^2(\mu)$ where $(X, \mathcal{M}, \mu)$ has $X$ compact, and $\mu$ is a regular complex measure on $\mathcal{M}$. Then $L^2(\mu) = \{f : X \to \mathbb{C} : f \text{ measurable}, \|f\|_2 < \infty\}$ and we have

$$\langle f, g \rangle = \int_X f \overline{g} \, \mathrm{d}\mu, \quad \|f\|_2 = \left( \int_X |f|^2 \, \mathrm{d}\mu \right)^{1/2}$$

is the inner product on this space. We also have

**Thm. 3.1.4 (Riesz-Fisher)** *$L^2(\mu)$ is complete (every Cauchy sequence of functions converges w.r.t. the $L^2$ norm).*

so this inner product space is indeed a Hilbert space. Given a continuous linear functional $\Lambda$ on $H$, the Riesz representation theorem states that

$$\Lambda f = \langle f, g \rangle = \int_X f \overline{g} \, \mathrm{d}\mu$$

for some $g \in \mathcal{M}$. But then define $\phi(E) = \int_E \overline{g} \, \mathrm{d}\mu$ as the image measure of $\overline{g}$, so

$$\int_X f \overline{g} \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\phi$$

In other words, $\Lambda f = \int_X f \, \mathrm{d}\phi$, which looks much more similar to the standard Riesz representation theorem.

$L^2(\mu) = \{f : X \to \mathbb{C} : f \text{ is measurable}, \|f\|_2 < \infty\}$ is a Hilbert space (complete) with the inner product $\langle f, g \rangle = \int_X f \cdot \overline{g} \, \mathrm{d}\mu$.

**Thm. 3.1.5** *If $H$ is a Hilbert space, $L : H \to \mathbb{C}$ is continuous and linear, then there exits a unique $y \in H$ so that $L(X) = \langle x, y \rangle$ for all $x \in H$.*

## 3.2 Complex Measures

Let $\mathcal{M}$ be a $\sigma$−algebra on $X$.

**Def'n. 3.2.1** *$\mu : \mathcal{M} \to \overline{\mathbb{R}}$ is called a* ***signed measure*** *if it is countably addivite and $+\infty$ and $-\infty$ are not in the range at the same time.*

**Def'n. 3.2.2** *$\mu : \mathcal{M} \to \mathbb{C}$ is called a **complex measure** if it is countably additive: if $E_i$ are disjoint, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

For now, it is not at all obvious how to define integration with respect to a complex measure. This will appear following the proof of the Lebesgue-Radon-Nikodym theorem.

**Def'n. 3.2.3** *For a set $E \in \mathcal{M}$, a **(measurable) partition** of $E$ is $\{E_i : i = 1, 2, \ldots\}$ so that $E_i \cap E_j = \emptyset$ and $\bigcup_{i=1}^{\infty} E_i = E$ and $E_i \in \mathcal{M}$ for all $I$.*

Let's try to motivate the following definition. Given a complex measure $\mu$, we want to find a positive measure $\lambda$ which satisfies $|\mu(E)| \leq \lambda(E)$ for every $E \in \mathcal{M}$. Such a measure, if it exists, must satisfy

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i) \geq \sum_{i=1}^{\infty} |\mu(E_i)|$$

for any partition $\{E_i\}$ of $E$; thus, it must be at least as large as the supremum. With this in mind, we can define the following set function:

**Def'n. 3.2.4** *Let $\mu$ be a complex or signed measure. Its total variation*

$$|\mu| : \mathcal{M} \to [0, +\infty] = \sup\left\{\sum_{i=1}^{\infty} |\mu(E_i)| : \{E_i\} \text{ is a partition of } E\right\}$$

Conveniently, $|\mu|$ is actually a positive measure (which we will see below). More surprisingly, $|\mu|(X) < \infty$; thus, for any $E \in \mathcal{M}$, $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X)$, so every complex measure is actually a bounded measure and its image is contained in a disc of finite radius. This property is sometimes summarized by saying "$\mu$ is of bounded variation".

**Thm. 3.2.5** *$|\mu|$ is a positive measure.*

Proof Let $E \in \mathcal{M}$, and $\{E_i\}$ an arbitrary partition of $E$. We first see that $\sum_{i=1}^{\infty} |\mu|(E_i) \leq |\mu|(E)$. Let $t_i < |\mu|(E_i)$, so there exists a partition $\{A_{ij} : j \in \mathbb{N}\}$ of $E_i$ so that

$$\sum_{j=1}^{\infty} |\mu(A_{ij})| > t_i$$

for all $i$. Then since $\{A_{ij} : (i, j) \in \mathbb{N}^2\}$ is a partition of $E$, and

$$\sum_{i=1}^{\infty} t_i \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_{ij})| \leq |\mu|(E)$$

and since this holds for all $i$, we have $\sum_{i=1}^{\infty} |\mu|(E_i) \leq \mu(E)$.

We now see the opposite direction. Let $\{A_j : j \in \mathbb{N}\}$ be an arbitrary partition of $E$. The set $\{A_j \cap E_i : j \in \mathbb{N}\}$ is a partition of $E_i$, while $\{A_j \cap E_i : i \in \mathbb{N}\}$ is a partition of $A_j$. Then

$$\begin{aligned}\sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_j \cap E_i)| \\ &\leq \sum_{i=1}^{\infty} |\mu|(E_i)\end{aligned}$$

where absolute convergence allows us to change the order of summation. Since this holds for an arbitrary partition $\{A_j\}$ of $E$, taking the supremum over all partitions gives the total variation. Thus equality holds. ∎

**Lemma 3.2.6** *Let $z_1, z_2, \ldots, z_N \in \mathbb{C}$. Then there exists $S \subset \{1, 2, \ldots, N\}$ so that*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

Proof Let $z_k = |z_k| e^{i\alpha_k}$, and for $\theta \in [-\pi, \pi]$, let $S(\theta) = \{k \in \{1, 2, \ldots, N\} : \cos(\alpha_k - \theta) > 0\}$. Then

$$\begin{aligned}\left| \sum_{k \in S(\theta)} z_k \right| &= \left| \sum_{k \in S(\theta)} |z_k| e^{-i\theta} \right| \\ &\geq \operatorname{Re} \sum_{k \in S(\theta)} e^{-i\theta} z_k \\ &= \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta) := h(\theta)\end{aligned}$$

where $\cos^+ = \max\{\cos, 0\}$ is the positive part of cos, and $h : [-\pi, \pi] \to \mathbb{R}$ is a continuous function on a compact set. Thus it has a maximum at some $\theta_0$. Fix $S = S(\theta_0)$ and

$$\begin{aligned}\left| \sum_{k \in S} z_k \right| = h(\theta_0) &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} h \, \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta) \, \mathrm{d}\theta \\ &= \frac{1}{2\pi} \sum_{k=1}^{N} |z_k| \int_{-\pi}^{\pi} \cos^+(\alpha_k - \theta) \, \mathrm{d}\theta \\ &= \frac{1}{\pi} \sum_{k=1}^{N} |z_k|\end{aligned}$$

since $\int_{-\pi}^{\pi}\cos^+(\alpha_k-\theta)=2$ for all $k$. ■

**Thm. 3.2.7** *If $\mu$ is a complex measure, then $|\mu|(X)<\infty$.*

Proof First, let $E\in\mathcal{M}$ be such that $|\mu|(E)=+\infty$. We show that there exists $A,B$ so $E=A\cup B$ and $|\mu(A)|\geq 1$ and $|\mu(B)|\geq 1$. Set $t=\pi(1+|\mu(E)|)$ and since $|\mu(E)|>t$, there exists a partition $\{E_i\}$ of $E$ such that

$$\sum_{i=1}^{N}|\mu(E_i)|>t$$

for some $N$. Then by the lemma with $z_k=\mu(E_k)$, get $S$ so that $\left|\sum_{k\in S}\mu(E_k)\right|\geq\frac{1}{\pi}\sum_{k=1}^{N}|\mu(E_k)|$ Thus

$$|\mu(A)|\geq\frac{1}{\pi}\sum_{k=1}^{N}\left|\mu(E_k)\right|>\frac{t}{\pi}\geq 1$$

and let $B=E\setminus A$. Then

$$|\mu(B)|\geq|\mu(A)|-|\mu(E)|\geq\frac{t}{\pi}-\left(\frac{t}{\pi}-1\right)=1$$

so that $E=A\cup B$ with $|\mu(A)|\geq 1$ and $|\mu(B)|\geq 1$.

