

Course Notes

Real Functions and Measures

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Chapter 1

Introduction

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1.1 Basic Topology

1.1.1 Examples and Definitions

We are given a set $X \neq \emptyset$ with a topology $\tau \subseteq \mathcal{P}(X)$ satisfying the following conditions:

1. $\emptyset \in \tau$ $X \in \tau$
2. $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3. $V_\alpha \in \tau$ for all $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

Then the pair (X, τ) is called a topological space, and the elements of τ are called open sets. We have some standard examples of topological spaces:

1. The open sets in \mathbb{R}
2. The discrete topology, $\tau = \mathcal{P}(X)$ for any $X \neq \emptyset$
3. The antidiscrete topology, $\tau = \{\emptyset, X\}$ for any $X \neq \emptyset$
4. Extended real line, $X = \mathbb{R} \cup \{-\infty, +\infty\}$. Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

The same can be done with a single symbol as well. In either case, the extended real line is a compact set.

5. Any metric spaces induces a topology. Consider a set $X \neq \emptyset$ arbitrary, and let $d : X \times X \rightarrow \mathbb{R}$ such that

- (a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$.
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$

Then $G \in \tau$ if and only if for any $x \in G$, there exists r so that $B_r(x) \subset G$. Classic examples of metric spaces include

- (a) $X = \mathbb{R}, d(x, y) = |x - y|$
- (b) $X = \mathbb{R}, d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
- (c) $X = \mathbb{R}^2, d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- (d) $X = \mathbb{R}^2, d(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$ for $p \geq 1$.
- (e) and similarly for $X = \mathbb{R}^n$
- (f) $X = C[0, 1], d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$.
- (g) normed space: X is a vector space over $\mathbb{R}, \|\cdot\| : X \rightarrow \mathbb{R}$ such that
 - i. $\|x\| = 0$ if and only if $x = 0$
 - ii. $\|cx\| = |c|\|x\|$
 - iii. $\|x + y\| \leq \|x\| + \|y\|$
 If $\|\cdot\|$ is a norm, then $d(x, y) = \|x - y\|$ is a metric.

6. The cofinite topology

1.1.2 Definitions

Def'n. 1.1.1 $U \subseteq X$ is a neighbourhood of $x \in X$ if there is some $G \in \tau$ such that $x \in G \subset U$.

Def'n. 1.1.2 $F \subseteq X$ is closed if F^c is open.

Def'n. 1.1.3 The closure of a set $E \subset X$ is the smallest closed set containing E (denoted \bar{E}).

Def'n. 1.1.4 x is an accumulation point of H if all neighbourhoods of x contains infinitely points of H . Equivalently, x is a limit point of $H \setminus \{x\}$.

Def'n. 1.1.5 If $H \subseteq X$, we have a natural subspace topology $\tau|_H = \{G \cap H : G \in \tau\}$. Homework: prove that $(H, \tau|_H)$ is a topological space. Homework: prove that $x \in \bar{E}$ if and only if for any neighbourhood U of x we have $U \cap E \neq \emptyset$.

1.2 Limits and Continuity

Def'n. 1.2.1 Let $(x_n) \subset X$ be a sequence and let $x \in X$. Then x is the limit of (x_n) if for any neighbourhood U of x , there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$.

Def'n. 1.2.2 Let $f : X \rightarrow Y$ be a function. Let $x \in X$ be an accumulation point of $D(f)$. The limit of f at x is $y \in Y$ if for any neighbourhood V of y there exists a neighbourhood U of x such that $f(z) \in V$ for all $z \in U \cap D(f) \setminus \{x\}$.

Def'n. 1.2.3 Let $f : X \rightarrow Y$ be a function, and let $x \in D(f)$. Then f is continuous at x if for any neighbourhood V of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of x .

Def'n. 1.2.4 $f : X \rightarrow Y$ is called continuous if it is continuous at every point.

Prop. 1.2.5 $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(G)$ is open for all G open.

1.3 Compact Sets

Let (X, τ) be a topological space.

Def'n. 1.3.1 A set $K \subset X$ is compact if every open cover of K contains a finite subcover.

Def'n. 1.3.2 A topological space is called locally compact if every point has a compact neighbourhood.

Prop. 1.3.3 $C[0, 1]$ with the sup norm is not locally compact.

PROOF Exercise. □

1. On the discrete topology, K is compact if and only if K is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of X).
3. On (\mathbb{R}, τ_e) , K is compact if and only if K is closed and bounded.
4. On (X, d) metric space, K is compact if and only if K is complete and totally bounded.

Def'n. 1.3.4 A topological space is called Hausdorff if for any $x \neq y$, there exists neighbourhoods $U \ni x$, $V \ni y$ so that $U \cap V = \emptyset$.

