

Course Notes

Real Functions and Measures

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Contents

1	Basics of Abstract Measure Theory	3
1.1	Review of Topology	3
1.1.1	Basic Definitions	3
1.1.2	Examples of Topological Spaces	3
1.1.3	Other Definitions	4
1.1.4	Functions and Continuity	5
1.2	Measure Theory	6
1.2.1	σ -algebras	6
1.2.2	Sequences of Measurable Functions	9
1.2.3	Measures	9
1.3	Towards Integration	11
1.3.1	Simple Functions	11
1.3.2	Integration of Positive Functions	12
1.3.3	Lebesgue's Monotone Convergence Theorem	14
1.4	Integration of Complex Valued Functions	17
1.4.1	Basic Properties	17
1.4.2	More Dominated Convergence	18
2	The Lebesgue measure	21
2.1	The Vector Space $L^1(\mu)$	21
2.1.1	Almost Everywhere	21
2.1.2	$L^1(\mu)$ as a normed space	22
2.1.3	Construction of the Lebesgue measure	23
2.2	The Riesz Representation Theorem	23
2.3	Regularity Properties of Borel Measures	29
2.4	Construction of the Lebesgue Measure	31
2.5	Complex Measures	34
2.6	Absolute Continuity and Singular Measures	36
2.7	$L^2(\mu)$	38
2.8	Hilbert Spaces	38

Chapter 1

Basics of Abstract Measure Theory

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1.1 Review of Topology

1.1.1 Basic Definitions

Def'n. 1.1.1 Let $X \neq \emptyset$ and $\tau \subseteq \mathcal{P}(X)$. We say that (X, τ) is a **topological space** if τ satisfies the following conditions:

1. $\emptyset \in \tau$ $X \in \tau$
2. $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3. $V_\alpha \in \tau$ for all $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

We call the elements of τ **open sets**.

Def'n. 1.1.2 $U \subseteq X$ is a **neighbourhood** of $x \in X$ if there is some $G \in \tau$ such that $x \in G \subset U$.

Def'n. 1.1.3 $F \subseteq X$ is **closed** if F^c is open.

Def'n. 1.1.4 The **closure** of a set $E \subset X$ is the smallest closed set containing E (denoted \bar{E}).

Def'n. 1.1.5 x is an **accumulation point** of H if all neighbourhoods of x contains infinitely points of H . Equivalently, x is a **limit point** of $H \setminus \{x\}$.

Def'n. 1.1.6 If $H \subseteq X$, we have a natural subspace topology $\tau|_H = \{G \cap H : G \in \tau\}$.

1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1. \mathbb{R} along with the open sets (denoted τ_e , the Euclidean topology).
2. The discrete topology, $\tau = \mathcal{P}(X)$ for any $X \neq \emptyset$. This is the “finest” topology.

3. The antidiscrete topology, $\tau = \{\emptyset, X\}$ for any $X \neq \emptyset$. This is the “coarsest” topology.
4. One can define the extended real line, $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

The same can be done with a single symbol as well. In either case, the extended real line is a compact set.

5. Any metric spaces induces a topology. Consider a set $X \neq \emptyset$ arbitrary, and let $d : X \times X \rightarrow \mathbb{R}$ such that

- (a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$.
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$

Then $G \in \tau$ if and only if for any $x \in G$, there exists r so that $B_r(x) \subset G$. There are many examples of metric spaces:

- (a) $X = \mathbb{R}$, $d(x, y) = |x - y|$
- (b) $X = \mathbb{R}$, $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
- (c) $X = \mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- (d) $X = \mathbb{R}^2$, $d(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$ for $p \geq 1$.
- (e) and similarly for $X = \mathbb{R}^n$
- (f) $X = C[0, 1]$, $d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$.
- (g) normed space: X is a vector space over \mathbb{R} , $\|\cdot\| : X \rightarrow \mathbb{R}$ such that
 - i. $\|x\| = 0$ if and only if $x = 0$
 - ii. $\|cx\| = |c| \|x\|$
 - iii. $\|x + y\| \leq \|x\| + \|y\|$

If $\|\cdot\|$ is a norm, then $d(x, y) = \|x - y\|$ is a metric.

6. The cofinite topology: $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$.

1.1.3 Other Definitions

Def’n. 1.1.7 $K \subset X$ is **compact** if every open cover of K contains a finite subcover.

Def’n. 1.1.8 A topological space is called **locally compact** if every point has a compact neighbourhood.

Prop. 1.1.9 $C[0, 1]$ with the sup norm is not locally compact.

PROOF I’ll do this later.

□

Def'n. 1.1.10 A topological space is called **Hausdorff** if for any $x \neq y$, there exists neighbourhoods $U \ni x$, $V \ni y$ so that $U \cap V = \emptyset$.

The anti-discrete topology is not Hausdorff.

1. On the discrete topology, K is compact if and only if K is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of X).
3. On (\mathbb{R}, τ_e) , K is compact if and only if K is closed and bounded.
4. On (X, d) metric space, K is compact if and only if K is complete and totally bounded.

Prop. 1.1.11 1. Let $K \subset X$ be compact, let $F \subset K$ closed. Then F is also compact.
2. Compact sets in a Hausdorff space are closed.

PROOF 1. Let $F \subset \bigcup V_\alpha$. Then $K \subset F^c \cup (\bigcup V_\alpha)$ is an open cover for K , so it has a finite subcover $F^c \cup V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$. But then since $F \cap F^c = \emptyset$, $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ is a finite subcover.
2. Let $K \subset X$ be compact, and prove that K^c is open. Thus let $x \in K^c$. For any $y \in K$, there exist U_y, V_y disjoint neighbourhoods of x and y respectively. Now consider the open cover $K \subset \bigcup_{y \in K} V_y$, and get our finite subcover $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. But then $U_{y_1} \cap \dots \cap U_{y_n} \cap K = \emptyset$ and is open since it is a finite intersection. \square

Def'n. 1.1.12 $\Gamma \subseteq \tau$ is a **base** for τ if every $U \in \tau$ can be written as a countable union of the elements of Γ . Γ is a **countable base** if Γ is countable.

Prop. 1.1.13 \mathbb{R} has a countable base of intervals.

PROOF Consider the collection $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$. To see this, for any open set U , one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left(\bigcup_{\{r: B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$ is obvious, so let $x \in U$ be arbitrary, and let s be maximal so that $B_s(x) \subseteq U$. Then choose $q \in \mathbb{Q}$ so that $|x - q| < s/3$ and $r \in \mathbb{Q}$ so that $0 < r < s/2$. Then by construction $B_r(q) \ni x$ and by the triangle inequality $B_{r/2}(q) \subseteq U$, so $x \in S$. Thus $U = S$ as desired. \square

Note that the exact same argument (with some work) can be generalized to show that \mathbb{R}^n has a countable base of open hyperrectangles.

Prop. 1.1.14 Every metric space which is a countable union of compact sets has a countable base.

PROOF See my PMATH 351 notes. \square

1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

Def'n. 1.1.15 Let $(x_n) \subset X$ be a sequence and let $x \in X$. Then x is the **limit** of (x_n) if for any neighbourhood U of x , there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.

Prop. 1.1.16 If $F \subset X$ is closed, then for all convergent sequences in F , the limit is also in F .

PROOF See Homework. □

Def'n. 1.1.17 Let $f : X \rightarrow Y$ be a function, and $x \in X$ an accumulation point of $D(f)$. The limit of f at x is $y \in Y$ if for any neighbourhood V of y there exists a neighbourhood U of x such that $f(U \cap D(f) \setminus \{x\}) \subseteq V$.

Def'n. 1.1.18 Let $f : X \rightarrow Y$ be a function, and let $x \in D(f)$. Then f is **continuous at x** if for any neighbourhood V of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of x .

Def'n. 1.1.19 $f : X \rightarrow Y$ is called **continuous** if it is continuous at every point.

Prop. 1.1.20 $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(G)$ is open for all G open.

PROOF Exercise. □

Thm. 1.1.21 Let $f : X \rightarrow Y$ be continuous and $K \subset X$ be compact. Then $f(K)$ is compact.

PROOF Recall that continuous functions pull back open sets. Let $f(K) \subset \bigcup U_\alpha$ be an open cover. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover for K , and has a finite subcover $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. But then $f(f^{-1}(U_{\alpha_1})) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$ is a subcover of $f(K)$. □

1.2 Measure Theory

1.2.1 σ -algebras

Def'n. 1.2.1 Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a **σ -algebra** if

1. $X \in \mathcal{M}$
2. $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
3. If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$

The pair (X, \mathcal{M}) is called a **measurable space**. The elements of \mathcal{M} are called **measurable sets**.

Def'n. 1.2.2 Let (X, \mathcal{M}) be a measurable space, (Y, τ) be a topological space. Then $f : X \rightarrow Y$ is called **measurable** if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.

Here are some simple examples of σ -algebras.

Ex. 1.2.3 1. $\mathcal{M} = \{\emptyset, X\}$ is a σ -algebra.

2. $\mathcal{P}(X) = \mathcal{M}$ is a σ -algebra.

3. $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable}\}$. To see this, given $A_n \in \mathcal{M}$, if everything is countable, then $\bigcup A_n$ is countable. If some A_i is countable, then $(\bigcup A_n)^c = \bigcap A_n^c$ is countable, so $\bigcup A_n \in \mathcal{M}$.

We will later see some proper examples, like the σ -algebra of Lebesgue measurable sets.

We have the following properties of σ -algebras.

Prop. 1.2.4 1. $\emptyset \in \mathcal{M}$

2. $A_1, A_2, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{M}$
3. $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$
4. $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$
5. f is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$.

PROOF 1. $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.

2. We can extend this to a countable union by introduction $A_{n+i} = \emptyset$ for $i \in \mathbb{N}$.
3. By DeMorgan's identities, $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.
4. $A \setminus B = A \cap B^c \in \mathcal{M}$.
5. H^c is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$. □

Prop. 1.2.5 Let $f : X \rightarrow Y$ be measurable, let $g : Y \rightarrow Z$ be continuous, then $g \circ f : X \rightarrow Z$ is measurable.

PROOF Let $V \subset Z$ be open, so $g^{-1}(V) \subset Y$ is open, so $f^{-1}(g^{-1}(V)) \in \mathcal{M}$ which is $(g \circ f)^{-1}(V)$. □

Prop. 1.2.6 Let (X, \mathcal{M}) be a measurable space, Y be a topological space. Let $\phi : \mathbb{R}^2 \rightarrow Y$ be continuous. If $u, v : X \rightarrow \mathbb{R}$ are measurable, then $h(x) = \phi(u(x), v(x))$ is measurable.

PROOF Define $f : X \rightarrow \mathbb{R}^2$ by $f(x) = (u(x), v(x))$. We will see that f is measurable, so that $h = \phi \circ f$ is measurable since ϕ is continuous. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals, so $R = I_1 \times I_2$ is an open rectangle. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Let $G \subset \mathbb{R}^2$ be an open set, so there exist R_n open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that f is measurable. □

Cor. 1.2.7 1. If $u, v : X \rightarrow \mathbb{R}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.

2. $u + iv : X \rightarrow \mathbb{C}$ is measurable.
3. $f : X \rightarrow \mathbb{C}$ is measurable, $f = u + iv \Rightarrow u, v, |f|$ are measurable.

Prop. 1.2.8 Define

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then χ_E is measurable if and only if $E \in \mathcal{M}$.

PROOF Naturally, $\chi_E^{-1}(1) = E$ and $\chi_E^{-1}(0) = E^c$, so χ_E is measurable if and only if $E, E^c \in \mathcal{M}$. □

Thm. 1.2.9 Let $\mathcal{F} \subset \mathcal{P}(X)$, then there exists a smallest σ -algebra containing \mathcal{F} . This is denoted by $S(\mathcal{F})$, the σ -algebra generated by \mathcal{F} .

PROOF Let $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra, } \mathcal{F} \subset \mathcal{M}\}$. Certainly $\Omega \neq \emptyset$ since $\mathcal{P}(X) \in \Omega$. Let $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$. We will see that $S(\mathcal{F})$ is a σ -algebra.

- (i) Since $X \in \mathcal{M}$, it follows that $X \in \bigcap \mathcal{M}$.

(ii) If $A \in S(\mathcal{F})$, then $A \in \mathcal{M}$ for all \mathcal{M} . Thus $A^c \in \mathcal{M}$ for all \mathcal{M} and $A^c \in \bigcap \mathcal{M}$.

(iii) In the same way, if $A_n \in S(\mathcal{F})$ for all n , then $A_n \in \mathcal{M}$ for all n, \mathcal{M} . Thus $\bigcup A_n \in \mathcal{M}$ for all \mathcal{M} so $\bigcup A_n \in \bigcap \mathcal{M} = S(\mathcal{F})$.

By definition, $\mathcal{F} \subset \bigcap \mathcal{M}$. Finally, $S(\mathcal{F})$ is minimal, since if $\mathcal{F} \subset \mathcal{N}$ is a σ -algebra, then $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$, so we are done. \square

Def'n. 1.2.10 Let (X, τ) be a topological space. Then $\mathcal{B} = S(\tau)$ is called the **Borel σ -algebra**. Borel sets are the elements of $S(\tau)$. A function $f : X \rightarrow Y$ is Borel measurable if $f^{-1}(G) \in \mathcal{B}$ for all $G \subset Y$ open.

Prop. 1.2.11 1. If $F \subset X$ is closed, then $F \in \mathcal{B}$.

