

# **Course Notes**

## **Real Functions and Measures**

*Alex Rutar*

BSM Fall 2018

# Contents

<b>1</b>	<b>Basics of Abstract Measure Theory</b>	<b>3</b>
1.1	Review of Topology . . . . .	3
1.1.1	Basic Definitions . . . . .	3
1.1.2	Examples of Topological Spaces . . . . .	3
1.1.3	Other Definitions . . . . .	4
1.1.4	Functions and Continuity . . . . .	5
1.2	Measure Theory . . . . .	6
1.2.1	$\sigma$ -algebras . . . . .	6
1.2.2	Sequences of Measurable Functions . . . . .	9
1.2.3	Measures . . . . .	9
1.3	Towards Integration . . . . .	11
1.3.1	Simple Functions . . . . .	11
1.3.2	Integration of Positive Functions . . . . .	12
1.3.3	Lebesgue's Monotone Convergence Theorem . . . . .	14
1.4	Integration of Complex Valued Functions . . . . .	17
1.4.1	Basic Properties . . . . .	17
1.4.2	More Dominated Convergence . . . . .	18
<b>2</b>	<b>The Lebesgue measure</b>	<b>21</b>
2.1	The Vector Space $L^1(\mu)$ . . . . .	21
2.1.1	Almost Everywhere . . . . .	21
2.1.2	$L^1(\mu)$ as a normed space . . . . .	22
2.1.3	Construction of the Lebesgue measure . . . . .	23
2.2	The Riesz Representation Theorem . . . . .	23
2.3	Regularity Properties of Borel Measures . . . . .	29
2.4	Construction of the Lebesgue Measure . . . . .	30



# Chapter 1

## Basics of Abstract Measure Theory

Prof contact: simonp@caesar.elte.hu Grading: HW each week for 25% Midterm 30% Final 45%

### 1.1 Review of Topology

#### 1.1.1 Basic Definitions

**Def'n. 1.1.1** Let  $X \neq \emptyset$  and  $\tau \subseteq \mathcal{P}(X)$ . We say that  $(X, \tau)$  is a **topological space** if  $\tau$  satisfies the following conditions:

1.  $\emptyset \in \tau$   $X \in \tau$
2.  $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3.  $V_\alpha \in \tau$  for all  $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

We call the elements of  $\tau$  **open sets**.

**Def'n. 1.1.2**  $U \subseteq X$  is a **neighbourhood** of  $x \in X$  if there is some  $G \in \tau$  such that  $x \in G \subset U$ .

**Def'n. 1.1.3**  $F \subseteq X$  is **closed** if  $F^c$  is open.

**Def'n. 1.1.4** The **closure** of a set  $E \subset X$  is the smallest closed set containing  $E$  (denoted  $\bar{E}$ ).

**Def'n. 1.1.5**  $x$  is an **accumulation point** of  $H$  if all neighbourhoods of  $x$  contains infinitely points of  $H$ . Equivalently,  $x$  is a **limit point** of  $H \setminus \{x\}$ .

**Def'n. 1.1.6** If  $H \subseteq X$ , we have a natural subspace topology  $\tau|_H = \{G \cap H : G \in \tau\}$ .

#### 1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1.  $\mathbb{R}$  along with the open sets (denoted  $\tau_e$ , the Euclidean topology).
2. The discrete topology,  $\tau = \mathcal{P}(X)$  for any  $X \neq \emptyset$ . This is the “finest” topology.

3. The antidiscrete topology,  $\tau = \{\emptyset, X\}$  for any  $X \neq \emptyset$ . This is the “coarsest” topology.
4. One can define the extended real line,  $X = \mathbb{R} \cup \{-\infty, +\infty\}$ . Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

The same can be done with a single symbol as well. In either case, the extended real line is a compact set.

5. Any metric spaces induces a topology. Consider a set  $X \neq \emptyset$  arbitrary, and let  $d : X \times X \rightarrow \mathbb{R}$  such that

- (a)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0 \Leftrightarrow x = y$ .
- (b)  $d(x, y) = d(y, x)$  for all  $x, y \in X$
- (c)  $d(x, y) \leq d(x, z) + d(z, y)$  for any  $x, y, z \in X$

Then  $G \in \tau$  if and only if for any  $x \in G$ , there exists  $r$  so that  $B_r(x) \subset G$ . There are many examples of metric spaces:

- (a)  $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$
- (b)  $X = \mathbb{R}$ ,  $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
- (c)  $X = \mathbb{R}^2$ ,  $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- (d)  $X = \mathbb{R}^2$ ,  $d(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$  for  $p \geq 1$ .
- (e) and similarly for  $X = \mathbb{R}^n$
- (f)  $X = C[0, 1]$ ,  $d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$ .
- (g) normed space:  $X$  is a vector space over  $\mathbb{R}$ ,  $\|\cdot\| : X \rightarrow \mathbb{R}$  such that
  - i.  $\|x\| = 0$  if and only if  $x = 0$
  - ii.  $\|cx\| = |c| \|x\|$
  - iii.  $\|x + y\| \leq \|x\| + \|y\|$

If  $\|\cdot\|$  is a norm, then  $d(x, y) = \|x - y\|$  is a metric.

6. The cofinite topology:  $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$ .

### 1.1.3 Other Definitions

**Def’n. 1.1.7**  $K \subset X$  is **compact** if every open cover of  $K$  contains a finite subcover.

**Def’n. 1.1.8** A topological space is called **locally compact** if every point has a compact neighbourhood.

**Prop. 1.1.9**  $C[0, 1]$  with the sup norm is not locally compact.

PROOF I’ll do this later.

□

**Def'n. 1.1.10** A topological space is called **Hausdorff** if for any  $x \neq y$ , there exists neighbourhoods  $U \ni x$ ,  $V \ni y$  so that  $U \cap V = \emptyset$ .

The anti-discrete topology is not Hausdorff.

1. On the discrete topology,  $K$  is compact if and only if  $K$  is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of  $X$ ).
3. On  $(\mathbb{R}, \tau_e)$ ,  $K$  is compact if and only if  $K$  is closed and bounded.
4. On  $(X, d)$  metric space,  $K$  is compact if and only if  $K$  is complete and totally bounded.

**Prop. 1.1.11** 1. Let  $K \subset X$  be compact, let  $F \subset K$  closed. Then  $F$  is also compact.  
2. Compact sets in a Hausdorff space are closed.

**PROOF** 1. Let  $F \subset \bigcup V_\alpha$ . Then  $K \subset F^c \cup (\bigcup V_\alpha)$  is an open cover for  $K$ , so it has a finite subcover  $F^c \cup V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ . But then since  $F \cap F^c = \emptyset$ ,  $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$  is a finite subcover.  
2. Let  $K \subset X$  be compact, and prove that  $K^c$  is open. Thus let  $x \in K^c$ . For any  $y \in K$ , there exist  $U_y, V_y$  disjoint neighbourhoods of  $x$  and  $y$  respectively. Now consider the open cover  $K \subset \bigcup_{y \in K} V_y$ , and get our finite subcover  $K \subset V_{y_1} \cup \dots \cup V_{y_n}$ . But then  $U_{y_1} \cap \dots \cap U_{y_n} \cap K = \emptyset$  and is open since it is a finite intersection.  $\square$

**Def'n. 1.1.12**  $\Gamma \subseteq \tau$  is a **base** for  $\tau$  if every  $U \in \tau$  can be written as a countable union of the elements of  $\Gamma$ .  $\Gamma$  is a **countable base** if  $\Gamma$  is countable.

**Prop. 1.1.13**  $\mathbb{R}$  has a countable base of intervals.

**PROOF** Consider the collection  $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$ . To see this, for any open set  $U$ , one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left( \bigcup_{\{r: B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$  is obvious, so let  $x \in U$  be arbitrary, and let  $s$  be maximal so that  $B_s(x) \subseteq U$ . Then choose  $q \in \mathbb{Q}$  so that  $|x - q| < s/3$  and  $r \in \mathbb{Q}$  so that  $0 < r < s/2$ . Then by construction  $B_r(q) \ni x$  and by the triangle inequality  $B_{r/2}(q) \subseteq U$ , so  $x \in S$ . Thus  $U = S$  as desired.  $\square$

Note that the exact same argument (with some work) can be generalized to show that  $\mathbb{R}^n$  has a countable base of open hyperrectangles.

**Prop. 1.1.14** Every metric space which is a countable union of compact sets has a countable base.

**PROOF** See my PMATH 351 notes.  $\square$

## 1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

**Def'n. 1.1.15** Let  $(x_n) \subset X$  be a sequence and let  $x \in X$ . Then  $x$  is the **limit** of  $(x_n)$  if for any neighbourhood  $U$  of  $x$ , there exists  $N \in \mathbb{N}$  such that  $n > N \Rightarrow x_n \in U$ .

**Prop. 1.1.16** If  $F \subset X$  is closed, then for all convergent sequences in  $F$ , the limit is also in  $F$ .

PROOF See Homework. □

**Def'n. 1.1.17** Let  $f : X \rightarrow Y$  be a function, and  $x \in X$  an accumulation point of  $D(f)$ . The limit of  $f$  at  $x$  is  $y \in Y$  if for any neighbourhood  $V$  of  $y$  there exists a neighbourhood  $U$  of  $x$  such that  $f(U \cap D(f) \setminus \{x\}) \subseteq V$ .

**Def'n. 1.1.18** Let  $f : X \rightarrow Y$  be a function, and let  $x \in D(f)$ . Then  $f$  is **continuous at  $x$**  if for any neighbourhood  $V$  of  $f(x)$ , then  $f^{-1}(V)$  is a neighbourhood of  $x$ .

**Def'n. 1.1.19**  $f : X \rightarrow Y$  is called **continuous** if it is continuous at every point.

**Prop. 1.1.20**  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(G)$  is open for all  $G$  open.

PROOF Exercise. □

**Thm. 1.1.21** Let  $f : X \rightarrow Y$  be continuous and  $K \subset X$  be compact. Then  $f(K)$  is compact.

PROOF Recall that continuous functions pull back open sets. Let  $f(K) \subset \bigcup U_\alpha$  be an open cover. Then  $\bigcup f^{-1}(U_\alpha)$  is an open cover for  $K$ , and has a finite subcover  $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ . But then  $f(f^{-1}(U_{\alpha_1})) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$  is a subcover of  $f(K)$ . □

## 1.2 Measure Theory

### 1.2.1 $\sigma$ -algebras

**Def'n. 1.2.1** Let  $X \neq \emptyset$  be a set.  $\mathcal{M} \subset \mathcal{P}(X)$  is called a  **$\sigma$ -algebra** if

1.  $X \in \mathcal{M}$
2.  $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
3. If  $A_n \in \mathcal{M}$  for all  $n \in \mathbb{N}$ , then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$

The pair  $(X, \mathcal{M})$  is called a **measurable space**. The elements of  $\mathcal{M}$  are called **measurable sets**.

**Def'n. 1.2.2** Let  $(X, \mathcal{M})$  be a measurable space,  $(Y, \tau)$  be a topological space. Then  $f : X \rightarrow Y$  is called **measurable** if  $f^{-1}(V) \in \mathcal{M}$  for all  $V \in \tau$ .

Here are some simple examples of  $\sigma$ -algebras.

**Ex. 1.2.3** 1.  $\mathcal{M} = \{\emptyset, X\}$  is a  $\sigma$ -algebra.

2.  $\mathcal{P}(X) = \mathcal{M}$  is a  $\sigma$ -algebra.

3.  $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable}\}$ . To see this, given  $A_n \in \mathcal{M}$ , if everything is countable, then  $\bigcup A_n$  is countable. If some  $A_i$  is countable, then  $(\bigcup A_n)^c = \bigcap A_n^c$  is countable, so  $\bigcup A_n \in \mathcal{M}$ .

We will later see some proper examples, like the  $\sigma$ -algebra of Lebesgue measurable sets.

We have the following properties of  $\sigma$ -algebras.

