

# **Course Notes**

## **Real Functions and Measures**

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# Chapter 1

## Basics of Abstract Measure Theory

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### 1.1 Review of Topology

#### 1.1.1 Basic Definitions

**Def'n. 1.1.1** Let  $X \neq \emptyset$  and  $\tau \subseteq \mathcal{P}(X)$ . We say that  $(X, \tau)$  is a **topological space** if  $\tau$  satisfies the following conditions:

1.  $\emptyset \in \tau$   $X \in \tau$
2.  $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
3.  $V_\alpha \in \tau$  for all  $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} V_\alpha \in \tau$

We call the elements of  $\tau$  **open sets**.

**Def'n. 1.1.2**  $U \subseteq X$  is a **neighbourhood** of  $x \in X$  if there is some  $G \in \tau$  such that  $x \in G \subset U$ .

**Def'n. 1.1.3**  $F \subseteq X$  is **closed** if  $F^c$  is open.

**Def'n. 1.1.4** The **closure** of a set  $E \subset X$  is the smallest closed set containing  $E$  (denoted  $\bar{E}$ ).

**Def'n. 1.1.5**  $x$  is an **accumulation point** of  $H$  if all neighbourhoods of  $x$  contains infinitely points of  $H$ . Equivalently,  $x$  is a **limit point** of  $H \setminus \{x\}$ .

**Def'n. 1.1.6** If  $H \subseteq X$ , we have a natural subspace topology  $\tau|_H = \{G \cap H : G \in \tau\}$ .

#### 1.1.2 Examples of Topological Spaces

Topological spaces are a very general construction, so here are some of the standard examples:

1.  $\mathbb{R}$  along with the open sets (denoted  $\tau_e$ , the Euclidean topology).
2. The discrete topology,  $\tau = \mathcal{P}(X)$  for any  $X \neq \emptyset$ . This is the “finest” topology.

3. The antidiscrete topology,  $\tau = \{\emptyset, X\}$  for any  $X \neq \emptyset$ . This is the “coarsest” topology.
4. One can define the extended real line,  $X = \mathbb{R} \cup \{-\infty, +\infty\}$ . Then

$$G \in \tau \Leftrightarrow \begin{cases} \forall x \in G \cap \mathbb{R} & \exists r > 0 \text{ s.t. } (x-r, x+r) \subset G \\ -\infty \in G & \exists b \in \mathbb{R} \text{ s.t. } (-\infty, b) \subset G \\ +\infty \in G & \exists a \in \mathbb{R} \text{ s.t. } (a, \infty) \subset G \end{cases}$$

The same can be done with a single symbol as well. In either case, the extended real line is a compact set.

5. Any metric spaces induces a topology. Consider a set  $X \neq \emptyset$  arbitrary, and let  $d : X \times X \rightarrow \mathbb{R}$  such that

- (a)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0 \Leftrightarrow x = y$ .
- (b)  $d(x, y) = d(y, x)$  for all  $x, y \in X$
- (c)  $d(x, y) \leq d(x, z) + d(z, y)$  for any  $x, y, z \in X$

Then  $G \in \tau$  if and only if for any  $x \in G$ , there exists  $r$  so that  $B_r(x) \subset G$ . There are many examples of metric spaces:

- (a)  $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$
- (b)  $X = \mathbb{R}$ ,  $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
- (c)  $X = \mathbb{R}^2$ ,  $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- (d)  $X = \mathbb{R}^2$ ,  $d(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$  for  $p \geq 1$ .
- (e) and similarly for  $X = \mathbb{R}^n$
- (f)  $X = C[0, 1]$ ,  $d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$ .
- (g) normed space:  $X$  is a vector space over  $\mathbb{R}$ ,  $\|\cdot\| : X \rightarrow \mathbb{R}$  such that
  - i.  $\|x\| = 0$  if and only if  $x = 0$
  - ii.  $\|cx\| = |c| \|x\|$
  - iii.  $\|x + y\| \leq \|x\| + \|y\|$
 If  $\|\cdot\|$  is a norm, then  $d(x, y) = \|x - y\|$  is a metric.

6. The cofinite topology:  $\tau = \{U \in \mathcal{P}(X) : U^c \text{ is finite}\}$ .

### 1.1.3 Other Definitions

**Def’n. 1.1.7**  $K \subset X$  is **compact** if every open cover of  $K$  contains a finite subcover.

**Def’n. 1.1.8** A topological space is called **locally compact** if every point has a compact neighbourhood.

**Prop. 1.1.9**  $C[0, 1]$  with the sup norm is not locally compact.

PROOF I’ll do this later.

□

**Def'n. 1.1.10** A topological space is called **Hausdorff** if for any  $x \neq y$ , there exists neighbourhoods  $U \ni x$ ,  $V \ni y$  so that  $U \cap V = \emptyset$ .

The anti-discrete topology is not Hausdorff.

1. On the discrete topology,  $K$  is compact if and only if  $K$  is finite.
2. On the anti-discrete topology, everything is compact (the only possible open cover consists of  $X$ ).
3. On  $(\mathbb{R}, \tau_e)$ ,  $K$  is compact if and only if  $K$  is closed and bounded.
4. On  $(X, d)$  metric space,  $K$  is compact if and only if  $K$  is complete and totally bounded.

**Prop. 1.1.11** 1. Let  $K \subset X$  be compact, let  $F \subset K$  closed. Then  $F$  is also compact.  
2. Compact sets in a Hausdorff space are closed.

