REPLACE

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1 Basic Definitions and Terminology

1.1 Iterated Function Systems

Throughout, we let λ denote the Lebesgue measure on $\mathbb{R}$.

Definition. An **IFS** is a finite set of contractions

$$S_i(x) = r_i x + a_i : \mathbb{R} \to \mathbb{R} \text{ for each } i = 0, 1, \ldots, k$$

with $k \geq 1$ and $0 < |r_i| < 1$.

Each IFS generates a unique invariant compact set K, known as its associated **self-similar set**, satisfying

$$K = \bigcup_{j=0}^{k} S_j(K).$$

Rescaling the r_i and a_i if necessary, we may assume the convex hull of K is $[0,1]$. We also associate probabilities $0 < p_i < 1$ for each $i = 0, \ldots, k$ which satisfy $\sum_i p_i = 1$. To these probabilities there is a unique self-similar measure μ with $\operatorname{supp} \mu = K$ satisfying

$$\mu = \sum_{j=0}^{k} p_j \mu \circ S_j^{-1}. \tag{1.1}$$

We are primarily interested in studying properties of this measure μ.

Let $\Sigma = \{0, 1, \ldots, k\}$ be our alphabet, let Σ^k denote the words of length k, and $\Sigma^* = \bigcup_{k=0}^{\infty} \Sigma_k$ denote the set of all the finite words on Σ. Given $\sigma = (\sigma_1, \ldots, \sigma_j) \in \Sigma$, we let

$$\sigma^- = (\sigma_1, \ldots, \sigma_{j-1})$$
$$S_\sigma = S_{\sigma_1} \circ \cdots \circ S_{\sigma_j}$$

and similarly,

$$r_\sigma = \prod_{i=1}^{j} r_{\sigma_i} \text{ and } p_\sigma = \prod_{i=1}^{j} p_{\sigma_i}.$$

Additionally, we set $r_{\max} = \max_i |r_i|$.

1.2 Generations

For any $0 < \alpha \leq 1$, we define the family

$$\Lambda_\alpha = \{\sigma \in \Sigma^* : |r_\sigma| < \alpha \leq |r_{\sigma^-}|\}$$

called the **words of generation α**. Given a word σ, we say that the **generation** $G(\sigma)$ is the interval $(|r_\sigma|, |r_{\sigma^-}|]$. By definition $\alpha \in G(\sigma)$ if and only if $\sigma \in \Lambda_\alpha$.

Let $h_1, \ldots, h_{s(\alpha)}$ be the collection of elements of the set $\{S_{\sigma(0)}, S_{\sigma(1)} : \sigma \in \Lambda_\alpha\}$ listed in increasing order. We set

$$\mathcal{F}_\alpha = \{[h_j, h_{j+1}] : 1 \leq j \leq s(\alpha) - 1 \text{ and } (h_j, h_{j+1}) \cap K \neq \emptyset\}$$

Elements of $\mathcal{F}_\alpha$ are called **net intervals of generation α**.

Definition. A pair (a, L) is a **neighbour** of $\Delta = [a_0, b_0] \in \mathcal{F}_\alpha$ if there is some $\sigma \in \Lambda_\alpha$ such that $S_\sigma(0,1) \cap \Delta \neq \emptyset$, $\lambda(\Delta)^{-1} r_\sigma = L$, and $\lambda(\Delta)^{-1}(a_0 - S_\sigma(0)) = a$, and we say that σ **generates** the neighbour (a, L). Then the **neighbour set** of Δ is the ordered tuple

$$V_\alpha(\Delta) = ((a_1, L_1), \dots, (a_j, L_j))$$

where each (a_i, L_i) is a (distinct) neighbour of Δ. We order these tuples so that $a_i \leq a_{i+1}$ and if $a_i = a_{i+1}$, then $L_i < L_{i+1}$.