Now assume $|\mu|(X)=\infty$ and apply the above procedure get $A_1,B_1$ with $|\mu(A_1)|\geq 1$ and $|\mu(B_1)|\geq 1$. As well, at least one of $|\mu|(A_1)$, $|\mu|(B_1)$ is infinity. Without loss of generality, it is $B_1$, so repeat this procedure to $B_1$. Get a sequence $A_1,A_2,\ldots$ with $|\mu(A_i)|\geq 1$ and $A_i$ disjoint. But then by countable additivity,

$$\sum_{i=1}^{\infty}\mu(A_i)=\mu\left(\bigcup_{i=1}^{\infty}A_i\right)<\infty$$

since $\mu$ is finite, a contradiction, since the LHS does not converge. ■

Recall that $\mu:\mathcal{M}\to\mathbb{C}$ is a complex measure if it is countably additive. Then the total variation of $\mu$ is given by

$$|\mu|(E)=\sup\left\{\sum_{i=1}^{\infty}|\mu(E_i)|:\{E_i\}\text{ is a partition}\right\}$$

Then $|\mu|$ is a positive measure and $|\mu|(X)<\infty$. If $\mu,\lambda:\mathcal{M}\to\mathbb{C}$ are complex measures, then $(\mu+\lambda)(E)=\mu(E)+\lambda(E)$ and $(c\cdot\mu)(E)=c\cdot\mu(E)$. Thus the set of complex measures is a vector space. Let $\left\|\mu\right\|:=|\mu|(X)$.

If $\mu$ is a signed measure ($\mu:\mathcal{M}\to\overline{R}$), then the total variation is defined in the same way.

**Def'n. 3.2.8** *Let $\mu$ be a signed measure. The* ***positive variation*** *of $\mu$ is $\mu^+:=\frac{1}{2}(|\mu|+\mu)$ and the* ***negative variation*** *of $\mu$ is $\mu^-:=\frac{1}{2}(|\mu|-\mu)$.*

These are positive measures since $|\mu|(E)\geq|\mu(E)|$. We have $\mu=\mu^+-\mu^-$; this is called the Jordan decomposition since we represent a signed measure as a difference between two positive measures. Additionally, we have $|\mu|=\mu_++\mu_-$.

## 3.3 Absolute Continuity and Singular Measures

**Def'n. 3.3.1** *Let $\mu$ be a positive measure and $\lambda$ be an arbitrary (positive, signed, or complex) measure. Then $\lambda$ is **absolutely continuous** with respect to $\mu$ if $\mu(E) = 0 \Rightarrow \lambda(E) = 0$. We write $\lambda \ll \mu$.*

**Def'n. 3.3.2** *$\lambda$ is concentrated on a set $A \in \mathcal{M}$ if $\lambda(E) = \lambda(E \cap A)$ for all $E \in \mathcal{M}$.*

**Prop. 3.3.3** *$\lambda$ is concentrated on $A$ if and only if $\lambda(E) = 0$ if $E \cap A = \emptyset$.*

Proof Let $E \cap A = \emptyset$. Then $\lambda(E) = \lambda(E \cap A) = \lambda(\emptyset) = 0$. Conversely, let $E \in \mathcal{M}$. Then $\lambda(E) = \lambda(E \cap A) + \lambda(E \cap A^c) = \lambda(E \cap A)$. ■

**Def'n. 3.3.4** *$\lambda_1$ and $\lambda_2$ are called **mutually singular** if there exist disjoint sets $A$ and $B$ such that $\lambda_1$ is concentrated on $A$ and $\lambda_2$ is contentrated on $B$. Then $\lambda_1 \perp \lambda_2$.*

Let's summarize a number of properties concerning the aforementioned definitions and the total variation of a measure.

**Prop. 3.3.5** *Let $\mu$ be a positive measure, $\lambda, \lambda_1, \lambda_2$ be arbitrary measures (positive, signed, or complex). Then*
1. *If $\lambda$ is concentrated on $A$, then $|\lambda|$ is also concentrated on $A$.*
2. *If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$*
3. *If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.*
4. *If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.*
5. *If $\lambda \ll \mu$, then $|\lambda| \ll \mu$*
6. *If$\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$*
7. *If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.*

Proof These are all exercises in the definitions. For simplicity of notation (and so that I don't repeat myself too much), let $\lambda$ be concentrated on $A$, $\lambda_1$ on $A_1$, $\lambda_2$ on $A_2$, and $\mu$ on $B$.
1. Suppose $E \cap A = \emptyset$; we want to show $|\lambda|(E) = 0$. Let $\{E_i\}$ be an arbitrary partition of $E$, so $E_i \cap A = \emptyset$ and $\lambda(E_i) = 0$ since $\lambda$ is concentrated on $A$. Thus $\sum_{i=1}^{n} |\lambda(E_i)| = 0$ and since the partition was arbitrary, $|\lambda|(E) = 0$.
2. By (1), $|\lambda_1|$ is also concentrated on $A_1$ and $|\lambda_2|$ is concentrated on $A_2$, and $A_1 \cap A_2 = \emptyset$.
3. Let $\mu$ concentrated on $B_1$ with $B_1 \cap A_1 = \emptyset$, and $\mu$ is concentated on $B_2$ with $B_2 \cap A_2 = \emptyset$. If $A_1 \cap B = \emptyset$ and $A_2 \cap B = \emptyset$, then $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$. Furthermore, $\mu$ is concentrated on $B_1 \cap B_2$ since if $E \cap B_1 = \emptyset$ or $E \cap B_2 = \emptyset$, then $\mu(E) = 0$.
4. Let $\mu(E) = 0$. Then $\lambda_1(E) = 0$ and $\lambda_2(E) = 0$ so $(\lambda_1 + \lambda_2)(E) = 0$.
5. Let $\mu(E) = 0$ and let $\{E_i\}$ be a partition of $E$. Then $\mu(E_i) = 0$ for all $i$, so $\lambda(E_i) = 0$ for all $i$. Thus $\sum_{i=1}^{\infty} |\lambda(E_i)| = 0$ for any partition, so $|\lambda|(E) = 0$.
6. Since $\lambda_2 \perp \mu$, get disjoint $A_2, B$ so $\lambda_2$ is concentrated on $A_2$ and $\mu$ is concentrated on $B$. Then $\lambda_1$ is also concentrated on $B$ since, if $E \cap B = \emptyset$, then $\mu(E) = 0$ so $\lambda_1(E) = 0$ since $\lambda_1 \ll \mu$.
7. By (6.), $\lambda \perp \lambda$, and $A \cap A = \emptyset$ implies $A = \emptyset$, so $\lambda = 0$. ■

**Prop. 3.3.6** *Let $\mu$ be a positive measure, $\lambda$ a complex measure. Then the following are equivalent:*

1. $\lambda \ll \mu$
2. *For any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\mu(E) < \delta$ *so* $|\lambda(E)| < \epsilon$.

Proof (2 ⇒ 1). Let $\epsilon > 0$ and choose $\delta$ satisfying the requirement. Then let $\mu(E) = 0$, so $\mu(E) < \delta$ and $|\lambda(E)| < \epsilon$. This holds for any $\epsilon > 0$ so $\lambda(E) = 0$.