**PROOF** 1. Let  $F \subset \bigcup V_\alpha$ . Then  $K \subset F^c \cup (\bigcup V_\alpha)$  is an open cover for  $K$ , so it has a finite subcover  $F^c \cup V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ . But then since  $F \cap F^c = \emptyset$ ,  $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$  is a finite subcover.  
2. Let  $K \subset X$  be compact, and prove that  $K^c$  is open. Thus let  $x \in K^c$ . For any  $y \in K$ , there exist  $U_y, V_y$  disjoint neighbourhoods of  $x$  and  $y$  respectively. Now consider the open cover  $K \subset \bigcup_{y \in K} V_y$ , and get our finite subcover  $K \subset V_{y_1} \cup \dots \cup V_{y_n}$ . But then  $U_{y_1} \cap \dots \cap U_{y_n} \cap K = \emptyset$  and is open since it is a finite intersection.  $\square$

**Def'n. 1.1.12**  $\Gamma \subseteq \tau$  is a **base** for  $\tau$  if every  $U \in \tau$  can be written as a countable union of the elements of  $\Gamma$ .  $\Gamma$  is a **countable base** if  $\Gamma$  is countable.

**Prop. 1.1.13**  $\mathbb{R}$  has a countable base of intervals.

**PROOF** Consider the collection  $\{B_r(q) : (r, q) \in \mathbb{Q} \times \mathbb{Q}\}$ . To see this, for any open set  $U$ , one can write

$$S := \bigcup_{r \in U \cap \mathbb{Q}} \left( \bigcup_{\{r: B_r(q) \subseteq U\}} B_r(q) \right)$$

$U \supseteq S$  is obvious, so let  $x \in U$  be arbitrary, and let  $s$  be maximal so that  $B_s(x) \subseteq U$ . Then choose  $q \in \mathbb{Q}$  so that  $|x - q| < s/3$  and  $r \in \mathbb{Q}$  so that  $0 < r < s/2$ . Then by construction  $B_r(q) \ni x$  and by the triangle inequality  $B_{r/2}(q) \subseteq U$ , so  $x \in S$ . Thus  $U = S$  as desired.  $\square$

Note that the exact same argument (with some work) can be generalized to show that  $\mathbb{R}^n$  has a countable base of open hyperrectangles.

**Prop. 1.1.14** Every metric space which is a countable union of compact sets has a countable base.

**PROOF** See my PMATH 351 notes.  $\square$

## 1.1.4 Functions and Continuity

Many of the standard notions of limits and continuity extend naturally to topological spaces.

**Def'n. 1.1.15** Let  $(x_n) \subset X$  be a sequence and let  $x \in X$ . Then  $x$  is the **limit** of  $(x_n)$  if for any neighbourhood  $U$  of  $x$ , there exists  $N \in \mathbb{N}$  such that  $n > N \Rightarrow x_n \in U$ .

**Prop. 1.1.16** If  $F \subset X$  is closed, then for all convergent sequences in  $F$ , the limit is also in  $F$ .

PROOF See Homework. □

**Def'n. 1.1.17** Let  $f : X \rightarrow Y$  be a function, and  $x \in X$  an accumulation point of  $D(f)$ . The limit of  $f$  at  $x$  is  $y \in Y$  if for any neighbourhood  $V$  of  $y$  there exists a neighbourhood  $U$  of  $x$  such that  $f(U \cap D(f) \setminus \{x\}) \subseteq V$ .

**Def'n. 1.1.18** Let  $f : X \rightarrow Y$  be a function, and let  $x \in D(f)$ . Then  $f$  is **continuous at  $x$**  if for any neighbourhood  $V$  of  $f(x)$ , then  $f^{-1}(V)$  is a neighbourhood of  $x$ .

**Def'n. 1.1.19**  $f : X \rightarrow Y$  is called **continuous** if it is continuous at every point.

**Prop. 1.1.20**  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(G)$  is open for all  $G$  open.

PROOF Exercise. □

**Thm. 1.1.21** Let  $f : X \rightarrow Y$  be continuous and  $K \subset X$  be compact. Then  $f(K)$  is compact.

PROOF Recall that continuous functions pull back open sets. Let  $f(K) \subset \bigcup U_\alpha$  be an open cover. Then  $\bigcup f^{-1}(U_\alpha)$  is an open cover for  $K$ , and has a finite subcover  $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ . But then  $f(f^{-1}(U_{\alpha_1})) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$  is a subcover of  $f(K)$ . □

## 1.2 Measure Theory

### 1.2.1 $\sigma$ -algebras

**Def'n. 1.2.1** Let  $X \neq \emptyset$  be a set.  $\mathcal{M} \subset \mathcal{P}(X)$  is called a  **$\sigma$ -algebra** if

1.  $X \in \mathcal{M}$
2.  $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
3. If  $A_n \in \mathcal{M}$  for all  $n \in \mathbb{N}$ , then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$

The pair  $(X, \mathcal{M})$  is called a **measurable space**. The elements of  $\mathcal{M}$  are called **measurable sets**.

**Def'n. 1.2.2** Let  $(X, \mathcal{M})$  be a measurable space,  $(Y, \tau)$  be a topological space. Then  $f : X \rightarrow Y$  is called **measurable** if  $f^{-1}(V) \in \mathcal{M}$  for all  $V \in \tau$ .

Here are some simple examples of  $\sigma$ -algebras.

**Ex. 1.2.3** 1.  $\mathcal{M} = \{\emptyset, X\}$  is a  $\sigma$ -algebra.

2.  $\mathcal{P}(X) = \mathcal{M}$  is a  $\sigma$ -algebra.

3.  $\mathcal{M} = \{A \subset X : A \text{ or } A^c \text{ is countable}\}$ . To see this, given  $A_n \in \mathcal{M}$ , if everything is countable, then  $\bigcup A_n$  is countable. If some  $A_i$  is countable, then  $(\bigcup A_n)^c = \bigcap A_n^c$  is countable, so  $\bigcup A_n \in \mathcal{M}$ .

We will later see some proper examples, like the  $\sigma$ -algebra of Lebesgue measurable sets.

We have the following properties of  $\sigma$ -algebras.

