Harmonic Analysis

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I. Harmonic Analysis

1 Locally Compact Groups

Definition. Let G be a group. A topology τ on G is a **group topology** provided that
- $x \mapsto x^{-1} : G \to G$ is continuous, and
- $(x,y) \mapsto xy : G \times G \to G$ is continuous.

We call (G,τ) a **topological group** where we omit τ when it is clear from context.

Equivalently, we may assert that $(x,y) \mapsto xy^{-1}$ is $\tau \times \tau - \tau$–continuous. Write $L_g(x) = gx$ and $R_g(x) = xg$ to denote the left and right multiplication maps; then it is easy to see that L_g and R_g are homeomorphisms. Similarly, $x \mapsto x^{-1}$ is a homeomorphism.

Definition. We say that a subset $A \subset G$ is **symmetric** if $A^{-1} = A$.

We have the following basic properties:

1.1 Proposition. *Let (G,τ) be a topological group.*
(i) If $\emptyset \neq A \subseteq G$ and U is open, then $AU = \{ay : a \in A, y \in U\}$ and likewise UA are open.
(ii) Given $U \in \tau$ and $x \in U$, then there is a symmetric $V \in \tau$ with $e \in V$ such that $VxV \subseteq U$. In particular, if $e \in U$, then we can find symmetric V so that $V^2 \subseteq U$.
(iii) If H is a subgroup of G, then $\overline{H}$ is also a subgroup.
(iv) An open subgroup is automatically closed.
(v) If $K, L \subseteq G$ are compact, then KL is compact.
(vi) If K is compact and C is closed in G, then KC is closed.

In $(\mathbb{R},+)$, then $\mathbb{Z}+\sqrt{2}\mathbb{Z}$ is not closed, so it is necessary to assume compactness in (vi).

Proof (i) $AU = \bigcup_{a\in A} L_a(U)$ is a union of open sets.

(ii) Consider the continuous map $(y,z) \mapsto yxz$. Since $exe = x \in U$, there is a $\tau \times \tau$–neighbourhood of (e,e) which maps into U have a basic neighbourhood $V_1 \times V_2$. Let $V = V_1 \cap V_2$. Moreover, we may replace V by $V^{-1} \cap V$. to attain symmetry.

(iii) Let $x,y \in \overline{H}$, $U \in \tau$ with $xy \in U$. Then (ii) provides V with $VxyV \subseteq U$. But $Vx \cap H \neq \emptyset$ and $\neq yV \cap H$ so there are ${}_1 \in Vx \cap H$, $h_2 \in yV \cap H$, and $h_1h_2 \in VxyV \subseteq U$. Thus $U \cap H \neq \emptyset$. Thus $xy \in \overline{H}$.
To use nets for inverses, if $x \in \overline{H}$, then $x = \lim_\alpha x_\alpha$ where $(x_\alpha)_{\alpha\in A} \subset H$ is a net. Then $x^{-1} = \lim_\alpha x_\alpha^{-1} \in \overline{H}$ as each $x_\alpha^{-1} \in H$.

(iv) If H is an open subgroup, then $H = G \setminus \bigcup_{x\in G\setminus H} xH$ is closed.

(v) $K \times L$ is compact, and hence so is its image under multiplication.

(vi) If $x \in \overline{KC}$, then $x = \lim_\alpha k_\alpha x_\alpha$ where $k_\alpha \in H$ and $x_\alpha \in C$. Since K is compact, we may assume (passing to a subnet if necessary) $k = \lim_\alpha k_\alpha$ exists in K. Then

$$k^{-1}x = \lim_\alpha k_\alpha^{-1} \cdot \lim_\alpha k_\alpha x_\alpha = \lim_\alpha k_\alpha^{-1} k_\alpha x_\alpha = \lim x_\alpha \in C$$

so $x = kk^{-1}x \in KC$. ∎

1.1 Homogenous Spaces

Let (G,τ) be a topological group, H a subgroup of G, and $G/H = \{xH; x \in G\}$. Let $\pi : G \to G/H$ be given by $\pi(x) = xH$ be the projection map. The **quotient topology** on G/H is $\tau_{G/H} = \{W \in G/H : \pi^{-1}(W) \in \tau\}$. Notice that if $U \in \tau \setminus \{\emptyset\}$, then $UH = \pi^{-1}(\pi(U))$ is open, so $\pi : G \to G/H$ is an open map.

1.2 Proposition. *Let (G,τ), H be as above.*

(i) The map $(x, yH) \mapsto xyH : G \times G/H \to G/H$ is $\tau \times \tau_{G/H} - \tau_{G/H}$ continuous and open.

(ii) If H is normal, then $(G/H, \tau_{G/H})$ is a topological group.

(iii) If H is closed, then $\tau_{G/H}$ is Hausdorff.

Proof (i) Let $x, y \in G$, $W \in \tau_{G/H}$ satisfy $xyH = \pi(xy) \in W$. Then $xy \in \pi^{-1}(W)$ and by Proposition 1.1 we have $V \in \tau$ with $e \in V$ such that $VxyV \subseteq \pi^{-1}(W)$. But then $(x, \pi(y)) \in Vx \times \pi(yV) \in \tau \times \tau_{G/H}$ and the latter set maps into $\pi(VxyV) \subseteq W$.
Also, if $U \in \tau \times \tau_{G/H}$, then $U = \bigcup_{(x,yH)\in U} V_x \times W_{yH}$ and

$$\pi(U) = \bigcup_{(x,yH)\in U} \pi(V_x \pi^{-1}(W_{yH}))$$

since π is open.

(ii) Recall that $(xH)(yH) = xyH$ is our multiplication operation on G/H and π is a group homomorphism. Then the following diagram commutes: We have that $\pi \times \mathrm{id}$ is open and $(x, yH) \mapsto xyH$ is open from (i), so the multiplication from $G/H \times G/H \to G/H$ must be open and continuous.

(iii) If $x, y \in G$ with $\pi(x) \neq \pi(y)$, then $e \notin xHy^{-1}$. Now $xHy^{-1} = L_x(R_{y^{-1}}(H))$ so xHy^{-1} is closed. Hence by the last proposition, there is a symmetric open V with $e \in V$ so $V^2 \subseteq G \setminus (xHy^{-1})$. But then $e \notin (VxH)(VyH)^{-1} = VxHy^{-1}V$: if we had $e = vxhy^{-1}u$ with $v, u \in V$ and $h \in H$, then $v^{-1}u^{-1} = xhy^{-1} \in V^2 \cap (xHy^{-1}) = \emptyset$, a contradiction. Hence $VxH \cap VyH = \emptyset$ so $\pi(Vx)$, $\pi(Vy)$ is a pair of separating neighbourhoods of $\pi(x)$, $\pi(y)$. ■

1.3 Corollary. *G is Hausdorff if and only if there exists $x \in G$ so that $\{x\}$ is closed.*

Proof In a Hausdorff space, points are closed. Conversely, if $\{x\}$ is closed, $\{e\} = L_{x^{-1}}(\{x\})$ is closed and a normal subgroup. Then $G \cong G/\{e\}$ is Hausdorff. ■

If (G,τ) is not Hausdorff, then $\{e\} \subsetneq \overline{\{e\}}$ is the smallest closed subgrup in G. Thus $\overline{\{e\}} \subseteq \bigcap_{x\in G} x\overline{\{e\}}x^{-1} \subseteq \overline{\{x\}}$ so $\overline{\{e\}}$ is normal. In particular, $G/\overline{\{e\}}$ is Hausdorff.

Definition. A **locally compact group** is a Hausdorff topological group (G,τ) which is locally compact.

(i) If there is any $U \in \tau \setminus \{\emptyset\}$ such that $\overline{U}$ is compact, then for any $x \in U$, we have $e \in x^{-1}U \subseteq L_{x^{-1}}(\overline{U})$ so $\overline{x^{-1}U}$ is compact. If $V \in \tau$ with $e \in V$ and $\overline{V}$ compact, then for any $x \in H$, $x \in xV$ and $\overline{xV} \subseteq L_x(\overline{V})$ and $\overline{xV}$ is compact. In particular, (G,τ) is locally compact if and only if there is some $U \in \tau \setminus \{\emptyset\}$ with $\overline{U}$ compact.

(ii) If (G, τ) is locally cmpact and N is a closed normal subgroup, then $({G/N}, \tau_{G/N})$ is locally compact. Indeed, $U \in \tau \setminus \{e\}$ with $\overline{U}$ compact, then $\overline{\pi(U)} \subseteq \pi(\overline{U})$ is compact.

