PMATH 465

Alex Rutar*
University of Waterloo

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*arutar@uwaterloo.ca
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Contents

I. Fundamentals of Manifolds

1 Introduction to Topology

Basic Constructions

Definition. A **topology** on a set X is a set τ of subsets of X such that
(i) $\emptyset \in \tau$ and $X \in \tau$
(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.
The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X.

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A. Similarly, we define the **closure** of A (denoted $\overline{A}$) as the intersction of all closed sets containing A. Then the **boundary** of A, denoted by ∂A, is the set $\partial A = \overline{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set $\mathcal{V}$ of subsets of X
(i) $\bigcup_{B \in \mathcal{B}} b = X$
(ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.
When $\mathcal{B}$ is a basis for a topology on X, the topology on X **generated** by $\mathcal{B}$ is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ, as above, is a topology on X since
(i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.
(ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{x \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.
(iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we cna choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since $\mathcal{B}$ is a basis, we can chose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.
Note that when τ is the topology on X generated by the basis $\mathcal{B}$, for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis $\mathcal{B}$ is the coarsest topology which contains $\mathcal{B}$.

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y, we define the **subspace topology** on X to be the topology for which as set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V.

If $\mathcal{C}$ is a basis for the topology on Y, then $\mathcal{B} = \{\, X \cap V \mid V \in \mathcal{C} \,\}$ is a basis for the subspace topology on X.

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generted by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where $\mathcal{C}$ and $\mathcal{D}$ are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \to \bigcup_{k \in K} X_k \,\middle|\, f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X. The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \,\middle|\, U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \,\middle|\, \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) $\mathbb{R}^n$ has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $\langle u, v \rangle = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on $\mathbb{R}^n$ for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{\langle u, v \rangle}$. This gives the standard metric on $\mathbb{R}^n$: for $a, \in \mathbb{R}^n$, $d(a, b) = \|b - a\|$.

Given a metric on a set Y, we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X, we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where $x \in X$, $r > 0$.

Maps on Topological Spaces

Definition. When X and Y are topological spaces and $f : X \to Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y. We say that $f : X \to Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

1.1 Theorem. (Glueing Lemma) *Let X and Y be topological spaces, and let $f : X \to Y$ be a function. Suppose either*
(i) $X = \bigcup_{k \in K} A_k$ where each A_k is open in X, or
(ii) $X = \bigcup_{k=1}^n A_k$ where each A_k is closed in X
and each restriction map $f_k : A_k \to Y$ is continuous, then f is continuous.

Proof Exercise. ∎

Definition. A topological space X is **compact** when it has the property that for every set $\mathcal{S}$ of open subsets of X with $X = \bigcup_{U \in S} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F \in \mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set $\mathcal{T}$ with $X \subseteq \bigcup_{T \in \mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ uch that $X \subseteq \bigcup_{G \in \mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \in X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspsace, then X if connected if and only if there do not exist open sets $U, V \in Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cap V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation $\sim$ on C by taking $a \sim b$ if and only if there exists a connected subspsace $A \subseteq X$ with $a \in A$ and $b \in B$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation $\approx$ on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0,1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X, and that these components are open.

Quotient Topology

Definition. (Quotient Topology) Let X be a topological space and let $\sim$ be an equivalence relation on X. The set of equivalence classes is denoted $X/\sim$, and $X/\sim$ is called the **quotient** of X by $\sim$. The map $\pi : X \to X/\sim$ given by $\pi(a) = [a]$ is called the natural **projection map** or **quotient map**. We define the **quotient topology** on $X/\sim$ by stipulating that for $W \subseteq X/\sim$, W is open in $X/\sim$ if and only if $\pi^{-1}(W)$ is open in X.

When a group G acts on a topological space X, we define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $b = g \cdot a$ for some $g \in G$. The equivalence classes are orbits. In this context, we also write $X/\sim$ as X/G.

When X, Y are any toplogical spaces and $\pi : X \to Y$ is surjective, we can define an equivalence relation X by $a \sim b$ if and only if $\pi(a) = \pi(b)$. We then have a natural bijection from Y to $X/\sim$ in which $y \in Y$ corresponds to the fibre $\pi^{-1}(y) \in X/\sim$.

If Y has the topology such that for $W \subseteq Y$, W is open in Y if and only if $q^{-1}(W)$ is open in X. In this case, we also use the terminology "quotient map" for π.

Remark. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let Y be any set. If $f : X \to Y$ is constant on the equivalence classes, then f induces a well-defined map $\overline{f} : X/\sim \to Y$ given by define $\overline{f}([a]) = f(a)$.

Example. Define an equivalence class on $[0,1] \subseteq \mathbb{R}$ by $s \sim t$ if and only if $s = t$ or $\{s,t\} = \{0,1\}$. Then $[0,1]\big/\sim \;\cong \mathbb{S}^1$. Define $f : [0,1] \to \S^1$ by $f(t) = e^{i2\pi t}$. Note that $f(0) = f(1)$, so f induces a continuous map $\overline{f} : [0,1]\big/\sim \;\to \mathbb{S}^1$. The inverse map can be constructed as follows. We define $g : \mathbb{S}^1 \to [0,1]\big/\sim$ by

$$g(x,y) = \begin{cases} \left[\frac{1}{2\pi}\cos^{-1} x\right] & : y \geq 0 \\ \left[1 - \frac{1}{2\pi}\cos^{-1} x\right] & : y \leq 0 \end{cases}$$

Then g is continuous by the Glueing lemma.

In particular, the same proof shows that $\mathbb{R}\big/\mathbb{Z}$ is homeomorphic to $\mathbb{S}^1$.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ can be defined in several ways. $\mathbb{P}^n$ is the set of all 1−dimensional vector subspaces of $\mathbb{R}^{n+1}$, or $\mathbb{P}^n = \mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$, or $\mathbb{P}^n = \mathbb{S}^n\big/\pm 1$ where $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$.

Let us show that $\mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times$ is homeomorphic to $\mathbb{S}^n\big/\pm 1$. Define $f : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{S}^n$ by $f(x) = x/|x|$, and $g = \pi \circ f$. Then g is given by $g(x) = \{\pm x/|x|\}$. Note that for $t \in \mathbb{R}^\times$,

$$g(tx) = \left[\frac{t}{|t|} \cdot \frac{x}{|x|}\right] = \left[\frac{x}{|x|}\right]$$

since $t/|t| = \pm 1$. Thus g induces a continuous map $\overline{g}$ on the quotient. We construct the inverse map in a similar way.

Definition. Let X be a topological space. Then

- X is **T1** when for all $a, b \in X$ there exists an open set U in X with $a \in U$ and $b \notin U$
- X is **T2** or **Hausdorff** when for all $a, b \in X$, there exist disjoint open sets $U, V \subseteq X$ with $a \in U$ and $v \in B$
- X is **T3** or **regular** when X is T1 and for every $a \in X$ and every closed set $B \subseteq X$ with $a \notin B$, there exist open sets $U, V \subseteq X$ with $a \in U$, $B \subseteq V$.
- X is **T4** or **normal** when X is T1 and for all disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

Definition. Let X be a topological space.

- X is **first countable** when for every $a \in X$, there exists a countable set B_a of open sets in X which contain a such that for every open set W in X with $a \in W$, there exists $U \in \mathcal{B}_a$ with $a \in U \subseteq W$.
- X is **second countable** when there exists a countable basis for the topology on X.

Example. (i) X is T1 if and only if every 1-point subset of X is closed in X

(ii) Every compact Hausdorff space is regular.

(iii) Every second countable regular space is normal.

(iv) Every metric space is normal.

(v) If X is second countable, then every open cover admits a countable subcover.

(vi) Every secound countable space X contains a countable dense subset.

1.2 Lemma. (Urysohn) *If X is normal and $A, B \subseteq X$ are disjoint and closed, then there is a countinuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

1.3 Theorem. (Tietze Extension) *If X is normal and $f : A \to \mathbb{R}$ is continuous for some $A \subseteq X$ closed, then there exists a continuous map $F : X \to \mathbb{R}$ such that $F|_A = f$ and $\sup_{a\in A} |f(a)| = \sup_{x\in X} |F(x)|$.*

1.4 Theorem. (Urysohn's Metrization) *If X is second countable and regular, then X is metrizable.*

Definition. An **n-dimensional topological manifold** is a Hausdorff, second countable topological space M which is **locally homeomorphic** to $\mathbb{R}^n$, meaning for every $p \in M$, there exists an open set $U \subseteq M$ with $p \in U$ and an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \subseteq M \to V \subseteq \mathbb{R}^n$. Such a homomorphism ϕ is called a **(local) coordinate chart** or **chart** on M at p. The domain U of a chart $\phi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^n$ is called a (local) **coordinate neighbourhood** at p. Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \to \phi_k(U_k) : k \in K\}$$

where K is any non-empty set such that $M = \bigcup_{k \in K} U_k$. Such a set of charts is called an **atlas** for M.

Definition. Two charts are called $\phi : U \to \phi(U)$ and $\psi : V \to \psi(V)$ are called **(smoothly) compatible** when either $U \cap V = \emptyset$ or $\phi^{-1} \circ \psi$ and $\psi \circ \phi^{-1}$ are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas $\mathcal{A}$ on M can be extend to a unique maximal smooth atlas $\mathcal{M}$ on M by adding to $\mathcal{A}$ every possible homeomorphism $\psi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^n$ which is compatible with all of the existing charts (since if ψ and χ are both compatible with every chart $\psi \in \mathcal{A}$, then ψ and χ will be compatible with each other). The maps $\psi \circ \phi^{-1}$ are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas $\mathcal{M}$ on M is called a **smooth structure** on M.

Definition. An $n-$**dimensional smooth (or C^∞) manifold** is an $n-$dimensional topological manifold with a smooth structure.

Remark. A topological manifold can have different smooth structures. For example, take $\mathcal{A} = \{\phi\}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is the identity map, and $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism given by $\psi(x)x^3$, since $\sqrt[3]{x}$ is not smooth at the origin.