**Prop. 1.2.4** 1.  $\emptyset \in \mathcal{M}$

2.  $A_1, A_2, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{M}$
3.  $A_n \in \mathcal{M}$  for all  $n \in \mathbb{N}$  then  $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$
4.  $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$
5.  $f$  is measurable,  $H \subset Y$  is closed, then  $f^{-1}(H) \in \mathcal{M}$ .

**PROOF** 1.  $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$ .

2. We can extend this to a countable union by introduction  $A_{n+i} = \emptyset$  for  $i \in \mathbb{N}$ .
3. By DeMorgan's identities,  $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$ .
4.  $A \setminus B = A \cap B^c \in \mathcal{M}$ .
5.  $H^c$  is open implies  $f^{-1}(H^c) \in \mathcal{M}$ . Then  $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$ . □

**Prop. 1.2.5** Let  $f : X \rightarrow Y$  be measurable, let  $g : Y \rightarrow Z$  be continuous, then  $g \circ f : X \rightarrow Z$  is measurable.

**PROOF** Let  $V \subset Z$  be open, so  $g^{-1}(V) \subset Y$  is open, so  $f^{-1}(g^{-1}(V)) \in \mathcal{M}$  which is  $(g \circ f)^{-1}(V)$ . □

**Prop. 1.2.6** Let  $(X, \mathcal{M})$  be a measurable space,  $Y$  be a topological space. Let  $\phi : \mathbb{R}^2 \rightarrow Y$  be continuous. If  $u, v : X \rightarrow \mathbb{R}$  are measurable, then  $h(x) = \phi(u(x), v(x))$  is measurable.

**PROOF** Define  $f : X \rightarrow \mathbb{R}^2$  by  $f(x) = (u(x), v(x))$ . We will see that  $f$  is measurable, so that  $h = \phi \circ f$  is measurable since  $\phi$  is continuous. Let  $I_1, I_2 \subset \mathbb{R}$  be open intervals, so  $R = I_1 \times I_2$  is an open rectangle. Then  $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$ . Let  $G \subset \mathbb{R}^2$  be an open set, so there exist  $R_n$  open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that  $f$  is measurable. □

**Cor. 1.2.7** 1. If  $u, v : X \rightarrow \mathbb{R}$  are measurable, then  $u + v$  and  $u \cdot v$  are measurable.

2.  $u + iv : X \rightarrow \mathbb{C}$  is measurable.
3.  $f : X \rightarrow \mathbb{C}$  is measurable,  $f = u + iv \Rightarrow u, v, |f|$  are measurable.

**Prop. 1.2.8** Define

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then  $\chi_E$  is measurable if and only if  $E \in \mathcal{M}$ .

**PROOF** Naturally,  $\chi_E^{-1}(1) = E$  and  $\chi_E^{-1}(0) = E^c$ , so  $\chi_E$  is measurable if and only if  $E, E^c \in \mathcal{M}$ . □

**Thm. 1.2.9** Let  $\mathcal{F} \subset \mathcal{P}(X)$ , then there exists a smallest  $\sigma$ -algebra containing  $\mathcal{F}$ . This is denoted by  $S(\mathcal{F})$ , the  $\sigma$ -algebra generated by  $\mathcal{F}$ .

**PROOF** Let  $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra, } \mathcal{F} \subset \mathcal{M}\}$ . Certainly  $\Omega \neq \emptyset$  since  $\mathcal{P}(X) \in \Omega$ . Let  $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$ . We will see that  $S(\mathcal{F})$  is a  $\sigma$ -algebra.

- (i) Since  $X \in \mathcal{M}$ , it follows that  $X \in \bigcap \mathcal{M}$ .



(ii) If  $A \in S(\mathcal{F})$ , then  $A \in \mathcal{M}$  for all  $\mathcal{M}$ . Thus  $A^c \in \mathcal{M}$  for all  $\mathcal{M}$  and  $A^c \in \bigcap \mathcal{M}$ .

(iii) In the same way, if  $A_n \in S(\mathcal{F})$  for all  $n$ , then  $A_n \in \mathcal{M}$  for all  $n, \mathcal{M}$ . Thus  $\bigcup A_n \in \mathcal{M}$  for all  $\mathcal{M}$  so  $\bigcup A_n \in \bigcap \mathcal{M} = S(\mathcal{F})$ .

By definition,  $\mathcal{F} \subset \bigcap \mathcal{M}$ . Finally,  $S(\mathcal{F})$  is minimal, since if  $\mathcal{F} \subset \mathcal{N}$  is a  $\sigma$ -algebra, then  $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$ , so we are done.  $\square$

**Def'n. 1.2.10** Let  $(X, \tau)$  be a topological space. Then  $\mathcal{B} = S(\tau)$  is called the **Borel  $\sigma$ -algebra**. Borel sets are the elements of  $S(\tau)$ . A function  $f : X \rightarrow Y$  is Borel measurable if  $f^{-1}(G) \in \mathcal{B}$  for all  $G \subset Y$  open.

**Prop. 1.2.11** 1. If  $F \subset X$  is closed, then  $F \in \mathcal{B}$ .

2.  $G_n \subset X$  are open, then  $\bigcap_{n=1}^{\infty} G_n \in \mathcal{B}$ . These are called  $G_\delta$ -sets.

3.  $F_n \subset X$  are closed, then  $\bigcup_{n=1}^{\infty} F_n \in \mathcal{B}$ . These are called  $F_\sigma$ -sets.

**PROOF** These follow directly from the definition of a  $\sigma$ -algebra.  $\square$

**Ex. 1.2.12**  $X = \mathbb{R}, \tau_e$ , then  $\mathcal{B} = S(\tau_e)$ . Let  $\Gamma_0 = \{(a, b) : a < b\}$  be a family of open intervals. We see that  $S(\Gamma_0) = \mathcal{B}$ . Since  $\Gamma_0 \subset \tau$ ,  $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$ . Conversely, let  $G \in \tau$ , then we have open intervals  $G = \bigcup_{n=1}^{\infty} I_n$  so that  $G \in S(\Gamma_0)$ . Thus  $S(\tau) \subset S(\Gamma_0)$  and  $S(\Gamma_0) = \mathcal{B}$ .

**Ex. 1.2.13** Let  $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$ . I claim that  $S(\Gamma_\infty) = \mathcal{B}$ . Certainly  $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$ . Then  $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$ . Similarly,  $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n] \in S(\Gamma_\infty)$ . Thus  $(a, \infty) \cap (-\infty, b) = (a, b) \in S(\Gamma_0)$ , and using the previous example,  $\mathcal{B} = S(\Gamma_\infty)$ .

**Prop. 1.2.14** Let  $(X, \mathcal{M})$  be a measurable space, and let  $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$  with the euclidean topology. If  $f^{-1}((\alpha, \infty]) \in \mathcal{M}$  for any  $\alpha \in \mathbb{R}$ , then  $f$  is measurable.

**PROOF** Recall that  $f$  is measurable if its inverse image takes open sets to measurable sets.

We have  $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty]))^c \in \mathcal{M}$ . Similarly,

$$f^{-1}([-\infty, \alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty} [-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha, \beta)) = f^{-1}([-\infty, \beta) \cap (\alpha, \infty]) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for  $\tau_e$ . Thus if  $G \subset \overline{\mathbb{R}}$  is open, then there exists open intervals so that  $G = \bigcup_{n=1}^{\infty} I_n$  and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired.  $\square$

### 1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the Dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

**Def'n. 1.2.15** Let  $(a_n)_{n \in \mathbb{N}} \subset \overline{\mathbb{R}}$  be a sequence, and  $b_k = \sup\{a_k, a_{k+1}, \dots\}$ . Then  $\beta = \inf_{k \in \mathbb{N}} b_k$  is called the  $\limsup$  of  $(a_n)$ . We can similarly define  $c_k = \inf\{a_k, a_{k+1}, \dots\}$  and  $\liminf = \sup_{k \in \mathbb{N}} c_k$ .

**Def'n. 1.2.16** Let  $f_n : X \rightarrow \overline{\mathbb{R}}$  be a sequence of functions. Then  $(\sup f_n) : X \rightarrow \overline{\mathbb{R}}$ ,  $(\sup f_n)(x) = \sup f_n(x)$  for all  $x \in X$ . Similarly,  $(\inf f_n) : X \rightarrow \overline{\mathbb{R}}$ ,  $(\inf f_n)(x) = \inf f_n(x)$  for all  $x \in X$ . Then  $(\liminf f_n)(x) = \liminf f_n(x)$ . If  $\lim f_n(x)$  exists for all  $x$ , then we say  $(\lim f_n)(x) = \lim f_n(x)$ .

**Thm. 1.2.17** Let  $f_n : X \rightarrow \overline{\mathbb{R}}$  be measurable. Then  $\sup f_n$ ,  $\inf f_n$ ,  $\limsup f_n$ ,  $\liminf f_n$  are measurable.

**PROOF** Let  $g = \sup f_n$ . It is enough to prove that  $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$  for all  $\alpha$ . Let  $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$ . Let  $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$ . We show that  $H = \bigcup_{n=1}^{\infty} H_n$ .

First let  $x \in H$ , so  $\sup f_n(x) > \alpha$ . Thus get  $N$  so that  $f_N(x) > \alpha$ , so  $x \in H_N$  and  $x$  is in the union. The converse is obvious.

Thus  $g$  is measurable. In the exact same way,  $\inf f_n$  is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k \geq i} f_k$$

is measurable. □

**Cor. 1.2.18** If  $\lim f_n$  exists, then it is measurable.

**PROOF** If  $\lim f_n$  exists, then  $\lim f_n = \limsup f_n$ . □

**Cor. 1.2.19** If  $f, g$  are measurable, then  $\max\{f, g\}$ ,  $\min\{f, g\}$  are measurable.

**Cor. 1.2.20** Let  $f$  be a function. Then  $f_+ = \max\{f, 0\}$  and  $f_- = -\min\{f, 0\}$  (the positive and negative parts of  $f$ ) are measurable. Similarly,  $|f| = f_+ + f_-$  is measurable.

### 1.2.3 Measures

**Def'n. 1.2.21** Let  $(X, \mathcal{M})$  be a measurable space. A function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  is called a **(positive) measure** if it is countably additive and not constant  $+\infty$ . In other words,

$$1. \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \text{ if } A_i \cap A_j = \emptyset$$

$$2. \exists A \in \mathcal{M} \text{ so that } \mu(A) < \infty$$

$(X, \mathcal{M}, \mu)$  is called a **measure space**.

**Prop. 1.2.22** 1.  $\mu(\emptyset) = 0$

2. If  $A_i \cap A_j = \emptyset$  then  $\mu\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$

3.  $A \subset B$  implies  $\mu(A) \leq \mu(B)$

4.  $A_1 \subset A_2 \subset A_3 \cdots$  then  $\lim_{n \rightarrow \infty} \mu A_n = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

5.  $A_1 \supset A_2 \supset A_3 \cdots$  and  $\mu(A_i) < \infty$  then  $\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

**PROOF** 1. Let  $A \in \mathcal{M}$  so that  $\mu(A) < \infty$ , and fix  $A_1 = A$ ,  $A_2 = A_3 = \cdots = \emptyset$ . Then  $\bigcup A_n = A$  so  $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$  so  $\mu(\emptyset) = 0$ .

2. Obvious

3. Note that  $B = A \cup (B \setminus A)$  is a disjoint union.

4. Define  $B_1 := A_1$  and  $B_i = A_i \setminus A_{i-1}$  for  $i \geq 2$ . Then  $B_i \cap B_j = \emptyset$  and  $\mu(A_n) = \mu\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mu(B_i)$ . Similarly,  $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ . Therefore,  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$ .