(1 ⇒ 2). For contradiction, assume there exists some $\epsilon > 0$ so that for each $\delta = 1/2^n$, there exists a set $E_n$ so that $\mu(E_n) < 1/2^n$ but $|\lambda(E_n)| \geq \epsilon$. As is standard, define

$$A_n = \bigcup_{k=n}^{\infty} E_k, \qquad A = \bigcap_{n=1}^{\infty} A_n$$

so that

$$\mu(A_n) \leq \sum_{k=n}^{\infty} \mu(E_k) \leq \sum_{k=n}^{\infty} \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

Thus since $\mu(A_1) < \infty$ and $A_1 \supset A_2 \supset \cdots$, $\mu(A) = 0$. Since $\lambda \ll \mu$, $|\lambda| \ll \mu$ so $|\lambda|(A) = 0$. However, $\lim_{n\to\infty} |\lambda|(A_n) = |\lambda|(A) = 0$ since $|\lambda|$ is a measure, while $|\lambda|(E_n) \geq |\lambda(E_n)| \geq \epsilon$, a contradiction. ■

If it important that $\lambda$ is finite; if not, then this may not hold. Set $f(x) = 1/|x|$, $\lambda(E) = \int_E f \,\mathrm{d}\mu$, and $\mu$ is the Lebesgue measure. However, for each $E = [-1/n, 1/n]$, and $\int_E f \,\mathrm{d}\mu = \infty$ while $\mu(E) = 1/2^n$.

### 3.3.1 The Lebesgue-Radon-Nikodym Theorem

**Lemma 3.3.7** *If* $\mu$ *is a positive,* $\sigma$*−finite measure (that is,* $X = \bigcup_{n=1}^{\infty} X_n$ *where* $\mu(X_n) < \infty$*), then there exists* $w \in L^1(\mu)$ *so that* $0 < w < 1$.

Proof Let $X = \bigcup\limits_{n=1}^{\infty} X_n$, and $\mu(X_n) < \infty$. Let

$$w_n(x) = \begin{cases} 0 & : x \in X \setminus X_n \\ \frac{1}{2^n(1+\mu(X_n))} & : x \in X_n \end{cases}$$

and $w(x) = \sum_{n=1}^{\infty} w_n(x)$. By construction, $0 < w < 1$ and

$$\int_X w \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int w_n \,\mathrm{d}\mu < \sum_{n=1}^{\infty} 1/2^n = 1$$

by Lebesgue's Monotone Convergence Theorem, so $w \in L^1(\mu)$. ■

**Thm. 3.3.8 (Lebesgue-Radon-Nikodym)** *Let* $\mu$ *be a positive,* $\sigma$*−finite measure,* $\lambda$ *be a complex measure on* $\mathcal{M}$.

(a) *There exists a unique decomposition of* $\lambda$ *as* $\lambda = \lambda_a + \lambda_s$ *such that* $\lambda_a \ll \mu$ *and* $\lambda_s \perp \mu$.

(b) *There exists a unique* $h \in L^1(\mu)$ *such that* $\lambda_a(E) = \int_E h \,\mathrm{d}\mu$ *for all* $E \in \mathcal{M}$. *This is the Radon-Nikodym derivative of* $\lambda_a$ *with respect to* $\mu$.

Proof Let's first see that the decomposition is unique. Assume $\lambda = \lambda_a + \lambda_s = \lambda_a' + \lambda_s'$, so $\lambda_a - \lambda_a' = \lambda_s' - \lambda_s$. Then since $\lambda_a - \lambda_a' \ll \mu$ and $\lambda_s' - \lambda_s \perp \mu$, we have $\lambda_a - \lambda_a' = 0 = \lambda_s' - \lambda_s$. We also see that $h$ is unique. Assume $h^*$ is another one, so $\lambda_a(E) = \int_E h^* \,\mathrm{d}\mu$. Write $h = h_1 + ih_2$, $h^* = h_1^* + ih_2^*$. Let $A_1 = \{x \in X : h_1(x) > h_1^*(x)\}$ and $A_2 = \{x \in X : h_1(x) < h_1^*(x)\}$. We will show $\mu(A_1) = \mu(A_2) = 0$ (so that $h_1 = h_1^*$ a.e.). In particular,

$$\int_{A_1} h \,\mathrm{d}\mu = \int_{A_1} h^* \,\mathrm{d}\mu \Rightarrow \int_{A_1} (h - h^*) \,\mathrm{d}\mu = 0$$

so $\mu(A_1) = 0$. We can argue similarly with $A_2$, so $h_1 = h_1^*$. In the exact same way, $h_2 = h_2^*$ a.e., showing uniqueness.

Now, for the hard part of the theorem, we show existence of $\lambda_a$, $\lambda_s$, $\mu$. Write $\lambda = \lambda_1^+ - \lambda_1^- + i(\lambda_2^+ - \lambda_2^i)$ and argue separately for each $\lambda_i^\pm$. We thus assume without loss of generality that $\lambda$ is a positive, finite measure. From the previous lemma, get $w \in L^1(\mu)$ such that $0 < w < 1$, and define $\phi(E) = \lambda(E) + \int_E w \,\mathrm{d}\mu$, so $\phi$ is a positive finite measure. We have $\int_X f \,\mathrm{d}\phi = \int_X f \,\mathrm{d}\lambda + \int_X f w \,\mathrm{d}\mu$; this holds for characteristic functions, and thus for simple functions, and finally for any non-negative measurable $f$. Let $f \in L^2(\phi)$, so

$$\begin{aligned} \left|\int_X f \,\mathrm{d}\lambda\right| &\leq \int_X |f| \,\mathrm{d}\lambda \\ &\leq \int_X |f| \,\mathrm{d}\phi = \int_X 1 \cdot |f| \,\mathrm{d}\phi \\ &= \langle 1, |f| \rangle_{L^2(\phi)} \\ &\leq \sqrt{\int_X 1 \,\mathrm{d}\phi} \cdot \sqrt{\int_X |f|^2 \,\mathrm{d}\phi} \\ &< \infty \end{aligned}$$

since $\phi(X) < \infty$ by finiteness of $\lambda, \mu$ and the second term since $f \in L^2(\phi)$. Let $T(f) = \int_X f \,\mathrm{d}\lambda$, so $T : L^2(\phi) \to \mathbb{C}$ is a bounded (and therefore continuous) linear functional. By the Riesz theorem for Hilbert spaces, there exists $\overline{g} \in L^2(\phi)$ so that $T(f) = \langle f, \overline{g} \rangle = \int_X f \cdot g \,\mathrm{d}\phi$. Thus

$$\int_X f \,\mathrm{d}\lambda = \int_X f g \,\mathrm{d}\phi \tag{1}$$

Thus substituting $f = \chi_E$, $\lambda(E) = \int_E g \,\mathrm{d}\phi$. Let's see that $0 \leq g \leq 1$ a.e.$[\phi]$. Define

$$A_1 = \{x \in X : g(x) < 0\}, \quad A_2 = \{x \in X : g(x) > 1\}$$

Then $0 \leq \lambda(A_1) = \int_{A_1} g \,\mathrm{d}\phi < 0$ if $\phi(A_1) > 0$, so $\phi(A_1) = 0$. Similarly, $\lambda(A_2) = \int_{A_2} g \,\mathrm{d}\phi > \phi(A_2) \geq \lambda(A_2)$, so $\phi(A_2) = 0$. We thus have by (1) and the definition of $\phi$,

$$\int_X f \,\mathrm{d}\lambda = \int_X f g \,\mathrm{d}\lambda + \int_X f g w \,\mathrm{d}\mu \quad \Longrightarrow \quad \int_X f(1-g) \,\mathrm{d}\lambda = \int_X f g w \,\mathrm{d}\mu \tag{2}$$

Let $A = \{x \in X : 0 \leq g(x) < 1\}$, $B = \{x \in X : g(x) = 1\}$, so $A \cap B = \emptyset$ and $\mu(X \setminus (A \cup B)) = 0$. Finally, we can define

$$\lambda_a(E) = \lambda(E \cap A), \quad \lambda_s(E) = \lambda(E \cap B)$$

Let's prove that they have the desired properties. Substitute $f = \chi_B$ into (2), so $\chi_B(1-g) = 0$ everywhere. We thus have $\int_B w \,\mathrm{d}\mu = 0$ so $\mu(B) = 0$. Thus $\mu$ is concentrated on $A$ and, by definition, $\lambda_s$ is concentrated on $B$, so $\lambda_s \perp \mu$.