What if we tried $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism which is not C^∞? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

Remark. We can give a smooth manifold M an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \to B(0,1)$
- $\phi : U \subseteq M \to (0,1)^n$
- $\phi : U \subseteq M \to \mathbb{R}^n$

Note that the maximal atlas $\mathcal{M}$ is determined from any subset $\mathcal{A} \subset \mathcal{M}$ such that the domains of the charts in $\mathcal{A}$ cover M.

Definition. Let M be an $m-$dimensional smooth manifold and N be an $n-$dimensional smooth manifold and let $f : M \to N$ be a function. Then we say f is smooth **smooth** at p when for some (hence for any) chart at ϕ on M at p and for some (hence any) chart ψ on N at $f(p)$, the map $\phi^{-1} \circ f \circ \psi$ is smooth at $x = \phi(p)$, and f is **smooth** if f is smooth at ever $p \in M$. We say that f is a **diffeomorphism** when f is invertible and both f and f^{-1} are smooth. We say that M and N are **diffeomorphic**, and write $M \cong N$, when there exists a diffeomorphism $f : M \to N$.

Remark. If is conceivable that a topological manifold M could be both of dimension n and of dimension m with $n \neq m$. To do this, we would need to have a homeomorphism from an open set in $\mathbb{R}^n$ to an open set in $\mathbb{R}^m$. In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When M is smooth, it is easy to see that this cannot happen. If $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ were smooth inverses, then the matrices $D(\psi \circ \phi^{-1})(\phi(p))$ and $D(\phi \circ \psi^{-1})(\psi(p))$ would be inverse matrices. But then a product of a matrix in $M_{m\times n}(\mathbb{R})$ and in $M_{n\times m}(\mathbb{R})$ cannot be inverses when $m \neq n$.

Remark. Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.

There are other kinds of manifolds (other than C^∞ manifolds); for example, one can define C^k manifolds, or analytic C^ω manifold has an atlas in which the transition maps are analytic.

Example.
1. $\mathbb{R}^n$ is a smooth n–dimensional manifold. It can be given an atlas consisting of 1 chart, the identity map.
2. Any n–dimensional vector space over $\mathbb{R}$ is a smooth n–dimensional manifold. It can be given an atlas with one chart. If $\{u_1, \dots, u_n\}$ is a basis for V, then one can define $\phi : V \to \mathbb{R}^n$ by $\phi(\sum t^i u_i) = (t^1, \dots, t^n) = t \in \mathbb{R}^n$.
3. Every open subset of a smooth n–dimensional manifold is also a smooth n–dimensional manifl.d
4. $M_{n\times m}(\mathbb{R})$ is an $n \cdot m$–dimensional manifold with pointwise $\mathbb{R}^{nm}$ structure.
5. $\{A \in M_{n\times m}(\mathbb{R}) : \mathrm{rank}(A) = \min\{n, m\}\}$ is a smooth manifold with one chart, since it is an open submanifold of $M_{n\times m}$. Suppose $n > m$; then take all $n \times n$ submatrices which have non-zero determinant (open by continuity of det), and maximal rank means that A is contained in one of these open subsets.
6. The disjoint union of countably many n–dimensional smooth manifolds.
7. The cartesian product of finitely many smooth manifolds is a smooth manifold. Let $\dim(M_k) = n_k$; the $\dim(M_1 \times \cdots \times M_\ell) = \sum_{k=1}^{\ell} n_k$. If $\phi_k : U_k \subseteq M_k \to \phi_k(U_k) \subseteq \mathbb{R}^{n_k}$ is a chart on M_k, then $\chi_k : \prod_{k=1}^{\ell} U_k \to \prod_{k=1}^{\ell} \mathbb{R}^{n_k}$ given by $\chi_k(p_1, \dots, p_\ell) = (\phi_1(p), \dots, \phi_\ell(p)$ is a chart in $M_1 \times \cdots \times M_\ell$.
8. One can show that $\mathbb{S}^n$ is a smooth n–dimensional manifold.

Remark. For $A \in M_{n\times m}(\mathbb{R})$, we denote the entry in the k^{th} row and ℓ^{th} column by A^k_ℓ.

Example. $\mathbb{S}^n$ is an example of an n–dimensional smooth manifold. It can, for example, be given a smooth atlas which contains $2(n+1)$ charts as follows. For $1 \leq k \leq n+1$, let

$$\begin{aligned} U_k &= \{x \in \mathbb{S}^n : x^k > 0\} \\ \phi_k &: U_k \to B(0,1) \subseteq \mathbb{R}^n \\ \phi_k(x) &= (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1}) \\ \phi_k^{-1}(t^1, \dots, t^n) &= \left(t_1, \dots, t^{k-1}, \sqrt{1 - \sum (t^i)^2}, t^k, \dots, t^n\right) \end{aligned}$$

and the corresponding opposite charts for $x^k < 0$. Note that $\mathbb{S}^n$ is a metric space. It has 2 standard metrics: eithre the one inherited from $\mathbb{R}^n$, or the arclength distance $d_s(U, v) = \cos^{-1}(u \cdot v)$.

We can also given $\mathbb{S}^n$ an atlas which only uses 2 charts, by stereographic projection from a north pole and a south pole.

This stereographic projection also shows that the rational points on the sphere are dense in $\mathbb{S}^n$, via the map

$$\phi(x) = \alpha\left(\frac{1}{1-x^{n+1}}\right) = \left(\frac{x^1}{1-x^{n+1}}, \ldots, \frac{x^n}{1-x^{n+1}}\right)$$

One can also find ϕ^{-1} and verify that they are both rational functions. In particular, $\phi^{-1}(\mathbb{Q}^n) \subseteq \mathbb{S}^n$ is dense.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ is commonly defined in at least 3 ways:

$$\begin{aligned}
\mathbb{P}^n &= \{1\text{-dimensional subspaces of } \mathbb{R}^{n+1}\} \\
\mathbb{P}^n &= \mathbb{R}^{n+1}\setminus\{0\}\big/\mathbb{R}^\times = \{[x] : 0 \neq x \in \mathbb{R}^{n+1}\}, [x] = \{tx : t \in \mathbb{R}^\times\} \\
\mathbb{P}^n &= \mathbb{S}^n\big/\pm 1
\end{aligned}$$

We can given $\mathbb{P}^n$ a smooth atlas with $n+1$ charts as follows: for $1 \leq k \leq n+1$, set

$$\begin{aligned}
&U_k = \{[x] \in \mathbb{P}^n : x^k \neq 0\} \\
&\phi_k : U_k \to \mathbb{R}^n, \phi_k([x]) = \left(\frac{x^1}{x^k}, \ldots, \frac{x^{k-1}}{x^{k-1}}, \frac{x^{k+1}}{x^{k+1}}, \ldots, x^{n+1}x^k\right)
\end{aligned}$$

with $\phi_k^{-1}(t_1, \ldots, t^n) = [(t_1, \ldots, t^{k-1}, 1, t^k, \ldots, t^n)]$.

Examples of Smooth Maps

- The inclusion $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ given by $f(x) = x$
- The quotient map $f : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{P}^n$
- The exponential map $f : \mathbb{R} \to \mathbb{S}^1$ given by $f(t) = e^{i2\pi t}$, or more generally $f : \mathbb{R}^n \to \mathbb{T}^n$ given by $f(t^1, \ldots, t^n) = (e^{2\pi i t^1}, \ldots, e^{2\pi i t^n})$
- The determinant map $f : M_n(\mathbb{R}) \to \mathbb{R}$ given by $f(A) = \det(A)$ is smooth
- For $A \in M_n(\mathbb{R})$, left and right multiplication by A, the transpose map, and the inverse map $f(A) = A^{-1}$ are smooth.

Partitions of Unity

1.5 Lemma. *Every open cover of a manifold has an (at most) countable subcover.*

Proof Let $\mathcal{S}$ be any open cover of M, and let $\mathcal{B}$ be a countable basis for the topology on M. For each $p \in M$, choose $U_p \in \mathcal{S}$ with $p \in U_p$, then choose $B_p \in \mathcal{B}$ with $p \in B_p \subseteq U_p$. Then $\{B_p : p \in M\} \subseteq B$ is an open cover of M, and it is a subset of $\mathcal{B}$, so it is (at most) countable; but then $\{U_p : p \in M\}$ gives an at most countable subcover of $\mathcal{S}$. ■

As a result, every manifold has a countable basis $\mathcal{B}$ such that for each $B \in \mathcal{B}$, there is a chart $\phi : U \to \phi(U)$ on M with $\phi(U) = B(0,2)$ and $\phi(B) = B(0,1)$.

1.6 Lemma. *Let M be a manifold, and let $\mathcal{S}$ be any open cover of M. Then there exists an at most countable open cover $\mathcal{B}$ of M such that*

1. *for each $B \in \mathcal{B}$ there is a chart $\phi_B : C_B \to \phi_B(C_B) = B(0,1)$ with $B \subseteq C_B \subseteq U_B \subseteq S$ for some $U_B \in S$ and $\phi_B(B) = B(0,1)$.*

2. *$\{C_B : B \in \mathcal{B}\}$ is locally finite, meaning that every point in M has an open neighbourhood which only intersects with finitely many of the sets C_B, $B \in \mathcal{B}$ (and hence also the sets $\overline{B}$, $B \in \mathcal{B}$).*

Proof Choose a countable set $\mathcal{V} = \{V_1, V_2, \ldots\}$ of regular coordinate balls which cover M with charts $\phi_i : W_i \to \phi_i(W_i) = B(0,2)$ such that $V_i = \phi_i^{-1}(B(0,1))$. We use the sets V_i to construct a strongly ascending chain of compact sets K_i in M with $K_i \subseteq H_{i+1}^{-1}$ for each i, and $M = \bigcup_{i=1}^{\infty} K_i$ as follows:

- Let $K_i = \overline{V_1}$; since K_1 is compact, we can choose $\ell_1 \in \mathbb{N}$ so that $K_1 \subseteq V_1 \cup \cdots \cup V_{\ell_1}$.
- Then we let $K_2 = \overline{V_1 \cup \cdots \cup V_{\ell_1}}$. Since K_2 is compact, we can choose $\ell_2 > \ell_1$ so that $K_2 \subseteq V_1 \cup \cdots \cup V_{\ell_2}$, and set $K_3 = \overline{V_1 \cup \cdots V_{\ell_2}}$.