5. Let  $C_n = A_1 \setminus A_n$ ,  $C_1 = \emptyset$ . Then  $C_1 \subset C_2 \subset \cdots$  and  $\mu(C_n) + \mu(A_n) = \mu(A_1)$ . Let  $A = \bigcap_{n=1}^{\infty} A_n$  so  $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$  and  $(\bigcup C_n) \cup A = A_1$  is a disjoint union. But then  $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$  so that

$$\mu(A_1) - \mu(A) = \mu\left(\bigcup C_n\right) = \lim_{n \rightarrow \infty} \mu(C_n) = \mu(A_n) - \lim_{n \rightarrow \infty} \mu(A_n)$$

Since  $\mu(A_1)$  is finite, we have  $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$ . □

**Ex. 1.2.23** Here are a few examples of measures that exist on arbitrary sets.

1.  $X$  arbitrary,  $\mathcal{M} = \mathcal{P}(X)$ , and

$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$

It is easy to verify it is countably additive.

2.  $X$  arbitrary,  $\mathcal{M} = \mathcal{P}(X)$ . Fix  $x_0 \in X$ . Then

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

## 1.3 Towards Integration

### 1.3.1 Simple Functions

**Def'n. 1.3.1**  $s : X \rightarrow \mathbb{R}$  or  $\mathbb{C}$  is called a **simple function** if its range is finite.

**Prop. 1.3.2** Let  $s$  be a simple function, so that  $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . Then  $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$  where  $A_i = s^{-1}(\{\alpha_i\})$  and  $s$  is **measurable** if and only if  $A_i \in \mathcal{M}$ .

PROOF Obvious. □

The following theorem is used later to define the intergral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function  $f$ .

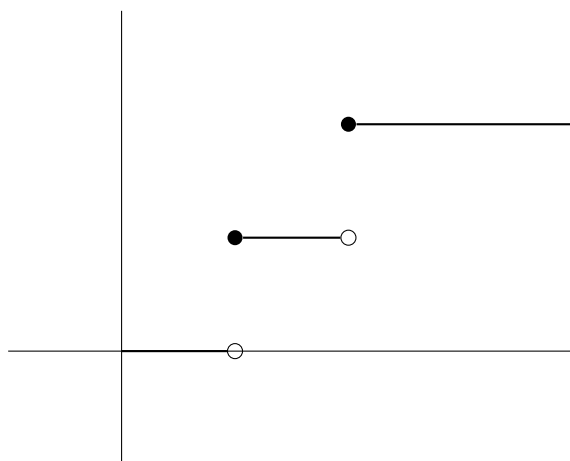
**Thm. 1.3.3** Let  $f : X \rightarrow [0, +\infty]$  be nonnegative measurable functions. Then there exists a sequence  $s_n : X \rightarrow [0, +\infty]$  of simple measurable functions with

1.  $(s_n)$  is increasing and bounded above by  $f$
2.  $\lim s_n = f$  pointwise.

PROOF Let  $n \in \mathbb{N}$ ,  $t \geq 0$ , and define  $k_n(t) = [2^n \cdot t]$  (i.e.  $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$ ). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn  $\phi_1$  below:



Then  $t - 2^{-n} \leq \phi_n(t) \leq t$ ,  $\lim \phi_n(t) = t$  uniformly, and  $\phi_n \leq \phi_{n+1}$ , so the sequence of functions is monotone. Define  $s_n = \phi_n \circ f$ , so for any  $x \in X$ ,  $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$ . Note that  $s_n$  is simple since it has finite range (from  $\phi_n$ ), and  $s_n \leq s_{n+1}$  because  $\phi_n \leq \phi_{n+1}$ , and  $s_n \leq f$  since  $\phi_n(t) \leq t$ . Furthermore,  $\phi_n$  is measurable since its level sets are intervals, so  $s_n = \phi_n \circ f$  is measurable. □

### 1.3.2 Integration of Positive Functions

**Def'n. 1.3.4** Let  $s : X \rightarrow [0, +\infty)$  be a measurable simple function  $s = \sum_{n=1}^N \alpha_i X_{A_i}$ . Let  $E \in \mathcal{M}$ . Then define the **integral** of  $s$  over  $E$  with respect to  $\mu$  as

$$\int_E s \, d\mu = \sum_{n=1}^N \alpha_i \mu(A_i \cap E)$$

where we define  $0 \cdot \infty = 0$ .

**Def'n. 1.3.5** Let  $f : X \rightarrow [0, +\infty]$  be a measurable function. Let  $E \in \mathcal{M}$ . Then the **(Lebesgue) integral** of  $f$  over  $E$  with respect to  $\mu$  is

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f; s \text{ is simple measurable} \right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only.

**Prop. 1.3.6** Let  $f, g : X \rightarrow [0, +\infty]$  be measurable functions. Let  $E, A, B \in \mathcal{M}$ .

1. If  $f \leq g$  then  $\int_E f \, d\mu \leq \int_E g \, d\mu$
2. If  $A \subset B$ , then  $\int_A f \, d\mu \leq \int_B f \, d\mu$
3.  $\int_E c \cdot f \, d\mu = c \cdot \int_E f \, d\mu$  for all  $c \geq 0$
4. If  $f(x) = 0$  for all  $x \in E$ , then  $\int_E f \, d\mu = 0$
5. If  $\mu(E) = 0$ , then  $\int_E f \, d\mu = 0$
6.  $\int_E f \, d\mu = \int_X f \cdot \chi_E \, d\mu$ .

**PROOF** 1. Note that

$$\left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\} \subset \left\{ \int_E s \, d\mu : 0 \leq s \leq g \right\}$$

2. Let  $0 \leq s \leq f$  be simple measurable. Then

$$\int_A s \, d\mu = \sum \alpha_i \mu(A \cap A_i) \leq \sum \alpha_i \mu(B \cap A_i) = \int_B s \, d\mu$$

Take the supremum for all  $0 \leq s \leq f$ , then the result follows.

3. Let  $S$  be simple and measurable, so  $s = \sum \alpha_i \chi_{A_i}$ . Then

$$\int_E c \cdot s \, d\mu = \sum_{i=1}^n \alpha_i \cdot c \cdot \mu(E \cap A_i) = c \cdot \sum \alpha_i \mu(E \cap A_i) = c \int_E s \, d\mu$$

Thus

$$\begin{aligned}\int_E c \cdot f \, d\mu &= \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq cf \right\} \\ &= \sup \left\{ \int_E c \cdot t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \sup \left\{ \int_E t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \int_E f \, d\mu\end{aligned}$$

4. If  $0 \leq s \leq f$ , then  $s = \sum \alpha_i \chi_{A_i}$ . If  $x \in A_i \cap E$ , then  $s(x) = \alpha_i$  and  $\alpha_i = 0$ . Then  $\alpha_i \mu(A_i \cap E) = 0$  for all  $i$ : either  $A_i \cap E = \emptyset$ , or  $A_i \cap E$  is not empty, and  $\alpha_i = 0$ . This is true for any  $0 \leq s \leq f$ , and taking supremums yields the result.
5. If  $\mu(E) = 0$  then  $\mu(A_i \cap E) = 0$ , and  $\int_E s \, d\mu = \sum \alpha_i \mu(A_i \cap E) = 0$  and taking supremums, the result holds.
6. Exercise. First prove if  $0 \leq s \leq f \cdot \chi_E$ , then  $\int_X s \, d\mu = \int_E s \, d\mu$ . Then prove

$$\left\{ \int_E s \, d\mu : 0 \leq s \leq f \cdot \chi_E \right\} = \left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\}$$

□

**Prop. 1.3.7** Let  $s$  be a simple and measurable. Then  $\phi(E) = \int_E s \, d\mu$  is a measure.

PROOF  $\phi(\emptyset) = 0$ , so  $\phi$  is not constant  $+\infty$ . Let  $E = \bigcup_{n=1}^{\infty} E_n$  be a disjoint union. Then

$$\begin{aligned}\phi(E) &= \sum_{i=1}^m \alpha_i \mu(A_i \cap E) \\ &= \sum_{i=1}^m \alpha_i \mu \left( A_i \cap \left( \bigcup_{n=1}^{\infty} E_n \right) \right) = \sum_{i=1}^m \alpha_i \mu \left( \bigcup_{n=1}^{\infty} (A_i \cap E_n) \right) \\ &= \sum_{i=1}^m \alpha_i \sum_{n=1}^{\infty} \mu(A_i \cap E_n) = \sum_{n=1}^{\infty} \sum_{i=1}^m \alpha_i \mu(A_i \cap E_n) \\ &= \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu = \sum_{n=1}^{\infty} \phi(E_n)\end{aligned}$$

□

**Prop. 1.3.8** Let  $s, t$  be nonnegative, measurable simple functions. Then

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

PROOF Write

$$s = \sum_{i=1}^m \alpha_i X_{A_i}, \quad t = \sum_{j=1}^n \beta_j X_{B_j}$$

and let  $E_{ij} = A_i \cap B_j$ , so  $X = \bigcup_{i,j} E_{ij}$  is a disjoint union. We now have

$$\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j) \mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu$$

Let  $\mu(E) = \int_E (s+t) d\mu$ , which is a measure as above. Thus

$$\begin{aligned} \int_X (s+t) d\mu &= \phi(X) = \phi\left(\bigcup_{i,j} E_{ij}\right) \\ &= \sum_{i,j} \phi(E_{ij}) = \sum_{i,j} \int_{E_{ij}} (s+t) d\mu \\ &= \sum_{i,j} \left( \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu \right) \\ &= \sum_{i,j} \varphi(E_{ij}) + \sum_{i,j} \theta(E_{ij}) \\ &= \int_X s d\mu + \int_X t d\mu \end{aligned}$$

where  $\varphi(E) = \int_E s d\mu$ ,  $\theta(X) = \int_X t d\mu$ . □

### 1.3.3 Lebesgue's Monotone Convergence Theorem

**Thm. 1.3.9 (Lebesgue's Monotone Convergence)** Let  $f_n : X \rightarrow [0, +\infty]$  be measurable, such that

(i)  $0 \leq f_1 \leq f_2 \leq \dots$

(ii)  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$  for all  $x \in X$

Then  $f$  is measurable, and  $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$ .

PROOF It was already proven that  $f$  is measurable. We have  $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$  for all  $n$ , so  $\alpha := \lim_{n \rightarrow \infty} \int_X f_n d\mu$  exists. We also have  $f_n \leq f$ , so  $\int f_n \leq \int f$  and  $\alpha \leq \int_X f_n d\mu$ . Thus we wish to show  $\alpha \geq \int_X f d\mu$ . It suffices to prove that  $\alpha \geq \int_X s d\mu$  for any simple  $s \leq f$ . Furthermore, if  $c \in (0, 1)$ , it suffices to show that  $\alpha \geq \int_X c \cdot s d\mu$ .

Define  $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\}$ . We have  $E_1 \subset E_2 \subset \dots$  so that  $\bigcup_{n=1}^{\infty} E_n = X$ . Then

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} c \cdot s d\mu$$

Let  $\phi(E) = \int_E s d\mu$ , so  $\int_{E_n} s d\mu = \phi(E_n)$ . Thus  $\lim_{n \rightarrow \infty} \phi(E_n) = \phi(X) = \int_X s d\mu$ . Thus

$$\alpha \geq c \cdot \lim_{n \rightarrow \infty} \phi(E_n) = c \cdot \int_X s d\mu = \int_X c \cdot s d\mu$$

as desired.  $\square$

**Ex. 1.3.10** Consider the function consisting of a triangle with base  $2/n$  and height  $n$ . Then  $\int_0^1 f_n = 1$  as a Riemannian integral. However,  $\lim f_n(x) = 0$  for any  $x$ , so  $\int_0^1 f = 0 \neq 1 = \lim \int_0^1 f_n$ .

**Thm. 1.3.11** Let  $f, g : X \rightarrow [0, +\infty]$  measurable, then  $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$ .

