Course Notes

Real Functions and Measures

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BSM Fall 2018

Contents

Chapter 1

Basics of Abstract Measure Theory

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1.1 Review of Topology

1.1.1 Basic Definitions

Def'n. 1.1.1 *Let $X \neq \emptyset$ and $\tau \subseteq \mathcal{P}(X)$. We say that (X, τ) is a **topological space** if τ satisfies the following conditions:*

1. $\emptyset \in \tau$ $X \in \tau$
2. $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3. $V_\alpha \in \tau$ *for all* $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

*We call the elements of τ **open sets**.*

Def'n. 1.1.2 *$U \subseteq X$ is a **neighbourhood** of $x \in X$ if there is some $G \in \tau$ such that $x \in G \subset U$.*

Def'n. 1.1.3 *$F \subseteq X$ is **closed** if F^c is open.*

Def'n. 1.1.4 *The **closure** of a set $E \subset X$ is the smallest closed set containing E (denoted $\overline{E}$).*

Def'n. 1.1.5 *x is an **accumulation point** of H if all neighbourhoods of x contains infinitely points of H. Equivalently, x is a limit point of $H \setminus \{x\}$.*

Def'n. 1.1.6 *If $H \subseteq X$, we have a natural subspace topology $\tau|_H = \{G \cap H : G \in \tau\}$.*

1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1. $\mathbb{R}$ along with the open sets (denoted τ_e, the Euclidean topology).
2. The discrete topology, $\tau = \mathcal{P}(X)$ for any $X \neq \emptyset$. This is the "finest" topology.

3. The antidiscrete topology, $\tau = \{\emptyset, X\}$ for any $X \neq \emptyset$ This is the "coarsest" topology.

4. One can define the extended real line, $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

   The same can be done with a single symbol as well. In either case, the extended real line is a compact set.

5. Any metric spaces induces a topology. Consider a set $X \neq 0$ arbitrary, and let $d : X \times X \to \mathbb{R}$ such that

   (a) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$.
   (b) $d(x,y) = d(y,x)$ for all $x, y \in X$
   (c) $d(x,y) \leq d(x,z) + d(z,y)$ for any $x, y, z \in X$

   Then $G \in \tau$ if and only if for any $x \in G$, there exists r so that $B_r(x) \subset G$. There are many examples of metric spaces:

   (a) $X = \mathbb{R}$, $d(x,y) = |x-y|$
   (b) $X = \mathbb{R}$, $d(x,y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
   (c) $X = \mathbb{R}^2$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$
   (d) $X = \mathbb{R}^2$, $d(x,y) = (|x_1-y_1|^p + |x_2-y_2|^p)^{1/p}$ for $p \geq 1$.
   (e) and similarly for $X = \mathbb{R}^n$
   (f) $X = C[0,1]$, $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$.
   (g) normed space: X is a vector space over $\mathbb{R}$, $\|\cdot\| : X \to \mathbb{R}$ such that
      i. $\|x\| = 0$ if and only if $X = 0$
      ii. $\|cx\| = |c| \|x\|$
      iii. $\|x+y\| \leq \|x\| + \|y\|$

      If $\|\cdot\|$ is a norm, then $d(x,y) = \|x-y\|$ is a metric.

6. The cofinite topology: $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$.

1.1.3 Other Definitions

Def'n. 1.1.7 *$K \subset X$ is* ***compact*** *if every open cover of K contains a finite subcover.*

Def'n. 1.1.8 *A topological space is called* ***locally compact*** *if every point has a compact neighbourhood.*

Prop. 1.1.9 *$C[0,1]$ with the sup norm is not locally compact.*

Proof I'll do this later. ■

Def'n. 1.1.10 *A topological space is called* ***Hausdorff*** *if for any $x \neq y$, there exists neighbourhoods $U \ni x$, $V \ni y$ so that $U \cap V = \emptyset$.*

The anti-discrete topology is not Hausdorff.

1. On the discrete topology, K is compact if and only if K is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of X).
3. On $(\mathbb{R}, \tau_e)$, K is compact if and only if K is closed and bounded.
4. On (X, d) metric space, K is compact if and only if K is complete and totally bounded.

Prop. 1.1.11
1. *Let $K \subset X$ be compact, let $F \subset K$ closed. Then F is also compact.*
2. *Compact sets in a Hausdorff space are closed.*

Proof
1. Let $F \subset \bigcup V_\alpha$. Then $K \subset F^c \cup (\bigcup V_\alpha)$ is an open cover for K, so it has a finite subcover $F^c \cup V_{\alpha_1} \cup \cdots V_{\alpha_n}$. But then since $F \cap F^c = \emptyset$, $F \subset V_{\alpha_1} \cup \cdots V_{\alpha_n}$ is a finite subcover.
2. Let $K \subset X$ be compact, and prove that K^c is open. Thus let $x \in K^c$. For any $y \in K$, there exist U_y, V_y disjoint neighbourhoods of x and y respectively. Now consider the open cover $K \subset \bigcup_{y \in K} V_y$, and get our finite subcover $K \subset V_{y_1} \cup \cdots \cup V_{y_n}$. But then $U_{y_1} \cap \cdots \cap U_{y_n} \cap K = \emptyset$ and is open since it is a finite intersection. ∎

Def'n. 1.1.12 *$\Gamma \subseteq \tau$ is a* ***base*** *for τ if every $U \in \tau$ can be written as a countable union of the elements of Γ. Γ is a* ***countable base*** *if Γ is countable.*

Prop. 1.1.13 *$\mathbb{R}$ has a countable base of intervals.*

Proof Consider the collection $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$. To see this, for any open set U, one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left(\bigcup_{\{r : B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$ is obvious, so let $x \in U$ be arbitrary, and let s be maximal so that $B_s(x) \subseteq U$. Then choose $q \in \mathbb{Q}$ so that $|x - q| < s/3$ and $r \in \mathbb{Q}$ so that $0 < r < s/2$. Then by construction $B_r(q) \ni x$ and by the triangle inequality $B_{r/2}(q) \subseteq U$, so $x \in S$. Thus $U = S$ as desired. ∎

Note that the exact same argument (with some work) can be generalized to show that $\mathbb{R}^n$ has a countable base of open hyperrectangles.

1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

Def'n. 1.1.14 *Let $(x_n) \subset X$ be a sequence and let $x \in X$. Then x is the* ***limit*** *of (x_n) if for any neighbourhood U of X, there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.*

Prop. 1.1.15 *If $F \subset X$ is closed, then for all convergent sequences in F, the limit is also in F.*

Proof See Homework. ∎

Def'n. 1.1.16 *Let $f : X \to Y$ be a function, and $x \in X$ an accumulation point of $D(f)$. The limit of f at x is $y \in Y$ if for any neighbourhood V of y there exists a neighbourhood U of x such that $f(U \cap D(f) \setminus \{x\}) \subseteq V$.*

Def'n. 1.1.17 *Let $f : X \to Y$ be a function, and let $x \in D(f)$. Then f is **continuous at** x if for any neighbourhood V of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of x.*

Def'n. 1.1.18 *$f : X \to Y$ is called **continuous** if it is continuous at every point.*

Prop. 1.1.19 *$f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is open for all G open.*

Proof Exercise. ∎

Thm. 1.1.20 *Let $f : X \to Y$ be continuous and $K \subset X$ be compact. Then $f(K)$ is compact.*

Proof Recall that continuous functions pull back open sets. Let $f(K) \subset \bigcup U_\alpha$ be an open cover. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover for K, and has a finite subcover $U_{\alpha_1} \cup \cdots U_{\alpha_n}$. But then $f(f^{-1}(U_{\alpha_1})) \cup \cdots \cup f(f^{-1}(U_{\alpha_n}))$ is a subcover of $f(K)$. ∎

1.2 Measure Theory

1.2.1 $\sigma-$algebras

Def'n. 1.2.1 *Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a $\sigma-$**algebra** if*

1. *$X \in \mathcal{M}$*
2. *$A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$*
3. *If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup\limits_{n \in \mathbb{N}} A_n \in \mathcal{M}$*

*The pair $(X, \mathcal{M})$ is called a **measurable space**. The elements of $\mathcal{M}$ are called **measurable sets**.*

Def'n. 1.2.2 *Let $(X, \mathcal{M})$ be a measurable space, (Y, τ) be a topological space. Then $f : X \to Y$ is called **measurable** if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.*

Here are some simple examples of $\sigma-$algebras.

Ex. 1.2.3

1. $\mathcal{M} = \{\emptyset, X\}$ is a $\sigma-$algebra.
2. $\mathcal{P}(X) = \mathcal{M}$ is a $\sigma-$algebra.
3. $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable.}\}$. To see this, given $A_n \in \mathcal{M}$, if everything is countable, then $\bigcup A_n$ is countable. If some A_i is countable, then $(\bigcup A_n)^c = \bigcap A_n^c$ is countable, so $\bigcup A_n \in \mathcal{M}$.

We will later see some proper exaples, like the $\sigma-$algebra of Lebesgue measurable sets.

We have the following properties of $\sigma-$algebras.

Prop. 1.2.4

1. *$\emptyset \in \mathcal{M}$*
2. *$A_1, A_2, \ldots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{M}$*
3. *$A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ then $\bigcap_{n=1}^\infty A_n \in \mathcal{M}$*

4. *$A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$*
5. *f is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$.*

Proof 1. $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.
2. We can extend this to a countable union by introduction $A_{n+i} = \emptyset$ for $i \in \mathbb{N}$.
3. By DeMorgan's identities, $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.
4. $A \setminus B = A \cap B^c \in \mathcal{M}$.
5. H^c is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$. ■

Prop. 1.2.5 *Let $f : X \to Y$ be measurable, let $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is measurable.*

Proof Let $V \subset Z$ be open, so $g^{-1}(V) \subset Y$ is open, so $f^{-1}(g^{-1}(V)) \in \mathcal{M}$ which is $(g \circ f)^{-1}(V)$.■

Prop. 1.2.6 *Let $(X, \mathcal{M})$ be a measurable space, Y be a topological space. Let $\phi : \mathbb{R}^2 \to Y$ be continuous. If $u, v : X \to \mathbb{R}$ are measurable, then $h(x) = \phi(u(x), v(x))$ is measurable.*

Proof Define $f : X \to \mathbb{R}^2$ by $f(x) = (u(x), v(x))$ We will see that f is measurable, so that $h = \phi \circ f$ is measurable since ϕ is continuous. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals, so $R = I_1 \times I_2$ is an open rectangle. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Let $G \subset \mathbb{R}^2$ be an open set, so there exist R_n open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that f is measurable. ■

Cor. 1.2.7 1. *If $u, v : X \to \mathbb{R}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.*
2. *$u + iv : X \to \mathbb{C}$ is measurable.*
3. *$f : X \to \mathbb{C}$ is measurable, $f = u + iv \Rightarrow u, v, |f|$ are measurable.*

Prop. 1.2.8 *Define*

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

Then χ_E is measurable if and only if $E \in \mathcal{M}$.

Proof Naturally, $\chi_E^{-1}(1) = E$ and $\chi_E^{-1}(0) = E^c$, so χ_E is measurable if and only if $E, E^c \in \mathcal{M}$. ■

Thm. 1.2.9 *Let $\mathcal{F} \subset \mathcal{P}(X)$, then there exists a smallest σ–algebra containing $\mathcal{F}$. This is denoted by $S(\mathcal{F})$, the σ–**algebra generated by** $\mathcal{F}$.*

Proof Let $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{–algebra}, \mathcal{F} \subset M\}$. Certainly $\Omega \neq \emptyset$ since $\mathcal{P}(X) \in \Omega$. Let $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$. We will see that $S(\mathcal{F})$ is a σ–algebra.
(i) Since $X \in \mathcal{M}$, it follows that $X \in \cap \mathcal{M}$.
(ii) If $A \in S(\mathcal{F})$, then $A \in \mathcal{M}$ for all $\mathcal{M}$. Thus $A^c \in \mathcal{M}$ for all $\mathcal{M}$ and $A^c \in \cap \mathcal{M}$.
(iii) In the same way, of $A_n \in S(\mathcal{F}$ for all n, then $A_n \in \mathcal{M}$ for all $n, \mathcal{M}$. Thus $\bigcup A_n \in \mathcal{M}$ for all $\mathcal{M}$ so $\bigcup A_n \in \mathcal{M} \in \bigcap \mathcal{M} = S(\mathcal{F})$.

By definition, $\mathcal{F} \subset \bigcap \mathcal{M}$. Finally, $S(\mathcal{F})$ is minimal, since if $\mathcal{F} \subset \mathcal{N}$ is a $\sigma-$algebra, then $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$, so we are done. ∎

Def'n. 1.2.10 *Let (X, τ) be a topological space. Then $\mathcal{B} = S(\tau)$ is called the **Borel $\sigma-$algebra**. Borel sets are the elements of $S(\tau)$. A function $f : X \to Y$ is Borel measurable if $f^{-1}(G) \in \mathcal{B}$ for all $G \subset Y$ open.*

Prop. 1.2.11
1. *If $F \subset X$ is closed, then $F \in \mathcal{B}$.*
2. *$G_n \subset X$ are open, then $\bigcap_{n=1}^{\infty} G_n \in B$. These are called $G_\delta-$sets.*
3. *$F_n \subset X$ are closed, then $\bigcup_{n=1}^{\infty} F_n \in B$. These are called $F_\sigma-$sets.*

Proof These follow directly from the definition of a $\sigma-$algebra. ∎

Ex. 1.2.12 $X = \mathbb{R}, \tau_e$, then $\mathcal{B} = S(\tau_e)$. Let $\Gamma_0 = \{(a,b) : a < b\}$ be a family of open intervals. We see that $S(\Gamma_0) = \mathcal{B}$. Since $\Gamma_0 \subset \tau$, $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$. Conversely, let $G \in \tau$, then we have open intervals $G = \bigcup_{n=1}^{\infty} I_n$ so that $G \in S(\Gamma_0)$. Thus $S(\tau) \subset S(\Gamma_0)$ and $S(\Gamma_0) = \beta$.

Ex. 1.2.13 Let $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$. I claim that $S(\Gamma_\infty) = \mathcal{B}$. Certainly $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$. Then $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$.Similarly, $(-\infty, a) = \bigcup_{n=1}^{\infty}(-\infty, a - 1/n] \in S(\Gamma_\infty)$.Thus $(a, \infty) \cap (-\infty, b) = (a,b) \in S(\gamma_0)$, and using the previous example, $\mathcal{B} = S(\Gamma_\infty)$.

Prop. 1.2.14 *Let $(X, \mathcal{M})$ be a measurable space, and let $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the eucildean topology. If $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for any $\alpha \in \mathbb{R}$, then f is measurable.*

Proof Recall that f is measurable if its inverse image takes open sets to measurable sets.
We have $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty])^c \in \mathcal{M}$. Similarly,

$$f^{-1}([-\infty, \alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty}[-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha, \beta) = f^{-1}([-\infty, \beta) \cap (\alpha, \infty]) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for τ_e. Thus if $G \subset \overline{\mathbb{R}}$ is open, then there exists open intervals so that $G = \bigcup_{n=1}^{\infty} I_n$ and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired. ∎

1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

Def'n. 1.2.15 *Let $(a_n)_{n\in\mathbb{N}} \subset \overline{R}$ be a sequence, and $b_k = \sup\{a_k, a_{k+1}, \ldots\}$. Then $\beta = \inf_{k\in\mathbb{N}} b_k$ is called the* $\limsup$ *of (a_n). We can similarly define $c_k = \inf\{a_k, a_{k+1}, \ldots\}$ and $\liminf = \sup_{k\in\mathbb{N}} c_k$.*

Def'n. 1.2.16 *Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of functions. Then $(\sup f_n) : X \to \overline{\mathbb{R}}$, $(\sup f_n)(x) = \sup f_n(x)$ for all $x \in X$. Similarly, $(\inf f_n) : X \to \overline{\mathbb{R}}$, $(\inf f_n)(x) = \inf f_n(x)$ for all $x \in X$. Then $(\liminf f_n)(x) = \liminf f_n(x)$. If $\lim f_n(x)$ exists for all x, then we say $(\lim f_n)(x) = \lim f_n(x)$.*

Thm. 1.2.17 *Let $f_n : X \to \overline{R}$ be measurable. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ are measurable.*

Proof Let $g = \sup f_n$. It is enough to prove that $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$ for all α. Let $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$. Let $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$. We show that $H = \bigcup\limits_{n=1}^{\infty} H_n$.