Example. (i) If G is any group and τ is the discrete topology, then (G, τ_d) is locally compact.

(ii) If $((\mathbb{R}, +), \tau_{\|\cdot\|})$ is locally compact.

(iii) If $\{G_i\}_{i \in I}$ is a family of locally compact groups, then $\prod_{i \in I} G_i$ is a locally compact group if and only if all but finitely many (G_i, τ_i) are compact.

(iv) $((\mathbb{R}^n, +), \tau_{\|\cdot\|})$ is a locally compact group

(v) Suppose $\{F_i\}_{i \in I}$ is an infinite family of finite groups (with discrete topologies), then $G = \prod_{i \in I} F_i$ is a compact group. If $F \subset I$ is finite, then $N_F = \{(x_i)_{i \in I} \in G : x_i = e \text{ for } i \in F\}$ is open and a normal subgroup. $\{N_F : F \subset I \text{ finite}\}$ is a base for the topology at e.

(vi) Let $(\Bbbk, \tau)$ be a locally compact field.. Then $\det^{-1}(\Bbbk \setminus \{0\}) = \mathrm{GL}_n(\Bbbk) \subseteq M_n(\Bbbk) \cong \Bbbk^{n^2}$ is an open subset and multiplicative subgroup, and hence locally compact. Notice that multiplication is governed by linear equations, and hence continuous, while inversion is continuous thanks to Cramer's rule.

Here are some common closed subgroups:

$$\overset{-1}{\det}(\{1\}) = \mathrm{SL}_n(\Bbbk)$$
$$O_n(\Bbbk) = \{x \in \mathrm{GL}_n(\Bbbk) : x^{-1} = X^T\}$$

As a special case, consider $U_n = \{x \in \mathrm{GL}_n(\mathbb{C}) : x^* = x^{-1}\}$ is governed by continuous equations, and thus closed in $M_n(\mathbb{C})$. Furthermore, U_n is bounded in $M_n(\mathbb{C})$, and hence compact.

1.2 *p*-adic Numbers

Let p be a prime in $\mathbb{N}$. We will establish product structures and topologies on

$$\mathbb{O}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : a_k \in \{0, 1, \ldots, p-1\} \right\} \cong \{0, 1, \ldots, p-1\}^{\mathbb{N}}$$
$$\mathbb{Q}_p = \left\{ \sum_{k=N}^{\infty} a_k p^k : N \in \mathbb{Z}, a_k \in \{0, 1, \ldots, p-1\} \right\}$$

are topological rings and a topological field respectively. Let $R_p = \prod_{n=0}^{\infty} {\mathbb{Z}/p^{n+1}\mathbb{Z}}$ which is a ring under pointwise operations.

1.4 Lemma. *The map $\rho : R_p \times R_p \to [0,1]$ given by*

$$\rho(x, y) = \sum_{n \in \mathbb{N}_0} \frac{\rho_n(x_n, y_n)}{p^n} \qquad \rho_n(x_n, y_n) = \begin{cases} 1 & : x_n = y_n \\ 0 & : x_n \neq y_n \end{cases}$$

is a metric on R_p which satisfies

- *(additively invariant): $\rho(x+z, y+z) = \rho(x, y)$ for $x, y, z \in R_p$*
- *τ_ρ is the product topology*

Proof Additive invariance is routine. Notice that if $\frac{1}{p^m} \geq \epsilon > \frac{1}{p^{m+1}}$, then the open ϵ–ball around a point x is $\{x_0\} \times \cdots \{x_m\} \times \prod_{n=m+1}^{\infty} \mathbb{Z}/p^{n+1}\mathbb{Z}$ is product-open. Conversely, any product-open set is a finite union of such ϵ–balls. ■

1.5 Corollary. *The function $\|x\|_p = \rho(x,0)$ in R_p satisfies*

- $\|x\|_p = 0$ *if and only if* $x = 0$
- $\|x+y\|_p \leq \|x\|_p + \|y\|_p$
- $\|xy\|_p \leq \|x\|_p \|y\|_p$
- $\|-x\|_p = \|x\|_p$

Hence (R_p, τ_ρ) is a compact topological ring.

Proof The properties follow directly using additive invariance. To see that R_p is a topological ring, if $(x_\alpha), (y_\alpha)$ have $x = \lim x_\alpha$ and $y = \lim y_\alpha$, then, for example,

$$\begin{aligned} \|xy - x_\alpha y_\alpha\|_p &\leq \|xy - x_\alpha y\|_p + \|x_\alpha y - x_\alpha y_\alpha\|_p \\ &\leq \|x - x_\alpha\|_p + \|y - y_\alpha\|_p \end{aligned}$$

as $\|y\|_p, \|x_\alpha\|_p \leq 1$. ■

We now view $\mathbb{O}_p$ as a closed subring of R_p. Define $\alpha : \mathcal{O}_p \to R_p$ be given on $a = \sum_{k=0}^{\infty} a_k p^k$ by

$$\alpha(a) = \left(\sum_{k=0}^{n} a_k p^k + p^{n+1}\mathbb{Z} \right)_{n=0}^{\infty}.$$

This map is an injection with range $\alpha(\mathcal{O}_p) = \{(x_n)_{n=0}^{\infty} \in R_p : x_n = \pi_n(x_{n+1}) \text{ for all } n\}$ where $\pi_n : \mathbb{Z}/p^{n+2}\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ is the canonical quotient map. In fact, this is called an inductive limit with respect to the maps π_n. Hence it is routine to show that

- $\alpha(\mathbb{O}_p)$ is a subring of R_p, and
- $\alpha(\mathbb{O}_p)$ is closed in R_p (just check net limits in product topology)

If $a, b \in \mathbb{O}_p$, define $a + b = \alpha^{-1}(\alpha(a) + \alpha(b))$.

Remark. (i) $1 + \sum_{k=1}^{\infty} 0 \cdot p^k$ is the multiplicative identity in $\mathbb{O}_p$. Then $-1 = \sum_{k=0}^{\infty}(p-1)p^k$.

(ii) If $n \in \mathbb{N}$, we can uniquely write $n = \sum_{k=0}^{m(n)} a_k p^k$ with $a_k \in \{0, \ldots, p-1\}$. Then $n \cdot 1 = \sum_{k=0}^{m(n)} a_k p^k \in \mathbb{O}_p$. In particular, $n \mapsto n \cdot 1 : \mathbb{N} \to \mathbb{O}_p$ is an additive semigroup homomorphism with dense ring. Hence $n \mapsto n \cdot 1 : \mathbb{Z} \to \mathbb{Q}_p$ has dense range. We call $\mathbb{O}_p$ the ***p*–adic integers**.

Let $a = \sum_{k=0}^{\infty} a_k p^k$ in $\mathbb{O}_p$. Let

$$\begin{aligned} \nu_p(a) &= \min\{k \in \mathbb{N}_0 : a_k \neq 0\}, \min\emptyset = -\infty \\ |a|_p &= p^{-\nu_p(a)}, p^{-\infty} = 0 \end{aligned}$$

and notice that $|a|_p = \|\alpha(a)\|_p$. However, $|a|_p$ has even nicer properties:

(i) $|a|_p = 0$ if and only if $a = 0$

(ii) $\nu_p(ab) = \nu_p(a) + \nu_p(b)$. Thus $|ab|_p = |a|_p |b|_p$

(iii) $\nu_p(a+b) \geq \min\{\nu_p(a), \nu_p(b)\}$. Thus $|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$

Notice that (i) and (ii) imply that $\mathbb{O}_p$ is an integral domain.

1.6 Proposition. *The multiplicative unit group of $\mathbb{O}_p$ is $\mathbb{O}_p \setminus p\mathbb{O}_p = \{a \in \mathbb{O}_p : |a|_p = 1\}$. Hence $\mathbb{O}_p^\times$ is open and a topological group.*

Proof The second equality is trivial. If $a \in \mathbb{O}_p^\times$, then $|a|_p, |a^{-1}|_p \leq 1$ and $1 = |1|_p = |aa^{-1}|_p = |a|_p|a^{=1}|_p$, so $|a|_p = 1$. If $|a|_p = 1$, let

$$x = \alpha(a) = \left(\sum_{k=0}^{n} a_k p^k + p^{n+1}\mathbb{Z}\right)_{n=0}^{\infty} \in R_p.$$

Then $x_n \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ since $p \nmid \sum_{k=0}^n a_k p^k$ in $\mathbb{Z}$. Hence there is $y_n \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ so $x_n y_n = 1 + p^{n+1}\mathbb{Z}$ and thus

$$1 + p^n\mathbb{Z} = \pi_N(1 + p^{n+2}\mathbb{Z}) = \pi_n(x_{n+1}y_{n+1}) = \pi(x_{n+1})\pi(y_{n+1}) = x_n\pi_n(y_{n+1})$$

so that $\pi_n(y_{n+1}) = y_n$. Thus if $y \in \alpha(\mathbb{O}_p)$, i.e. $y = \alpha(b)$ with $ab = \alpha^{-1}(\alpha(a)\alpha(b)) = \alpha^{-1}((1 + p^{n+1}\mathbb{Z})_{n=0}^{\infty}) = 1$ and $a \in \mathbb{O}_p^\times$.