**PROOF** We proved that there exists increasing sequences of simple functions  $s_n, t_n$  such that  $\lim s_n(x) = f(x)$ ,  $\lim t_n(x) = g(x)$ . Then  $s_n(x) + t_n(x) \rightarrow f(x) + g(x)$  monotonically. But then

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X \lim_{n \rightarrow \infty} (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \int_X (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \left( \int_X s_n d\mu + \int_X t_n d\mu \right) \\ &= \int_X \lim_{n \rightarrow \infty} s_n d\mu + \int_X \lim_{n \rightarrow \infty} t_n d\mu \\ &= \int_X f d\mu + \int_X g d\mu \end{aligned} \quad \square$$

**Cor. 1.3.12** If  $f_n : X \rightarrow [0, +\infty]$  is a sequence of measurable functions, then

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=1}^{\infty} f_n d\mu$$

**Ex. 1.3.13** Let  $X = \mathbb{N}$ ,  $\mathcal{M} = \mathcal{P}(X)$ ,  $\mu(E)$  is the counting measure. Let  $a : X \rightarrow [0, \infty)$  be a function. This is a sequence. Every function is measurable. Let  $s_n(i) = a(i)$  for  $i \leq n$  and 0 otherwise, which is a simple function, and  $s_n \leq s_{n+1}$ . Then  $\lim_{n \rightarrow \infty} s_n(i) = a(i)$  so  $s_n \rightarrow a$  pointwise, so by LMC  $\int_X s_n d\mu = \int_X a d\mu$ . Also,

$$\int_X s_n d\mu = \sum_{i=1}^n a(i) \mu(\{i\}) = \sum_{i=1}^n a(i)$$

$$\text{so } \int_X a d\mu = \sum_{n=1}^{\infty} a(n).$$

**Lemma 1.3.14 (Fatou)** Let  $f_n : X \rightarrow [0, \infty)$  be a sequence of measurable functions. Then

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu$$



**PROOF** Let  $g_k = \inf\{f_k, f_{k+1}, \dots\}$  so  $\liminf f_n = \lim_{n \rightarrow \infty} g_n$  and  $g_n$  is increasing. Note that  $g_k \leq f_k$  for any  $k$ , so  $\int_X g_k d\mu \leq \int_X f_k d\mu$ . Thus

$$\begin{aligned} \int_X \liminf f_n d\mu &= \int_X \lim g_n d\mu \\ &= \lim \int_X g_n d\mu \\ &= \liminf \int_X g_n d\mu \\ &\leq \liminf \int_X f_n d\mu \end{aligned} \quad \square$$

**Ex. 1.3.15** It is possible for the inequality to be strict. Define  $f_{2n} = \chi_{[0,1]}$  and  $f_{2n+1} = \chi_{[1,2]}$ . Thus  $\liminf f_n(x) = 0$  so  $\int_{[0,2]} \liminf f_n d\mu = 0$  but  $\inf_{[0,2]} \int_{[0,2]} f_n d\mu = 1$

**Thm. 1.3.16** Let  $f : X \rightarrow [0, \infty]$  be measurable. Let  $\phi(E) = \int_E f d\mu$ ,  $E \in \mathcal{M}$ . Then  $\phi$  is a measure and  $\int_X g d\phi = \int_X g \cdot f d\mu$ .

**PROOF** Certainly  $\phi(\emptyset) = 0$ , so  $\phi \neq +\infty$ . Thus let  $E = \bigcup_{i=1}^{\infty} E_i$  be a disjoint union. Then  $\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f$ . Thus we have

$$\begin{aligned} \phi(E) &= \int_E f d\mu \\ &= \int_X \chi_E f d\mu \\ &= \int_X \sum_{i=1}^{\infty} \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_X \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \phi(E_i) \end{aligned}$$

Now, we prove that  $\int_X g d\mu = \int_X g f d\mu$ .

First, we do this for  $g = \chi_E$ . Then  $\int_X \chi_E d\mu = \phi(E)$  on the left, and  $\int_X \chi_E f d\mu = \int_E f d\mu = \phi(E)$  and equality holds.

Now, let  $g = \sum_{i=1}^n \alpha_i \chi_{A_i}$  be a simple function. Then  $\int_X \sum \alpha_i \chi_{A_i} d\phi = \sum \alpha_i \int_X \chi_{A_i} d\phi$  on the left and  $\int_X \sum \alpha_i \chi_{A_i} f d\mu = \sum \alpha_i \int_X \chi_{A_i} f d\mu$ .

Finally, let  $g$  be an arbitrary measurable function, and let  $(s_n) \rightarrow g$  be an increasing sequence of simple functions. Note that  $s_n f \rightarrow g f$ . Thus

$$\begin{aligned}\int_X g \, d\phi &= \int_X \lim s_n \, d\phi = \lim \int_X s_n \, d\phi \\ &= \lim \int_X s_n f \, d\mu = \int_X \lim(s_n f) \, d\mu \\ &= \int_X g \cdot f \, d\mu\end{aligned}$$

as desired. □

## 1.4 Integration of Complex Valued Functions

**Def'n. 1.4.1** A function  $f : X \rightarrow \mathbb{C}$  is called *Lebesgue integrable* if  $\int_X |f| \, d\mu < \infty$ . The collection of such functions is  $L^1(\mu)$ .

### 1.4.1 Basic Properties

**Def'n. 1.4.2** Let  $f \in L^1(\mu)$ . Then  $f = u + iv$  and denote  $u = \operatorname{Re} f$ ,  $v = \operatorname{Im} f$ . Let  $E \in \mathcal{M}$ ; then the integral of  $f$  over  $E$  with respect to  $\mu$  is

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \left( \int_E v^+ \, d\mu - \int_E v^- \, d\mu \right)$$

**Thm. 1.4.3** Let  $f, g \in L^1(\mu)$ ,  $\alpha, \beta \in \mathbb{C}$ , so  $\alpha f + \beta g \in L^1(\mu)$  and

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

**PROOF** Note that  $\alpha f + \beta g$  is measurable, so  $\int_X |\alpha f + \beta g| \, d\mu \leq |\alpha| \int_X |f| \, d\mu + |\beta| \int_X |g| \, d\mu < \infty$ . For real measurable functions,  $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$  directly by expanding the definition and using additivity over positive functions. We thus show  $\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu$ . If  $\alpha \geq 0$ , then

$$\begin{aligned}\int_X \alpha f \, d\mu &= \int_X \alpha(u + iv) \, d\mu = \int_X (\alpha u^+ - \alpha u^- + i\alpha v^+ - i\alpha v^-) \, d\mu \\ &= \int_X ((\alpha u)^+ - (\alpha u)^- + (i\alpha v)^+ - (i\alpha v)^-) \, d\mu \\ &= \int_X (\alpha u)^+ \, d\mu - \int_X (\alpha u)^- \, d\mu + \int_X i(\alpha v)^+ \, d\mu - \int_X i(\alpha v)^- \, d\mu \\ &= \alpha \int_X u^+ \, d\mu - \alpha \int_X u^- \, d\mu + \alpha \int_X iv^+ \, d\mu - \alpha \int_X iv^- \, d\mu \\ &= \alpha \int_X f \, d\mu\end{aligned}$$

and similarly for  $\alpha = -1$ ,  $\alpha = i$ . □

**Thm. 1.4.4** Let  $f \in L^1(\mu)$ . Then  $\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$ .

**PROOF** Let  $z = \int_X f \, d\mu$ . Let  $\alpha = \frac{|z|}{z}$  if  $z \neq 0$ , and  $\alpha = 1$  otherwise. Then  $\alpha \int_X f \, d\mu = |z|$ . Let  $u = \operatorname{Re}(\alpha \cdot f) \leq |\alpha \cdot f| \leq |f|$  since  $|\alpha| = 1$ . Thus

$$\begin{aligned} \left| \int_X f \, d\mu \right| &= \alpha \cdot \int_X f \, d\mu \\ &= \int_X \alpha f \, d\mu \\ &= \int_X \operatorname{Re}(\alpha f) \, d\mu \\ &\leq \int_X |f| \, d\mu \end{aligned} \quad \square$$

## 1.4.2 More Dominated Convergence

Naturally, we want similar results as we have before. Indeed, we have the following theorem:

**Thm. 1.4.5 (Lebesgue's Dominated Convergence)** Let  $f_n : X \rightarrow \mathbb{C}$  be measurable functions such that  $f = \lim f_n$ . Assume that there is some  $g \in L^1(\mu)$  such that  $|f_n| \leq g$  for all  $n$ . Then  $f \in L^1(\mu)$  and  $\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$ .

**PROOF** We certainly know that  $f$  is measurable, and  $|f| \leq g$ , so  $f \in L^1(\mu)$ . As well, the triangle inequality show that  $|f - f_n| \leq 2g$  for any  $n$ . We will see that  $0 \leq \liminf \int_X |f - f_n| \, d\mu \leq \limsup \int_X |f - f_n| \, d\mu \leq 0$ . Assuming that this holds, then  $\lim \int_X |f - f_n| \, d\mu = 0$  and

$$0 \leq \lim \left| \int_X f \, d\mu - \int_X f_n \, d\mu \right| \leq \int_X |f - f_n| \, d\mu = 0$$

The first two inequalities are obvious: we must show that  $\limsup \int_X |f_n| \, d\mu \leq 0$ . Firstly, we have

$$\begin{aligned} \int_X 2g \, d\mu &= \int_X \left( 2g - \lim_{n \rightarrow \infty} |f - f_n| \right) d\mu \\ &= \int_X \liminf (2g - |f - f_n|) \, d\mu \\ &\leq \lim \int_X (2g - |f - f_n|) \, d\mu && \text{By Fatou's Lemma} \\ &= \int_X 2g + \liminf \left( - \int_X |f - f_n| \, d\mu \right) \\ &= \int_X 2g - \limsup \int_X |f - f_n| \, d\mu \end{aligned}$$

and since  $\int_X 2g \, d\mu$  is finite, we subtract and  $\limsup \int_X |f - f_n| \, d\mu \leq 0$ .  $\square$

**Ex. 1.4.6** Consider  $\lim_{n \rightarrow \infty} \int_0^n e^{-nx} dx$ . Define

$$f_n(x) = \begin{cases} e^{-nx} & \text{if } x \leq n \\ 0 & \text{if } x > n \end{cases}$$

Note that  $f_n(x) \leq g(x) = e^{-x}$  and  $\int_0^\infty e^{-x} dx < \infty$ . Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^n e^{-nx} dx &= \int_{[0, \infty)} \lim_{n \rightarrow \infty} f_n(x) dx \\ &= \int_{[0, \infty)} \chi_{\{0\}} dx \\ &= 0 \end{aligned}$$

**Rmk. 1.4.7** For the Riemann integral, we have  $\int \lim f_n = \lim \int f_n$  as long as the convergence of  $f_n$  is uniform.



# Chapter 2

## The Lebesgue measure

### 2.1 The Vector Space $L^1(\mu)$

#### 2.1.1 Almost Everywhere

Let  $(X, \mathcal{M}, \mu)$  be a measure space.

**Def'n. 2.1.1** Let  $E \in \mathcal{M}$ . We say that property  $P$  holds almost everywhere in  $E$  if there exists  $N \in \mathcal{M}$  such that  $\mu(N) = 0$ ,  $N \subset E$ , and  $P$  holds in  $E \setminus N$ .

**Ex. 2.1.2** Two functions  $f, g : X \rightarrow \mathbb{C}$  are equal almost everywhere if  $\exists N \subset X$  such that  $\mu(N) = 0$  and  $f(x) = g(x)$  on  $X \setminus N$ .

**Prop. 2.1.3** Let  $E \subset X$  be such that  $A_1, A_2, B_1, B_2 \in \mathcal{M}$  for which  $\int_X f d\mu = \int_X g d\mu$ . Then  $A_1 \subset E \subset B_1$ ,  $A_2 \subset E \subset B_2$ , and  $\mu(B_1 \setminus A_1) = 0$  and  $\mu(B_2 \setminus A_2) = 0$ . Then  $\mu(A_1) = \mu(A_2)$ .