Repeat the above process to obtain $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq K_3^\circ \subseteq \cdots$ with $\bigcup_{i=1}^{k} K_i = M$. For each $m \in \mathbb{N}$, note that $K_{m+1} \setminus K_m^\circ$ is compact and contained in the open set $K_{m+2} \setminus K_{m-1}$ (with $K_0 = \emptyset$). For each $p \in K_{m+1} \setminus K_m^\circ$, choose $U_p \in \mathcal{S}$ with $p \in U_p$ and then choose a regular coordinate ball B_p and a chart $\phi_p : C_p \subseteq M \to \phi_p(C_p) = B(0,2) \subseteq \mathbb{R}^n$ with $\phi_p(B_p) = B(0,1)$ and $C_p \subseteq U_p \cap (K_{m+2}^\circ \setminus K_{m-1})$. The coordinate balls B_p, $p \in K_{m+1} \setminus K_m^\circ$ cover the compact set $K_{m+1} \setminus K_m^\circ$, so we can choose a *finite* set $\mathcal{B}_m$ of such regular coordinate balls B_p so that $K_{m+1} \setminus K_m^\circ \subseteq \cup \mathcal{B}_m \subseteq K_{m+2}^\circ \setminus K_{m-1}$.

Now, the set $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ is a countable set of such regular coordinate balls. Note that for each $B \in \mathcal{B}$, we have chart $\phi_B : C_B \to \phi_B(C_B) = B(0,2)$ and the set $\{C_B : B \in \mathcal{B}\}$ is locally finite since every point in M is contained in one of the sets $K_{m+2}^\circ \setminus K_{m-1}$ and each of these sets only intersects with the coordinate balls from the finite sets $\mathcal{B}_l$ with $m-2 \le l \le m+2$.■

1.7 Theorem. (Partitions of Unity) *Let M be a smooth manifold, and let $\mathcal{S}$ be any open cover of M. There exists a set $\{\psi_u : u \in \mathcal{S}\}$ of smooth maps $\psi_u : M \to \mathbb{R}$ such that*

1. *$\psi_u(M) \subseteq [0,1]$ for each $u \in \mathcal{S}$*
2. *$\operatorname{supp}(\psi_u) \subseteq U$ for ech $U \in \mathcal{S}$*
3. *$\{\operatorname{supp}(\psi_u) : u \in \mathcal{S}\}$ is locally finite: every point in M contains an open neighbourhood whicl only intersects finitely many of the sets $\operatorname{supp}(\psi_n)$, $u \in \mathcal{S}$*
4. *$\sum_{u \in \mathcal{S}} \psi_u = 1$*

Such a set of functions $\{\psi_u : u \in \mathcal{S}\}$ is called a (smooth) **partition of unity** on M for $\mathcal{S}$ (or **subordinate** to $\mathcal{S}$).

Proof Let $\mathcal{B}$ be a countable set of regular coordinate balls as in the previous lemma. Recall that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{1/t} & : t < 0 \\ 0 & : t \ge 0 \end{cases}$$

is smooth, so the function $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = f(|x|^2 - 1)$ is smooth with $g(x) > 0$ for $|x| < 1$ and $g(x) = 0$ for $|x| \ge 1$. For each $B \in \mathcal{B}$, we define a smooth bump function $\sigma_B : M \to \mathbb{R}$ by

$$\sigma_B(p) = \begin{cases} g(\phi_B(p)) & : p \in B \\ 0 & : p \notin B \end{cases}$$

where $\phi_B : C_B \subseteq M \to \phi_B(C_B) = B(9,2)$ with $\phi_B(B) = B(0,1)$ as in the previous lemma. Note that $\sigma(B)$ is smooth with $\sigma_B(p) > 0$ for $p \in B$ and $\sigma_B(p) = 0$ for $p \notin B$. Now for each $B \in \mathcal{B}$, define $\tau_B' : M \to \mathbb{R}$ by

$$\tau_B = \frac{\sigma_B}{c \in \mathcal{B}\sigma_c}$$

Note that $\sum_{c\in\mathcal{B}} \sigma_c$ is well-defined by the local finiteness of $\mathcal{B}$ and $\sum_{c\in\mathcal{B}} \sigma_c(p) > 0$. Furthermore, note that $\tau_B(p) > 0$ for all $p \in B$, and $\tau_B(p) = 0$ for all $p \notin B$, and $\sum_{B\in\mathcal{B}} \tau_B = 1$. Then define $\rho_V : M \to \mathbb{R}$ by $\rho_V = \sum_{B\in\mathcal{B}_V} \tau_B$. ■

2 Immersions, Embedding, Submanifolds

2.1 Theorem. (Inverse Function Theorem) *Let $U \subseteq \mathbb{R}^n$ be open, $p \in U$, and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be smooth and suppose $Df(p)$ is invertible. Then f is a local diffeomorphism.*

2.2 Corollary. *Let $n < m$ and $U \subseteq \mathbb{R}^n$ be open, and let $p \in U$, and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be smooth and suppose $Df(p)$ has rank n. Then the range of f is locally equal to the graph of a smooth function. Such a map f is called a local* ***immersion*** *at p.*

Proof Since $Df(p)$ is an $m \times n$ matrix of rank n, some n rows of $Df(p)$ form an invertible submatrix. Reorder the variables in $\mathbb{R}^m$ (if necessary) so that the top n rows form an invertible matrix. Write elements in $U \subseteq \mathbb{R}^n$ as t and write elements of $\mathbb{R}^m$ as (x,y). Also write $(x,y) = f(t) = (u(t), v(t))$ so

$$Df = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

with $\frac{\partial u}{\partial t}(p)$ invertible. Then by the inverse function theorem, $u(t)$ is a local diffeomorphism. Say $u : U_0 \subseteq U \to V_0 \subseteq \mathbb{R}^n$ is the diffeomorphism, and let $g : V_0 \to U_0$ be its inverse. Then the range of f is locally equal to the graph of the function $y = v(g(x)) =: h(x)$. If $(x,y) \in \Gamma(f)$ with $(x,y) = f(t) = (u(t), v(t))$, then since $x = u(t)$ we have $t = g(x)$ so $y = v(t) = v(g(x)) = k(x)$. If $(x,y) \in \Gamma(k)$, then $y = k(x) = v(g(x))$ and we can choose $t = g(x)$ to get $x = u(t)$ and $y = v(g(x)) = v(t)$ so that $(x,y) = (u(x), v(t)) = f(t)$. ■

2.3 Theorem. (Implicit Function) *Let $n < m$, $U \subseteq \mathbb{R}^m$ be open, $p \in U$, and $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be smooth. Suppose $Df(p)$ has rank n and let $q = f(p)$. Then the level set $f^{-1}(q)$ is locally equal to a graph of a smooth function.*

2.4 Theorem. *Let $U \subseteq \mathbb{R}^n$ be open with $p \in U$, let $f : U \to \mathbb{R}^m$ be smooth with $f(p) = q$; and suppose that Df has constant rank r in U. Then the level set (or fibre) $f^{-1}(q)$ is locally equal to the graph of a smooth function (with $n - r$ independent variables and r dependent variables).*

Proof Since Df is an $m \times n$ matrix of rank r, there is some $r \times r$ submatrix of $Df(p)$ which is invertible; without loss of generality, it is the upper left submatrix. Write elements in $\mathbb{R}^n$ as (x,y) with $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^{n-r}$ and write elements in $\mathbb{R}^m$ as (u,v) with $u \in \mathbb{R}^r$ and $v \in \mathbb{R}^{m-r}$, with say $p = (a,b)$ and $q = f(p) = (c,d)$. Then we have $(u,v) = f(x,y) = \big(u(x,y), v(x,y)\big)$ so that

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

with $\frac{\partial u}{\partial x}(p) = \frac{\partial u}{\partial x}(a,b)$ being an invertible $r \times r$ matrix. Define $F : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ by $F(x,y) = (u(x,y), y)$. Then

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & I \end{pmatrix}$$

so that $DF(p)$ is inertible. By the IVT, F is a local diffeomorphism, say $F : U_0 \subseteq U \subseteq \mathbb{R}^n \to V_0 \subseteq \mathbb{R}^n$ is a diffeomorphism with U_0 an open rectangular box. Let $G : V_0 \to U_0$ denote the smooth inverse of F. Note that G is of the form $G(u,y) = (g(u,y), y)$ for some smooth function $g : V_0 \to \mathbb{R}^r$. We claim that $f^{-1}(q) = f^{-1}(c,d)$ is locally equal to the graph of $x = g(c,y)$. FIrst, note that

$$(u,y) = F(G(u,y)) = F(g(u,y),y) = \Big(u(g(u,y),y), y\Big)$$

so that, in particular, $u(g(u,y),y) = u$ and so

$$f(G(u,y)) = (u(g(u,y),y), v(g(u,y),y)) = (u, h(u,y))$$

where $h(u,y) = v(g(u,y),y)$. Thus

$$Df(x,y) \cdot DG(u,y) = D(f \circ G)(u,y) = \begin{pmatrix} I & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial y} \end{pmatrix}$$