**Prop. 1.2.4** 1.  $\emptyset \in \mathcal{M}$

2.  $A_1, A_2, \dots, A_n \in \mathcal{M} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{M}$
3.  $A_n \in \mathcal{M}$  for all  $n \in \mathbb{N}$  then  $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$
4.  $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$
5.  $f$  is measurable,  $H \subset Y$  is closed, then  $f^{-1}(H) \in \mathcal{M}$ .

**PROOF** 1.  $X \in \mathcal{M} \Rightarrow X^c \in \mathcal{M}$ .

2. We can extend this to a countable union by introduction  $A_{n+i} = \emptyset$  for  $i \in \mathbb{N}$ .
3. By DeMorgan's identities,  $(\bigcap A_n)^c = \bigcup A_n^c \in \mathcal{M}$ .
4.  $A \setminus B = A \cap B^c \in \mathcal{M}$ .
5.  $H^c$  is open implies  $f^{-1}(H^c) \in \mathcal{M}$ . Then  $f^{-1}(H) = (f^{-1}(H^c))^c \in \mathcal{M}$ . □

**Prop. 1.2.5** Let  $f : X \rightarrow Y$  be measurable, let  $g : Y \rightarrow Z$  be continuous, then  $g \circ f : X \rightarrow Z$  is measurable.

**PROOF** Let  $V \subset Z$  be open, so  $g^{-1}(V) \subset Y$  is open, so  $f^{-1}(g^{-1}(V)) \in \mathcal{M}$  which is  $(g \circ f)^{-1}(V)$ . □

**Prop. 1.2.6** Let  $(X, \mathcal{M})$  be a measurable space,  $Y$  be a topological space. Let  $\phi : \mathbb{R}^2 \rightarrow Y$  be continuous. If  $u, v : X \rightarrow \mathbb{R}$  are measurable, then  $h(x) = \phi(u(x), v(x))$  is measurable.

**PROOF** Define  $f : X \rightarrow \mathbb{R}^2$  by  $f(x) = (u(x), v(x))$ . We will see that  $f$  is measurable, so that  $h = \phi \circ f$  is measurable since  $\phi$  is continuous. Let  $I_1, I_2 \subset \mathbb{R}$  be open intervals, so  $R = I_1 \times I_2$  is an open rectangle. Then  $f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathcal{M}$ . Let  $G \subset \mathbb{R}^2$  be an open set, so there exist  $R_n$  open rectangles so that

$$G = \bigcup_{n=1}^{\infty} R_n \Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \mathcal{M}$$

so that  $f$  is measurable. □

**Cor. 1.2.7** 1. If  $u, v : X \rightarrow \mathbb{R}$  are measurable, then  $u + v$  and  $u \cdot v$  are measurable.

2.  $u + iv : X \rightarrow \mathbb{C}$  is measurable.
3.  $f : X \rightarrow \mathbb{C}$  is measurable,  $f = u + iv \Rightarrow u, v, |f|$  are measurable.

**Prop. 1.2.8** Define

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then  $\chi_E$  is measurable if and only if  $E \in \mathcal{M}$ .

**PROOF** Naturally,  $\chi_E^{-1}(1) = E$  and  $\chi_E^{-1}(0) = E^c$ , so  $\chi_E$  is measurable if and only if  $E, E^c \in \mathcal{M}$ . □

**Thm. 1.2.9** Let  $\mathcal{F} \subset \mathcal{P}(X)$ , then there exists a smallest  $\sigma$ -algebra containing  $\mathcal{F}$ . This is denoted by  $S(\mathcal{F})$ , the  $\sigma$ -algebra generated by  $\mathcal{F}$ .

**PROOF** Let  $\Omega = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra, } \mathcal{F} \subset \mathcal{M}\}$ . Certainly  $\Omega \neq \emptyset$  since  $\mathcal{P}(X) \in \Omega$ . Let  $S(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$ . We will see that  $S(\mathcal{F})$  is a  $\sigma$ -algebra.

- (i) Since  $X \in \mathcal{M}$ , it follows that  $X \in \bigcap \mathcal{M}$ .



(ii) If  $A \in S(\mathcal{F})$ , then  $A \in \mathcal{M}$  for all  $\mathcal{M}$ . Thus  $A^c \in \mathcal{M}$  for all  $\mathcal{M}$  and  $A^c \in \bigcap \mathcal{M}$ .

(iii) In the same way, if  $A_n \in S(\mathcal{F})$  for all  $n$ , then  $A_n \in \mathcal{M}$  for all  $n, \mathcal{M}$ . Thus  $\bigcup A_n \in \mathcal{M}$  for all  $\mathcal{M}$  so  $\bigcup A_n \in \bigcap \mathcal{M} = S(\mathcal{F})$ .

By definition,  $\mathcal{F} \subset \bigcap \mathcal{M}$ . Finally,  $S(\mathcal{F})$  is minimal, since if  $\mathcal{F} \subset \mathcal{N}$  is a  $\sigma$ -algebra, then  $\mathcal{N} \in \Omega \Rightarrow S(\mathcal{F}) \subset \mathcal{N}$ , so we are done.  $\square$

**Def'n. 1.2.10** Let  $(X, \tau)$  be a topological space. Then  $\mathcal{B} = S(\tau)$  is called the **Borel  $\sigma$ -algebra**. Borel sets are the elements of  $S(\tau)$ . A function  $f : X \rightarrow Y$  is Borel measurable if  $f^{-1}(G) \in \mathcal{B}$  for all  $G \subset Y$  open.

**Prop. 1.2.11** 1. If  $F \subset X$  is closed, then  $F \in \mathcal{B}$ .

2.  $G_n \subset X$  are open, then  $\bigcap_{n=1}^{\infty} G_n \in \mathcal{B}$ . These are called  $G_\delta$ -sets.

