Course Notes

Graph Theory

Alex Rutar

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Contents

Chapter 1

Basic Structure of Graphs

1.1 A Brief Introduction

1.1.1 Basic Definitions

Def'n. 1.1.1 *A **simple graph** $G = (V, E)$ consists of a vertex set V and edge set E where $E \subseteq \binom{V}{2}$.*

Note that we write $\binom{V}{2}$ instead of $V \times V$ to make it clear that we cannot have loops and multiple edges. If the graph is undirected, edges are unordered pairs $\{v_1, v_2\}$ of vertices; we write (v_1, v_2) for directed edges.

Def'n. 1.1.2 *Two graphs F and G are **isomorphic** if there exists a bijective mapping $f : V(F) \to V(G)$ such that for every $a, b \in V(F)$: $\{a, b\} \in E(F) \Leftrightarrow \{f(a), f(b)\} \in E(G)$.*

Def'n. 1.1.3 *The **degree** of a vertex $v \in V(G)$ is the number of edges having v as an endpoint. We denote this number by $d(v)$, and $\Delta(G)$ to denote the maximal degree in G, and $\delta(G)$ to denote the minimal degree in G.*

Def'n. 1.1.4 *The **vertex neighbourhood** of a graph U is denoted by $N(U) := \{v \in V(G) : \exists u \in U \text{ s.t. } \{v, u\} \in E(G)\}$.*

Def'n. 1.1.5 *A **path** is a sequence $v_1 e_1 v_2 e_2 \dots v_i e_i v_{i+1} \dots e_j v_{j+1}$ where each $v_i \in V(G)$ and $e_i = \{v_i, v_{i+1}\} \in E(G)$ where all v_i's are different. A **cycle** is a path in which $v_1 = v_{j+1}$.*

Def'n. 1.1.6 *The **complementary graph** of $G = (V, E)$ is $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$.*

Def'n. 1.1.7 *A graph G is called **connected** if for every $u, v \in V(G)$ there exists a path between u and v.*

Def'n. 1.1.8 *A connected graph that becomes disconnected with the removal of any edge is called a **tree**. Equivalently, a **tree** is a graph which is connected and contains no cycle.*

Prop. 1.1.9 *Any tree on at least two vertices contains at least two vertices of degree 1 ("leaves").*

Proof Consider a path of maximal length. We claim that both endpoints of P have degree one. Suppose for contradiction that an endpoint has greater than one. Then the endpoint has another neighbour on the path (in which case we have a cycle), or a unique neighbour (in which case the path is not maximal), a contradiction in either case. □

Prop. 1.1.10 *A tree on n vertices always has $n-1$ edges.*

Proof Delete the edges one by one. Each time, the number of connected components increases by one. After deleting all edges, we have n components, at the beginning, we have one, so we deleted $n-1$ edges. □

1.1.2 Prüfer Codes

We now have an interesting question: how many different trees can be given on n labelled vertices? To investigate this, we consider the Prüfer code. Delete the smallest labelled degree one vertex and write up its unique neighbour's label. Continue doing this until only one point remains. The obtained sequences of labels is the Prüfer code.

Properties:

- The length is $n-1$
- The last digit must be n

Thm. 1.1.11 (Cayley) *The number of different trees on n labelled vertices is n^{n-2}.*

Proof We will show that sequences $x \in \{1,2,\dots,n\}^{n-1}$ with $x_{n-1} = n$ are in bijective correspondence with the trees on n labelled vertices. First, given $a_1 a_2 \dots a_{n-2} a_{n-1}$ with $a_{n-1} = n$ we want to decode it. Let $b_1, b_2, \dots, b_{n-1}$ be the sequence of labels of vertices deleted in the order of the indices. If we "decode" $b_1 b_2 \dots b_{n-1}$, we know the tree, since we have the $n-1$ edges $\{a_i, b_i\}$.

1. $b_1 := \min\{k \in \{1,2,\dots,n\} : k \notin \{a_1,\dots,a_{n-1}\}\}$
2. $b_2 := \min\{k \in \{1,2,\dots,n\} : k \notin \{b_1, a_2,\dots,a_{n-1}\}\}$

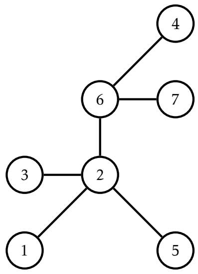
(∗) $\boxed{b_i := \min\{k \in \{1,2,\dots,n\} : k \notin \{b_1,\dots,b_{i-1},a_i,\dots,a_{n-1}\}\}}$ □

We show that taking any sequence $a_1 a_2 \dots a_{n-1}$ with $a_{n-1} = n$ and applying (∗) to obtain $b_1,\dots,b_{n-1}$, the graph we obtain on vertices $1,\dots,n$ with the $n-1$ edges $\{a_i, b_i\}$ (1) is a tree, and (2) has Prüfer code is just $a_1 a_2 \dots a_{n-1}$.

Note that (∗) implies that $\{b_1, b_2, \dots, b_{n-1}, a_{n-1}\} = \{1,2,\dots,n\}$. Define graphs T_i for $i = n-1, n-2, \dots, 2, 1$ on the graph spanned by the edges $\{a_{n-1}, b_{n-1}\}, \{a_{n-2}, b_{n-2}\}, \dots, \{a_i, b_i\}$. It suffices to prove that T_i is a tree for every i and b_i is its smallest labelled degree 1 vertex.

We do this by induction. Clearly, it is true for $i = n-1$. Once it is true for $i = n-1,\dots,j+1$, we prove this for $i = j$. We know that T_{j+1} is a tree, and we wish to add the edge $\{b_j, a_j\}$. Thus $b_j \notin V(T_{j+1}) = \{b_{j+1}, b_{j+2}, \dots, b_{n-1}, a_{n-1}\}$ so b_j is indeed degree one; $a_j \in V(T_{j+1})$ and T_j is a tree. If b_j was not the smallest degree 1 vertex, then there exists some $k > j$ such that $b_k < b_j$ and b_k has degree one in T_j. But then $b_k \notin \{b_1,\dots,b_{j-1},a_j,\dots,a_{n-1}\}$ so (∗) would have chosen it in place of b_j, a contradiction.

Here's an illustration of a particular tree along with the corresponding Prüfer code.:



$$\Longrightarrow \qquad (2,2,6,2,6,7)$$

We then have

$$\begin{aligned}
a_1 &= 2 & b_1 &= \min\{k \in [7], k \notin \{2,2,6,2,6,7\}\} = 1\\
a_2 &= 2 & b_2 &= \min\{k \in [7], k \notin \{1,2,6,2,6,7\}\} = 3\\
a_3 &= 6 & b_3 &= \min\{k \in [7], k \notin \{1,3,6,2,6,7\}\} = 4\\
a_4 &= 2 & b_4 &= \min\{k \in [7], k \notin \{1,3,4,2,6,7\}\} = 5\\
a_5 &= 6 & b_5 &= \min\{k \in [7], k \notin \{1,3,4,5,6,7\}\} = 2\\
a_6 &= 7 & b_6 &= \min\{k \in [7], k \notin \{1,3,4,5,2,7\}\} = 6
\end{aligned}$$

so that the edges are given by $E(G) = \{\{1,2\},\{3,2\},\{4,6\},\{5,2\},\{2,6\},\{6,7\}\}$.