Since Df has constant rank r and DG is invertible, the matrix on the right is of rank r for all $(u,v) \in V_0$. Thus it follows that $\frac{\partial h}{\partial y} = 0$ for all u, b, so that $h(u,y)$ is independent of y and $h(u,y) = h(u,b)$ for all y; let $k(u) = h(u,b)$. Let us calculate $k(c)$. We have

$$\begin{aligned} f(a,b) = (c,d) &\implies (u(a,b), v(a,b)) = (c,d) \\ &\implies u(a,b) = c \\ &\implies F(a,b) = (u(a,b), b) = (c,b) \\ &\implies (a,b) = G(c,b) \\ &\implies (c,d) = f(a,b) = f(G(c,b)) = (c, h(c,b)) = (c, k(c)) \\ &\implies k(c) = d \end{aligned}$$

Finally, let us show that $f^{-1}(c,d)$ is (locally) the graph of $x = g(c,y)$. We have

$$\begin{aligned} (x,y) = f^{-1}(c,d) &\implies f(x,y) = (c,d) \\ &\implies u(x,y) = c \text{ and } v(x,y) = d \\ &\implies F(x,y) = (u(x,y), y) = (c,y) \\ &\implies (x,y) = G(c,y) = (g(c,y), y) \\ &\implies x = g(c,y) \end{aligned}$$

We thus have

$$\begin{aligned} x = g(c,y) &\implies G(c,y) = (g(c,y), y) = (x,y) \\ &\implies f(x,y) = f(G(x,y)) = (c, h(c,y)) = (c, k(c)) = (c,d) \end{aligned}$$

as required. ∎

Definition. When N and M are smooth manifolds and $f : N \to M$ is a smooth map, we say that f has **rank r** at $p \in N$ when for some (hence for every) chart ϕ on N at p and for some (hence every) chart ψ on M at $f(p)$, the matrix $D(\psi f \phi^{-1})(\phi(p))$ has rank r.

2.5 Corollary. *Let N and M be smooth manifolds, with $p \in N$. Let $f : N \to M$ be smooth with $f(p) = q \in M$. Suppose f has constant rank r in an open neighbourhood of p. Then there exists a chart ϕ on N at p and a chart ψ on M at $q = f(p)$ such that $\phi(p) = 0$ and $\psi(q) = 0$ and*

$$(\psi \circ f \circ \phi^{-1})(x^1, \ldots, x^r, \ldots, x^n) = (x^1, \ldots, x^r, 0, \ldots, 0)$$

where $n = \dim(N)$ and $m = \dim(M)$.

Proof Choose any chart ϕ_0 on N at p and any chart ψ_0 on M at q with $\phi_0(p) = 0$ and $\psi_0(q) = 0$. Then $D(\psi_0 f \phi_0^{-1})$ has constant rank r near 0. Let ϕ_1 and ψ_1 be linear permutation maps so that the upper left $r \times r$ submatrix of $D(\psi, \psi_0, f\phi_0^{-1}\phi_1^{-1})(0)$. Say $f_1 = \psi_1\psi_0 f \phi_0^{-1}\phi_1^{-1}$. Let F, G, f_1 be as in the proof of the rank theorem (for the function f_1). Let us verify that for the charts $\phi = F\phi_1\phi_0$ and $\psi = H\psi_1\psi_0$ where $H(u,v) = (u, v - k(u))$ we have $(\psi f \phi^{-1})(u,y) = (u,0)$. ■

2.6 Corollary. *When $f : M \to N$ is a smooth map of smooth manifolds with constant rank r in M, for $q \in \operatorname{im} f$, the level set (fibre) $f^{-1}(q)$ can be given charts (obtainedd from canonical charts) to make it a smooth $(\dim M - r)$–dimnsional manifold.*

Definition. Let N and M be smooth manifolds (of dimensions m and n). A smooth map $f : N \to M$ is called a (smooth) **immersion** when f has rank n in N. An **immersed submanifold** of M is the image of an immersion $f : N \to M$ or the image of an injective immersion $f : N \to M$.

Note that when $f : N \to M$ is injective, we can give the image $f(N)$ a smooth atlas which mapes $f : N \to f(N)$ a diffeomorphism. When we do this, the resulting topology on $f(N) \subseteq M$ does not necssarily agree with the subspace topology of M.

Definition. An **embedded submanifold** of M is a subset $N \subseteq M$ which is a smoth manifold such that the inclusion map $f : N \to M$ (given by $f(p) = p$) is an immersion such that the topology in the previous remark agrees with the subspace topology.

When $f : M \to N$ is a smooth map of smooth manifolds of constant rank r and $q \in \operatorname{im} f$, the level set $f^{-1}(q)$ is an embedded submanifold of M.

Remark. When $N \subseteq M$ is an embedded submanifold,
- If $f : M \to K$ is smooth, then the restriction $f : N \to K$ is smooth
- If $f : K \to M$ is smooth and $f(K) \subseteq N$, then $f : K \to N$ is smooth

Example. $\mathrm{SL}_n(R)$ is a smooth manifold. Recall that $\mathrm{GL}_n(\mathbb{R})$ is a smooth n^2–dimensional manifold, since it is open in the n^2–dimensional vector space $M_n(\mathbb{R})$. We have $\mathrm{SL}_n(\mathbb{R}) = f^{-1}(\{1\})$ where f is the determinant evaluation map. Then for fixed ℓ, $\det X = \sum_{i=1}^n (-1)^{i+\ell} X_\ell^i \deg X_{(\ell)}^{(i)}$, where $X_{(l)}^{(i)}$ is the matrix obtained from X by removing row i and column j. We have

$$Df = \left(\mathbb{P} f x_1^1, \ldots, \frac{\partial f}{\partial x_n^n} \right) \in M_{1 \times n^2}(\mathbb{R})$$

with $\frac{\partial f}{\partial x_\ell^k} = (-1)^{k+\ell} \det X_{(\ell)}^{(k)}$, so that $Df = 0$ if and only if $\det X = 0$. Thus f has contant rank 1, so $\mathrm{SL}_n(\mathbb{R}) = f^{-1}(1)$ is an embedded submanifold of $M_n(\mathbb{R})$ of dimension.

3 Tangent Vectors

Definition. A vector u in $\mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ is an ordered pair (a, u).

Definition. Let M be a smooth manifold and let $p \in M$. A **tangent vector** on M at p is a set of vectors $X = \{\phi_* x : \phi \text{ is a chart on } M \text{ at } p\}$, where $\phi_* x$ is a vector in $\mathbb{R}^n$ at the point $x = \phi(p)$ such that when ϕ and ψ are two charts on M at p, we have $\psi_* X = D(\psi\phi^{-1})(\phi(p))\phi_* X$.

The set of all tangent vectors on M at p is denoted by T_pM. Note that T_pM is an n−dimensional vector space. When $I \subseteq \mathbb{R}$ is an open interval, $s \in I$, and $\alpha : I \subseteq \mathbb{R} \to M$ is a smooth map with $\alpha(s) = p$, we define $\alpha'(s)$ to be the tangent vector $\alpha'(s) \in T_pM$ given by $\phi_*\alpha'(s) = \beta'(s)$ where $\beta(t) = \phi(\alpha(t))$. Note that, by the chain rule, we do have $\phi_*\alpha'(s) = D(\psi\phi^{-1})\phi_*\alpha'(s)$.

When ϕ is a chart on M at p, we often write

$$x = x(p) = \phi(p) = (\phi^1(p), \ldots, \phi^n(p)) = (x^1(p), \ldots, x^n(p))$$

(so each $x^k = \phi^k$ is a function $x^k, \phi^k : U \subseteq M \to \mathbb{R}$). When ψ is another chart and we write $y = \psi(p)$, we often write $y = y(x) = (\psi\phi^{-1})(x) = (y^1(x), \ldots, y^n(x))$ and we write

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

With this notation, if $u = \phi_* X$ and $v = \psi_* X$, then $v = D(\psi\phi^{-1})u = \frac{\partial y}{\partial x}u$, so $V^k = \sum_{i=1}^n \frac{\partial y^k}{\partial x^i} u^i$.

Definition. Let $f : M \to N$ be a smooth map of smooth manifolds with $p \in M$. We define the **induced map** or the **pushforward** f_* or the **differential** df to be the map $f_* = df : T_pM \to T_{f(p)}N$ given as folows. Given $X \in T_pM$, choose $\alpha : (-\epsilon, \epsilon) \to M$ smooth with $\alpha(0) = p$, $\alpha'(0) = X$, when we let $\beta(t) = f(\alpha(t))$ and define $df(x) = f_*(x) = \beta'(0)$. Given a chart ϕ on M at p and ψ on N at $f(p)$, if $u = \phi_* X$ and $V = \psi_*(f_* x)$, then verify that $v = D(\psi f \phi^{-1})(\phi(p))u$.

1. When ϕ is a chart on M at p and ψ is a chart on N at $f(p)$, $\psi_* f_* X = D(\psi f \phi^{-1})_{\phi(p)}\phi_* X$
2. The map $df = f_*$ is linear
3. If $g : L \to M$ and $f : M \to N$ are smooth, then $(f \circ g)_* = f_* \circ g_*$.
4. When $\iota : M \to M$ is the identity map, $d\iota : T_pM \to T_pM$ is the identity map
5. If $f : M \to N$ is a diffeomorphism, then $f_* : T_pM \to T_pM$ is an isomorphism.
6. For $f : M \to N$ smooth, f is of rank r at p if and only if f_* is of rank r at p.

When $U \subseteq \mathbb{R}^n$ is open, U is a manifold with atlas $\{\emptyset\}$ where ϕ is the identity map. In this case, we identify $X \in T_pU$ with $\phi_* x \in \mathbb{R}^n$. With this convention, $\phi_* X$ is equal to $\phi_* X$ where the second ϕ_* is the pushforward. When $N \leq M$ is a submanifold (immersed or embedded), the inclusion map $\iota : N \to M$ is an injective immersion. Thus, the map $\iota_* : T_pN \to T_pM$. In this situation, we identify T_pN with the subspace $\iota_*(T_pN) \subseteq T_pM$.