**PROOF** Note that  $A_1 \setminus A_2 \subset E \setminus A_2 \subset B_2 \setminus A_2$ . As well,  $\mu(A_1 \setminus A_2) \leq \mu(B_2 \setminus A_2) = 0$ . Then

$$\begin{aligned}\mu(A_1) &= \mu(A_1 \cap A_2^c) + \mu(A_1 \cap A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2) \\ \mu(A_2) &= \mu(A_2 \cap A_1^c) + \mu(A_2 \cap A_1) = \mu(A_2 \setminus A_1) + \mu(A_2 \cap A_1) = \mu(A_1 \cap A_2)\end{aligned}\quad \square$$

**Prop. 2.1.4** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let

$$\mathcal{M}^* = \{E \subset X : \exists A, B \in \mathcal{M}, A \subset E \subset B, \mu(B \setminus A) = 0\}$$

Then  $\mathcal{M}^*$  is a  $\sigma$ -algebra, and  $\mu^* : \mathcal{M}^* \rightarrow [0, +\infty]$  defined by  $\mu^*(E) = \mu(A)$ .

**PROOF** We show that  $\mathcal{M}^*$  is a  $\sigma$ -algebra, and  $\mu$  is countably additive.

1.  $X \in \mathcal{M}$  so  $X \in \mathcal{M}^*$ .
2. If  $E \in \mathcal{M}^*$ , get  $A \subset E \subset B$  so  $B^c \subset E^c \subset A^c$ ,  $A^c, B^c \in \mathcal{M}$ . As well,  $\mu(A^c \setminus B^c) = \mu(A^c \cap B) = \mu(B \setminus A) = 0$ , so  $E^c \in \mathcal{M}^*$ .
3. If  $E_i \in \mathcal{M}^*$  is a countable collection, then get  $A_i \subset E_i \subset B_i$ . Fix  $A = \bigcup A_i$  and  $B = \bigcup B_i$ . Then  $B \setminus A = \bigcup (B_i \setminus A) \subset \bigcup (B_i \setminus A_i)$  so  $\mu(B \setminus A) = 0$  and  $A \subset \bigcup E_i \subset B$  so  $\bigcup E_i \in \mathcal{M}^*$ .
4. Let  $E_i$  be disjoint,  $E = \bigcup E_i$ , and  $E_i \in \mathcal{M}^*$ . Get  $A_i \subset E_i \subset B_i$ . Then  $\mu^*(\bigcup E_i) = \mu(\bigcup A_i) = \sum \mu(A_i) = \sum \mu(E_i)$ .  $\square$

**Def'n. 2.1.5** We call the space  $(X, \mathcal{M}^*, \mu^*)$  the **completion** of  $(X, \mathcal{M}, \mu)$ .

In particular, every subset of a set with measure 0 is measurable.

## 2.1.2 $L^1(\mu)$ as a normed space

**Prop. 2.1.6** 1. Let  $f : X \rightarrow [0, +\infty)$  be measurable,  $E \in \mathcal{M}$ . If  $\int_E f \, d\mu = 0$ , then  $f = 0$  almost everywhere in  $E$ .

2. Let  $f \in L^1(\mu)$ . If  $\int_E f \, d\mu = 0$  for all  $E \in \mathcal{M}$ , then  $f = 0$  almost everywhere in  $X$ .

**PROOF** 1. Let  $A_n = \{x \in E : f(x) > 1/n\}$ , so that

$$\frac{1}{n}\mu(A_n) \leq \int_{A_n} f \, d\mu \leq \int_E f \, d\mu = 0 \implies \mu(A_n) = 0$$

for all  $n$ . But then

$$N = \{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n \implies \mu(N) \leq \sum \mu(A_n) = 0$$

2. Write  $f = u + iv$  so that

$$\int_E f \, d\mu = \int_E u^+ \, d\mu - \int_E u^- \, d\mu + i \int_E v^+ \, d\mu - i \int_E v^- \, d\mu$$

We show that  $u^+ = 0$  almost everywhere (the other terms are identical). Let  $E = \{x \in X : u(x) \geq 0\}$ , so  $\int_E f \, d\mu = 0$ , so its real part is zero and  $\int_E u^+ \, d\mu = 0$ . Thus  $u^+ = 0$  almost everywhere in  $E$ . The result follows.  $\square$

**Def'n. 2.1.7** A **normed space** over  $\mathbb{R}$  is a vector space  $V$  over  $\mathbb{R}$  with a map  $\|\cdot\| : V \rightarrow \mathbb{R}$  such that

(i)  $x \in V \implies \|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x = 0$ .

(ii)  $\|\lambda x\| \leq |\lambda| \|x\|$  for all  $\lambda \in \mathbb{R}$  and  $x \in V$

(iii)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in V$ .

Now  $L^1(\mu) = \{f : X \rightarrow \mathbb{C} \text{ measurable and } \int_X |f| \, d\mu < \infty\}$ . We certainly have that  $L^1(\mu)$  is a vector space. We wish to define  $\|f\| = \int_X |f| \, d\mu$ . The only problem is that

$$\int_X |f| \, d\mu = 0 \implies f = 0 \text{ almost everywhere}$$

To deal with this problem, we quotient our space by the equivalence relation  $f \sim g$  if and only if  $f = g$  almost everywhere. With this in mind, define  $V = L^1(\mu)/\sim$  denote the set of equivalence classes. We need to define  $+, \cdot, \|\cdot\|$  on  $V$ . Let  $[f]$  denote the class of  $f$ . Then

$$[f] + [g] = [f + g]$$

$$c[f] = [cf]$$

$$\|[f]\| = \int_X |f| \, d\mu$$

Let's verify that this is well defined: if  $f_1 \sim f_2$  and  $g_1 \sim g_2$ , then  $f_1 + g_1 \sim f_2 + g_2$ . Indeed, this is true since the sums are equal except perhaps on a union of measure zero sets, so equality holds almost everywhere. The second definition is obviously well defined. Finally, by a homework assignment,  $\|[f]\|$  is also well defined. Now, let's verify the properties of the norm.

- (i) Certainly  $\|f\| \geq 0$ , and  $\|f\| = 0$  implies  $f = 0$  almost everywhere, so  $[f] = [0] = 0$ .
- (ii) We have  $\|\lambda \cdot f\| = \int_X |\lambda f| d\mu = |\lambda| \int_X |f| d\mu = |\lambda| \|f\|$
- (iii) We have  $\|f + g\| = \int_X |f + g| d\mu \leq \int_X |f| + \int_X |g| = \|f\| + \|g\|$

In  $L^1(\mu)$ , two functions are the same if they are equal almost everywhere. However, this can be a challenge: if  $f \in L^1(\mu)$  and  $x_0 \in X$ , then  $f(x_0)$  is not well defined. For example, it is challenging to give meaning to boundary conditions of functions.

### 2.1.3 Construction of the Lebesgue measure

We begin from the Riemann integral  $\int_a^b f(x) dx$  for a continuous function  $f$ . Define  $\text{supp } f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$ . For continuous functions with compact (bounded) support, define  $\Lambda f = \int_{\mathbb{R}} f(x) dx$  is the Riemann integral, which is a functional. In particular,

$$\text{measure}((a, b)) = \text{length}((a, b)) = \sup\{\Lambda f : f \text{ is continuous, compact support, } 0 \leq f \leq 1, \text{supp } f \subset (a, b)\}$$

We will extend this to a  $\sigma$ -algebra containing the Borel sets. In order to define these, for open sets,  $\mu(G) = \sup\{\Lambda f : 0 \leq f \leq 1, \text{supp } f \subset G\}$ , where  $\Lambda$  is the Riemann integral. For an arbitrary set,  $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$ . However, this “measure” is not countably additive: the  $\sigma$ -algebra  $\mathcal{P}(X)$  is too large (Vitali’s construction). Instead, we will define  $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$ , which means that  $E \cap K$  is regular for any  $K$  compact, and regular means that the outer measure and inner measure are equal. The outer measure is  $\sup\{\mu(K) : K \subset E \text{ compact}\} = \mu(E)$ .

## 2.2 The Riesz Representation Theorem

In this section, we assume that  $(X, \tau)$  be a locally compact, Hausdorff topological space.

**Def’n. 2.2.1** We denote the space of continuous functions with compact support by  $C_c(X) = \{f : X \rightarrow \mathbb{C} \mid f \in C(X), \text{supp } f \text{ is compact}\}$ .

**Def’n. 2.2.2** Let  $\Lambda : C_c(X) \rightarrow \mathbb{C}$  be a **linear functional**, i.e.  $\Lambda(cf + g) = c\Lambda f + \Lambda g$ .  $\Lambda$  is called a **positive linear functional** if  $f \geq 0 \Rightarrow \Lambda f \geq 0$ .

**Def’n. 2.2.3** We say that  $K < f$  if  $K$  is compact and  $f \in C_c(X)$ ,  $0 \leq f \leq 1$  implies that  $x \in K \Rightarrow f(x) = 1$ . We say that  $f < G$  if  $G$  is open,  $f \in C_c(X)$ ,  $0 \leq f \leq 1$ , and  $\text{supp } f \subset G$ .

**Lemma 2.2.4 (Urysohn)** Let  $G \in \tau$ ,  $K \subset G$  compact. Then there exists  $f \in C_c(X)$  such that  $K < f < G$ .

PROOF Will do later. □

**Lemma 2.2.5 (Partition of Unity)** Let  $G_1, G_2, \dots, G_n \in \tau$ , and let  $K \subset G_1 \cup \dots \cup G_n$  be compact. Then there are functions  $h_i \in C_c(X)$  such that  $h_i < G_i$  and  $K < \sum h_i$ .

PROOF Also will do later. □



How can we create a positive linear functional on  $C_c(X)$ ? If  $\mu$  is a measure, and functions on  $C_c(X)$  are measurable, then  $\Lambda f = \int_X f d\mu$  is a positive linear functional. The representation theorem says that there are no other examples.

**Thm. 2.2.6 (Riesz Representation)** *Let  $(X, \tau)$  be as above. If  $\Lambda : C_c(X) \rightarrow \mathbb{C}$  is a positive linear functional, then there exists a unique measure space  $(X, \mathcal{M}, \mu)$  such that  $\Lambda f = \int_X f d\mu$  for any  $f \in C_c(X)$ ,  $\mathcal{M} \supset \tau$ , and*

- (i)  $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$  for all  $E \in \mathcal{M}$ .
- (ii)  $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$  for all  $E \in \mathcal{M}$  with  $\mu(E) < \infty$ .
- (iii)  $\mu(K) < \infty$  for any  $K$  compact.
- (iv)  $\mathcal{M}$  is complete.

First, let's get some definitions out of the way. Fix the notation as above.

**Def'n. 2.2.7** *Fix a Borel measure  $\mu$ . The **Lebesgue outer measure** is defined  $\mu(E) = \inf\{\mu(G) : E \subset G \text{ open}\}$ .*

**Def'n. 2.2.8** *We say that  $E \subset X$  is **regular** if  $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ . Similarly,  $E \subset X$  is **locally regular** if there exists a compact  $K$  so that  $K \cap E$  is regular.*

**PROOF** For an open set  $G \in \tau$ , let  $\mu(G) = \sup\{\Lambda f : f < G\}$ . Then  $\mu(\emptyset) = 0$  and  $G_1 \subset G_2$  implies that  $\mu(G_1) \leq \mu(G_2)$ . Then extend  $\mu$  to arbitrary  $E \subset X$  as an outer measure.

Now let  $\mathcal{M} = \{E \subset X : E \text{ is locally regular}\}$ . Let's first see that  $\mathcal{M}$  satisfies the desired properties. We first see that  $\mathcal{M}$  is complete. Let  $E \in \mathcal{M}$ ,  $\mu(E) = 0$  and  $A \subset E$ . We want to show that  $A \in \mathcal{M}$ . Let  $K$  be compact, and consider  $K \cap A$  so that  $\mu(K \cap A) = 0$ . Then if  $F \subset K \cap A$  is compact,  $\mu(F) = 0$  implies  $\sup\{\mu(F) : F \subset K \cap A \text{ compact}\} = 0$ .

