REPLACE

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1 Basic Definitions and Terminology

1.1 Iterated Function Systems

Throughout, we let λ denote the Lebesgue measure on $\mathbb{R}$.

Definition. An **IFS** is a finite set of contractions

$$S_i(x) = r_i x + a_i : \mathbb{R} \to \mathbb{R} \text{ for each } i = 0, 1, \ldots, k$$

with $k \geq 1$ and $0 < |r_i| < 1$.

Each IFS generates a unique invariant compact set K, known as its associated **self-similar set**, satisfying

$$K = \bigcup_{j=0}^{k} S_j(K).$$

Rescaling the r_i and a_i if necessary, we may assume the convex hull of K is $[0,1]$. We also associate probabilities $0 < p_i < 1$ for each $i = 0, \ldots, k$ which satisfy $\sum_i p_i = 1$. To these probabilities there is a unique self-similar measure μ with $\operatorname{supp} \mu = K$ satisfying

$$\mu = \sum_{j=0}^{k} p_j \mu \circ S_j^{-1}. \tag{1.1}$$

We are primarily interested in studying properties of this measure μ.

Let $\Sigma = \{0, 1, \ldots, k\}$ be our alphabet, let Σ^k denote the words of length k, and $\Sigma^* = \bigcup_{k=0}^{\infty} \Sigma_k$ denote the set of all the finite words on Σ. Given $\sigma = (\sigma_1, \ldots, \sigma_j) \in \Sigma$, we let

$$\sigma^- = (\sigma_1, \ldots, \sigma_{j-1})$$
$$S_\sigma = S_{\sigma_1} \circ \cdots \circ S_{\sigma_j}$$

and similarly,

$$r_\sigma = \prod_{i=1}^{j} r_{\sigma_i} \text{ and } p_\sigma = \prod_{i=1}^{j} p_{\sigma_i}.$$

Additionally, we set $r_{\max} = \max_i |r_i|$.

1.2 Generations

For any $0 < \alpha \leq 1$, we define the family

$$\Lambda_\alpha = \{\sigma \in \Sigma^* : |r_\sigma| < \alpha \leq |r_{\sigma^-}|\}$$

called the **words of generation α**. Given a word σ, we say that the **generation** $G(\sigma)$ is the interval $(|r_\sigma|, |r_{\sigma^-}|]$. By definition $\alpha \in G(\sigma)$ if and only if $\sigma \in \Lambda_\alpha$.

Let $h_1, \ldots, h_{s(\alpha)}$ be the collection of elements of the set $\{S_{\sigma(0)}, S_{\sigma(1)} : \sigma \in \Lambda_\alpha\}$ listed in increasing order. We set

$$\mathcal{F}_\alpha = \{[h_j, h_{j+1}] : 1 \leq j \leq s(\alpha) - 1 \text{ and } (h_j, h_{j+1}) \cap K \neq \emptyset\}$$

Elements of $\mathcal{F}_\alpha$ are called **net intervals of generation α**.

Definition. A pair (a, L) is a **neighbour** of $\Delta = [a_0, b_0] \in \mathcal{F}_\alpha$ if there is some $\sigma \in \Lambda_\alpha$ such that $S_\sigma(0,1) \cap \Delta \neq \emptyset$, $\lambda(\Delta)^{-1} r_\sigma = L$, and $\lambda(\Delta)^{-1}(a_0 - S_\sigma(0)) = a$, and we say that σ **generates** the neighbour (a, L). Then the **neighbour set** of Δ is the ordered tuple

$$V_\alpha(\Delta) = ((a_1, L_1), \ldots, (a_j, L_j))$$

where each (a_i, L_i) is a (distinct) neighbour of Δ. We order these tuples so that $a_i \leq a_{i+1}$ and if $a_i = a_{i+1}$, then $L_i < L_{i+1}$.

Abusing notation slightly, we say that $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ if either the net intervals are distinct, or if they are the same, then the neighbour set of some net interval is different. (TODO: is this actually abusing notation? Or are the notions equivalent?) We can generalize the notion of generation to net intervals. Let $\Delta \in \mathcal{F}_\alpha$ have neighbour set $((a_1, L_1), \ldots, (a_j, L_j))$. For each i, let σ_i generate (a_i, L_i). Then we call

$$G(\Delta) = \bigcap_{i=1}^{j} G(\sigma_i)$$

the **generation** of some $\Delta \in \mathcal{F}_\alpha$. Note that

(i) $\alpha \in G(\Delta)$

(ii) For any $\beta \in G(\Delta)$, $\Delta \in \mathcal{F}_\beta$ and $V_\beta(\Delta) = V_\alpha(\Delta)$

(iii) If $\gamma \notin G(\Delta)$, either $\Delta \notin \mathcal{F}_\gamma$ or $V_\gamma(\Delta) \neq V_\alpha(\Delta)$.