Let X be the vector on $\mathbb{S}^2$ at p with $\phi_* X = (1, 0)$. Let $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion map. We have $\phi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ with $u = \phi_* X = (1, 0)$. Then $\iota_* X = D(\psi\eta\phi^{-1})_{\phi(p)}\phi_* X$ where ψ is the identity on $\mathbb{R}^3$.

Tangent Vectors as Differential Operators

Recall that a vector $u \in \mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ acts as a differential operator on smooth maps $f : \mathbb{R}^n \to \mathbb{R}$ by directional derivative. Choose any smooth map $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \to \mathbb{R}^n$ with

$\alpha(0) = a$ and $\alpha'(0) = u$, and define $u(f) = u_a(f) = D_u f(a) = \beta'(0)$ where $\beta(t) = f(\alpha(t))$. Since $\beta(t) = f(\alpha(t))$, we have $\beta'(t) = Df(\alpha(t)) \cdot \alpha'(t)$ so

$$\begin{aligned} u(f) &= D_u f(a) = \beta'(0) = Df(a) \cdot u \\ &= \left(\frac{\partial f}{\partial x^1}(a), \ldots, \frac{\partial f}{\partial x^n}(a) \right) \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) \cdot u^i \end{aligned}$$

or as a differential operator, $u = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}$.

Definition. When M is a smooth manifold, $p \in M$, and $X \in T_pM$, X acts as a differential operator on a smooth function $f : M \to \mathbb{R}$ as follows: choose a smooth map $\alpha(-\epsilon, \epsilon) \subseteq \mathbb{R} \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$, and define $X(f) = X_p(f) = \beta'(0)$ where $\beta(t) = f(\alpha(t))$.

When ϕ is a chart on M at p, then

$$\begin{aligned} X(f) &= (\phi_* X)(f \circ \phi^{-1}) = D_{\phi_* X}(f \circ \phi^{-1})(\phi(p)) \\ &= D(f \circ \phi^{-1})(\phi(p)) \cdot (\phi_* X) \\ &= \sum_{i=1}^{n} \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p)) \cdot u^i \end{aligned}$$

where $u = \phi_* X \in \mathbb{R}^n$. So when $u = \phi_* X \in \mathbb{R}^n$, X acts as the differential operator $X = \sum_{i=1}^n u^i \left.\frac{\partial}{\partial x^i}\right|_p$ where $\left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p))$. With this notation,

$$T_pM = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

If ϕ and ψ are two charts at p on M, then T_pM has to representations as differential operators. Let us determine how $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial y^\ell}$ are related. When $X \in T_pM$, $u = \phi_* X \in \mathbb{R}^n$ and $v = \psi_* X \in \mathbb{R}^n$, we have $V = D(\psi \circ \phi^{-1})(\phi(p)) \cdot u = \left(\frac{\partial y}{\partial x}\right)(\phi(p)) \cdot u$. When $u = \frac{\partial}{\partial x^j}$,

$$v = \left(\frac{\partial y}{\partial x}\right) \cdot e_k$$

so $v^\ell = \left(\frac{\partial y}{\partial x}\right)_k^\ell = \frac{\partial y^\ell}{\partial x^k}$ so that

$$v = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^k} \frac{\partial}{\partial y^i}$$

Definition. A **derivation** on M at p is a linear map $L : C^\infty(M) \to \mathbb{R}$ or $L : C_p^\infty(M) \to \mathbb{R}$ where $C_p^\infty(M)$ is the vector space of **germs** of smooth functions on M at p, which satisfies the product rule at p:

$$L(fg) = L(f) \cdot g(p) + f(p) \cdot L(g)$$

Every $X \in T_pM$ gives a derivation on M at p. Moreover, it can be shown that every derivation on M at p is of this form. Thus allows us to give an alternate definition for T_pM as the space of derivations on M at p.

Definition. Let TM be the disjoint union of all the tangent spaces. A **vector field** on M is a function $X : M \to TM$ such that $X(p) \in T_pM$.

Given a chart $\phi : U \to \phi(U)$ on M, the restriction of X to U determines and is determine by the vetor field $\phi_* X$ on $\phi(u) \subseteq \mathbb{R}^n$ by $(\phi_* X)(\phi(p)) = \phi_*(X(p))$, or $(\phi_* X)(x) = \phi_*(X(\phi^{-1}(x))) \in \mathbb{R}^n$. We say that X is **smooth** at p when for some chart ϕ on M at p, the vector field $\phi_* X$ is smooth at $\phi(p)$. When X is a smooth vector field on M, X acts as a differential operator $X : C^\infty(M) \to C^\infty(M)$ by $X(f)(p) = X_p(f)$.

The space of smooth vector fields on M is $\Gamma(M, TM) = \Gamma(TM) = \mathcal{X}(M)$.

The Pushforward or Differential

If X is a smooth vector field on a smooth manifold N and $f : N \to M$ is a smooth map, for each point $p \in N$, we have the linear map $df = f_* : T_pN \to T_{f(p)}M$.

Note that f_* does not in general give a map $f_* : \Gamma(TN) \to \Gamma(TM)$, if f is not surjective, or f is not injective with $p, q \in N$ with $p \neq q$ and $f(p) = f(q)$ and $f_* X_p \neq f_* X_q$.

If $f : N \to M$ is a diffeomorphism, then $f)*$ dos give a well-defined bijective map $f_* : \Gamma(TN) \to \Gamma(TM)$. If f is an injective immersion, then $f : N \to f(N)$ is a diffeomorphism.

The Lie Bracket of Vector Fields

Definition. When X and Y are two smooth vector fields on M, we define the **Lie bracket** of X and Y, denoted by $[X,Y](f)$, by $[X,Y]f = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.

Note that $[X,Y]$ satisfies the product rule since

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(f \cdot Y(g) + g \cdot Y(f)) - Y(f \cdot X(g) + g \cdot X(f)) \\
&= f \cdot X(Y(g)) + X(f) \cdot Y(g) + g \cdot X(Y(f)) + X(g) \cdot Y(f) - y \cdot Y(X(g)) - Y(f) \cdot X(g) - g \cdot Y(X(f)) - Y(g) \cdot \\
&= g[X,Y](g) + g[X,Y](f)
\end{aligned}$$

Given a chart $\phi : U \to \phi(U)$ on M at p, we can calculate a formula for the Lie bracket: say $u = \phi_* X$ and $v = \phi_* Y$ $(u(x) = \phi_*(X_{\phi^{-1}(x)}), v(x) = \phi_*(Y_{\phi^{-1}(x)}))$. Then for $f \in C^\infty(M)$,

$$\begin{aligned}
[X,Y]_p(f) &= X_p(Y(f)) - Y_p(X(f)) \\
&= \sum_i u^i \frac{\partial}{\partial x^i}\left(\sum) j v \frac{\partial g}{\partial x^j}\right) - \sum_i v^i \frac{\partial}{\partial x^i}\left(\sum_j u^j \frac{\partial g}{\partial x^j}\right) \\
&= \sum_{i,j}\left(u^i \frac{\partial v^j}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} + u^i g^j \frac{\partial^2 g}{\partial x^i \partial x^j} - v^i \frac{\partial u^j}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} - v^i u^j \frac{\partial^2 g}{\partial x^i \partial x^j}\right) \\
&= \sum_{i,j}\left(\frac{\partial v^j}{\partial x^i} \cdot u^i - \frac{\partial u^j}{\partial x^i} \cdot v^i\right)\frac{\partial g}{\partial x^j}
\end{aligned}$$

Thus $[X,Y]_p$ is a vector in T_pM. It is the vector given by $w^j = \sum_i \left(\frac{\partial v^j}{\partial x^i} u^i - \frac{\partial u^j}{\partial x^i} v^i\right)$ and $w = \sum_j w^j \frac{\partial}{\partial x^j} = Dv \cdot u - Du \cdot v$.

Integral Curves and Flows

Given a smooth vector field X on a smooth manifold M, and given $p \in M$, the existence and uniqueness theorem for (systems) of ODEs guarantees that there is a unique smooth map (or curve) $\alpha_p : I_p \subseteq \mathbb{R} \to M$ where I is the (unique) maximal open interval α and $\alpha(0) = p$ and $\alpha'(t) = X_{\alpha(t)}$. A stronger version of the existence and uniqueness theorem also guarantees that $\alpha_p(t)$ varies smoothly with p to give a unique smooth map $\theta : U \subseteq M \times \mathbb{R} \to M$ where U is the (unique) maximal open connected domain given by $\theta(p,t) = \alpha_p(t)$.

Example. (i) Find a vector field which is a parabola at each point.
(ii) Find a smooth vector field so that the solution curves have vertical asymptote.

When a vector field X on a 2 dimensional manifold M, we define the **index** of X at p as follows. Choose a chart $\phi : C \to \phi(C) = B(0,2)$ on M at p. Thus $U = \phi_* X$ is a vector field in $\mathbb{R}^2$ with no zeros in $B(0,2)$ except at 0.

When we restrict u to the circle $\mathbb{S}^1$ and we define the index of X at p to be the winding number of this map $u : \mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$. When a vector field on X has finitely many isolated zeros, the index of X is the sum of the indices at the zeros of X.

3.1 Theorem. *When X is a smooth vector field with isolated zeros on a* ***compact*** *2-dimensional manifold M,* $\operatorname{Ind} X = \chi(M)$*, the Euler characteristic of M.*

4 Lie Groups

Definition. A Lie group G is both a smooth manifold and a group such that the group operations $\mu : G \times G \to G$ and inversion $\nu : G \to G$ are smooth maps.