1.2 Paths, Circuits, and Cycles

1.2.1 Eulerian Circuits

Def'n. 1.2.1 *An **Eulerian circuit** is a closed walk in a graph that contains every edge exactly once. An **Eulerian path** is a walk containing every edge exactly once and not (necessarily) ending at the same vertex.*

Note that we do allow multiple edges between vertices. We also assume that our graph is connected.

Thm. 1.2.2 (Euler) *A graph contains an Eulerian circuit if and only if every vertex has even degree.*

Cor. 1.2.3 *A graph contains an Eulerian path if and only if all but two vertices have even degree.*

In both cases, necessity is obvious: every time a path arrives at a vertex, it adds two to the degree since there is a unique edge in and out from the vertex. Thus for the corollary the only odd vertices can be the endpoints, and for the theorem, there are no endpoints in the path.

Proof (cor.) To see the corollary, first add an edge connecting the two odd degree vertices. Then by the theorem, we have an Eulerian circit walk through it so that the added edge is the last one traversed. Delete it, and the remaining part of the walk gives our Eulerian path □

Proof (thm.) We can now prove the theorem. Consider a maximal walk P on G that does not repeat edges. Because of the evenness of all degrees, it must be closed. If every edge is contained, it is an Eulerian circuit and we are done. If there exists some $e \in E(G)$ that is not in the walk, then by connectedness, there must be a path from a vertex in the walk to an endpoint of e. Considering a shortest such path (perhaps containing 0 edges), all edges on it are unused so far. Now take the starting point of this path on our walk, go through the closed path on our walk, go through the closed walk we have starting here and thus contine on the said path and include e, a contradiction. □

1.2.2 Hamiltonian Cycles

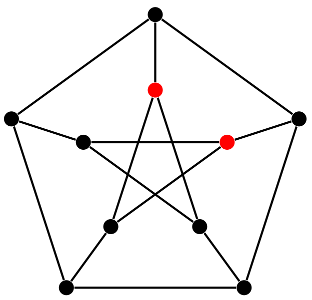
Def'n. 1.2.4 *A **Hamiltonian cycle** in a graph is a cycle containing every vertex exactly once. A **Hamiltonian path** is a path that contains every vertex exactly once.*

1.2.3 Necessary Conditions

Prop. 1.2.5 *If G contains a Hamiltonian cycle, then after deleting any k of its vertices, the remaining graph cannot have more than k components. Similarly, if G contains a Hamiltonian path, then deleting any k of it vertices yields a graph with at most $k+1$ components.*

Proof This can be easily proven by induction. □

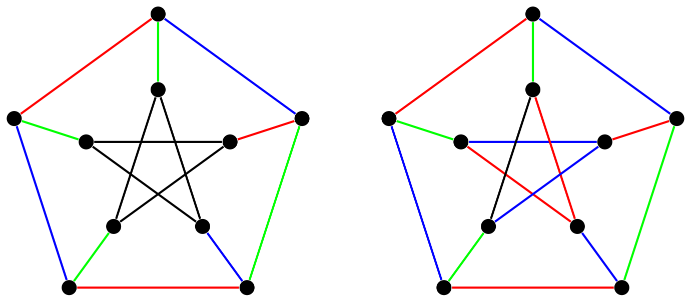
Here is a graph for which this condition holds, but does not have a Hamiltonian cycle.



The condition holds: delete $k = l_1 + l_2$ vertices, where l_1 is in the "outer cycle" and l_2 is in the "inner cycle". (Above, $l_1 = 0$ and $l_2 = 2$.) Then the outer cycle may not fall apart into more than l_1 components, the inner cycle may not fall apart into more than l_2 component, so if $l_1, l_2 > 0$, then there are at most $l_1 + l_2$ components. If l_1 or l_2 is 0, then the graph remains connected.

Furthermore, there does not exist a Hamiltonian cycle. Suppose such a cycle exists, then it is a cycle containing 10 edges. Colour the alternating edges red and blue. Thus every vertex is adjacent to a red edge and a blue edge. Colour the remaining 5 edges white, so every vertex will be the end vertex of exactly 1 white edge. However, such an edge-coloring is impossible.

Up to isomorphism, the outer edges must be coloured with two of colour 1, two of colour 2, and one of colour 3. We then fill in the interior edges as required, until the inner cycle, in which we are forced to draw the following edges, yielding our contradiction.



1.2.4 Sufficient Conditions

Thm. 1.2.6 (Dirac, 1952) *If G is a graph on n vertices, with every degree being at least $\frac{n}{2}$, then G contains a Hamiltonian cycle.*

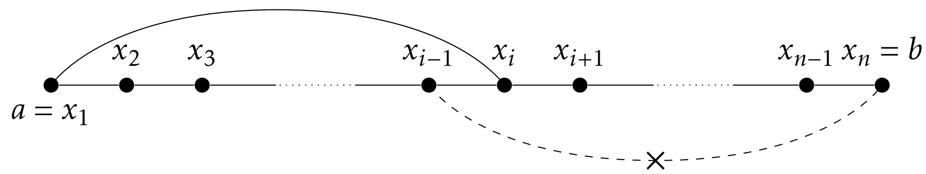
Proof Follows from Ore's Theorem. □

This bound is tight: for every graph on n vertices, $f(n) \geq n/2$ is the minimal requirement since the graph composed of two disconnected components $K_{n/2}$ has degree $n/2-1$ everywhere but does not have a Hamiltonian cycle. As a fun fact, this is not Paul Dirac, but rather Gabriel Andrew Dirac (Paul Dirac's stepson).

Thm. 1.2.7 (Ore, 1960) *If G is a graph on n vertices such that for every nonadjacent pair of vertices $u, v \in V(G)$, $d(u)+d(v) \geq n$ is satisfied, then H contains a Hamiltonian cycle.*

Proof Assume for contradiction we have G_0 satisfying the condition but does not have a Hamiltonian cycle. Saturate G_0 go obtain G: if two vertices are non-adjacent and connecting them does not create a Hamiltonian cycle, then connect them. This new graph is still a counterexample since whenever we increase the degree of some vertices, they are no longer non-adjacent. Do this maximally, so that the addition of any edge would create a Hamiltonian cycle and G still satisfies Ore's condition.

Now, if $a, b \in V(G)$ are non-adjacent in G, then there exists a Hamiltonian path starting at a and ending at b (since adding $\{a, b\}$ would create a Hamiltonian cycle). Consider such a Hamiltonian path $(x_1, x_2, \ldots, x_n)$.



