Course Notes

Graph Theory

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Chapter 1

Introduction

1.1 Introduction

1.1.1 Basic Definitions

Def'n. 1.1.1 *A **graph** $G = (V, E)$ consists of a vertex set V and edge set E where $E \subseteq \binom{V}{2}$.*

Note that we write $\binom{V}{2}$ instead of $V \times V$ to make it clear that we cannot have loops and multiple edges.

Def'n. 1.1.2 *Two graphs F and G are **isomorphic** if there exists a bijective mapping $f : V(F) \to V(G)$ such that for every $a, b \in V(F)$: $\{a, b\} \in E(F) \Leftrightarrow \{F(a), f(b)\} \in E(G)$.*

Def'n. 1.1.3 *The **degree** of a vertex $v \in V(G)$ is the number of edges having v as an endpoint.*

Def'n. 1.1.4 *A **path** is asequence $v_1 e_1 v_2 e_2 \ldots v_i e_i v_{i+1} \ldots e_j v_{j+1}$ where each $v_i \in V(G)$ and $e_i = \{v_i, v_{i+1}\} \in E(G)$ where all v_i's are different. A **cycle** is a path in which $v_1 = v_{j+1}$.*

Def'n. 1.1.5 *The **complementary graph** of $G = (V, E)$ is $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$.*

Def'n. 1.1.6 *A graph G is called **connected** if for every $u, v \in V(G)$ there exists a path between u and v.*

Def'n. 1.1.7 *A connected graph that becomes disconnected with the removal of any edge is called a **tree**. Equivalently, a **tree** is a graph which is connected and contains no cycle.*

Prop. 1.1.8 *Any tree on at least two vertices contains at least two vertices of degree 1 ("leaves").*

Proof Consider a path of maximal length. We claim that both endpoints of P have degree one. Suppose for contradiction that an endpoint has greater than one. Then the endpoint has another neighbour on the path (in which case we have a cycle), or a unique neighbour (in which case the path is not maximal), a contradiction in either case. $\square$

Prop. 1.1.9 *A tree on n vertices always has $n - 1$ edges.*

Proof Delete the edges one by one. Each time, the number of connected components increases by one. After deleting all edges, we have n components, at the beginning, we have one, so we deleted $n-1$ edges. □

We now have an interesting question: how many different trees can be given on n labelled vertices? To investigate this, we consider the Prüfer code. Delete the smallest labelled degree one vertex and write up its unique neighbour's label. Continue doing this until only one point remains. The obtained sequences of labels is the Prüfer code.

Properties:

- The length is $n-1$
- The last digit must be n

Thm. 1.1.10 (Cayley) *The number of different trees on n labelled vertices is* n^{n-2}.

Proof We will show that sequences $x \in \{1,2,\dots,n\}^{n-1}$ with $x_{n-1} = n$ are in bijective correspondence with the trees on n labelled vertices. First, given $a_1 a_2 \dots a_{n-2} a_{n-1}$ with $a_{n-1} = n$ we want to decode it. Let $b_1, b_2, \dots, b_{n-1}$ be the sequence of labels of vertices deleted in the order of the indices. If we "decode" $b_1 b_2 \dots b_{n-1}$, we know the tree, since we have the $n-1$ edges $\{a_i, b_i\}$.

1. $b_1 := \min\{k \in \{1,2,\dots,n\} : k \notin \{a_1,\dots,a_{n-1}\}\}$
2. $b_2 := \min\{k \in \{1,2,\dots,n\} : k \notin \{b_1, a_2,\dots,a_{n-1}\}\}$

(∗) $\boxed{b_i := \min\{k \in \{1,2,\dots,n\} : k \notin \{b_1,\dots,b_{i-1}, a_i,\dots,a_{n-1}\}\}}$

We show that taking any sequence $a_1 a_2 \dots a_{n-1}$ with $a_{n-1} = n$ and applying (∗) to obtain $b_1,\dots,b_{n-1}$, the graph we obtain on vertices $1,\dots,n$ with the $n-1$ edges $\{a_i, b_i\}$ (1) is a tree, and (2) has Prüfer code is just $a_1 a_2 \dots a_{n-1}$.

Note that (∗) implies that $\{b_1, b_2, \dots, b_{n-1}, a_{n-1}\} = \{1,2,\dots,n\}$. Define graphs T_i for $i = n-1, n-2, \dots, 2, 1$ on the graph spanned by the edges $\{a_{n-1}, b_{n-1}\}, \{a_{n-2}, b_{n-2}\}, \dots, \{a_i, b_i\}$. It suffices to prove that T_i is a tree for every i and b_i is its smallest labelled degree 1 vertex.

We do this by induction. Clearly, it is true for $i = n-1$. Once it is true for $i = n-1, \dots, j+1$, we prove this for $i = j$. We know that T_{j+1} is a tree, and we wish to add the edge $\{b_j, a_j\}$. Thus $b_j \notin V(T_{j+1}) = \{b_{j+1}, b_{j+2}, \dots, b_{n-1}, a_{n-1}\}$ so b_j is indeed degree one; $a_j \in V(T_{j+1})$ and T_j is a tree. If b_j was not the smallest degree 1 vertex, then there exists some $k > j$ such that $b_k < b_j$ and b_k has degree one in T_j. But then $b_k \notin \{b_1, \dots, b_{j-1}, a_j, \dots, a_{n-1}\}$ so (∗) would have chosen it in place of b_j, a contradiction. □