1.3 Transition Types

Let $0 < \alpha \leq 1$ and $\Delta \in \mathcal{F}_\alpha$ and suppose Δ has neighbour set $((a_1, L_1), \ldots, (a_n, L_n))$ Set

$$L_{\max} = \max\{|L_i| : i = 1, \ldots, n\} \text{ and } \gamma = \lambda(\Delta) \cdot L_{\max};$$

in other words, γ is the largest value achieved by $|r_\sigma|$ where σ generates some neighbour of Δ. Let Δ have children $(\Delta_1, \ldots, \Delta_n) \in \mathcal{F}_\gamma$. As in the proof of Proposition 1.1, we see that either $n > 1$ or if $n = 1$, then $V_\alpha(\Delta) \neq V_\gamma(\Delta_1)$. We call the tuple $(\Delta_1, \ldots, \Delta_n)$ the **children** of $\Delta \in \mathcal{F}_\alpha$. Note that it suffices to take any γ' such that

$$\max\Big\{r_{\max}\gamma, \lambda(\Delta) \cdot \max\{|L_i| : i = 1, \ldots, n; |L_i| \neq L_{\max}\}\Big\} < \gamma' \leq \gamma$$

where the inner maximum is taken to be 0 if the set is empty.

Definition. Suppose $\Delta = [a, b] \in \mathcal{F}_\alpha$ has children $(\Delta_1, \ldots, \Delta_n)$ in generation γ. Write $\Delta_i = [a_i, a_i + L_i]$. We define the **transition type** of Δ, denoted $\mathcal{C}_\alpha(\Delta)$, to be the tuple

$$\left(\Big(\frac{a_1 - a}{\lambda(\Delta)}, \frac{L_1}{\lambda(\Delta)}, V_\gamma(\Delta_1)\Big), \ldots, \Big(\frac{a_n - a}{\lambda(\Delta)}, \frac{L_n}{\lambda(\Delta)}, V_\gamma(\Delta_n)\Big)\right)$$

Remark. To compute the children, it is not sufficient to consider $\gamma = |r_i|\alpha$ for some $0 \leq i \leq k$. For example, in the main IFS example, take $\Delta = [4/15, 1/3] \in \mathcal{F}_\alpha$ where $\alpha = 1/3$. Now $\Delta \in \mathcal{F}_{1/5}$, but has different neighbour set, and $\Delta \notin \mathcal{F}_{1/9}$ at all. However, the largest possible value of $|r_i|$ misses the children.

1.4 Basic Properties of IFS

Here are some properties of the generations $\{\Lambda_\alpha\}_{0<\alpha\leq 1}$.

1.1 Proposition. *Let $0 < \alpha < \beta \leq 1$.*

(i) $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ if and only if there exists some $a_i \in \{0\} \cup \mathbb{N}$ such that $\rho := \prod_{i=0}^{k} |r_i|^{a_i}$ satisfies $\alpha < \rho \leq \beta$.

(ii) If σ is any infinite word, there exists a unique index i_α such that $(\sigma_1, \ldots, \sigma_{i_\alpha}) \in \Lambda_\alpha$. In particular, when $\alpha < \beta$, $i_\alpha \geq i_\beta$.

Proof Recall that $\sigma \in \Lambda_\alpha$ for any $\alpha \in (|r_\sigma|, |r_{\sigma^-}|]$. This is sufficient for the forward implication of (i) and (ii).

To see the reverse implication of (i), suppose such a ρ exists and let ω be a word such that $|r_\omega| = \rho$. Let σ be a prefix of ω such that $\sigma \in \Lambda_\beta$; then since $\omega \notin \Lambda_\alpha$, $\sigma \notin \Lambda_\alpha$ as well. Let $\Delta \subseteq S_\sigma[0,1]$ with $\Delta \in \mathcal{F}_\alpha$ be any net interval, so that σ generates some neighbour (a, L) of Δ where $L = r_\sigma \lambda(\Delta)^{-1}$. If $\Delta \notin \mathcal{F}_\beta$, we are done, so let's suppose $\Delta = \Delta' \in \mathcal{F}_\beta$. Now suppose (a', L') is any neighbour of Δ' generated by $\tau \in \Lambda_\beta$; it suffices to show that $(a', L') \neq (a, L)$. Suppose $a = a'$; then if τ generates (a', L'), we have $\sigma\tau$, and since $\sigma \in \Lambda_\alpha$ and $\tau \in \Lambda_\beta$ with $\Lambda_\alpha \neq \Lambda_\beta$, we have $r_\sigma \neq r_\tau$ so $L \neq L'$. ■