Abusing notation slightly, we say that $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ if either the net intervals are distinct, or if they are the same, then the neighbour set of some net interval is different. (TODO: is this actually abusing notation? Or are the notions equivalent?) We can generalize the notion of generation to net intervals. Let $\Delta \in \mathcal{F}_\alpha$ have neighbour set $((a_1, L_1), \dots, (a_j, L_j))$. For each i, let σ_i generate (a_i, L_i). Then we call

$$G(\Delta) = \bigcap_{i=1}^{j} G(\sigma_i)$$

the **generation** of some $\Delta \in \mathcal{F}_\alpha$. Note that

(i) $\alpha \in G(\Delta)$
(ii) For any $\beta \in G(\Delta)$, $\Delta \in \mathcal{F}_\beta$ and $V_\beta(\Delta) = V_\alpha(\Delta)$
(iii) If $\gamma \notin G(\Delta)$, either $\Delta \notin \mathcal{F}_\gamma$ or $V_\gamma(\Delta) \neq V_\alpha(\Delta)$.

1.3 Transition Types

Let $0 < \alpha \leq 1$ and $\Delta \in \mathcal{F}_\alpha$ and suppose Δ has neighbour set $((a_1, L_1), \dots, (a_n, L_n))$ Set

$$L_{\max} = \max\{|L_i| : i = 1, \dots, n\} \text{ and } \gamma = \lambda(\Delta) \cdot L_{\max};$$

in other words, γ is the largest value achieved by $|r_\sigma|$ where σ generates some neighbour of Δ. Let Δ have children $(\Delta_1, \dots, \Delta_n) \in \mathcal{F}_\gamma$. As in the proof of Proposition 1.1, we see that either $n > 1$ or if $n = 1$, then $V_\alpha(\Delta) \neq V_\gamma(\Delta_1)$. We call the tuple $(\Delta_1, \dots, \Delta_n)$ the **children** of $\Delta \in \mathcal{F}_\alpha$. Note that it suffices to take any γ' such that

$$\max\Big\{r_{\max}\gamma, \lambda(\Delta) \cdot \max\{|L_i| : i = 1, \dots, n; |L_i| \neq L_{\max}\}\Big\} < \gamma' \leq \gamma$$

where the inner maximum is taken to be 0 if the set is empty.

Definition. Suppose $\Delta = [a, b] \in \mathcal{F}_\alpha$ has children $(\Delta_1, \dots, \Delta_n)$ in generation γ. Write $\Delta_i = [a_i, a_i + L_i]$. We define the **transition type** of Δ, denoted $\mathcal{C}_\alpha(\Delta)$, to be the tuple

$$\Big(\Big(\frac{a_1 - a}{\lambda(\Delta)}, \frac{L_1}{\lambda(\Delta)}, V_\gamma(\Delta_1)\Big), \dots, \Big(\frac{a_n - a}{\lambda(\Delta)}, \frac{L_n}{\lambda(\Delta)}, V_\gamma(\Delta_n)\Big)\Big)$$

In the natural way, we define the **parent** of $\Delta \in \mathcal{F}_\alpha$ to be the net interval $\widehat{\Delta} \in \mathcal{F}_\beta$ with $\beta > \alpha$, so that Δ is a child of $\widehat{\Delta}$.

Given $x \in K$, there exists a nested sequence of intervals $(\Delta_i)_{i=1}^\infty$ such that for any $i \in \mathbb{N}$

- $[0,1] = \Delta_1 \supseteq \Delta_2 \supseteq \cdots$,
- $x \in \Delta_i$, and
- Δ_{i+1} is the child of Δ_i.

We call this sequence the **representative intervals** (TODO think of something else to call this) of $x \in K$.

Remark. To compute the children, it is not sufficient to consider $\gamma = |r_i|\alpha$ for some $0 \leq i \leq k$. For example, in the main IFS example, take $\Delta = [4/15, 1/3] \in \mathcal{F}_\alpha$ where $\alpha = 1/3$. Now $\Delta \in \mathcal{F}_{1/5}$, but has different neighbour set, and $\Delta \notin \mathcal{F}_{1/9}$ at all. However, the largest possible value of $|r_i|$ misses the children.

