Harmonic Analysis

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I. Harmonic Analysis

1 Locally Compact Groups

Definition. Let G be a group. A topology τ on G is a **group topology** provided that

- $x \mapsto x^{-1} : G \to G$ is continuous, and
- $(x,y) \mapsto xy : G \times G \to G$ is continuous.

We call (G,τ) a **topological group** where we omit τ when it is clear from context.

Equivalently, we may assert that $(x,y) \mapsto xy^{-1}$ is $\tau \times \tau - \tau$–continuous. Write $L_g(x) = gx$ and $R_g(x) = xg$ to denote the left and right multiplication maps; then it is easy to see that L_g and R_g are homeomorphisms. Similarly, $x \mapsto x^{-1}$ is a homeomorphism.

Definition. We say that a subset $A \subset G$ is **symmetric** if $A^{-1} = A$.

We have the following basic properties:

1.1 Proposition. *Let (G,τ) be a topological group.*

(i) If $\emptyset \neq A \subseteq G$ and U is open, then $AU = \{ay : a \in A, y \in U\}$ and likewise UA are open.
(ii) Given $U \in \tau$ and $x \in U$, then there is a symmetric $V \in \tau$ with $e \in V$ such that $VxV \subseteq U$. In particular, if $e \in U$, then we can find symmetric V so that $V^2 \subseteq U$.
(iii) If H is a subgroup of G, then $\overline{H}$ is also a subgroup.
(iv) An open subgroup is automatically closed.
(v) If $K, L \subseteq G$ are compact, then KL is compact.
(vi) If K is compact and C is closed in G, then KC is closed.

In $(\mathbb{R},+)$, then $\mathbb{Z}+\sqrt{2}\mathbb{Z}$ is not closed, so it is necessary to assume compactness in (vi).

Proof (i) $AU = \bigcup_{a\in A} L_a(U)$ is a union of open sets.

(ii) Consider the continuous map $(y,z) \mapsto yxz$. Since $exe = x \in U$, there is a $\tau \times \tau$–neighbourhood of (e,e) which maps into U have a basic neighbourhood $V_1 \times V_2$. Let $V = V_1 \cap V_2$. Moreover, we may replace V by $V^{-1} \cap V$. to attain symmetry.

(iii) Let $x, y \in \overline{H}$, $U \in \tau$ with $xy \in U$. Then (ii) provides V with $VxyV \subseteq U$. But $Vx \cap H \neq \emptyset$ and $\neq yV \cap H$ so there are ${}_1 \in Vx \cap H$, $h_2 \in yV \cap H$, and $h_1 h_2 \in VxyV \subseteq U$. Thus $U \cap H \neq \emptyset$. Thus $xy \in \overline{H}$.
To use nets for inverses, if $x \in \overline{H}$, then $x = \lim_\alpha x_\alpha$ where $(x_\alpha)_{\alpha\in A} \subset H$ is a net. Then $x^{-1} = \lim_\alpha x_\alpha^{-1} \in \overline{H}$ as each $x_\alpha^{-1} \in H$.

(iv) If H is an open subgroup, then $H = G \setminus \bigcup_{x\in G\setminus H} xH$ is closed.

(v) $K \times L$ is compact, and hence so is its image under multiplication.

(vi) If $x \in \overline{KC}$, then $x = \lim_\alpha k_\alpha x_\alpha$ where $k_\alpha \in H$ and $x_\alpha \in C$. Since K is compact, we may assume (passing to a subnet if necessary) $k = \lim_\alpha k_\alpha$ exists in K. Then

$$k^{-1}x = \lim_\alpha k_\alpha^{-1} \cdot \lim_\alpha k_\alpha x_\alpha = \lim_\alpha k_\alpha^{-1} k_\alpha x_\alpha = \lim x_\alpha \in C$$

so $x = kk^{-1}x \in KC$. ∎

1.1 Homogenous Spaces

Let (G,τ) be a topological group, H a subgroup of G, and ${G}/{H} = \{xH; x \in G\}$. Let $\pi : G \to {G}/{H}$ be given by $\pi(x) = xH$ be the projection map. The **quotient topology** on ${G}/{H}$ is $\tau_{G/H} = \{W \in {G}/{H} : \pi^{-1}(W) \in \tau\}$. Notice that if $U \in \tau \setminus \{\emptyset\}$, then $UH = \pi^{-1}(\pi(U))$ is open, so $\pi : G \to {G}/{H}$ is an open map.

1.2 Proposition. *Let (G,τ), H be as above.*
(i) The map $(x,yH) \mapsto xyH : G \times {G}/{H} \to {G}/{H}$ is $\tau \times \tau_{G/H} - \tau_{G/H}$ continuous and open.
(ii) If H is normal, then $({G}/{H}, \tau_{G/H})$ is a topological group.
(iii) If H is closed, then $\tau_{G/H}$ is Hausdorff.