**Claim 1:**  $\mu$  is  $\sigma$ -subadditive.

**PROOF** If  $\mu(E_j) = \infty$  for some  $j$ , then we are done. Thus assume  $\mu(E_j) < \infty$  for all  $j$ . Let  $\epsilon > 0$ ,  $\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$  be arbitrary. Let  $G_j \supset E_j$  be open, such that  $\mu(G_j) \leq \mu(E_j) + \frac{\epsilon}{2^j}$ . Then

$$\gamma < \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \mu\left(\bigcup_{j=1}^{\infty} G_j\right)$$

so there exists some  $f < \bigcup_{j=1}^{\infty} G_j$  so  $\gamma < \Lambda f$ . Let  $K = \text{supp } f$  and  $K \subset \bigcup_{j=1}^{\infty} G_j$  and by compactness

there exists some  $n$  so  $K \subset \bigcup_{j=1}^n G_j$ . Apply our partition of unity and get some  $h_j < G_j$  for each

$j = 1, \dots, n$  such that if  $x \in K$ , then  $\sum_{j=1}^n h_j(x) = 1$ . Then  $f \cdot h_j < G_j$  so  $f = f \cdot \sum_{j=1}^n h_j$ .

Thus

$$\begin{aligned}\gamma < \Lambda f &= \Lambda \left( \sum_{j=1}^n f h_j \right) = \sum_{j=1}^n \Lambda(f h_j) \\ &\leq \sum_{j=1}^n \mu(G_j) \leq \sum_{j=1}^n \left( \mu(E_j) + \frac{\epsilon}{2^j} \right) \\ &\leq \sum_{j=1}^{\infty} \left( \mu(E_j) \right) + \epsilon\end{aligned}$$

which holds for all  $\epsilon > 0$  if and only if

$$\gamma \leq \sum_{j=1}^{\infty} \mu(E_j)$$

for all  $\gamma \leq \mu \left( \bigcup_{j=1}^{\infty} E_j \right)$  and the result follows.  $\square$

**Claim 2:** If  $K < f < G$ , then  $\mu(K) \leq \Lambda f \leq \mu(G)$ . Thus if  $K$  is compact,  $\mu(K) = \inf\{\Lambda f : K < f\}$ , so  $\mu(K) < \infty$ .

PROOF It is obvious that  $\Lambda f \leq \mu(G)$ . Thus let  $\gamma < \mu(K)$  and  $\alpha \in (0, 1)$ . Let  $V_\alpha := \{x \in X : f(x) > \alpha\}$  and  $K \subset V_\alpha$ . Now  $\gamma < \mu(K) \leq \mu(V_\alpha)$ , so we have some  $h < V_\alpha$  such that  $\gamma < \Lambda h$ . Then  $\alpha \cdot h \leq f$  since in  $V_\alpha$ ,  $\alpha \cdot h \leq \alpha < f$  and in  $V_\alpha^c$ ,  $\alpha \cdot h = 0 \leq f$ . Now  $\alpha \cdot \Lambda h = \Lambda(\alpha h) \leq \Lambda f$  so  $\gamma < \Lambda f / \alpha$ . This is true for all  $\alpha \in (0, 1)$  and  $\gamma \leq \Lambda f$ . Since this holds for all  $\gamma < \mu(K)$ , we have  $\mu(K) \leq \Lambda f$  as required.

Now, let  $K$  be compact. Since  $\mu(K) \leq \Lambda f$  for all  $K < f$ . Let  $\epsilon > 0$ , so we have  $G \in \tau$ ,  $G \supset K$  such that  $\mu(G) \leq \mu(K) + \epsilon$ . Then by Urysohn's lemma, get some  $f$  so that  $\mu(K) \leq \Lambda f \leq \mu(G)$ , so  $\Lambda f \leq \mu(K) + \epsilon$  and the result holds.  $\square$

**Claim 3:** If  $0 \leq f \leq 1$ , then  $\Lambda f \leq \mu(\text{supp } f)$ .

PROOF Let  $G \supset \text{supp } f$  be open, so  $f < G$  and  $\mu(G) \geq \Lambda f$ . Then  $\mu(\text{supp } f) = \inf\{\mu(G) : E \subset G \in \tau\} \geq \Lambda f$ .  $\square$

**Claim 4:** If  $G \in \tau$ , then  $G$  is regular.

PROOF We must show  $\mu(G) = \sup\{\mu(K) : K \subset G \text{ compact}\}$ . Take  $\gamma < \mu(G)$ . We know that  $\sup\{\mu(K) : K \subset G \text{ compact}\} \leq \mu(G)$ , so we prove the  $\geq$  case. We need  $K$  compact so that  $\mu(K) > \gamma$ . Let  $f < G$  be such that  $\Lambda f > \gamma$ . Then  $\mu(\text{supp } f) > \gamma$  is compact, as desired.  $\square$

**Claim 5:** If  $E_i$  are disjoint regular, then  $\mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i)$ .

PROOF We first prove this for two compact sets. Thus let  $K_1, K_2$  be disjoint compact sets. Then  $K_2^c$  is open and  $K_2^c \supset K_1$ . By Urysohn's lemma, get  $f \in C_c(X)$  so that  $K_1 < f < K_2^c$  and  $x \in K_1$  implies  $f(x) = 1$ , and  $x \in K_2$  implies  $f(x) = 0$ . Since  $K_1 \cup K_2$  is compact, for all  $\epsilon > 0$ ,

get  $g < K_1 \cup K_2$  such that  $\mu(K_1 \cup K_2) + \epsilon > \Lambda g$ . Note that  $K_1 < f \cdot g$  and  $K_2 < (1 - f) \cdot g$ . Thus  $\mu(K_1) + \mu(K_2) \leq \Lambda(f \cdot g) + \Lambda((1 - f) \cdot g) = \Lambda g < \mu(K_1 \cup K_2) + \epsilon$  which is true for any  $\epsilon > 0$ . Thus  $\mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$  as required.

We now prove that  $\mu(\cup E_i) \geq \sum \mu(E_i)$ . If  $\mu(\cup E_i) = +\infty$ , we are done, so assume  $\mu(\cup E_i) < +\infty$ . If the  $E_i$  are regular, then there is a compact set  $H_i \subset E_i$  so that

$$\mu(H_i) > \mu(E_i) - \frac{\epsilon}{2^i}$$

Let  $K_n = \bigcup_{i=1}^n H_i$ . Now

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) &\geq \mu(K_n) \\ &= \sum_{i=1}^n \mu(H_i) \\ &> \sum_{i=1}^n \mu(E_i) - \epsilon \end{aligned}$$

As well, this holds for any  $\epsilon > 0$  and  $n \in \mathbb{N}$ , so we are done.  $\square$

**Claim 6:** If the  $E_i$  are regular, then  $\bigcup_{i=1}^{\infty} E_i$  is regular when  $\mu(\cup E_i) < \infty$ .

PROOF We have

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum_{i=1}^N \mu(E_i) + \epsilon \\ &\leq \mu(K_N) + 2\epsilon \end{aligned}$$

Thus for any  $\epsilon > 0$ , get  $N$  so that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) - 2\epsilon \leq \mu(K_N)$$

so  $\cup E_i$  is regular.  $\square$

**Claim 6(?):**  $E$  is regular and  $\mu(E) < \infty$  if and only if for any  $\epsilon > 0$ , there exists  $K$  compact,  $G$  open so that  $K \subset E \subset G$  and  $\mu(G \setminus K) < \epsilon$ .

PROOF There exists by regularity (and the definition of the outer measure)  $K, G$  so that

$$\mu(E) - \frac{\epsilon}{2} \leq \mu(K) \leq \mu(G) \leq \mu(E) + \epsilon/2$$

As well,  $\mu(G) = \mu(K \cup (G \setminus K)) = \mu(K) + \mu(G \setminus K)$  and  $\mu(G \setminus K) = \mu(G) - \mu(K) < \epsilon$ .

Conversely, let  $K \subset E \subset G$  and  $\mu(G \setminus K) < \epsilon$ . Then

$$\mu(E) \leq \mu(G) = \mu(K) + \mu(G \setminus K) < \mu(K) + \epsilon$$

so  $\mu(E) < \infty$  and  $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ , and  $E$  is regular.  $\square$

**Claim 7:**

1. Let  $A, B$  be regular with  $\mu(A), \mu(B) < \infty$ . Then  $A \setminus B, A \cup B, A \cap B$  are regular and have finite measure.
2. If  $E$  is regular and  $\mu(E) < \infty$ , then  $E$  is locally regular.
3. If  $E_i$  are regular, then  $\bigcup_{i=1}^{\infty} E_i$  is regular.

**PROOF** Recall that for any  $\epsilon > 0$ , there exists  $K_1 \subset A \subset G_1$  and  $K_2 \subset B \subset G_2$  such that  $\mu(G_1 \setminus K_1) < \epsilon$  and  $\mu(G_2 \setminus K_2) < \epsilon$ .

1. Note that  $A \setminus B \subset G_1 \setminus K_2 \subset (G_1 \setminus K_1) \cup (K_1 \setminus G_2) \cup (G_2 \setminus K_2)$ , where  $K_1 \setminus G_2$  is compact. Thus  $\mu(A \setminus B) \leq \epsilon + \mu(K_1 \setminus G_1) + \epsilon < \infty$  and  $\mu(A \setminus B) - 2\epsilon \leq \mu(K_1 \setminus G_2)$  so  $A \setminus B$  is regular. Finally since  $A \cup B = (A \setminus B) \cup B$ ,  $A \cup B$  is regular and  $\mu(A \cup B) < \infty$ . Thus  $A \cap B = (A \cup B) \setminus ((A \setminus B) \cup (B \setminus A))$  is regular and has measure less than infinity.
2. Let  $E$  be regular,  $\mu(E) < \infty$ , and  $K$  be a compact set. Then  $\mu(K) < \infty$ ,  $K$  is regular,  $E \cap K$  is regular and  $E$  is locally regular.
3. Set  $F_1 = E_1, F_n = E_n \setminus \left( \bigcup_{i=1}^{n-1} E_i \right)$  so  $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$  and the  $F_i$  are disjoint. By Claim 5,  $\bigcup F_i$  is regular and  $F_i$  are regular. □

**Claim 8: If  $E$  is locally regular and  $\mu(E) < \infty$ , then  $E$  is regular.**

**PROOF** Let  $\epsilon > 0$  and  $G \supset E$  be open so that  $\mu(G) < \mu(E) + 1 < \infty$ . As well,  $G$  is regular, so there exists  $K$  with  $\mu(G) < \mu(K) + \epsilon/2$ . Now,

$$\begin{aligned} \mu(E) &= \mu((E \setminus K) \cup (E \cap K)) \leq \mu(E \setminus K) + \mu(E \cap K) \\ &\leq \mu(G \setminus K) + \mu(E \cap K) \\ &< \frac{\epsilon}{2} + \mu(E \cap K) \end{aligned}$$

so  $\mu(E \cap K) > \mu(E) - \epsilon/2$ . Then since  $E$  is locally regular,  $E \cap K$  is regular and get a compact set  $L \subset E \cap K$  such that  $\mu(L) > \mu(E \cap K) - \epsilon/2 > \mu(E) - \epsilon$ . Thus  $E$  is regular. □

**Claim 9:  $\mathcal{M}$  is a  $\sigma$ -algebra,  $\mathcal{M} \subset \tau$ , and  $\mu$  is countably additive on  $\mathcal{M}$ .**

**PROOF** Let  $A \in \mathcal{M}$ : we see that  $A^c \in \mathcal{M}$ . For any  $K$  compact,  $A \cap K$  is regular. Let  $K$  be compact and take  $A^c \cap K = K \setminus (A \cap K)$  is regular by Claim 7.

Now let  $A_n \in \mathcal{M}$ : we see that  $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ . Let  $K$  be an arbitrary compact set, so

$$A \cap K = \bigcup_{n=1}^{\infty} (A_n \cap K)$$

is regular by Claim 7.