First let $x \in H$, so $\sup f_n(x) > \alpha$. Thus get N so that $f_N(x) > \alpha$, so $x \in H_N$ and x is in the union. The converse is obvious.

Thus g is measureable. In the exact same way, $\inf f_n$ is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k\geq i} f_k$$

is measurable. ∎

Cor. 1.2.18 *If $\lim f_n$ exists, then it is measurable.*

Proof If $\lim f_n$ exists, then $\lim f_n = \limsup f_n$. ∎

Cor. 1.2.19 *If f, g are measurable, then $\max\{f, g\}$, $\min\{f, g\}$ are measurable.*

Cor. 1.2.20 *Let f be a function. Then $f_+ = \max\{f, 0)\}$ and $f_- = -\min\{f, 0\}$ (the positive and negative parts of f) are measurable. Similarly, $|f| = f_+ + f_i$ is measurable.*

1.2.3 Measures

Def'n. 1.2.21 *Let $(X, \mathcal{M})$ be a measurable space. A function $\mu : \mathcal{M} \to [0, +\infty]$ is called a **(positive) measure** if it is countably additive and not constant $+\infty$. In other words,*

1. $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ *if* $A_i \cap A_j = \emptyset$
2. $\exists A \in \mathcal{M}$ *so that* $\mu(A) < \infty$

*$(X, \mathcal{M}, \mu)$ is called a **measure space**.*

Prop. 1.2.22
1. $\mu(\emptyset) = 0$
2. *If* $A_i \cap A_j = \emptyset$ *then* $\mu\left(\sum\limits_{i=1}^{n} A_i\right) = \sum\limits_{i=1}^{n} \mu(A_i)$
3. $A \subset B$ *implies* $\mu(A) \leq \mu(B)$
4. $A_1 \subset A_2 \subset A_3 \cdots$ *then* $\lim_{n\to\infty} \mu A_n = \mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$

5. $A_1 \supset A_2 \supset A_3 \cdots$ *and* $\mu(A_i) < \infty$ *then* $|lim_{n\to\infty}\mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

PROOF
1. Let $A \in \mathcal{M}$ so that $\mu(A) < \infty$, and fix $A_1 = A$, $A_2 = A_3 = \cdots = \emptyset$. Then $\bigcup A_n = A$ so $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$ so $\mu(\emptyset) = 0$.

2. Obvious

3. Note that $B = A \cup (B \setminus A)$ is a disjoint union.

4. Define $B_1 := A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i \geq 2$. Then $B_i \cap B_j = \emptyset$ and $\mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$. Similary, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ Therefore, $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$.

5. Let $C_n = A_1 \setminus A_n$, $C_1 = \emptyset$. Then $C_1 \subset C_2 \subset \cdots$ and $\mu(C_n) + \mu(A_n) = \mu(A_1)$. Let $A = \bigcap_{n=1}^{\infty} A_n$ so $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$ and $(\bigcup C_n) \cup A = A_1$ is a disjoint union. But then $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$ so that
$$\mu(A_1) - \mu(A) = \mu(\bigcup C_n) = \lim_{n\to\infty} \mu(C_n) = \mu(A_n) - \lim \mu(A_n)$$
Since $\mu(A_1)$ is finite, we have $\mu(A) = \lim \mu(A_n)$. ■

Ex. 1.2.23 Here are a few examples of measures that exist on arbitrary sets.

1. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$, and
$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$
It is easy to verify it is countably additive.

2. X arbitrary, $\mathcal{M} = \mathcal{P}(X)$. Fix $x_0 \in X$. Then
$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

1.3 Towards Integration

1.3.1 Simple Functions

Def'n. 1.3.1 $s : X \to \mathbb{R}$ *or* $\mathbb{C}$ *is called a* ***simple function*** *if its range is finite.*

Prop. 1.3.2 *Let s be a simple function, so that $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i = s^{-1}(\{\alpha_i\})$ and s is* ***measurable*** *if and only if $A_i \in \mathcal{M}$.*

Proof Obvious. ■

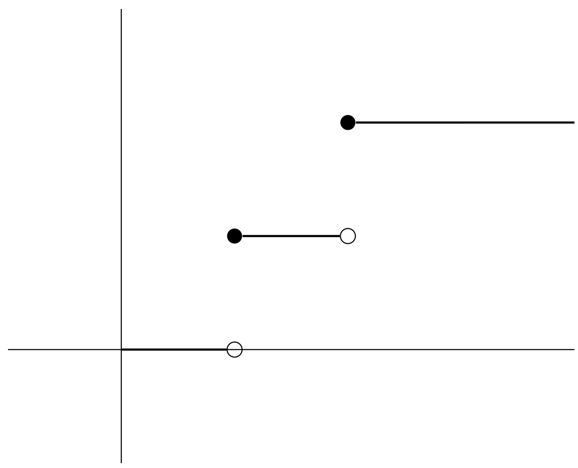
The following theorem is used later to define the intergral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function f.

Thm. 1.3.3 *Let $f : X \to [0, +\infty]$ be nonnegative measurable functions. Then there exists a sequence $s_n : X \to [0, +\infty]$ of simple measurable functions with*
1. *(s_n) is increasing and bounded above by f*
2. *$\lim s_n = f$ pointwise.*

Proof Let $n \in \mathbb{N}$, $t \geq 0$, and define $k_n(t) = [2^n \cdot t]$ (i.e. $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn ϕ_1 below:



Then $t - 2^{-n} \leq \phi_n(t) \leq t$, $\lim \phi_n(t) = t$ uniformly, and $\phi_n \leq \phi_{n+1}$, so the sequence of functions is monotone. Define $s_n = \phi_n \circ f$, so for any $x \in X$, $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$. Note that s_n is simple since it has finite range (from ϕ_n), and $s_n \leq s_{n+1}$ because $\phi_n \leq \phi_{n+1}$, and $s_n \leq f$ since $\phi_n(t) \leq t$. Furthermore, ϕ_n is measurable since its level sets are intervals, so $s_n = \phi_n \circ f$ is measurable. ■

1.3.2 Integration of Positive Functions

Def'n. 1.3.4 *Let $s : X \to [0, +\infty)$ be a measurable simple function $s = \sum_{n=1}^{N} \alpha_i X_{A_i}$. Let $E \in \mathcal{M}$. Then define the* ***integral*** *of s over E with respect to μ as*

$$\int_E s \, \mathrm{d}\mu = \sum_{n=1}^{N} \alpha_i \mu(A_i \cap E)$$

where we define $0 \cdot \infty = 0$.

Def'n. 1.3.5 *Let $f : X \to [0,+\infty]$ be a measurable function. Let $E \in \mathcal{M}$. Then the* ***(Lebesgue) integral*** *of f over E with respect to μ is*

$$\int_E f \, \mathrm{d}\mu = \sup\left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le f;\ s \text{ is simple measurable}\right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only.

Prop. 1.3.6 *Let $f, g : X \to [0,+\infty]$ be measurable functions. Let $E, A, B \in \mathcal{M}$.*

1. *If $f \le g$ then $\int_E f \, \mathrm{d}\mu \le \int_E g \, \mathrm{d}\mu$*
2. *If $A \subset B$, then $\int_A f \, \mathrm{d}\mu \le \int_B f \, \mathrm{d}\mu$*
3. *$\int_E c \cdot f \, \mathrm{d}\mu = c \cdot \int_E f \, \mathrm{d}\mu$ for all $c \ge 0$*
4. *If $f(x) = 0$ for all $x \in E$, then $\int_E f \, \mathrm{d}\mu = 0$*
5. *If $\mu(E) = 0$, then $\int_E f \, \mathrm{d}\mu = 0$*
6. *$\int_E f \, \mathrm{d}\mu = \int_X f \cdot \chi_E \, \mathrm{d}\mu$.*

Proof

1. This follows directly since

$$\left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le f\right\} \subset \left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le g\right\}$$

2. Let $0 \le s \le f$ be simple measurable. Then

$$\int_A s \, \mathrm{d}\mu = \sum \alpha_i \mu(A \cap A_i) \le \sum \alpha_i \mu(B \cap A_i) = \int_B s \, \mathrm{d}\mu$$

Take the supremum for all $0 \le s \le f$, then the result follows.

3. Let S be simple and measurable, so $s = \sum \alpha_i \chi_{A_i}$. Then

$$\int_E c \cdot s \, \mathrm{d}\mu = \sum_{i=1}^{n} \alpha_I \cdot c \cdot \mu(E \cap A_i) = c \cdot \sum \alpha_i \mu(E \cap A_i) = c \int_E s \, \mathrm{d}\mu$$

Thus

$$\begin{aligned}
\int_E c \cdot f \, \mathrm{d}\mu &= \sup\left\{\int_E s \, \mathrm{d}\mu : 0 \le s \le cf\right\} \\
&= \sup\left\{\int_E c \cdot t \, \mathrm{d}\mu : 0 \le t \le f\right\} \\
&= c \cdot \sup\left\{\int_E t \, \mathrm{d}\mu : 0 \le t \le f\right\} \\
&= c \cdot \int_E f \, \mathrm{d}\mu
\end{aligned}$$

4. If $0 \le s \le f$, then $s = \sum \alpha_i \chi_{A_i}$. If $x \in A_i \cap E$, then $s(x) = \alpha_i$ and $\alpha_i = 0$. Then $\alpha_i \mu(A_i \cap E) = 0$ for all i: either $A_i \cap E = \emptyset$, or $A_i \cap E$ is not empty, and $\alpha_i = 0$. This is true for any $0 \le s \le f$, and taking supremums yields the result.

5. If $\mu(E) = 0$ then $\mu(A_i \cap E) = 0$, and $\int_E s \,\mathrm{d}\mu = \sum \alpha_i \mu(A_i \cap E) = 0$ and taking supremums, the result holds.

6. Exercise. First prove if $0 \le s \le f \cdot \chi_E$, then $\int_X s \,\mathrm{d}\mu = \int_E s \,\mathrm{d}\mu$. Then prove

$$\left\{ \int_E s \,\mathrm{d}\mu : 0 \le s \le f \cdot \chi_E \right\} = \left\{ \int_E s \,\mathrm{d}\mu : 0 \le s \le f \right\} \qquad \blacksquare$$

Prop. 1.3.7 *Let s be a simple and measurable. Then $\phi(E) = \int_E s \,\mathrm{d}\mu$ is a measure.*

Proof $\phi(\emptyset) = 0$, so ϕ is not constant $+\infty$. Let $E = \bigcup_{n=1}^\infty E_n$ be a disjoint union. Then

$$\begin{aligned}
\phi(E) &= \sum_{i=1}^m \alpha_i \mu(A_i \cap E) \\
&= \sum_{i=1}^m \alpha_i \mu\left(A_i \cap \left(\bigcup_{n=1}^\infty E_n \right)\right) = \sum_{i=1}^m \alpha_i \mu\left(\bigcup_{n=1}^\infty (A_i \cap E_n)\right) \\
&= \sum_{i=1}^m \alpha_i \sum_{n=1}^\infty \mu(A_i \cap E_n) = \sum_{n=1}^\infty \sum_{i=1}^m \alpha_i \mu(A_i \cap E_n) \\
&= \sum_{n=1}^\infty \int_{E_n} s \,\mathrm{d}\mu = \sum_{n=1}^\infty \phi(E_n)
\end{aligned} \qquad \blacksquare$$

Prop. 1.3.8 *Let s, t be nonnegative, measurable simple functions. Then*

$$\int_X (s+t) \,\mathrm{d}\mu = \int_X s \,\mathrm{d}\mu + \int_X t \,\mathrm{d}\mu$$

Proof Write

$$s = \sum_{i=1}^m \alpha_i \chi_{A_i}, \quad t = \sum_{j=1}^n \beta_j \chi_{\beta_j}$$

and let $E_{ij} = A_i \cap B_j$, so $X = \bigcup_{i,j} E_{ij}$ is a disjoint union. We now have

$$\int_{E_{ij}} (s+t) \,\mathrm{d}\mu = (\alpha_i + \beta_j)\mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s \,\mathrm{d}\mu + \int_{E_{ij}} t \,\mathrm{d}\mu$$

Let $\phi(E) = \int_E (s+t)\,\mathrm{d}\mu$, which is a measure as above. Thus

$$\begin{aligned}
\int_X (s+t)\,\mathrm{d}\mu = \phi(X) = \phi\left(\bigcup_{i,j} E_{ij}\right) \\
&= \sum_{i,j} \phi(E_{ij}) = \sum_{i,j} \int_{E_{ij}} (s+t)\,\mathrm{d}\mu \\
&= \sum_{i,j} \left(\int_{E_{ij}} s\,\mathrm{d}\mu + \int_{E_{ij}} t\,\mathrm{d}\mu \right) \\
&= \sum_{i,j} \varphi(E_{ij}) + \sum_{i,j} \theta(E_{ij}) \\
&= \int_X s\,\mathrm{d}\mu + \int_X t\,\mathrm{d}\mu
\end{aligned}$$

where $\varphi(E) = \int_E s\,\mathrm{d}\mu$, $\theta(X) = \int_E t\,\mathrm{d}\mu$. ■

To extend this result to general functions, we need some stronger machinery. In particular, we have the first of many convergence theorems.

1.4 Lebesgue's Monotone Convergence Theorem

Thm. 1.4.1 (Lebesgue's Monotone Convergence) *Let $f_n : X \to [0, +\infty]$ be measurable, such that*

(i) $0 \le f_1 \le f_2 \le \cdots$

(ii) $f(x) := \lim_{n\to\infty} f_n(x)$ *for all* $x \in X$

Then f is measurable, and $\int_X f\,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n\,\mathrm{d}\mu$.

Proof It was already proven that f is measurable. We have $\int_X f_n\,\mathrm{d}\mu \le \int_X f_{n+1}\,\mathrm{d}\mu$ for all n, so $\alpha := \lim_{n\to\infty} \int_X f_n\,\mathrm{d}\mu$ exists. We also have $f_n \le f$, so $\int_X f_n\,\mathrm{d}\mu \le \int_X f\,\mathrm{d}\mu$ and $\alpha \le \int_X f_n\,\mathrm{d}\mu$. Thus we wish to show $\alpha \ge \int_X f\,\mathrm{d}\mu$. It suffices to prove that $\alpha \ge \int_X s\,\mathrm{d}\mu$ for any simple $s \le f$. Furthermore, if $c \in (0,1)$, it suffices to show that $\alpha \ge \int_X c \cdot s\,\mathrm{d}\mu$.

Define $E_n = \{x \in X : f_n(x) \ge c \cdot s(x)\}$. We have $E_1 \subset E_2 \subset \cdots$ so that $\bigcup_{n=1}^{\infty} E_n = X$. Then

$$\int_X f_n\,\mathrm{d}\mu \ge \int_{E_n} f_n\,\mathrm{d}\mu \ge \int_{E_n} c \cdot s\,\mathrm{d}\mu$$

Let $\phi(E) = \int_E s\,\mathrm{d}\mu$, so $\int_{E_n} s\,\mathrm{d}\mu = \phi(E_n)$. Thus $\lim_{n\to\infty} \phi(E_n) = \phi(X) = \int_X s\,\mathrm{d}\mu$. Thus

$$\alpha \ge c \cdot \lim_{n\to\infty} \phi(E_n) = c \cdot \int_X s\,\mathrm{d}\mu = \int_X c \cdot s\,\mathrm{d}\mu$$

as desired. ■

Note that in the proof, choosing $c \in (0,1)$ was important since it lets us define the E_n so that $\bigcup_{n=1}^{\infty} E_n = X$ - this would not work by defining $E_n = \{x \in X : f_n(x) \ge s(x)\}$.

1.4.1 Applications of Monotone Convergence

Now, as advertised, we can prove linearity of the integral for general measurable functions using the Monotone Convergence Theorem.