Proof (i) Let $x,y \in G$, $W \in \tau_{G/H}$ satisfy $xyH = \pi(xy) \in W$. Then $xy \in \pi^{-1}(W)$ and by Proposition 1.1 we have $V \in \tau$ with $e \in V$ such that $VxyV \subseteq \pi^{-1}(W)$. But then $(x,\pi(y)) \in Vx \times \pi(yV) \in \tau \times \tau_{G/H}$ and the latter set maps into $\pi(VxyV) \subseteq W$.
Also, if $U \in \tau \times \tau_{G/H}$, then $U = \bigcup_{(x,yH)\in U} V_x \times W_{yH}$ and

$$\pi(U) = \bigcup_{(x,yH)\in U} \pi(V_x \pi^{-1}(W_{yH}))$$

since π is open.

(ii) Recall that $(xH)(yH) = xyH$ is our multiplication operation on ${G}/{H}$ and π is a group homomorphism. Then the following diagram commutes: We have that $\pi \times \mathrm{id}$ is open and $(x,yH) \mapsto xyH$ is open from (i), so the multiplication from ${G}/{H} \times {G}/{H} \to {G}/{H}$ must be open and continuous.

(iii) If $x,y \in G$ with $\pi(x) \neq \pi(y)$, then $e \notin xHy^{-1}$. Now $xHy^{-1} = L_x(R_{y^{-1}}(H))$ so xHy^{-1} is closed. Hence by the last proposition, there is a symmetric open V with $e \in V$ so $V^2 \subseteq G \setminus (xHy^{-1})$. But then $e \notin (VxH)(VyH)^{-1} = VxHy^{-1}V$: if we had $e = vxhy^{-1}u$ with $v,u \in V$ and $h \in H$, then $v^{-1}u^{-1} = xhy^{-1} \in V^2 \cap (xHy^{-1}) = \emptyset$, a contradiction. Hence $VxH \cap VyH = \emptyset$ so $\pi(Vx)$, $\pi(Vy)$ is a pair of separating neighbourhoods of $\pi(x)$, $\pi(y)$. ■

1.3 Corollary. *G is Hausdorff if and only if there exists $x \in G$ so that $\{x\}$ is closed.*

Proof In a Hausdorff space, points are closed. Conversely, if $\{x\}$ is closed, $\{e\} = L_{x^{-1}}(\{x\})$ is closed and a normal subgroup. Then $G \cong {G}/{\{e\}}$ is Hausdorff. ■

If (G,τ) is not Hausdorff, then $\{e\} \subsetneq \overline{\{e\}}$ is the smallest closed subgrup in G. Thus $\overline{\{e\}} \subseteq \bigcap_{x\in G} x\overline{\{e\}}x^{-1} \subseteq \overline{\{x\}}$ so $\overline{\{e\}}$ is normal. In particular, ${G}/{\overline{\{e\}}}$ is Hausdorff.

Definition. A **locally compact group** is a Hausdorff topological group (G,τ) which is locally compact.

(i) If there is any $U \in \tau \setminus \{\emptyset\}$ such that $\overline{U}$ is compact, then for any $x \in U$, we have $e \in x^{-1}U \subseteq L_{x^{-1}}(\overline{U})$ so $\overline{x^{-1}U}$ is compact. If $V \in \tau$ with $e \in V$ and $\overline{V}$ compact, then for any $x \in H$, $x \in xV$ and $\overline{xV} \subseteq L_x(\overline{V})$ and $\overline{xV}$ is compact. In particular, (G,τ) is locally compact if and only if there is some $U \in \tau \setminus \{\emptyset\}$ with $\overline{U}$ compact.

(ii) If (G,τ) is locally cmpact and N is a closed normal subgroup, then $({}^{G}\!/\!_{N}, \tau_{G/N})$ is locally compact. Indeed, $U \in \tau \setminus \{e\}$ with $\overline{U}$ compact, then $\overline{\pi(U)} \subseteq \pi(\overline{U})$ is compact.

Example. (i) If G is any group and τ is the discrete topology, then (G, τ_d) is locally compact.

(ii) If $((\mathbb{R},+), \tau_{\|\cdot\|})$ is locally compact.

(iii) If $\{G_i\}_{i\in I}$ is a family of locally compact groups, then $\prod_{i\in I} G_i$ is a locally compact group if and only if all but finitely many (G_i, τ_i) are compact.

(iv) $((\mathbb{R}^n,+), \tau_{\|\cdot\|})$ is a locally compact group

(v) Suppose $\{F_i\}_{i\in I}$ is an infinite family of finite groups (with discrete topologies), then $G = \prod_{i\in I} F_i$ is a compact group. If $F \subset I$ is finite, then $N_F = \{(x_i)_{i\in I} \in G : x_i = e \text{ for } i \in F\}$ is open and a normal subgroup. $\{N_F : F \subset I \text{ finite}\}$ is a base for the topology at e.