Observe that if $\{a, x_i\} \in E(G)$, then $\{a, x_{i-1}\} \notin E(G)$ or we would have a Hamiltonian cycle given by $(a, x_i, x_{i+1}, \ldots, b, x_{i-1}, x_{i-2}, \ldots, a)$. This implies that $d(b) \leq n-1-d(a)$: for each neighbour of a, there is a distinct non-neighbour of b. But this is a contradiction since $d(a)+d(b) \leq n-1$ while $\{a, b\} \notin E(G)$ and G does not satisfy the condition. $\square$

Thm. 1.2.8 (Pósa, 1962) *Let G be a graph on n vertiecs with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Then for every $k < n/2$, we have $d_k \geq k+1$, then G contains a Hamiltonian cycle.*

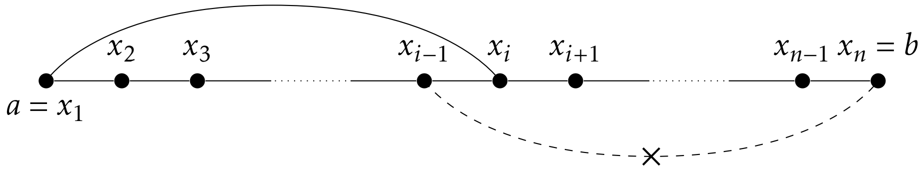
Proof Again, follows from Chvátal's Theorem below. $\square$

Thm. 1.2.9 (Chvátal, 1972)

(i) Let G be a graph on n vertices with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Assume that whenever for some $k < n/2$, if we have $d_k \leq k$, $d_{n-k} \geq n-k$. Then G contains a Hamiltonian cycle.

(ii) Assume that the number $d'_1 \leq d'_2 \leq \cdots d'_n$ do not satisfy the implication. Then there exists a graph G with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ such that $d_i \geq d'_i$ and G has no Hamiltonian cycle.

Proof (i) Assume that the statement is false, consider a counterexample G_0, and saturate it to obtain G. This works: adding edges only increases the degrees of the vertices, and thus the values of d_k and d_{n-k}, so the condition still holds. Now, from Ore's proof, we know that for every $a, b \in V(G)$ that are non-adjacent, there exists a Hamiltonian path from a to b and $d(a)+d(b) \leq n-1$. Consider an $a, b \in V(G)$, $\{a, b\} \notin E(G)$ pair for which $d(a)+d(b)$ is maximal



We may assume $d(a) \leq d(b)$ so that $d(a) \leq \frac{n-1}{2} < \frac{n}{2}$. For notation, fix $h = d(a)$; we claim that $d_h \leq h < \frac{n}{2}$. We show $d_h \leq h$.

Now consider the vertices x which are not neighbours of b. Since $\{x, b\}$ is a non-adjacent pair of b which is not maximal, we must have $d(x) \leq d(a) = h$. Furthermore, from the Ore proof argument, we know that the number of non-neighbours of b is at least as large as $d(a) = h$, so there are at least h vertices with $d(x) \leq h$. Thus $d_h \leq h$ and we have our claim.

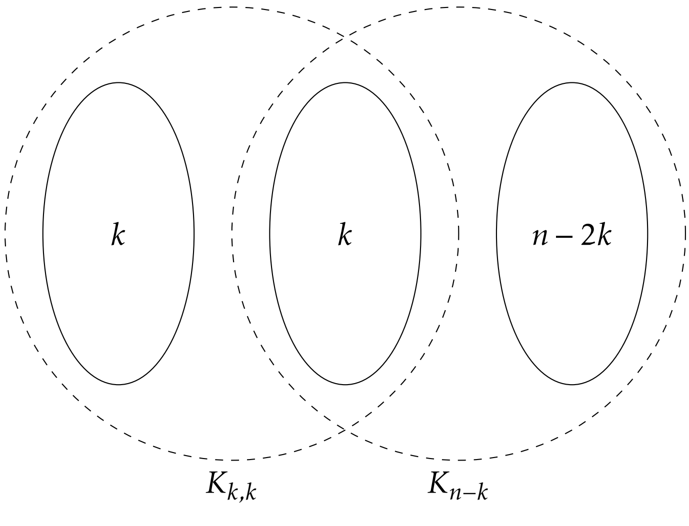
Thus by our condition, $d_{n-h} \geq n-h$, so there exist $h+1$ vertices with degree at least $n-h$. Since $d(a) = h$, there exists some non-neighbour y of a with $d(y) \geq n-h$. But then $\{a, y\}$ is a non-adjacent pair with $d(a) + d(y) \geq h + n - h = n > d(a) + d(b)$, a contradiction to the choice of $\{a, b\}$.

(ii) Assume $d'_1 \leq \cdots \leq d'_n$ violates our condition. Then we have k so that $d'_k \leq k < n/2$ and $d'_{n-k} \leq n-k-1$. Fix such a k, then

$$d'_1 \leq d'_2 \leq \cdots \leq d'_k \leq k$$

$$d'_{k+1} \leq d'_{k+2} \leq \cdots \leq d'_n \leq n-k-1$$

and clearly, $d'_{n-k+1} \leq \cdots \leq d'_n \leq n-1$. Set $d_1 = d_2 = \cdots = d_k := k$, $d_{k+1}, d_{k+2} = \cdots = d_{n-k} := n-k-1$ and $d_{n-k+1} = \cdots = d_n := n-1$. Thus it is enough to show a graph with these degrees and no Hamiltonian cycle. Here is such a graph:



It can be verified that the degrees in the first compoent are k, second are $n-1$ and third are $n-k-1$. Furthermore, the graph has no Hamiltonian cycle. Remove the k vertices of degree $N-1$, and we are left with $k+1$ components, so the necessary condiion for Hamiltonicity we had is violated. $\square$

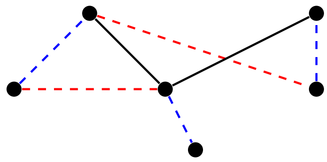
1.3 Matchings

Def'n. 1.3.1 *Two edges e, f are* ***independent*** *if $e \cap f = \emptyset$; i.e. they do not share any endpoints.*

Def'n. 1.3.2 *A* ***matching*** *in a graph is a subset $M \subset E(G)$ such that all edges in $E(G)$ are independent.*

Def'n. 1.3.3 *A matching is* ***maximal*** *if no edges can be added to it,* ***maximum*** *if there is no larger matching, and* ***perfect*** *if it covers all the vertices.*

All perfect matchings are maximum, and all maximum matchings are maximal. Below we have two matchings: a maximal matching in red, and a perfect and maximum matching in blue.



1.3.1 Bipartite Matchings

Def'n. 1.3.4 *A graph $G = (V, E)$ is* ***bipartite*** *if $V = A \cup B$ with $A \cap B = \emptyset$, and for all $e \in E(G)$, $|e \cap A| = |e \cap B| = 1$ (every edge has an endpoint in A and an endpoint in B).*

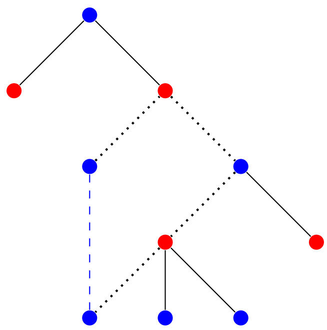
Bipartite graphs have a natural characterization:

Thm. 1.3.5 *A graph is bipartite if and only if it contains no odd cycles.*

Proof The forwards direction is easy. Thus suppose G contains no odd cycle. Observe that we can argue separately for each component, so we can assume G is connected. Thus G has a spanning tree $T \subseteq G$ such that $V(T) = V(G)$.

Note that T is bipartite: choose a starting node to belong to A, then there is a unique way to partition the vertices into A and B. Now put back the edges. If there is an edge between two vertices of the same class, then the endpoints have even distance from each other, so that when adding the edge we have an odd cycle.