Thm. 1.4.2 *Let $f, g : X \to [0, +\infty]$ measurable, then $\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu$.*

Proof We proved that there exists increasing sequences of simple functions s_n, t_n such that $\lim s_n(x) = f(x)$, $\lim t_n(x) = g(x)$ monotonically. Then $s_n(x) + t_n(x) \to f(x) + g(x)$ monotonically so that

$$\begin{aligned}\int_X (f+g)\,\mathrm{d}\mu &= \int_X \lim_{n\to\infty}(s_n + t_n)\,\mathrm{d}\mu \\ &= \lim_{n\to\infty}\int_X (s_n + t_n)\,\mathrm{d}\mu \\ &= \lim_{n\to\infty}\left(\int_X s_n\,\mathrm{d}\mu + \int_X t_n\,\mathrm{d}\mu\right) \\ &= \int_X \lim_{n\to\infty} s_n\,\mathrm{d}\mu + \int_X \lim_{n\to\infty} t_n\,\mathrm{d}\mu \\ &= \int_X f\,\mathrm{d}\mu + \int_X g(\mathrm{d}\mu)\end{aligned}$$

relying on linearity of the integral over simple functions. ■

Cor. 1.4.3 *If $f_n : X \to [0, +\infty]$ is a sequence of measurable functions, then*

$$\sum_{n=1}^{\infty}\int_X f_n\,\mathrm{d}\mu = \int_X \sum_{n=1}^{\infty} f_n\,\mathrm{d}\mu$$

Proof Define $g_n = \sum_{i=1}^{n} f_i$ is monotonically increasing. ■

Ex. 1.4.4 Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(X)$, $\mu(E)$ is the counting measure. Let $a : X \to [0, \infty)$ be a function. This is a sequence. Every function is measurable. Let $s_n(i) = a(i)$ for $i \leq n$ and 0 otherwise, which is a simple function, and $s_n \leq s_{n+1}$. Then $\lim_{n\to\infty} s_n(i) = a(i)$ so $s_n \to a$ pointwise, so by LMC $\int_X s_n\,\mathrm{d}\mu = \int_X a\,\mathrm{d}\mu$. Also,

$$\int_X s_n\,\mathrm{d}\mu = \sum_{i=1}^{n} a(i)\mu(\{i\}) = \sum_{i=1}^{n} a(i)$$

so $\int_X a\,\mathrm{d}\mu = \sum\limits_{n=1}^{\infty} a(n)$.

For the following Lemma, we will see an application of it during the section on Complex measures (to follow).

Lemma 1.4.5 (Fatou) *Let $f_n : X \to [0, \infty)$ be a sequence of measurable functions. Then*

$$\int_X \liminf f_n\,\mathrm{d}\mu \leq \liminf \int_X f_n\,\mathrm{d}\mu$$

Proof Let $g_k = \inf\{f_k, f_{k+1}, \ldots\}$ so $\liminf f_n = \lim_{n\to\infty} g_n$ and g_n is increasing. Note that $g_k \leq f_k$ for any k, so $\int_X g_k \,\mathrm{d}\mu \leq \int_X f_k \,\mathrm{d}\mu$. Thus

$$\begin{aligned}\int_X \liminf f_n \,\mathrm{d}\mu &= \int_X \lim_{n\to\infty} g_n \,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu \\ &= \liminf \int_X g_n \,\mathrm{d}\mu \\ &\leq \liminf \int_X f_n \,\mathrm{d}\mu\end{aligned}$$

Note that the inequality can hold strictly (exercise). ■

Our next application is a strengthening of the previous result for simple functions.

Thm. 1.4.6 *Let $f : X \to [0,\infty]$ be measurable. Let $\phi(E) = \int_E f \,\mathrm{d}\mu$, $E \in \mathcal{M}$. Then ϕ is a measure and $\int_X g \,\mathrm{d}\phi = \int_X g \cdot f \,\mathrm{d}\mu$.*

Proof Certainly $\phi(\emptyset) = 0$, so $\phi \neq +\infty$. Thus let $E = \bigcup_{i=1}^\infty E_i$ be a disjoint union and $\chi_E f = \sum_{i=1}^\infty \chi_{E_i} f$. Thus we have

$$\begin{aligned}\phi(E) &= \int_E f \,\mathrm{d}\mu = \int_X \chi_E f \,\mathrm{d}\mu \\ &= \int_X \sum_{i=1}^\infty \chi_{E_i} f \,\mathrm{d}\mu = \sum_{i=1}^\infty \int_X \chi_{E_i} f \,\mathrm{d}\mu \\ &= \sum_{i=1}^\infty \int_{E_i} \mathrm{d}\mu = \sum_{i=1}^\infty \phi(E_i)\end{aligned}$$

where we interchange the summation using the previous corollary. Now, we prove that $\int_X g \,\mathrm{d}\phi = \int_X gf \,\mathrm{d}\mu$.

- *Let $g = \chi_E$ be a characteristic function.* Then

$$\int_X \chi_E \,\mathrm{d}\phi = \phi(E) = \int_E f \,\mathrm{d}\mu = \int_X \chi_E f \,\mathrm{d}\mu$$

- *Let $g = \sum\limits_{i=1}^n \alpha_i \chi_{A_i}$ be a simple function.* Then

$$\begin{aligned}\int_X \sum_{i=1}^n \alpha_i \chi_{A_i} \,\mathrm{d}\phi &= \sum_{i=1}^n \alpha_i \int_X \chi_{A_i} \,\mathrm{d}\phi \\ &= \sum_{i=1}^n \alpha_i \int_X \chi_{A_i} f \,\mathrm{d}\mu \\ &= \int_X \sum_{i=1}^n \alpha_i \chi_{A_i} f \,\mathrm{d}\mu\end{aligned}$$

  applying the result for characteristic functions.

- *Let g be an arbitrary measurable function.* For the final step, we apply monotone convergence. Let $(s_n) \to g$ be an increasing sequence of simple functions, and note that $s_n f \to gf$. Thus

$$\begin{aligned}\int_X g\,\mathrm{d}\phi &= \int_X \lim s_n\,\mathrm{d}\phi = \lim \int_X s_n\,\mathrm{d}\phi \\ &= \lim \int_X s_n f\,\mathrm{d}\mu = \int_X \lim(s_n f)\,\mathrm{d}\mu \\ &= \int_X g\cdot f\,\mathrm{d}\mu\end{aligned}$$

  as desired.

Thus we have our result. ■

1.5 Integration of Complex Valued Functions

Def'n. 1.5.1 *A function $f : X \to \mathbb{C}$ is called* ***Lebesgue integrable*** *if $\int_X |f|\,\mathrm{d}\mu < \infty$. The collection of such functions is $L^1(\mu)$.*

1.5.1 Basic Properties

Def'n. 1.5.2 *Let $f \in L^1(\mu)$. Then $f = u + iv$ and denote $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Let $E \in \mathcal{M}$; then the integral of f over E with respect to μ is*

$$\int_E f\,\mathrm{d}\mu = \int_E u^+\,\mathrm{d}\mu - \int_E u^-\,\mathrm{d}\mu + i\left(\int_E v^+\,\mathrm{d}\mu - \int_E v^-\,\mathrm{d}\mu\right)$$

with the integral as defined for positive functions.

Thm. 1.5.3 *Let $f, g \in L^1(\mu)$, $\alpha, \beta \in \mathbb{C}$, so $\alpha f + \beta g = L^1(\mu)$ and*

$$\int_X (\alpha f + \beta g)\,\mathrm{d}\mu = \alpha \int_X f\,\mathrm{d}\mu + \beta \int_X g\,\mathrm{d}\mu$$

Proof Note that $\alpha f + \beta g$ is measurable, so $\int_X |\alpha f + \beta g|\,\mathrm{d}\mu \le |\alpha| \int_X |f|\,\mathrm{d}\mu + |\beta| \int_X |g|\,\mathrm{d}\mu < \infty$. For real measurable functions, $\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu$ directly by expanding the definition and using additivity over positive functions. We thus show $\int_X \alpha f\,\mathrm{d}\mu = \alpha \int_X f\,\mathrm{d}\mu$. If $\alpha \ge 0$, then

$$\begin{aligned}\int_X \alpha f\,\mathrm{d}\mu = \int_X \alpha(u+iv) &= \int_X (\alpha u^+ - \alpha u^- + i\alpha v^+ - i\alpha v^-)\,\mathrm{d}\mu \\ &= \int_X ((\alpha u)^+ - (\alpha u)^- + (i\alpha v)^+ - (i\alpha v)^-)\,\mathrm{d}\mu \\ &= \int_X (\alpha u)^+\,\mathrm{d}\mu - \int_X (\alpha u)^-\,\mathrm{d}\mu + \int_X i(\alpha v)^+\,\mathrm{d}\mu - \int_X i(\alpha v)^-\,\mathrm{d}\mu \\ &= \alpha \int_X u^+\,\mathrm{d}\mu - \alpha \int_X u^-\,\mathrm{d}\mu + \alpha \int_X iv^+\,\mathrm{d}\mu - \alpha \int_X iv^-\,\mathrm{d}\mu \\ &= \alpha \int_X f\,\mathrm{d}\mu\end{aligned}$$

and similarly for $\alpha = -1$, $\alpha = i$. ■

As usual, we have a triangle-type inequality for complex integration.

Thm. 1.5.4 *Let $f \in L^1(\mu)$. Then $\left|\int_X f \,\mathrm{d}\mu\right| \leq \int_X |f| \,\mathrm{d}\mu$.*

Proof Let $z = \int_X f \,\mathrm{d}\mu$. Let $\alpha = \frac{|z|}{z}$ if $z \neq 0$, and $\alpha = 1$ otherwise. Then $\alpha \int_X f \,\mathrm{d}\mu = |z|$. Let $\mathrm{Re}(\alpha \cdot f) \leq |\alpha \cdot f| \leq |f|$ since $|\alpha| = 1$ so that

$$\begin{aligned}
\left|\int_X f \,\mathrm{d}\mu\right| &= \alpha \cdot \int_X f \,\mathrm{d}\mu \\
&= \int_X \alpha f \,\mathrm{d}\mu \\
&= \int_X \mathrm{Re}(\alpha f) \,\mathrm{d}\mu + i \int_X \mathrm{Im}(\alpha f) \,\mathrm{d}\mu \\
&= \int_X \mathrm{Re}(\alpha f) \,\mathrm{d}\mu \\
&\leq \int_X |f| \,\mathrm{d}\mu
\end{aligned}$$

since $\alpha \int_X f \,\mathrm{d}\mu \in \mathbb{R}$, we have $\int_X \mathrm{Im}(\alpha f) \,\mathrm{d}\mu = 0$. ■

1.5.2 Lebesgue's Dominated Convergence Theorem

The next theorem is a natural generalization of the Monotone Convergence Theorem to complex measurable functions (which do not have a standard partial ordering). Note this follows from the Monotone Convergence Theorem since we apply Fatou's Lemma. The Monotone Convergence Theorem is a special case of this theorem.

Thm. 1.5.5 (Lebesgue's Dominated Convergence) *Let $f_n : X \to \mathbb{C}$ be measurable functions such that $f = \lim f_n$. Assume that there is some $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all n. Then $f \in L^1(\mu)$ and $\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$.*

Proof We certainly know that f is measurable, and $|f| \leq g$, so $f \in L^1(\mu)$. As well, the triangle inequality show that $|f - f_n| \leq 2g$ for any n. We will see that $0 \leq \liminf \int_X |f - f_n| \,\mathrm{d}\mu \leq \limsup \int_X |f - f_n| \,\mathrm{d}\mu \leq 0$. Assuming that this holds, then $\lim_{n\to\infty} \int_X |f - f_n| \,\mathrm{d}\mu = 0$ and

$$0 \leq \lim_{n\to\infty} \left|\int_X f \,\mathrm{d}\mu - \int_X f_n \,\mathrm{d}\mu\right| \leq \lim_{n\to\infty} \int_X |f - f_n| \,\mathrm{d}\mu = 0$$

and the desired result follows directly.

The first two inequalities are obvious: we must show that $\limsup \int_X |f_n| \, \mathrm{d}\mu \leq 0$. Firstly, we have

$$\begin{aligned}
\int_X 2g \, \mathrm{d}\mu &= \int_X \left(2g - \lim_{n\to\infty} |f - f_n|\right) \mathrm{d}\mu \\
&= \int_X \liminf (2g - |f - f_n|) \, \mathrm{d}\mu \\
&\leq \liminf \int_X (2g - |f - f_n|) \, \mathrm{d}\mu && \text{By Fatou's Lemma} \\
&= \int_X 2g + \liminf \left(- \int_X |f - f_n| \, \mathrm{d}\mu\right) \\
&= \int_X 2g - \limsup \int_X |f - f_n| \, \mathrm{d}\mu
\end{aligned}$$

and since $\int_X 2g \, \mathrm{d}\mu$ is finite, we subtract and $\limsup \int_X |f - f_n| \, \mathrm{d}\mu \leq 0$. ■

Ex. 1.5.6 Consider $\lim_{n\to\infty} \int_0^n e^{-nx} \, \mathrm{d}x$. Define

$$f_n(x) = \begin{cases} e^{-nx} & \text{if } x \leq n \\ 0 & \text{if } x > n \end{cases}$$

Note that $f_n(x) \leq g(x) = e^{-x}$ and $\int_0^\infty e^{-x} \, \mathrm{d}x < \infty$. Thus

$$\begin{aligned}
\lim_{n\to\infty} \int_0^n e^{-nx} \, \mathrm{d}x &= \int_{[0,\infty)} \lim_{n\to\infty} f_n(x) \, \mathrm{d}x \\
&= \int_{[0,\infty)]} \chi_{\{0\}} \, \mathrm{d}x \\
&= 0
\end{aligned}$$

Rmk. 1.5.7 For the Riemann integral, we have $\int \lim f_n = \lim \int f_n$ as long as the convergence of f_n is uniform.

Chapter 2

Construction of Regular Measures

2.1 The Vector Space $L^1(\mu)$

2.1.1 Almost Everywhere

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Def'n. 2.1.1 *Let $E \in \mathcal{M}$. We say that property P holds almost everywhere in E if there exists $N \in \mathcal{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds in $E \setminus N$.*

Ex. 2.1.2 Two functions $f, g : X \to \mathbb{C}$ are equal almost everywhere if $\exists N \subset X$ such that $\mu(N)$ and $f(x) = g(x)$ on $X \setminus N$.

Prop. 2.1.3 *Let $E \subset X$ be such that $A_1, A_2, B_1, B_2 \in \mathcal{M}$ for which $\int_X f \,\mathrm{d}\mu = \int_X g \,\mathrm{d}\mu$. Then $A_1 \subset E \subset B_1$, $A_2 \subset E \subset B_2$, and $\mu(B_1 \setminus A_1) = 0$ and $\mu(B_2 \setminus A_2) = 0$. Then $\mu(A_1) = \mu(A_2)$.*

Proof Note that $A_1 \setminus A_2 \subset E \setminus A_2 \subset B_2 \setminus A_2$. As well, $\mu(A_1 \setminus A_2) \leq \mu(B_2 \setminus A_2) = 0$. Then

$$\mu(A_1) = \mu(A_1 \cap A_2^c) + \mu(A_1 \cap A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2)$$
$$\mu(A_2) = \mu(A_2 \cap A_1^c) + \mu(A_2 \cap A_1) = \mu(A_2 \setminus A_1) + \mu(A_2 \cap A_1) = \mu(A_1 \cap A_2) \qquad \blacksquare$$

Prop. 2.1.4 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let*

$$\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, A \subset E \subset B, \mu(B \setminus A) = 0\}$$

Then $\mathcal{M}^*$ is a σ–algebra, and $\mu^* : \mathcal{M}^* \to [0, +\infty]$ defined by $\mu^*(E) = \mu(A)$.

Proof We show that $\mathcal{M}^*$ is a σ–algebra, and μ is countably additive.