Example. (p−adic numbers) Let p be a prime in $\mathbb{N}$. We will establish product structures and topologies on

$$\mathbb{O}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : a_k \in \{0,1,\dots,p-1\} \right\} \cong \{0,1,\dots,p-1\}^{\mathbb{N}}$$
$$\mathbb{Q}_p = \left\{ \sum_{k=N}^{\infty} a_k p^k : N \in \mathbb{Z}, a_k \in \{0,1,\dots,p-1\} \right\}$$

are topological rings and a topological field respectively. Let $R_p = \prod_{n=0}^{\infty} \mathbb{Z}\big/p^{n+1}\mathbb{Z}$ which is a ring under pointwise operations.

1.4 Lemma. *The map $\rho : R_p \times R_p \to [0,1]$ given by*

$$\rho(x,y) = \sum_{n\in\mathbb{N}_0} \frac{\rho_n(x_n,y_n)}{p^n} \qquad\qquad \rho_n(x_n,y_n) = \begin{cases} 1 & : x_n = y_n \\ 0 & : x_n \neq y_n \end{cases}$$

is a metric on R_p which satisfies

- *(additively invariant): $\rho(x+z, y+z) = \rho(x,y)$ for $x,y,z \in R_p$*
- *τ_ρ is the product topology*

Proof Additive invariance is routine. Notice that if $\frac{1}{p^m} \geq \epsilon > \frac{1}{p^{m+1}}$, then the open ϵ−ball around a point x is $\{x_0\} \times \cdots \{x_m\} \times \prod_{n=m+1}^{\infty} \mathbb{Z}\big/p^{n+1}\mathbb{Z}$ is product-open. Conversely, any product-open set is a finite union of such ϵ−balls. ■

1.5 Corollary. *The function $\|x\|_p = \rho(x,0)$ in R_p satisfies*

- $\|x\|_p = 0$ *if and only if* $x = 0$
- $\|x+y\|_p \leq \|x\|_p + \|y\|_p$
- $\|xy\|_p \leq \|x\|_p \|y\|_p$
- $\|-x\|_p = \|x\|_p$

Hence (R_p, τ_ρ) is a compact topological ring.

Proof The properties follow directly using additive invariance. To see that R_p is a topological ring, if $(x_\alpha),(y_\alpha)$ have $x=\lim x_\alpha$ and $y=\lim y_\alpha$, then, for example,

$$\begin{aligned}\left\|xy-x_\alpha y_\alpha\right\|_p &\le \left\|xy-x_\alpha y\right\|_p+\left\|x_\alpha y-x_\alpha y_\alpha\right\|_p\\ &\le \|x-x_\alpha\|_p+\left\|y-y_\alpha\right\|_p\end{aligned}$$

as $\left\|y\right\|_p, \|x_\alpha\|_p\le 1$. ■

We now view $\mathbb{O}_p$ as a closed subring of R_p. Define $\alpha:\mathcal{O}_p\to R_p$ be given on $a=\sum_{k=0}^\infty a_kp^k$ by

$$\alpha(a)=\left(\sum_{k=0}^n a_kp^k+p^{n+1}\mathbb{Z}\right)_{n=0}^\infty.$$

This map is an injection with range $\alpha(\mathcal{O}_p)=\{(x_n)_{n=0}^\infty\in R_p : x_n=\pi_n(x_{n+1})\text{ for all } n\}$ where $\pi_n:\mathbb{Z}/p^{n+2}\mathbb{Z}\to\mathbb{Z}/p^{n+1}\mathbb{Z}$ is the canonical quotient map. In fact, this is called an inductive limit with respect to the maps π_n. Hence it is routine to show that

- $\alpha(\mathbb{O}_p)$ is a subring of R_p, and
- $\alpha(\mathbb{O}_p)$ is closed in R_p (just check net limits in product topology)

If $a,b\in\mathbb{O}_p$, define $a+b=\alpha^{-1}(\alpha(a)+\alpha(b))$.

Remark. (i) $1+\sum_{k=1}^\infty 0\cdot p^k$ is the multiplicative identity in $\mathbb{O}_p$. Then $-1=\sum_{k=0}^\infty(p-1)p^k$.

(ii) If $n\in\mathbb{N}$, we can uniquely write $n=\sum_{k=0}^{m(n)}a_kp^k$ with $a_k\in\{0,\dots,p-1\}$. Then $n\cdot 1=\sum_{k=0}^{m(n)}a_kp^k\in\mathbb{O}_p$. In particular, $n\mapsto n\cdot 1:\mathbb{N}\to\mathbb{O}_p$ is an additive semigroup homomorphism with dense ring. Hence $n\mapsto n\cdot 1:\mathbb{Z}\to\mathbb{Q}_p$ has dense range. We call $\mathbb{O}_p$ the ***p*–adic integers**.

2 Abelian Locally Compact Groups

3 Compact Groups

4 Introduction to Amenability Theory