Now apply (2) with $f = (1+g+g^2+\cdots+g^n)\chi_E$ for any $E \in \mathcal{M}$ and every $n \in \mathbb{N}$. On the LHS of (2), we have

$$\begin{aligned}
\lim_{n\to\infty}\int_E (1-g^{n+1})\,\mathrm{d}\lambda &= \lim_{n\to\infty}\int_{E\cap A}(1-g^{n+1})\,\mathrm{d}\lambda + \lim_{n\to\infty}\int_{E\cap B}(1-g^{n+1})\,\mathrm{d}\lambda\\
&= \int_{E\cap A}\lim_{n\to\infty}(1-g^{n+1})\,\mathrm{d}\lambda\\
&= \int_{E\cap A} 1\,\mathrm{d}\lambda\\
&= \lambda(E\cap A) = \lambda_a(E)
\end{aligned}$$

by the monotone convergence theorem, since $1-g^{n+1} = 0$ on $B$ and $1-g^{n+1} \to 1$ on $A$ monotonically. On the RHS of (2), we have

$$\lim_{n\to\infty}\int_E g(1+g+\cdots+g^n)w\,\mathrm{d}\mu = \int_E \lim_{n\to\infty} g(1+g+\cdots+g^n)w\,\mathrm{d}\mu = \int_E h\,\mathrm{d}\mu$$

monotonically, where $h = \lim_{n\to\infty} g(1+g+\cdots+g^n)2$. Thus $\lambda_a(E) = \int_E h\,\mathrm{d}\lambda$ for all $E \in \mathcal{M}$. But then it follows directly that $\lambda_a \ll \mu$ and $h \in L^1(\mu)$, so $\lambda_a(X) < \infty$. ■

### 3.3.2 Applications of Radon-Nikodym

**Thm. 3.3.9** *Let $\mu$ be a complex measure. Then there exists a measurable $h$ such that $|h| = 1$ and $\mu(E) = \int_E h\,\mathrm{d}|\mu|$. This is called the polar decomposition of $\mu$.*

Proof Note that $\mu \ll |\mu|$ since if $|\mu|(E) = 0$ then $|\mu(E)| = 0$. Thus get $h \in L^1(|\mu|)$ so that $\mu(E) = \int_E h\,\mathrm{d}|\mu|$. We will see that $|h| \le 1$ a.e. and $|h| \ge 1$ a.e. Let $A_r = \{x \in X : |h(x)| < r\}$. Let $\{E_j\}$ be a partition of $A_r$. Then

$$\sum_{j=1}^{\infty}|\mu(E_j)| = \sum_{j=1}^{\infty}\left|\int_{E_j} h\,\mathrm{d}|\mu|\right| \le \sum_{j=1}^{\infty}\int_{E_j}|h|\,\mathrm{d}|\mu| \le \sum_{j=1}^{\infty} r\cdot|\mu|(E_j) = r\cdot|\mu|(A_r)$$

for any partition, so taking the supremum over all partitions, $|\mu|(A_r) \le r\cdot|\mu|(A_r)$, so $|\mu|(A_r) = 0$ when $r < 1$.

Otherwise, set $S = \mathbb{C}\setminus\overline{B_1(0)}$. Let $B_r(z) \subset S$ for some $z$, and define $E = h^{-1}(B_r(z))$. We want to show that $|\mu|(E) = 0$. Assuming so, write $S$ as a countable union of open balls $B_1, B_2, \ldots$ so that

$$\mu(\{x : |h(x)| > 1\}) = \mu(h^{-1}(S)) = \sum_{i=1}^{\infty}\mu(h^{-1}(B_i)) = 0$$

Thus let's show that $\mu(E) = 0$. Assume not; then, for any $E$, we have

$$\left|\frac{1}{|\mu|(E)}\cdot\int_E h\,\mathrm{d}|\mu|\right| = \frac{|\mu(E)|}{|\mu|(E)} \le 1$$

so that

$$\begin{aligned}\left|\frac{1}{|\mu|(E)}\int_E h\,\mathrm{d}|\mu| - z\right| &= \left|\frac{1}{|\mu|(E)}\cdot\int_E (h-z)\,\mathrm{d}|\mu|\right| \\ &\leq \frac{1}{|\mu|(E)}\int_E |h-z|\,\mathrm{d}|\mu| \\ &< \frac{r\cdot|\mu|(E)}{|\mu|(E)} = r\end{aligned}$$

a contradiction, since $B_r(z)\subset S$. ∎

**Thm. 3.3.10** *Let $\mu$ be a positive measure, and $g\in L^1(\mu)$. Let $\lambda(E)=\int_E g\,\mathrm{d}\mu$. Then $|\lambda|(E)=\int_E |g|\,\mathrm{d}\mu$.*

Proof By the previous theorem, there exists $h$ so that $|h|=1$ and $\lambda(E)=\int_E h\,\mathrm{d}\lambda$. Then

$$\int_E g\,\mathrm{d}\mu = \int_E h\,\mathrm{d}\lambda \quad\Longrightarrow\quad \int_X fg\,\mathrm{d}\mu = \int_X fh\,\mathrm{d}\lambda$$

by monotone convergence, for any measurable $f$. In particular, consider $f=\overline{h}\cdot\chi_E$. Then

$$\int_E \overline{h}g\,\mathrm{d}\mu = \int_E \mathrm{d}|\lambda| = |\lambda|(E)$$

and since the RHS is real and non-negative and $E$ is an arbitrary set, $\overline{h}g$ is real and non-negative almost everywhere. Let $x\in X$ be such that $\overline{h}(x)\cdot g(x)\geq 0$, and let $g(x)=r\cdot e^{i\phi}$, $\overline{h}=e^{i\alpha}$. Then $\overline{h}(x)g(x)=re^{i(\alpha+\phi)}=r=|g|$ since it is non-negative and real, so $\overline{h}\cdot h=|h|$ a.e.$[\mu]$. ∎

With these theorems, we are in place to prove the following decomposition theorem for measures. Recall that

$$\mu^+=\frac{1}{2}(|\mu|+\mu),\quad \mu^-=\frac{1}{2}(|\mu|-\mu)$$

so $\mu=\mu^+-\mu^-$ and $|\mu|=\mu^++\mu^-$.