See the diagram below for an illustration of this idea (the path is dotted, the added edge is blue and dashed). □



Thm. 1.3.6 (Hall) *A bipartite graph $G = (A, B; E)$ admits a matching covering A if and only if for every $U \subseteq A$, $V \subseteq B$, $|N(U)| \geq |U|$.*

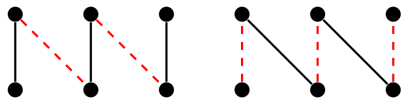
Cor. 1.3.7 (Frobenius) *A bipartite graph $G = (A, B; E)$ contains a perfect matching if and only if Hall's condition holds and $|A| = |B|$.*

Here are some definitions for matchings that will be useful in the proof:

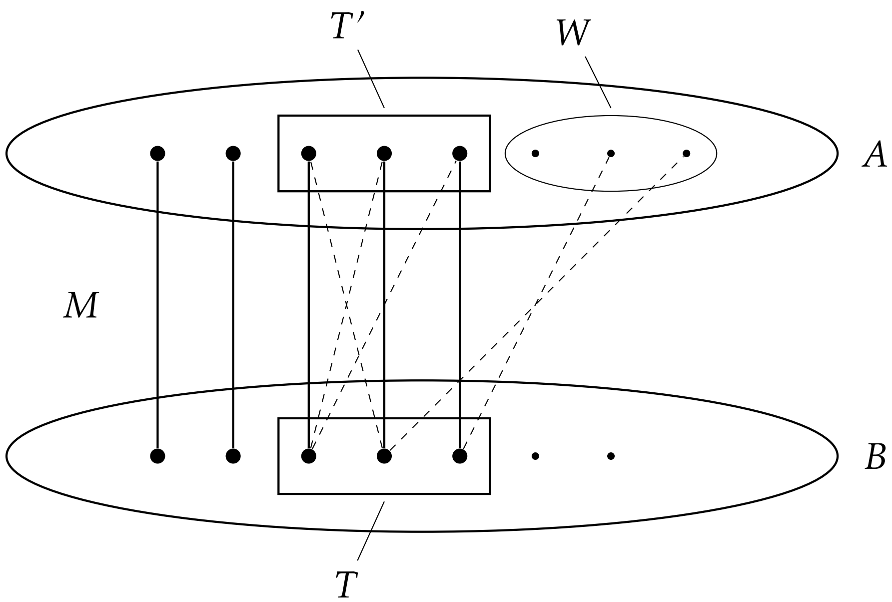
Def'n. 1.3.8 *Given a matching $M \subseteq E(G)$, an* ***alternating path*** *(with respect to M) is a path such that every second edge belongs to M.*

Def'n. 1.3.9 *An alternating path beginning and ending at vertices not covered by any matching edge (from M) is called an* ***augumenting path****.*

The intuitive reason to consider an augumenting path is that it is one such that the matching can be extended to a larger one by considering the complement of the edges in the matching with respect to the path. Below is an augumenting path along with a new matching that can be constructed from it (both in red).



Proof (Hall) Let $M \subseteq E(G)$ be a matching in a bipartite graph $G = (A, B; E)$ for which there exists no augumenting path. Assume M does not cover A, let the set of vertices in A not covered by M be called W, so $W \neq \emptyset$. Let $T \subseteq B$ be the set of vertices one can go to from W via an alternating path (w.r.t. M).



Every $v \in T$ is covered by M, otherwise there would exist an augumenting path w.r.t. M, but we assumed it does not exist. Let T' be the set of vertices in A that are the pairs of the vertices in T according to M. We have $|T'| = |T|$, and we claim that $U = T' \cup W$ violates the condition. In particular, we show that $N(T' \cup W) \subseteq T$. Write $N(T' \cup W) = N(T') \cup N(W)$ and we show each part separately.

1. $N(W) \subseteq T$ since any single edge starting at some $w \in W$ must be adjacent to a matching, so it is part of an alternating path and its other end must be in T by definition (of T).

2. $N(T') \subseteq T$. Let $v' \in T'$; then v' has a matched pair $v \in T$ (i.e., $\{v, v'\} \in M$) and since $v \in T$, there exists an alternating path from W to v. This path does not contain the matching edge $\{v, v'\}$, so $P \cup \{v, v'\}$ is still an alternating path and if z is a neighbour of v' not yet in the path, then $P \cup \{v, v'\} \cup \{v', z\}$ is as an alternating path that goes from W to z. Since $z \in T$ by definition, we have $N(T') \subseteq T$.

Thus $N(T' \cup W) \subseteq T$. But $|T| = |T'| < |T'| + |W| = |T' \cup W|$, violating Hall's condition. Thus if no subset of A violates the condition, then there still exists an augumenting path, so one can increase the size of M. If we cannot do this anymore, then either the condition is violated, or $W = \emptyset$ and M contains every vertex in A. □

1.3.2 Minimax: Matchings and Coverings

Def'n. 1.3.10 *The **matching number** $\nu(G)$ of a graph G is the size of the number of edges in a maximum matching.*

Def'n. 1.3.11 *The **(vertex) covering number** $\tau(G)$ of a graph G is the minimum number of vertices such that every edge contains a vertex.*

Note that $\nu(G) \leq \tau(G)$ since every edge in a matching needs a separate point in a vertex covering. Strict inequality can occur; for example any odd cycle. This is an example of a "minimax" property: if equality holds, then both values must be optimal.

Thm. 1.3.12 (König) *If G is bipartite, then $\tau(G) = \nu(G)$.*

Proof Let M be a maximum matching in a bipartite graph G, i.e. $|M| = \nu(G)$. Define W, T, T' as in the proof of Hall's Theorem. We proved that $N(T' \cup W) = T$. This implies that the vertices in T cover all edges with an endpoint in $T' \cup W$. Since every edge has an endpoint in A, the remaining edges are covered by the vertices in $A \setminus (T' \cup W)$. Thus $\tau(G) \leq |T| + |A \setminus (T' \cup W)| = |M|$ and equality holds. □

1.3.3 Tutte's Theorem and Applications

Let c_{odd} denotes the number of components with odd size; that is, the number of components with an odd number of vertices.

Thm. 1.3.13 (Tutte) *A graph G contains a perfect matching iff for any $S \subseteq V(G)$, we must have $c_{odd}(G \setminus S) \leq |S|$.*

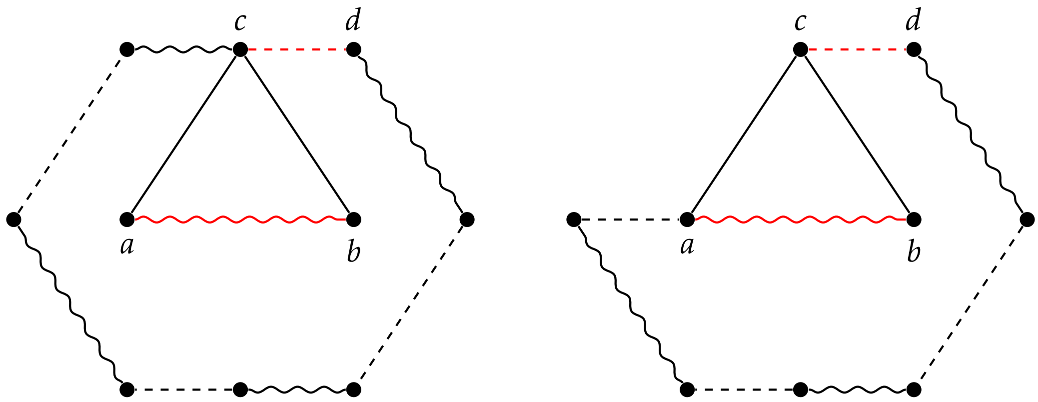
Proof (Lovász) ($\Rightarrow$) In each odd component of $G \setminus S$ at least one vertex must be matched to a vertex in S. Thus $|S|$ cannot be smaller than the number of such components, which is $c_{\text{odd}}(G \setminus S)$.