Example. $O_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I\}$ is a Lie group. Define $F : \mathrm{GL}_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $F(X) = X^T X$. Thus $O_n(\mathbb{R}) = F^{-1}(I)$. When $n = 2$, $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ we have $F(X) = \begin{pmatrix} x^2 + y^2 & xz + yw \\ xz + yw & z^2 + w^2 \end{pmatrix}$ so that

$$DF = \begin{pmatrix} 2x & 2y & 0 & 0 \\ z & w & x & y \\ z & w & x & y \\ 0 & 0 & 2z & 2w \end{pmatrix}$$

In general, for $A \in \mathrm{GL}_n(\mathbb{R})$, $F(R_A(X)) = F(XA) = A^T X^T X A = L_{A^T} R_A F(x)$. Thus by the chain rule, $DF(XA) \cdot DR_A(X) = DL_{A^T}(X^T XA) \cdot DR_A(X^T X) \cdot DF(X)$, so we can identify $T_p \mathrm{GL}_n(\mathbb{R})$ or $T_p M_n(\mathbb{R})$ with the vectr space $M_n(R)$. Note that L_{A^T} and R_A are diffeomorphisms of $\mathrm{GL}_n(\mathbb{R})$, so DL_{A^T} and DR_A are invertible. Thus $\operatorname{rank} DF(XA) = \operatorname{rank} DF(X)$. In particular, taking $X = I$, $\operatorname{rank} DF(A) = \operatorname{rank} DF(I)$, so F has consant rank. Let us calculate $\operatorname{rank} DF : T_I \mathrm{GL}_n(\mathbb{R}) \to T_I M_n(\mathbb{R})$. Let $A \in T_i \mathrm{GL}_n(\mathbb{R})$, so $A \in M_n(\mathbb{R})$, and let $\alpha(t) = I + tA$ so that $\alpha(0) = I$ and $\alpha'(0) = A$. Then $DF(I) \cdot A = \beta'(0)$ where $\beta(t) = F(\alpha(t)) = (I + tA)^T (I + t(A + A^T) + t^2 A^T A$. Then $\beta'(t) = A + A^T + 2tA^T A$, so $\beta'(0) = A + A^T$ so that $DF(I) \cdot A = A + A^T$. The range of DF at I is the set of matrices B of the form $B = A + A^T$ for some matrix $A \in M_n(\mathbb{R})$, or equivalently, the set of symmetric matrices in $M_n(\mathbb{R})$. Thus the dimension of the range of DF is $(n^2 + n)/2$, so F has constant rank $r = (n^2 + n)/2$ and thus $\dim O_n(\mathbb{R}) = n^2 - r = \frac{n^2 - n}{2}$.

Thus by the constant rank theorem, $O_n(\mathbb{R})$ is a regular embedded submanifold of $\mathrm{GL}_n(\mathbb{R})$. In fact, $T_I O_n(\mathbb{R})$ can be identified with $\ker DF(I) \subseteq T_I \mathrm{GL}_n(\mathbb{R}) = M_n(\mathbb{R})$, which is

$\{A \in M_n(\mathbb{R}^n) : A^T + A = 0\}$. One can do the same for $U_n(\mathbb{C}) = \{A \in \mathrm{GL}(\mathbb{C}) : A^*A = I\}$ and $A^* = \overline{A}^T$.

Definition. When $f : M \to M$ is a diffeomorphisn and $X \in \Gamma(M, TM)$, we say that X is **invariant** under f when $f_*X = X$ (where $f_*(X_p) = X_{f(p)}$ for all $p \in M$). When G is a Lie group and $X \in \Gamma(G, TG)$, we say that X is **left-invariant** when X is invariant under the left multiplication map $\ell_a : G \to G$ where $\ell_a(p) = ap$ for all $a \in G$.

Note that $(\ell_a)_*(X) = X$ for all $a \in G$.

On the other hand, if we define a vector field X on G by the formula $X_a = (\ell_a)_*A$ where $A \in T_eG$, then X is left invariant since for all $a, b \in G$,

$$(\ell_a)_*X_b = ((\ell_a)_* \circ (\ell_b)_*)(X_e) = X_{ab}$$

Definition. A **Lie algebra** is a vector space V with an alternating bilinear map $[,] : V \times V \to V$ which satisfies the Jacobi identity $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$.

Example. $M_n(\mathbb{R})$ is a Lie algebra using $[A,B] = AB - BA$, as one can verify directly. More generally, when V is a vector space, End V is a Lie algebra with Lie bracket $[A,B] = AB - BA$. For example, when M is a smooth manifold, $X(M) = \Gamma(M, TM)$ is a vector space with Lie bracket $[X,Y](f) = X(Y(f)) - Y(X(f))$.

Given $A \in T_eG$, there is a unique left invariant vector field X on G with $X_e = A$, and X is given by $X_p = (\ell_p)_*A$. By the assignment if X and Y are left-invariant vector fields on a Lie group G, then $[X,Y]$ is left invariant since $(\ell_a)_*[X,Y] = [(\ell_a)_*X, (\ell_a)_*Y] = [X,Y]$.

Definition. For a Lie group G, the **Lie algebra** of G, denoted by $\mathfrak{g}$, is the Lie algebra of left-invariant vector fields on G.

Equivalently, we may define $\mathfrak{g} = T_eG$ with the corresponding Lie algebra given by $[A,B] = [X,Y]_e$, where $A, B \in T_eG = \mathfrak{g}$, and X, Y are the left invariant vector fields on G with $X_e = A$ and $Y_e = B$.

Definition. A **Lie subgroup** of a Lie group G is a subgroup $H \subseteq G$ that is also an immersed (or embedded) submanifold.

Let G be a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. We identify $T_p\mathrm{GL}_n(\mathbb{R})$ with $M_n(\mathbb{R})$, and we identify T_pG with a subspace of $M_n(\mathbb{R})$.

Example.

1. Given $A \in T_IG \subseteq M_n(\mathbb{R})$, find a formula for $U_p = U(P)$, where $P \in G \subseteq M_n(\mathbb{R})$ and U is the left-invariant vector field on G with $U_i = A$.
   We have $U_p = (L_P)_*A$, where $L_P : G \to G$ is given by $L_P(X) = PX$. Note that L_P is the restriction of the map $L_P : M_n(\mathbb{R}) \to M_n(\mathbb{R})$. This map L_p is linear, so DL_P is equal to L_P as a linear map on $M_n(\mathbb{R})$. Thus we have $U_P = (P_P)_*(A) = (DL_P)A = PA$.
2. Given $A, B \in \mathfrak{G} = T_IG \subseteq M_n(\mathbb{R})$, let U and V be given by $U(P) = PA$ and $V(P) = PB$. Note that $U = R_A$, $V = R_B$, so $DU = R_A$ and $DV = R_B$ as inear maps on $M_n(\mathbb{R})$, and we have
$$[A,B] = [U,V]_I = DV(I)U(I) - DU(I)V(I) = R_B(A) - R_A(B) = AB - BA$$
3. Let $A \in \mathfrak{G} = T_I(G) \subseteq M_n(\mathbb{R})$, let $U(P) = PA$. We need to find the integral curve $\alpha : I \subseteq \mathbb{R} \to G$ with $\alpha(0) = I$. Then we want $\alpha'(t) = U(\alpha(t)) = \alpha(t)A$ for all t. The solution to this DE is given by $\alpha(t) = e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \cdots$ so that $\alpha'(t) = (e^{tA})A$. As a consequence of the above formula, note that $\mathfrak{g} = \{A \in M_n(\mathbb{R}) : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$.

Thus formula allows us to give an explicit description of the Lie algebras of many Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$.

Given $A \in M_n(\mathbb{R})$, $\det e^A = e^{\operatorname{tr} A}$. By Schur's Theorem or the Jordan Normal Form, there is a matrix $P \in \mathrm{GL}_n(\mathbb{C})$ so that $P^{-1}AP = T$ where T is upper triangular, so that

$$\det e^A = \det(Pe^T P^{-1}) = \det e^T = e^{\operatorname{tr} A}$$

Recall when G is a Lie subgroup of $\mathrm{GL}_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ and if $J = T_I G \subseteq T_I \mathrm{GL}_n(\mathbb{R})$, the left invariant vctor field U on G with $U(I) = A \in J$ is given by $U(P) = PA$. The Lie bracket on J is given by $[A,B] = AB - BA$, and the integral curve of $U(P) = PA$ is given by $\alpha : \mathbb{R} \to G$ where $\alpha(t) = e^{tA}$, and hence

$$J = \{A \in M_n(\mathbb{R}) : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$$

For example, the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$ is

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : e^{tA} \in \mathrm{SL}_n(\mathbb{R}) \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \det e^{tA} = 1 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : e^{\operatorname{tr} tA} = 1 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \operatorname{tr}(tA) = 0 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \operatorname{tr}(A) = 0\}\end{aligned}$$

The Lie algebra of $O_n(\mathbb{R})$ is

$$\begin{aligned}\mathfrak{o}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : e^{tA} \in O_n(\mathbb{R}) \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : (e^{tA})^T (e^{tA}) = I \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : (e^{tA^T})(e^{tA}) = I \forall t\}\end{aligned}$$

If $(e^{tA^T})(e^{tA}) = I$ for all $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}t}(e^{tA^T})(e^{tA}) = \frac{\mathrm{d}}{\mathrm{d}t} I$ so that

$$(e^{tA^T} A^T (e^{tA}) + (e^{tA^T})(e^{tA}) \cdot A = 0$$

and taking $t = 0$ gives $A^T + A = 0$. Then $A^T = -A$ so $tA^T = -tA$ so $e^{tA^T} = e^{-tA} = (e^{tA})^{-1}$ for all t, so $e^{tA^T} \cdot e^{tA} = I$ for all t. Thus $\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$.

