

# PMATH 465

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# I. Fundamentals of Manifolds

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## 1 INTRODUCTION TO TOPOLOGY

### BASIC CONSTRUCTIONS

**Definition.** A **topology** on a set  $X$  is a set  $\tau$  of subsets of  $X$  such that

- (i)  $\emptyset \in \tau$  and  $X \in \tau$
- (ii) If  $U_\alpha \in \tau$  for all  $\alpha \in A$ , then  $\bigcup_{\alpha \in A} U_\alpha \in \tau$ .
- (iii) If  $n \in \mathbb{N}$  and  $U_i \in \tau$  for each  $1 \leq i \leq n$ , then  $\bigcap_{i=1}^n U_i \in \tau$ .

The sets  $U \in \tau$  are called the **open sets** in  $X$ , and sets of the form  $X \setminus U$  for some open set  $U$  are called the **closed sets** in  $X$ .

**Definition.** When  $X$  is a topological space and  $A \subseteq X$ , the **interior** of  $A$  (denoted  $A^\circ$ ) is the union of all open sets contained in  $A$ . Similarly, we define the **closure** of  $A$  (denoted  $\bar{A}$ ) as the intersection of all closed sets containing  $A$ . Then the **boundary** of  $A$ , denoted by  $\partial A$ , is the set  $\partial A = \bar{A} \setminus A^\circ$ .

*Example.* Let  $X$  be any set. The **discrete topology** on  $X$  is the topology  $\tau = \mathcal{P}(X)$ , and the **trivial topology** on  $X$  is the topology  $\tau = \{\emptyset, X\}$ .

**Definition.** A **basis** for a topology on a set  $X$  is a set  $\mathcal{B}$  of subsets of  $X$

- (i)  $\bigcup_{B \in \mathcal{B}} B = X$
- (ii) for all  $a \in X$  and  $U, V \in \mathcal{B}$  such that  $a \in U \cap V$ , then there exists  $W \in \mathcal{B}$  with  $a \in W \subseteq U \cap V$ .

When  $\mathcal{B}$  is a basis for a topology on  $X$ , the topology on  $X$  **generated** by  $\mathcal{B}$  is the set  $\tau$  of subsets of  $X$  such that for  $W \subseteq X$ ,  $W \in \tau$  if and only if for all  $a \in W$ , there exists  $U \in \mathcal{B}$  such that  $a \in U \subseteq W$ .

Note that  $\tau$ , as above, is a topology on  $X$  since

- (i)  $\emptyset \in \tau$  vacuously and  $X \in \tau$  obviously.
- (ii) If  $A_k \in \tau$  for all  $k \in K$  (where  $K$  is any set of indices), then given  $a \in \bigcup_{k \in K} A_k$ , we can choose  $\ell \in K$  so that  $a \in A_\ell$ . Then since  $A_\ell \in \tau$ , we can choose  $U_\ell \in \mathcal{B}$  so that  $a \in U_\ell \subseteq A_\ell$ . Thus  $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$ .
- (iii) By induction, it suffices to prove that if  $A, B \in \tau$ , then  $A \cap B \in \tau$ . Suppose  $A, B \in \tau$ , and let  $a \in A \cap B$ . Since  $A \in \tau$ , we can choose  $U \in \mathcal{B}$  so that  $a \in U \subseteq A$ . Since  $B \in \tau$ , we can choose  $V \in \mathcal{B}$  so that  $a \in V \subseteq B$ . Then we have  $a \in U \cap V$ . Since  $\mathcal{B}$  is a basis, we can choose  $W \in \mathcal{B}$  with  $a \in W \subseteq U \cap V$ , so  $a \in W \subseteq U \cap V \subseteq A \cap B$ .

Note that when  $\tau$  is the topology on  $X$  generated by the basis  $\mathcal{B}$ , for  $A \subseteq X$ ,  $A \in \tau$  if and only if there exists some  $S \subseteq \mathcal{B}$  such that  $A = \bigcup_{s \in S} s$ . In this sense, the topology  $\tau$  on  $X$  generated by the basis  $\mathcal{B}$  is the coarsest topology which contains  $\mathcal{B}$ .

**Definition. (Subspace Topology)** When  $Y$  is a topological space and  $X \subseteq Y$  is a subset of  $Y$ , we define the **subspace topology** on  $X$  to be the topology for which a set  $U \subseteq X$  is open if and only if  $U = X \cap V$  for some open set  $V$ .

If  $\mathcal{C}$  is a basis for the topology on  $Y$ , then  $\mathcal{B} = \{X \cap V \mid V \in \mathcal{C}\}$  is a basis for the subspace topology on  $X$ .