1.4 Basic Properties of IFS

Here are some properties of the generations $\{\Lambda_\alpha\}_{0<\alpha\leq 1}$.

1.1 Proposition. *Let* $0<\alpha<\beta\leq 1$.

(i) $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ *if and only if there exists some* $a_i \in \{0\}\cup\mathbb{N}$ *such that* $\rho := \prod_{i=0}^k |r_i|^{a_i}$ *satisfies* $\alpha<\rho\leq\beta$.

(ii) If σ *is any infinite word, there exists a unique index* i_α *such that* $(\sigma_1,\ldots,\sigma_{i_\alpha})\in\Lambda_\alpha$. *In particular, when* $\alpha<\beta$, $i_\alpha \geq i_\beta$.

Proof Recall that $\sigma\in\Lambda_\alpha$ for any $\alpha\in(|r_\sigma|,|r_{\sigma^-}|]$. This is sufficient for the forward implication of (i) and (ii).

To see the reverse implication of (i), suppose such a ρ exists and let ω be a word such that $|r_\omega|=\rho$. Let σ be a prefix of ω such that $\sigma\in\Lambda_\beta$; then since $\omega\notin\Lambda_\alpha$, $\sigma\notin\Lambda_\alpha$ as well. Let $\Delta\subseteq S_\sigma[0,1]$ with $\Delta\in\mathcal{F}_\alpha$ be any net interval, so that σ generates some neighbour (a,L) of Δ where $L=r_\sigma\lambda(\Delta)^{-1}$. If $\Delta\notin\mathcal{F}_\beta$, we are done, so let's suppose $\Delta=\Delta'\in\mathcal{F}_\beta$. Now suppose (a',L') is any neighbour of Δ' generated by $\tau\in\Lambda_\beta$; it suffices to show that $(a',L')\neq(a,L)$. Suppose $a=a'$; then if τ generates (a',L'), we have $\sigma\tau$, and since $\sigma\in\Lambda_\alpha$ and $\tau\in\Lambda_\beta$ with $\Lambda_\alpha\neq\Lambda_\beta$, we have $r_\sigma\neq r_\tau$ so $L\neq L'$. ∎

Remark. One way to think about the children of an interval as follows. Enumerate the points $\left\{\prod_{i=0}^k |r_i|^{a_i} : a_i\in\{0\}\cup\mathbb{N}\right\}$ in decreasing order $(\rho_i)_{i=1}^\infty$. As in Proposition 1.1, the $\mathcal{F}_\alpha$ change on transitions of intervals $[\rho_{i+1},\rho_i)$. However, if $\Delta\in\mathcal{F}_\alpha$ with $\alpha\in[\rho_{k+1},\rho_k)$, it may be that $\Delta\in\mathcal{F}_\beta$ for any $\beta\in[\rho_{k+2},\rho_{k+1})$, with $V_\beta(\Delta)=V_\alpha(\Delta)$. The children are the net intervals in generation ρ_m where $m>k+1$ is minimal such that either $\Delta\notin\mathcal{F}_{\rho_m}$ or $V_{\rho_m}(\Delta)\neq V_\alpha(\Delta)$.

1.2 Proposition. *Consider the IFS* $\{S_i\}_{i=0}^m$ *and let* $0<\alpha\leq 1$. *Then* $\mathcal{C}_\alpha(\Delta)$ *depends only on the neighbour set* $V_\alpha(\Delta)$.