Since $p\mathbb{O}_p$ is closed, we see that $\mathbb{O}_p^\times$ is open in $\mathbb{O}_p$. Continuity of multiplication follows (ii). Finally, if $a, b \in \mathbb{O}_p$,

$$|a^{-1} - b^{-1}|_p = |a|_p|a^{-1} - b^{-1}|_p|b|_p = |b - a|_p \qquad \blacksquare$$

1.7 Corollary. *Each ideal in $\mathbb{O}_p$ is of the form $p^k\mathbb{O}_p$ for $k \in \mathbb{N}_0$.*

Proof If I is an ideal in $\mathbb{O}_p$, then let $k(I) = \min\{k \in \mathbb{N}_0 : \nu_p(a) = k \text{ for some } a \in I\}$. Then there is $a \in I$ with $\nu_p(a) = k(I)$, so $p^{-k(I)} \in a\mathbb{O}_p^\times \subseteq a\mathbb{O}_p \subseteq I$. Thus $p^{-k(I)}\mathbb{O}_p \subseteq I$. Clearly $I \subseteq p^{-k(I)}\mathbb{O}_p$ as well. $\blacksquare$

We now extend the valuation and norm to $\mathbb{Q}_p$. If $a = \sum_{k\in\mathbb{Z}} a_k p^k \in \mathbb{Q}_p$, let $\nu_p(a) = \min\{k \in \mathbb{Z} : a_k \neq 0\}$ and $|a|_p = p^{-\nu_p(a)}$. Then for $a \in \mathbb{Q}_p \setminus\{0\}$ admits (formal) factorization

$$a = \sum_{k=\nu_p(a)}^{\infty} a_k p^k = p^{\nu_p(a)} \sum_{k=\nu_p(a)}^{\infty} a_k p^{k-\nu_p(a)} = p^{\nu_p(a)} \underbrace{\sum_{k=0}^{\infty} a_{k+\nu_p(a)} p^k}_{:=a'\in\mathbb{O}_p^\times}$$

Thus, if $a, b \in \mathbb{Q}_p \setminus\{0\}$, we define multiplication and addition by $ab = p^{\nu_p(a)+\nu_p(b)}a'b'$ and $a^{-1} = p^{-\nu_p(a)}(a')^{-1}$. Furthermore, assuming $\nu_p(a) \leq \nu_p(b)$, we define addition by

$$a + b = p^{\nu_p(a)}(a' + p^{\nu_p(b)-\nu_p(a)}b')$$

and $0 + a = a$, $0a = 0$. Notice that $|ab|_p = |a|_p|b|_p$, $|1/a|_p = 1/|a|_p$ and if $\nu_p(a) \leq \nu_p(b)$, $|a + b|_p = p^{-\nu_p(a)}|a' + p^{\nu_p(b)-\nu_p(a)}b'|_p \leq |a|_p$ so, generally, $|a + b|_p \leq \max\{|a|_p, |b|_p\}$. Also, if $|a|_p = 0$, then $|a| = 0$. Thus $(\mathbb{Q}_p, |\cdot|_p)$ is a "normed field", and hence a topological field.

Note that

$$\mathbb{O}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\} = \{a \in \mathbb{Q}_p : |a|_p < p\}$$

is a compact open neighbourhood of 0, so $\mathbb{Q}_p$ is locally compact. Moreover, each $p^k\mathbb{Q}_p = \{a \in \mathbb{Q}_p : |a|_p < p^{k-1}\}$ is a closed and open ball about 0.

1.3 Haar Integral and Haar Measure

Let G be a locally compact group. Define for $f : G \to \mathbb{C}$, $x \in G$, $f \cdot x = f \circ L_x$, and $x \cdot f = f \circ R_x$. Notice that $(f,x) \mapsto f \cdot x$, as an adjoint action, is a right (group) action of G on functions. Likewise, $(x,f) \mapsto x \cdot f$ is a left action.

1.8 Proposition. *Given $f \in C_c(G)$, then*

$$\lim_{x \to e} \|f \cdot x - f\|_\infty = 0 = \lim_{x \to e} \|x \cdot f - f\|_\infty .$$

Proof Let $\epsilon > 0$, $W = W^{-1}$ a relatively compact neighbourhood of e, and let $K = \overline{W} \operatorname{supp} f$. Given $y \in V$, $x \mapsto |f(xy) - f(y)|$ is continuous, so there is a neighbourhood U_y of e so $|f(xy) - f(y)| < \epsilon$ whenever $x \in U_y$. Then find $V_y^{-1} = V_y$ of e so $V_y^2 \subseteq U_y$. Then $K \subseteq \bigcup_{y \in K} V_y y$ so by compactness get some finite subcover $\bigcup_{j=1}^n V_{y_j} y_j \supseteq K$. Let $V = \left(\bigcap_{j=1}^n V_{y_j} \right) \cap W$, so $V^{-1} = V$.

If $x \in V$, then for $y \in K$ we have $y \in V_{y_j} y_j$ for some j, i.e. $y y_j^{-1} \in V_{y_j}$, and hence

$$xy = x y y_j^{-1} y_j \in V V_{y_j} y_j \subseteq V_{y_j}^2 y_j \subseteq U_{y_j} y_j$$

so that

$$|f(xy) - f(y)| \le |f(xy) - f(y_j)| + |f(y_j) - f(y)| < 2\epsilon.$$

If $y \notin K$, then $Wy \cap \operatorname{supp}(f) = \emptyset$, so for $x \in V \subseteq W$, we have $f(xy) = 0 = f(y)$. Thus if $x \in V$, then $\|f \cdot x - f\|_\infty < \epsilon$. ∎

1.9 Theorem. (Existence of Haar Integral) *There exists a linear functional $I : C_c(G) \to \mathbb{C}$ satisfying*

- *(positivity): $I(f) > 0$ if $f \in C_c^+(G) = \{g \in C_c(G) \setminus \{0\} : g \ge 0\}$.*
- *(left invariance): $I(f \cdot x) = I(f)$ for $f \in C_c(G)$, $x \in G$.*

Let for $f, \phi \in C_c^+(G)$

$$(f : \phi) = \inf \left\{ \sum_{j=1}^n c_j : \text{there are } x_1, \ldots, x_n \in G, c_i > 0, n \in \mathbb{N} \text{ s.t. } f \le \sum_{j=1}^n c_j \phi \cdot x_j \right.$$

Notive that $0 < \frac{\|f\|_\infty}{\|\phi\|_\infty} \le (f : \phi)$. Also, if $U = \{x \in G : \phi(x) > \frac{1}{2} \|\phi\|_\infty\}$, then $\operatorname{supp} f$ is covered by finitely many $x^{-1}U$, $x \in G$, and thus $(f : \phi) < \infty$.

Claim I *For f, g in $C_c^+(G)$, we have*

(i) $(f \cdot x : \phi) = (f : \phi)$ *for x in G*
(ii) $(cf : \phi) = c(f : \phi) = (f : \frac{1}{c}\phi)$ *for $c > 0$*
(iii) $(f + g, \phi) \le (f : \phi) + (g : \phi)$.
(iv) $(f : \phi) \le (f : g)(g : \phi)$

Proof Note that (i) and (ii) are straightforward. To see (iii) and (iv), consider

$$f \le \sum_{j=1}^n c_j \phi \cdot x_j \qquad g \le \sum_{j=n+1}^N c_j \phi \cdot x \qquad f \le \sum_{k=1}^m b_k g \cdot y_k$$

so that $f + g \le \sum_{j=1}^N c_j \phi \cdot x_k$ and $(f + g : \phi) \le \sum_{j=1}^n c_j + \sum_{j=n+1}^N c_j$, giving (iii). To get (iv), note $f \le \sum_{k=1}^m b_k \sum_{j=n+1}^N c_j \phi \cdot (x_j y_k)$ so $(f : \phi) \le \sum_{k=1}^m b_k \sum_{j=k+1}^N c_j$, giving (iv).