**Thm. 3.3.11 (Hahn Decomposition)** *Let $\mu$ be a signed measure. Then there exists $A,B\in\mathcal{M}$ so that $A\cup B=X$, $A\cap B=\emptyset$, and $\mu^+(E)=\mu(E\cap A)$ and $\mu^-(E)=-\mu(E\cap B)$.*

Proof By the first theorem, get $h$ so that $|h|=1$ and $\mu(E)=\int_E h\,\mathrm{d}|\mu|$. Since $h$ is real-valued, $h=\pm 1$. Let $A=h^{-1}(\{1\})$, $B=h^{-1}(\{-1\})$. Then

$$\frac{1}{2}(h(x)+1)=\begin{cases} h(x) & : x\in A\\ 0 & : x\in B\end{cases}$$

so that since $\mu^+=\frac{1}{2}(|\mu|+\mu)$,

$$\mu^+(E)=\frac{1}{2}\int_E(1+h)\,\mathrm{d}|\mu| = \int_{E\cap A} h\,\mathrm{d}|\mu| = \mu(E\cap A)$$

The same approach works with $\mu^-$. ∎

**Cor. 3.3.12** *If $\mu=\lambda_1-\lambda_2$, where $\lambda_1,\lambda_2$ are positive measures, then $\lambda_1\geq\mu^+$ and $\lambda_2\geq\mu^-$.*

Proof Since $\mu(E)\leq\lambda_1(E)$ for all $E$,

$$\mu^+(E)=\mu(E\cap A)\leq\lambda_1(E\cap A)\leq\lambda_1(E)$$

and similarly for $\mu^-$. ∎

## 3.4 Representation Theorem for Bounded Linear Functionals

Let $B$ be a Banach space (a complete, normed vector space).

**Def'n. 3.4.1** *$L : B \to \mathbb{C}$ is called a **bounded linear functional** if $L(\alpha x + y) = \alpha L(x) + L(y)$ for $\alpha \in \mathbb{C}$, $x, y \in B$ and*

$$\|L\| := \sup_{\|x\|\leq 1} |L(x)| < \infty$$

*Then $|L(x)| \leq \|L\| \cdot \|x\|$. Let $B^*$ denote the collection of bounded linear functionals, called the **dual space** of $B$. This is a normed vector space with pointwise addition and scalar multiplication.*

**Prop. 3.4.2** *Let $B$ be any normed space. Then $B^*$ is a Banach space.*

A general question in functional analysis is the characterization of the dual soace of a banach space $B$. When $B$ is a Hilbert space, the inner product provides this representation via the standard Riesz representation theorem (see Hilbert space section in notes). If $B = L^p(\mu)$, it was shown that $B^* = L^q(\mu)$ where $1/q + 1/p = 1$.

To frame our general approach, note that $C_c(X)$ is not complete (for non-compact $C$). For example, when $X = \mathbb{R}$, $1/(1+x^2)$ is not in $C_c(\mathbb{R})$, but it is the limit of a sequence of functions in $C_c(X)$.

**Def'n. 3.4.3** *We say $f : X \to \mathbb{C}$ **vanishes at infinity** if for all $\epsilon > 0$, there exists $K \subset X$ compact such that for all $x \notin K$, $|f(x)| < \epsilon$.*

Let $C_0(X)$ be the set of comples functions which vanish at infinity, and the norm $\|f\| = \sup_{x\in X} |f(x)|$. Note that $C_c(X) \subset C_0(X)$.

**Prop. 3.4.4** *$C_0(X)$ is complete.*

Proof Let $(f_n) \subset C_0(X)$ be Cauchy. Then for every $\epsilon > 0$, get $N$ so that $\|f_n - f_m\| < \epsilon$ for all $n, m \geq N$. Thus $(f_n(x)) \subset \mathbb{C}$ is Cauchy, so $(f_n) \to f$ pointwise. Let's first show that $f$ is continuous. Certainly $\|f - f_n\| \leq \epsilon$ for all $n \geq N$, but then $f_n \to f$ uniformly, so $f$ is continuous.

Furthermore, $f$ vanishes at infinity since to any $\epsilon > 0$, get $N$ so that $\|f - f_N\| < \epsilon$, and since $f_N$ vanishes at infinity, get $K$ compact so that $|f_N(x)| < \epsilon$ if $x \notin K$. Thus $|f(x)| < 2\epsilon$ if $x \notin K$.

Note that $f_n(x) - \epsilon < f_m(x) < f_m(x) + \epsilon$ for all $n, m > N$ so that $|f(x) - f_n(x)| \leq \epsilon$ for all $n \geq N$. Let's show that the limit is actually uniform. For any $\epsilon > 0$, since $f_n$ vanishes at infinity, get $K_n$ compact so that $f_n < \epsilon$ outside $K_n$. ■

In fact, $C_c(X)$ is not complete; its closure is $C_0(X)$.

**Def'n. 3.4.5** *Let $\mathcal{M}$ be a $\sigma-$algebra on $X$, and let $\mu : \mathcal{M} \to \mathbb{C}$ be a complex measure. Then there exists $h : X \to \mathbb{C}$ measurable so that $\mu(E) = \int_E h \,\mathrm{d}|\mu|$. Now for any $f : X \to \mathbb{C}$, we define*

$$\int_E f \,\mathrm{d}\mu = \int_E f h \,\mathrm{d}|\mu|$$

*where $|\mu|$ is a positive measure.*

**Def'n. 3.4.6** *A complex measure $\mu$ is regular of $|\mu|$ is regular.*

An example of a bounded linear functional on $C_0(X)$ is the map $\phi : C_0(X) \to \mathbb{C}$ given by

$$\phi(f) = \int_X \mathrm{d}\mu$$

with a given complex measure $\mu$.

**Prop. 3.4.7** *$\phi$ is bounded, linear, and $\|\phi\| \leq \|\mu\|$.*

Proof Linearity is clear, so let $f \in C_0(X)$ with $\|f\| \leq 1$. Then $|f(x)| \leq 1$ for all $x$, so

$$|\phi(f)| = \left|\int_X f \cdot h \,\mathrm{d}|\mu|\right| \leq \int_X |f| \,\mathrm{d}|\mu| \leq \int_X 1 \,\mathrm{d}|\mu| = |\mu|(X) = \|\mu\| \qquad \blacksquare$$

**Thm. 3.4.8** *Every bounded linear function $\phi : C_0(X) \to \mathbb{C}$ can be given uniquely in the form $\phi(f) = \int_X f \,\mathrm{d}\mu$ and $\|\phi\| = \|\mu\|$. In other words, $C_0(X)^* = \mathrm{Meas}(X, \mathbb{C})$.*

Proof To see uniqueness, assume $\mu_1, \mu_2 \in \mathrm{Meas}(X, \mathbb{C})$ such that $\phi(f) = \int_X f \,\mathrm{d}\mu_i$ for $i = 1, 2$. Let $\mu = \mu_1 - \mu_2$; we will see that $\mu = 0$. Get $h$ so that $\int_X f \,\mathrm{d}\mu = \int_X fh \,\mathrm{d}|\mu|$, so $\int_X f \cdot h \,\mathrm{d}|\mu| = 0$ for all $f \in C_0(X)$. Recall that

$$|\mu|(X) = \int_X |h| \,\mathrm{d}|\mu| = \int_X h(f_n - \overline{h}) \,\mathrm{d}|\mu| \leq \int_X |h||f_n \overline{h}| \,\mathrm{d}|\mu| \to 0$$

since $C_0(X)$ is dense in $L^1(|\mu|)$, we can get $f_n \to \overline{h}$ in $L^1$. Thus $|\mu|(X) = 0$ so $|\mu|(E) = 0$ for all $E$ so $\mu(E) = 0$.

The real content of this theorem is uniqueness. For $f \geq 0$, define

$$\Lambda f = \sup\{|\phi(h)| : h \in C_c(X), |h| \leq f\}$$

If $f \in C_c(X)$, then $f = f_1^+ - f_1^- + i(f_2^+ - f_2^-)$ and define $\Lambda f = \Lambda f_1^+ - \Lambda f_1^- + i\left(\lambda f_2^+ - \Lambda f_2^-\right)$. We will see that $\Lambda : C_c(X) \to \mathbb{C}$ is a positive, linear functional. It suffices to show this for positive functions. Let's see that $\Lambda(f + g) \geq \Lambda f + \Lambda g$. To $f$ and $g$, there exists $h_1, h_2 \in C_c(X)$ so that $\Lambda f \leq |\phi(h_1)| + \epsilon$, $\Lambda g \leq |\phi(h_2)| + \epsilon$, and $|h_1| \leq f$, $|h_2| \leq g$. Let $\alpha_i$ be such that $|\phi(h_i)| = \alpha_i \phi(h_i)$, so $|\alpha_i| = 1$. Then

$$\begin{aligned}
\Lambda f + \Lambda g &\leq |\phi(h_1)| + |\phi(h_2)| + 2\epsilon \\
&= \phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\epsilon \\
&\leq \Lambda(|\alpha_1 h_1 + \alpha_2 h_2|) + 2\epsilon \\
&\leq \Lambda(|h_1| + |h_2|) + 2\epsilon \\
&\leq \Lambda(f + g) + 2\epsilon
\end{aligned}$$

for any $\epsilon > 0$, as required.

