REPLACE

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Contents

I. Finite Type IV

1 Basic Definitions and Terminology

Iterated Function Systems

Throughout, we let λ denote the Lebesgue measure on $\mathbb{R}$.

Definition. An **IFS** is a finite set of contractions

$$S_i(x) = r_i x + a_i : \mathbb{R} \to \mathbb{R} \text{ for each } i = 0, 1, \dots, k$$

with $k \geq 1$ and $0 < |r_i| < 1$.

Each IFS generates a unique invariant compact set K, known as its associated self-similar set, satisfying

$$K = \bigcup_{j=0}^{k} S_j(K).$$

Rescaling the r_i and a_i if necessary, we may assume the convex hull of K is $[0,1]$. We also associate probabilities $0 < p_i < 1$ for each $i = 0, \dots, k$ which satisfy $\sum_i p_i = 1$. To these probabilities there is a unique self-similar measure μ with $\operatorname{supp} \mu = K$ satisfying

$$\mu = \sum_{j=0}^{k} p_j \mu \circ S_j^{-1}. \tag{1.1}$$

We are primarily interested in studying properties of this measure μ.

Let $\Sigma = \{0, 1, \dots, k\}$ be our alphabet, let Σ^k denote the words of length k, and $\Sigma^* = \bigcup_{k=0}^{\infty} \Sigma_k$ denote the set of all the finite words on Σ. Given $\sigma = (\sigma_1, \dots, \sigma_j) \in \Sigma$, we let

$$\begin{aligned} \sigma^- &= (\sigma_1, \dots, \sigma_{j-1}) \\ S_\sigma &= S_{\sigma_1} \circ \cdots \circ S_{\sigma_j} \end{aligned}$$

and similarly,

$$r_\sigma = \prod_{i=1}^{j} r_{\sigma_i} \text{ and } p_\sigma = \prod_{i=1}^{j} p_{\sigma_i}.$$

For any $0 < \alpha < 1$, we define the family

$$\Lambda_\alpha = \{\sigma \in \Sigma^* : |r_\sigma| \leq \alpha < |r_{\sigma^-}|\}$$

called the **words of generation α**. Given a word σ, we say that the **generation** $G(\sigma)$ is the interval $[r_\sigma, r_{\sigma^-})$. By definition $\alpha \in G(\sigma)$ if and only if $\sigma \in \Lambda_\alpha$.

Let $h_1, \dots, h_{s(\alpha)}$ be the collection of elements of the set $\{S_{\sigma(0)}, S_{\sigma(1)} : \sigma \in \Lambda_\alpha\}$ listed in increasing order. We set

$$\mathcal{F}_\alpha = \{[h_j, h_{j+1}] : 1 \leq j \leq s(\alpha) - 1 \text{ and } (h_j, h_{j+1}) \cap K \neq \emptyset\}$$

Elements of $\mathcal{F}_\alpha$ are called **net intervals of generation *n***.

Definition. A pair (a, L) is a **neighbour** of $\Delta \in \mathcal{F}_n$ if there is some $\sigma \in \Lambda_\alpha$ such that $S_\sigma(0,1) \cap \Delta \neq \emptyset$, $\lambda(\Delta)^{-1} r_\sigma = L$, and $\lambda(\Delta)^{-1}(a - S_\sigma(0)) = a$, and we say that σ **generates** the neighbour (a, L). Then the **neighbour set** of Δ is the ordered tuple

$$V_\alpha(\Delta) = ((a_1, L_1), \ldots, (a_j, L_j))$$

where each (a_i, L_i) is a (distinct) neighbour of Δ. We order these tuples so that $a_i \leq a_{i+1}$ and if $a_i = a_{i+1}$, then $L_i < L_{i+1}$.

Definition. We say that the IFS $\{S_i\}_{i=1}^{\infty}$ is **finite type** if there are only finitely many neighbourhood sets.

We can generalize the notion of generation to net intervals. Let $\Delta \in \mathcal{F}_\alpha$ have neighbour set $((a_1, L_1), \ldots, (a_j, L_j))$. For each i, let σ_i generate (a_i, L_i). Then we call

$$G(\Delta) = \bigcap_{i=1}^{j} G(\sigma_i)$$

the **generation** of some $\Delta \in \mathcal{F}_\alpha$. Note that

(i) $\alpha \in G(\Delta)$

(ii) For any $\beta \in G(\Delta)$, $\Delta \in \mathcal{F}_\beta$ and $V_\beta(\Delta) = V_\alpha(\Delta)$

(iii) If $\gamma \notin G(\Delta)$, either $\Delta \notin \mathcal{F}_\gamma$ or $V_\gamma(\Delta) \neq V_\alpha(\Delta)$.

Write $G(\Delta) = [|r_{\sigma_u}|, |r_{\sigma_v^-}|)$ and let ω such that $\sigma_i \preccurlyeq \omega$ for some i, $|r_\omega| < |r_{\sigma_u}|$, and $\gamma := |r_\omega|$ is maximal.