Proof *[Sketch.]* Let $\Delta\in\mathcal{F}_\alpha$ have neighbour set $\big((a_1,L_1),\ldots,(a_n,L_n)\big)$ and let $\{i_1,\ldots,i_m\}$ be the set of indices such that $L_{i_1}=\cdots=L_{i_m}=:L$ are maximal. Let γ be such that $(\Delta_1,\ldots,\Delta_n)$ with $\Delta_i\in\mathcal{F}_\gamma$ are the children of Δ. Let $\Gamma\subseteq\Lambda_\alpha$ be the set of words which generate some (a_{i_j},L_{i_j}) for $1\leq j\leq m$. Note the following facts:

- If $\sigma\in\Lambda_\alpha\setminus\Gamma$ has $S_\sigma[0,1]\supseteq\Delta$, then $\sigma\in\Lambda_\gamma$
- If $\sigma\in\Gamma$, then $\sigma\notin\Lambda_\gamma$ but $\sigma l\in\Lambda_\gamma$ for any $0\leq l\leq m$.

But then the words $\tau\in\Lambda_\gamma$ such that $S_\tau[0,1]\supseteq\Delta$ are precisely the words

$$\tau\in\Lambda_\alpha\setminus\Gamma \text{ with } S_\tau[0,1]\supseteq\Delta$$

or

$$\tau=\sigma l \text{ with } \sigma\in\Gamma \text{ and } l\in\{0,\ldots,k\}.$$

Thus the set $\{\tau\in\Lambda_\gamma : S_\tau[0,1]\supseteq\Delta\}$ depends only on $V_\alpha(\Delta)$, which fully determines $C_\alpha(\Delta)$. ∎

2 Iterated Function Systems of Finite Type

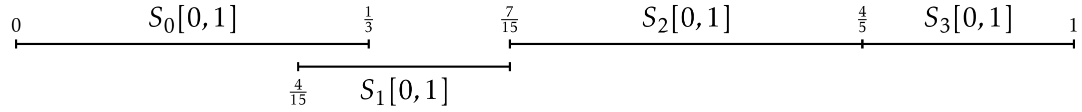
2.1 Finite Type

Definition. We say that the IFS $\{S_i\}_{i=0}^k$ is **finite type** if there are only finitely many neighbourhood sets.

Example. The governing example throughout this section is the IFS given by

$$S_0(x)=\frac{1}{3}x \qquad S_1(x)=\frac{1}{5}x+\frac{4}{15}$$
$$S_2(x)=\frac{1}{3}x+\frac{7}{15} \qquad S_3(x)=\frac{1}{5}x+\frac{4}{5}$$

which is perhaps better summarized by the diagram $S_i[0,1]$ for $i=0,\ldots,3$:



2.1 Corollary. *$\{S_i\}_{i=0}^k$ is finite type if and only if there are finitely many transition types.*

Proof Follows from Proposition 1.2. ∎

Remark. Computationally, one can prove that an IFS is of finite type as follows. Starting with $\Delta = [0,1]$, the children $(\Delta_1, \dots, \Delta_n)$ in generation γ and their neighbour sets $V_\gamma(\Delta_1), \dots, V_\gamma(\Delta_n)$. Recursively repeat the process for any Δ_i in which $V_\gamma(\Delta_i)$ has not yet been observed. If this process terminates, then the IFS is of finite type.

2.2 Open Set Condition and Weak Separation

Recall that an IFS $\{S_i\}_{i=0}^k$ satisfies the **open set condition** if there exists a non-empty bounded open set $U \subseteq \mathbb{R}$ such that $\dot\bigcup_{i=0}^k S_i(U) \subseteq U$ disjointly. In this case, it follows that the $S_i(0,1)$ must be disjoint: otherwise, for any interval neighbourhood $V \subseteq U$, some $S_i(V) \cap S_j(V) \neq \emptyset$, violating the disjoint union. In this case, there is exactly one neighbour set and the only transition type is realized at the first generation. Thus every IFS which satisfies the open set condition is finite type.

In [some paper], the authors introduce the notion of the **weak separation property**. Note that we have slightly rephrased the definition here, due to the differences the definition of generation, but the notions are equivalent.