($\Leftarrow$) Assume the statement is false and let G_0 be a counterexample. Saturate G_0: add a maximal number of edges without creating a perfect matching. If the edge joins two even components, then we get a new even component; or two odd components, and we have a new even component; or an even and an odd component, and we have a new odd component. In any case, the number of odd components stays the same and the graph still satisfies the condition. Denote the saturated graph by G. As well, define $S := \{v \in V(G) : \{u, v\} \in E(G) \forall u \in V(G) \setminus \{v\}\}$ to be the set of vertices connected to every other vertex.

First assume that $G \setminus S$ results in a union of vertex disjoint complete graphs. Then in $G \setminus S$, we can create a perfect matching within each even component and every vertex except one in the odd components (since the components are complete). We can then pair these components arbitrarily in S, since S is connected to everything. Then there must be only an even number of vertices in S remaining, or G violates the condition with $S = \emptyset$; and these vertices are mutually connected.

Now assume not. Then there is some component that is not complete and get a, a' with $\{a, a'\} \notin E(G)$. Now consider a shortest path from a to a' within the component, and let a, c, b denote the first three vertices in the path. Then $c \notin S$ and $\{a, b\} \notin E(G)$. Thus we have some $d \in V(G)$ so that $\{c, d\} \notin E(G)$. Since G is saturated, there exists a perfect matching M_1 in $G \cup \{\{a, b\}\}$ and M_2 in $G \cup \{\{c, d\}\}$. Start a walk from d along alternating edges in M_1 and M_2. Let P be a longest such path. Then P can end only in a, b, or c: since the matching is perfect, every other vertex has an edge in each matching, and we can never return to an earlier vertex in the path. Then there are two cases.

The matching M_1 is drawn with squiggles, and the matching M_2 is drawn with dashes. The additional edges are drawn in red.

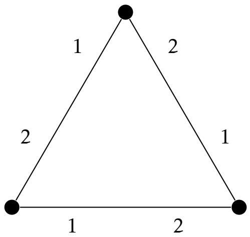


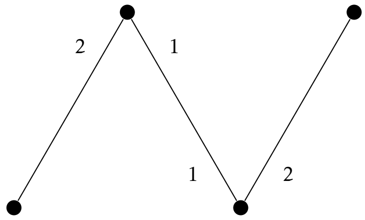
Suppose we end in c. Then the edge set $M_2 \setminus P \cup (M_1 \cap P)$ forms a perfect matching. If P ends in a or b, then without loss of generality we may assume it ends in a. Now consider the cycle $C = P \cup \{a, c\} \cup \{c, d\}$. Then $M_2 \setminus C \cup (M_1 \cap C)$ is a perfect matching. Thus a perfect matching exists in G, so the claim is true, so the theorem holds. □

1.3.4 Stable Matchings

Def'n. 1.3.14 *Let G be a graph with a linear order of the neighbours of $v \in V(G)$ for every $v \in V(G)$ (these are called **preference lists**). A matching $M \subseteq E(G)$ is said to be **stable** with respect to these preference lists if there exists no edge in $e \in E(G) \setminus M$, $e = \{a, b\}$, such that both a and b prefer each other more than it current pair according to M.*

A stable matching may not exist: for example, cyclic preferences in K_3 (see below). As well, a stable matching may not be maximum matching (though it must be maximal).





Thm. 1.3.15 (Gale-Shapley) *If G is bipartite, then there always exists (i.e. for any preference lists) a stable matching.*

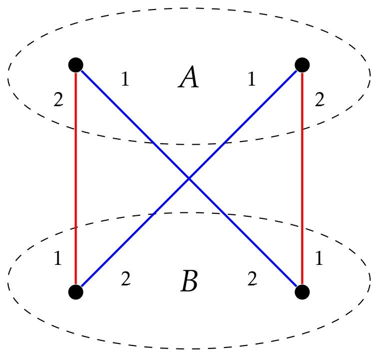
Proof We give an algorithm that produces a stable matching. It works in rounds. During the first round, each $b \in B$ "offers" a pairing to the neighbour that is first on their preference list. Then each $w \in W$ chooses the neighbour that is highest on their preference list, offers a wait to that neighbour, and denies all other offers. During the second round, each $b \in B$ who got an answer of "wait" will do nothing, and those who got denied make an offer to the second on their preference list. Now for each $w \in W$, do as in round 1: say "wait" for the best, and "no" to the others.

Repeat this process until there are no more new offers made. Clearly, this is a matching, so we show that it is stable as well. To see this, let $e \in E(G) \setminus M$ where M is the matching obtained. There can be two reasons $e = \{b, w\} \notin M$:

1. b never offered to w, but then it is because b got accepted by someone he prefers. Thus e is not a "destabalizing" edge.
2. b offered to w but got rejected. But then this happens because w got a better offer (according to her preference list), so she has a better partner. Thus e is not a "destabalizing" edge.

Thus the matching is stable. □

Interestingly, this leads to an optimal matching for the set B, but not for the set W. For example:



Both the red and blue matchings are stable, but the red matching favours B, while the blue matching favours A.

1.4 Colourings

We will get the full Canadian™ experience and spell "colour" properly throughout.

1.4.1 Vertex Colourings

Def'n. 1.4.1 *A **proper colouring** of a graph is a function $c : V(G) \to S$ (a set of "available colours") such that, for any $\{u,v\} \in E(G)$, $c(u) \neq c(v)$.*

Def'n. 1.4.2 *The minimum number of colours needed for a proper colouring of G is called the **chromatic number** $\chi(G)$ of G.*

Def'n. 1.4.3 *The **clique number** of G is denoted $\omega(G) := \max\{r : K_r \subseteq G\}$, in other words the size of the maximal complete subgraph of G.*

Rmk. 1.4.4 A graph is bipartite if and only if $\chi(G) = 2$.

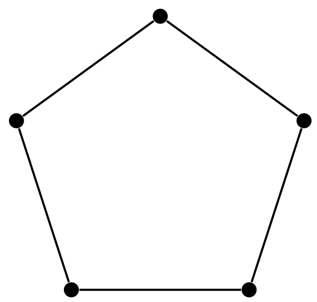
Recall that $\Delta(G)$ and $\delta(G)$ denote the largest and smallest degree elements, respectively. We certainly have $\chi(G) \leq \Delta(G) + 1$ by choosing colours greedily. However, greedy colourings certainly do not yield optimal colourings. We can refine this argument so that $\chi(G) \leq \max\{\delta(F) : F \subseteq G\} + 1$. We also have $\chi(G) \geq \omega(G)$. It is possible that $\chi(G) > \omega(G)$; for example, in the 5-cycle.