We wish to "homogonize" the effect of ϕ. Fix $\psi_0 \in C_c^+(G)$ and let $I_\phi(f) = \frac{(f:\phi)}{(\psi_0:\phi)}$. Then $I_\phi : C_c^+(G) \to \mathbb{R}_{\geq 0}$ is

(i') left translation invariant

(ii') $\mathbb{R}_{\geq 0}$ –homogenous

(iii') subadditive.

by using the claim above directly. Thus by (iv), $(\psi_0 : \phi) \leq (\psi_0 : f)(f : \phi)$ and $(f : \phi) \leq (f : \psi_0)(\psi_0 : \phi)$, giving

iv' $0 < \frac{1}{(\psi_0:f)} \leq I_\phi(f) \leq (f : \psi_0)$.

Claim II *If $f, g \in C_c^+(G)$, $\epsilon > 0$, there is a neighbourhood V of e such that*

$$I_\phi(f) + I_\phi(g) < I_\phi(f+g) + \epsilon$$

whenever $\phi \in C_c^+(G)$ with $\mathrm{supp}(\phi) \subseteq V$.

Proof Let $k \in C_c^+(G)$ be so $k|_{\mathrm{supp}(f+g)} = 1$ and let $\delta > 0$. Take $h = f + g + \delta k$ and $f' = \frac{f}{h}$, $g' = \frac{g}{h} \in C_c^+(G)$. Uniform continuity of f', g' provides a neighbourhood v of e such that $|f'(x) - f'(y)| < \delta$, $|g'(x) - g'(y)| < \delta$ if $y^{-1}x \in V$. If $\phi \in C_c^+(G)$, $\mathrm{supp}(\phi) \subseteq V$, and $x_1, \ldots, x_n$ in G, $c_1, \ldots, c_n > 0$ satisfy

$$h \leq \sum_{j=1}^{n} c_j \phi_j \cdot x_j^{-1}$$

then for x in G

$$\begin{aligned} f(x) = f'(x)h(x) &\leq \sum_{j=1}^{n} f'(x) c_j \phi(x_j^{-1}x) \\ &\leq \sum_{j=1}^{n} [f'(x_j) + \delta] c_j \phi(x_j^{-1}x) \end{aligned}$$

by properties of f', g' proven above and $\mathrm{supp}(\phi) \subseteq C$. Likewise,

$$g \leq \sum_{j=1}^{n} [g'(x_j) + \delta] c_j \phi \cdot x_j^{-1}.$$

Now $f' + g' = (f+g)/h = \frac{f+g}{f+g+\delta k} \leq 1$ so

$$\begin{aligned} (f : \phi) + (g : \phi) &\leq \sum_{j=1}^{n} [f'(x_j) + \delta] c_j + \sum_{j=1}^{n} [g'(x_j) + \delta] c_j \\ &\leq \sum_{j=1}^{n} [1 + 2\delta] c_j \end{aligned}$$

and $(f : \phi) + (g : \phi) \leq (1 + 2\delta)(h : \phi)$. Divide by $(\psi_0 : \phi)$ and (iii') and (iv') above to get

$$I_\phi(f) + I_\phi(g) \leq (1 + 2\delta) I_\phi(h) \leq (1 + 2\delta)[I_\phi(f+h) + \delta I_\phi(k)]$$

Thus with sufficiently small δ, $I_\phi(f) + I_\phi(g) < I_\phi(f+g) + \epsilon$.

We are now in position to complete the proof.

Claim III *Construction of the functional I.*

Proof Inequality (iv') tells us that

$$x_\phi = (I_\phi(f))_{f\in C_c^+(G)} \in \prod_{f\in C_c^+(G)} [\frac{1}{(\psi_0 : f)}, (f:\psi_0)] = X$$

which, by Tychonoff, is compact. For ϕ, ϕ' in $\Phi = \{\psi \in C_c^+(G) : \psi(e) = 1\}$, $\phi \le \phi'$ if $\phi \ge \phi'$ pointwise, which is a preorder. Notice that $\phi\phi' \le \phi \wedge \phi'$ (pointwise minimum), so that $(\Phi, \le)$ is directed. Hence $(x_\phi)_{\phi\in\Phi}$ admits a converging subnet $x = \lim_{\mu\in M} x_{\phi_\mu}$ in X.

Write $x = (I(f))_{f\in C_c^+(G)}$, so $I(f) = \lim_{\mu\in M} I_{\phi_\mu}(f)$ for each $f \in C_c^+(G)$. Then it follows that from (i'), (ii'), and (iii') that for f, g in $C_c^+(G)$, we have

$$I(F\cdot x) = I(f) \qquad I(cf) = cI(f) \qquad I(f+g) \le I(f) + I(g)$$

for $x \in G$, $c > 0$. Moreover, by cofinality, if V is a neighbourhood of e, then $\operatorname{supp}(\phi_\mu) \subseteq V$ for μ sufficiently large in M. Hence given $\epsilon > 0$, by Claim II, $I_{\phi_\mu}(f) + I_{\phi_\mu}(g) < I_{\phi_\mu}(f+g) + \epsilon$ for μ sufficiently large in M. Since $\epsilon > 0$ as arbitrary, we have $I(f) + I(g) \le I(f+g)$.

Let $I(0) = 0$. If $f \in C_c^{\mathbb{R}}(G)$ and $f = f_1 - f_2 = g_1 - g_2$ with $f_1, f_2, g_1, g_2 \ge 0$, then $h = f_1 + g_2 = g_1 + f_2$ satisfies that $I(h) = I(f_1) + I(g_2) = I(g_1) + I(f_2)$ and hence we may define $I(f) = I(f_1) - I(f_2)$, which do not depend on the choice of f_1, f_2. One may check that $I : C_c^{\mathbb{R}}(G) \to \mathbb{R}$ is $\mathbb{R}-$linear. Finally, if $f \in C_c(G)$, let $I(f) = I(\operatorname{Re} f) + iI(\operatorname{Im} f)$. It is left as an exercise to verify that $I : C_c(G) \to \mathbb{C}$ is $\mathbb{C}-$linear.

Finally, the fact that $I(f\cdot x) = I(f)$ for $f \in C_c(G)$ and $x \in G$ follows for f in $C_c^+(G)$ as above. If $f \in C_c^+(G)$, then (iv') tellus us that $I(f) \ge \frac{1}{(\psi_0 : f)} > 0$. ■

Remark. (i) In Claim III, $I_\phi(\psi_0) = 1$ so $I(\psi_0) = 1$.

(ii) If G is discrete, then $\psi_0 = 1_{\{e\}} = \min \Phi$. Then $I_{\psi_0}(f) = \frac{(f:\psi_0)}{(\psi_0:\psi_0)} = \sum_{x\in G} f(x)$ for $f \in C_c^+(G)$.

(iii) If $G = \mathbb{R}$, let ψ_0 be the linear function which is 0 on $(-\infty, -1/2-\delta) \cup (1/2+\delta, \infty)$, 1 on $(-1/2+\delta, 1/2-\delta)$, and continuied linearly on the remainder. Notice that $(\psi_0, \phi_n) \approx n$, so $\frac{(f:\phi_n)}{(\psi_0:\phi_n)}$ is approximately the Riemann-Darboux upper sum.

(iv) Examine $\mathbb{O}_p$, $\psi_0 = 1_{\mathbb{O}_0}$, $\psi_n = 1_{p^n\mathbb{O}_p}$.