Definition. The IFS $\{S_i\}_{i=0}^k$ has the **weak separation property** if there exists some $x_0 \in \mathbb{R}$ and $\ell \in \mathbb{N}$ such that for any $\tau \in \Sigma^*$ and $0 < \alpha \leq 1$, any closed ball with radius α contains no more than ℓ distinct points of the form $S_\sigma(S_\tau(x_0))$ where $\sigma \in \Lambda_\alpha$.

They also prove that the weak separation property is strictly weaker than the open set condition. We will see that an IFS of finite type trivially satisfies the weak separation property.

2.2 Lemma. (Bounds on Interval Width) *Let $\{S_i\}_{i=0}^m$ be an IFS of finite type. Then there exists some $\epsilon > 0$ such that for any $0 < \alpha \leq 1$ and $\Delta \in \mathcal{F}_\alpha$,*

$$0 < \epsilon \leq \frac{\lambda(\Delta)}{\alpha} \leq 1$$

Proof Let $S \subseteq \Lambda_\alpha$ denote the set of words such that for any $\sigma \in S$, $\Delta \subseteq S_\sigma[0,1]$. By definition of finite type, $\lambda(\Delta)^{-1} r_\sigma$ takes one of finitely many values L_i; let $\lambda(\Delta)^{-1} r_\omega = L$ denote the maximum of such values. Then $\lambda(\Delta) \geq r_\omega / L$, so that

$$\frac{1}{r_{\max} L} \leq \frac{r_\omega}{L\alpha} \leq \frac{\lambda(\Delta)}{\alpha}$$

where $r_{\max} = \max_{0 \leq i \leq m} |r_i|$. The upper inequality follows since $r_\sigma / \alpha \leq 1$ for any $\sigma \in \Lambda_\alpha$. ∎

2.3 Corollary. *Any IFS $\{S_i\}_{i=0}^k$ of finite type has the weak separation property.*

Proof We may take any x_0 and τ arbitrary. From Lemma 2.2, by counting the number of net intervals contained in $B_\alpha(S_\tau(x_0))$, it suffices to take $\ell > \frac{1}{\epsilon} + 1$. ∎

TODO: is the converse true? Can I find a counterexample?

2.3 Other Notions of Finite Type

Definition. We say that the IFS is **generalized finite type** if (this is a long definition, see "A generalized finite type condition for iterated function systems" by Ka-Sing Lau and Sze-Man Ngai.)

Definition. Let $\{S_i\}_{i=0}^k$ be an IFS as above. Let $r_{\min} = \min\{|r_i| : i = 0,\dots,k\}$, and define $\Lambda_n = \{\sigma \in \Sigma^* : |r_\sigma| \leq r_{\min}^n < |r_{\sigma^-}|\}$. The words $\sigma, \tau \in \Lambda_n$ are said to be neighbours if $S_\sigma(0,1) \cap S_\tau(0,1) \neq \emptyset$. Denote by $N(\sigma)$ the set of all neighbours of σ. We say that $\sigma \in \Lambda_n$ and $\tau \in \Lambda_m$ have the same neighbourhood type if there is a map $f(x) = \pm r_{\min}^{n-m} x + c$ such that

$$\{f \circ S_\eta : \eta \in N(\sigma)\} = \{S_\nu : \nu \in N(\tau)\} \text{ and } f \circ S_\sigma = S\tau.$$

If there are only finitely many neighbour hood type, we say that $\{S_i\}_{i=0}^k$ is **finite type III**.

2.4 Proposition. *If the IFS $\{S_i\}_{i=0}^k$ is generalized finite type, then it is finite type.*

Proof We proceed by contrapositive. Suppose $\{S_i\}_{i=0}^k$ is not finite type. Then there exists some $x \in K$ with representative intervals $(\Delta_i)_{i=1}^\infty$ with infinitely many neighbour sets. Let $\{\mathcal{M}_k\}_{k=0}^\infty$ be any sequence of nested index sets. ...TODO, pretty sure this is true. ∎

Remark. It was proven in that paper that that the IFS is of finite type, in our current notion. (TODO: actually do this. It shouldn't be hard, but there might be a bit of work to show for arbitrary α there are only finitely possible many neighbour sets)

Is the converse true? Can I think of something with WSP but not finite type?