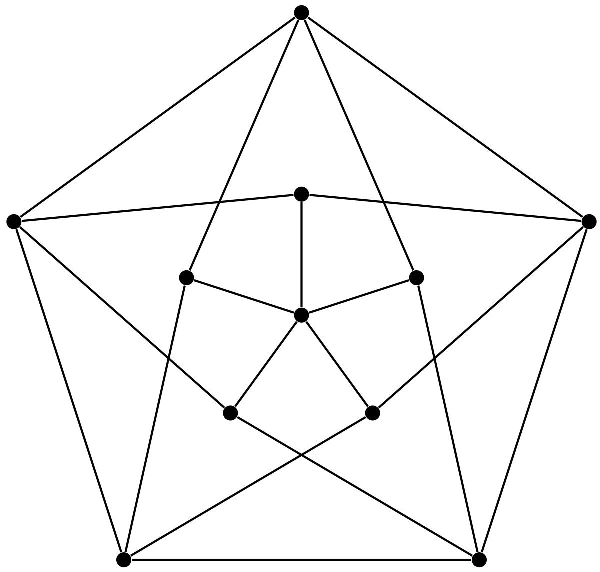
Thm. 1.4.5 *For all $k \in \mathbb{N}$, there exists G_k such that $\omega(G_k) = 2$ and $\chi(G_k) = k$.*

Proof (Mycielski's Construction) We construct a sequence of graphs $M_2, M_3, \ldots, M_k, \ldots$ such that for all i, $\omega(G_i) = 2$, $\chi(G_i) = i$. Define $M_2 = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$. Now define

$$V(M_{k+1}) := V(M_k) \cup \{u_i : i = 1, \ldots, n = |V(M_k)|\} \cup \{z\}$$
$$E(M_{k+1}) := E(M_k) \cup \Big\{\{u_i, u_j\} : \{v_i, v_j\} \in E(M_k)\Big\} \cup \{\{z, u_i\} : i = 1, \ldots, n = |V(M_k)|\}$$

Connect u_i to every neighbour of v_i, and z to every u_i.





We prove that M_k has clique number 2 by induction. Assume there exists K_3 in M_{k+1}. But then it cannot have two vertices in $U = \{u_1, u_2, \dots, u_n\}$ since U is independent, and cannot have vertex z since its only neighbours are in U. Thus K_3 must have vertices u_i, v_l, v_m; but then, v_i, v_l, v_m would be K_3, a contradiction.

Now we show that $\chi(M_{k+1}) = \chi(M_k) + 1 = k + 1$. We certainly have $\chi(M_{k+1}) \leq k + 1$ by colouring every vertex in U by a new colour, and the new vertex any of the original colours. Furthermore, $\chi(M_k) \geq k + 1$. Suppose for contradiction we have a k−colouring of M_{k+1}. We then get a colouring of the "M_k−part" of M_{k+1} as follows.

$$c'(v_i) = \begin{cases} c(v_i) & : c(v_i) \neq c(z) \\ c(u_i) & : c(v_i) = c(z) \end{cases}$$

We see that this is a valid colouring, so let $\{v_i, v_j\} \in E(M_k)$ so that $\{u_i, v_j\} \in E(M_{k+1})$.

If $\{v_i, v_j\} \in E(M_k)$, then $\{u_i, v_j\} \in E(M_{k+1})$ so that $c'(v_i) \neq c'(v_j)$. If $c(v_i) \neq c(z) \neq c(v_j)$, $c'(v_i) = c(v_i) \neq c(v_j) = c'(v_j)$. If $c(v_i) = c(z) = c(u_i)$, then $\{v_i, v_j\} \notin M$.

In any case, c' is a proper colouring, a contradiction so equality must hold. □

A stronger result was proven by Erdös using the "probabilistic method".

Thm. 1.4.6 (Erdös) *For all k, l, there exists G such that simultaneously $\chi(G) > k$ and $g(G) > l$, where $g(G)$ is the "girth", that is the length of the shortest cycle in G.*

Def'n. 1.4.7 *Given a graph G and a list $L(v)$ of available colours at every vertex $v \in V(G)$. A proper list colouring is a proper colouring such that for all $v \in V(G)$, $c(v) \in L(v)$.*

1.4.2 Edge Colourings

Def'n. 1.4.8 *A **proper edge colouring** of a graph G is a function $f : E(G) \to S$ such that if $e, e' \in E(G)$ and $e \cap e' \neq \emptyset$, then $f(e) \neq f(e')$. The minimum number of colours needed for a proper edge colouring of G is the **edge chromatic number** $\chi_e(G)$ or $\chi'(G)$.*

We certainly have $\chi_e(G) \geq \Delta(G)$. Furthermore, $3 = \chi_E(C_{2k+1}) > \Delta(C_{2k+1}) = 2$, so this inequality can hold strictly. However, it turns out that this is the worst case scenario, as described in the following theorem:

Thm. 1.4.9 (Vizing) *For any finite simple G, $\Delta(G) \leq \chi_e(G) \leq \Delta(G) + 1$.*

We also have the following theorem (which we will not prove).

Thm. 1.4.10 (Shannon) *For any finite G (perhaps with parallel edges), $\chi_e(G) \leq \frac{3}{2}\Delta(G)$.*

Before we prove Vizing's theorem, consider the following construction.

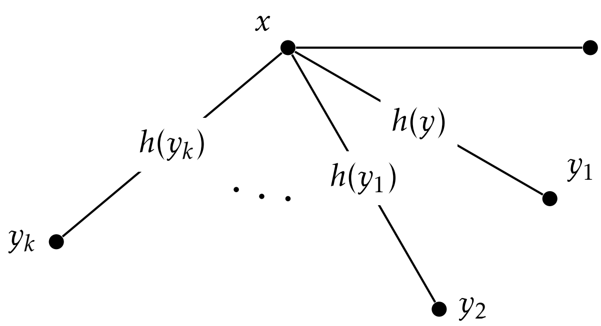
Def'n. 1.4.11 *For a graph G, its **line graph** $L(G)$ is defined as follows:*

$$\begin{aligned} V(L(G)) &= E(G) \\ E(L(G)) &= \{\{e, e'\} : e, e' \in E(G), e \cap e' \neq \emptyset\} \end{aligned}$$

In this sense, an edge colouring is a special case of vertex colouring. A proper vertex colouring of $L(G)$ is equivalent to a proper edge colouring of G so $\chi_e(G) = \chi(L(G))$. In fact, $\omega(L(G)) = \Delta(G)$ unless $\Delta(G) \leq 2$ and $K_3 \subseteq G$ (we claim this without proof). These graphs are collections of vertex disjoint paths and cycles. Now let's prove Vizing's theorem.

Proof (Vizing) Let G be finite and simple with $\Delta(G) =: \Delta$ for simplicity. We provide a construction for colouring the edges of G with $\Delta + 1$ colours. Assume we start colouring the edges of G from a set of $\Delta + 1$ colours and if we colour all edges, we are done. If we get stuck, we can resolve the situation as follows.

First, set some notation: for $v \in V(G)$, let $H(v)$ denote the set of "missing" colours of v. Since $\Delta + 1 > \Delta$, $H(v) \neq \emptyset$. Thus let $h(v)$ denote a "well-chosen" element of $H(v)$.