We now show  $\mathcal{M} \supset \tau$ . It suffices by closure under complement that all the closed sets are in  $\mathcal{M}$ . If  $A$  is closed, then  $A \cap K$  is compact and thus regular, so  $A \in \mathcal{M}$ .

Finally, let  $E_i \in \mathcal{M}$  be disjoint: we see that  $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ . It suffices to show  $\geq$ . If  $\mu(E_i) = +\infty$ , we are done, so assume  $\mu(E_i) < \infty$  for all  $i$ . But then by Claim 8,  $E_i$  are regular, and the result holds by Claim 5.  $\square$

**Claim 10:**  $\Lambda f = \int_X f d\mu$  for all  $f \in C_c(X)$ .

**PROOF** It suffices to prove this for real valued functions. If  $f = u + iv$ , then  $\Lambda f = \Lambda u + i\Lambda v = \int_X u d\mu + i \int_X v d\mu = \int_X f d\mu$ . Furthermore, it suffices to show  $\Lambda f \leq \int_X f d\mu$  since  $\Lambda(-f) \leq \int_X -f d\mu$  so that  $-\Lambda f \leq -\int_X f d\mu$  and  $\Lambda f \geq \int_X f d\mu$ , implying equality.

As well, it is enough to prove that  $\Lambda \leq \int f$  for  $f \geq 0$ . Let  $K = \text{supp } f$  be compact, and  $a = \min f$ ,  $b = \max f$ . Let  $\epsilon > 0$  be arbitrary. For every  $K$ , there exists  $G \supset K$  so that  $\mu(G) \leq \mu(K) + \epsilon$ . Then by Urysohn's lemma, there exists  $h \in C_c(X)$  so that  $K < g < G$ . Thus  $|a| \cdot h(x) = |a|$  for all  $x \in K$ , so  $F = f + |a|h \geq 0$  since  $f \geq -|a|$ . Now

$$\Lambda f \leq \int_X F d\mu = \int_X f + |a| \int_X h$$

so  $\Lambda f \leq \int_X f + |a|(\int_X h - \Lambda g)$ . As well, by Claim 2,

$$\mu(K) \leq \Lambda h \leq \mu(G)$$

so that

$$\int_X \chi_K \leq \int_X h d\mu \leq \int_X \chi_G = \mu(G)$$

and  $|\Lambda h - \int h| < \epsilon$ . Thus  $\Lambda f \leq \int f + |a|\epsilon$  for all  $\epsilon > 0$ , so  $\Lambda f \leq \int f$ .

It now remains to show  $\Lambda f \leq \int_X f d\mu$  for  $f \geq 0$ . Since  $f = Mf/M$  where  $M = \max f$ , we can assume  $0 \leq f \leq 1$ . Fix  $K = \text{supp } f$ , let  $\epsilon > 0$  be arbitrary. Let  $0 = c_0 < c_1 < c_2 < \dots < c_n = 1$  with  $c_k - c_{k-1} < \epsilon$  for all  $k$  and  $\mu(f^{-1}(c_k)) = 0$  for all  $k = 1, \dots, n-1$ . The existence of such a set follows from Assignment 6. Let  $K_j = K \cap f^{-1}([c_{j-1}, c_j])$  for  $j = 1, 2, \dots, n$  and  $L_j = K \cap f^{-1}([c_{j-1}, c_j])$  for  $j = 1, 2, \dots, n-1$ . To  $K_j$  and  $\epsilon$ , there exists  $G_j \supset K_j$  such that  $\mu(G_j) \leq \mu(K_j) + \frac{\epsilon}{2^j}$ . By Urysohn's lemma, get  $h_j$  so that  $K_h < h_j < G_j$ , so  $f \leq \sum_{j=1}^n c_j h_j$  since for  $x \notin \text{supp } f = K$ ,  $f = 0$ , and otherwise, there exists  $j$  so that  $x \in K_j$  implies  $h_j = 1$  and  $f(x) \leq c_j = c_j h_j(x) \leq \sum c_i h_i$ . Then

$$\begin{aligned} \Lambda f &\leq \Lambda\left(\sum_{j=1}^n c_j h_j\right) = \sum_{j=1}^n c_j \Lambda h_j \\ &\leq \sum_{j=1}^n c_j \mu(G_j) \\ &\leq \sum_{j=1}^n c_j \mu(K_j) + \sum_{j=1}^n c_j \frac{\epsilon}{2^j} \\ &\leq \sum_{j=1}^n (c_{j-1} + c_j - c_{j-1}) \mu(L_j) + \epsilon \\ &\leq \sum_{j=1}^n c_{j-1} \mu(L_j) + \epsilon \cdot \mu(K) + \epsilon \end{aligned}$$

where  $g$  is a simple function such that  $g(x) = c_{j-1}$  if  $x \in L_j$  and  $g \leq f$ . Then

$$\begin{aligned} &= \int_X g \, d\mu + \epsilon(1 + \mu(K)) \\ &\leq \int_X f \, d\mu + \epsilon(1 + \mu(K)) \end{aligned}$$

for any  $\epsilon > 0$ , and we are done!  $\square$

## 2.3 Regularity Properties of Borel Measures

We have the Riesz representation theorem in a locally compact Hausdorff space. Our aim is to introduce the Lebesgue measure in  $\mathbb{R}^k$  which respects translation as well.

**Def'n. 2.3.1** A measure defined on the family of Borel sets is called a **Borel measure**.

**Def'n. 2.3.2** Let  $\mu : \mathcal{B} \rightarrow [0, +\infty]$  be a Borel measure.

1.  $E$  is called **outer regular** if  $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$ .
2.  $E$  is called **inner regular** if  $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$
3.  $\mu$  is called **regular** if every  $E \in \mathcal{B}$  is inner and outer regular.

**Def'n. 2.3.3** A set  $E \subset X$  is called  $\sigma$ -**compact** if  $E = \bigcup_{n=1}^{\infty} E_n$ , for  $E_n$  compact.

**Def'n. 2.3.4** A  $G_\delta$  set is one of the form  $\bigcap_{n=1}^{\infty} A_n$  with  $A_n$  open, and a  $F_\sigma$  set is one of the form  $\bigcup_{n=1}^{\infty} B_n$  for  $B_n$  closed.

**Thm. 2.3.5** Let  $X$  be a locally compact,  $\sigma$ -compact Hausdorff space. Let  $\mathcal{M} \supset \mathcal{B}$  be a  $\sigma$ -algebra,  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  be a measure such that

- (i)  $\mu(E) = \inf\{\mu(G) : E \subset G \in \tau\}$
- (ii)  $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ ,  $\mu(E) < \infty$
- (iii)  $\mu(K) < \infty$  for  $K$  compact.

Then

1. For all  $E \in \mathcal{M}$  and  $\epsilon > 0$ , there exists  $F$  closed and  $G$  open so that  $F \subset E \subset G$  and  $\mu(G \setminus F) < \epsilon$ .
2.  $\mu$  is regular
3. For all  $E \in \mathcal{M}$ , there exists a  $F_\sigma$  set  $A$  and a  $G_\delta$  set  $B$  so  $A \subset E \subset B$  and  $\mu(B \setminus A) = 0$ .

**PROOF** Since  $X$  is  $\sigma$ -compact,  $X = \bigcup_{n=1}^{\infty} K_n$ ,  $K_n$  compact.

1. By (iii), we have  $\mu(K_n \cap E) < \infty$ . Thus by (i), get  $G_n$  open so that  $G_n \supset K_n \cap E$  with  $\mu(G_n \setminus (K_n \cap E)) < \frac{\epsilon}{2^{n+1}}$ . Let  $G = \bigcup_{n=1}^{\infty} G_n$  be open, so that  $G \setminus E \subset \bigcup (G_n \setminus (K_n \cap E))$  and  $\mu(G \setminus E) < \frac{\epsilon}{2}$ . Repeat this for  $E^c$ : get an open set  $H$  such that  $\mu(H \setminus E^c) < \frac{\epsilon}{2}$ . Then  $F = H^c \subset E$  satisfies  $\mu(E \setminus F) = \mu(F^c \setminus E^c) = \mu(H \setminus E^c) < \frac{\epsilon}{2}$ . Then  $\mu(G \setminus F) \leq \mu(G \setminus E) + \mu(E \setminus F) < \epsilon$ . Then  $\mu(G \setminus F) \leq \mu(G \setminus E) + \mu(E \setminus F) < \epsilon$ .

2.  $E$  is outer regular by (i). If  $\mu < \infty$ , then  $E$  is inner regular by (ii). If  $\mu(E) = \infty$ , let  $F \subset E$  be given by 1. Then  $\mu(F) = +\infty$ , or  $\mu(E)$  would be finite. Let  $H_n = \bigcup_{k=1}^n (F \cap K_k)$  compact,  $H_n \subset F$ . Then  $\bigcup_{n=1}^{\infty} H_n = F$ , and  $\mu(H_n) \rightarrow \mu(F) = \infty$ . Thus  $\mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$ .
3. Apply 1 with  $\epsilon = 1/j$  for  $j \in \mathbb{N}$ . Then there exists  $F_j \subset E \subset G_j$  so  $\mu(G_j \setminus F_j) < \frac{1}{j}$ . Let  $A = \bigcup_{j=1}^{\infty} F_j$  and  $B = \bigcap_{j=1}^{\infty} G_j$ . Then  $A \subset E \subset B$  and  $\mu(B \setminus A) \leq \mu(G_j \setminus F_j) < \frac{1}{j}$ , so  $\mu(B \setminus A) = 0$ .  $\square$

**Thm. 2.3.6** Let  $X$  be locally compact and Hausdorff, and assume that every open set is  $\sigma$ -compact. Let  $\lambda : \mathcal{B} \rightarrow [0, \infty]$  be a Borel measure such that  $\lambda(K) < \infty$ . Then  $\lambda$  is regular.

PROOF Let  $\Lambda f = \int_X f d\lambda$ . Then  $\Lambda : C_c(X) \rightarrow \mathbb{C}$  is a positive linear functional. By the Riesz representation theorem, there exists  $\mu : \mathcal{M} \rightarrow [0, \infty]$  such that  $\int_X f d\mu = \Lambda f = \int_X f d\lambda$ . We see that  $\lambda = \mu$  on  $\mathcal{B}$ . We first prove this for open sets. Let  $G \in \tau$ ; then there exists  $K_n$  so  $G = \bigcup_{n=1}^{\infty} K_n$ . By Urysohn's lemma, there exists  $f_i$  such that  $K_i < f_i < G$ . Let  $g_n = \max\{f_1, f_2, \dots, f_n\}$ , so  $g_n \in C_c(X)$ , and  $g_n \rightarrow \chi_G$  pointwise. But then

$$\begin{aligned} \lambda(G) &= \int_X \chi_G d\lambda \\ &= \int_X \lim_{n \rightarrow \infty} g_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\mu \\ &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \int_X \chi_G d\mu \\ &= \mu(G) \end{aligned}$$

Now for any  $E \in \mathcal{B}$ , apply (i). Then  $F \subset E \subset G$ ,  $\mu(G \setminus F) < \epsilon$ . Since  $G \setminus F$  is open,  $\lambda(G \setminus F) = \mu(G \setminus F) < \epsilon$  so  $\lambda(G) \leq \lambda(E) + \epsilon$ . Thus  $|\mu(E) - \lambda(E)| < \epsilon$  for all  $\epsilon > 0$  so  $\lambda(E) = \mu(E)$ .  $\square$

## 2.4 Construction of the Lebesgue Measure

We have the Riesz Representation Theorem in a locally compact Hausdorff space.

**Def'n. 2.4.1** Let  $E \subset \mathbb{R}^k$ ,  $x \in \mathbb{R}^k$ . Then  $E + x = \{y + x : y \in E\}$  is the **translate** of  $E$ .

**Def'n. 2.4.2** We define a  $k$ -cell in  $\mathbb{R}^k$  by  $W = I_1 \times I_2 \times \dots \times I_k$  where  $I_j$  is an interval. We also define  $(W) = (b_1 - a_1)(b_2 - a_1) \dots (b_k - a_k)$  where  $a_j, b_j$  are the endpoints of the  $I_j$ .