2. $G_n \subset X$ are open, then $\bigcap_{n=1}^{\infty} G_n \in \mathcal{B}$. These are called G_δ -sets.

3. $F_n \subset X$ are closed, then $\bigcup_{n=1}^{\infty} F_n \in \mathcal{B}$. These are called F_σ -sets.

PROOF These follow directly from the definition of a σ -algebra. \square

Ex. 1.2.12 $X = \mathbb{R}, \tau_e$, then $\mathcal{B} = S(\tau_e)$. Let $\Gamma_0 = \{(a, b) : a < b\}$ be a family of open intervals. We see that $S(\Gamma_0) = \mathcal{B}$. Since $\Gamma_0 \subset \tau$, $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$. Conversely, let $G \in \tau$, then we have open intervals $G = \bigcup_{n=1}^{\infty} I_n$ so that $G \in S(\Gamma_0)$. Thus $S(\tau) \subset S(\Gamma_0)$ and $S(\Gamma_0) = \mathcal{B}$.

Ex. 1.2.13 Let $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$. I claim that $S(\Gamma_\infty) = \mathcal{B}$. Certainly $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$. Then $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$. Similarly, $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n] \in S(\Gamma_\infty)$. Thus $(a, \infty) \cap (-\infty, b) = (a, b) \in S(\Gamma_0)$, and using the previous example, $\mathcal{B} = S(\Gamma_\infty)$.

Prop. 1.2.14 Let (X, \mathcal{M}) be a measurable space, and let $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the euclidean topology. If $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for any $\alpha \in \mathbb{R}$, then f is measurable.

PROOF Recall that f is measurable if its inverse image takes open sets to measurable sets.

We have $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty]))^c \in \mathcal{M}$. Similarly,

$$f^{-1}([-\infty, \alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty} [-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha, \beta)) = f^{-1}([-\infty, \beta) \cap (\alpha, \infty]) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for τ_e . Thus if $G \subset \overline{\mathbb{R}}$ is open, then there exists open intervals so that $G = \bigcup_{n=1}^{\infty} I_n$ and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired. \square

1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the Dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

Def'n. 1.2.15 Let $(a_n)_{n \in \mathbb{N}} \subset \overline{\mathbb{R}}$ be a sequence, and $b_k = \sup\{a_k, a_{k+1}, \dots\}$. Then $\beta = \inf_{k \in \mathbb{N}} b_k$ is called the \limsup of (a_n) . We can similarly define $c_k = \inf\{a_k, a_{k+1}, \dots\}$ and $\liminf = \sup_{k \in \mathbb{N}} c_k$.

Def'n. 1.2.16 Let $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of functions. Then $(\sup f_n) : X \rightarrow \overline{\mathbb{R}}$, $(\sup f_n)(x) = \sup f_n(x)$ for all $x \in X$. Similarly, $(\inf f_n) : X \rightarrow \overline{\mathbb{R}}$, $(\inf f_n)(x) = \inf f_n(x)$ for all $x \in X$. Then $(\liminf f_n)(x) = \liminf f_n(x)$. If $\lim f_n(x)$ exists for all x , then we say $(\lim f_n)(x) = \lim f_n(x)$.

Thm. 1.2.17 Let $f_n : X \rightarrow \overline{\mathbb{R}}$ be measurable. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ are measurable.

PROOF Let $g = \sup f_n$. It is enough to prove that $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$ for all α . Let $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$. Let $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$. We show that $H = \bigcup_{n=1}^{\infty} H_n$.

First let $x \in H$, so $\sup f_n(x) > \alpha$. Thus get N so that $f_N(x) > \alpha$, so $x \in H_N$ and x is in the union. The converse is obvious.

Thus g is measurable. In the exact same way, $\inf f_n$ is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k \geq i} f_k$$

is measurable. □

Cor. 1.2.18 If $\lim f_n$ exists, then it is measurable.

PROOF If $\lim f_n$ exists, then $\lim f_n = \limsup f_n$. □

Cor. 1.2.19 If f, g are measurable, then $\max\{f, g\}$, $\min\{f, g\}$ are measurable.

Cor. 1.2.20 Let f be a function. Then $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$ (the positive and negative parts of f) are measurable. Similarly, $|f| = f_+ + f_-$ is measurable.

1.2.3 Measures

Def'n. 1.2.21 Let (X, \mathcal{M}) be a measurable space. A function $\mu : \mathcal{M} \rightarrow [0, +\infty]$ is called a **(positive) measure** if it is countably additive and not constant $+\infty$. In other words,

$$1. \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \text{ if } A_i \cap A_j = \emptyset$$

$$2. \exists A \in \mathcal{M} \text{ so that } \mu(A) < \infty$$

(X, \mathcal{M}, μ) is called a **measure space**.

Prop. 1.2.22 1. $\mu(\emptyset) = 0$

2. If $A_i \cap A_j = \emptyset$ then $\mu\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$

3. $A \subset B$ implies $\mu(A) \leq \mu(B)$

4. $A_1 \subset A_2 \subset A_3 \cdots$ then $\lim_{n \rightarrow \infty} \mu A_n = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

5. $A_1 \supset A_2 \supset A_3 \cdots$ and $\mu(A_i) < \infty$ then $\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

PROOF 1. Let $A \in \mathcal{M}$ so that $\mu(A) < \infty$, and fix $A_1 = A$, $A_2 = A_3 = \cdots = \emptyset$. Then $\bigcup A_n = A$ so $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$ so $\mu(\emptyset) = 0$.

2. Obvious

3. Note that $B = A \cup (B \setminus A)$ is a disjoint union.

4. Define $B_1 := A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then $B_i \cap B_j = \emptyset$ and $\mu(A_n) = \mu\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mu(B_i)$. Similarly, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$. Therefore, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$.

5. Let $C_n = A_1 \setminus A_n$, $C_1 = \emptyset$. Then $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) + \mu(A_n) = \mu(A_1)$. Let $A = \bigcap_{n=1}^{\infty} A_n$ so $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$ and $(\bigcup C_n) \cup A = A_1$ is a disjoint union. But then $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$ so that

$$\mu(A_1) - \mu(A) = \mu\left(\bigcup C_n\right) = \lim_{n \rightarrow \infty} \mu(C_n) = \mu(A_n) - \lim_{n \rightarrow \infty} \mu(A_n)$$

Since $\mu(A_1)$ is finite, we have $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$. \square

Ex. 1.2.23 Here are a few examples of measures that exist on arbitrary sets.

1. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$, and

$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$

It is easy to verify it is countably additive.

2. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$. Fix $x_0 \in X$. Then

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

1.3 Towards Integration

1.3.1 Simple Functions

Def'n. 1.3.1 $s : X \rightarrow \mathbb{R}$ or \mathbb{C} is called a **simple function** if its range is finite.

Prop. 1.3.2 Let s be a simple function, so that $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $A_i = s^{-1}(\{\alpha_i\})$ and s is **measurable** if and only if $A_i \in \mathcal{M}$.

PROOF Obvious. □

The following theorem is used later to define the intergral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function f .

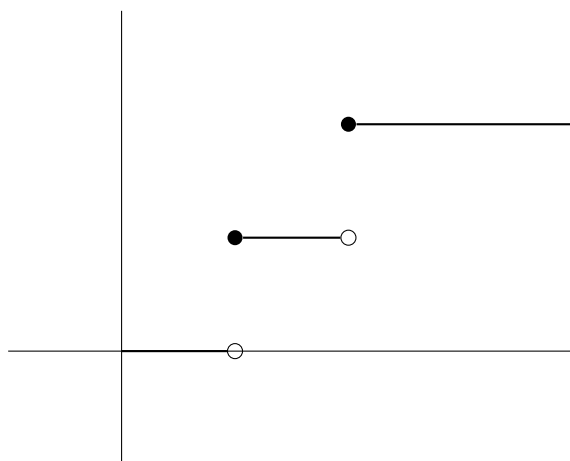
Thm. 1.3.3 Let $f : X \rightarrow [0, +\infty]$ be nonnegative measurable functions. Then there exists a sequence $s_n : X \rightarrow [0, +\infty]$ of simple measurable functions with

1. (s_n) is increasing and bounded above by f
2. $\lim s_n = f$ pointwise.

PROOF Let $n \in \mathbb{N}$, $t \geq 0$, and define $k_n(t) = [2^n \cdot t]$ (i.e. $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn ϕ_1 below:



Then $t - 2^{-n} \leq \phi_n(t) \leq t$, $\lim \phi_n(t) = t$ uniformly, and $\phi_n \leq \phi_{n+1}$, so the sequence of functions is monotone. Define $s_n = \phi_n \circ f$, so for any $x \in X$, $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$. Note that s_n is simple since it has finite range (from ϕ_n), and $s_n \leq s_{n+1}$ because $\phi_n \leq \phi_{n+1}$, and $s_n \leq f$ since $\phi_n(t) \leq t$. Furthermore, ϕ_n is measurable since its level sets are intervals, so $s_n = \phi_n \circ f$ is measurable. □

1.3.2 Integration of Positive Functions

Def'n. 1.3.4 Let $s : X \rightarrow [0, +\infty)$ be a measurable simple function $s = \sum_{n=1}^N \alpha_i X_{A_i}$. Let $E \in \mathcal{M}$. Then define the **integral** of s over E with respect to μ as

$$\int_E s \, d\mu = \sum_{n=1}^N \alpha_i \mu(A_i \cap E)$$

where we define $0 \cdot \infty = 0$.

Def'n. 1.3.5 Let $f : X \rightarrow [0, +\infty]$ be a measurable function. Let $E \in \mathcal{M}$. Then the **(Lebesgue) integral** of f over E with respect to μ is

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f; s \text{ is simple measurable} \right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only.

Prop. 1.3.6 Let $f, g : X \rightarrow [0, +\infty]$ be measurable functions. Let $E, A, B \in \mathcal{M}$.

1. If $f \leq g$ then $\int_E f \, d\mu \leq \int_E g \, d\mu$
2. If $A \subset B$, then $\int_A f \, d\mu \leq \int_B f \, d\mu$
3. $\int_E c \cdot f \, d\mu = c \cdot \int_E f \, d\mu$ for all $c \geq 0$
4. If $f(x) = 0$ for all $x \in E$, then $\int_E f \, d\mu = 0$
5. If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$
6. $\int_E f \, d\mu = \int_X f \cdot \chi_E \, d\mu$.

PROOF 1. Note that

$$\left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\} \subset \left\{ \int_E s \, d\mu : 0 \leq s \leq g \right\}$$

2. Let $0 \leq s \leq f$ be simple measurable. Then

$$\int_A s \, d\mu = \sum \alpha_i \mu(A \cap A_i) \leq \sum \alpha_i \mu(B \cap A_i) = \int_B s \, d\mu$$

Take the supremum for all $0 \leq s \leq f$, then the result follows.

3. Let S be simple and measurable, so $s = \sum \alpha_i \chi_{A_i}$. Then

$$\int_E c \cdot s \, d\mu = \sum_{i=1}^n \alpha_i \cdot c \cdot \mu(E \cap A_i) = c \cdot \sum \alpha_i \mu(E \cap A_i) = c \int_E s \, d\mu$$

Thus

$$\begin{aligned}\int_E c \cdot f \, d\mu &= \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq cf \right\} \\ &= \sup \left\{ \int_E c \cdot t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \sup \left\{ \int_E t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \int_E f \, d\mu\end{aligned}$$

4. If $0 \leq s \leq f$, then $s = \sum \alpha_i \chi_{A_i}$. If $x \in A_i \cap E$, then $s(x) = \alpha_i$ and $\alpha_i = 0$. Then $\alpha_i \mu(A_i \cap E) = 0$ for all i : either $A_i \cap E = \emptyset$, or $A_i \cap E$ is not empty, and $\alpha_i = 0$. This is true for any $0 \leq s \leq f$, and taking supremums yields the result.
5. If $\mu(E) = 0$ then $\mu(A_i \cap E) = 0$, and $\int_E s \, d\mu = \sum \alpha_i \mu(A_i \cap E) = 0$ and taking supremums, the result holds.
6. Exercise. First prove if $0 \leq s \leq f \cdot \chi_E$, then $\int_X s \, d\mu = \int_E s \, d\mu$. Then prove

$$\left\{ \int_E s \, d\mu : 0 \leq s \leq f \cdot \chi_E \right\} = \left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\}$$

□

Prop. 1.3.7 Let s be a simple and measurable. Then $\phi(E) = \int_E s \, d\mu$ is a measure.