Let Δ have children $(\Delta_1, \ldots, \Delta_n) \in \mathcal{F}_\gamma$. Note that either $n > 1$ or if $n = 1$, then $V_\alpha(\Delta) \neq V_\gamma(\Delta_1)$. Furthermore, if $\gamma \leq \gamma' < |r_{\sigma_u}|$ is arbitrary and Δ has children $(\Delta_1', \ldots, \Delta_{n'}')$ in $\mathcal{F}_{\gamma'}$, then $n = n'$ and $\Delta_i' = \Delta_i$ for each $i = 1, \ldots, n$. We call the tuple $(\Delta_1, \ldots, \Delta_n)$ the **immediate children** of $\Delta \in \mathcal{F}_\alpha$.

Definition. Suppose $\Delta \in \mathcal{F}_\alpha$ has immediate children $(\Delta_1, \ldots, \Delta_n)$ in generation γ. Write $\Delta_i = [a_i, a_i + L_i]$. We define the **characteristic** of Δ, denoted $\mathcal{C}_\alpha(\Delta)$, to be the tuple

$$\Big((a_1, \lambda(\Delta)^{-1} L_1, V_\gamma(\Delta_1)), \ldots, (a_n, \lambda(\Delta)^{-1} L_n, V_\gamma(\Delta_n))\Big)$$

From the definition of finite type, one can see that there are only finitely many possible characteristics. Intuitively, the characteristic defines how Δ transitions from one generation to the next.

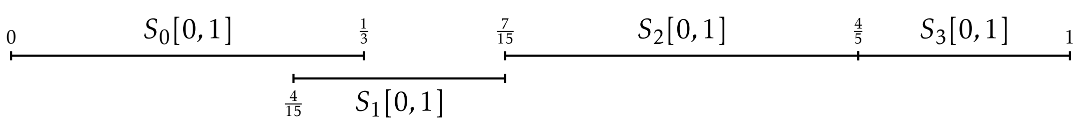
Remark. To compute the immediate children, it is not sufficient to consider $\gamma = |r_i|\alpha$ for some $0 \leq i \leq k$. For example, in the main IFS example, take $\Delta = [4/15, 1/3] \in \mathcal{F}_\alpha$ where $\alpha = 1/3$. Now $\Delta \in \mathcal{F}_{1/5}$, but has different neighbour set, and $\Delta \notin \mathcal{F}_{1/9}$ at all. However, the largest possible value of $|r_i|$ misses the immediate children.

Main Example

The governing example throughout this section is the IFS given by

$$S_0(x) = \frac{1}{3}x \qquad S_1(x) = \frac{1}{5}x + \frac{4}{15}$$
$$S_2(x) = \frac{1}{3}x + \frac{7}{15} \qquad S_3(x) = \frac{1}{5}x + \frac{4}{5}$$

which is perhaps better summarized by the diagram $S_i[0,1]$ for $i = 0, \ldots, 3$:



Fundamental Properties of Finite Type

Here are some properties of the generations $\{\Lambda_\alpha\}_{0<\alpha<1}$.

1.1 Proposition. *Let* $0 < \alpha < \beta < 1$.

(i) $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ *if and only if there exists some* $a_i \in \{0\} \cup \mathbb{N}$ *such that* $\rho := \prod_{i=0}^{k} |r_i|^{a_i}$ *satisfies* $\alpha \leq \rho < \beta$.

(ii) If σ *is any infinite word, there exists a unique* i_α *such that* $(\sigma_1, \ldots, \sigma_{i_\alpha}) \in \Lambda_\alpha$. *In particular, when* $\alpha < \beta$, $i_\alpha \geq i_\beta$.

Proof Recall that $\sigma \in \Lambda_\alpha$ for any $\alpha \in [|r_\sigma|, |r_{\sigma^-}|)$. This is sufficient for the forward implication of (i) and (ii).

To see the reverse implication of (i), suppose such a ρ exists and let ω be a word such that $|r_\omega| = \rho$. Let σ be a prefix of ω such that $\sigma \in \Lambda_\beta$; then since $\omega \notin \Lambda_\alpha$, $\sigma \notin \Lambda_\alpha$ as well. Let i be such that $|r_i|$ is minimal, and fix $\tau = \sigma i \ldots i$ be such that $\tau \in \Lambda_\alpha$. TODO...I am pretty sure this is true. Definitely true that the neighbour types of the new generation are different, even if the intervals are the same. ■

Remark. One way to think about the immediate children of an interval as follows. Enumerate the points $\left\{\prod_{i=0}^{k} |r_i|^{a_i} : a_i \in \{0\} \cup \mathbb{N}\right\}$ in decreasing order $(\rho_i)_{i=1}^{\infty}$. As in Proposition 1.1, the $\mathcal{F}_\alpha$ change on transitions of intervals $[\rho_{i+1}, \rho_i)$, but the Δ may be in both generations, and have the same neighbourhood type.

TODO: is this actually necessary? We could just use the generations ρ_i; even though some transitions will just be the identity map, it doesn't really matter. Need to think which will lead to more predictable symbolic representations (eg. for proof of below).