3.  $F_n \subset X$  are closed, then  $\bigcup_{n=1}^{\infty} F_n \in \mathcal{B}$ . These are called  $F_\sigma$ -sets.

**PROOF** These follow directly from the definition of a  $\sigma$ -algebra.  $\square$

**Ex. 1.2.12**  $X = \mathbb{R}, \tau_e$ , then  $\mathcal{B} = S(\tau_e)$ . Let  $\Gamma_0 = \{(a, b) : a < b\}$  be a family of open intervals. We see that  $S(\Gamma_0) = \mathcal{B}$ . Since  $\Gamma_0 \subset \tau$ ,  $S(\Gamma_0) \subset S(\tau) = \mathcal{B}$ . Conversely, let  $G \in \tau$ , then we have open intervals  $G = \bigcup_{n=1}^{\infty} I_n$  so that  $G \in S(\Gamma_0)$ . Thus  $S(\tau) \subset S(\Gamma_0)$  and  $S(\Gamma_0) = \mathcal{B}$ .

**Ex. 1.2.13** Let  $\Gamma_\infty = \{(a, \infty) : a \in \mathbb{R}\}$ . I claim that  $S(\Gamma_\infty) = \mathcal{B}$ . Certainly  $S(\Gamma_\infty) \subset S(\tau) = \mathcal{B}$ . Then  $(-\infty, a] = (a_1, \infty)^c \in S(\Gamma_\infty)$ . Similarly,  $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n] \in S(\Gamma_\infty)$ . Thus  $(a, \infty) \cap (-\infty, b) = (a, b) \in S(\Gamma_0)$ , and using the previous example,  $\mathcal{B} = S(\Gamma_\infty)$ .

**Prop. 1.2.14** Let  $(X, \mathcal{M})$  be a measurable space, and let  $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$  with the euclidean topology. If  $f^{-1}((\alpha, \infty]) \in \mathcal{M}$  for any  $\alpha \in \mathbb{R}$ , then  $f$  is measurable.

**PROOF** Recall that  $f$  is measurable if its inverse image takes open sets to measurable sets.

We have  $f^{-1}([-\infty, \alpha]) = (f^{-1}((\alpha, \infty]))^c \in \mathcal{M}$ . Similarly,

$$f^{-1}([-\infty, \alpha)) = f^{-1}\left(\bigcap_{n=1}^{\infty} [-\infty, \alpha - 1/n]\right) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, \alpha - 1/n]) \in \mathcal{M}$$

We then have

$$f^{-1}((\alpha, \beta)) = f^{-1}([-\infty, \beta) \cap (\alpha, \infty]) = f^{-1}([-\infty, \beta)) \cap f^{-1}((\alpha, \infty]) \in \mathcal{M}$$

Recall that the open intervals are a base for  $\tau_e$ . Thus if  $G \subset \overline{\mathbb{R}}$  is open, then there exists open intervals so that  $G = \bigcup_{n=1}^{\infty} I_n$  and

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathcal{M}$$

as desired.  $\square$

### 1.2.2 Sequences of Measurable Functions

Our goal is to prove that the pointwise limit of measurable functions is measurable. This does not hold for Riemann integrability! For example, a function with a finite number of discontinuities is Riemann integrable, but the Dirichlet function is not Riemann integrable and is discontinuous only at a countable number of points.

**Def'n. 1.2.15** Let  $(a_n)_{n \in \mathbb{N}} \subset \overline{\mathbb{R}}$  be a sequence, and  $b_k = \sup\{a_k, a_{k+1}, \dots\}$ . Then  $\beta = \inf_{k \in \mathbb{N}} b_k$  is called the  $\limsup$  of  $(a_n)$ . We can similarly define  $c_k = \inf\{a_k, a_{k+1}, \dots\}$  and  $\liminf = \sup_{k \in \mathbb{N}} c_k$ .

**Def'n. 1.2.16** Let  $f_n : X \rightarrow \overline{\mathbb{R}}$  be a sequence of functions. Then  $(\sup f_n) : X \rightarrow \overline{\mathbb{R}}$ ,  $(\sup f_n)(x) = \sup f_n(x)$  for all  $x \in X$ . Similarly,  $(\inf f_n) : X \rightarrow \overline{\mathbb{R}}$ ,  $(\inf f_n)(x) = \inf f_n(x)$  for all  $x \in X$ . Then  $(\liminf f_n)(x) = \liminf f_n(x)$ . If  $\lim f_n(x)$  exists for all  $x$ , then we say  $(\lim f_n)(x) = \lim f_n(x)$ .

**Thm. 1.2.17** Let  $f_n : X \rightarrow \overline{\mathbb{R}}$  be measurable. Then  $\sup f_n$ ,  $\inf f_n$ ,  $\limsup f_n$ ,  $\liminf f_n$  are measurable.

**PROOF** Let  $g = \sup f_n$ . It is enough to prove that  $g^{-1}((\alpha, +\infty]) \in \mathcal{M}$  for all  $\alpha$ . Let  $H = g^{-1}((\alpha, +\infty]) = \{x \in X : \sup f_n(x) > \alpha\}$ . Let  $H_n = f_n^{-1}((\alpha, +\infty]) = \{x \in X : f_n(x) > \alpha\} \in \mathcal{M}$ . We show that  $H = \bigcup_{n=1}^{\infty} H_n$ .

First let  $x \in H$ , so  $\sup f_n(x) > \alpha$ . Thus get  $N$  so that  $f_N(x) > \alpha$ , so  $x \in H_N$  and  $x$  is in the union. The converse is obvious.