Table of Lie algebras:

G	$\mathfrak{g}$
$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$\mathrm{GL}_n^+(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A > 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A = 1\}$	$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \operatorname{tr} A = 0\}$
$\emptyset_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I\}$	$\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$
$\mathrm{SO}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I, \det A = 1\}$	$\mathfrak{so}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$
$\mathrm{GL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{C})$
$\mathrm{SL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$	$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{C}) : \operatorname{tr} A = 0\}$
$O_n(\mathbb{C}) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^T A = I\}$	$\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{C}) : A + A^T = 0\}$
$\mathrm{SO}_n(\mathbb{C}) = \{A \in \mathrm{SL}_n(\mathbb{C}) : A^T A = I, \det A = 1\}$	$\mathfrak{so}_n(\mathbb{C}) = \{A \in M_n(\mathbb{R}) : \operatorname{tr} A = 0, A + A^T = 0\}$
$U_n(\mathbb{C}) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^* A = 1\}$	$\mathfrak{so}_n(\mathbb{R}) = \{A \in M_n(\mathbb{C}) : A^* + A = 0\}$
$SU_n(\mathbb{C}) = \{A \in \mathrm{SL}_n(\mathbb{C}) : A^* A = 1\}$	$\mathfrak{so}_n(\mathbb{R}) = \{A \in M_n(\mathbb{C}) : \operatorname{tr} A = 0, A^* + A = 0\}$

5 Smooth k–forms

Suppose $\alpha : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^3$ and let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$, then the length of α is

$$\int_C dL = \int_\alpha dL = \int_{t \in I} |\alpha'(t)| dt$$

and

$$\int_C f dL = \int_\alpha f dL = \int_{t \in I} f(\alpha(t)) |\alpha'(t)| dt$$

Given $\sigma : R \subseteq \mathbb{R}^2 \to U \subseteq \mathbb{R}^3$, $f : U \subseteq \mathbb{R}^3 \to R$, the area of $\operatorname{im} \sigma$ is given by

$$\sigma(s,t) = (x(s,t), y(s,t), z(s,t))$$
$$D\sigma = \begin{pmatrix} \frac{\partial}{\partial s} x(s,t) & \frac{\partial}{\partial t} x(s,y) \\ \frac{\partial}{\partial s} y(s,t) & \frac{\partial}{\partial t} y(s,y) \\ \frac{\partial}{\partial s} z(s,t) & \frac{\partial}{\partial t} z(s,y) \end{pmatrix}$$

and denote σ_s, σ_t as the respective columns, so

$$A = \int_S dA = \int_\sigma dA = \iint_{(s,t) \in R} |\sigma_s(s,t) \times \sigma_t(st)| ds dt$$

and

$$\int_S f dA = \int_\sigma f dA = \iint_{(s,t) \in R} f(\sigma(s,t)) |\sigma_s \times \sigma_t| ds dt$$

For $\alpha : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^3$, $F : U \to \mathbb{R}^3$, say $F = (P, Q, R)$, then

$$\begin{aligned} W &= \int_C F \cdot T dL = \int_\alpha F \cdot T dL \\ &= \int_{t \in I} F(\alpha(t)) \cdot \frac{\alpha'(t)}{|\alpha'(t)|} |\alpha'(t)| dt \\ &= \int_{t \in I} (P(\alpha(t)) x'(t) + Q(\alpha(t)) y'(t) + R(\alpha(t)) z'(t)) dt \\ &= \int_\alpha P dx + Q dy + R dz \end{aligned}$$

Definition. A **smooth** k–**form** in $U \subseteq \mathbb{R}^n$ is an expression of the form $a(x) = \sum_I a_I(x) dx^I$ where the sum is taken over multi-indices $I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq n$ and each $a_I : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.
For a smooth map $\sigma : R \subseteq \mathbb{R}^k \to \mathbb{R}^n$ and $s = \operatorname{im} \sigma$ and for $a(x) = \sum_I a_I(x) dx^I$, we define

$$\int_S a = \int_\sigma a := \sum_I \int_R a_I(\sigma(t)) \det\left(\frac{\partial x^I}{\partial t}\right) dt^{i_1} \cdots t^k$$

where

$$\frac{\partial x^I}{\partial x} = \begin{pmatrix} \frac{\partial x^{i_1}}{\partial t^1} & \cdots & \frac{\partial x^1}{\partial k} \\ \vdots & & \vdots \\ \frac{\partial x^{i_k}}{\partial t^1} & \cdots & \frac{\partial x^{i_k}}{\partial t^k} \end{pmatrix}$$

For $a(x) = \sum_I a_I(x)dx$, we define $da = \sum_I \sum_j \frac{\partial a_I}{\partial x_j} dx^j \wedge dx^I$, using the rule $dx^j \wedge dx^i = -dx^i \wedge dx^j$. With this notation, Gauss' Theorem and Stoke's Theorem become

$$\int_S d\alpha = \int_{\delta S} \alpha$$

where $S = \operatorname{im}\sigma$, $\sigma : R \subseteq R^{k+1} \to \mathbb{R}^n$, $\alpha = \sum a_I dx^I$ is a k–form, and $d\alpha$ is a $(k+1)$–form.

The Exterior Algebra

If V is a vector space with basis $\{u_1, \ldots, u_n\}$, then the dual space $V^* = \{\text{linear maps } g : V \to \mathbb{R}\}$ has dual basis $\{f^1, \ldots, f^k\}$ where each $f^k : V \to \mathbb{R}$ and $f^k(u_\ell) = \delta^k_\ell$.

We have a canonical evaluation map $E : V \to V^{**}$ given by $E(v)(g) = g(v)$, which is an isomorphism.

The space $\Lambda^k V = \{\text{alternating } k\text{–linear maps } L : V^* \times \cdots \times V^* \to \mathbb{R}\}$ has a basis

$$\{U_i = U_{i_1} \wedge \cdots \wedge U_{i_k} : I \text{is an increasing multi-index}\}$$

where for $v^i \in V$ and $g^i \in V^*$,

$$(v_1 \wedge \cdots \wedge v_k)(g^1, \ldots, v^k) = \det \begin{pmatrix} g^1(v_1) & \cdots & g^1(v_k) \\ \vdots & & \vdots \\ g^k(v_1) & \cdots & g^k(v_k) \end{pmatrix}$$

Also $\Lambda^k V^*$ has basis given similarly.

$\mathbb{R}^n$ has standard basis $\{e_1, \ldots, e_n\}$, which we can consider as differential operators $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$. The dual basis for $(\mathbb{R}^n)^*$ is denoted by $\{dx^1, \ldots, dx^n\}$ where $dx^k\left(\frac{\partial}{\partial x^\ell}\right) = \delta^k_\ell$. So for example,

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \begin{cases} (-1)^\sigma & : J = \sigma(I) \\ 0 & : \text{otherwise} \end{cases}$$

A smooth k–form on $U \subseteq \mathbb{R}^n$ is a smooth map $\alpha : U \subseteq \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)^*$. Note that $dx^I : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ has

$$\begin{aligned} dx^I(u_1, \ldots, u_k) &= dx^I\left(\sum_{j_1=1}^n u_1^{j_1} \frac{\partial}{\partial x^{j_1}}, \ldots, \sum_{j_k=1}^n u_k^{j_j} \frac{\partial}{\partial x^{j_k}}\right) \\ &= \sum_{\text{all } J} u_1^{j_1} \cdots u_k^{j_k} \underbrace{dx^I\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right)}_{(-1)^\sigma \text{ if } J=\sigma(I); 0 \text{ otherwise}} \\ &= \sum_{\sigma \in S_n} (-1)^\sigma u_1^{i_{\sigma(1)}} \cdots u_k^{i_{\sigma(k)}} \\ &= \det(A^I) \end{aligned}$$

where A^I consists of the rows $i_1, \ldots, i_k$ of the matrix $A = (u_1, \ldots, u_k)$.

k-forms at a point on a Manifold

Let M be a smooth manifold and fix a point $p \in M$. Given a chart ϕ on M at p, T_pM has a basis $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$. We denote the ual basis for $T_p^*M = (T_pM)^*$ by $\{dx^1, \dots, dx^n\}$. Then $\Lambda^k T_p^* M$ has basis $\{dx^I : I \text{ increasing}\}$ (it is $\binom{n}{k}$ dimensional). If $X_j = \sum_{i=1}^n u_j^i \frac{\partial}{\partial x^i} \in T_pM$, then

$$dx^I(X_1, \dots, X_k) = \det \begin{pmatrix} u_1^{i_1} & \cdots & u_k^{i_1} \\ \vdots & & \vdots \\ u_1^{i_k} & \cdots & u_k^{i_k} \end{pmatrix}$$

An element $\alpha \in \Lambda^k T_p^* M$ can be written uniquely as $\alpha = \sum_{I \text{ increasing}} a_I dx^I$ with $A_I \in \mathbb{R}$, and α is called a k–form on M at p.