Assume we get stuck on some edge $\{x, y\}$. This means $H(x) \cap H(y) = \emptyset$, and choose $h(x), h(y)$ so $h(x) \neq h(y)$. Since $h(y) \notin H(x)$, there exists y_1 so that $\{x, y_1\} \in E(G)$ with colour $h(y)$. Suppose also that $h(y_1) \notin H(y_i)$, and get y_2 so that $\{x, y_2\} \in E(G)$ with colour $h(y_1)$.

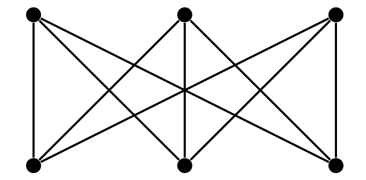
Repeat this, until we arrive at one of two terminal cases. If we arrive at some y_k so that $H(y_k) \cap H(x) \neq \emptyset$, then we can recursively recolour the previous vertices: colour $\{x, y_k\}$ with the available colour c_k. But then $\{x, y_{k-1}\}$ can be re-coloured $c_{k-1} = h(y_k)$; and continue this until we colour $\{x, y\}$ with $h(y_1)$ and the situation is resolved.

We can also find a "loop": we arrive at a y_k such that $H(y_k) \cap H(x) = \emptyset$ but $h(y_k)$ is not a new colour in the process and $h(y_k) = h(y_i)$ for some $0 \leq i < k$. Let F denote the subgraph of edges that are already coloured with either $h(x)$ or $h(y_k)$. We have $\Delta(F) \leq 2$, so F is the union of vertex disjoint paths or a cycles. Also, $d_F(x) = d_F(y_i) = d_F(y_k) = 1$, so they are all endpoints of some path component of F. In particular, they cannot all be in the same component, so y_i or y_k is in a different path component of F than the one containing x. Without loss of generality, assume it is y_k. Then flip the two colours $h(x)$, $h(y_k)$ in the component of y_k in the component of y_k, so $h(x)$ becomes available at y_k while it is still available at x. Then apply the recursive process: recolour $\{x, y_k\}$ with $h(x)$, and repeat until $\{x, y\}$ is coloured. □

1.4.3 List Colourings

Def'n. 1.4.12 *The* ***list-chromatic number*** *or* ***choice number*** ch(G) *of G is the minimum $k \in \mathbb{N}$ such that, whenever $v \in V(G)$, $|L(v)| \geq k$, we always have a proper list colouring.*

We certainly have $\text{ch}(G) \geq \chi(G)$: if we had $\text{ch}(G) < \chi(G)$, then there is no possible solution for identical lists all of size $\chi(G) - 1$. Also, we can have $\text{ch}(G) > \chi(G)$.



Con. 1.4.13 *For every finite loopless F,* $\mathrm{ch}(L(F)) = \chi(L(F))$.

Thm. 1.4.14 *For all $k \geq 2$, there exists G such that $\chi(G) = 2$ and* $\mathrm{ch}(G) > k$.

Proof Fix $n_k = \binom{2k-1}{k}$ and let $G_k = K_{n_k,n_k}$. Now for every vertex in A, assign a unique $l \in \binom{[2k-1]}{k}$ to each vertex; and similarly for B. We claim there is no proper list coloring with respect to these lists. For contradiction, assume there is a proper list coloring c. Then $|c(A)| := |\{c(v) : v \in A\}| \geq k$ since if $c(A) < k$, then there is a vertex v with a list not containing any of these colours. Similarly, $|c(G) \geq k|$. However, for a proper colouring, we need $c(A) \cap c(B) = \emptyset$ so $|c(A) \cup c(B)| \geq 2k$, a contradiction since we only have $2k - 1$ colours in all the lists. □

We now introduce the "Dinitz problem". Given an $n \times n$ matrix with a set of n numbers at each entry, is it always possible to choose for every entry one of the elements of the set so that we obtain a Latin square? A Latin square has the property that every row and column has distinct entries.

Equivalently, we ask the question: is it true that $\mathrm{ch}(L(K_{n,n})) = n$?

Thm. 1.4.15 (Galvin) *If F is bipartite and $G = L(F)$, then* $\mathrm{ch}(G) = \chi(G)$.

Cor. 1.4.16 *The answer to Dinitz's question is affirmative.*

Thm. 1.4.17 (König) *If F is bipartite, then $\chi_e(F) = \Delta(F)$.*

Note that we do not require that F is simple.

Proof If F is is regular of degree d, then this is true: get a perfect matching, colour all the edges the same, delete those edges, and we are left with a regular bipartite graph with degrees all $d - 1$. Repeat this $d - 1$ times to get a colouring. Furthermore, this also holds when we have multiple edges between vertices.

If F is not regular and has $\Delta(F) = d$, we extend to a (not necessarily simple) d−regular graph. We can assume $|A| = |B|$: if there are additional isolated vertices, we can ignore them. But then if F is not d−regular, there must be some vertex in A and a vertex in B with degree less than d. Join them, and repeat this until we are done. □

1.4.4 Galvin's Theorem

We are now in a position to prove Galvin's Theorem.

Def'n. 1.4.18 *Let G be a digraph. A subset $U \subseteq V(G)$ is called a* ***kernel*** *if the following conditions are satisfied:*

1. *U is an independent set*

2. *For every $v \in V(G) \setminus U$, there exists $u \in U$ so that $(v,u) \in E(G)$.*

Lemma 1.4.19 *Let G be a graph with a list of available colors $L(v)$ assigned to every vertex $v \in V(G)$. If G has an orientation satisfying the following two conditions, then G is properly list colorable from the given lists.*

1. *For all $v \in V(G)$, $|L(v)| \geq d_+(v)+1$.*
2. *Every subgraph of our oriented G has a kernel.*

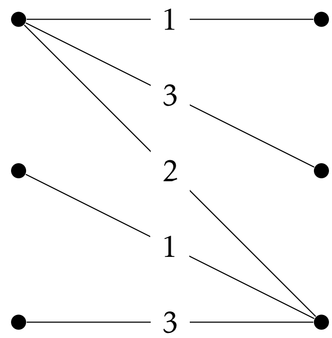
Proof Consider all vertices that have color 1 in their list and label them Y_1. The subgraph induced by Y_1 (in the oriented version of G) contains a kernel U_1. Colour vertices in U_1 with colour 1 and delete them and delete color 1 from every list of the remaining vertices.

Now for the remaining points, both conditions still hold. Condition 1 holds since either $L(v)$ remained the same or decreased by 1 color. In the latter case, since U_1 is a kernel, v must have at least 1 edge sent to U_1, so $d_+(v)$ is at least 1 smaller. Condition 2 also holds since the induced subgraphs are induced subgraphs of G. As well, our colouring so far is proper, and we will never violate the colouring later since we deleted colour 1 from all other vertices.

Now repeat this construction for all the colours in all the lists. We will never have isolated vertices since $|L(v)| \geq d_+(v)+1$. Eventually, the entire graph is colored: any vertex having only 1 color on its list already at some state of the process must have $d_+(v) = 0$ by condition 2, so when its only colour is considered, it will be a member of the corresponding kernel and thus become coloured with its only available colour. □

Let's restate Galvin's theorem here.

Thm. 1.4.20 (Galvin) *If F is bipartite and $G = L(F)$, then* $\mathrm{ch}(G) = \chi(G)$.