PROOF $\phi(\emptyset) = 0$, so ϕ is not constant $+\infty$. Let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union. Then

$$\begin{aligned}\phi(E) &= \sum_{i=1}^m \alpha_i \mu(A_i \cap E) \\ &= \sum_{i=1}^m \alpha_i \mu \left(A_i \cap \left(\bigcup_{n=1}^{\infty} E_n \right) \right) = \sum_{i=1}^m \alpha_i \mu \left(\bigcup_{n=1}^{\infty} (A_i \cap E_n) \right) \\ &= \sum_{i=1}^m \alpha_i \sum_{n=1}^{\infty} \mu(A_i \cap E_n) = \sum_{n=1}^{\infty} \sum_{i=1}^m \alpha_i \mu(A_i \cap E_n) \\ &= \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu = \sum_{n=1}^{\infty} \phi(E_n)\end{aligned}$$

□

Prop. 1.3.8 Let s, t be nonnegative, measurable simple functions. Then

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

PROOF Write

$$s = \sum_{i=1}^m \alpha_i X_{A_i}, \quad t = \sum_{j=1}^n \beta_j X_{B_j}$$

and let $E_{ij} = A_i \cap B_j$, so $X = \bigcup_{i,j} E_{ij}$ is a disjoint union. We now have

$$\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j) \mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu$$

Let $\mu(E) = \int_E (s+t) d\mu$, which is a measure as above. Thus

$$\begin{aligned} \int_X (s+t) d\mu &= \phi(X) = \phi\left(\bigcup_{i,j} E_{ij}\right) \\ &= \sum_{i,j} \phi(E_{ij}) = \sum_{i,j} \int_{E_{ij}} (s+t) d\mu \\ &= \sum_{i,j} \left(\int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu \right) \\ &= \sum_{i,j} \varphi(E_{ij}) + \sum_{i,j} \theta(E_{ij}) \\ &= \int_X s d\mu + \int_X t d\mu \end{aligned}$$

where $\varphi(E) = \int_E s d\mu$, $\theta(X) = \int_X t d\mu$. □

1.3.3 Lebesgue's Monotone Convergence Theorem

Thm. 1.3.9 (Lebesgue's Monotone Convergence) Let $f_n : X \rightarrow [0, +\infty]$ be measurable, such that

(i) $0 \leq f_1 \leq f_2 \leq \dots$

(ii) $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in X$

Then f is measurable, and $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$.

PROOF It was already proven that f is measurable. We have $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$ for all n , so $\alpha := \lim_{n \rightarrow \infty} \int_X f_n d\mu$ exists. We also have $f_n \leq f$, so $\int f_n \leq \int f$ and $\alpha \leq \int_X f_n d\mu$. Thus we wish to show $\alpha \geq \int_X f d\mu$. It suffices to prove that $\alpha \geq \int_X s d\mu$ for any simple $s \leq f$. Furthermore, if $c \in (0, 1)$, it suffices to show that $\alpha \geq \int_X c \cdot s d\mu$.

Define $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\}$. We have $E_1 \subset E_2 \subset \dots$ so that $\bigcup_{n=1}^{\infty} E_n = X$. Then

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} c \cdot s d\mu$$

Let $\phi(E) = \int_E s d\mu$, so $\int_{E_n} s d\mu = \phi(E_n)$. Thus $\lim_{n \rightarrow \infty} \phi(E_n) = \phi(X) = \int_X s d\mu$. Thus

$$\alpha \geq c \cdot \lim_{n \rightarrow \infty} \phi(E_n) = c \cdot \int_X s d\mu = \int_X c \cdot s d\mu$$

as desired. \square

Ex. 1.3.10 Consider the function consisting of a triangle with base $2/n$ and height n . Then $\int_0^1 f_n = 1$ as a Riemannian integral. However, $\lim f_n(x) = 0$ for any x , so $\int_0^1 f = 0 \neq 1 = \lim \int_0^1 f_n$.

Thm. 1.3.11 Let $f, g : X \rightarrow [0, +\infty]$ measurable, then $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$.

PROOF We proved that there exists increasing sequences of simple functions s_n, t_n such that $\lim s_n(x) = f(x)$, $\lim t_n(x) = g(x)$. Then $s_n(x) + t_n(x) \rightarrow f(x) + g(x)$ monotonically. But then

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X \lim_{n \rightarrow \infty} (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \int_X (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \left(\int_X s_n d\mu + \int_X t_n d\mu \right) \\ &= \int_X \lim_{n \rightarrow \infty} s_n d\mu + \int_X \lim_{n \rightarrow \infty} t_n d\mu \\ &= \int_X f d\mu + \int_X g d\mu \end{aligned} \quad \square$$

Cor. 1.3.12 If $f_n : X \rightarrow [0, +\infty]$ is a sequence of measurable functions, then

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=1}^{\infty} f_n d\mu$$

Ex. 1.3.13 Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(X)$, $\mu(E)$ is the counting measure. Let $a : X \rightarrow [0, \infty)$ be a function. This is a sequence. Every function is measurable. Let $s_n(i) = a(i)$ for $i \leq n$ and 0 otherwise, which is a simple function, and $s_n \leq s_{n+1}$. Then $\lim_{n \rightarrow \infty} s_n(i) = a(i)$ so $s_n \rightarrow a$ pointwise, so by LMC $\int_X s_n d\mu = \int_X a d\mu$. Also,

$$\int_X s_n d\mu = \sum_{i=1}^n a(i) \mu(\{i\}) = \sum_{i=1}^n a(i)$$

$$\text{so } \int_X a d\mu = \sum_{n=1}^{\infty} a(n).$$

Lemma 1.3.14 (Fatou) Let $f_n : X \rightarrow [0, \infty)$ be a sequence of measurable functions. Then

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu$$

PROOF Let $g_k = \inf\{f_k, f_{k+1}, \dots\}$ so $\liminf f_n = \lim_{n \rightarrow \infty} g_n$ and g_n is increasing. Note that $g_k \leq f_k$ for any k , so $\int_X g_k d\mu \leq \int_X f_k d\mu$. Thus

$$\begin{aligned} \int_X \liminf f_n d\mu &= \int_X \lim g_n d\mu \\ &= \lim \int_X g_n d\mu \\ &= \liminf \int_X g_n d\mu \\ &\leq \liminf \int_X f_n d\mu \end{aligned} \quad \square$$

Ex. 1.3.15 It is possible for the inequality to be strict. Define $f_{2n} = \chi_{[0,1]}$ and $f_{2n+1} = \chi_{[1,2]}$. Thus $\liminf f_n(x) = 0$ so $\int_{[0,2]} \liminf f_n d\mu = 0$ but $\inf_{[0,2]} \int_{[0,2]} f_n d\mu = 1$

Thm. 1.3.16 Let $f : X \rightarrow [0, \infty]$ be measurable. Let $\phi(E) = \int_E f d\mu$, $E \in \mathcal{M}$. Then ϕ is a measure and $\int_X g d\phi = \int_X g \cdot f d\mu$.

PROOF Certainly $\phi(\emptyset) = 0$, so $\phi \neq +\infty$. Thus let $E = \bigcup_{i=1}^{\infty} E_i$ be a disjoint union. Then $\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f$. Thus we have

$$\begin{aligned} \phi(E) &= \int_E f d\mu \\ &= \int_X \chi_E f d\mu \\ &= \int_X \sum_{i=1}^{\infty} \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_X \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \phi(E_i) \end{aligned}$$

Now, we prove that $\int_X g d\mu = \int_X g f d\mu$.

First, we do this for $g = \chi_E$. Then $\int_X \chi_E d\mu = \phi(E)$ on the left, and $\int_X \chi_E f d\mu = \int_E f d\mu = \phi(E)$ and equality holds.

Now, let $g = \sum_{i=1}^n \alpha_i \chi_{A_i}$ be a simple function. Then $\int_X \sum \alpha_i \chi_{A_i} d\phi = \sum \alpha_i \int_X \chi_{A_i} d\phi$ on the left and $\int_X \sum \alpha_i \chi_{A_i} f d\mu = \sum \alpha_i \int_X \chi_{A_i} f d\mu$.

Finally, let g be an arbitrary measurable function, and let $(s_n) \rightarrow g$ be an increasing sequence of simple functions. Note that $s_n f \rightarrow g f$. Thus

$$\begin{aligned}\int_X g \, d\phi &= \int_X \lim s_n \, d\phi = \lim \int_X s_n \, d\phi \\ &= \lim \int_X s_n f \, d\mu = \int_X \lim(s_n f) \, d\mu \\ &= \int_X g \cdot f \, d\mu\end{aligned}$$

as desired. □

1.4 Integration of Complex Valued Functions

Def'n. 1.4.1 A function $f : X \rightarrow \mathbb{C}$ is called *Lebesgue integrable* if $\int_X |f| \, d\mu < \infty$. The collection of such functions is $L^1(\mu)$.

1.4.1 Basic Properties

Def'n. 1.4.2 Let $f \in L^1(\mu)$. Then $f = u + iv$ and denote $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Let $E \in \mathcal{M}$; then the integral of f over E with respect to μ is

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \left(\int_E v^+ \, d\mu - \int_E v^- \, d\mu \right)$$

Thm. 1.4.3 Let $f, g \in L^1(\mu)$, $\alpha, \beta \in \mathbb{C}$, so $\alpha f + \beta g \in L^1(\mu)$ and

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

PROOF Note that $\alpha f + \beta g$ is measurable, so $\int_X |\alpha f + \beta g| \, d\mu \leq |\alpha| \int_X |f| \, d\mu + |\beta| \int_X |g| \, d\mu < \infty$. For real measurable functions, $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$ directly by expanding the definition and using additivity over positive functions. We thus show $\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu$. If $\alpha \geq 0$, then

$$\begin{aligned}\int_X \alpha f \, d\mu &= \int_X \alpha(u + iv) \, d\mu = \int_X (\alpha u^+ - \alpha u^- + i\alpha v^+ - i\alpha v^-) \, d\mu \\ &= \int_X ((\alpha u)^+ - (\alpha u)^- + (i\alpha v)^+ - (i\alpha v)^-) \, d\mu \\ &= \int_X (\alpha u)^+ \, d\mu - \int_X (\alpha u)^- \, d\mu + \int_X i(\alpha v)^+ \, d\mu - \int_X i(\alpha v)^- \, d\mu \\ &= \alpha \int_X u^+ \, d\mu - \alpha \int_X u^- \, d\mu + \alpha \int_X iv^+ \, d\mu - \alpha \int_X iv^- \, d\mu \\ &= \alpha \int_X f \, d\mu\end{aligned}$$

and similarly for $\alpha = -1$, $\alpha = i$. □

Thm. 1.4.4 Let $f \in L^1(\mu)$. Then $\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$.

PROOF Let $z = \int_X f \, d\mu$. Let $\alpha = \frac{|z|}{z}$ if $z \neq 0$, and $\alpha = 1$ otherwise. Then $\alpha \int_X f \, d\mu = |z|$. Let $u = \operatorname{Re}(\alpha \cdot f) \leq |\alpha \cdot f| \leq |f|$ since $|\alpha| = 1$. Thus

$$\begin{aligned} \left| \int_X f \, d\mu \right| &= \alpha \cdot \int_X f \, d\mu \\ &= \int_X \alpha f \, d\mu \\ &= \int_X \operatorname{Re}(\alpha f) \, d\mu \\ &\leq \int_X |f| \, d\mu \end{aligned} \quad \square$$

1.4.2 More Dominated Convergence

Naturally, we want similar results as we have before. Indeed, we have the following theorem:

Thm. 1.4.5 (Lebesgue's Dominated Convergence) Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions such that $f = \lim f_n$. Assume that there is some $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all n . Then $f \in L^1(\mu)$ and $\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$.

PROOF We certainly know that f is measurable, and $|f| \leq g$, so $f \in L^1(\mu)$. As well, the triangle inequality show that $|f - f_n| \leq 2g$ for any n . We will see that $0 \leq \liminf \int_X |f - f_n| \, d\mu \leq \limsup \int_X |f - f_n| \, d\mu \leq 0$. Assuming that this holds, then $\lim \int_X |f - f_n| \, d\mu = 0$ and

$$0 \leq \lim \left| \int_X f \, d\mu - \int_X f_n \, d\mu \right| \leq \int_X |f - f_n| \, d\mu = 0$$

The first two inequalities are obvious: we must show that $\limsup \int_X |f_n| \, d\mu \leq 0$. Firstly, we have

$$\begin{aligned} \int_X 2g \, d\mu &= \int_X \left(2g - \lim_{n \rightarrow \infty} |f - f_n| \right) d\mu \\ &= \int_X \liminf (2g - |f - f_n|) \, d\mu \\ &\leq \lim \int_X (2g - |f - f_n|) \, d\mu && \text{By Fatou's Lemma} \\ &= \int_X 2g + \liminf \left(- \int_X |f - f_n| \, d\mu \right) \\ &= \int_X 2g - \limsup \int_X |f - f_n| \, d\mu \end{aligned}$$

and since $\int_X 2g \, d\mu$ is finite, we subtract and $\limsup \int_X |f - f_n| \, d\mu \leq 0$. \square

Ex. 1.4.6 Consider $\lim_{n \rightarrow \infty} \int_0^n e^{-nx} dx$. Define

$$f_n(x) = \begin{cases} e^{-nx} & \text{if } x \leq n \\ 0 & \text{if } x > n \end{cases}$$

Note that $f_n(x) \leq g(x) = e^{-x}$ and $\int_0^\infty e^{-x} dx < \infty$. Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^n e^{-nx} dx &= \int_{[0, \infty)} \lim_{n \rightarrow \infty} f_n(x) dx \\ &= \int_{[0, \infty)} \chi_{\{0\}} dx \\ &= 0 \end{aligned}$$

Rmk. 1.4.7 For the Riemann integral, we have $\int \lim f_n = \lim \int f_n$ as long as the convergence of f_n is uniform.

Chapter 2

The Lebesgue measure

2.1 The Vector Space $L^1(\mu)$

2.1.1 Almost Everywhere

Let (X, \mathcal{M}, μ) be a measure space.

Def'n. 2.1.1 Let $E \in \mathcal{M}$. We say that property P holds almost everywhere in E if there exists $N \in \mathcal{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds in $E \setminus N$.

Ex. 2.1.2 Two functions $f, g : X \rightarrow \mathbb{C}$ are equal almost everywhere if $\exists N \subset X$ such that $\mu(N) = 0$ and $f(x) = g(x)$ on $X \setminus N$.