1.10 Theorem. (Harr Measure) *Let $\mathcal{B}(G)$ denote the Borel $\sigma-$algebra on G. Then there is a Radon measure $m : \mathcal{B}(G) \to [0, \infty]$ such that*

- *m is left invariant: $m(xE) = m(E)$ for $x \in G$, $E \in \mathcal{B}(G)$*
- *$m(U) > 0$ for $U \in \tau \setminus \{\emptyset\}$.*

Proof The Riesz Representation Theorem provides a measure $m : \mathcal{B}(G) \to [0, \infty]$ for which

$$\int_G f \,\mathrm{d}m = I(f)$$

for $f \in C_c(G)$. Notice that

$$\int_G f\cdot x \,\mathrm{d}m = I(f\cdot x) = \int_G f$$

for any $x \in G$, $f \in C_c(G)$. Thus if $U \in \tau$, $\operatorname{supp} f \subseteq U$ if and only if $\operatorname{supp}(f \cdot x) \subseteq x^{-1}U$ for $x \in G$ and $f \in C_c(G)$. Thus

$$\begin{aligned} m(U) &= \sup\{I(f) : f \in C_c(G), 0 \le f \le 1 \text{ and } \operatorname{supp}(f) \subseteq U\} \\ &= \sup\{I(f \cdot x) : f \in C_c(G), 0 \le f \le 1 \text{ and } \operatorname{supp}(f) \subseteq U\} \\ &= \sup\{I(g) : g \in C_c(G), 0 \le g \le 1, \operatorname{supp}(g) \subseteq x^{-1}U \\ &= m(x^{-1}U). \end{aligned}$$

Therefore, for any $E \in \mathcal{B}(G)$, we have

$$\begin{aligned} m(E) &= \inf\{m(U) : E \in U, U \in \tau\} \\ &= \inf\{m(xU) : E \subseteq U, U \in \tau\} \\ &= \inf\{m(xU) : xE \subseteq xU, U \in \tau\} = m(xE). \end{aligned}$$

Finally, if $U \in \tau \setminus \{\emptyset\}$, there is $f \in C_c^+(G)$ with $0 \le f \le 1$ and $\operatorname{supp}(f) \subseteq U$, so $m(U) \ge I(f) > 0$. ■

Remark. If $E \in \mathcal{G}(G)$, $m(E) < \infty$, then $m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$. Inner regularity need not hold on infinite measure sets: taking $G = \mathbb{R}_d \times \mathbb{R}$, then $\mathbb{R}_d \times \{0\}$ is closed, and thus Borel. However, $m(E) = \infty$ while $m(K) = 0$ for each compact $K \subset E$.

1.11 Theorem. (Uniqueness of Haar Measure) *Let $m' : \mathcal{B}(G) \to [0, \infty]$ be any Radon measure such that $m(xE) = m'(E)$ for $x \in G$ and $E \in \mathcal{B}(G)$. Then there is $c \ge 0$ such that $m' = cm$.*

Proof Fix a symmetric neighbourhood $W = W^{-1}$ of e with $\overline{W}$ compact. Given $f \in C_c^+(G)$, $\epsilon > 0$, and U a neighbourhood of e such that $\|f - x \cdot f\|_\infty < \epsilon$. Let $V = U \cap W$. Then let $x \in G$, and for any $x' \in G$ with $x'x^{-1} \in V$, we have

$$\begin{aligned} \left| \int_G f(yx)dm'(y) - \int_G f(yx')dm'(y) \right| &\le \left\| x \cdot f - x' \cdot f \right\|_\infty m(\operatorname{supp}(f)x^{-1} \cup \operatorname{supp}(f)Vx^{-1}) \\ &< \epsilon m(\operatorname{supp}(f)x^{-1} \cup \operatorname{supp}(f)Wx^{-1}) \end{aligned}$$

and hence $x \mapsto \int_G x \cdot f \, dm'$ is continuous at each point in G. Thus

$$D_f(x) = \frac{\int_G x \cdot f \, dm'}{\int_G f \, dm}$$

defines a continuous function on G.

If $f,g \in C_c^+(G)$, then $(x,y) \mapsto f(x)g(y^{-1})$ is non-negative, continuous, Borel measurable, and compactly supported on $G \times G$. Then by left-invariance and Tonelli's theorem,

$$\begin{aligned}\left(\int_G f\,dm\right)\cdot\left(\int_G g(y^{-1})dm'(y)\right) &= \int_G\int_G f(x)g(y^{-1})dm'(y)dm(x) \\ &= \int_G\int_G f(x)g((x^{-1}y)^{-1})dm'(y)dm(x) \\ &= \int_G\int_G f(x)g(y^{-1}x)dm(x)dm'(y) \\ &= \int_G\int_G f(yx)g(x)dm(x)dm'(y) \\ &= \int_G g(y)\left(\int_G f(yx)dm'(y)\right)dm(x)\end{aligned}$$

Thus,

$$\int_G g(y^{-1})dm'(y) = \int_G g(x)D_f(x)dm(x).$$

But if we have any other $f' \in C_c^+(G)$, then we would have

$$\int_G g(x)D_f(x)dm(x) = \int_G g(y^{-1})dm'(y) = \int_G g(x)D_{f'}(x)dm(x)$$

so it follows that $D_f = D_{f'}$ m–a.e. Since D_f, $D_{f'}$ are continuous, we see that $D_f = D_{f'}$ everywhere. In particular, $D_f(e) = D_{f'}(e)$, or that

$$\frac{\int_G f\,dm'}{\int_G f\,dm} = D_f(e) = D_{f'}(e) = \frac{\int_G f'dm'}{\int_G f'dm}$$

Let c denote this common value, so $c\int_G f\,dm = \int_G f\,dm'$ for $f \in C_c^+(G)$. Hence $m' = cm$. ■

Example. (i) If G is discrete, then $C_c(G)$ is composed of functions with finite support, and $m(E)$ is (a multiple of) the counting measure. In the finite case, we normalize by $|G|$.

(ii) If $G = \mathbb{R}^n$, $I(f) = \int_{\mathbb{R}^n} f$ and m is n-dimensional Lebesgue measure.

(iii) Let $G = \mathrm{GL}_n(\mathbb{R})$.

(a) If $t \in \mathrm{GL}_n(\mathbb{R})$, then for $f \in C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f \circ t(y)dy = \frac{1}{|\det t|}\int_{\mathbb{R}^n} f(y)dy.$$

Indeed, show that this holds for an elementary matrix t, and $\mathrm{GL}_n(\mathbb{R})$ is the algebra generated by these elements.

(b) If $X \in \mathrm{GL}_n(\mathbb{R})$, then $L_X : M_n(\mathbb{R}) \to M_n(R)$ is isomorphic to

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} xy_1 \\ \vdots \\ xy_n \end{pmatrix} : (\mathbb{R}^n)^n \to (\mathbb{R}^n)^n$$

and hence $\det L_X = \det X^n$. Thus if $f \in C_c(M_n(\mathbb{R}))$, we have that

$$\int_{M_n(\mathbb{R})} f(xy)dy = \int_{M_n(\mathbb{R})} f \circ L_X(y)dy = \frac{1}{|\det X|^n}\int_{M_n(\mathbb{R})} f(y)dy.$$

Now since $\mathrm{GL}_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$, so $C_c(\mathrm{GL}_n(\mathbb{R})) \subset C_c(M_n(\mathbb{R}))$, and we define for $f \in C_c(\mathrm{GL}_n(\mathbb{R}))$

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{R})} f(y)\frac{1}{|\det y|^n}dy.$$

Then for $x \in \mathrm{GL}_n(\mathbb{R})$, we have

$$\begin{aligned} I(f \cdot x) &= \int_{\mathrm{GL}_n(\mathbb{R})} f(xy)\frac{1}{|\det xy|^n}\cdot |\det x|^n dy \\ &= \int_{\mathrm{GL}_n(\mathbb{R})} f(y)\frac{1}{|\det y|^n}\cdot\frac{|\det x|^n}{|\det x|^n}dy = I(f) \end{aligned}$$

If $M_{\mathrm{GL}_n(\mathbb{R})}$ is the measure associated with I, then with m the Lebesgue measure on $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we have

$$\frac{dm_{\mathrm{GL}_n(\mathbb{R})}}{dm}(y) = \frac{1}{|\det y|^n}$$

(c) If we take $\mathbb{R}^\times \cong \mathrm{GL}_1(\mathbb{R})$, then

$$I(f) = \int_{\mathbb{R}^\times} f(y)\frac{dy}{|y|}$$

(d) Consider $\mathbb{C}^\times = \mathbb{C}\setminus\{0\} \subseteq \mathbb{C} \cong \mathbb{R}^2$. Then $[L_{x+iy}]_{(1,i)} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ so that $\det L_{x+iy} = |x+iy|^2$. Thus we get an integral on $G = \mathbb{C}^\times$ by

$$I(f) = \int_{\mathbb{C}^\times} f(z)\frac{dz}{|z|^2}$$

(e) On $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{R})$, we likewise find Haar integral

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{C})} f(y)\frac{1}{|\det y|^{2n}}dy.$$

(iv) Suppose G admits an open (hence closed) subgroup H. If m is a Haar measure on G, then $m_H = m|_{\mathcal{B}(H)}$ is a Haar measure on H. Let T be a transversal for left cosets of H (A of C), so $G = \bigcup_{t\in T} tH$. If $U \subset G$ is open with $m(U) < \infty$, then

$$\begin{aligned} \{t \in T : U \cap tH \neq \emptyset\} &= \{t \in T : M(U \cap tH) > 0\} \\ &= \bigcup_{n=1}^{\infty}\{t \in T : m(U \cap tH) < \frac{1}{n}\} \end{aligned}$$

is countable, so if $E \in \mathcal{B}(G)$, $m(E) < \infty$, $E \subseteq \bigcup_{j=1}^\infty t_j H$ and then

$$\begin{aligned} m(E) &= \sum_{j=1}^\infty m(E \cap t_j H) = \sum_{j=1}^\infty m(t_j^{-1}(E \cap t_j H)) \\ &= \sum_{j=1}^\infty m_H((t_j^{-1}E) \cap H) \end{aligned}$$

(v) Suppose H is a closed, non-open subgroup of G. We wish to see that for compact $K \subseteq H$, $m(K) = 0$.