Change of Coordinates

Suppose that ϕ and ψ are two charts on M at p, so that T_pM has bases $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ and $\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\}$, and T_p^*M has dual bases $\{dx^1, \dots, dx^n\}$ and $\{dy^1, \dots, dy^n\}$ with corresponding bases for $\Lambda^k T_p^* M$. Let $\alpha \in \Lambda^k T_pM$. Say $\alpha = \sum_I a_I dx^I = \sum_J b_J dx^J$. If $X = \sum_i u^i \frac{\partial}{\partial x^i} = \sum_j v^j \frac{\partial}{\partial y^j}$, then $v = D(\psi\phi^{-1})$ is

$$\begin{aligned} v^j &= \sum_i \frac{\partial y^j}{\partial x^i} u^i \\ \frac{\partial}{\partial x^i} &= \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \end{aligned}$$

where $y(x) = \psi \circ \phi^{-1}(x)$. Then for each increasing multi-index I,

$$\begin{aligned} a_I &= \alpha\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right) \\ &= \left(\sum_J b_J dy^J\right)\left(\sum_{\ell_1} \frac{\partial y^{\ell_1}}{\partial x^{\ell_1}} \frac{\partial}{\partial y^{\ell_1}} \dots, \sum_{\ell_k} \frac{\partial y^{\ell_k}}{\partial x^{\ell_k}} \frac{\partial}{\partial y^{\ell_k}}\right) \\ &= \sum_{\text{incr } J} \sum_{\text{all } L} b_J \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\ell_k}}{\partial x^{i_k}} dy^J \left(\frac{\partial}{\partial y^{\ell_1}}, \dots, \frac{\partial}{\partial y^{\ell_k}}\right) \\ &= \sum_{\text{incr } J} \sum_{\sigma \in S_n} (-1)^\sigma b_J \frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_{\sigma(k)}}}{\partial x^{i_k}} \\ &= \sum_{\text{incr J}} b_J \det\left(\frac{\partial y^J}{\partial x^I}\right) \end{aligned}$$

where

$$\frac{\partial y^J}{\partial x^I} = \begin{pmatrix} \frac{\partial y^{j_1}}{\partial x^{i_1}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{i_1}} \\ \vdots & & \vdots \\ \frac{\partial y^{j_1}}{\partial x^{i_k}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{i_k}} \end{pmatrix}$$

When $\alpha = \sum_I a_I dx^I = \sum_J b_J dy^J$,

$$a_I = \sum_J b_J \det\left(\frac{\partial y^J}{\partial x^I}\right)$$

For an increasing multi-index L, taking $b_K = 1$ and $b_J = 0$ for $J \leq L$, we obtain

$$dy^L = \sum_I a_I dx^I, a_I = 1 \cdot \det\left(\frac{\partial y^L}{\partial x^I}\right)$$

so

$$dy^L = \sum_I \det\left(\frac{\partial y^L}{\partial x^I}\right) dx^I$$

The Wedge Product or Exterior Product

When ϕ is a chart on M at p and $\alpha = \sum_I a_I dx^I \in \Lambda^k T_p^* M$ and $\beta = \sum_J b_J dx^J \in \Lambda^\ell T_p^* M$, we would like to define $\alpha \wedge \beta \in \Lambda^{k+\ell} T_p^* M$ by $\alpha \wedge \beta = \sum_{I,J} a_I b_J dx^I \wedge dx^J$ (where we can use the rule $dx^i \wedge dx^j = -dx^j \wedge dx^i$ to put the multi-index in increasing order). We need to make sure that the definition does not depend on the choice of the chart ϕ. Note that for vectors $X_1, \ldots, X_{k+\ell} \in T_p M$, given (in the chart ϕ) by $X_j = \sum_{i=1}^n u_j^i \frac{\partial}{\partial x^i}$ we have

$$\begin{aligned}
(dx^I \wedge dx^J)(X_1, \ldots, X_{k+\ell}) &= \det \begin{pmatrix} u_1^{i_1} & \cdots & u^{i_1} u_{k+\ell} \\ \vdots & & \vdots \\ u_1^{i_k} & \cdots & u^{i_k} u_{k+\ell} \\ u_j^{i_1} & \cdots & u^{j_1} u_{k+\ell} \\ \vdots & & \vdots \\ u_\ell^{i_1} & \cdots & u^{j_\ell} u_{k+\ell} \end{pmatrix} \\
&= \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma u^{i_1} u_{\sigma(1)} \cdots u_{\sigma(k)}^{i_k} u_{\sigma(k+1)}^{j_1} \cdots u_{\sigma(k+\ell)}^{j_\ell} \\
&= \sum_\tau \sum_\mu \sum_\nu (-1)^\tau (-1)^\mu (-1)^\nu \mu_{\mu(\tau(1))}^{i_1} \cdots u_{\mu(\tau(k))}^{i_k} u_{\nu(\tau(k+1))}^{j_1} \cdots u_{\nu(\tau(k+1))}^{j_\ell}
\end{aligned}$$

where the sums are over τ a permutation of $\{1, \ldots, k+\ell\}$ so that $\tau(1) < \cdots < \tau(k)$ and $\tau(k+1) < \cdots, \tau(k+\ell)$, μ is a permutation of $\{(\tau(1), \ldots, \tau(k)\}$ and ν is a permutation of $\{\tau(k+1), \ldots, \tau(k+\ell)\}$, so that

$$\begin{aligned}
&= \sum_\tau (-1)^\tau \det \begin{pmatrix} u_{\tau(1)}^{i_1} & \cdots & u_{\tau(1)}^{i_k} \\ \vdots & & \vdots \\ u_{\tau(k)}^{i_1} & \cdots & u_{\tau(k)}^{i_k} \end{pmatrix} \det \begin{pmatrix} u_{\tau(k+1)}^{j_1} & \cdots & u_{\tau(k+1)}^{i_\ell} \\ \vdots & & \vdots \\ u_{\tau(k+\ell)}^{i_1} & \cdots & u_{\tau(k+\ell)}^{i_\ell} \end{pmatrix} \\
&= \sum_\tau (-1)^\tau dx^I (X_{\tau(1)}, \ldots, X_{\tau(k)}) dx^J (X_{\tau(k+1)}, \ldots, X_{\tau(k+\ell)})
\end{aligned}$$

Thus for $\alpha = \sum a_I dx^I$, $\beta = \sum b_J dx^J$, $\gamma = \sum_{I,J} a_I b_J dx^I \wedge dx^J$ we have

$$\gamma(X_1, \ldots, X_{k+\ell}) = \sum_{\tau \in T_{k,\ell}} (-1)^\tau \alpha(X_{\tau(1)}, \ldots, X_{\tau(k)}) \cdot \beta(X_{\tau(k+1)}, \ldots, X_{\tau(k+\ell)})$$

where $T_{k,l}$ is the set of permutations τ of $\{1,\ldots,k+\ell\}$ such that $\tau(1) < \cdots < \tau(k)$ and $\tau(k+1) < \cdots \tau(k+\ell)$.

Definition. When $f : M \to N$ is a smooth map of smooth manifolds and $p \in M$, we define the **pullback**

$$f^* = f^*(p) : \Lambda^k T^*_{f(p)} N \to \Lambda^k T_p M$$

by $f^*(\beta)(X_1,\ldots,X_k) = \beta(f_* X_1,\ldots,f_* X_k)$ where $\beta \in \Lambda^k T^*_{f(p)} N$ and each $X_j \in T_p M$ so that $f_* X_j \in T_{f(p)} N$.

Let M be a chart on M at p and ψ a chart on N at $f(p)$. Let $\beta = \sum b_J dx^J$, write $X_j = \sum_i u^j_j \frac{\partial}{\partial x^i}$ and say $\alpha = f^*\beta = \sum_I a_I dx^I$.

$$\begin{aligned}
a_I &= \alpha\left(\frac{\partial}{\partial x^{i_1}},\ldots,\frac{\partial}{\partial x^{i_k}}\right) \\
&= \beta\left(f_*\frac{\partial}{\partial x^{i_1}},\ldots,f_*\frac{\partial}{\partial x^{i_k}}\right) \\
&= \left(\sum_J b_J dy^J\right)\left(\sum_{\ell_1}\frac{\partial y^{\ell_1}}{\partial x^{i_1}},\ldots,\sum_{\ell_k}\frac{\partial y^{\ell_k}}{\partial x^{i_k}}\right) \\
&= \sum_{J\text{ incr}}\sum_{\text{all } L} B)J \frac{\partial y^{\ell_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{\ell_k}}{\partial x^{i_k}} dy^J\left(\frac{\partial}{\partial y^{\ell_1}},\frac{\partial}{\partial y^{\ell_k}}\right) \\
&= \sum_{J\text{ incr}}\sum_{\sigma\in S_k}(-1)^\sigma b_J \frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}}\cdots\frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}} \\
&= \sum_J b_J \det\left(\frac{\partial y^J}{\partial x^I}\right)
\end{aligned}$$

where $y = y(x) = (\psi f \phi^{-1})(x)$.

Definition. A k–form at each point p on a smooth manifold M is given by a map $\alpha : M \to \dot{\bigcup}_{p\in M}\Lambda^k T^*_p M$, where $\alpha(p) \in \Lambda^k T^*_p M$ for all $p \in M$. We say that sucha map α is **smooth** at $p \in M$ when for some (hence for every) chart $\phi : U \subseteq M \to \phi(U) \subseteq \mathbb{R}^m$ on M at p, when we write the restriction of α to U as $\alpha(p) = \sum_I \alpha_I(p) dx^I$, the coefficient functions $\alpha_I : U \subseteq M \to \mathbb{R}$ are smooth. Such a map $\alpha : M \to \dot{\bigcup}_{p\in M}\Lambda^k T^*_p M$ is called smooth (on M) when it is smooth at every point $p \in M$.

Another way to think about smooth k–forms is as follows. Consider $\alpha : M \to \dot{\bigcup}_{p\in M}\Lambda^k T^*_p M$ with $\alpha(p) \in \Lambda^k T^*_p M$ for all $p \in M$. Let $\Lambda^k T^* M = \dot{\bigcup}_{p\in M}\Lambda^* T^*_p M$ and define the **projection map** $\pi : \Lambda^k T^* M \to M$ by $\pi(\alpha_p) = p$, when $\alpha_p \Lambda^k T^*_p M$. We give $\Lambda^k T^* M$ the structure of a smooth vector bundle of rank $\binom{n}{k}$ on M as follows. For each chart $\phi : U \to \phi(U)$ on M, we define a chart

$$\Phi : \pi^{-1}(U) = \dot{\bigcup_{p\in U}}\Lambda^j T^*_p M \to \phi(U) \times \Lambda^k(\mathbb{R}^n)^* \equiv \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$

by $\Phi(\alpha_p) = (\phi(p), \sum_I a_I dx^I)$, where the restriction of α to U is given by $\alpha(p) = \sum_I a_I(\phi(p)) dx^I$.

With this definition, a smooth k–form on M is a smooth map $\alpha : M \to \Lambda^k T^* M$ such that $\pi(\alpha(p)) = p$. We denote the space of all k–forms on M by $\Omega^k(M)$ or $\Gamma(M, \Lambda^k T^* M)$ (sections) or $\Gamma(\Lambda^k T^* M)$.