Prop. 2.1.3 Let $E \subset X$ be such that $A_1, A_2, B_1, B_2 \in \mathcal{M}$ for which $\int_X f d\mu = \int_X g d\mu$. Then $A_1 \subset E \subset B_1$, $A_2 \subset E \subset B_2$, and $\mu(B_1 \setminus A_1) = 0$ and $\mu(B_2 \setminus A_2) = 0$. Then $\mu(A_1) = \mu(A_2)$.

PROOF Note that $A_1 \setminus A_2 \subset E \setminus A_2 \subset B_2 \setminus A_2$. As well, $\mu(A_1 \setminus A_2) \leq \mu(B_2 \setminus A_2) = 0$. Then

$$\begin{aligned}\mu(A_1) &= \mu(A_1 \cap A_2^c) + \mu(A_1 \cap A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2) \\ \mu(A_2) &= \mu(A_2 \cap A_1^c) + \mu(A_2 \cap A_1) = \mu(A_2 \setminus A_1) + \mu(A_2 \cap A_1) = \mu(A_1 \cap A_2)\end{aligned}\quad \square$$

Prop. 2.1.4 Let (X, \mathcal{M}, μ) be a measure space. Let

$$\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, A \subset E \subset B, \mu(B \setminus A) = 0\}$$

Then \mathcal{M}^* is a σ -algebra, and $\mu^* : \mathcal{M}^* \rightarrow [0, +\infty]$ defined by $\mu^*(E) = \mu(A)$.

PROOF We show that \mathcal{M}^* is a σ -algebra, and μ is countably additive.

1. $X \in \mathcal{M}$ so $X \in \mathcal{M}^*$.
2. If $E \in \mathcal{M}^*$, get $A \subset E \subset B$ so $B^c \subset E^c \subset A^c$, $A^c, B^c \in \mathcal{M}$. As well, $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \setminus A) = 0$, so $E^c \in \mathcal{M}^*$.
3. If $E_i \in \mathcal{M}^*$ is a countable collection, then get $A_i \subset E_i \subset B_i$. Fix $A = \bigcup A_i$ and $B = \bigcup B_i$. Then $B \setminus A = \bigcup (B_i \setminus A) \subset \bigcup (B_i \setminus A_i)$ so $\mu(B \setminus A) = 0$ and $A \subset \bigcup E_i \subset B$ so $\bigcup E_i \in \mathcal{M}^*$.
4. Let E_i be disjoint, $E = \bigcup E_i$, and $E_i \in \mathcal{M}^*$. Get $A_i \subset E_i \subset B_i$. Then $\mu^*(\bigcup E_i) = \mu(\bigcup A_i) = \sum \mu(A_i) = \sum \mu(E_i)$. \square

Def'n. 2.1.5 We call the space $(X, \mathcal{M}^*, \mu^*)$ the **completion** of (X, \mathcal{M}, μ) .

In particular, every subset of a set with measure 0 is measurable.

2.1.2 $L^1(\mu)$ as a normed space

Prop. 2.1.6 1. Let $f : X \rightarrow [0, +\infty)$ be measurable, $E \in \mathcal{M}$. If $\int_E f \, d\mu = 0$, then $f = 0$ almost everywhere in E .

2. Let $f \in L^1(\mu)$. If $\int_E f \, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ almost everywhere in X .

PROOF 1. Let $A_n = \{x \in E : f(x) > 1/n\}$, so that

$$\frac{1}{n}\mu(A_n) \leq \int_{A_n} f \, d\mu \leq \int_E f \, d\mu = 0 \implies \mu(A_n) = 0$$

for all n . But then

$$N = \{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n \implies \mu(N) \leq \sum \mu(A_n) = 0$$

2. Write $f = u + iv$ so that

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \int_E v^+ \, d\mu - i \int_E v^- \, d\mu$$

We show that $u^+ = 0$ almost everywhere (the other terms are identical). Let $E = \{x \in X : u(x) \geq 0\}$, so $\int_E f \, d\mu = 0$, so its real part is zero and $\int_E u^+ \, d\mu = 0$. Thus $u^+ = 0$ almost everywhere in E . The result follows. \square

Def'n. 2.1.7 A **normed space** over \mathbb{R} is a vector space V over \mathbb{R} with a map $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

(i) $x \in V \implies \|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| \leq |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in V$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Now $L^1(\mu) = \{f : X \rightarrow \mathbb{C} \text{ measurable and } \int_X |f| \, d\mu < \infty\}$. We certainly have that $L^1(\mu)$ is a vector space. We wish to define $\|f\| = \int_X |f| \, d\mu$. The only problem is that

$$\int_X |f| \, d\mu = 0 \implies f = 0 \text{ almost everywhere}$$

To deal with this problem, we quotient our space by the equivalence relation $f \sim g$ if and only if $f = g$ almost everywhere. With this in mind, define $V = L^1(\mu)/\sim$ denote the set of equivalence classes. We need to define $+, \cdot, \|\cdot\|$ on V . Let $[f]$ denote the class of f . Then

$$[f] + [g] = [f + g]$$

$$c[f] = [cf]$$

$$\|[f]\| = \int_X |f| \, d\mu$$

Let's verify that this is well defined: if $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$. Indeed, this is true since the sums are equal except perhaps on a union of measure zero sets, so equality holds almost everywhere. The second definition is obviously well defined. Finally, by a homework assignment, $\|[f]\|$ is also well defined. Now, let's verify the properties of the norm.

- (i) Certainly $\|f\| \geq 0$, and $\|f\| = 0$ implies $f = 0$ almost everywhere, so $[f] = [0] = 0$.
- (ii) We have $\|\lambda \cdot f\| = \int_X |\lambda f| d\mu = |\lambda| \int_X |f| d\mu = |\lambda| \|f\|$
- (iii) We have $\|f + g\| = \int_X |f + g| d\mu \leq \int_X |f| + \int_X |g| = \|f\| + \|g\|$

In $L^1(\mu)$, two functions are the same if they are equal almost everywhere. However, this can be a challenge: if $f \in L^1(\mu)$ and $x_0 \in X$, then $f(x_0)$ is not well defined. For example, it is challenging to give meaning to boundary conditions of functions.

2.1.3 Construction of the Lebesgue measure

We begin from the Riemann integral $\int_a^b f(x) dx$ for a continuous function f . Define $\text{supp } f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. For continuous functions with compact (bounded) support, define $\Lambda f = \int_{\mathbb{R}} f(x) dx$ is the Riemann integral, which is a functional. In particular,

$$\text{measure}((a, b)) = \text{length}((a, b)) = \sup\{\Lambda f : f \text{ is continuous, compact support, } 0 \leq f \leq 1, \text{supp } f \subset (a, b)\}$$

We will extend this to a σ -algebra containing the Borel sets. In order to define these, for open sets, $\mu(G) = \sup\{\Lambda f : 0 \leq f \leq 1, \text{supp } f \subset G\}$, where Λ is the Riemann integral. For an arbitrary set, $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$. However, this “measure” is not countably additive: the σ -algebra $\mathcal{P}(X)$ is too large (Vitali’s construction). Instead, we will define $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$, which means that $E \cap K$ is regular for any K compact, and regular means that the outer measure and inner measure are equal. The outer measure is $\sup\{\mu(K) : K \subset E \text{ compact}\} = \mu(E)$.

2.2 The Riesz Representation Theorem

In this section, we assume that (X, τ) be a locally compact, Hausdorff topological space.

Def’n. 2.2.1 We denote the space of continuous functions with compact support by $C_c(X) = \{f : X \rightarrow \mathbb{C} \mid f \in C(X), \text{supp } f \text{ is compact}\}$.

Def’n. 2.2.2 Let $\Lambda : C_c(X) \rightarrow \mathbb{C}$ be a **linear functional**, i.e. $\Lambda(cf + g) = c\Lambda f + \Lambda g$. Λ is called a **positive linear functional** if $f \geq 0 \Rightarrow \Lambda f \geq 0$.

By positivity, if $f \leq g$, then $g - f \geq 0$ so $\Lambda g - \Lambda f = \Lambda(g - f) \geq 0$ and $\Lambda f \leq \Lambda g$.

Def’n. 2.2.3 We say that $K < f$ if K is compact and $f \in C_c(X)$, $0 \leq f \leq 1$ implies that $x \in K \Rightarrow f(x) = 1$. We say that $f < G$ if G is open, $f \in C_c(X)$, $0 \leq f \leq 1$, and $\text{supp } f \subset G$.

Lemma 2.2.4 (Urysohn) Let $G \in \tau$, $K \subset G$ compact. Then there exists $f \in C_c(X)$ such that $K < f < G$.

PROOF Will do later. □

Lemma 2.2.5 (Partition of Unity) Let $G_1, G_2, \dots, G_n \in \tau$, and let $K \subset G_1 \cup \dots \cup G_n$ be compact. Then there are functions $h_i \in C_c(X)$ such that $h_i < G_i$ and $K < \sum h_i$.

PROOF Also will do later. □

How can we create a positive linear functional on $C_c(X)$? If μ is a measure, and functions on $C_c(X)$ are measurable, then $\Lambda f = \int_X f d\mu$ is a positive linear functional. The representation theorem says that there are no other examples.

Thm. 2.2.6 (Riesz Representation) *Let (X, τ) be as above. If $\Lambda : C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional, then there exists a unique measure space (X, \mathcal{M}, μ) such that $\Lambda f = \int_X f d\mu$ for any $f \in C_c(X)$, $\mathcal{M} \supset \tau$, and*

- (i) $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$ for all $E \in \mathcal{M}$.
- (ii) $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$.
- (iii) $\mu(K) < \infty$ for any K compact.
- (iv) \mathcal{M} is complete.

First, let's get some definitions out of the way. Fix the notation as above.

Def'n. 2.2.7 *Fix a Borel measure μ . The **Lebesgue outer measure** is defined $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$.*

Def'n. 2.2.8 *We say that $E \subset X$ is **regular** if $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$. Similarly, $E \subset X$ is **locally regular** for every compact K , $K \cap E$ is regular.*

Claim 0: Definition of μ and \mathcal{M} ; completeness of \mathcal{M} .

PROOF For an open set $G \in \tau$, let $\mu(G) = \sup\{\Lambda f : f < G\}$. Then $\mu(\emptyset) = 0$ and $G_1 \subset G_2$ implies that $\mu(G_1) \leq \mu(G_2)$. Then extend μ to arbitrary $E \subset X$ as an outer measure. Now let $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$. Note that \mathcal{M} contains compact sets, since they are regular. This is direct from the definition, since $\mu(F) \leq \mu(K)$ for any compact $F \subseteq K$ and the supremum occurs exactly at K .

As well, \mathcal{M} is complete: let $E \in \mathcal{M}$, $\mu(E) = 0$ and $A \subset E$. We want to show that $A \in \mathcal{M}$. Let K be an arbitrary compact set; then $\mu(K \cap A) = 0$. Now if $F \subset K \cap A$ is compact, $\mu(F) = 0$. Thus $\sup\{\mu(F) : F \subset K \cap A \text{ compact}\} = 0$, so $K \cap A$ is regular and A is locally regular and an element of \mathcal{M} . □

Claim 1: μ is σ -subadditive. In other words, if E_1, E_2, \dots are arbitrary subsets of X , then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

PROOF If $\mu(E_j) = \infty$ for some j , then we are done. Thus assume $\mu(E_j) < \infty$ for all j . Let $\epsilon > 0$, $\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ be arbitrary. We will show that $\gamma \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Let $G_j \supset E_j$ be open, such that $\mu(G_j) \leq \mu(E_j) + \frac{\epsilon}{2^j}$. Then

$$\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \mu\left(\bigcup_{j=1}^{\infty} G_j\right)$$

so there exists some $f < \bigcup_{j=1}^{\infty} G_j$ so $\gamma < \Lambda f$ by the definition of μ on open sets. Let $K = \text{supp } f$ so that

$$K \subset \bigcup_{j=1}^{\infty} G_j \implies K \subset \bigcup_{j=1}^n G_j$$

since $\{G_j\}$ are an open cover for K and K is compact. Get a partition of unity $h_j < G_j$ for each $j = 1, \dots, n$ which satisfies $\sum_{j=1}^n h_j(x) = 1$ for any $x \in K$. Then $f \cdot h_j < G_j$ and $f = f \cdot \sum_{j=1}^n h_j$ so that

$$\begin{aligned} \gamma < \Lambda f &= \Lambda \left(\sum_{j=1}^n f h_j \right) = \sum_{j=1}^n \Lambda(f h_j) \\ &\leq \sum_{j=1}^n \mu(G_j) \leq \sum_{j=1}^n \left(\mu(E_j) + \frac{\epsilon}{2j} \right) \\ &\leq \sum_{j=1}^{\infty} \left(\mu(E_j) \right) + \epsilon \end{aligned}$$

which holds for all $\epsilon > 0$ if and only if $\gamma \leq \sum_{j=1}^{\infty} \mu(E_j)$. This holds for any $\gamma \leq \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ and the result follows. \square

Claim 2: If $K < f < G$, then $\mu(K) \leq \Lambda f \leq \mu(G)$. Thus if K is compact, $\mu(K) < \infty$.