Thus  $g$  is measurable. In the exact same way,  $\inf f_n$  is measurable. As well,

$$\limsup f_n = \inf_i \sup_{k \geq i} f_k$$

is measurable. □

**Cor. 1.2.18** If  $\lim f_n$  exists, then it is measurable.

**PROOF** If  $\lim f_n$  exists, then  $\lim f_n = \limsup f_n$ . □

**Cor. 1.2.19** If  $f, g$  are measurable, then  $\max\{f, g\}$ ,  $\min\{f, g\}$  are measurable.

**Cor. 1.2.20** Let  $f$  be a function. Then  $f_+ = \max\{f, 0\}$  and  $f_- = -\min\{f, 0\}$  (the positive and negative parts of  $f$ ) are measurable. Similarly,  $|f| = f_+ + f_-$  is measurable.

### 1.2.3 Measures

**Def'n. 1.2.21** Let  $(X, \mathcal{M})$  be a measurable space. A function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  is called a **(positive) measure** if it is countably additive and not constant  $+\infty$ . In other words,

$$1. \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \text{ if } A_i \cap A_j = \emptyset$$

$$2. \exists A \in \mathcal{M} \text{ so that } \mu(A) < \infty$$

$(X, \mathcal{M}, \mu)$  is called a **measure space**.

**Prop. 1.2.22** 1.  $\mu(\emptyset) = 0$

2. If  $A_i \cap A_j = \emptyset$  then  $\mu\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$

3.  $A \subset B$  implies  $\mu(A) \leq \mu(B)$

4.  $A_1 \subset A_2 \subset A_3 \cdots$  then  $\lim_{n \rightarrow \infty} \mu A_n = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

5.  $A_1 \supset A_2 \supset A_3 \cdots$  and  $\mu(A_i) < \infty$  then  $\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

**PROOF** 1. Let  $A \in \mathcal{M}$  so that  $\mu(A) < \infty$ , and fix  $A_1 = A$ ,  $A_2 = A_3 = \cdots = \emptyset$ . Then  $\bigcup A_n = A$  so  $\mu(A) = \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset)$  so  $\mu(\emptyset) = 0$ .

2. Obvious

3. Note that  $B = A \cup (B \setminus A)$  is a disjoint union.

4. Define  $B_1 := A_1$  and  $B_i = A_i \setminus A_{i-1}$  for  $i \geq 2$ . Then  $B_i \cap B_j = \emptyset$  and  $\mu(A_n) = \mu\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mu(B_i)$ . Similarly,  $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ . Therefore,  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) = \sum_{n=1}^{\infty} \mu(B_n)$ .

5. Let  $C_n = A_1 \setminus A_n$ ,  $C_1 = \emptyset$ . Then  $C_1 \subset C_2 \subset \cdots$  and  $\mu(C_n) + \mu(A_n) = \mu(A_1)$ . Let  $A = \bigcap_{n=1}^{\infty} A_n$  so  $A_1 \setminus A = \bigcup_{n=1}^{\infty} C_n$  and  $(\bigcup C_n) \cup A = A_1$  is a disjoint union. But then  $\mu(\bigcup A_n) + \mu(A) = \mu(A_1)$  so that

$$\mu(A_1) - \mu(A) = \mu\left(\bigcup C_n\right) = \lim_{n \rightarrow \infty} \mu(C_n) = \mu(A_n) - \lim_{n \rightarrow \infty} \mu(A_n)$$

Since  $\mu(A_1)$  is finite, we have  $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$ . □

**Ex. 1.2.23** Here are a few examples of measures that exist on arbitrary sets.

1.  $X$  arbitrary,  $\mathcal{M} = \mathcal{P}(X)$ , and

$$\mu(E) = \begin{cases} |E| & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is not finite} \end{cases}$$

It is easy to verify it is countably additive.

2.  $X$  arbitrary,  $\mathcal{M} = \mathcal{P}(X)$ . Fix  $x_0 \in X$ . Then

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

## 1.3 Towards Integration

### 1.3.1 Simple Functions

**Def'n. 1.3.1**  $s : X \rightarrow \mathbb{R}$  or  $\mathbb{C}$  is called a simple function if its range is finite.

**Prop. 1.3.2** Let  $s$  be a simple function, so that  $R(s) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . Then  $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$  where  $A_i = s^{-1}(\{\alpha_i\})$  and  $s$  is measurable if and only if  $A_i \in \mathcal{M}$ .

PROOF Obvious. □

The following theorem is used later to define the integral. It is clear that we should define the integral of a simple function as the sum of the integrals of its characteristic functions, and this allows us to extend the integral by limits to the function  $f$ .

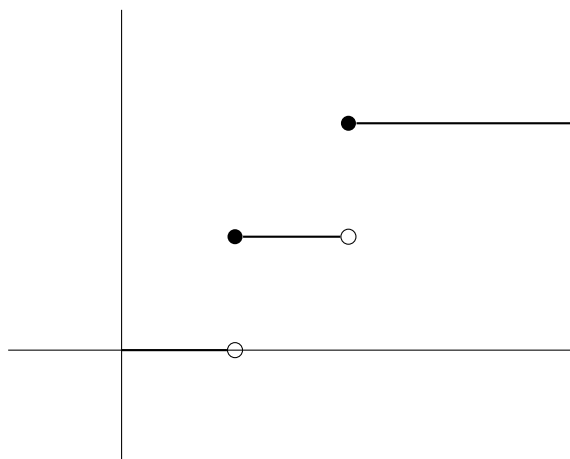
**Thm. 1.3.3** Let  $f : X \rightarrow [0, +\infty]$  be nonnegative measurable functions. Then there exists a sequence  $s_n : X \rightarrow [0, +\infty]$  of simple measurable functions with

1.  $(s_n)$  is increasing and bounded above by  $f$
2.  $\lim s_n = f$  pointwise.