(a) Given open U with $K \subseteq U$, then there is open V with $e \in V$ so $VK \subseteq U$. Indeed, for $x \in K$, find open U_x with $e \in U_x$, so $U_x x \subseteq U$. Find open V_x, $e \in V_x$, so $V_x^2 \subseteq U_x$, then $K \subseteq \bigcup_{j=1}^n V_{x_j} x_j$ where $x_1, \dots, x_j \in K$. Let $V = \bigcap_{j=1}^n V_{x_k}$. If $x \in K$, $x \in V_{x_j} x_j$ for soe j so $VxVV_{x_j}x_j \subseteq V_{x_j}^2 x_j \subseteq U_{x_j}x_j \subseteq U$, i.e. $VK = \bigcup_{x \in K} V_x \subseteq U$.

(b) Suppose we had compact $K \subseteq H$ such that $m(K) > 0$. We may find open U so $K \subseteq U$ and $m(U) < 2m(K)$ (by outer regularity). Take V as above. Since H is non-open, there is $x \in V \setminus H$. Then
- $K \cap xK = \emptyset$ as $K \subseteq H$, while
- $K \cup xK \subseteq U$.

. Thus $2m(K) = m(K \cup xK) \leq m(U) < 2m(K)$, a contradiction.

Thus, a closed non-open subgroup H of G is always m-locally null. Hence, if G is σ-compact, then closed non-open H are m-null. However, if $G = \mathbb{R} \times \mathbb{R}_d$, $H = \{0\} \times \mathbb{R}_d$ is closed, m-locally null, but not m-null.

(vi) The measure on $(\mathbb{Q}_p, +)$ is determined by $(\mathbb{O}_p, +)$. Likewise, the measure $\mathrm{GL}_n(\mathbb{Q}_p)$is determined by $\mathrm{GL}_n(\mathbb{O}_p) = \{a \in M_n(\mathbb{O}_p) : \det a \in \mathbb{O}_p^\times\}$

(vii) On $\mathrm{GL}_n(\mathbb{O}_p)$, we have Haar integral

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} f(y) \frac{1}{|\det y|_p^n} dy$$

(viii) G is compact if and only if $m(G) < \infty$. The forward is clear since m is Radon. If G is not compact, let U be a open neighbourhood of e so $\overline{U}$ is compact, so $0 < m(U) < \infty$. For any compact set K, $KU \subseteq K\overline{U}$ is compact, hence $KU \subsetneq G$. Inductively find $(x_n)_{n=1}^\infty \subset G$ so $x_{n+1} \notin \{x_1, \dots, x_n\}U$. Notice that $x_j V \cap x_k V = \emptyset$ for $j \neq k$ for V a neighbourhood of e with $V = V^{-1}, V^2 \subset U$. Then $m(G) \geq nm(V)$ for any $n \in \mathbb{N}$, so $m(G) = \infty$.

1.4 The Space $L^1(G)$L1G

Let G be a locally compact group equipped with Haar measure m. Then

$$L^1(G) = \{f : G \to \mathbb{C} : f \text{ measurable}, \|f\|_1 = \int_G |f| dm < \infty\} / \sim_m \quad a.e.$$

This is a Banach space. Recall that by definition of the Lebesgue integral

$$S^1(G) = \mathrm{span}\{\chi_E : E \in \mathcal{B}(G), m(E) < \infty\} / \sim_m \quad a.e.$$

If $0 < m(E) < \infty$, then, given $\epsilon > 0$, there are compact $K \subseteq E$ and open $U \supseteq E$. Hence Urysohn's lemma provides $f \in C_c^+(G)$ such that $f|_L = 1$, $\mathrm{supp}(f) \subseteq U$, and $0 \leq f \leq 1$. Hence $\|f - 1_E\|_1 < \epsilon$. Note that if $f, g \in C_c(G)$, then $f = g$ m a.e. if and only if $f = h$. Thus $C_c(G) \subseteq L^1(G)$ is dense.

1.5 The Modular Function

If $x \in G$, define $m_x : \mathcal{B}(G) \to [0, \infty]$ by $m_x(E) = m(Ex)$. Recall $Ex = R_{x^{-1}}^{-1}(E) \in \mathcal{B}(G)$. Then
- m_x is left invariant

- $m_x(U) = m(Ux) > 0$ if U is non-empty and open
- $m_x(K) = m(Kx) < \infty$ if K is compact.

Hence by uniqueness of Har measure, $m_x = \Delta(x)m$ for some $\Delta(x) \in (0,\infty)$. Notice if $E \in \mathcal{B}(G)$ and $0 < m(E) < \infty$ and $x,y \in G$, then

$$\Delta(xy)m(E) = m(Exy) = \Delta(y)m(Ex) = \Delta(x)\Delta(y)(E)$$

so $\Delta : G \to (0,\infty) \subset \mathbb{R}^\times$ is a homomorphism.

Recall that $x \cdot f(y) = f(yx)$.

1.12 Proposition. *(i) For $f \in L^1(G)$, $x \in G$, we have $x \cdot f \in L^1(G)$, and*

$$\int_G f\,dm = \Delta(x)\int_G x \cdot f\,dm$$

(ii) $\Delta : G \to (0,\infty)$ is continuous.

Proof (i) Let $E \in \mathcal{B}(G)$ with $m(E) < \infty$. Then

$$\Delta(x)\int_G \mathbf{1}_E\,\mathrm{d}m = \Delta(x)m(E) = m(Ex) = \int_G \mathbf{1}_{E_x}\,\mathrm{d}m = \int_G x^{-1}\cdot\mathbf{1}_E\,\mathrm{d}m$$

and replacing x by x^{-1}, we have

$$\int_G \mathbf{1}_E\,\mathrm{d}m = \Delta(x)\int_G x\cdot\mathbf{1}_E\,\mathrm{d}m.$$

If $f \in L^1_+(G)$, so there is $(\phi_n)_{n=1}^\infty \subset S^1_+(G)$ such that $\phi_n \le \phi_{n+1}$ and $(\phi_n)_{n=1}^\infty \to f$ m-a.e. Then by monotone convergence,

$$\Delta(x)\int_G x\cdot f\,\mathrm{d}m = \lim_{n\to\infty}\Delta(x)\int_G x\cdot\phi_n\,\mathrm{d}m = \lim_{n\to\infty}\int_G \phi_n\,\mathrm{d}m = \int_G f\,\mathrm{d}m.$$

Since $L^1(G) = \operatorname{span} L^1_+(G)$, we are done.

(ii) Let $f \in C_c^+(G)$, $\epsilon > 0$, and $V = V^{-1}$ be a relatively compact neighbourhood of e so $\|x\cdot f - f\|_\infty < \epsilon$ for $\in V$. Then for $x \in V$,

$$\begin{aligned}|\Delta(x)-1| &= \frac{|\Delta(x)\int_G f\,\mathrm{d}m - \int_G f\,\mathrm{d}m|}{\int_G f\,\mathrm{d}m}\\ &\le \frac{1}{\int_G f\,\mathrm{d}m}\int_G |x^{-1}\cdot f - f|\,\mathrm{d}m < \epsilon\frac{m(\operatorname{supp}(f)V}{\int_G f\,\mathrm{d}m}\end{aligned}$$

Notice we can arrange for V to be decreasing as $\epsilon \to 0$, so this gives continuity of Δ at e. Now if $x,y \in G$,

$$|\Delta(x,y) - \Delta(y)| = |\Delta(x)-1|\Delta(y)$$

which shows that Δ is continuous at y. ■

1.13 Proposition. *(i) The integral $f \mapsto \int_G f(x)\frac{1}{\Delta(x)}\,\mathrm{d}x$ on $C_c(G)$ is right invariant.*