Remark. One way to think about the children of an interval as follows. Enumerate the points $\left\{\prod_{i=0}^k |r_i|^{a_i} : a_i \in \{0\} \cup \mathbb{N}\right\}$ in decreasing order $(\rho_i)_{i=1}^\infty$. As in Proposition 1.1, the $\mathcal{F}_\alpha$ change on transitions of intervals $[\rho_{i+1}, \rho_i)$. However, if $\Delta \in \mathcal{F}_\alpha$ with $\alpha \in [\rho_{k+1}, \rho_k)$, it may be that $\Delta \in \mathcal{F}_\beta$ for any $\beta \in [\rho_{k+2}, \rho_{k+1})$, with $V_\beta(\Delta) = V_\alpha(\Delta)$. The children are the net intervals in generation ρ_m where $m > k+1$ is minimal such that either $\Delta \notin \mathcal{F}_{\rho_m}$ or $V_{\rho_m}(\Delta) \neq V_\alpha(\Delta)$.

1.2 Proposition. *Consider the IFS $\{S_i\}_{i=0}^m$ and let $0 < \alpha \leq 1$. Then $C_\alpha(\Delta)$ depends only on the neighbour set $V_\alpha(\Delta)$.*

Proof *[Sketch.]* Let $\Delta \in \mathcal{F}_\alpha$ have neighbour set $\big((a_1, L_1), \ldots, (a_n, L_n)\big)$ and let $\{i_1, \ldots, i_m\}$ be the set of indices such that $L_{i_1} = \cdots = L_{i_m} =: L$ are maximal. Let γ be such that $(\Delta_1, \ldots, \Delta_n)$ with $\Delta_i \in \mathcal{F}_\gamma$ are the children of Δ. Let $\Gamma \subseteq \Lambda_\alpha$ be the set of words which generate some (a_{i_j}, L_{i_j}) for $1 \leq j \leq m$. Note the following facts:

- If $\sigma \in \Lambda_\alpha \setminus \Gamma$ has $S_\sigma[0,1] \supseteq \Delta$, then $\sigma \in \Lambda_\gamma$
- If $\sigma \in \Gamma$, then $\sigma \notin \Lambda_\gamma$ but $\sigma l \in \Lambda_\gamma$ for any $0 \leq l \leq m$.

But then the words $\tau \in \Lambda_\gamma$ such that $S_\tau[0,1] \supseteq \Delta$ are precisely the words

$$\tau \in \Lambda_\alpha \setminus \Gamma \text{ with } S_\tau[0,1] \supseteq \Delta$$

or

$$\tau = \sigma l \text{ with } \sigma \in \Gamma \text{ and } l \in \{0, \ldots, k\}.$$

Thus the set $\{\tau \in \Lambda_\gamma : S_\tau[0,1] \supseteq \Delta\}$ depends only on $V_\alpha(\Delta)$, which fully determines $C_\alpha(\Delta)$. ■

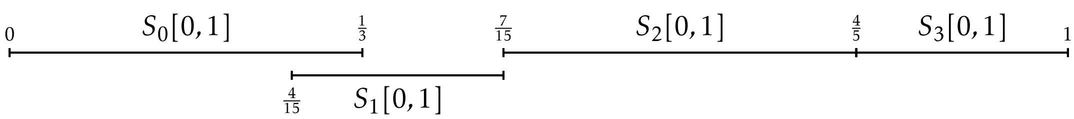
1.5 Iterated Function Systems of Finite Type

Definition. We say that the IFS $\{S_i\}_{i=0}^k$ is **finite type** if there are only finitely many neighbourhood sets.