Now, let's see that $\Lambda(f + g) \leq \Lambda f + \Lambda g$. Let $h \in C_c(X)$, such that $|h| \leq f + g$. Let $V = \{x \in X : f(x) + g(x) > 0\}$ and define

$$h_1 = \frac{f \cdot h}{f + g}, \qquad h_2 = \frac{g \cdot h}{f + g}$$

on $V$, and 0 on $V^c$. One can verify that $h_1, h_2 \in C_c(X)$. Then

$$|\phi(h)| = |\phi(h_1) + \phi(h_2)| \leq |\phi(h_1)| + |\phi(h_2)| \leq \Lambda f + \Lambda g$$

and since this is true for all $|h| \leq f + g$, taking supremums, we have $\Lambda(f+g) \leq \Lambda f + \Lambda g$.

Furthermore, we have

$$|\phi(f)| \leq \Lambda(|f|) \leq \|f\|$$

for all $f \in C_c(X)$. This follows since for all $|h| \leq f + g$, $|\phi(h)| \leq \|h\| \leq \|f\|$. Thus $\Lambda$ is a positive linear functional on $C_c(X)$, and by the Riesz representation theorem, there exists a positive Borel measure $\lambda$ so that

$$\Lambda f = \int_X f \, \mathrm{d}\lambda$$

for all $f \in C_c(X)$. Since $\lambda(X) = \sup\{\Lambda f : f \prec X\}$, it follows that$\Lambda f \leq \|f\| = 1$. Then

$$|\phi(f)| \leq \Lambda(|f|) = \int_X |f| \, \mathrm{d}\lambda = \|f\|_{L^1(\lambda)}$$

Note that $\phi$ is defined in $C_0(X)$. Since $C_0(X)$ is dense in $L^1(\lambda)$, If $f \in L^1(\lambda)$, then there exists a sequence $(f_n) \subset C_0(X)$ such that $\|f_n - f\|_{L^1(\lambda)} \to 0$. Thus let $\widetilde{\phi}(f) = \lim_{n\to\infty} \phi(f_n)$ be an extension of $\phi$ to $L^1(\lambda)$. Thus $\widetilde{\phi} : L^1(\lambda) \to \mathbb{C}$ is a bounded linear functional.

Since $\widetilde{\phi} \in L^1(\lambda)^*$. Then by the Riesz representation theorem for $L^p$, we have $(L^p)^* = L^q$, so when $p = q$, we have $\tilde{\phi} \in L^\infty(\lambda)$. Thus there exists a bounded measurable function such that

$$\widetilde{\phi}(f) = \int_f \cdot g \, \mathrm{d}\lambda$$

for any $f \in L^1(\lambda)$, and $\left\|\widetilde{\phi}\right\| = \sup|g|$, so $|g| \leq 1$. Now define $\mu(E) = \int_E g \, \mathrm{d}\lambda$.

If $f = \chi_E$, then $\widetilde{\phi}(\chi_E) = \int_E g \, \mathrm{d}\lambda = \mu(E) = \int_X \chi_E \, \mathrm{d}\mu$. Thus $\widetilde{\phi}(s) = \int_X s \, \mathrm{d}\mu$, for simple functions, so by LMC, $\widetilde{\phi}(f) = \int_X f \, \mathrm{d}\mu$ for non-negative functions. Thus $\Lambda f = \int_X f \, \mathrm{d}\lambda$ for all $f \in C_c(X)$.

It remains to prove that $1 = \left\|\phi\right\| = \left\|\mu\right\| = |\mu|(X)$. Thus suppose $|f| \leq 1$. Then

$$|\phi(f)| = \left|\int_X f \cdot g \, \mathrm{d}\lambda\right| \leq \int_X |f| \cdot |g| \, \mathrm{d}\lambda \leq \int_X |g| \, \mathrm{d}\lambda$$

and taking the supremum over all $|f| \leq 1$, we have

$$1 = \left\|\phi\right\| \leq \int_X |g| \, \mathrm{d}\lambda \leq \int_X 1 \, \mathrm{d}\lambda = \lambda(X) \leq 1$$

We thus have that $|\mu|(X) = \int_X |g| \, \mathrm{d}\lambda = 1$, since $\mu(X) = \int_X g \, \mathrm{d}\lambda$ (absolute value by some theorem before).

Thus for $\phi \in C_0(X)^*$, there exists a unique $\mu \in \mathrm{Meas}(X, \mathbb{C})$ so that $\phi(g) = \int_X f \, \mathrm{d}\mu$, and $\left\|\phi\right\| = \left\|\mu\right\|$. ∎

**Cor. 3.4.9** $\mathrm{Meas}(X, \mathbb{C})$ *is complete.*

Proof The dual space of a normed space is complete. ∎

# Chapter 4

# Differentation of Measures

Let $f \in L^1(\mathbb{R})$ and set $\mu(E) = \int_E f\,\mathrm{d}\mu$. If $f$ is continuous at $x$, then

$$\lim_{r\to 0} \frac{\mu((x-r,x+r))}{m((x-r,r+r))} = f(x)$$

To see this, by continuity, for every $\epsilon > 0$ get $\delta$ so that whenever $|x-y| < \delta$, $|f(x)-f(y)| < \epsilon$. Let $r < \delta$, and then

$$\begin{aligned}
\left|\frac{\mu((x-r,x+r))}{2r} - f(x)\right| &= \frac{1}{2r}\left|\int_{x-r}^{x+r} f(y)\,\mathrm{d}y - \int_{x-r}^{x+r} f(x)\,\mathrm{d}y\right| \\
&\leq \frac{1}{2}\int_{x-r}^{x+r} |f(x)-f(y)|\,\mathrm{d}y \\
&< \frac{1}{2r}\cdot\epsilon\cdot 2r = \epsilon
\end{aligned}$$

**Def'n. 4.0.1** *Let $f \in L^1(\mathbb{R}^k)$. Then $x \in \mathbb{R}^k$ is called a* ***Lebesgue point*** *of $f$ if*

$$\lim_{r\to 0} \frac{1}{m(B_r(x))}\int_{B_r(x)} |f(x)-f(y)|\,\mathrm{d}y = 0$$

**Prop. 4.0.2** *If $f$ is continuous at $x$, then $x$ is a Lebesgue point.*

**Prop. 4.0.3** *Let $f \in L^1(\mathbb{R}^k)$, and let $\mu(E) = \int_E f\,\mathrm{d}m$. If $x$ is a Lebesgue point of $f$, then* $\lim_{r\to 0} \frac{\mu(B_r(x))}{m(B_r(x))} = f(x)$.