(ii) For $f \in L^1(G)$,

$$\int_G f(x^{-1})\frac{1}{\Delta(x)}\,\mathrm{d}x = \int_G f(x)\,\mathrm{d}x$$

Proof (i) If $y \in G$ and $f \in C_c(G)$, then

$$\begin{aligned}\int_G y\cdot f(x)\frac{1}{\Delta(x)}\,\mathrm{d}x &= \int_G f(xy)\frac{1}{\Delta(x)}\,\mathrm{d}x = \int_G f(xy)\frac{1}{\Delta(xy)}\Delta(y)\,\mathrm{d}x \\ &= \int_G f(x)\frac{1}{\Delta(x)}\,\mathrm{d}x\end{aligned}$$

(ii) If $f \in C_c^+(G)$, then for any $y \in G$,

$$\begin{aligned}\int_G f\cdot y(x^{-1})\frac{1}{\Delta(x)}\,\mathrm{d}x &= \int_G f((xy^{-1})^{-1})\frac{1}{\Delta(x)}\,\mathrm{d}x \\ &= \int_G f(x^{-1})\frac{1}{\Delta(x)}\,\mathrm{d}x\end{aligned}$$

by the proof above. Hence by uniqueness of left Haar integral, there is $c > 0$ so that

$$\int_G f(x^{-1})\frac{1}{\Delta(x)}\,\mathrm{d}x = c\int_G f(x)\,\mathrm{d}x$$

for $f \in C_c(G)$. Notice, by continuity of $f \mapsto \int_G f\,\mathrm{d}m$ on $L^1(G)$, this holds for $f \in L^1(G)$. Now, if $c \neq 1$, there is a relatively compact neighbourhood $U = U^{-1}$ of e such that $|\Delta(x) - 1| < \frac{1}{2}|c-1|$ for $x \in U$. Then we have

$$\begin{aligned}0 &= \left|\int_G \mathbf{1}_U(x^{-1})\frac{1}{\Delta(x)}\,\mathrm{d}x - c\int_G \mathbf{1}_U(x)\,\mathrm{d}x\right| \\ &= \left|\int_U \left(\frac{1}{\Delta(x)} - c\right)\mathrm{d}x\right| \\ &= \left|\int_U \left(\frac{1}{\Delta(x)} - 1 + 1 - c\right)\mathrm{d}x\right| \\ &\geq m(U)\left||1-c| - \frac{1}{2}|c-1|\right| = \frac{m(U)}{2}|1-c| > 0\end{aligned}$$

which is a contradiction. ■

For $f \in L^1(G)$, $x \in G$, we let

$$\begin{aligned}x * f(y) &= f(x^{-1}y) \\ f * x(y) &= f(yx^{-1})\frac{1}{\Delta(x)} \\ f^*(x) &= \overline{f(x^{-1})}\frac{1}{\Delta(x)}\end{aligned}$$

Notice that $\|f\|_1 = \|x * f\|_1 = \|f * x\|_1 = \|f^*\|_1$. Moreover,
- $x' * (x * f) = (x'x) * f$ and $(f * x) * x' = f * (xx')$
- $f^{**} = f$.

- $(x * f)^* = f^* * x^{-1}$. Indeed, for m-a.e. y, we have

$$
\begin{aligned}
(x * f)^*(y) &= \overline{[x * f](y^{-1})}\frac{1}{\Delta(y)} \\
&= \overline{f(x^{-1}y^{-1})}\frac{\Delta(y)}{}
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{f((yx)^{-1})}\tfrac{1}{\Delta(yx)}\tfrac{1}{\Delta(x^{-1})} \\
&= f^* * x^{-1}(y)
\end{aligned}
$$

1.14 Proposition. *For $f \in L^1(G)$, $\lim_{x\to e} \|x * f - f\|_1 = 0 = \lim_{x\to e} \|f * x - f\|_1$.*

Proof First, consider $g \in C_c(G)$ and $\epsilon > 0$, and let $V = V^{-1}$ be a relatively compact neighbourhood of e so $\|g \cdot x - g\|_\infty < \epsilon$ for $x \in V$. Then

$$
\|x * g - g\|_1 = \int_G |g \cdot x^{-1} \,\mathrm{d}m \le \left\|g \cdot x^{-1} - g\right\|_\infty m(V \operatorname{supp}(g)) < \epsilon m(V \operatorname{supp}(g))
$$

so $\lim_{x\to e} \|x * g - g\|_1 = 0$. If $f \in L^1(G)$, $\epsilon > 0$, find $g \in C_c(G)$ such that $\|f - g\|_1 < \epsilon$. Then

$$
\begin{aligned}
\|x * f - f\|_1 &\le \|x * f - x * g\|_1 + \|x * g - g\|_1 + \|g - f\|_1 \\
&< 2\epsilon + \|x * g - g\|_1
\end{aligned}
$$

where $\|x * g - g\|_1 \to 0$ as $x \to e$. Since $\epsilon > 0$ was arbitrary, we are done.

Now, for f, x as above,

$$
\|f_x - f\|_1 = \|(f * x - f)^*\|_1 = \left\|x^{-1} * f^* - f^*\right\|_1
$$

where $x^{-1} \to e$ as $x \to e$. ∎

1.15 Theorem. (Weil Integral Formula) *Let N be a closed normal subgroup of G.*

(i) If $f \in C_c(G)$, then $x \mapsto \int_N f(xn)dn : G \to \mathbb{C}$ is constant on cosets and hence defines a function $T_n f : G/N \to \mathbb{C}$. Furthermore, $T_N(C_c^*(G)) \subseteq C_c^+(G/N)$ and the operator $T_N : C_c(G) \to C_c(G/N)$ is linear and covariant:

$$
(T_N f) \cdot (yN) = T_N(f \cdot y)
$$

for $f \in C_c(G)$ and $y \in G$.

(ii) The functional on $C_c(G)$ given by $f \mapsto \int_{G/N} T_n f(xN)\,\mathrm{d}xN$ is hence a Haar integral on G, so we may write

$$
\int_{G/N} \int_N f(xn)\,\mathrm{d}n\,\mathrm{d}xN = \int_G f(x)\,\mathrm{d}x
$$

Proof (ii) is a direct consequence of (i); let's see the proof of (i).

The N-invariance of the first function is evident. Let $f \in C_c(G)$. We inspect the continuity of $T_N f$ at x in G. Given $\epsilon > 0$, let $V = V^{-1}$ be a real compact neighbourhood of e, so $\left\|f\cdot y - f\right\|_\infty < \epsilon$ for $y \in V$. Let $g \in C_c^+(G)$ satisfy that $0 \le g \le 1$ and $g|_{Vx^{-1}\operatorname{supp}(f)} = 1$. For $y \in V$, $yN = q_N(y) \in q_N(V)$ so

$$\begin{aligned}|T_N f(yxN) - T_N f(xN)| &\le \int_N |f(yx) - f(xn)| g(n)\,\mathrm{d}n \\ &< \epsilon m_N(\operatorname{supp}(g)\cap N)\end{aligned}$$

Notice as $\epsilon \to 0$, we may shrink V and hence $\operatorname{supp}(g)$. Hence $T_N f$ is continuous at xN. Now, $\operatorname{supp}(T_N f) \subseteq q_N(\operatorname{supp} f)$ so $T_N f \in C_c({G/N})$.

If $g \in C_c^+(G)$, $x \in G$ is such that $f(x) > 0$, let U be a neigbourhood of e, $f(xy) > \frac{1}{2}f(x)$ for $y \in U$. Then

$$T_N f(xN) = \int_N f(xn)\,\mathrm{d}n \ge \frac{1}{2} f(x) m_N(U\cap N) > 0$$ ■

1.16 Corollary. *If N is closed and normal in G, Then $\Delta_G|_N = \Delta_N$.*

Proof Let $n' \in N$ and $f \in C_c^+(G)$. Then

$$\begin{aligned}\int_G n'\cdot f(x)\,\mathrm{d}x &= \int_{G/N}\int_N f(xnn')\,\mathrm{d}n\,\mathrm{d}xN \\ &= \int_{G/N} \frac{1}{\Delta_N(n')}\int_N f(xn)\,\mathrm{d}n\,\mathrm{d}xN = \frac{1}{\Delta_N(n')}\int_G f(x)\,\mathrm{d}x\end{aligned}$$

so that $\Delta_G(n') = \Delta_N(n')$. ■

Definition. We say that G is **unimodular** if $\Delta = 1$ on G.

1.17 Proposition. *G is unimodular in any of the following cases:*

(i) G is abelian, discrete, or compact
(ii) G is perfect: $G = \overline{[G,G]}$
(iii) ${G/Z(G)}$ is unimodular.
(iv) There is a closed, unimodular normal subgroup N such that ${G/N}$ is compact.

Proof (i) This is (nearly) obvious for G abelian or discrete. If G is compact, then $\log\circ\Delta : G \to (\mathbb{R},+)$ is a continuous homomorphism whose range is a compact subgroup.