1. $X \in \mathcal{M}$ so $X \in \mathcal{M}^*$.
2. If $E \in \mathcal{M}^*$, get $A \subset E \subset B$ so $B^c \subset E^c \subset A^c$, $A^c, B^c \in \mathcal{M}$. As well, $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \setminus A) = 0$, so $E^c \in \mathcal{M}^*$.
3. If $E_i \in \mathcal{M}^*$ is a countable collection, then get $A_i \subset E_i \subset B_i$. Fix $A = \bigcup A_i$ and $B = \bigcup B_i$. Then $B \setminus A = \bigcup(B_i \setminus A) \subset U(B_i \subset A_i)$ so $\mu(B \setminus A) = 0$ and $A \subset \bigcup E_i \subset B$ so $\bigcup E_i \in \mathcal{M}^*$.
4. Let E_i be disjoint, $E = \bigcup E_i$, and $E_i \in \mathcal{M}^*$. Get $A_i \subset E_i \subset B_i$. Then $\mu^*(\bigcup E_i) = \mu(\bigcup A_i) = \sum \mu(A_i) = \sum \mu(E_i)$. $\blacksquare$

Def'n. 2.1.5 *We call the space $(X, \mathcal{M}^*, \mu^*)$ the* ***completion*** *of $(X, \mathcal{M}, \mu)$.*

In particular, every subset of a set with measure 0 is measurable.

2.1.2 $L^1(\mu)$ as a normed space

Prop. 2.1.6 *1. Let $f : X \to [0, +\infty)$ be measurable, $E \in \mathcal{M}$. If $\int_E f \,\mathrm{d}\mu = 0$, then $f = 0$ almost everywhere in E.*

2. Let $f \in L^1(\mu)$. If $\int_E f \,\mathrm{d}\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ almost everywhere in X.

Proof 1. Let $A_n = \{x \in E : f(x) > 1/n\}$, so that

$$\frac{1}{n}\mu(A_n) \le \int_{A_n} \mathrm{d}\mu \le \int_E f \,\mathrm{d}\mu = 0 \Longrightarrow \mu(A_n) = 0$$

for all n. But then

$$N = \{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n \to \mu(N) \le \sum \mu(A_n) = 0$$

2. Write $f = u + iv$ so that

$$\int_E f \,\mathrm{d}\mu = \int_E u^+ \,\mathrm{d}\mu - \int_E u^- \,\mathrm{d}\mu + i\int_E v^+ \,\mathrm{d}\mu - i\int_E v^- \,\mathrm{d}\mu$$

We show that $u^+ = 0$ almost everywhere (the other terms are identical). Let $E = \{x \in X : u(x) \ge 0\}$, so $\int_E f \,\mathrm{d}\mu = 0$, so its real part is zero and $\int_E u^+ \,\mathrm{d}\mu = 0$. Thus $u^+ = 0$ almost everywhere in E. The result follows. ■

Def'n. 2.1.7 *A **normed space** over $\mathbb{R}$ is a vector space V over $\mathbb{R}$ with a map $\|\cdot\| : V \to \mathbb{R}$ such that*

(i) $x \in V \Rightarrow \|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| \le |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in V$

(iii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in V$.

Now $L^1(\mu) = \{f : X \to \mathbb{C}$ measurable and $\int_X |f| \mathrm{d}\mu < \infty\}$. We certainly have that $L^1(\mu)$ is a vector space. We wish to define $\|f\| = \int_X |f| \mathrm{d}\mu$. The only problem is that

$$\int_X |f| \mathrm{d}\mu = 0 \Longrightarrow f = 0 \text{ almost everywhere}$$

To deal with this problem, we quotient our space by the equivalence relation $f \sim g$ if and only if $f = g$ almost everywhere. With this in mind, define $V = L^1(\mu)/\sim$ denote the set of equivalence classes. We need to define $+, \cdot, \|\cdot\|$ on V. Let $[f]$ denote the class of f. Then

$$[f] + [g] = [f + g]$$
$$c[f] = [cf]$$
$$\|[f]\| = \int_X |f| \mathrm{d}\mu$$

Let's verify that this is well defined: if $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$. Indeed, this is true since the sums are equal except perhaps on a union of measure zero sets, so equality holds almost everywhere. The second definition is obviously well defined. Finally, by a homework assignment, $\|[f]\|$ is also well defined. Now, let's verify the properties of the norm.

(i) Certainly $\|[f]\| \geq 0$, and $\|[f]\| = 0$ implies $f = 0$ almost everywhere, so $[f] = [0] = 0$.

(ii) We have $\|\lambda \cdot [f]\| = \int_X |\lambda f| \, \mathrm{d}\mu = |\lambda| \int_X |f| \, \mathrm{d}\mu = |\lambda| \|[f]\|$

(iii) We have $\|[f] + [g]\| = \int_X |f + g| \, \mathrm{d}\mu \leq \int_X |f| + \int_X |g| = \|[f]\| + \|[g]\|$

In $L^1(\mu)$, two functions are the same if they are equal almost everywhere. However, this can be a challenge: if $f \in L^1(\mu)$ and $x_0 \in X$, then $f(x_0)$ is not well defined. For example, it is challenging to give meaning to boundary conditions of functions.

2.1.3 Construction of the Lebesgue measure

We begin from the Riemann integral $\int_a^b f(x) \, \mathrm{d}x$ for a continuous function f. Define $\operatorname{supp} f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. For continuous functions with compact (bounded) support, define $\Lambda f = \int_{\mathbb{R}} f(x) \, \mathrm{d}x$ is the Riemann integral, which is a functional. In particular,

$$\text{measure}((a,b)) = \text{length}((a,b)) = \sup\{\Lambda f : f \text{ is continuous, compact support, } 0 \leq f \leq 1, \operatorname{supp} f \subset (a,b)\}$$

We will extend this to a σ−algebra containing the Borel sets. In order to define these, for open sets, $\mu(G) = \sup\{\Lambda f : 0 \leq f \leq 1, \operatorname{supp} f \subset G\}$, where Λ is the Riemann integral. For an arbitrary set, $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$. However, this "measure" is not countably additive: the σ−algebra $\mathcal{P}(X)$ is too large (Vitali's construction). Instead, we will define $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$, which means that $E \cap K$ is regular for any K compact, and regular means that the outer measure and inner measure are equal. The outer measure is $\sup\{\mu(K) : K \subset E \text{ compact}\} = \mu(E)$.

2.2 The Riesz Representation Theorem

In this section, we assume that (X, τ) be a locally compact, Hausdorff topological space.

Def'n. 2.2.1 *We denote the space of continuous functions with compact support by* $C_c(X) = \{f : X \to \mathbb{C} \mid f \in C(X), \operatorname{supp} f \text{ is compact}\}$.

Def'n. 2.2.2 *Let* $\Lambda : C_c(X) \to \mathbb{C}$ *be a **linear functional**, i.e.* $\Lambda(cf + g) = c\Lambda f + \Lambda g$. Λ *is called a **positive** linear functional if* $f \geq 0 \Rightarrow \Lambda f \geq 0$.

By positivity, if $f \leq g$, then $g - f \geq 0$ so $\Lambda g - \Lambda g = \Lambda(g - f) \geq 0$ and $\Lambda f \leq \Lambda g$.

Def'n. 2.2.3 *We say that* $K \prec f$ *if* K *is compact and* $f \in C_c(X)$, $0 \leq f \leq 1$ *implies that* $x \in K \Rightarrow f(x) = 1$. *We say that* $f \prec G$ *if* G *is open,* $f \in C_c(X)$, $0 \leq f \leq 1$, *and* $\operatorname{supp} f \subset G$.

Lemma 2.2.4 (Urysohn) *Let* $G \in \tau$, $K \subset G$ *compact. Then there exists* $f \in C_c(X)$ *such that* $K \prec f \prec G$.

Proof Will do later. ∎

Lemma 2.2.5 (Partition of Unity) *Let* $G_1, G_2, \ldots, G_n \in \tau$, *an let* $K \subset G_1 \cup \cdots \cup G_n$ *be compact. Then there are functions* $h_i \in C_c(X)$ *such that* $h_i \prec G_i$ *and* $K \prec \sum h_i$.

Proof Also will do later. ■

How can we create a positive linear functional on $C_c(X)$? If μ is a measure, and functions on $C_c(X)$ are measurable, then $\Lambda f = \int_X f \,\mathrm{d}\mu$ is a positive linear functional. The representation theorem says that there are no other examples.

Thm. 2.2.6 (Riesz Representation) *Let (X,τ) be as above. If $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique measure space $(X,\mathcal{M},\mu)$ such that $\Lambda f = \int_X f \,\mathrm{d}\mu$ for any $f \in C_c(X)$, $\mathcal{M} \supset \tau$, and*

(i) $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$ for all $E \in \mathcal{M}$.
(ii) $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$.
(iii) $\mu(K) < \infty$ for any K compact.
(iv) $\mathcal{M}$ is complete.

First, let's get some definitions out of the way. Fix the notation as above.

Def'n. 2.2.7 *Fix a Borel measure μ. The **Lebesgue outer measure** is defined $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$.*

Def'n. 2.2.8 *We say that $E \subset X$ is **regular** if $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$. Similarly, $E \subset X$ is **locally regular** for every compact K, $K \cap E$ is regular.*

Claim 0: Definition of μ and $\mathcal{M}$; completeness of $\mathcal{M}$.

Proof For an open set $G \in \tau$, let $\mu(G) = \sup\{\Lambda f : f \prec G\}$. Then $\mu(\emptyset) = 0$ and $G_1 \subset G_2$ implies that $\mu(G_1) \leq \mu(G_2)$. Then extend μ to arbitrary $E \subset X$ as an outer measure. Now let $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$. Note that $\mathcal{M}$ contains compact sets, since they are regular. This is direct from the definition, since $\mu(F) \leq \mu(K)$ for any compact $F \subseteq K$ and the supremum occurs exactly at K.

As well, $\mathcal{M}$ is complete: let $E \in \mathcal{M}$, $\mu(E) = 0$ and $A \subset E$. We want to show that $A \in \mathcal{M}$. Let K be an arbitrary compact set; then $\mu(K \cap A) = 0$. Now if $F \subset K \cap A$ is compact, $\mu(F) = 0$. Thus $\sup\{\mu(F) : F \subset K \cap A \text{ compact}\} = 0$, so $K \cap A$ is regular and A is locally regular and an element of $\mathcal{M}$. ■

Claim 1: μ is σ−subadditive. In other words, if $E_1, E_2, \ldots$ are arbitrary subsets of X, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

Proof If $\mu(E_j) = \infty$ for some j, then we are done. Thus assume $\mu(E_j) < \infty$ for all j. Let $\epsilon > 0$, $\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ be arbitrary. We will show that $\gamma \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Let $G_j \supset E_j$ be open, such that $\mu(G_j) \leq \mu(E_j) + \frac{\epsilon}{2^j}$. Then

$$\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \mu\left(\bigcup_{j=1}^{\infty} G_j\right)$$

so there exists some $f \prec \bigcup_{j=1}^{\infty} G_j$ so $\gamma < \Lambda f$ by the definition of μ on open sets. Let $K = \operatorname{supp} f$ so that

$$K \subset \bigcup_{j=1}^{\infty} G_j \quad \Longrightarrow \quad K \subset \bigcup_{j=1}^{n} G_j$$

since $\{G_j\}$ are an open cover for K and K is compact. Get a partition of unity $h_j \prec G_j$ for each $j = 1, \dots, n$ which satisfies $\sum_{j=1}^{n} h_j(x) = 1$ for any $x \in K$. Then $f \cdot h_j \prec G_j$ and $f = f \cdot \sum_{j=1}^{n} h_j$ so that

$$\begin{aligned}\gamma < \Lambda f = \Lambda\left(\sum_{j=1}^{n} f h_j\right) &= \sum_{j=1}^{n} \Lambda(f h_j) \\ &\leq \sum_{j=1}^{n} \mu(G_j) \leq \sum_{j=1}^{n}\left(\mu(E_j) + \frac{\epsilon}{2^j}\right) \\ &\leq \sum_{j=1}^{\infty}\Big(\mu(E_j)\Big) + \epsilon\end{aligned}$$

which holds for all $\epsilon > 0$ if and only if $\gamma \leq \sum_{j=1}^{\infty} \mu(E_j)$. This holds for any $\gamma \leq \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$ and the result follows. ■

Claim 2: If $K \prec f \prec G$, then $\mu(K) \leq \Lambda f \leq \mu(G)$. Thus if K is compact, $\mu(K) < \infty$.

Proof It is direct from the definition of μ that $\Lambda f \leq \mu(G)$. Thus let $\gamma < \mu(K)$ and $\alpha \in (0,1)$. Let $V_\alpha := \{x \in X : f(x) > \alpha\}$ and $K \subset V_\alpha$ since $f \equiv 1$ on K. Since f is continuous, $V_\alpha = f^{-1}((\alpha, \infty))$ is the preimage of an open set and thus open.

Now $\gamma < \mu(K) \leq \mu(V_\alpha)$, so we have some $h \prec V_\alpha$ such that $\gamma < \Lambda h$. Then $\alpha \cdot h \leq f$ since in V_α, $\alpha \cdot h \leq \alpha < f$ and in V_α^c, $\alpha \cdot h = 0 \leq f$. Now $\alpha \cdot \Lambda h = \Lambda(\alpha h) \leq \Lambda f$ so $\gamma < \Lambda f / \alpha$. This is true for all $\alpha \in (0,1)$ and $\gamma \leq \Lambda f$. Since this holds for all $\gamma < \mu(K)$, we have $\mu(K) \leq \Lambda f$ as required.

Now, let K be compact so that $\mu(K) \leq \Lambda f$ for all $K \prec f$. Let $\epsilon > 0$ and get $G \in \tau$, $G \supset K$ such that $\mu(G) \leq \mu(K) + \epsilon$. Then by Urysohn's lemma, get some $K \prec f \prec G$ so that $\mu(K) \leq \Lambda f \leq \mu(G)$, so $\Lambda f \leq \mu(K) + \epsilon$ and the result holds. Now suppose K is compact, so $\mu(K) = \inf\{\mu(G) : K \subset G \in \tau\}$. By Urysohn's Lemma, get f with $K \prec f \prec G$, and by (1.), $\mu(K) \leq \Lambda f \leq \mu(G)$ so that $\mu(K) = \inf\{\Lambda f : K \prec f\}$. As a corollary, we have that $\mu(K) < \infty$ (since Λ is a positive linear functional). Besides demonstrating one of our properties, this provides a convenient way of computing the measure of compact sets. ■

Claim 3: If $G \in \tau$, then G is regular.

Proof We first show that if $0 \leq f \leq 1$, then $\Lambda f \leq \mu(\operatorname{supp} f)$. Let $G \supset \operatorname{supp} f$ be open, so $f \prec G$ and $\mu(G) \geq \Lambda f$. Then $\mu(\operatorname{supp} f) = \inf\{\mu(G) : E \subset G \in \tau\} \geq \Lambda f$.

Now we want to show $\mu(G) = \sup\{\mu(K) : K \subset G \text{ compact}\}$. It suffices to show that $\sup\{\mu(K) : K \subset G \text{ compact}\} \geq \mu(G)$, so let $\gamma < \mu(G)$ and we want K compact so that $\mu(K) > \gamma$. Let $f \prec G$ be such that $\Lambda f > \gamma$. Then $\mu(\operatorname{supp} f) > \gamma$ by the previous claim is compact, as desired. ■

Claim 4: Suppose $E_1, E_2, \dots$ are disjoint regular. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

If we assume additionally that $\mu(\cup E_i) < \infty$, then $\bigcup_{i=1}^{\infty} E_i$ is regular.

Proof We first prove this for two compact sets. Thus let K_1, K_2 be disjoint compact sets. Then K_2^c is open and $K_2^c \supset K_1$. By Urysohn's lemma, get $f \in C_c(X)$ so that $K_1 \prec f \prec K_2^c$ and $x \in K_1$ implies $f(x) = 1$, and $x \in K_2$ implies $f(x) = 0$.

Since $K_1 \cup K_2$ is compact, for all $\epsilon > 0$, get $g \prec K_1 \cup K_2$ such that $\mu(K_1 \cup K_2) + \epsilon > \Lambda g$ (by Claim 2). Furthermore, $K_1 \prec f \cdot g$ and $K_2 \prec (1-f)\cdot g$. Thus $\mu(K_1)+\mu(K_2) \leq \Lambda(f\cdot g)+\Lambda((1-f)\cdot g = \Lambda g < \mu(K_1 \cup K_2)+\epsilon$ which is true for any $\epsilon > 0$. Thus $\mu(K_1)+\mu(K_2) \leq \mu(K_1 \cup K_2) \leq \mu(K_1)+\mu(K_2)$ and equality holds, as required.