PROOF It is direct from the definition of μ that $\Lambda f \leq \mu(G)$. Thus let $\gamma < \mu(K)$ and $\alpha \in (0, 1)$. Let $V_\alpha := \{x \in X : f(x) > \alpha\}$ and $K \subset V_\alpha$ since $f \equiv 1$ on K . Since f is continuous, $V_\alpha = f^{-1}((\alpha, \infty))$ is the preimage of an open set and thus open.

Now $\gamma < \mu(K) \leq \mu(V_\alpha)$, so we have some $h < V_\alpha$ such that $\gamma < \Lambda h$. Then $\alpha \cdot h \leq f$ since in V_α , $\alpha \cdot h \leq \alpha < f$ and in V_α^c , $\alpha \cdot h = 0 \leq f$. Now $\alpha \cdot \Lambda h = \Lambda(\alpha h) \leq \Lambda f$ so $\gamma < \Lambda f / \alpha$. This is true for all $\alpha \in (0, 1)$ and $\gamma \leq \Lambda f$. Since this holds for all $\gamma < \mu(K)$, we have $\mu(K) \leq \Lambda f$ as required.

Now, let K be compact so that $\mu(K) \leq \Lambda f$ for all $K < f$. Let $\epsilon > 0$ and get $G \in \tau$, $G \supset K$ such that $\mu(G) \leq \mu(K) + \epsilon$. Then by Urysohn's lemma, get some $K < f < G$ so that $\mu(K) \leq \Lambda f \leq \mu(G)$, so $\Lambda f \leq \mu(K) + \epsilon$ and the result holds. Now suppose K is compact, so $\mu(K) = \inf\{\mu(G) : K \subset G \in \tau\}$. By Urysohn's Lemma, get f with $K < f < G$, and by (1.), $\mu(K) \leq \Lambda f \leq \mu(G)$ so that $\mu(K) = \inf\{\Lambda f : K < f\}$. As a corollary, we have that $\mu(K) < \infty$ (since Λ is a positive linear functional). Besides demonstrating one of our properties, this provides a convenient way of computing the measure of compact sets. \square

Claim 3: If $G \in \tau$, then G is regular.

PROOF We first show that if $0 \leq f \leq 1$, then $\Lambda f \leq \mu(\text{supp } f)$. Let $G \supset \text{supp } f$ be open, so $f < G$ and $\mu(G) \geq \Lambda f$. Then $\mu(\text{supp } f) = \inf\{\mu(G) : E \subset G \in \tau\} \geq \Lambda f$.

Now we want to show $\mu(G) = \sup\{\mu(K) : K \subset G \text{ compact}\}$. It suffices to show that $\sup\{\mu(K) : K \subset G \text{ compact}\} \geq \mu(G)$, so let $\gamma < \mu(G)$ and we want K compact so that $\mu(K) > \gamma$. Let $f < G$ be such that $\Lambda f > \gamma$. Then $\mu(\text{supp } f) > \gamma$ by the previous claim is compact, as desired. \square

Claim 4: Suppose E_1, E_2, \dots are disjoint regular. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

If we assume additionally that $\mu(\cup E_i) < \infty$, then $\cup_{i=1}^{\infty} E_i$ is regular.

PROOF We first prove this for two compact sets. Thus let K_1, K_2 be disjoint compact sets. Then K_2^c is open and $K_2^c \supset K_1$. By Urysohn's lemma, get $f \in C_c(X)$ so that $K_1 < f < K_2^c$ and $x \in K_1$ implies $f(x) = 1$, and $x \in K_2$ implies $f(x) = 0$.

Since $K_1 \cup K_2$ is compact, for all $\epsilon > 0$, get $g < K_1 \cup K_2$ such that $\mu(K_1 \cup K_2) + \epsilon > \Lambda g$ (by Claim 2). Furthermore, $K_1 < f \cdot g$ and $K_2 < (1-f) \cdot g$. Thus $\mu(K_1) + \mu(K_2) \leq \Lambda(f \cdot g) + \Lambda((1-f) \cdot g) = \Lambda g < \mu(K_1 \cup K_2) + \epsilon$ which is true for any $\epsilon > 0$. Thus $\mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$ and equality holds, as required.

Now, by Claim 1, it remains to show that $\mu(\cup E_i) \geq \sum \mu(E_i)$. If $\mu(\cup E_i) = +\infty$, we are done, so assume $\mu(\cup E_i) < +\infty$. Since the E_i are regular, there is a compact set $H_i \subset E_i$ so that $\mu(H_i) > \mu(E_i) - \frac{\epsilon}{2^i}$ for each $i \in \mathbb{N}$. Let $K_n = \cup_{i=1}^n H_i$. Now

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \mu(K_n) = \sum_{i=1}^n \mu(H_i) > \sum_{i=1}^n \mu(E_i) - \epsilon$$

Taking the limit as n goes to infinity gives $\mu(\cup E_i) \geq \sum \mu(E_i) - \epsilon$ for any $\epsilon > 0$, so we are done.

Let's now see the second part. For any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^N \mu(E_i) + \epsilon \leq \mu(K_N) + 2\epsilon$$

with K_N compact defined in the same way as above. Since $\epsilon > 0$ was arbitrary, the result follows directly. \square

Claim 5: E is regular and $\mu(E) < \infty$ if and only if for any $\epsilon > 0$, there exists K compact, G open so that $K \subset E \subset G$ and $\mu(G \setminus K) < \epsilon$.

PROOF There exists by regularity (and the definition of the outer measure) $K \subset E \subset G$ so that

$$\mu(E) - \frac{\epsilon}{2} \leq \mu(K) \leq \mu(G) \leq \mu(E) + \epsilon/2$$

As well, $\mu(G) = \mu(K \cup (G \setminus K)) = \mu(K) + \mu(G \setminus K)$ and $\mu(G \setminus K) = \mu(G) - \mu(K) < \epsilon$.

Conversely, let $K \subset E \subset G$ and $\mu(G \setminus K) < \epsilon$. Then

$$\mu(E) \leq \mu(G) = \mu(K) + \mu(G \setminus K) < \mu(K) + \epsilon$$

so $\mu(E) < \infty$ and $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$, so E is regular. \square

Claim 6:

1. Let A, B be regular with $\mu(A), \mu(B) < \infty$. Then $A \setminus B, A \cup B, A \cap B$ are regular and have finite measure.
2. If $\mu(E) < \infty$, then E is regular if and only if E is locally regular.
3. If E_i are regular, then $\cup_{i=1}^{\infty} E_i$ is regular.

PROOF Recall that for any $\epsilon > 0$, there exists $K_1 \subset A \subset G_1$ and $K_2 \subset B \subset G_2$ such that $\mu(G_1 \setminus K_1) < \epsilon$ and $\mu(G_2 \setminus K_2) < \epsilon$.

1. Note that $A \setminus B \subset G_1 \setminus K_2 \subset (G_1 \setminus K_1) \cup (K_1 \setminus G_2) \cup (G_2 \setminus K_2)$, where $K_1 \setminus G_2$ is compact. Thus $\mu(A \setminus B) \leq \epsilon + \mu(K_1 \setminus G_1) + \epsilon < \infty$ and $\mu(A \setminus B) - 2\epsilon \leq \mu(K_1 \setminus G_2)$ so $A \setminus B$ is regular. Finally since $A \cup B = (A \setminus B) \cup B$, $A \cup B$ is regular and $\mu(A \cup B) < \infty$. Thus $A \cap B = (A \cup B) \setminus ((A \setminus B) \cup (B \setminus A))$ is regular and has measure less than infinity.
2. Let $\mu(E) < \infty$, and first suppose E is regular. Let K be a compact set. Then $\mu(K) < \infty$ and K is regular, so $E \cap K$ is regular (by 1.) so E is locally regular.

Conversely, suppose E is locally regular. Let $\epsilon > 0$ and $G \supset E$ be open so that $\mu(G) < \mu(E) + 1 < \infty$. As well, G is regular, so there exists K with $\mu(G) < \mu(K) + \epsilon/2$. Now,

$$\begin{aligned} \mu(E) &= \mu((E \setminus K) \cup (E \cap K)) \leq \mu(E \setminus K) + \mu(E \cap K) \\ &\leq \mu(G \setminus K) + \mu(E \cap K) \\ &< \frac{\epsilon}{2} + \mu(E \cap K) \end{aligned}$$

so $\mu(E \cap K) > \mu(E) - \epsilon/2$. Then since E is locally regular, $E \cap K$ is regular and get a compact set $L \subset E \cap K$ such that $\mu(L) > \mu(E \cap K) - \epsilon/2 > \mu(E) - \epsilon$. This holds for any $\epsilon > 0$, so E is regular.

3. Set $F_1 = E_1$, $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right)$ so $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and the F_i are disjoint. By Claim 4, $\bigcup F_i$ is regular and F_i are regular (TODO: finiteness requirement?) \square

Claim 7: \mathcal{M} is a σ -algebra, $\mathcal{M} \subset \tau$, and μ is countably additive on \mathcal{M} .

PROOF We demonstrate the requirements:

- Let $A \in \mathcal{M}$: we see that $A^c \in \mathcal{M}$. If K is an arbitrary compact set, then $A^c \cap K = K \setminus (A \cap K)$ is regular by Claim 7 since K is regular (and thus locally regular), and $A \cap K$ is regular since A is locally regular.
- Now let $A_n \in \mathcal{M}$; we will show that $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$. Indeed, if K is an arbitrary compact set, then

$$A \cap K = \bigcup_{n=1}^{\infty} (A_n \cap K)$$

is regular by Claim 6.

- We now show $\mathcal{M} \supset \tau$. It suffices by closure under complements to show that all closed sets are in \mathcal{M} . If A is closed, then $A \cap K$ is compact and thus regular, so $A \in \mathcal{M}$.
- Finally, let $E_i \in \mathcal{M}$ be locally regular and disjoint; it suffices to show that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

If $\mu(E_i) = +\infty$, we are done, so assume $\mu(E_i) < \infty$ for all i . But then by Claim 6.2, the E_i are regular, so the result holds by Claim 6.3. \square

Claim 8: $\int_X f d\mu = \int f d\mu$ for all $f \in C_c(X)$.

PROOF We are finally almost done: we just need to show that μ , as defined, actually represents Λ . Let's start by simplifying f as much as possible.

- It suffices to do this for real valued functions. If $f = u + iv$, then $\Lambda f = \Lambda u + i\Lambda v = \int_X u \, d\mu + i \int_X v \, d\mu = \int_X f \, d\mu$.
- It suffices to show $\Lambda f \leq \int_X f \, d\mu$. If this holds for all f , then $\Lambda(-f) \leq \int_X -f \, d\mu$ so that $-\Lambda f \leq -\int_X f \, d\mu$ and $\Lambda f \geq \int_X f \, d\mu$ and equality holds.
- It is enough to prove that $\Lambda f \leq \int_X f \, d\mu$ for $f \geq 0$. Assuming so, let f be arbitrary and let $K = \text{supp } f$ be compact, and $a = \min f$, $b = \max f$. The general idea of the proof is to translate f by the value $|a|$ so that it is positive. However, we cannot do this directly since $f + |a|$ is not compactly supported; however, we can use Urysohn's Lemma to translate it on its support. Now, let $\epsilon > 0$ be arbitrary. Fix $K = \text{supp } f$ and get $G \supset K$ so that $\mu(G) \leq \mu(K) + \epsilon$. By Urysohn's lemma, there exists $h \in C_c(X)$ so that $K \subset h \subset G$. Thus $|a| \cdot h(x) = |a|$ for all $x \in K$, so $F := f + |a|h \geq 0$ since $f \geq -|a|$. Now by assumption,

$$\Lambda F \leq \int_X F \, d\mu = \int_X f \, d\mu + |a| \int_X h \, d\mu$$

so that

$$\begin{aligned} \Lambda f &= \Lambda F - |a|\Lambda h \\ &\leq \int_X f \, d\mu + |a| \int_X h \, d\mu - |a|\Lambda h \\ &\leq \int_X f \, d\mu + |a| \left(\int_X h \, d\mu - \Lambda h \right) \end{aligned}$$

We now want to show $|\int_X h \, d\mu - \Lambda h| < \epsilon$, and the result will follow. By Claim 2, $\mu(K) \leq \Lambda h \leq \mu(G)$. As well, $\int_X h \, d\mu \leq \mu(G)$ since $h \leq \chi_G$. Thus since $h \geq 0$, by assumption

$$\mu(K) \leq \Lambda h \leq \int_X h \, d\mu \leq \mu(G)$$

and the result follows since $\mu(G) - \mu(K) < \epsilon$. Thus $\Lambda f \leq \int f + |a|\epsilon$ for all $\epsilon > 0$, so $\Lambda f \leq \int f$ as desired.

It now remains to show $\Lambda f \leq \int_X f \, d\mu$ for $f \geq 0$. Since $f = Mf/M$ where $M = \max f$, we can assume $0 \leq f \leq 1$. Fix $K = \text{supp } f$, let $\epsilon > 0$ be arbitrary. Let $0 = c_0 < c_1 < c_2 < \dots < c_n = 1$ with $c_k - c_{k-1} < \epsilon$ for all k and $\mu(f^{-1}(c_k)) = 0$ for all $k = 1, \dots, n-1$. The existence of such a set follows from Assignment 6. Let $K_j = K \cap f^{-1}([c_{j-1}, c_j])$ for $j = 1, 2, \dots, n$ and $L_j = K \cap f^{-1}([c_{j-1}, c_j])$ for $j = 1, 2, \dots, n-1$.