**Definition. (Disjoint Union Topology)** If  $X$  and  $Y$  are topological spaces with  $X \cap Y = \emptyset$ , then the **disjoint union topology** on  $X \cup Y$  is the topology in which a subset  $U \subseteq X \cup Y$  is open in  $X \cup Y$  if and only if  $U \cap X$  is open in  $X$  and  $U \cap Y$  is open in  $Y$ .

**Definition. (Product Topology)** If  $X$  and  $Y$  are topological spaces, the **product topology** on  $X \times Y$  is the topology generated by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are bases for the topologies on  $X, Y$  respectively.

**Definition. (Infinite Product Topology)** We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \rightarrow \bigcup_{k \in K} X_k \mid f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on  $X$ . The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

*Example. (Metric Topology)*  $\mathbb{R}^n$  has a standard **inner product**, and for  $u, v \in \mathbb{R}^n$ ,  $\langle u, v \rangle = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$ . This gives the standard norm on  $\mathbb{R}^n$  for  $u \in \mathbb{R}^n$ ,  $\|u\| = \sqrt{\langle u, u \rangle}$ . This gives the standard metric on  $\mathbb{R}^n$ : for  $a, b \in \mathbb{R}^n$ ,  $d(a, b) = \|b - a\|$ .

Given a metric on a set  $Y$ , we obtain (by restriction) an induced metric on any subset  $X \subseteq Y$ . Given a metric space  $X$ , we define the **metric topology** on  $X$  to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where  $x \in X, r > 0$ .

## MAPS ON TOPOLOGICAL SPACES

**Definition.** When  $X$  and  $Y$  are topological spaces and  $f : X \rightarrow Y$ , we say that  $f$  is **continuous** when it has the property that  $f^{-1}(V)$  is open in  $X$  for every open set  $V$  in  $Y$ . We say that  $f : X \rightarrow Y$  is a **homeomorphism** when  $f$  is bijective and both  $f$  and  $f^{-1}$  are continuous. Then  $X, Y$  are **homeomorphic** if there exists a homeomorphism  $f : X \rightarrow Y$ .

**1.1 Theorem. (Glueing Lemma)** Let  $X$  and  $Y$  be topological spaces, and let  $f : X \rightarrow Y$  be a function. Suppose either

(i)  $X = \bigcup_{k \in K} A_k$  where each  $A_k$  is open in  $X$ , or

(ii)  $X = \bigcup_{k=1}^n A_k$  where each  $A_k$  is closed in  $X$

and each restriction map  $f_k : A_k \rightarrow Y$  is continuous, then  $f$  is continuous.

PROOF Exercise. ■

**Definition.** A topological space  $X$  is **compact** when it has the property that for every set  $\mathcal{S}$  of open subsets of  $X$  with  $X = \bigcup_{U \in \mathcal{S}} U$ , there exists a finite subset  $\mathcal{F} \subseteq \mathcal{S}$  such that  $X = \bigcup_{F \in \mathcal{F}} F$ .

Note that when  $X \subseteq Y$  is a subspace,  $X$  is compact if and only if  $X$  has the property that for every set  $\mathcal{T}$  with  $X \subseteq \bigcup_{T \in \mathcal{T}} T$ , there exists a finite subset  $\mathcal{G} \subseteq \mathcal{T}$  such that  $X \subseteq \bigcup_{G \in \mathcal{G}} G$ .

**Definition.** A topological space  $X$  is **connected** when there do not exist non-empty disjoint open sets  $U, V \subseteq X$  such that  $X = U \cup V$ .

Note that if  $Y$  is a metric space and  $X \subseteq Y$  is a subspace, then  $X$  is connected if and only if there do not exist open sets  $U, V \subseteq Y$  such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cup V$$

**Definition.** A topological space  $X$  is called **path connected** when it has the property that for all  $a, b \in X$ , there exists a continuous map  $\alpha : [0, 1] \rightarrow X$  with  $\alpha(0) = a$  and  $\alpha(1) = b$ .

It is easy to see that if  $X$  is path connected, then  $X$  is connected.

**Definition.** Let  $X$  be a topological space. If we define a relation  $\sim$  on  $X$  by taking  $a \sim b$  if and only if there exists a connected subspace  $A \subseteq X$  with  $a \in A$  and  $b \in A$ .

It is clear that this is an equivalence relation. Note that when  $X$  is a topological space, its connected components are connected, and each connected subspace of  $X$  is contained in one of its connected components.

**Definition.** Let  $X$  be a topological space. Define a relation  $\approx$  on  $X$  by  $a \approx b$  if and only if there exists a continuous map  $\alpha : [0, 1] \rightarrow X$  with  $\alpha(0) = a$  and  $\alpha(1) = b$ . Such a map  $\alpha$  is called a **continuous path**.