Now, by Claim 1, it remains to show that $\mu(\cup E_i) \geq \sum \mu(E_i)$. If $\mu(\cup E_i) = +\infty$, we are done, so assume $\mu(\cup E_i) < +\infty$. Since the E_i are regular, there is a compact set $H_i \subset E_i$ so that $\mu(H_i) > \mu(E_i) - \frac{\epsilon}{2^i}$ for each $i \in \mathbb{N}$. Let $K_n = \bigcup_{i=1}^{n} H_i$. Now

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \epsilon$$

Taking the limit as n goes to infinity gives $\mu(\cup E_i) \geq \sum \mu(E_i) - \epsilon$ for any $\epsilon > 0$, so we are done.

Let's now see the second part. For any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{N} \mu(E_i) + \epsilon \leq \mu(K_N) + 2\epsilon$$

with K_N compact defined in the same way as above. Since $\epsilon > 0$ was arbitrary, the result follows directly. ■

Claim 5: E is regular and $\mu(E) < \infty$ if and only if for any $\epsilon > 0$, there exists K compact, G open so that $K \subset E \subset G$ and $\mu(G \setminus K) < \epsilon$.

Proof There exists by regularity (and the definition of the outer measure) $K \subset E \subset G$ so that

$$\mu(E) - \frac{\epsilon}{2} \leq \mu(K) \leq \mu(G) \leq \mu(E) + \epsilon/2$$

As well, $\mu(G) = \mu(K \cup (G \setminus K)) = \mu(K) + \mu(G \setminus K)$ and $\mu(G \setminus K) = \mu(G) - \mu(K) < \epsilon$.

Conversely, let $K \subset E \subset G$ and $\mu(G \setminus K) < \epsilon$. Then

$$\mu(E) \leq \mu(G) = \mu(K) + \mu(G \setminus K) < \mu(K) + \epsilon$$

so $\mu(E) < \infty$ and $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$, so E is regular. ■

Claim 6:

1. **Let A, B be regular with $\mu(A), \mu(B) < \infty$. Then $A \setminus B$, $A \cup B$, $A \cap B$ are regular and have finite measure.**
2. **If $\mu(E) < \infty$, then E is regular if and only if E is locally regular.**
3. **If E_i are regular, then $\bigcup_{i=1}^{\infty} E_i$ is regular.**

Proof Recall that for any $\epsilon > 0$, there exists $K_1 \subset A \subset G_1$ and $K_2 \subset B \subset G_2$ such that $\mu(G_1 \setminus K_1) < \epsilon$ and $\mu(G_2 \setminus K_2) < \epsilon$.

1. Note that $A \setminus B \subset G_1 \setminus K_2 \subset (G_1 \setminus K_1) \cup (K_1 \setminus G_2) \cup (G_2 \setminus K_2)$, where $K_1 \setminus G_2$ is compact. Thus $\mu(A \setminus B) \leq \epsilon + \mu(K_1 \setminus G_1) + \epsilon < \infty$ and $\mu(A \setminus B) - 2\epsilon \leq \mu(K_1 \setminus G_2)$ so $A \setminus B$ is regular. Finally since $A \cup B = (A \setminus B) \cup B$, $A \cup B$ is regular and $\mu(A \cup B) < \infty$. Thus $A \cap B = (A \cup B) \setminus ((A \setminus B) \cup (B \setminus A))$ is regular and has measure less than infinity.

2. Let $\mu(E) < \infty$, and first suppose E is regular. Let K be a compact set. Then $\mu(K) < \infty$ and K is regular, so $E \cap K$ is regular (by 1.) so E is locally regular.

   Conversely, suppose E is locally regular. Let $\epsilon > 0$ and $G \supset E$ be open so that $\mu(G) < \mu(E) + 1 < \infty$. As well, G is regular, so there exists K with $\mu(G) < \mu(K) + \epsilon/2$. Now,

$$\begin{aligned} \mu(E) &= \mu((E \setminus K) \cup (E \cap K)) \leq \mu(E \setminus K) + \mu(E \cap K) \\ &\leq \mu(G \setminus K) + \mu(E \cap K) \\ &< \frac{\epsilon}{2} + \mu(E \cap K) \end{aligned}$$

   so $\mu(E \cap K) > \mu(E) - \epsilon/2$. Then since E is locally regular, $E \cap K$ is regular and get a compact set $L \subset E \cap K$ such that $\mu(L) > \mu(E \cap K) - \epsilon/2 > \mu(E) - \epsilon$. This holds for any $\epsilon > 0$, so E is regular.

3. Set $F_1 = E_1$, $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right)$ so $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ and the F_i as disjoint. By Claim 4, $\cup F_i$ is regular and F_i are regular (TODO: finiteness requirement?) ■

Claim 7: $\mathcal{M}$ is a σ−algebra, $M \subset \tau$, and μ is countably additive on $\mathcal{M}$.

Proof We demonstrate the requirements:

- Let $A \in \mathcal{M}$: we see that $A^c \in \mathcal{M}$. If K is an arbitrary compact set, then $A^c \cap K = K \setminus (A \cap K)$ is regular by Claim 7 since K is regular (and thus locally regular), and $A \cap K$ is regular since A is locally regular.
- Now let $A_n \in \mathcal{M}$; we will show that $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$. Indeed, if K is an arbitrary compact set, then

$$A \cap K = \bigcup_{n=1}^{\infty} (A_n \cap K)$$

  is regular by Claim 6.
- We now show $\mathcal{M} \supset \tau$. It suffices by closure under complements to show that all closed sets are in $\mathcal{M}$. If A is closed, then $A \cap K$ is compact and thus regular, so $A \in \mathcal{M}$.
- Finally, let $E_i \in \mathcal{M}$ be locally regular and disjoint; it suffices to show that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

  If $\mu(E_i) = +\infty$, we are done, so assume $\mu(E_i) < \infty$ for all i. But then by Claim 6.2, the E_i are regular, so the result holds by Claim 6.3. ■

Claim 8: $\Lambda f = \int_X f \, \mathrm{d}\mu$ for all $f \in C_c(X)$.

Proof We are finally almost done: we just need to show that μ, as defined, actually represents Λ. Let's start by simplifying f as much as possible.

- *It suffices to do this for real valued functions.* If $f = u + iv$, then $\Lambda f = \Lambda u + i\Lambda v = \int_X u\,\mathrm{d}\mu + i\int_X v\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu$.
- *It suffices to show $\Lambda f \leq \int_X f\,\mathrm{d}\mu$.* If this holds for all f, then $\Lambda(-f) \leq \int_X -f\,\mathrm{d}\mu$ so that $-\Lambda f \leq -\int_X f\,\mathrm{d}\mu$ and $\Lambda f \geq \int_X f\,\mathrm{d}\mu$ and equality holds.
- *It is enough to prove that $\Lambda f \leq \int_X f\,\mathrm{d}\mu$ for $f \geq 0$.* Assuming so, let f be arbitrary and let $K = \operatorname{supp} f$ be compact, and $a = \min f$, $b = \max f$. The general idea of the proof is to translate f by the value $|a|$ so that it is positive. However, we cannot do this directly since $f + |a|$ is not compactly supported; however, we can use Urysohn's Lemma to translate it on its support. Now, let $\epsilon > 0$ be arbitrary. Fix $K = \operatorname{supp} f$ and get $G \supset K$ so that $\mu(G) \leq \mu(K) + \epsilon$. By Urysohn's lemma, there exists $h \in C_c(X)$ so that $K \prec h \prec G$. Thus $|a| \cdot h(x) = |a|$ for all $x \in K$, so $F := f + |a|h \geq 0$ since $f \geq -|a|$. Now by assumption,

  $$\Lambda F \leq \int_X F\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + |a|\int_X h\,\mathrm{d}\mu$$

  so that

  $$\begin{aligned}\Lambda f &= \Lambda F - |a|\Lambda h \\ &\leq \int_X f\,\mathrm{d}\mu + |a|\int_X h\,\mathrm{d}\mu - |a|\Lambda h \\ &\leq \int_X f\,\mathrm{d}\mu + |a|\left(\int_X h\,\mathrm{d}\mu - \Lambda h\right)\end{aligned}$$

  We now want to show $|\int_X h\,\mathrm{d}\mu - \Lambda h| < \epsilon$, and the result will follow. By Claim 2, $\mu(K) \leq \Lambda h \leq \mu(G)$. As well, $\int_X h\,\mathrm{d}\mu \leq \mu(G)$ since $h \prec G$ implies $h \leq \chi_G$. Thus since $h \geq 0$, by assumption

  $$\mu(K) \leq \Lambda h \leq \int_X h\,\mathrm{d}\mu \leq \mu(G)$$

  and the result follows since $\mu(G) - \mu(K) < \epsilon$. Thus $\Lambda f \leq \int f + |a|\epsilon$ for all $\epsilon > 0$, so $\Lambda f \leq \int f$ as desired.

It now remains to show $\Lambda f \leq \int_X f\,\mathrm{d}\mu$ for $f \geq 0$. Since $f = Mf/M$ where $M = \max f$, we can assume $0 \leq f \leq 1$. Fix $K = \operatorname{supp} f$, let $\epsilon > 0$ be arbitrary. Let $0 = c_0 < c_1 < c_2 < \cdots < c_n = 1$ with $c_k - c_{k-1} < \epsilon$ for all k and $\mu(f^{-1}(c_k)) = 0$ for all $k = 1, \ldots, n-1$. The existence of such a set follows from Assignment 6. Let $K_j = K \cap f^{-1}([c_{j-1}, c_j])$ for $j = 1, 2, \ldots, n$ and $L_j = K \cap f^{-1}([c_{j-1}, c_j))$ for $j = 1, 2, \ldots, n-1$.

For each K_j and any $\epsilon > 0$, there exists $\tau \ni G_j \supset K_j$ such that $\mu(G_j) \leq \mu(K_j) + \frac{\epsilon}{2^j}$. By Urysohn's lemma, get h_j so that $K_j \prec h_j \prec G_j$. Then $f \leq \sum_{j=1}^n c_j h_j$: if $x \in K^c$, $f = 0$. Otherwise, if $x \in K$, then $x \in K_j$ for some j. Since $h_j = 1$ and $f(x) \leq c_j$ on K_j, we have $f(x) \leq c_j = c_j h_j(x) \leq \sum_{i=1}^n c_i h_i$.

Now, there is just a lot of algebra.

$$\begin{aligned}
\Lambda f &\leq \Lambda\left(\sum c_j h_j\right) = \sum_{i=1}^n c_j \Lambda h_j && \text{(linearity and positivity)}\\
&\leq \sum_{j=1}^n c_j \mu(G_j) && (h_j \prec G_j)\\
&\leq \sum_{j=1}^n c_j \mu(K_j) + \sum_{j=1}^n c_j \frac{\epsilon}{2^j} && \text{(choice of } K_j)\\
&\leq \sum_{j=1}^n (c_{j-1} + c_j - c_{j-1})\mu(L_j) + \epsilon && (L_j \subset K_j, |c_j| \leq 1)\\
&\leq \sum_{j=1}^n c_{j-1}\mu(L_j) + \epsilon \cdot \mu(K) + \epsilon && (L_j \text{ disjoint}, c_j - c_{j-1} < \epsilon)
\end{aligned}$$

Now define g so $g(x) = c_{j-1}$ if $x \in L_j$, and $g \equiv 0$ outside K. Then g is a simple function, so that the summation above is precisely the integral of g. Furthermore, $g \leq f$ so $\int_X g \, d\mu \leq \int_X f \, d\mu$ and

$$\begin{aligned}
\Lambda f &\leq \int_X g \, d\mu + \epsilon + \epsilon\mu(K)\\
&\leq \int_X f \, d\mu + \epsilon(1 + \mu(K))
\end{aligned}$$

and, since $\mu(K) < \infty$, because this holds for any $\epsilon > 0$, we are done! ■

2.3 Regularity Properties of Borel Measures

At the beginning of the Riesz Representation Theorem, we introduced a variety of conditions which we will summarize here independetly.

Def'n. 2.3.1 *A measure defined on the family of Borel sets is called a **Borel measure**.*

Def'n. 2.3.2 *Let $\mu : \mathcal{B} \to [0, +\infty]$ be a Borel measure.*

1. *E is called **outer regular** if $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$.*
2. *E is called **inner regular** if $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$*
3. *μ is called **regular** if every $E \in \mathcal{B}$ is inner and outer regular.*

The next condition is a finiteness condition: naturally, we like spaces that aren't too big.

Def'n. 2.3.3 *A set $E \subset X$ is called σ*−***compact*** *if $E = \bigcup_{n=1}^{\infty} E_n$, for E_n compact.*

The sets in the next definition are standard in real analysis.

Def'n. 2.3.4 *A G_δ set is one of the form $\bigcap_{n=1}^{\infty} A_n$ with A_n open, and a F_σ set is one of the form $\bigcup_{n=1}^{\infty} B_n$ for B_n closed.*

Measure spaces $(X, \mathcal{M}, \mu)$ which satisfy these properties are particularly nice. To be precise, by "nice", we have the following theorem:

Thm. 2.3.5 *Let X be a locally compact, $\sigma-$compact Hausdorff space. Let $\mathcal{M} \supset \mathcal{B}$ be a $\sigma-$algebra, $\mu : \mathcal{M} \to [0, +\infty]$ be a measure such that*

(i) $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$ (outer regularity)
(ii) $\mu(E) = \sup\{\mu(K) : K \subset E$ compact$\}$, $\mu(E) < \infty$ (inner regularity for finite measure sets)
(iii) $\mu(K) < \infty$ for K compact (finite on compact sets)

Then

1. *For all $E \in \mathcal{M}$ and $\epsilon > 0$, there exists F closed and G open so that $F \subset E \subset G$ and $\mu(G \setminus F) < \epsilon$.*
2. *μ is regular*
3. *For all $E \in \mathcal{M}$, there exists a F_σ set A and a G_δ set B so $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.*

Thankfully, the proof is not too hard.

Proof Since X is $\sigma-$compact, write $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact.

1. By (iii), we have $\mu(K_n \cap E) < \infty$. Thus by (i), get G_n open so that $G_n \supset K_n \cap E$ with $\mu(G_n \setminus (K_n \cap E)) < \frac{\epsilon}{2^{n+1}}$. Let $G = \bigcup_{n=1}^{\infty} G_n$ be open, so that
$$G \setminus E \subset \bigcup_{n=1}^{\infty} G_n \setminus (K_n \cap E)$$
and
$$\mu(G \setminus E) \leq \sum_{n=1}^{\infty} \mu(G_n \setminus (K_n \cap E)) < \frac{\epsilon}{2}$$
Repeat this for E^c: get an open set H such that $\mu(H \setminus E^c) < \frac{\epsilon}{2}$. Then $F = H^c \subset E$ satisfies $\mu(E \setminus F) = \mu(F^c \setminus E^c) = \mu(H \setminus E^c) < \frac{\epsilon}{2}$. Thus $\mu(G \setminus F) \leq \mu(G \setminus E) + \mu(E \setminus F) < \epsilon$.
2. E is outer regular by (i). If $\mu(E) < \infty$, then E is inner regular by (ii), so E is regular; thus suppose $\mu(E) = \infty$.
Let $F \subset E$ be given by 1, so that $\mu(F) = +\infty$ (or $\mu(E)$ would be finite). Note that $H_n := \bigcup_{k=1}^{n} (F \cap K_k)$ is a compact set, so that $H_n \subset F$. Then $\bigcup_{n=1}^{\infty} H_n = F$, and $\mu(H_n) \to \mu(F) = \infty$. Thus $\mu(E) = \sup\{\mu(K) : K \subset E$ compact$\}$.
3. Apply 1 wih $\epsilon = 1/j$ for $j \in \mathbb{N}$. Then there exists $F_j \subset E \subset G_j$ so $\mu(G_j \setminus F_j) < \frac{1}{j}$. Define
$$A = \bigcup_{j=1}^{\infty} F_j, \qquad B = \bigcap_{j=1}^{\infty} G_j$$
Then $A \subset E \subset B$ and $\mu(B \setminus A) \leq \mu(G_j \setminus F_j) < \frac{1}{j}$ for any $j \in \mathbb{N}$, so $\mu(B \setminus A) = 0$. ■

As a corollary to this, if we assume that X is a locally compact and $\sigma-$compact space and Λ is a positive linear functional on $C_c(X)$, then the measure μ representing Λ is a regular measure. More generally, if we assume that *every* open set is $\sigma-$compact, we have the following theorem:

Thm. 2.3.6 *Let X be locally compact and Hausdorff, and assume that every open set is σ−compact. Let $\lambda : \mathcal{B} \to [0,\infty]$ be a Borel measure such that $\lambda(K) < \infty$ for any compact set K. Then λ is regular.*

Proof Let $\Lambda f = \int_X f \,\mathrm{d}\lambda$. Then $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional. By the Riesz representation theorem, there exists $\mu : \mathcal{M} \to [0,\infty]$ such that $\int_X f \,\mathrm{d}\mu = \Lambda f = \int_X f \,\mathrm{d}\lambda$. We see that $\lambda = \mu$ on $\mathcal{B}$, so that λ is regular since μ is.