The anti-discrete topology is not Hausdorff.

Prop. 1.3.5 1. Let $K \subset X$ be compact, let $F \subset K$ closed. Then F is also compact.

2. Compact sets in a Hausdorff space are closed.

PROOF 1. Let $F \subset \bigcup V_\alpha$. Then $K \subset F^c \cup (\bigcup V_\alpha)$ is an open cover for K , so it has a finite subcover $F^c \cup V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$. But then since $F \cap F^c = \emptyset$, $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ is a finite subcover.

2. Let $K \subset X$ be compact, and prove that K^c is open. Thus let $x \in K^c$. For any $y \in K$, there exist U_y, V_y disjoint neighbourhoods of x and y respectively. Now consider the open cover $K \subset \bigcup_{y \in K} V_y$, and get our finite subcover $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. But then $U_{y_1} \cap \dots \cap U_{y_n} \cap K = \emptyset$ and is open since it is a finite intersection. □

Thm. 1.3.6 Let $f : X \rightarrow Y$ be continuous and $K \subset X$ be compact. Then $f(K)$ is compact.

PROOF Recall that continuous functions pull back open sets. Let $f(K) \subset \bigcup U_\alpha$ be an open cover. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover for K , and has a finite subcover $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. But then $f(f^{-1}(U_{\alpha_1})) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$ is a subcover of $f(K)$. □

1.4 Measure Theory

Def'n. 1.4.1 Let $X \neq \emptyset$ be a set. $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ -algebra if

1. $X \in \mathcal{M}$
2. $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
3. If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$

The pair (X, \mathcal{M}) is called a measurable space. The elements of \mathcal{M} are called measurable sets.

Def'n. 1.4.2 Let (X, \mathcal{M}) be a measurable space, (Y, τ) be a topological space. Then $f : X \rightarrow Y$ is called measurable if $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$.

Ex. 1.4.3 1. $\mathcal{M} = \{\emptyset, X\}$ is a σ -algebra.

2. $\mathcal{P}(X) = \mathcal{M}$ is a σ -algebra.

3. $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable.}\}$. To see this, given $A_n \in \mathcal{M}$, if everything is countable, then $\bigcup A_n$ is countable. If some A_i is countable, then $(\bigcup A_n)^c = \bigcap A_n^c$ is countable, so $\bigcup A_n \in \mathcal{M}$.

We will later see some proper examples, like the σ -algebra of Lebesgue measurable sets.

Prop. 1.4.4 1. $\emptyset \in \mathcal{M}$ since $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$.

2. $A_1, A_2, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{M}$ since we can take $A_{n+1} = \emptyset$, etc.

3. $A_n \in \mathcal{M} \forall n \in \mathbb{N}$ so $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$. This can be seen since $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$.

4. $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$ since $A \setminus B = A \cap B^c \in \mathcal{M}$.

5. f is measurable, $H \subset Y$ is closed, then $f^{-1}(H) \in \mathcal{M}$. To see this, H^c is open implies $f^{-1}(H^c) \in \mathcal{M}$. Then $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$.

Prop. 1.4.5 Let $f : X \rightarrow Y$ be measurable, let $g : Y \rightarrow Z$ be continuous, then $g \circ f : X \rightarrow Z$ is measurable.

PROOF Let $V \subset Z$ be open, so $g^{-1}(V) \subset Y$ is open, so $f^{-1}(g^{-1}(V)) \in \mathcal{M}$ which is $(g \circ f)^{-1}(V)$. \square

Prop. 1.4.6 Let (X, \mathcal{M}) be a measurable space, Y be a topological space. Let $\phi : \mathbb{R}^2 \rightarrow Y$ be continuous. If $u, v : X \rightarrow \mathbb{R}$ are measurable, then $h(x) = \phi(u(x), v(x))$ is measurable.

PROOF Let $f(x) = (u(x), v(x))$, $f : X \rightarrow \mathbb{R}^2$. We will see that f is measurable, so that $h = \phi \circ f$ is measurable. Let $I_1, I_2 \subset \mathbb{R}$ be open intervals, so $R = I_1 \times I_2$ is a rectangle. Then $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$. Let $G \subset \mathbb{R}^2$ be an open set, so there exist R_n open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that f is measurable. \square

Cor. 1.4.7 1. If $u, v : X \rightarrow \mathbb{R}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.

2. $u + iv : X \rightarrow \mathbb{C}$ is measurable.

3. $f : X \rightarrow \mathbb{C}$ is measurable, $f = u + iv \Rightarrow u, v, |f|$ are measurable.

Prop. 1.4.8 Define

$$\chi_E(X) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then χ_E is measurable if and only if $E \in \mathcal{M}$.