For each K_j and any $\epsilon > 0$, there exists $\tau \ni G_j \supset K_j$ such that $\mu(G_j) \leq \mu(K_j) + \frac{\epsilon}{2j}$. By Urysohn's lemma, get h_j so that $K_j \subset h_j \subset G_j$. Then $f \leq \sum_{j=1}^n c_j h_j$: if $x \in K^c$, $f = 0$. Otherwise, if $x \in K$, then $x \in K_j$ for some j . Since $h_j = 1$ and $f(x) \leq c_j$ on K_j , we have $f(x) \leq c_j = c_j h_j(x) \leq \sum_{i=1}^n c_i h_i$.

Now, there is just a lot of algebra.

$$\begin{aligned}
\Lambda f &\leq \Lambda \left(\sum_{i=1}^n c_j h_j \right) = \sum_{i=1}^n c_j \Lambda h_j && \text{(linearity and positivity)} \\
&\leq \sum_{j=1}^n c_j \mu(G_j) && (h_j \leq G_j) \\
&\leq \sum_{j=1}^n c_j \mu(K_j) + \sum_{j=1}^n c_j \frac{\epsilon}{2^j} && \text{(choice of } K_j) \\
&\leq \sum_{j=1}^n (c_{j-1} + c_j - c_{j-1}) \mu(L_j) + \epsilon && (L_j \subset K_j, |c_j| \leq 1) \\
&\leq \sum_{j=1}^n c_{j-1} \mu(L_j) + \epsilon \cdot \mu(K) + \epsilon && (L_j \text{ disjoint, } c_j - c_{j-1} < \epsilon)
\end{aligned}$$

Now define g so $g(x) = c_{j-1}$ if $x \in L_j$, and $g \equiv 0$ outside K . Then g is a simple function, so that the summation above is precisely the integral of g . Furthermore, $g \leq f$ so $\int_X g \, d\mu \leq \int_X f \, d\mu$ and

$$\begin{aligned}
\Lambda f &\leq \int_X g \, d\mu + \epsilon + \epsilon \mu(K) \\
&\leq \int_X f \, d\mu + \epsilon(1 + \mu(K))
\end{aligned}$$

and, since $\mu(K) < \infty$, because this holds for any $\epsilon > 0$, we are done! \square

2.3 Regularity Properties of Borel Measures

We have the Riesz representation theorem in a locally compact Hausdorff space. Our aim is to introduce the Lebesgue measure in \mathbb{R}^k which respects translation as well.

Def'n. 2.3.1 A measure defined on the family of Borel sets is called a **Borel measure**.

Def'n. 2.3.2 Let $\mu : \mathcal{B} \rightarrow [0, +\infty]$ be a Borel measure.

1. E is called **outer regular** if $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$.
2. E is called **inner regular** if $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$
3. μ is called **regular** if every $E \in \mathcal{B}$ is inner and outer regular.

Def'n. 2.3.3 A set $E \subset X$ is called **σ -compact** if $E = \bigcup_{n=1}^{\infty} E_n$, for E_n compact.

Def'n. 2.3.4 A G_δ set is one of the form $\bigcap_{n=1}^{\infty} A_n$ with A_n open, and a F_σ set is one of the form $\bigcup_{n=1}^{\infty} B_n$ for B_n closed.

Thm. 2.3.5 Let X be a locally compact, σ -compact Hausdorff space. Let $\mathcal{M} \supset \mathcal{B}$ be a σ -algebra, $\mu : \mathcal{M} \rightarrow [0, +\infty]$ be a measure such that

- (i) $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$
- (ii) $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}, \mu(E) < \infty$
- (iii) $\mu(K) < \infty$ for K compact.

Then

- 1. For all $E \in \mathcal{M}$ and $\epsilon > 0$, there exists F closed and G open so that $F \subset E \subset G$ and $\mu(G \setminus F) < \epsilon$.
- 2. μ is regular
- 3. For all $E \in \mathcal{M}$, there exists a F_σ set A and a G_δ set B so $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.

PROOF Since X is σ -compact, $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact.

- 1. By (iii), we have $\mu(K_n \cap E) < \infty$. Thus by (i), get G_n open so that $G_n \supset K_n \cap E$ with $\mu(G_n \setminus (K_n \cap E)) < \frac{\epsilon}{2^{n+1}}$. Let $G = \bigcup_{n=1}^{\infty} G_n$ be open, so that $G \setminus E \subset \bigcup_{n=1}^{\infty} (G_n \setminus (K_n \cap E))$ and $\mu(G \setminus E) < \frac{\epsilon}{2}$. Repeat this for E^c : get an open set H such that $\mu(H \setminus E^c) < \frac{\epsilon}{2}$. Then $F = H^c \subset E$ satisfies $\mu(E \setminus F) = \mu(F^c \setminus E^c) = \mu(H \setminus E^c) < \frac{\epsilon}{2}$. Then $\mu(G \setminus F) \leq \mu(G \setminus E) + \mu(E \setminus F) < \epsilon$. Then $\mu(G \setminus F) \leq \mu(G \setminus E) + \mu(E \setminus F) < \epsilon$.
- 2. E is outer regular by (i). If $\mu < \infty$, then E is inner regular by (ii). If $\mu(E) = \infty$, let $F \subset E$ be given by 1. Then $\mu(F) = +\infty$, or $\mu(E)$ would be finite. Let $H_n = \bigcup_{k=1}^n (F \cap K_k)$ compact, $H_n \subset F$. Then $\bigcup_{n=1}^{\infty} H_n = F$, and $\mu(H_n) \rightarrow \mu(F) = \infty$. Thus $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$.
- 3. Apply 1 with $\epsilon = 1/j$ for $j \in \mathbb{N}$. Then there exists $F_j \subset E \subset G_j$ so $\mu(G_j \setminus F_j) < \frac{1}{j}$. Let $A = \bigcup_{j=1}^{\infty} F_j$ and $B = \bigcap_{j=1}^{\infty} G_j$. Then $A \subset E \subset B$ and $\mu(B \setminus A) \leq \mu(G_j \setminus F_j) < \frac{1}{j}$, so $\mu(B \setminus A) = 0$. \square

Thm. 2.3.6 Let X be locally compact and Hausdorff, and assume that every open set is σ -compact. Let $\lambda : \mathcal{B} \rightarrow [0, \infty]$ be a Borel measure such that $\lambda(K) < \infty$. Then λ is regular.

PROOF Let $\Lambda f = \int_X f d\lambda$. Then $\Lambda : C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional. By the Riesz representation theorem, there exists $\mu : \mathcal{M} \rightarrow [0, \infty]$ such that $\int_X f d\mu = \Lambda f = \int_X f d\lambda$. We see that $\lambda = \mu$ on \mathcal{B} . We first prove this for open sets. Let $G \in \tau$; then there exists K_n so $G = \bigcup_{n=1}^{\infty} K_n$. By Urysohn's lemma, there exists f_i such that $K_i < f_i < G$. Let $g_n = \max\{f_1, f_2, \dots, f_n\}$, so

$g_n \in C_c(X)$, and $g_n \rightarrow \chi_G$ pointwise. But then

$$\begin{aligned}\lambda(G) &= \int_X \chi_G d\lambda \\ &= \int_X \lim_{n \rightarrow \infty} g_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\mu \\ &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \int_X \chi_G d\mu \\ &= \mu(G)\end{aligned}$$

Now for any $E \in \mathcal{B}$, apply (i). Then $F \subset E \subset G$, $\mu(G \setminus F) < \epsilon$. Since $G \setminus F$ is open, $\lambda(G \setminus F) = \mu(G \setminus F) < \epsilon$ so $\lambda(G) \leq \lambda(E) + \epsilon$. Thus $|\mu(E) - \lambda(E)| < \epsilon$ for all $\epsilon > 0$ so $\lambda(E) = \mu(E)$. \square

2.4 Construction of the Lebesgue Measure

We have the Riesz Representation Theorem in a locally compact Hausdorff space.

Def'n. 2.4.1 Let $E \subset \mathbb{R}^k$, $x \in \mathbb{R}^k$. Then $E + x = \{y + x : y \in E\}$ is the **translate** of E .

Def'n. 2.4.2 We define a k -cell in \mathbb{R}^k by $W = I_1 \times I_2 \times \cdots \times I_k$ where I_j is an interval. We also define $(W) = (b_1 - a_1)(b_2 - a_1) \cdots (b_k - a_k)$ where a_j, b_j are the endpoints of the I_j .

We know that $(W + x) = (W)$ for any k -cell W and $x \in \mathbb{R}$.

Thm. 2.4.3 There exists a σ -algebra \mathcal{M} in \mathbb{R}^k and a complete measure $m : \mathcal{M} \rightarrow [0, +\infty]$ satisfying

1. $m(W) = (W)$ for any k -cell W .
2. $\mathcal{M} \supset \mathcal{B}$ and $E \in \mathcal{M}$ if and only if there exists $A \in F_\sigma, B \in G_\delta$ such that $A \subset E \subset B$ and $m(B \setminus A) = 0$.
3. m is translation invariant: $m(E + x) = m(E)$.
4. If μ is a translation invariant Borel measure, and $\mu(K) < \infty$ for all K compact, then there exists $c \in \mathbb{R}$ so that $\mu(E) = c \cdot m(E)$.
5. If $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is linear, then there exists $\Delta(T) \in \mathbb{R}$ such that $m(T(E)) = \Delta(T) \cdot m(E)$.

PROOF For $f \in C_c(\mathbb{R}^k)$, let $\Lambda f = \int_{\mathbb{R}^k} f(x) dx$ (the Riemann Integral). Then $\Lambda : C_c(\mathbb{R}^k) \rightarrow \mathbb{C}$ is a positive linear functional, so by the Riesz representation theorem, there exists a unique measure m and $\mathcal{M} \supset \mathcal{B}$ so for all $f \in C_c(\mathbb{R}^k)$, $\Lambda f = \int_{\mathbb{R}^k} f dm$.

We first see (i). By the definition of m , for an open k -cell W , $m(W) = \sup\{\Lambda f : f < W\} = (W)$ (by definition of the Riemann integral). If W is an arbitrary k -cell, then there exist open k -cells W_n such that $W = \bigcap_{n=1}^{\infty} W_n$. Then $(W_n) \rightarrow (W)$, so $m(W_n) \rightarrow m(W)$, and $(W_n) = m(W_n)$. Thus $(W) = m(W)$.

Let λ be a Borel measure. If $\lambda(W) = m(W)$ for all W k -cells, then $\lambda(E) = m(E)$ for all $E \in \mathcal{B}$. For any G open, $G = \bigcup_{n=1}^{\infty} W_n$ disjointly, so $\lambda(G) = m(W)$. Then since λ and m are regular, $\lambda(E) = \inf\{\lambda(G) : E \subset G \in \tau\} = \inf\{m(G) : E \subset G \in \tau\} = m(E)$ for all $E \in \mathcal{B}$.

We now see (iii). Define $\lambda(E) = m(E + X)$. If W is a box, then $\lambda(W) = m(W + x) = (W + x) = (W) = m(W)$, so by the lemma, $\lambda(E) = m(E)$ for all $E \in \mathcal{B}$. Then regularity implies $\lambda(E) = m(E)$ for all measurable E .

We have (iv): let $c = \mu([0, 1]^k) = c \cdot ([0, 1]^k)$. Translation invariance of μ implies $\mu(W) = c \cdot (W)$.

We have (v). If $\dim(\text{Im}(T)) < k$, then $m(\text{Im}(T)) = 0$ so $\Delta(T) = 0$. Otherwise, T is a homeomorphism so $T(E) \in \mathcal{B}$ for all $E \in \mathcal{B}$. Let $\mu(E) = m(T(E))$. Then $\mu(E + x) = m(T(E) + T(x)) = m(T(E)) = \mu(E)$, so μ is translation invariant. Then by (iv), $\mu(E) = c \cdot m(E)$ and set $\Delta(T) = c$. \square

Thm. 2.4.4 If $A \subset \mathbb{R}$ for which every set is Lebesgue measurable, then $m(A) = 0$.

PROOF Partition \mathbb{R} into cosets by \mathbb{Q} ; let E be a set containing exactly one element of each class (axiom of choice). Now if $r \neq s$, $r, s \in \mathbb{Q}$, then $(E + r) \cap (E + s) = \emptyset$. But then $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r)$ disjointly. Given A , define $A_t = A \cap (E + t)$ for $t \in \mathbb{Q}$. Now let $K \subset A_t$, so $K \subset E + t$. Since $(K + r_1) \cap (K + r_2) = \emptyset$, define $H = \bigcup_{r \in \mathbb{Q} \cap [0, 1]} (K + r)$ is a countable disjoint union. But then $\infty > m(H) = \sum_r m(K)$ so $m(K) = 0$ and $m(A_t) = 0$. But then

$$\bigcup_{t \in \mathbb{Q}} A_t = \bigcup_{t \in \mathbb{Q}} (A \cap (E + t)) = A \cap \left(\bigcup_{t \in \mathbb{Q}} (E + t) \right) = A \cap \mathbb{R} = A$$

so $m(A) = 0$ as well. \square

Let X be a locally compact, Hausdorff topological space. Let \mathcal{M} be a σ -algebra, μ be a measure satisfying the properties in the Riesz representation theorem. We then have

Thm. 2.4.5 (Lusin) Let $f : X \rightarrow \mathbb{C}$ be a measurable function, with $\text{supp } f \subset A$ and $\mu(A) < \infty$. Then for any $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$ and $\sup_X |G| \leq \sup_X |f|$.