PROOF Let  $n \in \mathbb{N}$ ,  $t \geq 0$ , and  $k_n(t) = [2^n \cdot t]$  (i.e.  $k_n(t) \leq 2^n \cdot t < k_n(t) + 1$ ). Then define

$$\phi_n(t) = \begin{cases} k_n(t) \cdot 2^{-n} & \text{if } t \leq n \\ n & \text{if } t > n \end{cases}$$

I've drawn  $\phi_1$  below:



Then  $t - 2^{-n} \leq \phi_n(t) \leq t$ ,  $\lim \phi_n(t) = t$ , and  $\phi_n \leq \phi_{n+1}$ . Define  $s_n = \phi_n \circ f$ , so for any  $x \in X$ ,  $\lim s_n(x) = \lim \phi_n \circ f(x) = f(x)$ . Note that  $s_n$  is simple since it has finite range (from  $\phi_n$ ), and  $s_n \leq s_{n+1}$  because  $\phi_n \leq \phi_{n+1}$ , and  $s_n \leq f$  since  $\phi_n(t) \leq t$ . Furthermore,  $\phi_n$  is measurable since its level sets are intervals, so  $\phi_n \circ f$  is measurable. □

### 1.3.2 Integration of Positive Functions

Let  $(X, \mathcal{M}, \mu)$  be a measure space.

**Def'n. 1.3.4** Let  $S : X \rightarrow [0, +\infty)$  be a measurable simple function  $s = \sum_{n=1}^n \alpha_i X_{A_i}$ . Let  $E \in \mathcal{M}$ . Then define the **integral of  $s$  over  $E$**  be with respect to  $\mu$  as

$$\int_E s \, d\mu = \sum_{n=1}^n \alpha_i \mu(A_i \cap E)$$

where we define  $0 \cdot \infty = 0$ .

**Def'n. 1.3.5** Let  $f : X \rightarrow [0, +\infty]$  be a measurable function. Let  $E \in \mathcal{M}$ . Then the **(Lebesgue) integral of  $f$  over  $E$**  with respect to  $\mu$  is

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f; s \text{ is simple measurable} \right\}$$

Unlike the Riemann integral, we take the supremum over lower sums only. Let  $f, g : X \rightarrow [0, +\infty]$  be measurable functions. Let  $E, A, B \in \mathcal{M}$ .

1. If  $f \leq g$  then  $\int_E f \, d\mu$  and  $\int_E g \, d\mu$
2. If  $A \subset B$ , then  $\int_A f \, d\mu \leq \int_B f \, d\mu$
3.  $\int_E c \cdot f \, d\mu = c \cdot \int_E f \, d\mu$  for all  $c \geq 0$
4. If  $f(x) = 0$  for all  $x \in E$ , then  $\int_E f \, d\mu = 0$
5. If  $\mu(E) = 0$ , then  $\int_E f \, d\mu = 0$
6.  $\int_E f \, d\mu = \int_X f \cdot \chi_E \, d\mu$ .

**PROOF** 1. Note that

$$\left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\} \subset \left\{ \int_E s \, d\mu : 0 \leq s \leq g \right\} \text{ Let}$$

$0 \leq s \leq f$  be simple measurable. Then

$$\int_A s \, d\mu = \sum \alpha_i \mu(A \cap A_i) \leq \sum \alpha_i \mu(B \cap A_i) = \int_B s \, d\mu$$

Take the supremum for all  $0 \leq s \leq f$ , then the result follows.

3. Let  $S$  be simple and measurable, so  $s = \sum \alpha_i \chi_{A_i}$ . Then

$$\int_E c \cdot s \, d\mu = \sum_{i=1}^n \alpha_i \cdot c \cdot \mu(E \cap A_i) = c \cdot \sum \alpha_i \mu(E \cap A_i) = c \int_E s \, d\mu$$

Thus

$$\begin{aligned}\int_E c \cdot f \, d\mu &= \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq cf \right\} \\ &= \sup \left\{ \int_E c \cdot t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \sup \left\{ \int_E t \, d\mu : 0 \leq t \leq f \right\} \\ &= c \cdot \int_E f \, d\mu\end{aligned}$$

4. If  $0 \leq s \leq f$ , then  $s = \sum \alpha_i \chi_{A_i}$ . If  $x \in A_i \cap E$ , then  $s(x) = \alpha_i$  and  $\alpha_i = 0$ . Then  $\alpha_i \mu(A_i \cap E) = 0$  for all  $i$ : either  $A_i \cap E = \emptyset$ , or  $A_i \cap E$  is not empty, and  $\alpha_i = 0$ . This is true for any  $0 \leq s \leq f$ , and taking supremums yields the result.
5. If  $\mu(E) = 0$  then  $\mu(A_i \cap E) = 0$ , and  $\int_E s \, d\mu = \sum \alpha_i \mu(A_i \cap E) = 0$  and taking supremums, the result holds.
6. Exercise. First prove if  $0 \leq s \leq f \cdot \chi_E$ , then  $\int_X s \, d\mu = \int_E s \, d\mu$ . Then prove  $\left\{ \int_E s \, d\mu : 0 \leq s \leq f \cdot \chi_E \right\} = \left\{ \int_E s \, d\mu : 0 \leq s \leq f \right\}$ .  $\square$

**Prop. 1.3.6** Let  $s$  be a simple and measurable. Then  $\phi(E) = \int_E s \, d\mu$  is a measure.

PROOF  $\phi(\emptyset) = 0$ , so  $\phi$  is not constant  $+\infty$ . Let  $E = \bigcup_{n=1}^{\infty} E_n$  be a disjoint union. Then

$$\begin{aligned}\phi(E) &= \sum_{i=1}^m \alpha_i \mu(A_i \cap E) \\ &= \sum_{i=1}^m \alpha_i \mu \left( A_i \cap \left( \bigcup_{n=1}^{\infty} E_n \right) \right) = \sum_{i=1}^m \alpha_i \mu \left( \bigcup_{n=1}^{\infty} (A_i \cap E_n) \right) \\ &= \sum_{i=1}^m \alpha_i \sum_{n=1}^{\infty} \mu(A_i \cap E_n) = \sum_{n=1}^{\infty} \sum_{i=1}^m \alpha_i \mu(A_i \cap E_n) \\ &= \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu = \sum_{n=1}^{\infty} \phi(E_n)\end{aligned}$$