Thm. 1.4.9 Let $\mathcal{F} \subset \mathcal{P}(X)$, then there exists a smallest σ -algebra containing \mathcal{F} . This is denoted by $S(\mathcal{F})$, the σ -algebra generated by \mathcal{F} .

PROOF Let $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra, } \mathcal{F} \subset \mathcal{M}\}$. Certainly $\Omega \neq \emptyset$ since $\mathcal{P}(X) \in \Omega$. Let $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$. We will see that $S(\mathcal{F})$ is a σ -algebra.

(i) $X \in \mathcal{M} \forall \mathcal{M} \Rightarrow X \in \bigcap \mathcal{M}$.

(ii) $A \in S(\mathcal{F}) \Rightarrow A \in \mathcal{M} \forall \mathcal{M} \Rightarrow A^c \in \mathcal{M} \forall \mathcal{M} \Rightarrow A^c \in \bigcap \mathcal{M}$.

(iii) $A_n \in S(\mathcal{F}) \forall n \Rightarrow A_n \in \mathcal{M} \forall \mathcal{M} \Rightarrow \bigcup A_n \in \mathcal{M} \forall \mathcal{M} \Rightarrow \bigcup A_n \in \bigcap \mathcal{M}$.

It is obvious that $\mathcal{F} \subset \bigcap \mathcal{M}$. $S(\mathcal{F})$ is minimal, since if $\mathcal{F} \subset \mathcal{N}$ is a σ -algebra, then $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$. \square

Def'n. 1.4.10 Let (X, τ) be a topological space. Then $\mathcal{B} = S(\tau)$ is called the Borel σ -algebra. Borel sets are the elements of $S(\tau)$. A function $f : X \rightarrow Y$ is Borel measurable if $f^{-1}(G) \in \mathcal{B}$ for all $G \subset Y$ open.

Prop. 1.4.11 1. If $F \subset X$ is closed, then $F \in \mathcal{B}$. To see this, if $F^c \in \tau$ then $F \in S(\tau)$.

2. $G_n \subset X$ are open, then $\bigcap_{n=1}^{\infty} G_n \in \mathcal{B}$. These are called G_δ -sets.

3. $F_n \subset X$ are closed, then $\bigcup_{n=1}^{\infty} F_n \in \mathcal{B}$. These are called F_σ -sets.

Ex. 1.4.12 $X = \mathbb{R}, \tau_e$, then $\mathcal{B} = S(\tau_e)$. Let $\Gamma_0 = \{(a, b) : a < b\}$ be a family of open intervals. We see that $S(\Gamma_0) = \mathcal{B}$. Since $\Gamma_0 \subset \tau$, $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$. Conversely, let $G \in \tau$, then we have open intervals $G = \bigcup_{n=1}^{\infty} I_n$ so that $G \in S(\Gamma_0)$. Thus $S(\tau) \subset S(\Gamma_0)$ and $S(\Gamma_0) = \mathcal{B}$.

Ex. 1.4.13 Let $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$. I claim that $S(\Gamma_\infty) = \mathcal{B}$. Certainly $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$. Then $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$. Similarly, $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n] \in S(\Gamma_\infty)$. Thus $(a, \infty) \cap (-\infty, b) = (a, b) \in S(\Gamma_0)$, and using the previous example, $\mathcal{B} = S(\Gamma_\infty)$.

Prop. 1.4.14 Let (X, \mathcal{M}) be a measurable space, and let $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the euclidean topology. If $f^{-1}((\alpha, \infty]) \in \mathcal{M} \forall \alpha \in \mathbb{R}$, then f is measurable.

PROOF We have $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty)))^c \in \mathcal{M}$. Similarly, $f^{-1}([-\infty, \alpha)) = f^{-1}(\bigcap [-\infty, \alpha - 1/n]) = \bigcup f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$.

We then have

$$f^{-1}((\alpha, \beta))f^{-1}([-\infty, \beta) \cap (\alpha, \infty)) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty)) \in \mathcal{M}$$

Thus if $G \subset \overline{\mathbb{R}}$ is open, then there exists open intervals so that $G = \bigcup_{n=1}^{\infty} I_n$ satisfies

$$f^{-1}(G) = f^{-1}\left(\bigcup I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

□

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

Def'n. 1.4.15 Let $(a_n)_{n \in \mathbb{N}} \subset \overline{\mathbb{R}}$ be a sequence, and $b_k = \sup\{a_k, a_{k+1}, \dots\}$. Then $\beta = \inf_{k \in \mathbb{N}} b_k$ is called the \limsup of (a_n) . We can similarly define $c_k = \inf\{a_k, a_{k+1}, \dots\}$ and $\liminf = \sup_{k \in \mathbb{N}} c_k$.