Proof (Galvin 1995) Consider an optimal colouring of $G = L(F)$, i.e. the edges of F with colour $1,2,\ldots,\chi(G)[=\chi_e(F) = \Delta(F)]$. We create the following orientation of G: for $e,f \in V(G) = E(F)$ with $e \cap f \neq \emptyset$, orient the edge $\{e,f\} \in E(G)$ as (e,f) if either $e \cap f \in A$ and $c(f) < c(f)$ or $e \cap f \in B$ and $c(f) < c(e)$. By the Lemma, it suffices to show that the two conditions hold for this orientation and lists of size $\chi(G)$.

To see condition 1 of the lemma, consider $e \in E(F) = V(G)$. Then $d_+(e) \leq \chi(G) - c(e) + c(e) - 1 = \chi(G) - 1 = |L(e)| - 1$.

To see condition 2, consider the linear order consistent with our orientation at every vertex $v \in V(G)$ as a preference list of the neighbours of v in F, and realize that a stable matching is the equivalent to a kernel in the oriented G. Since every induced subgraph of G belongs to a subgraph of F, which is certainly bipartite, we have a stable matching by Gale-Shapley, our induced subgraph will have a kernel. □

1.5 Planarity

In order to transition to the notion of planarity, let's talk about the 4-colour theorem.

Def'n. 1.5.1 *A graph is called **planar** if it can be drawn on the plane such that edges do not cross.*

A graph is planar if and only if it can be drawn on the sphere S_2 without edge crossings. This follows by the stereographic projection. In general, our goal is to characterize planarity.

1.5.1 Euler's Formula

If G is a connected planar graph with n vertices, e edges, and it is drawn on the plane with f faces, then $n+f=e+2$. Note that f is counted so that the infinite fase is also taken into account.

Now observe that if L is a convex polyhedron, its graph (where the vertices are the vertices of L, and the edges are the edges of L) is always planar. Therefore Euler's formula has the consequence that if L is a convex polyhedron with n vertices, e edges, and f faces, then $n+f=e+2$.

Proof Consider a connected graph G with a planar embedding on n vertices, e edges, and f faces. If G has no cycles, then it is a tree, so $f=1$ and $e=n-1$ so $n+f=e+2$. If G contains a cycle, consider a cycle that is a boundary of some face and delete one of its edges. Then n remains the same, e and f both decreases by 1. Thus the truth value of the statement remains the same, so we continue this process until we have no cycles and are left with a tree. For a tree, the statement holds, so the statement holds for the initial graph. □

Thm. 1.5.2 *If G is a planar graph with n vertices and e edges, then $e \leq 3n-6$.*

Proof Let G be embedded on the plane and let the number of faces with exactly i edges at their boundary be denoted f_i. We use the convention of counting pendant edges twice. Then we can write $f=f_3+f_4+\cdots+f_k$. As well, $3f_3+4f_4+\cdots+kf_k=2e$ and

$$2e = 3f_3+4f_4+\cdots+kf_k \geq 3(f_3+f_4+\cdots+f_k) = 3f$$

so $f \leq 2e/3$. Now applying Euler's Formula gives $n+\frac{2}{3}e \geq e+2$ so that $3n-6 \geq e$. □

Cor. 1.5.3 *K_5 is not planar.*

Proof We have 5 vertices and 10 edges, contradicting the above inequality. □

Thm. 1.5.4 *If G is planar and contains no C_3, then $|E(G)| \leq 2|V(G)-4$.*

Proof Consider a planar drawing of G. Let the number of faces with exactly i edges on their boundary be f_i. Then $f=f_3+f_4+\cdots+f_k$ for some k, where for G $f_3=0$ by $C_3 \not\subseteq G$. Then by $2e=\sum_{i=3}^{k} if_i = \sum_{i=4}^{k} if_i \geq \sum_{i=4}^{k} 4f_i = 4f$. Thus $f \leq e/2$, so $n+e/2 \geq e+2$ and $2n-4 \geq e$. □

Cor. 1.5.5 *$K_{3,3}$ is not planar.*

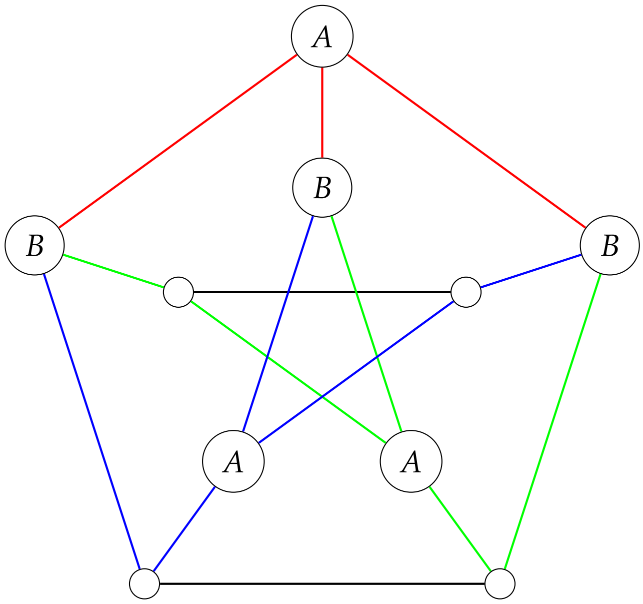
Proof $C_3 \not\subseteq K_{3,3}$, but $g = |E(K_{3,3})| \not\leq 2|V(K_{3,3}) - 4| = 8$. □

We know that G cannot be planar if K_5 or $K_{3,3}$ or any edge subdivision is a subgraph of G. Surprisingly, the converse holds as well:

Thm. 1.5.6 (Kuratowski) *G is planar if and only if it has no subgraph isomorphic to a possibly subdivided version of K_5 or $K_{3,3}$.*

Def'n. 1.5.7 *Two graphs are called **topologically isomorphic** if they can be obtained from each other by edge subdivision and its inverse operation.*

Is the Petersen graph planar? No, it contains an edge-subdivided $K_{3,3}$:



Def'n. 1.5.8 *A graph F is a **minor** of another graph G is F can be obtained from G by subsequently performing the following operations:*

- *Deleting a vertex*
- *Deleting an edge*
- *Contracting an edge*

Only allowing the first operation results in an induced subgraph, and allowing the first two operations results in an arbitrary subgraph.

Furthermore, if F is a topological subgraph of G, then F is also a minor of G.

Thm. 1.5.9 (Wagner) *G is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.*

One can check that indeed the following two properties for a graph are equivalent:

- G does not contain a topological K_5 or $K_{3,3}$
- G does not contain a K_5 or $K_{3,3}$ minor.

One can think of a topological minor as a connected component (equiv a tree) that was contracted to obtain each point, along with unique edges between the disjoint trees. Then one can find branching points which can be used to form the topological minor.

However this does not necessarily work for K_5. There are two cases: there is a vertex of degree 4 which branches to every other vertex in each component (and we have K_5 as a topological minor), or there are two vertices which branch to 3 trees each. Then label these two branching points differently, and then join the remaining branching points to finish $K_{3,3}$.

Thm. 1.5.10 (Robertson-Seymour) *In any infinite sequence of finite simple graphs, there are two such that one is a minor of the other.*