PROOF If $\text{supp } g \subset A$, then $f(x) = 0 = g(x)$ if $x \in A^c$ and $\{x \in X : f(x) \neq g(x)\} = \{x \in A : f(x) \neq g(x)\}$. Then $\mu(A) < \infty$ so there exists $K \subset A$ compact, with $\mu(A \setminus K) < \epsilon/2$. Then

$$\mu(\{x \in A : f(x) \neq g(x)\}) \leq \mu(\{x \in K : f(x) \neq g(x)\}) + \frac{\epsilon}{2}$$

We may assume that A is compact.

We first prove this for $0 \leq f \leq 1$. For $t \geq 0$ and each $N \in \mathbb{N}$, let $k_n(t)$ satisfy $k_n(t) \leq t \cdot 2^n < k_n(t) + 1$, and define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n}, & 0 \leq t \leq n \\ n, & t > n \end{cases}$$

Let $s_n(x) = \phi_n(f(x))$ and $t_n = s_n - s_{n-1}$. Then since

$$k_{n-1}(t) \leq t \cdot 2^{n-1} < k_{n-1}(t) + 1 \implies 2k_{n-1}(t) \leq t \cdot 2^n < 2k_{n-1}(t) + 2$$

is the largest even number below $t \cdot 2^n$. Then $k_n - 2k_{n-1} \in \{0, 1\}$ for all t . Since $0 \leq f \leq 1$,

$$t_n(x) = \phi_n(f(x)) - \phi_{n-1}(f(x)) = 2^{-n} \cdot k_n(f(x)) - 2^{-(n-1)} \cdot k_{n-1}(f(x))$$

this becomes

$$2^n \cdot t_n(x) = k_n(f(x)) - 2 \cdot k_{n-1}(f(x)) \in \{0, 1\}$$

which is the characteristic function of some set, which we denote T_n . Furthermore, $f(x) = \sum_{n=1}^{\infty} t_n(x)$. Let $V \supset A$ be open so that \bar{V} is compact. Let K_n be compact, V_n be an open set such that $K_n \subset T_n \subset V_n \subset V$ with $\mu(V_n \setminus K_n) < \epsilon/2^n$. By Urysohn's lemma, there exists $h_n \in C_c(X)$ with $K_n \subset h_n \subset V_n$. Let $g = \sum_{n=1}^{\infty} 2^{-n} \cdot h_n$, which converges uniformly so that g is continuous and $\text{supp } g \subset \bar{V}$. If $x \in K_n$, then $h_n(x) = 1$ and $t_n(x) = 2^{-n}$, so $2^{-n} \cdot h_n(x) = t_n(x)$. If $x \notin V_n$, then $h_n(x) = 0$ so $t_n(x) = 0$, so $2^{-n} \cdot h_n(x) = t_n(x)$. Thus

$$S = \{x \in A : f(x) \neq g(x)\} \subset \bigcup_{n=1}^{\infty} (V_n \setminus K_n)$$

and $\mu(S) \leq \sum \mu(V_n \setminus K_n) < \epsilon$.

Now if f is bounded, $0 \leq f/M \leq 1$ follows directly by the above.

For f bounded and real valued, note that $0 \leq f + K \leq 2K$ and get $g \in C_c(X)$ near $f + K$. Then $g - K$ satisfies the required values.

Now for any real valued function, let $B_n = \{x \in X : |f(x)| > n\} \implies \bigcap_{n=1}^{\infty} B_n = \emptyset$, with $\mu(B_1) \leq \mu(\text{supp } f) < \infty$ and $B_{n+1} \subset B_n$ for all n . Thus $\mu(B_n) \rightarrow \mu(\bigcap B_n) = 0$. Let N be such that $\mu(B_N) < \epsilon/2$, so if $x \notin B_N$, $f(x) \leq N$ and define $\tilde{f}(x) = (1 - \chi_{B_N(x)})f(x)$. Then \tilde{f} is bounded, and get $g \in C_c(X)$ to \tilde{f} . In particular, $\mu(\tilde{f} \neq g) < \epsilon/2$. Then let $g = \tilde{g}$ and

$$\mu(g \neq f) \leq \mu(f \neq \tilde{f}) + \mu(\tilde{f} \neq g) = \epsilon$$

Finally, let f be complex valued and $f = f_1 + if_2$. Then get $g_1, g_2 \in C_c(X)$ and set $g = g_1 + ig_2$. We will prove that $\sup |G| \leq \sup |f|$. If $\sup |f| = \infty$ we are done, so let $R = \sup_X |f|$. Let

$$\phi(z) = \begin{cases} z & : |z| \leq R \\ \frac{R \cdot z}{|z|} & : |z| > R \end{cases}$$

then ϕ is continuous and $|\phi| \leq R$. We already have $g \in C_c(X)$ so that $\mu(f \neq g) < \epsilon$. Let $\tilde{g} = \phi \circ g$, which is also continuous and $|\tilde{g}| \leq R$. Finally, $\mu(\tilde{g} \neq f) = \mu\{\phi \circ f \neq \phi \circ g\} \leq \mu\{g \neq f\} < \epsilon$. \square

Cor. 2.4.6 In the same context, let $f : X \rightarrow \mathbb{C}$ be measurable, $\text{supp } f \subset A$, and $\mu(A) < \infty$ and $|f| \leq 1$. Then there exists $g_n \in C_c(X)$ with $|g_n| \leq 1$ and $\lim g_n(x) = f_n(x)$ almost everywhere.

2.5 Complex Measures

Let \mathcal{M} be a σ -algebra in X .

Def'n. 2.5.1 $\mu : \mathcal{M} \rightarrow \bar{\mathbb{R}}$ is called a **signed measure** if it is countably additive, and $+\infty$ and $-\infty$ are not in the range at the same time.

Def'n. 2.5.2 $\mu : \mathcal{M} \rightarrow \mathbb{C}$ is called a **complex measure** if it is countably additive: if E_i are disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Def'n. 2.5.3 For a set $E \in \mathcal{M}$, a partition of E is $\{E_i : i = 1, 2, \dots\}$ so that $E_i \cap E_j = \emptyset$ and $\bigcup_{i=1}^{\infty} E_i = E$ and $E_i \in \mathcal{M}$ for all i .

Def'n. 2.5.4 Let μ be a complex or signed measure. Its total variation

$$|\mu| : \mathcal{M} \rightarrow [0, +\infty] = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : \{E_i\} \text{ is a partition of } E \right\}$$

Thm. 2.5.5 $|\mu|$ is a positive measure.

PROOF Let $E \in \mathcal{M}$, and $\{E_i\}$ an arbitrary partition of E . We first see that $\sum |\mu(E_i)| \leq |\mu|(E)$. Let $t_i < |\mu|(E_i)$, so there exists a partition $\{A_{ij} : j\}$ of E_i so that

$$\sum_{j=1}^{\infty} |\mu(A_{ij})| > t_i$$

for all i . Then $\{A_{ij}\}_{i,j}$ is a partition of E , and

$$\sum_{i=1}^{\infty} t_i \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_{ij})| \leq |\mu|(E)$$

and since this holds for all i , we must have

$$\sum_{i=1}^{\infty} |\mu|(E_i) \leq |\mu|(E)$$

We now see the opposite direction. Let $\{A_j\}_j$ be an arbitrary partition of E . The set $\{A_j \cap E_i\}_j$ is a partition of E_i , while $\{A_j \cap E_i\}_i$ is a partition of A_j . Then

$$\begin{aligned} \sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\ &\leq \sum_j \sum_i \sum_j |\mu(A_j \cap E_i)| \\ &= \sum_i \sum_j |\mu(A_j \cap E_i)| \\ &\leq \sum_i |\mu|(E_i) \end{aligned}$$

and since this holds for an arbitrary partition $\{A_j\}$ of E , taking the supremum over all partitions gives the total variation. Thus equality holds. \square

Lemma 2.5.6 *Let $z_1, z_2, \dots, z_N \in \mathbb{C}$. Then there exists $S \subset \{1, 2, \dots, N\}$ so that*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^N |z_k|$$

PROOF Let $z_k = |z_k|e^{i\alpha_k}$, and for $\Theta \in [-\pi, \pi]$, let $S(\Theta) = \{k \in \{1, 2, \dots, N\} : \cos(\alpha_k - \Theta) > 0\}$. Then

$$\begin{aligned} \left| \sum_{k \in S(\theta)} z_k \right| &= \left| \sum_{k \in S(\theta)} |z_k| e^{i\theta} \right| \\ &\geq \operatorname{Re} \sum_{k \in S(\theta)} e^{-i(\alpha_k - \theta)} \\ &= \sum_{k \in S(\theta)} |z_k| \cos(\alpha_k - \theta) \\ &= \sum_{k=1}^N |z_k| \cos^+(\alpha_k - \theta) := h(\theta) \end{aligned}$$

and $h : [-\pi, \pi] \rightarrow \mathbb{R}$ is a continuous function. It has a maximum at some θ_0 . Fix $S = S(\theta_0)$ and

$$\begin{aligned} \left| \sum_{k \in S} z_k \right| &= h(\theta_0) \\ &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} h \, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^N |z_k| \cos^+(\alpha_k - \theta) \, d\theta \\ &= \frac{1}{2\pi} \sum_{k=1}^N |z_k| \int_{-\pi}^{\pi} \cos^+(\alpha_k - \theta) \, d\theta \\ &= \frac{1}{\pi} \sum_{k=1}^N |z_k| \end{aligned} \quad \pi \sum_{k=1}^N |z_k| \cos^+(\alpha_k - \theta) \, d\theta$$

since $\int_{-\pi}^{\pi} \cos^+(\alpha_k - \theta) \, d\theta = 2$. \square

Thm. 2.5.7 *If μ is a complex measure, then $|\mu|(X) < \infty$.*

PROOF Let $E \in \mathcal{M}$ such that $|\mu|(E) = +\infty$. Set $t = \pi(1 + |\mu(E)|)$ and since $|\mu(E)| > t$, there exists a partition $\{E_i\}$ of E such that

$$\sum_{i=1}^N |\mu(E_i)| > t$$

for some N . Then by the lemma with $z_k = \mu(E_k)$, let $A = \bigcup_{k \in S} E_k$. By the lemma,

$$|\mu(A)| \geq \frac{1}{\pi} \sum_{k=1}^N |\mu(E_k)| > \frac{t}{\pi} \geq 1$$

and let $B = E \setminus A$. Then

$$|\mu(B)| \geq |\mu(A)| - |\mu(E)| \geq \frac{t}{\pi} - \left(\frac{t}{\pi} - 1\right) = 1$$

so $E = A \cup B$, $\mu(A) > 1$ and $\mu(B) > 1$.

Now assume $|\mu|(X) = \infty$ and get A_1, B_1 with $|\mu(A_1)| \geq 1$ and $|\mu(B_1)| \geq 1$. As well, at least one of $|\mu|(A_1)$, $|\mu|(B_1)$ is infinity. Without loss of generality, it is B_1 , so repeat this procedure to B_1 . Get a sequence A_1, A_2, \dots with $|\mu(A_i)| \geq 1$ and A_i disjoint. As well, $\mu(\bigcup A_i) = \sum \mu(A_i)$ where the LHS is finite, but the RHS does not converge, a contradiction. \square

Recall that $\mu : \mathcal{M} \rightarrow \mathbb{C}$ is a complex measure if it is countably additive. Then the total variation of μ is given by

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : \{E_i\} \text{ is a partition} \right\}$$

Then $|\mu|$ is a positive measure and $|\mu|(X) < \infty$. If $\mu, \lambda : \mathcal{M} \rightarrow \mathbb{C}$ are complex measures, then $(\mu + \lambda)(E) = \mu(E) + \lambda(E)$ and $(c \cdot \mu)(E) = c \cdot \mu(E)$. Thus the set of complex measures is a vector space. Let $\|\mu\| := |\mu|(X)$.

If μ is a signed measure ($\mu : \mathcal{M} \rightarrow \overline{\mathbb{R}}$), then the total variation is defined in the same way.

Def'n. 2.5.8 Let μ be a signed measure. The **positive variation** of μ is $\mu_+ := \frac{1}{2}(|\mu| + \mu)$ and the **negative variation** of μ is $\mu_- := \frac{1}{2}(|\mu| - \mu)$.

These are positive measures since $|\mu|(E) \geq |\mu(E)|$. We have $\mu = \mu_+ - \mu_-$; this is called the Jordan decomposition, and $|\mu| = \mu_+ + \mu_-$.

2.6 Absolute Continuity and Singular Measures

Def'n. 2.6.1 Let μ be a positive measure and λ be an arbitrary (positive, signed, or complex) measure. Then λ is **absolutely continuous** with respect to μ if $\mu(E) = 0 \Rightarrow \lambda(E) = 0$. We write $\lambda \ll \mu$.

Def'n. 2.6.2 λ is concentrated on a set $A \in \mathcal{M}$ if $\lambda(E) = \lambda(E \cap A)$ for all $E \in \mathcal{M}$.

Prop. 2.6.3 λ is concentrated on A if and only if $\lambda(E) = 0$ if $E \cap A = \emptyset$.

PROOF Let $E \cap A = \emptyset$. Then $\lambda(E) = \lambda(E \cap A) = \lambda(\emptyset) = 0$.

Conversely, let $E \in \mathcal{M}$. Then $\lambda(E) = \lambda(E \cap A) + \lambda(E \cap A^c) = \lambda(E \cap A)$. \square

Def'n. 2.6.4 λ_1 and λ_2 are called **mutually singular** if there exist disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is concentrated on B . Then $\lambda_1 \perp \lambda_2$.