We know that  $(W + x) = (W)$  for any  $k$ -cell  $W$  and  $x \in \mathbb{R}$ .

**Thm. 2.4.3** *There exists a  $\sigma$ -algebra  $\mathcal{M}$  in  $\mathbb{R}^k$  and a complete measure  $m : \mathcal{M} \rightarrow [0, +\infty]$  satisfying*

1.  $m(W) = (W)$  for any  $k$ -cell  $W$ .
2.  $\mathcal{M} \supset \mathcal{B}$  and  $E \in \mathcal{M}$  if and only if there exists  $A \in F_\sigma, B \in G_\delta$  such that  $A \subset E \subset B$  and  $m(B \setminus A) = 0$ .
3.  $m$  is translation invariant:  $m(E + x) = m(E)$ .
4. If  $\mu$  is a translation invariant Borel measure, and  $\mu(K) < \infty$  for all  $K$  compact, then there exists  $c \in \mathbb{R}$  so that  $\mu(E) = c \cdot m(E)$ .
5. If  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is linear, then there exists  $\Delta(T) \in \mathbb{R}$  such that  $m(T(E)) = \Delta(T) \cdot m(E)$ .

**PROOF** For  $f \in C_c(\mathbb{R}^k)$ , let  $\Lambda f = \int_{\mathbb{R}^k} f(x) dx$  (the Riemann Integral). Then  $\Lambda : C_c(\mathbb{R}^k) \rightarrow \mathbb{C}$  is a positive linear functional, so by the Riesz representation theorem, there exists a unique measure  $m$  and  $\mathcal{M} \supset \mathcal{B}$  so for all  $f \in C_c(\mathbb{R}^k)$ ,  $\Lambda f = \int_{\mathbb{R}^k} f dm$ .

We first see (i). By the definition of  $m$ , for an open  $k$ -cell  $W$ ,  $m(W) = \sup\{\Lambda f : f < W\} = (W)$  (by definition of the Riemann integral). If  $W$  is an arbitrary  $k$ -cell, then there exist open  $k$ -cells  $W_n$  such that  $W = \bigcap_{n=1}^{\infty} W_n$ . Then  $(W_n) \rightarrow (W)$ , so  $m(W_n) \rightarrow m(W)$ , and  $(W_n) = m(W_n)$ . Thus  $(W) = m(W)$ .

Let  $\lambda$  be a Borel measure. If  $\lambda(W) = m(W)$  for all  $W$   $k$ -cells, then  $\lambda(E) = m(E)$  for all  $E \in \mathcal{B}$ . For any  $G$  open,  $G = \bigcup_{n=1}^{\infty} W_n$  disjointly, so  $\lambda(G) = m(W)$ . Then since  $\lambda$  and  $m$  are regular,  $\lambda(E) = \inf\{\lambda(G) : E \subset G \in \tau\} = \inf\{m(G) : E \subset G \in \tau\} = m(E)$  for all  $E \in \mathcal{B}$ .

We now see (iii). Define  $\lambda(E) = m(E + X)$ . If  $W$  is a box, then  $\lambda(W) = m(W + x) = (W + x) = (W) = m(W)$ , so by the lemma,  $\lambda(E) = m(E)$  for all  $E \in \mathcal{B}$ . Then regularity implies  $\lambda(E) = m(E)$  for all measurable  $E$ .

We have (iv): let  $c = \mu([0, 1]^k) = c \cdot ([0, 1]^k)$ . Translation invariance of  $\mu$  implies  $\mu(W) = c \cdot (W)$ .

We have (v). If  $\dim(\text{Im}(T)) < k$ , then  $m(\text{Im}(T)) = 0$  so  $\Delta(T) = 0$ . Otherwise,  $T$  is a homeomorphism so  $T(E) \in \mathcal{B}$  for all  $E \in \mathcal{B}$ . Let  $\mu(E) = m(T(E))$ . Then  $\mu(E + x) = m(T(E) + T(x)) = m(T(E)) = \mu(E)$ , so  $\mu$  is translation invariant. Then by (iv),  $\mu(E) = c \cdot m(E)$  and set  $\Delta(T) = c$ .  $\square$

**Thm. 2.4.4** *If  $A \subset \mathbb{R}$  for which every set is Lebesgue measurable, then  $m(A) = 0$ .*

**PROOF** Partition  $\mathbb{R}$  into cosets by  $\mathbb{Q}$ ; let  $E$  be a set containing exactly one element of each class (axiom of choice). Now if  $r \neq s$ ,  $r, s \in \mathbb{Q}$ , then  $(E + r) \cap (E + s) = \emptyset$ . But then  $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r)$  disjointly. Given  $A$ , define  $A_t = A \cap (E + t)$  for  $t \in \mathbb{Q}$ . Now let  $K \subset A_t$ , so  $K \subset E + t$ . Since  $(K + r_1) \cap (K + r_2) = \emptyset$ , define  $H = \bigcup_{r \in \mathbb{Q} \cap [0, 1]} (K + r)$  is a countable disjoint union. But then  $\infty > m(H) = \sum_r m(K)$  so  $m(K) = 0$  and  $m(A_t) = 0$ . But then

$$\bigcup_{t \in \mathbb{Q}} A_t = \bigcup_{t \in \mathbb{Q}} (A \cap (E + t)) = A \cap \left( \bigcup_{t \in \mathbb{Q}} (E + t) \right) = A \cap \mathbb{R} = A$$

so  $m(A) = 0$  as well.  $\square$



Let  $X$  be a locally compact, Hausdorff topological space. Let  $\mathcal{M}$  be a  $\sigma$ -algebra,  $\mu$  be a measure satisfying the properties in the Riesz representation theorem. We then have

**Thm. 2.4.5 (Lusin)** *Let  $f : X \rightarrow \mathbb{C}$  be a measurable function, with  $\text{supp } f \subset A$  and  $\mu(A) < \infty$ . Then for any  $\epsilon > 0$ , there exists  $g \in C_c(X)$  such that  $\mu(\{x \in X : f(x) \neq g(x)\}) < \epsilon$  and  $\sup_X |g| \leq \sup_X |f|$ .*

**PROOF** If  $\text{supp } g \subset A$ , then  $f(x) = 0 = g(x)$  if  $x \in A^c$  and  $\{x \in X : f(x) \neq g(x)\} = \{x \in A : f(x) \neq g(x)\}$ . Then  $\mu(A) < \infty$  so there exists  $K \subset A$  compact, with  $\mu(A \setminus K) < \epsilon/2$ . Then

$$\mu(\{x \in A : f(x) \neq g(x)\}) \leq \mu(\{x \in K : f(x) \neq g(x)\}) + \frac{\epsilon}{2}$$

We may assume that  $A$  is compact.

We first prove this for  $0 \leq f \leq 1$ . For  $t \geq 0$  and each  $N \in \mathbb{N}$ , let  $k_n(t)$  satisfy  $k_n(t) \leq t \cdot 2^n < k_n(t) + 1$ , and define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n}, & 0 \leq t \leq n \\ n, & t > n \end{cases}$$

Let  $s_n(x) = \phi_n(f(x))$  and  $t_n = s_n - s_{n-1}$ . Then since

$$k_{n-1}(t) \leq t \cdot 2^{n-1} < k_{n-1}(t) + 1 \implies 2k_{n-1}(t) \leq t \cdot 2^n < 2k_{n-1}(t) + 2$$

is the largest even number below  $t \cdot 2^n$ . Then  $k_n - 2k_{n-1} \in \{0, 1\}$  for all  $t$ . Since  $0 \leq f \leq 1$ ,

$$t_n(x) = \phi_n(f(x)) - \phi_{n-1}(f(x)) = 2^{-n} \cdot k_n(f(x)) - 2^{-(n-1)} \cdot k_{n-1}(f(x))$$

this becomes

$$2^n \cdot t_n(x) = k_n(f(x)) - 2 \cdot k_{n-1}(f(x)) \in \{0, 1\}$$

which is the characteristic function of some set, which we denote  $T_n$ . Furthermore,  $f(x) = \sum_{n=1}^{\infty} t_n(x)$ . Let  $V \supset A$  be open so that  $\bar{V}$  is compact. Let  $K_n$  be compact,  $V_n$  be an open set such that  $K_n \subset T_n \subset V_n \subset V$  with  $\mu(V_n \setminus K_n) < \epsilon/2^n$ . By Urysohn's lemma, there exists  $h_n \in C_c(X)$  with  $K_n \subset h_n \subset V_n$ . Let  $g = \sum_{n=1}^{\infty} 2^{-n} \cdot h_n$ , which converges uniformly so that  $g$  is continuous and  $\text{supp } g \subset \bar{V}$ . If  $x \in K_n$ , then  $h_n(x) = 1$  and  $t_n(x) = 2^{-n}$ , so  $2^{-n} \cdot h_n(x) = t_n(x)$ . If  $x \notin V_n$ , then  $h_n(x) = 0$  so  $t_n(x) = 0$ , so  $2^{-n} \cdot h_n(x) = t_n(x)$ . Thus

$$S = \{x \in A : f(x) \neq g(x)\} \subset \bigcup_{n=1}^{\infty} (V_n \setminus K_n)$$

and  $\mu(S) \leq \sum \mu(V_n \setminus K_n) < \epsilon$ .

Now if  $f$  is bounded,  $0 \leq f/M \leq 1$  follows directly by the above.

For  $f$  bounded and real valued, note that  $0 \leq f + k \leq 2K$  and get  $g \in C_c(x)$  near  $f + K$ . Then  $g - K$  satisfies the required values.

Now for any real valued function, let  $B_n = \{x \in X : |f(x)| > n\} \implies \bigcap_{n=1}^{\infty} B_n = \emptyset$ , with  $\mu(B_1) \leq \mu(\text{supp } f) < \infty$  and  $B_{n+1} \subset B_n$  for all  $n$ . Thus  $\mu(B_n) \rightarrow \mu(\bigcap B_n) = 0$ . Let  $N$  be such that

$\mu(B_N) < \epsilon/2$ , so if  $x \notin B_N$ ,  $f(x) \leq N$  and define  $\tilde{f}(x) = (1 - \chi_{B_N(x)})f(x)$ . Then  $\tilde{f}$  is bounded, and get  $g \in C_c(x)$  to  $\tilde{f}$ . In particular,  $\mu(\tilde{f} \neq \tilde{g}) < \epsilon/2$ . Then let  $g = \tilde{g}$  and

$$\mu(g \neq f) \leq \mu(f \neq \tilde{f}) + \mu(\tilde{f} \neq \tilde{g}) = \epsilon$$

Finally, let  $f$  be complex valued and  $f = f_1 + if_2$ . Then get  $g_1, g_2 \in C_c(X)$  and set  $g = g_1 + ig_2$ . We will prove that  $\sup |G| \leq \sup |f|$ . If  $\sup |f| = \infty$  we are done, so let  $R = \sup_X |f|$ . Let

$$\phi(z) = \begin{cases} z & : |z| \leq R \\ \frac{R \cdot z}{|z|} & : |z| > R \end{cases}$$

then  $\phi$  is continuous and  $|\phi| \leq R$ . We already have  $g \in C_c(X)$  so that  $\mu(f \neq g) < \epsilon$ . Let  $\tilde{g} = \phi \circ g$ , which is also continuous and  $|\tilde{g}| \leq R$ . Finally,  $\mu(\tilde{g} \neq f) = \mu\{\phi \circ f \neq \phi \circ g\} \leq \mu\{g \neq f\} < \epsilon$ .  $\square$

**Cor. 2.4.6** *In the same context, let  $f : X \rightarrow \mathbb{C}$  be measurable,  $\text{supp } f \subset A$ , and  $\mu(A) < \infty$  and  $|f| \leq 1$ . Then there exists  $g_n \in C_c(X)$  with  $|g_n| \leq 1$  and  $\lim g_n(x) = f_n(x)$  almost everywhere.*