We first prove this for open sets. Let $G \in \tau$; then there exists compact K_n so $G = \bigcup_{n=1}^{\infty} K_n$. By Urysohn's lemma, there exists f_i such that $K_i \prec f_i \prec G$. Let $g_n = \max\{f_1, f_2, \ldots, f_n\}$, so $g_n \in C_c(X)$, and $g_n \to \chi_G$ pointwise. But then applying Lebesgue's Monotone Convergence theorem (and the fact that $\lambda = \mu$ on $C_c(X)$),

$$\begin{aligned}\lambda(G) &= \int_X \chi_G \,\mathrm{d}\lambda = \int_X \lim_{n\to\infty} g_n \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\lambda \\ &= \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} g_n \,\mathrm{d}\mu = \int_X \chi_G \,\mathrm{d}\mu \\ &= \mu(G)\end{aligned}$$

Now for any $E \in \mathcal{B}$, get F closed, G open so that $F \subset E \subset G$ and $\mu(G \setminus F) < \epsilon$. Since $G \setminus F$ is open, $\lambda(G \setminus F) = \mu(G \setminus F) < \epsilon$ so $\lambda(G) \le \lambda(E) + \epsilon$. Thus $|\mu(E) - \lambda(E)| < \epsilon$ for all $\epsilon > 0$ so $\lambda(E) = \mu(E)$. ■

2.4 Construction of the Lebesgue Measure

We have the Riesz Representation Theorem in a locally compact Hausdorff space.

Def'n. 2.4.1 *Let $E \subset \mathbb{R}^k$, $x \in \mathbb{R}^k$. Then $E + x = \{y + x : y \in E\}$ is the* ***translate*** *of E.*

Def'n. 2.4.2 *We define a k−cell in $\mathbb{R}^k$ by $W = I_1 \times I_2 \times \cdots \times I_k$ where I_j is an interval. We also define* $\mathrm{vol}(W) = (b_1 - a_1)(b_2 - a_1)\cdots(b_k - a_k)$ *where a_j, b_j are the endpoints of the I_j.*

We know that $\mathrm{vol}(W + x) = \mathrm{vol}(W)$ for any k−cell W and $x \in \mathbb{R}$.

Thm. 2.4.3 *There exists a σ−algebra $\mathcal{M}$ in $\mathbb{R}^k$ and a complete measure $m : \mathcal{M} \to [0,+\infty]$ satisfying*
1. *$m(W) = \mathrm{vol}(W)$ for any k−cell W.*
2. *$\mathcal{M} \supset \mathcal{B}$ and $E \in \mathcal{M}$ if and only if there exists $A \in F_\sigma, B \in G_\delta$ such that $A \subset E \subset B$ and $m(B \setminus A) = 0$.*
3. *m is translation invarient: $m(E + x) = m(E)$.*
4. *If μ is a translation invariant Borel measure, and $\mu(K) < \infty$ for all K compact, then there exists $c \in \mathbb{R}$ so that $\mu(E) = c \cdot m(E)$.*
5. *If $T : \mathbb{R}^k \to \mathbb{R}^k$ is linear, then there exists $\Delta(T) \in \mathbb{R}$ such that $m(T(E)) = \Delta(T) \cdot m(E)$.*

Proof For $f \in C_c(\mathbb{R}^k)$, let $\Lambda f = \int_{R^k} f(x)\,\mathrm{d}x$ (the Riemann Integral). Then $\Lambda : C_c(\mathbb{R}^k) \to \mathbb{C}$ is a positive linear functional, so by the Riesz representation theorem, there exists a unique measure m and $\mathcal{M} \supset \mathcal{B}$ so for all $f \in C_c(\mathbb{R}^k)$, $\Lambda f = \int_{\mathbb{R}^k} f \,\mathrm{d}m$. Let's prove that this measure has the appropriate properties:
1. By the definition of m, for an open k−cell W, $m(W) = \sup\{\Lambda f : f \prec W\} = \mathrm{vol}(W)$ (by definition of the Riemann integral). If W is an arbitrary k−cell, then there exist open k−cells W_n such that $W = \bigcap\limits_{n=1}^{\infty} W_n$. Then $\mathrm{vol}(W_n) \to \mathrm{vol}(W)$, so $m(W_n) \to m(W)$, and $\mathrm{vol}(W_n) = m(W_n)$. Thus $\mathrm{vol}(W) = m(W)$.

Let λ be a Borel measure. If $\lambda(W) = m(W)$ for all W $k-$cells, then $\lambda(E) = m(E)$ for all $E \in \mathcal{B}$. For any G open, $G = \bigcup_{n=1}^{\infty} W_n$ disjointly, so $\lambda(G) = m(W)$. Then since λ and m are regular, $\lambda(E) = \inf\{\lambda(G) : E \subset G \in \tau\} = \inf\{m(G) : E \subset G \in \tau\} = m(E)$ for all $E \in \mathcal{B}$.

We now see (iii). Dfine $\lambda(E) = m(E + X)$. If W is a box, then $\lambda(W) = m(W + x) = \text{vol}(W + x) = \text{vol}(W) = m(W)$, so by the lemma, $\lambda(E) = m(E)$ for all $E \in \mathcal{B}$. Then regularity implies $\lambda(E) = m(E)$ for all measurable E.

We have (iv): let $c = \mu([0,1]^k) = c \cdot \text{vol}([0,1]^k)$. Translation invariance of vol implies $\mu(W) = c \cdot \text{vol}(W)$.

We have (v). If $\dim(\text{Im}(T)) < k$, then $m(\text{Im}(T)) = 0$ so $\Delta(T) = 0$. Otherwise, T is a homeomorphism so $T(E) \in \mathcal{B}$ for all $E \in \mathcal{B}$. Let $\mu(E) = m(T(E))$. Then $\mu(E + x) = m(T(E) + T(x)) = m(T(E)) = \mu(E)$, so μ is translation invariant. Then by (iv), $\mu(E) = c \cdot m(E)$ and set $\Delta(T) = c$. ■

Thm. 2.4.4 *If $A \subset \mathbb{R}$ for which every set is Lebesgue measurable, then $m(A) = 0$.*

Proof Partition $\mathbb{R}$ into cosets by $\mathbb{Q}$; let E be a set containing exactly one element of each class (axiom of choice). Now if $r \neq s$, $r, s \in \mathbb{Q}$, then $(E + r) \cap (E + s) = \emptyset$. But then $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r)$ disjointly. Given A, define $A_t = A \cap (E + t)$ for $t \in \mathbb{Q}$. Now let $K \subset A_t$, so $K \subset E + t$. Since $(K + r_1) \cap (K + r_2) = \emptyset$, define $H = \bigcup_{r \in \mathbb{Q} \cap [0,1]} (K + r)$ is a countable disjoint union. But then $\infty > m(H) = \sum_r m(K)$ so $m(K) = 0$ and $m(A_t) = 0$. But then

$$\bigcup_{t \in \mathbb{Q}} A_t = \cup(A \cap (E + t)) = A \cap \left(\bigcup_{t \in \mathbb{Q}} (E + t) \right) = A \cap \mathbb{R} = A$$

so $m(A) = 0$ as well. ■

2.5 Measurability and Continuity

Let X be a locally compact, Hausdorff topological space. Let $\mathcal{M}$ be a $\sigma-$algebra, μ be a measure satisfying the properties in the Riesz representation theorem. We then have

Thm. 2.5.1 (Lusin) *Let $f : X \to \mathbb{C}$ be a measurable function, with $\text{supp} f \subset A$ and $\mu(A) < \infty$. Then for any $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$ and $\sup_X |G| \leq \sup_X |f|$.*

Proof It suffices to assume that we can do this for compact K. Assuming so, let $\mu(A) < \infty$ and get $K \subset A$ compact with $\mu(A \setminus K) < \epsilon/2$. Define $\hat{f}$ so that $\hat{f} = f$ on K and $\hat{f} = 0$ otherwise, so $\sup \hat{f} \subset K$ and f' is measurable. By assumption, get g so that $\mu(\{x : g(x) \neq f'(x)\}) < \epsilon/2$. Then

$$\mu(\{x \in X : f(x) \neq g(x)\}) \leq \mu(A \setminus K) + \mu\Big(\{x \in K \cup A^c : \hat{f}(x) \neq g(x)\}\Big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since $f' = f$ on $(A \setminus K)^c$. Now, let's prove the statement for A compact.

We first assume that $0 \leq f \leq 1$. For $t \geq 0$ and each $N \in \mathbb{N}$, define $k_n(t) = [2^n \cdot t]$, so $k_n(t) \in \mathbb{Z}$ and $k_n(t) \leq t \cdot 2^n < k_n(t) + 1$. Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n}, & 0 \leq t \leq n \\ n, & t > n \end{cases}$$

Let $s_n(x) = \phi_n(f(x))$ and $t_n = s_n - s_{n-1}$. Observe that $f = \sum_{n=1}^{\infty} t_n$; I claim that $2^n \cdot t_n \in \{0,1\}$. To see this, first note that

$$k_{n-1}(t) \leq t \cdot 2^{n-1} < k_{n-1}(t) + 1 \implies 2k_{n-1}(t) \leq t \cdot 2^n < 2k_{n-1}(t) + 2$$

so $2k_{n-1}(t)$ is the largest even number below $t \cdot 2^n$. Thus $k_n - 2k_{n-1} \in \{0,1\}$ for all t. Since $0 \leq f \leq 1$, for all n and x,

$$\begin{aligned} 2^n \cdot t_n(x) &= 2^n \cdot (\phi_n(f(x)) - \cdot \phi_{n-1}(f(x))) \\ &= 2^n \left(2^{-n} \cdot k_n(f(x)) - 2^{-(n-1)} \cdot k_{n-1}(f(x))\right) \\ &= k_n(f(x)) - 2k_{n-1}(f(x)) \in \{0,1\} \end{aligned}$$

as required. Thus $2^n \cdot t_n$ is the characteristic function of some set $T_n \subset A$, so $\mu(T_n) < \infty$.

Let $V \supset A$ be open so that $\overline{V}$ is compact; this set exists since X is locally compact and A is compact (by assumption). *To construct it, for each $x \in A$, let $V_x \subset F_x$ be a compact neighbourhood of x. Since $\{V_x\}_{x \in A}$ is an open cover for A, there exists a subcover $\{V_{x_i}\}_{i=1}^n$. Then $A \subset \overline{\bigcup_{i=1}^n V_x} \subset \bigcup_{i=1}^n F_{x_i}$ is a closed subset of a compact set, and thus compact.* Since $\mu(T_n) < \infty$, get K_n compact, V_n open, so that $K_n \subset T_n \subset V_n$ with $\mu(V_n \setminus K_n) < \epsilon/2^n$. We can assume $V_n \subset V$ since we can always take $V_n \cap V$, which is open.

By Urysohn's lemma, there exists $h_n \in C_c(X)$ with $K_n \prec h_n \prec V_n$. Define $g = \sum_{n=1}^{\infty} 2^{-n} \cdot h_n$ is a uniform limit, so g is continuous and $\operatorname{supp} g \subset \overline{V}$. If $x \in K_n$, then $h_n(x) = 1$ and $t_n(x) = 2^{-n}$, so $2^{-n} \cdot h_n(x) = t_n(x)$. If $x \notin V_n$, then $h_n(x) = 0$ so $t_n(x) = 0$ and $2^{-n} \cdot h_n(x) = t_n(x)$. Thus

$$S = \{x \in A : f(x) \neq g(x)\} \subset \bigcup_{n=1}^{\infty} (V_n \setminus K_n)$$

and $\mu(S) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus K_n) < \epsilon$.

If $-A \leq f \leq A$, then $0 \leq f + A \leq 2A$ and apply the above theorem to $(f+A)/(2A)$ and get some $\hat{g}$. Then $2A\hat{g} - A$ has the desired properties. Additionally, for any real valued function, let $B_n = \{x \in X : |f(x)| > n\}$. Then $\bigcap_{n=1}^{\infty} B_n = \emptyset$, $\mu(B_1) \leq \mu(\operatorname{supp} f) < \infty$, and $B_{n+1} \subset B_n$ for all n. Thus $\mu(B_n) \to \mu(\bigcap B_n) = 0$. Let N be such that $\mu(B_N) < \epsilon/2$, so if $x \notin B_N$, $f(x) \leq N$, and define $\tilde{f}(x) = (1 - \chi_{B_n(x)})f(x)$. Then $\tilde{f}$ is bounded, and apply the above to get $g \in C_c(x)$ so that $\mu\left(\{x : \tilde{f}(x) \neq g(x)\}\right) < \epsilon/2$. But then

$$\mu(\{x : g(x) \neq f(x)\}) \leq \mu(\{x : f(x) \neq \tilde{f}(x)\}) + \mu(\{x : \tilde{f}(x) \neq g(x)\}) = \epsilon$$

All that is left is to do this for complex valued functions, satisfying the additional constraint. To be precise, let f be complex valued and write $f = f_1 + if_2$. Then for $\epsilon > 0$, get $g_1, g_2 \in C_c(X)$

satisfying the requirements for $\epsilon/2$ and set $g = g_1 + ig_2$. We will prove that $\sup|G| \leq \sup|f|$. If $\sup|f| = \infty$ we are done, so let $R = \sup_X |f|$. Let

$$\phi(z) = \begin{cases} z & : |z| \leq R \\ \frac{R\cdot z}{|z|} & : |z| > R \end{cases}$$

then ϕ is continuous and $|\phi| \leq R$. We already have $g \in C_c(X)$ so that $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$. Let $\tilde{g} = \phi \circ g$, which is also continuous and $|\tilde{g}| \leq R$. Finally, if $\tilde{f} \neq \tilde{g}$, then certainly $f \neq g$, so

$$\mu\{\tilde{g} \neq f\} = \mu\{\phi \circ f \neq \phi \circ g\} \leq \mu\{g \neq f\} < \epsilon$$

and we are done. ■

For the following corollary, we require the same context for $(X, \mathcal{M}, \mu)$ as in Lusin's Theorem.

Cor. 2.5.2 *Let* $f : X \to \mathbb{C}$ *be measurable,* $\operatorname{supp} f \subset A$*, and* $\mu(A) < \infty$ *and* $|f| \leq 1$*. Then there exists* $g_n \in C_c(X)$ *with* $|g_n| \leq 1$ *and* $\lim g_n(x) = f_n(x)$ *almost everywhere.*

Proof Apply Lusin's theorem with $\epsilon = 1/n$ for each g_n. ■

Chapter 3

Complex Measures

3.1 Hilbert Spaces

3.1.1 Basic Definitions

A complex vector space H is called an inner product space if there is a pairing $\langle \cdot, \cdot \rangle : H \to \mathbb{C}$ satisfying, for any $x, y \in H$ and $\alpha \in \mathbb{C}$

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$
3. $\langle x, x \rangle \geq 0$
4. $\langle x, x \rangle = 0$ if and only if $x = 0$

One can define a norm by $\|x\| = \sqrt{\langle x, x \rangle}$. The fact that this is acutally a norm is a consequence of the following proposition:

Prop. 3.1.1 *On an inner product space H, the following inequalities hold:*

1. ***Cauchy-Schwarz:*** $|\langle x, y \rangle| \leq \|x\| \|y\|$
2. ***Triangle:*** $\|x + y\| \leq \|x\| + \|y\|$

From the triangle inequality, we can define a metric $d(x, y) = \|x - y\|$. Thus we call H a Hilbert space if it is also complete with respect to this metric.