$\square$

**Prop. 1.3.7** Let  $s, t$  be nonnegative, measurable simple functions. Then

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

PROOF Write

$$s = \sum_{i=1}^m \alpha_i \chi_{A_i}, \quad t = \sum_{j=1}^n \beta_j \chi_{B_j}$$

and let  $E_{ij} = A_i \cap B_j$ , so  $X = \bigcup_{i,j} E_{ij}$  is a disjoint union. We now have

$$\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j) \mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu$$

Let  $\mu(E) = \int_E (s+t) d\mu$ , which is a measure as above. Thus

$$\begin{aligned} \int_X (s+t) d\mu &= \phi(X) = \phi\left(\bigcup_{i,j} E_{ij}\right) \\ &= \sum_{i,j} \phi(E_{ij}) = \sum_{i,j} \int_{E_{ij}} (s+t) d\mu \\ &= \sum_{i,j} \left( \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu \right) \\ &= \sum_{i,j} \varphi(E_{ij}) + \sum_{i,j} \theta(E_{ij}) \\ &= \int_X s d\mu + \int_X t d\mu \end{aligned}$$

where  $\varphi(E) = \int_E s d\mu$ ,  $\theta(X) = \int_X t d\mu$ . □

### 1.3.3 Lebesgue's Monotone Convergence Theorem

**Thm. 1.3.8 (Lebesgue's Monotone Convergence)** *Let  $f_n : X \rightarrow [0, +\infty]$  be measurable, such that*

(i)  $0 \leq f_1 \leq f_2 \leq \dots$

(ii)  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$  for all  $x \in X$

*Then  $f$  is measurable, and  $\int_X f d\mu = \lim \int_X f_n d\mu$ .*

**PROOF** It was already proven that  $f$  is measurable. We have  $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$  for all  $n$ , so  $\alpha := \lim_{n \rightarrow \infty} \int_X f_n d\mu$  exists. We also have  $f_n \leq f$ , so  $\int f_n \leq \int f$  and  $\alpha \leq \int_X f_n d\mu$ . Thus we wish to show  $\alpha \geq \int_X f d\mu$ . It suffices to prove that  $\alpha \geq \int_X s d\mu$  for any simple  $s \leq f$ . Let  $c \in (0, 1)$ ; it suffices to show that  $\alpha \geq \int_X c \cdot s d\mu$ . Define  $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\}$ . We have  $E_1 \subset E_2 \subset \dots$  so that  $\bigcup E_n = X$ . Then

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} c \cdot s d\mu$$

Let  $\phi(E) = \int_E s d\mu$ , so  $\int_{E_n} s d\mu = \phi(E_n) \rightarrow \phi(\bigcup E_n) = \phi(X) = \int_X s d\mu$ . Thus

$$\alpha \geq c \cdot \lim_{n \rightarrow \infty} \phi(E_n) = c \cdot \int_X s d\mu = \int_X cs d\mu$$

as desired. □

**Ex. 1.3.9** Consider the function consisting of a triangle with base  $2/n$  and height  $n$ . Then  $\int_0^1 f_n = 1$  as a Riemannian integral. However,  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for any  $x$ , so  $\int_0^1 f = 0 \neq 1 = \lim \int_0^1 f_n$ .

**Thm. 1.3.10** Let  $f, g : X \rightarrow [0, +\infty]$  measurable, then  $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$ .

PROOF We proved that there exists increasing sequences of simple functions  $s_n, t_n$  such that  $\lim s_n(x) = f(x)$ ,  $\lim t_n(x) = g(x)$ . Then  $s_n(x) + t_n(x) \rightarrow f(x) + g(x)$  monotonically. But then

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X \lim_{n \rightarrow \infty} (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \int_X (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \left( \int_X s_n d\mu + \int_X t_n d\mu \right) \\ &= \int_X \lim_{n \rightarrow \infty} s_n d\mu + \int_X \lim_{n \rightarrow \infty} t_n d\mu \\ &= \int_X f d\mu + \int_X g d\mu \end{aligned} \quad \square$$

**Cor. 1.3.11** If  $f_n : X \rightarrow [0, +\infty]$  is a sequence of measurable functions, then

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=1}^{\infty} f_n d\mu$$

**Ex. 1.3.12** Let  $X = \mathbb{N}$ ,  $\mathcal{M} = \mathcal{P}(X)$ ,  $\mu(E)$  is the counting measure. Let  $a : X \rightarrow [0, \infty)$  be a function. This is a sequence. Every function is measurable. Let  $s_n(i) = a(i)$  for  $i \leq n$  and 0 otherwise, which is a simple function, and  $s_n \leq s_{n+1}$ . Then  $\lim_{n \rightarrow \infty} s_n(i) = a(i)$  so  $s_n \rightarrow a$  pointwise, so by LMC  $\int_X s_n d\mu = \int_X a d\mu$ . Also,

$$\int_X s_n d\mu = \sum_{i=1}^n a(i) \mu(\{i\}) = \sum_{i=1}^n a(i)$$

$$\text{so } \int_X a d\mu = \sum_{n=1}^{\infty} a(n).$$

**Lemma 1.3.13 (Fatou)** Let  $f_n : X \rightarrow [0, \infty)$  be a sequence of measurable functions. Then

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu$$

PROOF Let  $g_k = \inf\{f_k, f_{k+1}, \dots\}$  so  $\liminf f_n = \lim_{n \rightarrow \infty} g_n$  and  $g_n$  is increasing. Note that  $g_k \leq f_k$  for any  $k$ , so  $\int_X g_k d\mu \leq \int_X f_k d\mu$ . Thus

$$\begin{aligned} \int_X \liminf f_n d\mu &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \lim \int_X g_n d\mu \\ &= \liminf \int_X g_n d\mu \\ &\leq \liminf \int_X f_n d\mu \end{aligned} \quad \square$$