Example. The governing example throughout this section is the IFS given by

$$S_0(x) = \frac{1}{3}x \qquad S_1(x) = \frac{1}{5}x + \frac{4}{15}$$
$$S_2(x) = \frac{1}{3}x + \frac{7}{15} \qquad S_3(x) = \frac{1}{5}x + \frac{4}{5}$$

which is perhaps better summarized by the diagram $S_i[0,1]$ for $i = 0, \ldots, 3$:



1.3 Corollary. *$\{S_i\}_{i=0}^k$ is finite type if and only if there are finitely many transition types.*

Proof Follows from Proposition 1.2. ■

Remark. Computationally, one can prove that an IFS is of finite type as follows. Starting with $\Delta = [0,1]$, the children $(\Delta_1,\dots,\Delta_n)$ in generation γ and their neighbour sets $V_\gamma(\Delta_1),\dots,V_\gamma(\Delta_n)$. Recursively repeat the process for any Δ_i in which $V_\gamma(\Delta_i)$ has not yet been observed. If this process terminates, then the IFS is of finite type.

1.4 Proposition. (Bounds on Interval Width) *Let $\{S_i\}_{i=0}^m$ be an IFS of finite type.*

(i) There exists a constant $M \geq 1$ such that for any $0 < \alpha \leq 1$ and $\Delta \in \mathcal{F}_\alpha$,

$$\frac{1}{M} \leq \frac{\lambda(\Delta)}{\alpha} \leq 1$$

(ii) There exists a constant $M \geq 1$ such that for any $0 < \alpha \leq 1$, for any $\Delta \in \mathcal{F}_\alpha$ with neighbours Δ^- and Δ^+,

$$\frac{1}{M} \leq \frac{\lambda(\Delta)}{\lambda(\Delta^-)} \leq M \text{ and } \frac{1}{M} \leq \frac{\lambda(\Delta)}{\lambda(\Delta^+)} \leq M.$$

Proof (i) Let $S \subseteq \Lambda_\alpha$ denote the set of words such that for any $\sigma \in S$, $\Delta \subseteq S_\sigma[0,1]$. By definition of finite type, $\lambda(\Delta)^{-1} r_\sigma$ takes one of finitely many values L_i; let $\lambda(\Delta)^{-1} r_\omega = L$ denote the maximum of such values. Then $\lambda(\Delta) \geq r_\omega / L$, so that

$$\frac{1}{r_{\max} L} \leq \frac{r_\omega}{L\alpha} \leq \frac{\lambda(\Delta)}{\alpha}$$

where $r_{\max} = \max_{0 \leq i \leq m} |r_i|$. The upper inequality follows since $r_\sigma / \alpha \leq 1$ for any $\sigma \in \Lambda_\alpha$.

(ii) Immediate from (i). ∎

2 Transition Matrices

Let $0 < \alpha < \beta \leq 1$ be arbitrary and suppose $\Delta = [a,b] \in \mathcal{F}_\alpha$ is arbitrary. Let $\widehat{\Delta} = [c,d] \in \mathcal{F}_\beta$ be the parent of Δ, and suppose

$$V_\alpha(\Delta) = ((a_1, L_1), \dots, (a_I, L_I))$$
$$V_\beta(\Delta) = ((c_1, M_1), \dots, (c_I, M_J))$$

Then the **transition matrix** $T_{\beta \to \alpha}(\Delta)$ is the $I \times J$ matrix defined as follows for fixed i, j. Let $\sigma \in \Lambda_\beta$ such that $S_\sigma(x) = \lambda(\widehat{\Delta}) \cdot (M_j x - c_j) + c$. Then $T_{ij} = \sum_{\omega \in S} p_\omega$ where

$$S = \{\omega : \sigma\omega \in \Lambda_\alpha \text{ and } S_{\sigma\omega}(x) = \lambda(\Delta) \cdot (L_i x - a_i) + a\}$$

and the empty sum is understood to be 0. It is straightforward to see that transition matrices are well-defined, since if σ and σ' satisfy $S_\sigma(x) = S_{\sigma'}(x)$, then $S_{\sigma\omega}(x) = S_{\sigma'\omega}(x)$ for any word ω.

2.1 Proposition. (Properties of Transition Matrices) *The following hold:*

(i) Suppose $0 < \alpha < \gamma < \beta \leq 1$ and $\Delta \in \mathcal{F}_\alpha$ has parent $\widehat{\Delta} \in \mathcal{F}_\gamma$. Then $T_{\beta \to \alpha}(\Delta) = T_{\beta \to \gamma}(\widehat{\Delta}) \cdot T_{\gamma \to \alpha}(\Delta)$.

3 Misc ideas

Define characteristic vector, so that the transition matrix is uniquely defined? To compute transition matrix, we need

- the neighbour type of the parent
- the neighbour type of the child
- the index of the child (there can be multiple children of the same neighbour type)