**Def'n. 4.0.4** *Let $x \in \mathbb{R}^k$. A sequence $\{E_i : i \in \mathbb{N}\}$ of sets (in $\mathbb{R}^k$) shrinks regularly to $x$ if for all $\alpha > 0$, there exists $(r_i)$ of positive numbers with $r_i \to 0$ and $E_i \subset B_{r_i}(x)$ and $m(E_i) \geq \alpha \cdot m(B_{r_i}(x))$.*

**Ex. 4.0.5**

1. $k = 1$, $E_i = (x-1/i, x+1/i)$. Then $\alpha = 1$ and $r_i = 1/i$.
2. $k = 1$, $E_i = (x, x+1/i)$. Then $\alpha = 1/2$ and $r_i = 1/i$.
3. $k = 2$, $x = (0,0)$, $E_i = (-1/i, 1/i)\times(-1/i, 1/i)$, $r_i = \sqrt{2}/i$, $\alpha = \pi/2$.
4. $k = 2$, $x = (0,0)$, $E_i = (-1/i, 1/i)\times(-1/i^2, 1/i^2)$. Then $r_i \geq 1/i$ and $m(B_{r_i}(x)) = \pi/i^2$ while $m(E_i) = 4/i^3$. But then there does not exist an $\alpha > 0$.

**Def'n. 4.0.6** *Let $\mu : \mathcal{B} \to \mathbb{C}$ be a complex Borel measure in $\mathbb{R}^k$. Then $\mu$ is **differentiable** at a point $x \in \mathbb{R}^k$ if for any sequence $\{E_n : n \in \mathbb{N}\}$ shrinking regularly to $x$, the limit*

$$\lim_{n\to\infty} \frac{\mu(E_n)}{m(E_n)}$$

*exists and is finite. This value is called the **derivative** of $\mu$ at $x$.*

Let's quickly see that this value is well-defined (i.e. does not depend on $\{E_n\}$).

Proof Let $\{E_n\}$ and $\{F_n\}$ shrink regular to $x$, and let $G_1 = E_1$, $G_2 = F_1$, $G_3 = E_2$, etc. Then $\{G_n\}$ shrinks regularly to $x$ so

$$\lim_{n\to\infty} \frac{\mu(G_n)}{m(G_n)}$$

exists and has $\mu(E_n)/m(E_n)$ as a subsequence (and similarly for $F_n$). ■

**Thm. 4.0.7** *If $\mu \ll m$, then $\mu$ is almost everywhere differentiable and $\mu' = f$ is the Radon-Nikodym derivative of $\mu$.*

Proof Recall that almost every point is a Lebesgue point of $f$. For such a Lebesgue point $x$, we show $\mu'(x) = f(x)$. Let $\{E_n\}$ be a sequence shrinking regularly to $x$. Then

$$\begin{aligned}
\left|\frac{\mu(E_n)}{m(E_n)} - f(x)\right| &= \frac{1}{m(E_n)} \cdot \left|\int_{E_n} f(y)\,\mathrm{d}y - \int_{E_n} f(x)\,\mathrm{d}y\right| \\
&\leq \frac{1}{\alpha} \cdot m(B_{r_n}(x)) \int_{E_n} |f(y) - f(x)|\,\mathrm{d}y \\
&\leq \frac{1}{\alpha} \cdot \frac{1}{m(B_{r_n}(x))} \int_{B_{r_n}(x)} |f(x) - f(y)|\,\mathrm{d}y
\end{aligned}$$

which tends to 0 as $n$ goes to infinity (and $r_n \to 0$) since $x$ is a Lebesgue point of $f$. ■

**Thm. 4.0.8** *If $\mu \perp m$, then $\mu$ is almost everywhere diferentiable and $\mu' = 0$ almost everywhere.*

**Def'n. 4.0.9** *Let $f \in L^1(\mathbb{R}^k)$. Let $Mf : \mathbb{R}^k \to [0, +\infty]$ be defined by*

$$(Mf)(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|\,\mathrm{d}m$$

**Thm. 4.0.10** *If $f \in L^1(\mathbb{R}^k)$, then almost every $x$ is a Lebesgue point.*

**Lemma 4.0.11** *$Mf$ is measurable and $m(\{x \in \mathbb{R}^k : (Mf)(x) > \alpha) \leq \frac{3^k}{\alpha} \cdot \|f\|_1$.*

Proof Let $\mu(E) = \int_E f\,\mathrm{d}m$, so $|\mu|(E) = \int_E |f|\,\mathrm{d}m$. Then

$$(Mf)(x) = \sup_{r>0} \frac{|\mu|(B_r(x))}{m(B_r(x))} =: (M\mu)(x)$$

so $M\mu = Mf$. We need to prove this for a general Borel measure $\mu$. ■

**Lemma 4.0.12** *Let $x_1, x_2, \ldots, x_N \in \mathbb{R}^k$, $r_1, r_2, \ldots, r_N > 0$. Let $W = \bigcup_{n=1}^N B_{r_n}(x_n)$. Then there exists $S \subset \{1, 2, \ldots, N\}$ such that*

1. *$B_{r_i}(x_i)$ for $i \in S$ are disjoint*
2. *$W \subset \bigcup_{i \in S} B_{3r_i}(x_i)$*
3. *$m(W) \leq 3^k \cdot \sum_{i \in S} m(B_{r_i}(x_i))$.*

Proof Construct such a set inductively. Let $r_{i_1}$ be the set with largest radius. Let $r_{i_2}$ be the set with largest radius disjoint from $B_{r_{i_1}}(x_{i_1})$, etc. ■

**Lemma 4.0.13** $m(\{x \in \mathbb{R}^k : (M\mu)(x) > \alpha\}) \leq 3^k \alpha^{-1} \|\mu\|$

Proof Do this for compact subsets, and take supremums. Since $K$ is covered by open balls with $|\mu|(B_r(x)) > \alpha \cdot m(B_r(x))$. ■

Proof Define

$$(T_r f)(x) = \frac{1}{B_r(x)} \int_{B_r(x)} |f(y) - f(x)| \, \mathrm{d}y$$

and

$$(Tf)(x) = \limsup_{r \to 0} T_r f(x)$$

We show that $Tf = 0$ almost everywhere, to the limit exists and is 0. Let $\alpha > 0$, $n \in \mathbb{N}$ be arbitrary; we prove that $\{Tf > 2\alpha\}$ is measurable and $m(\{Tf > 2\alpha\}) = 0$. Let $g \in C(\mathbb{R}^k)$ such that $\|f - g\|_1 < 1/n$ by Lusin's theorem, and note that $Tg = 0$ since it is continuous. Let $h = f - g$. Then $T_r h \leq Mh + |h|$ and $T_r f < T_r g + T_r$ so as $r \to 0$, $Tf \leq 0 + Th \leq Mh + |h|$. Thus $\{Tf > 2\alpha\} \subset \{Mh > \alpha\} \cup \{|h| > \alpha\}$. By choice of $g$, $m(\{|h| > \alpha\}) < \frac{1}{\alpha \cdot n}$. Similarly, $m(\{Mh > \alpha\}) \leq 3^k \alpha^{-1} \cdot \frac{1}{n}$. Thus

$$m(\{Tf > 2\alpha\}) \leq \frac{3^k + 1}{\alpha \cdot n}$$

But then since the inclusion holds for all $n$, it must be 0. Then since $m$ is complete, $\{Tf > 2\alpha\}$ is a measurable because t is a subset of a zero measure set. ■

## 4.1 Theory of Real Functions

**Prop. 4.1.1** *Let $f \in L^1(\mathbb{R})$, $a \in [-\infty, +\infty)$. Let $F(x) = \int_a^x f \, \mathrm{d}m$. Then $F$ is continuous.*

Proof We have

$$|F(x+h) - f(X)| = \left| \int_x^{x+h} f \, \mathrm{d}m \right| \leq \int_x^{x+h} |f| \, \mathrm{d}m$$

Define $\mu(E) = \int_E |f| \, \mathrm{d}m$ so that $\mu \ll m$. Thus to every $\epsilon > 0$, get $\delta > 0$ so that $m(E) < \delta$ implies $\mu(E) < \epsilon$. But then if $|h| < \delta$,

$$\int_x^{x+h} |f| \, \mathrm{d}m < \epsilon \Longrightarrow |F(x+h) - f(x)| < \epsilon$$ ■