1.2 Proposition. *Let* $0 < \alpha < 1$ *and suppose* $\Delta \in \mathcal{F}_\alpha$ *has characteristic* $\mathcal{C}_\alpha(\Delta)$. *Let* Δ *have immediate children* $(\Delta_1, \ldots, \Delta_n)$. *Then the characteristic of any* Δ_i *depends only on* $\mathcal{C}_\alpha(\Delta)$.

2 Transition Matrices

Let $0 < \alpha < \beta < 1$ be arbitrary and suppose $\Delta = [a, b] \in \mathcal{F}_\alpha$ is arbitrary. Let $\widehat{\Delta} = [c, d] \in \mathcal{F}_\beta$ be the parent of Δ, and suppose

$$V_\alpha(\Delta) = ((a_1, L_1), \ldots, (a_I, L_I))$$
$$V_\beta(\Delta) = ((c_1, M_1), \ldots, (c_I, M_J))$$

Then the **transition matrix** $T_{\beta\to\alpha}(\Delta)$ is the $I \times J$ matrix defined as follows for fixed i, j. Let $\sigma \in \Lambda_\beta$ such that $S_\sigma(x) = \lambda(\widehat{\Delta}) \cdot (M_j x - c_j) + c$. Then $T_{ij} = \sum_{\omega \in S} p_\omega$ where

$$S = \{\omega : \sigma\omega \in \Lambda_\alpha \text{ and } S_{\sigma\omega}(x) = \lambda(\Delta) \cdot (L_i x - a_i) + a\}$$

and the empty sum is understood to be 0. It is straightforward to see that transition matrices are well-defined, since if σ and σ' satisfy $S_\sigma(x) = S_{\sigma'}(x)$, then $S_{\sigma\omega}(x) = S_{\sigma'\omega}(x)$ for any word ω.

3 Misc ideas

3.1 Proposition. (Properties of Transition Matrices) *The following hold:*

(i) Suppose $0 < \alpha < \gamma < \beta < 1$ *and* $\Delta \in \mathcal{F}_\alpha$ *has parent* $\widehat{\Delta} \in \mathcal{F}_\gamma$. *Then* $T_{\beta\to\alpha}(\Delta) = T_{\beta\to\gamma}(\widehat{\Delta}) \cdot T_{\gamma\to\alpha}(\Delta)$.

3.2 Proposition. *Let* $0 < \alpha < 1$ *and* $\Delta \in \mathcal{F}_\alpha$ *arbitrary. Let* $r = |r_i|$ *for some* i, *so that* Δ *has children* $(\Delta_1, \ldots, \Delta_n)$ *ordered left to right in* $\mathcal{F}_{r\alpha}$. *Then the sequence* $(\mu(\Delta_i), V_{r\alpha}(\Delta_i))_{i=1}^{n}$ *depends only on* α, $\lambda(\Delta)$ *and* $V_\alpha(\Delta)$.

Remark. One might hope to drop the requirement on knowing α; however, this does not hold if so. However, this is may be sufficient to prove that there are only finitely many types:

3.3 Proposition. (Bounds on Interval Width) *(i) There exists a constant $M \geq 1$ such that for any $0 < \alpha < 1$ and $\Delta \in \mathcal{F}_\alpha$,*

$$\frac{1}{M} \leq \frac{\lambda(\Delta)}{\alpha} \leq 1$$

(ii) There exists a constant $M \geq 1$ such that for any $0 < \alpha < 1$, for any $\Delta \in \mathcal{F}_\alpha$ with neighbours Δ^- and Δ^+,

$$\frac{1}{M} \leq \frac{\lambda(\Delta)}{\lambda(\Delta^-)} \leq M \text{ and } \frac{1}{M} \leq \frac{\lambda(\Delta)}{\lambda(\Delta^+)} \leq M.$$

Proof (i) Let $S \subseteq \Lambda_\alpha$ denote the set of words such that for any $\sigma \in S$, $\Delta \subseteq S_\sigma[0,1]$. By definition of finite type, $\lambda(\Delta)^{-1} r_\sigma$ takes one of finitely many values L_i; let $\lambda(\Delta)^{-1} r_\omega = L$ denote the maximum of such values. Then $\lambda(\Delta) \geq r_\omega / L$, so that

$$\frac{1}{r_{\max} L} \leq \frac{r_\omega}{L\alpha} \leq \frac{\lambda(\Delta)}{\alpha}$$

where $r_{\max} = \max_{0 \leq i \leq m} |r_i|$. The upper inequality follows since $r_\sigma / \alpha \leq 1$ for any $\sigma \in \Lambda_\alpha$.

(ii) Immediate from (i). ■

Note: there are infinitely many normalized lengths possible (even when normalizing with a value of $\prod_{i=0}^m |r_i|^{a_i}$).

How hard to characterize transition maps $\Lambda_\alpha \to \Lambda_{r_i \alpha}$? If there are only finitely many such transitions, we have finiteness requirement.

May need to do some sort of local argument? Fix a value of x, and consider a monotonic sequence $(\alpha_i)_{i=1}^\infty$ with $0 < \alpha_i < 1$ and $(\alpha_i) \to 0$. Then look at the sequence of character.

How hard to prove that something is actually finite type?