**Ex. 1.3.14** It is possible for the inequality to be strict. Define  $f_{2n} = \chi_{[0,1]}$  and  $f_{2n+1} = \chi_{[1,2]}$ . Thus  $\liminf f_n(x) = 0$  so  $\int_{[0,2]} \liminf f_n d\mu = 0$  but  $\inf_{[0,2]} \int_{[0,2]} f_n d\mu = 1$

**Thm. 1.3.15** Let  $f : X \rightarrow [0, \infty]$  be measurable. Let  $\phi(E) = \int_E f d\mu$ ,  $E \in \mathcal{M}$ . Then  $\phi$  is a measure and  $\int_X g d\phi = \int_X g \cdot f d\mu$ .

**PROOF** Certainly  $\phi(\emptyset) = 0$ , so  $\phi \neq +\infty$ . Thus let  $E = \bigcup_{i=1}^{\infty} E_i$  be a disjoint union. Then  $\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f$ . Thus we have

$$\begin{aligned} \phi(E) &= \int_E f d\mu \\ &= \int_X \chi_E f d\mu \\ &= \int_X \sum_{i=1}^{\infty} \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_X \chi_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \int_{E_i} f d\mu \\ &= \sum_{i=1}^{\infty} \phi(E_i) \end{aligned}$$

Now, we prove that  $\int_X g d\phi = \int_X g f d\mu$ .

First, we do this for  $g = \chi_E$ . Then  $\int_X \chi_E d\phi = \phi(E)$  on the left, and  $\int_X \chi_E f d\mu = \int_E f d\mu = \phi(E)$  and equality holds.

Now, let  $g = \sum_{i=1}^n \alpha_i \chi_{A_i}$  be a simple function. Then  $\int_X \sum \alpha_i \chi_{A_i} d\phi = \sum \alpha_i \int_X \chi_{A_i} d\phi$  on the left and  $\int_X \sum \alpha_i \chi_{A_i} f d\mu = \sum \alpha_i \int_X \chi_{A_i} f d\mu$ .

Finally, let  $g$  be an arbitrary measurable function, and let  $(s_n) \rightarrow g$  be an increasing sequence of simple functions. Note that  $s_n f \rightarrow g f$ . Thus

$$\begin{aligned} \int_X g d\phi &= \int_X \lim s_n d\phi = \lim \int_X s_n d\phi \\ &= \lim \int_X s_n f d\mu = \int_X \lim (s_n f) d\mu \\ &= \int_X g \cdot f d\mu \end{aligned}$$

as desired. □

## 1.4 Integral of Complex Valued Functions

**Def'n. 1.4.1** A function  $f : X \rightarrow \mathbb{C}$  is called **Lebesgue integrable** if  $\int_X |f| d\mu < \infty$ . The collection of such functions is  $L^1(\mu)$ .

**Def'n. 1.4.2** Let  $f \in L^1(\mu)$ . Then  $f = u + iv$  and denote  $u = \operatorname{Re} f$ ,  $v = \operatorname{Im} f$ . Let  $E \in \mathcal{M}$ ; then the integral of  $f$  over  $E$  with respect to  $\mu$  is

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \left( \int_E v^+ d\mu - \int_E v^- d\mu \right)$$

**Thm. 1.4.3** Let  $f, g \in L^1(\mu)$ ,  $\alpha, \beta \in \mathbb{C}$ , so  $\alpha f + \beta g \in L^1(\mu)$  and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$$

**PROOF** Note that  $\alpha f + \beta g$  is measurable, so  $\int_X |\alpha f + \beta g| d\mu \leq |\alpha| \int_X |f| d\mu + |\beta| \int_X |g| d\mu < \infty$ . For real measurable functions,  $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$  directly by expanding the definition and using additivity over positive functions. We thus show  $\int_X \alpha f d\mu = \alpha \int_X f d\mu$ . If  $\alpha \geq 0$ , then

$$\begin{aligned} \int_X \alpha f d\mu &= \int_X \alpha(u + iv) d\mu = \int_X (\alpha u^+ - \alpha u^- + i\alpha v^+ - i\alpha v^-) d\mu \\ &= \int_X ((\alpha u)^+ - (\alpha u)^- + i(\alpha v)^+ - i(\alpha v)^-) d\mu \\ &= \int_X (\alpha u)^+ d\mu - \int_X (\alpha u)^- d\mu + \int_X i(\alpha v)^+ d\mu - \int_X i(\alpha v)^- d\mu \\ &= \alpha \int_X u^+ d\mu - \alpha \int_X u^- d\mu + \alpha \int_X iv^+ d\mu - \alpha \int_X iv^- d\mu \\ &= \alpha \int_X f d\mu \end{aligned}$$

and similarly for  $\alpha = -1$ ,  $\alpha = i$ . □

**Thm. 1.4.4** Let  $f \in L^1(\mu)$ . Then  $\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$ .

**PROOF** Let  $z = \int_X f d\mu$ . Let  $\alpha = \frac{|z|}{z}$  if  $z \neq 0$ , and  $\alpha = 1$  otherwise. Then  $\alpha \int_X f d\mu = |z|$ . Let  $u = \operatorname{Re}(\alpha \cdot f) \leq |\alpha \cdot f| \leq |f|$  since  $|\alpha| = 1$ . Thus

$$\begin{aligned} \left| \int_X f d\mu \right| &= \alpha \cdot \int_X f d\mu \\ &= \int_X \alpha f d\mu \\ &= \int_X \operatorname{Re}(\alpha f) d\mu \\ &\leq \int_X |f| d\mu \end{aligned}$$