3 Transitions

3.1 Characteristic Vectors and Symbolic Representations

Suppose $\Delta \in \mathcal{F}_\alpha$ has parent $\widehat{\Delta} \in \mathcal{F}_\beta$. Let $(\Delta_1,\dots,\Delta_m)$ denote the ordered children of $\widehat{\Delta}$ which satisfy $V_\alpha(\Delta_i) = V_\alpha(\Delta)$. Let j be the index so that $\Delta_j = \Delta$. Then we say that the **characteristic vector** of Δ is the pair $(V_\beta(\widehat{\Delta}), j)$.

Remark. TODO: I don't like this definition, because it feels very artificial. The point of the characteristic vector is that, given the sequence of characteristic vectors, it fully encodes the necessary information to compute the transition matricies between children and parents. In a sense, this is the notion of 'transition type', that the neighbour type of the parent fully determines what happens with the children.

We say that the characteristic vector of $[0,1]$ is $(((0,1)),1)$.

Given a point $x \in K$, consider the nested sequence of net intervals $[0,1] = \Delta_1 \supseteq \Delta_2 \supseteq \cdots$ for each i $x \in \Delta_i$ and Δ_{i+1} is a child of Δ_i. Then the **symbolic representation** of $x \in K$ is the sequence $\{\mathcal{C}(\Delta_i)\}_{i=1}^\infty$. It is straightforward to see that

- The symbolic representation uniquely determines the point x.
- The symbolic representation is unique unless x is the endpoint of some net interval Δ, in which case there may be two symbolic representations.

3.2 Transition Matrices

Let $0 < \alpha < \beta \leq 1$ be arbitrary and suppose $\Delta = [a,b] \in \mathcal{F}_\alpha$ is arbitrary. Let $\widehat{\Delta} = [c,d] \in \mathcal{F}_\beta$ be the parent of Δ, and suppose

$$V_\alpha(\Delta) = ((a_1,L_1),\dots,(a_I,L_I))$$
$$V_\beta(\Delta) = ((c_1,M_1),\dots,(c_I,M_J))$$

Then the **transition matrix** $T_{\beta\to\alpha}(\Delta)$ is the $I \times J$ matrix defined as follows for fixed i, j. Let $\sigma \in \Lambda_\beta$ such that $S_\sigma(x) = \lambda(\widehat{\Delta}) \cdot (M_j x - c_j) + c$. Then $T_{ij} = \sum_{\omega \in S} p_\omega$ where

$$S = \{\omega : \sigma\omega \in \Lambda_\alpha \text{ and } S_{\sigma\omega}(x) = \lambda(\Delta) \cdot (L_i x - a_i) + a\}$$

and the empty sum is understood to be 0. It is straightforward to see that transition matrices are well-defined, since if σ and σ' satisfy $S_\sigma(x) = S_{\sigma'}(x)$, then $S_{\sigma\omega}(x) = S_{\sigma'\omega}(x)$ for any word ω. More importantly, we have the following properties of transition matrices:

3.1 Proposition. (Properties of Transition Matrices) *The following hold:*

(i) Suppose $0 < \alpha < \gamma < \beta \leq 1$ and $\Delta \in \mathcal{F}_\alpha$ has parent $\widehat{\Delta} \in \mathcal{F}_\gamma$. Then $T_{\beta\to\alpha}(\Delta) = T_{\beta\to\gamma}(\widehat{\Delta}) \cdot T_{\gamma\to\alpha}(\Delta)$.

4 Misc ideas

Define characteristic vector, so that the transition matrix is uniquely defined? To compute transition matrix, we need

- the neighbour type of the parent
- the neighbour type of the child
- the index of the child (there can be multiple children of the same neighbour type)