There are some standard continuous functions on H:

Prop. 3.1.2 *For a fixed $y \in H$, the mappings*

$$x \mapsto \langle x, y \rangle, \quad x \mapsto \langle y, x \rangle, \quad x \mapsto \|x\|$$

are continuous functions.

The first map is particularly important since, to some $x \in H$, we can associate the continuous linear functional ϕ_x defined by $\phi_x(y) = \langle x, y \rangle$. More importantly, in Hilbert spaces, the converse holds as well. The first part is the content of (the original) Riesz representation theorem:

Thm. 3.1.3 *Let H be a Hilbert space, and H^* be the vector space of continuous linear functions on H^*.*

1. *For each $L \in H^*$, then there exists a unique $y \in H$ such that*

$$Lx = \langle x, y \rangle$$

for all $x \in H$.

2. *H^* is a Hilbert space with inner product*

$$\langle \phi_u, \phi_v \rangle = \langle v, u \rangle$$

and is isomorphic (as a Hilbert space) to H, where $\phi_x(y) = \langle x, y \rangle$.

3.1.2 The space $L^2(\mu)$

In particular, consider the space $L^2(\mu)$ where $(X, \mathcal{M}, \mu)$ has X compact, and μ is a regular complex measure on $\mathcal{M}$. Then $L^2(\mu) = \{f : X \to \mathbb{C} : f \text{ measurable}, \|f\|_2 < \infty\}$ and we have

$$\langle f, g \rangle = \int_X f \overline{g} \, \mathrm{d}\mu, \quad \|f\|_2 = \left(\int_X |f|^2 \, \mathrm{d}\mu \right)^{1/2}$$

is the inner product on this space. We also have

Thm. 3.1.4 (Riesz-Fisher) *$L^2(\mu)$ is complete (every Cauchy sequence of functions converges w.r.t. the L^2 norm).*

so this inner product space is indeed a Hilbert space. Given a continuous linear functional Λ on H, the Riesz representation theorem states that

$$\Lambda f = \langle f, g \rangle = \int_X f \overline{g} \, \mathrm{d}\mu$$

for some $g \in \mathcal{M}$. But then define $\phi(E) = \int_E \overline{g} \, \mathrm{d}\mu$ as the image measure of $\overline{g}$, so

$$\int_X f \overline{g} \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\phi$$

In other words, $\Lambda f = \int_X f \, \mathrm{d}\phi$, which looks much more similar to the standard Riesz representation theorem.

$L^2(\mu) = \{f : X \to \mathbb{C} : f \text{ is measurable}, \|f\|_2 <\}$ is a Hilbert space (complete) with the inner product $\langle f, g \rangle = \int_X f \cdot \overline{g} \, \mathrm{d}\mu$.

Thm. 3.1.5 *If H is a Hilbert space, $L : H \to \mathbb{C}$ is continuous and linear, then there exits a unique $y \in H$ so that $L(X) = \langle x, y \rangle$ for all $x \in H$.*

3.2 Complex Measures

Let $\mathcal{M}$ be a $\sigma-$algebra on X.

Def'n. 3.2.1 *$\mu : \mathcal{M} \to \overline{\mathbb{R}}$ is called a* ***signed measure*** *if it is countably addivite and $+\infty$ and $-\infty$ are not in the range at the same time.*

Def'n. 3.2.2 *$\mu : \mathcal{M} \to \mathbb{C}$ is called a* ***complex measure*** *if it is countably additive: if E_i are disjoint, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

For now, it is not at all obvious how to define integration with respect to a complex measure. This will appear following the proof of the Lebesgue-Radon-Nikodym theorem.

Def'n. 3.2.3 *For a set $E \in \mathcal{M}$, a* ***(measurable) partition*** *of E is $\{E_i : i = 1, 2, \dots\}$ so that $E_i \cap E_j = \emptyset$ and $\bigcup_{i=1}^{\infty} E_i = E$ and $E_i \in \mathcal{M}$ for all I.*

Let's try to motivate the following definition. Given a complex measure μ, we want to find a positive measure λ which satisfies $|\mu(E)| \leq \lambda(E)$ for every $E \in \mathcal{M}$. Such a measure, if it exists, must satisfy

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i) \geq \sum_{i=1}^{\infty} |\mu(E_i)|$$

for any partition $\{E_i\}$ of E; thus, it must be at least as large as the supremum. With this in mind, we can define the following set function:

Def'n. 3.2.4 *Let μ be a complex or signed measure. Its total variation*

$$|\mu| : \mathcal{M} \to [0, +\infty] = \sup\left\{\sum_{i=1}^{\infty} |\mu(E_i)| : \{E_i\} \text{ is a partition of } E\right\}$$

Conveniently, $|\mu|$ is actually a positive measure (which we will see below). More surprisingly, $|\mu|(X) < \infty$; thus, for any $E \in \mathcal{M}$, $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X)$, so every complex measure is actually a bounded measure and its image is contained in a disc of finite radius. This property is sometimes summarized by saying "μ is of bounded variation".

Thm. 3.2.5 *$|\mu|$ is a positive measure.*

Proof Let $E \in \mathcal{M}$, and $\{E_i\}$ an arbitrary partition of E. We first see that $\sum_{i=1}^{\infty} |\mu|(E_i) \leq |\mu|(E)$. Let $t_i < |\mu|(E_i)$, so there exists a partition $\{A_{ij} : j \in \mathbb{N}\}$ of E_i so that

$$\sum_{j=1}^{\infty} |\mu(A_{ij})| > t_i$$

for all i. Then since $\{A_{ij} : (i,j) \in \mathbb{N}^2\}$ is a partition of E, and

$$\sum_{i=1}^{\infty} t_i \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_{ij})| \leq |\mu|(E)$$

and since this holds for all i, we have $\sum_{i=1}^{\infty} |\mu|(E_i) \leq \mu(E)$.

We now see the opposite direction. Let $\{A_j : j \in \mathbb{N}\}$ be an arbitrary partition of E. The set $\{A_j \cap E_i : j \in \mathbb{N}\}$ is a partition of E_i, while $\{A_j \cap E_i : i \in \mathbb{N}\}$ is a partition of A_j. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\
&\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_j \cap E_i)| \\
&\leq \sum_{i=1}^{\infty} |\mu|(E_i)
\end{aligned}$$

where absolute convergence allows us to change the order of summation. Since this holds for an arbitrary partition $\{A_j\}$ of E, taking the supremum over all partitions gives the total variation. Thus equality holds. ∎

Lemma 3.2.6 *Let $z_1, z_2, \ldots, z_N \in \mathbb{C}$. Then there exists $S \subset \{1, 2, \ldots, N\}$ so that*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

Proof Let $z_k = |z_k| e^{i\alpha_k}$, and for $\theta \in [-\pi, \pi]$, let $S(\theta) = \{k \in \{1, 2, \ldots, N\} : \cos(\alpha_k - \theta) > 0\}$. Then

$$\begin{aligned}
\left| \sum_{k \in S(\theta)} z_k \right| &= \left| \sum_{k \in S(\theta)} |z_k| e^{-i\theta} \right| \\
&\geq \operatorname{Re} \sum_{k \in S(\theta)} e^{-i\theta} z_k \\
&= \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta) := h(\theta)
\end{aligned}$$

where $\cos^+ = \max\{\cos, 0\}$ is the positive part of , and $h : [-\pi, \pi] \to \mathbb{R}$ is a continuous function on a compact set. Thus it has a maximum at some θ_0. Fix $S = S(\theta_0)$ and

$$\begin{aligned}
\left| \sum_{k \in S} z_k \right| &= h(\theta_0) \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} h \, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta) \, \mathrm{d}\theta \\
&= \frac{1}{2\pi} \sum_{k=1}^{N} |z_k| \int_{-\pi}^{\pi} \cos^+(\alpha_k - \theta) \, \mathrm{d}\theta \\
&= \frac{1}{\pi} \sum_{k=1}^{N} |z_k|
\end{aligned}$$

since $\int_{-\pi}^{\pi}\cos^+(\alpha_k-\theta)=2$ for all k. ■

Thm. 3.2.7 *If μ is a complex measure, then $|\mu|(X)<\infty$.*

Proof First, let $E\in\mathcal{M}$ be such that $|\mu|(E)=+\infty$. We show that there exists A,B so $E=A\cup B$ and $|\mu(A)|\geq 1$ and $|\mu(B)|\geq 1$. Set $t=\pi(1+|\mu(E)|)$ and since $|\mu(E)|>t$, there exists a partition $\{E_i\}$ of E such that

$$\sum_{i=1}^{N}|\mu(E_i)|>t$$

for some N. Then by the lemma with $z_k=\mu(E_k)$, get S so that $\left|\sum_{k\in S}\mu(E_k)\right|\geq\frac{1}{\pi}\sum_{k=1}^{N}|\mu(E_k)|$ Thus

$$|\mu(A)|\geq\frac{1}{\pi}\sum_{k=1}^{N}\left|\mu(E_k)\right|>\frac{t}{\pi}\geq 1$$

and let $B=E\setminus A$. Then

$$|\mu(B)|\geq|\mu(A)|-|\mu(E)|\geq\frac{t}{\pi}-\left(\frac{t}{\pi}-1\right)=1$$

so that $E=A\cup B$ with $|\mu(A)|\geq 1$ and $|\mu(B)|\geq 1$.

Now assume $|\mu|(X)=\infty$ and apply the above procedure get A_1,B_1 with $|\mu(A_1)|\geq 1$ and $|\mu(B_1)|\geq 1$. As well, at least one of $|\mu|(A_1)$, $|\mu|(B_1)$ is infinity. Without loss of generality, it is B_1, so repeat this procedure to B_1. Get a sequence $A_1,A_2,\dots$ with $|\mu(A_i)|\geq 1$ and A_i disjoint. But then by countable additivity,

$$\sum_{i=1}^{\infty}\mu(A_i)=\mu\left(\bigcup_{i=1}^{\infty}A_i\right)<\infty$$

since μ is finite, a contradiction, since the LHS does not converge. ■

Recall that $\mu:\mathcal{M}\to\mathbb{C}$ is a complex measure if it is countably additive. Then the total variation of μ is given by

$$|\mu|(E)=\sup\left\{\sum_{i=1}^{\infty}|\mu(E_i)|:\{E_i\}\text{ is a partition}\right\}$$

Then $|\mu|$ is a positive measure and $|\mu|(X)<\infty$. If $\mu,\lambda:\mathcal{M}\to\mathbb{C}$ are complex measures, then $(\mu+\lambda)(E)=\mu(E)+\lambda(E)$ and $(c\cdot\mu)(E)=c\cdot\mu(E)$. Thus the set of complex measures is a vector space. Let $\left\|\mu\right\|:=|\mu|(X)$.

If μ is a signed measure ($\mu:\mathcal{M}\to\overline{R}$), then the total variation is defined in the same way.

Def'n. 3.2.8 *Let μ be a signed measure. The* ***positive variation*** *of μ is $\mu^+:=\frac{1}{2}(|\mu|+\mu)$ and the* ***negative variation*** *of μ is $\mu^-:=\frac{1}{2}(|\mu|-\mu)$.*

These are positive measures since $|\mu|(E)\geq|\mu(E)|$. We have $\mu=\mu^+-\mu^-$; this is called the Jordan decomposition since we represent a signed measure as a difference between two positive measures. Additionally, we have $|\mu|=\mu_++\mu_-$.

3.3 Absolute Continuity and Singular Measures

Def'n. 3.3.1 *Let μ be a positive measure and λ be an arbitrary (positive, signed, or complex) measure. Then λ is* ***absolutely continuous*** *with respect to μ if $\mu(E) = 0 \Rightarrow \lambda(E) = 0$. We write $\lambda \ll \mu$.*

Def'n. 3.3.2 *λ is concentrated on a set $A \in \mathcal{M}$ if $\lambda(E) = \lambda(E \cap A)$ for all $E \in \mathcal{M}$.*

Prop. 3.3.3 *λ is concentrated on A if and only if $\lambda(E) = 0$ if $E \cap A = \emptyset$.*

Proof Let $E \cap A = \emptyset$. Then $\lambda(E) = \lambda(E \cap A) = \lambda(\emptyset) = 0$. Conversely, let $E \in \mathcal{M}$. Then $\lambda(E) = \lambda(E \cap A) + \lambda(E \cap A^c) = \lambda(E \cap A)$. ∎

Def'n. 3.3.4 *λ_1 and λ_2 are called* ***mutually singular*** *if there exist disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is contentrated on B. Then $\lambda_1 \perp \lambda_2$.*

Let's summarize a number of properties concerning the aforementioned definitions and the total variation of a measure.

Prop. 3.3.5 *Let μ be a positive measure, , λ_1, λ_2 be arbitrary measures (positive, signed, or complex). Then*

1. *If λ is concentrated on A, then $|\lambda|$ is also concentrated on A.*
2. *If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$*
3. *If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.*
4. *If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.*
5. *If $\lambda \ll \mu$, then $|\lambda| \ll \mu$*
6. *If$\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$*
7. *If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.*

Proof These are all exercises in the definitions. For simplicity of notation (and so that I don't repeat myself too much), let λ be concentrated on A, λ_1 on A_1, λ_2 on A_2, and μ on B.

1. Suppose $E \cap A = \emptyset$; we want to show $|\lambda|(E) = 0$. Let $\{E_i\}$ be an arbitrary partition of E, so $E_i \cap A = \emptyset$ and $\lambda(E_i) = 0$ since λ is concentrated on A. Thus $\sum_{i=1}^{n} |\lambda(E_i)| = 0$ and since the partition was arbitrary, $|\lambda|(E) = 0$.
2. By (1), $|\lambda_1|$ is also concentrated on A_1 and $|\lambda_2|$ is concentrated on A_2, and $A_1 \cap A_2 = \emptyset$.
3. Let μ concentrated on B_1 with $B_1 \cap A_1 = \emptyset$, and μ is concentated on B_2 with $B_2 \cap A_2 = \emptyset$. If $A_1 \cap B = \emptyset$ and $A_2 \cap B = \emptyset$, then $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$. Furthermore, μ is concentrated on $B_1 \cap B_2$ since if $E \cap B_1 = \emptyset$ or $E \cap B_2 = \emptyset$, then $\mu(E) = 0$.
4. Let $\mu(E) = 0$. Then $\lambda_1(E) = 0$ and $\lambda_2(E) = 0$ so $(\lambda_1 + \lambda_2)(E) = 0$.
5. Let $\mu(E) = 0$ and let $\{E_i\}$ be a partition of E. Then $\mu(E_i) = 0$ for all i, so $\lambda(E_i) = 0$ for all i. Thus $\sum_{i=1}^{\infty} |\lambda(E_i)| = 0$ for any partition, so $|\lambda|(E) = 0$.
6. Since $\lambda_2 \perp \mu$, get disjoint A_2, B so λ_2 is concentrated on A_2 and μ is concentrated on B. Then λ_1 is also concentrated on B since, if $E \cap B = \emptyset$, then $\mu(E) = 0$ so $\lambda_1(E) = 0$ since $\lambda_1 \ll \mu$.
7. By (6.), $\lambda \perp \lambda$, and $A \cap A = \emptyset$ implies $A = \emptyset$, so $\lambda = 0$. ∎

Prop. 3.3.6 *Let μ be a positive measure, λ a complex measure. Then the following are equivalent:*

1. *$\lambda \ll \mu$*
2. *For any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ so $|\lambda(E)| < \epsilon$.*

Proof (2 $\Rightarrow$ 1). Let $\epsilon > 0$ and choose δ satisfying the requirement. Then let $\mu(E) = 0$, so $\mu(E) < \delta$ and $|\lambda(E)| < \epsilon$. This holds for any $\epsilon > 0$ so $\lambda(E) = 0$.