(ii) Any commutator $[x,y] \in xyx^{-1}y^{-1} \in \ker\Delta$.

(iii) Let $Z = Z(G)$. If $y \in G$ and $f \in C_c(G)$, we have by Weyl's integral formula

$$\begin{aligned}\int_G y \cdot f(x)dx &= \int_{G/Z}\int_Z f(xz)\,\mathrm{d}z\,\mathrm{d}xZ \\ &= \int_{G/Z}\int_Z f(xyz)\,\mathrm{d}z\,\mathrm{d}xZ \\ &= \int_{G/Z} T_Z f(xyZ)\,\mathrm{d}xZ \\ &= \int_{G/Z} T_Z f(xZyZ)dxZ \\ &= \int_{G/Z} T_Z f(xZ)dxZ = \int_G f(x)dx\end{aligned}$$

(iv) We have $\Delta_G|_N = \Delta_N = 1$ by assumption, i.e. $N \in \ker\Delta_G$, so Δ_G induces a homomorphism $\Delta : G/N \to (0,\infty)$ where $\Delta_G = \overline{\Delta}\circ\pi_N$. Verify that $\overline{\Delta}$ is continuous, so $\log\circ\overline{\Delta} : G/N \to \mathbb{R}^+$ is a continuous homomorphism, whose range is a closed subgroup. It follows that $\Delta_G = 1$ on G. ■

Example. Here are some examples of unimodular groups.

(i) Let $\Bbbk$ be a locally compact field with $|\Bbbk| > 3$. Then $\mathrm{SL}_n(\Bbbk)$ is perfect.

Let $\{E_{ij}\}_{i,j=1}^n$ be a matrix unit for $M_n(\Bbbk)$, so $E_{ij}E_{k\ell} = \delta_{jk}E_{il}$.

If $\lambda \in \Bbbk$, i,j,k distinct (i.e. $n \geq 3$), then

$$[e+\lambda E_{ik}, e+E_{kj}] = (e+\lambda E_{ik})(e+E_{kj})(e-\lambda E_{ik})(e-E_{kj}) = e+\lambda E_{ij}.$$

If $n = 2$, then

$$\left[\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}, \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right] = e+\lambda E_{12}$$

where $\lambda = (1-\alpha^2)\beta$.

If $S = \langle e+\lambda E_{ij} : \lambda \in \Bbbk, i \neq j\rangle$. Using only elemnentary operations induced by multiplying by elements of S on the left, and element a of $\mathrm{SL}_n(\Bbbk)$ satisfies (see pic)

By an evident induction, we see that there are $s_1, s_2 \in S$ so $s_1 s a s_2 = e$. Thus $a = s^{-1}s_1^{-1}s_2^{-1} \in S$.

(ii) Let $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. Consider $G = \mathrm{GL}_n(\Bbbk)$. Notice that $Z = Z(G) = \Bbbk^\times e$. From (i), $\mathrm{SL}_n(\Bbbk)$ is perfect.

Let $H = Z\cdot\mathrm{SL}_n(\Bbbk)$, which is closed (check!) and $H/Z \cong \mathrm{SL}_n(\Bbbk)/Z\cap\mathrm{SL}_n(\Bbbk)$ is perfect, being the quotient of a perfect group, hence unimodular.

If $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{R}$ and n is odd, $H = G$. Else if $\Bbbk = \mathbb{R}$ and n is even, then $H = \mathrm{GL}_n(\mathbb{R})_e = \det^{-1}((0,\infty))$ (connnected component of e) and $G = \mathrm{GL}_n(\mathbb{R})_e \,\dot\cup\, (-e)\mathrm{GL}_n(\mathbb{R})_e$, so $G/H \cong \{-1,1\}$ is compact.

(iii) $E(n) = \mathbb{R}^n \rtimes (n)$. Since $N = \mathbb{R}^n \rtimes \{e\}$ is closed, normal, and abelian, and $G/N = (n)$ is compact.

(iv) Consider

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

then

$$Z(\mathbb{H}) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

has $\mathbb{H}/Z \cong \mathbb{R}^2$.

Remark. In A1, a "Braconnier" modular function $\delta : \mathrm{Aut}(G) \to (0, \infty)$ is defined.

(i) If $\gamma : G \to \mathrm{Aut}(G)$ has $\gamma(x)(y) = xyx^{-1}$, so γ is a homomorphism. Then $\delta(\gamma(x)) = \frac{1}{\Delta(x)}$.

(ii) If G is compact and $\alpha \in \mathrm{Aut}(G)$, then $\alpha(G) = G$ so $1 = m(G) = m(\alpha(G))$ and $\delta(\alpha) = 1$.

(iii) If G is discrete and $\alpha \in \mathrm{Aut}(G)$, then for any non-empty finite $F \subsetneq G$, we have $|F| = |\alpha(F)|$, and it follows that $\delta(\alpha) = 1$.

(iv) If G is unimodular and H an open subgroup of G, then H is unimodular However, there is some subtlety here:

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

is closed in $\mathrm{SL}_2(\mathbb{R})$ and $H \cong \mathbb{R} \rtimes (0, \infty)$ is not unimodular. Moreover,

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}, a > 0 \right\}$$

is open, normal and abelian in $G = \left\{ \begin{pmatrix} 2^n & r \\ 0 & 1 \end{pmatrix} n \in \mathbb{Z}, r \in \mathbb{R} \right\}$ and G is closed in $\mathrm{GL}_2(\mathbb{R})$.

1.6 The Convolution Algebra of Measures

Let G be a locally compact group. Let

$$M(G) = \{\mu : \mathcal{B}(G) \to \mathbb{C} : \mu \text{ Radon measure}\} = \operatorname{span} M_+(G)$$
$$M_+(G) = \{\mu : \mathcal{B}(G) \to [0, \infty) | \mu \text{ Radon}]$$

If $\mu \in M_+(G)$ with $\mu(G) < \infty$ so μ is finite.

Recall the Hahn-Jordan Decomposition: each μ in $M(G)$ admits a decomposition $\mu = \sum_{k=0}^{3} i^k \mu_k$ where each $\mu_i \in M_+(G)$, $\mu_0 \perp \mu_2$, and $\mu_1 \perp \mu_3$. Any measures satisfying this decomposition are unique.

Definition. If $\mu \in M(G)$, we define the $|\mu| : \mathcal{B}(G) \to [0, \infty)$ by

$$|\mu|(E) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(E_k)| : E = \dot{\bigcup_{k=1}^{\infty}} E_k, E_k \in \mathcal{B}(G) \right\}$$

and $|\mu| = M_+(G)$. If $\mu = \sum_{k=0}^{3} i^k \mu_k$ as in Hahn-Jordan, then $|\mu_0 - \mu_2| = \mu_0 + \mu_2$ and $|\mu_1 - \mu_3| = \mu_1 + \mu_3$. Furthermore,

$$|\mu| \le |\mu_0 - \mu_2| + |\mu_1 - \mu_3| \text{ and } |\mu_0 - \mu_2||\mu_1 - \mu_3| \le |\mu|$$

1.18 Theorem. (Riesz-Markov Duality) *Let $C_0(G) = \overline{C_c(G)} \subseteq C_b(G)$ with the uniform topology. Then $C_0(G)^* \cong M(G)$ through the map $\mu \mapsto \langle \mu, \cdot \rangle$ where $\langle \mu, f \rangle = \int_G f \,\mathrm{d}\mu$. Moreover, $\left\| \langle \mu, \cdot \rangle \right\|_{op} = |\mu|(G)$.*

Remark. Let $\mathcal{B}^\infty(G) = \{f : G \to \mathbb{C} : f \text{ bounded and Borel measurable}\}$, which is a Banach space under the uniform norm. Note that $\mathcal{B}^\infty(G) = \overline{\operatorname{span}}\{1_E : E \in \mathcal{B}(G)\}$. We have

$$\left| \int_G f \,\mathrm{d}\mu \right| \leq \int_G |f| d|\mu| \leq \|f\|_\infty \left\|\mu\right\|_1$$

If $\mu \in M(G)$ and $\epsilon > 0$, then inner regularity provides compact $K \subseteq G$ such that $|\mu|(K) > |\mu(G) - \epsilon$. Hence $|\mu|(G \setminus K) < \epsilon$. Then $\left\|\mu - \mu_k\right\|_1 = \left\|\mu_{G \setminus K}\right\|_1 = |\mu|(G \setminus K) < \epsilon$.

2 Abelian Locally Compact Groups

3 Compact Groups

4 Introduction to Amenability Theory