**Prop. 4.1.2** *Let $f, a, F$ be as above. Then $F$ is almost everywhere differentiable, and $F' = f$ almost everywhere.*

Proof Let $x$ be a Lebesgue point of $f$. Since almost every point is a Lebesgue point, it suffices to prove that $F$ is differentiable at $x$ and $F'(x) = f(x)$. We have

$$\begin{aligned}
\left|\frac{F(x+h)-f(X)}{h} - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(y)\,\mathrm{d}y - \frac{1}{h}\int_x^{x+h} f(x)\,\mathrm{d}y\right| \\
&= \frac{1}{|h|}\int_x^{x+h} |f(y)-f(x)|\,\mathrm{d}y \\
&\le \frac{2}{2|h|}\int_{x-h}^{x+h} |f(y)-f(x)|\,\mathrm{d}y \\
&\to 0
\end{aligned}$$

since $x$ is a Lebesgue point. ■

**Prop. 4.1.3** *Let $f, a, F$ be as above. Then if $F$ is differentiable at $x$, then $x$ is a Lebesgue point of $f$.*

*What can we say about the integral of the derivative? When does it hold that $f(x) - f(a) = \int_a^x f'\,\mathrm{d}m$?*

**Ex. 4.1.4**
1. Consider $f(x) = x^2\sin(1/x^2)$, a differentiable function. Then $f(0) = 0$ and $f'(x) = 2x\sin(1/x^2) - \cos(1/x^2)\cdot 2/x$, so $f' \notin L^1$.
2. Suppose $f$ is the Cantor function, and let $f(0) = 0$, $f(1) = 1$. We say

$$\begin{aligned}
f|_{(1/3,2/3)} &= \frac{1}{2} \\
f|_{1/9,2/9} = \frac{1}{4}, f|_{7/9,8/9} &= \frac{3}{4}
\end{aligned}$$

etc and if $F$ denotes the cantor set, define $f(x) = \sup\{f(\xi) : \xi < x, \xi \in [0,1] \setminus F\}$.

*Let's try to characterize functions for which this holds. Now, let's assume that $f' \in L^1$, $f(x) - f(a) = \int_a^x f'\,\mathrm{d}m$. If $\mu(E) = \int_E f'\,\mathrm{d}m$, then $\mu \ll$ so for every $\epsilon > 0$, there exists $\delta > 0$ so that whenever $m(E) < \delta$, $\mu(E) < \epsilon$. If $E$ is a union of disjoint open intervals $(\alpha_i, \beta_i)$, then*

$$\mu(E) = \sum_{i=1}^n \int_{\alpha_i}^{\beta_i} f\,\mathrm{d}m = \sum_{i=1}^n (f(\beta_i) - f(\alpha_i)) < \epsilon$$

*whenever $\sum(\beta_i - \alpha_i) < \delta$.*

**Def'n. 4.1.5** *We say that $f : [a,b] \to \mathbb{C}$ is called* ***absolutely continuous*** *if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $n$ and any family $(\alpha_i, \beta_i)$ for $i = 1, \dots, n$ of disjoint open intervals,*

$$\sum_{i=1}^n (\beta_i - \alpha_i) < \delta \Longrightarrow \sum_{i=1}^n |f(\beta_i) - f(\alpha_i)| < \epsilon$$

*Note that when $n = 1$, we have uniform continuity. We already proved the following:*

**Prop. 4.1.6** *Let $f$ be a non-decreasing function. If $f$ is almost everywhere differentiable and $f' \in L^1$, then $\int_a^c f' \, \mathrm{d}m = f(x) - f(a)$, then $f$ is absolutely continuous.*

*We will prove the converse, first for non-decreasing functions.*

**Thm. 4.1.7** *Let $f$ be non-decreasing and continuous. Then the following statements are equivalent:*
1. *$f$ is AC on $[a,b]$*
2. *If $E \in \mathcal{M}$ and $m(E) = 0$, then $m(f(E)) = 0$.*
3. *$f$ is almost everywhere differentiable, $f' \in L^1$, and $f(x) - f(a) = \int_a^x f' \, \mathrm{d}m$ for all $x \in [a,b]$.*

Proof We showed earlier that (c)⇒(a).

(a)⇒(b). Let $E \subset [a,b]$, $m(E) = 0$. Take $\epsilon > 0$, and let $\delta$ be given by AC. Then there exists $V \subset [a,b]$ open with $m(V) < \delta$ and $E \subset V$. Write $V = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$ and for all $N$, we have

$$\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta \Longrightarrow \sum_{i=1}^{N} f(\beta_i) - f(\alpha_i) < \epsilon$$

so that $m(f(V)) < \epsilon$. Take $\epsilon = 1/n$ and let $H_n = \bigcup_{i=1}^{n} [f(\alpha_i), f(\beta_i)] \supset f(V) \supset f(E)$. Then $m(H_n) < 1/n$, so $f(E) \subset \bigcap_{n=1}^{\infty} H_n$ so by completeness of $E$, $m(f(E)) = 0$.

(b)⇒(c). Let $g(x) = f(x) + x$, so $g$ is strictly increasing. It can be shown that $g$ satisfies (b), i.e. $m(g(E)) = 0$ if $m(E) = 0$. Let $\mu(E) = m(g(E))$, which is countably additive (since $g$ is bijective). Then (b) implies that $\mu \ll m$, so by Radon-Nikodym, get $h \in L^1$ so $\mu(E) = \int_E h \, \mathrm{d}m$. Apply this to $E = [a,x]$ so that $g(E) = [g(a), g(x)]$. Then $g(x) - g(a) = m(g(E)) = \mu(E) = \int_a^x h \, \mathrm{d}m$ so that $f(x) - f(a) = \int_a^x (h - a) \, \mathrm{d}m$. We have thus proved $f'(x) = h(x) - 1$. ■

*Let's extend to monotone functions.*

**Def'n. 4.1.8** *Let $f : [a,b] \to \mathbb{R}$. The **total variation** function of $f$ is*

$$F_a(X) = \sup \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : a = t_0 < t_1 < t_2 < \cdots < t_n = x, n \in \mathbb{N} \right\}$$

*We say that $f$ has **bounded variation** on $[a,b]$ of $F_a(b) < \infty$.*

**Ex. 4.1.9** If $f$ is nondecreasing and bounded, then $f$ has bounded variation.

**Prop. 4.1.10** *Let $f$ have BV on $[a,b]$. Then $F_a$, $F_a + f$, $F_a - f$ are non-decreasing. Let $x < y$, so $t_0 = a < t_1 < \cdots < t_n = x < y$. Then*

$$F_a(y) \geq |f(y) - f(x)| + \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

*Take the supremum in $\{t_i\}$, so*

$$F_a(y) \geq |f(y) - f(x)| + F_a(x)$$

*so $F_a(y) \geq F_a(x)$ and $F_a$ is nondecreasing. If $|f(y) - f(x)| = f(y) - f(x)$, then $(F_a - f)(y) \geq (F_a - f)(x)$. If $|f(y) - f(x)| = f(x) - f(y)$, then $(F_a + f)(y) \geq (F_a + f)(x)$.*

**Prop. 4.1.11 (Jordan Decomposition)** *If $f$ has BV, then it is the difference of two nondecreasing functions.*

Proof Take $f = F_a - (F_a - f)$. ■

**Prop. 4.1.12** *If $f$ is AC, then $F_a$ is AC.*

*The Jordan decomposition gives two non-decreasing AC functions.*

**Thm. 4.1.13** *The following are equivalent:*

1. $f : [a,b] \to \mathbb{R}$ *is AC.*
2. $f$ *is differentiable almost wverywhee,* $f' \in L^1$*, and* $f(x) - f(a) = \int_a^x f' \, \mathrm{d}m$.

Proof Write $f = f_1 - f_2$ where $f_1, f_2$ are non-deceasing, and apply the previous theorem. ■