Prop. 2.6.5 Let μ be a positive measure, λ_1, λ_2 be arbitrary measures (positive, signed, or complex). Then

Prop. 2.6.6 1. λ is concentrated on A implies $|\lambda|$ is also concentrated on A .

2. $\lambda_1 \perp \lambda_2 \Rightarrow |\lambda_1| \perp |\lambda_2|$
3. $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ implies $\lambda_1 + \lambda_2 \perp \mu$.
4. $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$ implies $\lambda_1 + \lambda_2 \ll \mu$.
5. $\lambda \ll \mu$ implies $|\lambda| \ll \mu$
6. $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$ implies $\lambda_1 \perp \lambda_2$
7. $\lambda \ll \mu$ and $\lambda \perp \mu$ implies $\lambda = 0$.

PROOF 1. $\lambda(E) = 0$ if $E \cap A = \emptyset$. Let $E \cap A = \emptyset$ and let $E = \cup E_i$ be a partition. Then $E_i \cap A = \emptyset$ so $\lambda(E_i) = 0$ and $\sum |\lambda(E_i)| = 0$.

- 2.
- 3.
4. Let $\mu(E) = 0$. Then $\lambda_1(E) = 0$ and $\lambda_2(E) = 0$ so $(\lambda_1 + \lambda_2)(E) = 0$.
5. Let $\mu(E) = 0$ and let $E = \cup E_i$. Then $\mu(E_i) = 0$ for all i , so $\lambda(E_i) = 0$ for all i . Otherwise, if $\sum |\lambda(E_i)| = 0$ for all partitions, then $|\lambda|(E) = 0$ so $|\lambda| \ll \mu$.
6. $\lambda_2 \perp \mu$ implies that there exist disjoint sets A, B so λ_2 is concentrated on A and μ is concentrated on B . We will see that λ_1 is also concentrated on B . Let $E \cap B = \emptyset$ so $\mu(E) = 0$ and $\lambda_1(E) = 0$, so λ_1 is concentrated on B .
7. λ is concentrated on A , μ is concentrated on B , and let $E \in \mathcal{M}$ be arbitrary. Then $\lambda(E) = \lambda(E \cap A) = 0$ since $\mu(E \cap A) = 0$ and $\lambda \ll \mu$. Thus $(E \cap A) \cap B = \emptyset$. \square

Prop. 2.6.7 Let μ be a positive measure, λ a complex measure. Then the following are equivalent:

1. $\lambda \ll \mu$
2. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ so $|\lambda(E)| < \epsilon$.

PROOF (2 \Rightarrow 1). Let $\epsilon > 0$ and choose δ satisfying the requirement. Then let $\mu(E) = 0$, so $\mu(E) < \delta$ and $|\lambda(E)| < \epsilon$. This holds for any $\epsilon > 0$ so $\lambda(E) = 0$.

(1 \Rightarrow 2). Assume the opposite: get $\epsilon > 0$ so that for each $\delta = 1/2^n$, there exists a set E_n so that $\mu(E_n) < 1/2^n$ but $|\lambda(E_n)| \geq \epsilon$. Let $A_n = \bigcup_{k=n}^{\infty} E_k$ and $A = \bigcap_{n=1}^{\infty} A_n$. Then

$$\mu(A_n) \leq \sum_{k=n}^{\infty} \mu(E_k) \leq \sum_{k=n}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{2^{n-1}}$$

so $\mu(A) = 0$ since $\mu(A_1) < \infty$. Since $\lambda \ll \mu$, then $|\lambda| \ll \mu$ so $|\lambda|(A) = 0$. However, $\lim |\lambda|(A_n) = |\lambda|(A) = 0$ while $|\lambda(E_n)| \geq \epsilon$ implies $|\lambda|(E_n) \geq \epsilon$ implies $|\lambda|(A_n) \geq \epsilon$ implies $\lim |\lambda|(A_n) \geq \epsilon$, a contradiction. If λ is not finite, this may not hold. Set $f(x) = 1/|x|$, $\lambda(E) = \int_E f d\mu$, and μ is the Lebesgue measure. However, for each $E = [-1/n, 1/n]$, and $\int_E f d\mu = \infty$ while $\mu(E) = 1/2^n$.

Lemma 2.6.8 *If μ is a positive, σ -finite measure ($X = \cup X_n, \mu(X_n) < \infty$), then there exists $w \in L^1(\mu)$ so that $0 < w < 1$.*

PROOF Let $X = \bigcup_{n=1}^{\infty} X_n$, and $\mu(X_n) < \infty$. Let

$$w_n(x) = \begin{cases} 0 & : x \in X \setminus X_n \\ \frac{1}{2^n(1+\mu(X_n))} & : x \in X_n \end{cases}$$

and $w(x) = \sum_{n=1}^{\infty} w_n(x)$. By construction, $0 < w < 1$ and $\int_X w d\mu = \sum \int w_n d\mu < \sum 1/2^n = 1$ so $w \in L^1(\mu)$. \square

2.7 $L^2(\mu)$

Let (X, \mathcal{M}, μ) be a measure space, and set $\|f\|_2 = \left(\int_X |f|^2 d\mu\right)^{1/2}$. Let $L^2(\mu) = \{f : X \rightarrow \mathbb{C} : f \text{ measurable, } \|f\|_2 < \infty\}$. This is a normed space if functions which are equal almost everywhere are identified. We also define

$$\langle x, y \rangle = \int_X f \bar{g} d\mu$$

L^2 is the infinite dimensional generalization of \mathbb{R}^k .

Thm. 2.7.1 (Riesz-Fisher) $L^2(\mu)$ is complete (every Cauchy sequence of functions converges w.r.t. the L^2 norm).

2.8 Hilbert Spaces

$L^2(\mu) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable, } \|f\|_2 < \infty\}$ is a Hilbert space (complete) with the inner product $\langle f, g \rangle = \int_X f \cdot \bar{g} d\mu$.

Thm. 2.8.1 *If H is a Hilbert space, $L : H \rightarrow \mathbb{C}$ is continuous and linear, then there exists a unique $y \in H$ so that $L(x) = \langle x, y \rangle$ for all $x \in H$.*

Thm. 2.8.2 (Lebesgue-Radon-Nikodym) *Let μ be a positive, σ -finite measure, λ be a complex measure on \mathcal{M} .*

- (a) *There exists a unique decomposition of λ as $\lambda = \lambda_a + \lambda_s$ such that $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$.*
- (b) *There exists a unique $h \in L^1(\mu)$ such that $\lambda_a(E) = \int_E h d\mu$ for all $E \in \mathcal{M}$. This is the Radon-Nikodym derivative of λ_a with respect to μ .*

PROOF Let's first see that the decomposition is unique. Assume $\lambda = \lambda_a + \lambda_s = \lambda'_a + \lambda'_s$, so $\lambda_a - \lambda'_a = \lambda'_s - \lambda_s$. We proved that $\lambda_a - \lambda'_a \ll \mu$ and $\lambda'_s - \lambda_s \perp \mu$, so $\lambda_a - \lambda'_a = 0 = \lambda'_s - \lambda_s$. We also see that h is unique. Assume h^* is another one, so $\lambda_a(E) = \int_E h^* d\mu$. Write $h = h_1 + ih_2$,

$h^* = h_1^* + ih_2^*$. Let $A_1 = \{x \in X : h_1(x) > h_1^*(x)\}$ and $A_2 = \{x \in X : h_1(x) < h_1^*(x)\}$. We will show $\mu(A_1) = \mu(A_2) = 0$ (so that $h_1 = h_1^*$ a.e.). In particular,

$$\int_{A_1} h \, d\mu = \int_{A_1} h^* \, d\mu \Rightarrow \int_{A_1} (h - h^*) \, d\mu = 0$$

so $\mu(A_1) = 0$. We can argue similarly with A_2 .

We now show existence of $\lambda_a, \lambda_s, \mu$. Write $\lambda = \lambda_1^+ - \lambda_1^- + i(\lambda_2^+ - \lambda_2^-)$ and argue separately for each λ_i^\pm . We thus assume without loss of generality that λ is a positive, finite measure. From the previous lemma, get $w \in L^1(\mu)$ such that $0 < w < 1$, and introduce $\phi(E) = \lambda(E) + \int_E w \, d\mu$, so ϕ is a positive finite measure. We have $\int_X f \, d\mu = \int_X f \, d\lambda + \int_X f w \, d\mu$. This holds for $f = \chi_E$, so it holds for simple functions as well, and for arbitrary functions by Lebesgue Monotone Convergence. Let $f \in L^2(\phi)$, so

$$\begin{aligned} \left| \int_X f \, d\lambda \right| &\leq \int_X |f| \, d\lambda \\ &\leq \int_X |f| \, d\phi = \int_X 1 \cdot |f| \, d\phi \\ &= \langle 1, |f| \rangle_{L^2(\phi)} \\ &\leq \sqrt{\int_X 1 \, d\phi} \cdot \sqrt{\int_X |f|^2 \, d\phi} \end{aligned}$$

Let $T(f) = \int_X f \, d\lambda$, so $T : L^2(\phi) \rightarrow \mathbb{C}$ is a bounded linear functional. By the Riesz theorem, there exists $g \in L^2(\phi)$ so that $T(f) = \langle f, g \rangle = \int_X f \cdot g \, d\phi$. Thus

$$\int_X f \, d\lambda = \int_X f \cdot g \, d\phi$$

Let's see that $0 \leq g \leq 1$ a.e. $[\phi]$. Let $f = \chi_E$ so $\lambda(E) = \int_E f \, d\phi$. Let $A_1 = \{x \in X : g(x) < 0\}$. Then $0 \leq \lambda(A_1) = \int_{A_1} g \, d\phi < 0$ if $\phi(A_1) > 0$. Similarly, let $A_2 = \{x \in X : g(x) > 1\}$, so $\lambda(A_2) = \int_{A_2} g \, d\phi > \phi(A_2) \geq \lambda(A_2)$, so $\phi(A_2) = 0$. We thus have

$$\int_X f \, d\lambda = \int_X f g \, d\lambda + \int_X f g w \, d\mu \Rightarrow \int_X f(1 - g) \, d\lambda = \int_X f g w \, d\mu$$

Let $A = \{x \in X : 0 \leq g(x) < 1\}$, $B = \{x \in X : g(x) = 1\}$. Then $A \cup B = X$, $A \cap B = \emptyset$. Define $\lambda_a(E) = \lambda(E \cap A)$, $\lambda_s(E) = \lambda(E \cap B)$. Apply the above with $f = \chi_B$, so $\chi_B(1 - g) = 0$ everywhere. We thus have $\int_B w \, d\mu = 0$ so $\mu(B) = 0$. Thus μ is concentrated on A and by definition, λ_s is concentrated on B . Thus $\lambda_s \perp \mu$.

Now apply (*) with $f = (1 + g + g^2 + \dots + g^n)\chi_E$ with $E \in \mathcal{M}$ for each $n \in \mathbb{N}$. We thus have

$$\int_E (1 - g^{n+1}) \, d\lambda = \int_{E \cap A} (1 - g^{n+1}) \, d\lambda + \int_{E \cap B} (1 - g^{n+1}) \, d\lambda$$

so $\lambda(E \cap A) = \lambda_a(E)$. Thus by LMC, the RHS goes to $\lambda_E h \, d\lambda$. Thus $\lambda_a(E) = \int_E h \, d\lambda$ for all $E \in \mathcal{M}$, so $\lambda_a \ll \mu$ and $h \in L^1(\mu)$, $\lambda_a(X) < \infty$. \square

Thm. 2.8.3 *Let μ be a complex measure. Then there exists a measurable h such that $|h| = 1$ and $\mu(E) = \int_E h d|\mu|$. This is called the polar decomposition of μ .*

PROOF Note that $\mu \ll |\mu|$ since if $|\mu|(E) = 0$ then $|\mu(E)| = 0$. Thus get $h \in L^1(|\mu|)$ so that $\mu(E) = \int_E h d|\mu|$. We will see that $|h| \leq 1$ a.e. and $|h| \geq 1$ a.e. Let $A_r = \{x \in X : |h(x)| < r\}$. Let $\{E_j\}$ be a partition of A_r . Then

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h d|\mu| \right| \leq \sum_j \int_{E_j} |h| d|\mu| \leq \sum_j r \cdot |\mu|(E_j) = r |\mu|(A_r)$$

for any partition, so taking the supremum, $|\mu|(A_r) \leq r \cdot |\mu|(A_r)$, so $|\mu|(A_r) = 0$ when $r < 1$. Let $B_r(z) \subset (\overline{B_1(0)})^c$. Let $E = \{x \in X : h(x) \in B_r(z)\}$. Then

$$\left| \frac{1}{|\mu|(E)} \cdot \int_E h d|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1$$

Thus

$$\begin{aligned} \left| \frac{1}{|\mu|(E)} \int_E h d|\mu| - z \right| &= \left| \frac{1}{|\mu|(E)} \cdot \int_E (h - z) d|\mu| \right| \\ &\leq \frac{1}{|\mu|(E)} \int_E |h - z| d|\mu| \\ &< \frac{r \cdot |\mu|(E)}{|\mu|(E)} = r \end{aligned}$$

Since we can cover $(\overline{B_1(0)})^c$ with countably many balls, $|h| \leq 1$ almost everywhere. \square