(1 $\Rightarrow$ 2). For contradiction, assume there exists some $\epsilon > 0$ so that for each $\delta = 1/2^n$, there exists a set E_n so that $\mu(E_n) < 1/2^n$ but $|\lambda(E_n)| \geq \epsilon$. As is standard, define

$$A_n = \bigcup_{k=n}^{\infty} E_k, \qquad A = \bigcap_{n=1}^{\infty} A_n$$

so that

$$\mu(A_n) \leq \sum_{k=n}^{\infty} \mu(E_k) \leq \sum_{k=n}^{\infty} \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

Thus since $\mu(A_1) < \infty$ and $A_1 \supset A_2 \supset \cdots$, $\mu(A) = 0$. Since $\lambda \ll \mu$, $|\lambda| \ll \mu$ so $|\lambda|(A) = 0$. However, $\lim_{n\to\infty} |\lambda|(A_n) = |\lambda|(A) = 0$ since $|\lambda|$ is a measure, while $|\lambda|(E_n) \geq |\lambda(E_n)| \geq \epsilon$, a contradiction. ■

If it important that λ is finite; if not, then this may not hold. Set $f(x) = 1/|x|$, $\lambda(E) = \int_E f \, d\mu$, and μ is the Lebesgue measure. However, for each $E = [-1/n, 1/n]$, and $\int_E f \, d\mu = \infty$ while $\mu(E) = 1/2^n$.

3.3.1 The Lebesgue-Radon-Nikodym Theorem

Lemma 3.3.7 *If μ is a positive, σ–finite measure (that is, $X = \bigcup_{n=1}^{\infty} X_n$ where $\mu(X_n) < \infty$), then there exists $w \in L^1(\mu)$ so that $0 < w < 1$.*

Proof Let $X = \bigcup\limits_{n=1}^{\infty} X_n$, and $\mu(X_n) < \infty$. Let

$$w_n(x) = \begin{cases} 0 & : x \in X \setminus X_n \\ \frac{1}{2^n(1+\mu(X_n))} & : x \in X_n \end{cases}$$

and $w(x) = \sum_{n=1}^{\infty} w_n(x)$. By construction, $0 < w < 1$ and

$$\int_X w \, d\mu = \sum_{n=1}^{\infty} \int w_n \, d\mu < \sum_{n=1}^{\infty} 1/2^n = 1$$

by Lebesgue's Monotone Convergence Theorem, so $w \in L^1(\mu)$. ■

Thm. 3.3.8 (Lebesgue-Radon-Nikodym) *Let μ be a positive, σ–finite measure, λ be a complex measure on $\mathcal{M}$.*

(a) *There exists a unique decomposition of λ as $\lambda = \lambda_a + \lambda_s$ such that $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$.*

(b) *There exists a unique $h \in L^1(\mu)$ such that $\lambda_a(E) = \int_E h \, d\mu$ for all $E \in \mathcal{M}$. This is the Radon-Nikodym derivative of λ_a with respect to μ.*

PROOF Let's first see that the decomposition is unique. Assume $\lambda = \lambda_a + \lambda_s = \lambda_a' + \lambda_s'$, so $\lambda_a - \lambda_a' = \lambda_s' - \lambda_s$. Then since $\lambda_a - \lambda_a' \ll \mu$ and $\lambda_s' - \lambda_s \perp \mu$, we have $\lambda_a - \lambda_a' = 0 = \lambda_s' - \lambda_s$. We also see that h is unique. Assume h^* is another one, so $\lambda_a(E) = \int_E h^* \,\mathrm{d}\mu$. Write $h = h_1 + ih_2$, $h^* = h_1^* + ih_2^*$. Let $A_1 = \{x \in X : h_1(x) > h_1^*(x)\}$ and $A_2 = \{x \in X : h_1(x) < h_1^*(x)\}$. We will show $\mu(A_1) = \mu(A_2) = 0$ (so that $h_1 = h_1^*$ a.e.). In particular,

$$\int_{A_1} h \,\mathrm{d}\mu = \int_{A_1} h^* \,\mathrm{d}\mu \Rightarrow \int_{A_1} (h - h^*) \,\mathrm{d}\mu = 0$$

so $\mu(A_1) = 0$. We can argue similarly with A_2, so $h_1 = h_1^*$. In the exact same way, $h_2 = h_2^*$ a.e., showing uniqueness.

Now, for the hard part of the theorem, we show existence of λ_a, λ_s, μ. Write $\lambda = \lambda_1^+ - \lambda_1^- + i(\lambda_2^+ - \lambda_2^i)$ and argue separately for each $\lambda_i^\pm$. We thus assume without loss of generality that λ is a positive, finite measure. From the previous lemma, get $w \in L^1(\mu)$ such that $0 < w < 1$, and define $\phi(E) = \lambda(E) + \int_E w \,\mathrm{d}\mu$, so ϕ is a positive finite measure. We have $\int_X f \,\mathrm{d}\phi = \int_X f \,\mathrm{d}\lambda + \int_X f w \,\mathrm{d}\mu$; this holds for characteristic functions, and thus for simple functions, and finally for any non-negative measurable f. Let $f \in L^2(\phi)$, so

$$\begin{aligned}
\left|\int_X f \,\mathrm{d}\lambda\right| &\leq \int_X |f| \,\mathrm{d}\lambda \\
&\leq \int_X |f| \,\mathrm{d}\phi = \int_X 1 \cdot |f| \,\mathrm{d}\phi \\
&= \langle 1, |f| \rangle_{L^2(\phi)} \\
&\leq \sqrt{\int_X 1 \,\mathrm{d}\phi} \cdot \sqrt{\int_X |f|^2 \,\mathrm{d}\phi} \\
&< \infty
\end{aligned}$$

since $\phi(X) < \infty$ by finiteness of λ, μ and the second term since $f \in L^2(\phi)$. Let $T(f) = \int_X f \,\mathrm{d}\lambda$, so $T : L^2(\phi) \to \mathbb{C}$ is a bounded (and therefore continuous) linear functional. By the Riesz theorem for Hilbert spaces, there exists $\overline{g} \in L^2(\phi)$ so that $T(f) = \langle f, \overline{g} \rangle = \int_X f \cdot g \,\mathrm{d}\phi$. Thus

$$\int_X f \,\mathrm{d}\lambda = \int_X f g \,\mathrm{d}\phi \tag{1}$$

Thus substituting $f = \chi_E$, $\lambda(E) = \int_E g \,\mathrm{d}\phi$. Let's see that $0 \leq g \leq 1$ a.e.$[\phi]$. Define

$$A_1 = \{x \in X : g(x) < 0\}, \quad A_2 = \{x \in X : g(x) > 1\}$$

Then $0 \leq \lambda(A_1) = \int_{A_1} g \,\mathrm{d}\phi < 0$ if $\phi(A_1) > 0$, so $\phi(A_1) = 0$. Similarly, $\lambda(A_2) = \int_{A_2} g \,\mathrm{d}\phi > \phi(A_2) \geq \lambda(A_2)$, so $\phi(A_2) = 0$. We thus have by (1) and the definition of ϕ,

$$\int_X f \,\mathrm{d}\lambda = \int_X f g \,\mathrm{d}\lambda + \int_X f g w \,\mathrm{d}\mu \quad \Longrightarrow \quad \int_X f(1-g) \,\mathrm{d}\lambda = \int_X f g w \,\mathrm{d}\mu \tag{2}$$

Let $A = \{x \in X : 0 \leq g(x) < 1\}$, $B = \{x \in X : g(x) = 1\}$, so $A \cap B = \emptyset$ and $\mu(X \setminus (A \cup B)) = 0$. Finally, we can define

$$\lambda_a(E) = \lambda(E \cap A), \quad \lambda_s(E) = \lambda(E \cap B)$$

Let's prove that they have the desired properties. Substitute $f = \chi_B$ into (2), so $\chi_B(1-g) = 0$ everywhere. We thus have $\int_B w \,\mathrm{d}\mu = 0$ so $\mu(B) = 0$. Thus μ is concentrated on A and, by definition, λ_s is concentrated on B, so $\lambda_s \perp \mu$.

Now apply (2) with $f = (1 + g + g^2 + \cdots + g^n)\chi_E$ for any $E \in \mathcal{M}$ and every $n \in \mathbb{N}$. On the LHS of (2), we have

$$\begin{aligned}\lim_{n\to\infty}\int_E (1-g^{n+1})\,\mathrm{d}\lambda &= \lim_{n\to\infty}\int_{E\cap A}(1-g^{n+1})\,\mathrm{d}\lambda + \lim_{n\to\infty}\int_{E\cap B}(1-g^{n+1})\,\mathrm{d}\lambda\\ &= \int_{E\cap A}\lim_{n\to\infty}(1-g^{n+1})\,\mathrm{d}\lambda\\ &= \int_{E\cap A} 1\,\mathrm{d}\lambda\\ &= \lambda(E\cap A) = \lambda_a(E)\end{aligned}$$

by the monotone convergence theorem, since $1 - g^{n+1} = 0$ on B and $1 - g^{n+1} \to 1$ on A monotonically. On the RHS of (2), we have

$$\lim_{n\to\infty}\int_E g(1+g+\cdots+g^n)w\,\mathrm{d}\mu = \int_E \lim_{n\to\infty} g(1+g+\cdots+g^n)w\,\mathrm{d}\mu = \int_E h\,\mathrm{d}\mu$$

monotonically, where $h = \lim_{n\to\infty} g(1+g+\cdots+g^n)2$. Thus $\lambda_a(E) = \int_E h\,\mathrm{d}\lambda$ for all $E \in \mathcal{M}$. But then it follows directly that $\lambda_a \ll \mu$ and $h \in L^1(\mu)$, so $\lambda_a(X) < \infty$. ■

3.3.2 Applications of Radon-Nikodym

Thm. 3.3.9 *Let μ be a complex measure. Then there exists a measurable h such that $|h| = 1$ and $\mu(E) = \int_E h\,\mathrm{d}|\mu|$. This is called the polar decomposition of μ.*

Proof Note that $\mu \ll |\mu|$ since if $|\mu|(E) = 0$ then $|\mu(E)| = 0$. Thus get $h \in L^1(|\mu|)$ so that $\mu(E) = \int_E h\,\mathrm{d}|\mu|$. We will see that $|h| \le 1$ a.e. and $|h| \ge 1$ a.e. Let $A_r = \{x \in X : |h(x)| < r\}$. Let $\{E_j\}$ be a partition of A_r. Then

$$\sum_{j=1}^\infty |\mu(E_j)| = \sum_{j=1}^\infty \left|\int_{E_j} h\,\mathrm{d}|\mu|\right| \le \sum_{j=1}^\infty \int_{E_j}|h|\,\mathrm{d}|\mu| \le \sum_{j=1}^\infty r\cdot|\mu|(E_j) = r\cdot|\mu|(A_r)$$

for any partition, so taking the supremum over all partitions, $|\mu|(A_r) \le r\cdot|\mu|(A_r)$, so $|\mu|(A_r) = 0$ when $r < 1$.

Otherwise, set $S = \mathbb{C}\setminus\overline{B_1(0)}$. Let $B_r(z) \subset S$ for some z, and define $E = h^{-1}(B_r(z))$. We want to show that $|\mu|(E) = 0$. Assuming so, write S as a countable union of open balls $B_1, B_2, \ldots$ so that

$$\mu(\{x : |h(x)| > 1\}) = \mu(h^{-1}(S)) = \sum_{i=1}^\infty \mu(h^{-1}(B_i)) = 0$$

Thus let's show that $\mu(E) = 0$. Assume not; then, for any E, we have

$$\left|\frac{1}{|\mu|(E)}\cdot\int_E h\,\mathrm{d}|\mu|\right| = \frac{|\mu(E)|}{|\mu|(E)} \le 1$$

so that

$$\begin{aligned}\left|\frac{1}{|\mu|(E)}\int_E h\,\mathrm{d}|\mu| - z\right| &= \left|\frac{1}{|\mu|(E)}\cdot\int_E (h-z)\,\mathrm{d}|\mu|\right| \\ &\leq \frac{1}{|\mu|(E)}\int_E |h-z|\,\mathrm{d}|\mu| \\ &< \frac{r\cdot|\mu|(E)}{|\mu|(E)} = r\end{aligned}$$

a contradiction, since $B_r(z) \subset S$. ■

Thm. 3.3.10 *Let μ be a positive measure, and $g \in L^1(\mu)$. Let $\lambda(E) = \int_E g\,\mathrm{d}\mu$. Then $|\lambda|(E) = \int_E |g|\,\mathrm{d}\mu$.*

Proof By the previous theorem, there exists h so that $|h| = 1$ and $\lambda(E) = \int_E h\,\mathrm{d}\lambda$. Then

$$\int_E g\,\mathrm{d}\mu = \int_E h\,\mathrm{d}\lambda \quad\Longrightarrow\quad \int_X fg\,\mathrm{d}\mu = \int_X fh\,\mathrm{d}\lambda$$

by monotone convergence, for any measurable f. In particular, consider $f = \overline{h}\cdot\chi_E$. Then

$$\int_E \overline{h}g\,\mathrm{d}\mu = \int_E \mathrm{d}|\lambda| = |\lambda|(E)$$

and since the RHS is real and non-negative and E is an arbitrary set, $\overline{h}g$ is real and non-negative almost everywhere. Let $x \in X$ be such that $\overline{h}(x)\cdot g(x) \geq 0$, and let $g(x) = r\cdot e^{i\phi}$, $\overline{h} = e^{i\alpha}$. Then $\overline{h}(x)g(x) = re^{i(\alpha+\phi)} = r = |g|$ since it is non-negative and real, so $\overline{h}\cdot h = |h|$ a.e.$[\mu]$. ■

With these theorems, we are in place to prove the following decomposition theorem for measures. Recall that

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \mu^- = \frac{1}{2}(|mu| - \mu)$$

so $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$.

Thm. 3.3.11 (Hahn Decomposition) *Let μ be a signed measure. Then there exists $A, B \in \mathcal{M}$ so that $AB = X$, $A \cap B = \emptyset$, and $\mu^+(E) = \mu(E\cap A)$ and $\mu^-(E) = -\mu(E \cap B)$.*

Proof By the first theorem, get h so that $|h| = 1$ and $\mu(E) = \int_E h\,\mathrm{d}|\mu|$. Since h is real-valued, $h = \pm 1$. Let $A = h^{-1}(\{1\})$, $B = h^{-1}(\{-1\})$. Then

$$\frac{1}{2}(h(x)+1) = \begin{cases} h(x) & : x \in A \\ 0 & : x \in B\end{cases}$$

so that since $\mu^+ = \frac{1}{2}(|\mu| + \mu)$,

$$\mu^+(E) = \frac{1}{2}\int_E (1+h)\,\mathrm{d}|\mu| = \int_{E\cap A} h\,\mathrm{d}|\mu| = \mu(E\cap A)$$

The same approach works with μ^-. ■

Cor. 3.3.12 *If $\mu = \lambda_1 - \lambda_2$, where λ_1, λ_2 are positive measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.*

Proof Since $\mu(E) \leq \lambda_1(E)$ for all E,

$$\mu^+(E) = \mu(E\cap A) \leq \lambda_1(E\cap A) \leq \lambda_1(E)$$

and similarly for μ^-. ■

3.4 Representation Theorem for Bounded Linear Functionals

Let B be a Banach space (a complete, normed vector space).

Def'n. 3.4.1 *$L : B \to \mathbb{C}$ is called a* ***bounded linear functional*** *if $L(\alpha x + y) = \alpha L(x) + L(y)$ for $\alpha \in \mathbb{C}$, $x, y \in B$ and*

$$\|L\| := \sup_{\|x\| \leq 1} |L(x)| < \infty$$

Then $|L(x)| \leq \|L\| \cdot \|x\|$. Let B^* denote the collection of bounded linear functionals, called the ***dual space*** *of B. This is a normed vector space with pointwise addition and scalar multiplication.*

Prop. 3.4.2 *Let B be any normed space. Then B^* is a Banach space.*

A general question in functional analysis is the characterization of the dual soace of a banach space B. When B is a Hilbert space, the inner product provides this representation via the standard Riesz representation theorem (see Hilbert space section in notes). If $B = L^p(\mu)$, it was shown that $B^* = L^q(\mu)$ where $1/q + 1/p = 1$.

To frame our general approach, note that $C_c(X)$ is not complete (for non-compact C). For example, when $X = \mathbb{R}$, $1/(1 + x^2)$ is not in $C_c(\mathbb{R})$, but it is the limit of a sequence of functions in $C_c(X)$.