One can show that if  $X$  is **locally path connected** (which means that  $X$  has a basis for its topology which consists of path connected sets), then the path components of  $X$  are equal to the connected components of  $X$ , and that these components are open.

## QUOTIENT TOPOLOGY

**Definition. (Quotient Topology)** Let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . The set of equivalence classes is denoted  $X/\sim$ , and  $X/\sim$  is called the **quotient** of  $X$  by  $\sim$ . The map  $\pi : X \rightarrow X/\sim$  given by  $\pi(a) = [a]$  is called the **natural projection map** or **quotient map**. We define the **quotient topology** on  $X/\sim$  by stipulating that for  $W \subseteq X/\sim$ ,  $W$  is open in  $X/\sim$  if and only if  $\pi^{-1}(W)$  is open in  $X$ .

When a group  $G$  acts on a topological space  $X$ , we define an equivalence relation  $\sim$  on  $X$  by  $a \sim b$  if and only if  $b = g \cdot a$  for some  $g \in G$ . The equivalence classes are orbits. In this context, we also write  $X/\sim$  as  $X/G$ .

When  $X, Y$  are any topological spaces and  $\pi : X \rightarrow Y$  is surjective, we can define an equivalence relation  $\sim$  on  $X$  by  $a \sim b$  if and only if  $\pi(a) = \pi(b)$ . We then have a natural bijection from  $Y$  to  $X/\sim$  in which  $y \in Y$  corresponds to the fibre  $\pi^{-1}(y) \in X/\sim$ .

If  $Y$  has the topology such that for  $W \subseteq Y$ ,  $W$  is open in  $Y$  if and only if  $\pi^{-1}(W)$  is open in  $X$ . In this case, we also use the terminology “quotient map” for  $\pi$ .

**Remark.** Let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . Let  $Y$  be any set. If  $f : X \rightarrow Y$  is constant on the equivalence classes, then  $f$  induces a well-defined map  $\bar{f} : X/\sim \rightarrow Y$  given by define  $\bar{f}([a]) = f(a)$ .

*Example.* Define an equivalence class on  $[0, 1] \subseteq \mathbb{R}$  by  $s \sim t$  if and only if  $s = t$  or  $\{s, t\} = \{0, 1\}$ . Then  $[0, 1]/\sim \cong \mathbb{S}^1$ . Define  $f : [0, 1] \rightarrow \mathbb{S}^1$  by  $f(t) = e^{i2\pi t}$ . Note that  $f(0) = f(1)$ , so  $f$  induces a continuous map  $\bar{f} : [0, 1]/\sim \rightarrow \mathbb{S}^1$ . The inverse map can be constructed as follows. We define  $g : \mathbb{S}^1 \rightarrow [0, 1]/\sim$  by

$$g(x, y) = \begin{cases} \left[ \frac{1}{2\pi} \cos^{-1} x \right] & : y \geq 0 \\ \left[ 1 - \frac{1}{2\pi} \cos^{-1} x \right] & : y \leq 0 \end{cases}$$

Then  $g$  is continuous by the Glueing lemma.

In particular, the same proof shows that  $\mathbb{R}/\mathbb{Z}$  is homeomorphic to  $\mathbb{S}^1$ .

*Example.* The projective space  $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$  can be defined in several ways.  $\mathbb{P}^n$  is the set of all 1-dimensional vector subspaces of  $\mathbb{R}^{n+1}$ , or  $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$ , or  $\mathbb{P}^n = \mathbb{S}^n / \pm 1$  where  $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$ .

Let us show that  $\mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$  is homeomorphic to  $\mathbb{S}^n / \pm 1$ . Define  $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{S}^n$  by  $f(x) = x/|x|$ , and  $g = \pi \circ f$ . Then  $g$  is given by  $g(x) = \{\pm x/|x|\}$ . Note that for  $t \in \mathbb{R}^\times$ ,

$$g(tx) = \left[ \frac{t}{|t|} \cdot \frac{x}{|x|} \right] = \left[ \frac{x}{|x|} \right]$$

since  $t/|t| = \pm 1$ . Thus  $g$  induces a continuous map  $\bar{g}$  on the quotient. We construct the inverse map in a similar way.

**Definition.** Let  $X$  be a topological space. Then

- $X$  is **T1** when for all  $a, b \in X$  there exists an open set  $U$  in  $X$  with  $a \in U$  and  $b \notin U$
- $X$  is **T2** or **Hausdorff** when for all  $a, b \in X$ , there exist disjoint open sets  $U, V \subseteq X$  with  $a \in U$  and  $b \in V$
- $X$  is **T3** or **regular** when  $X$  is T1 and for every  $a \in X$  and every closed set  $B \subseteq X$  with  $a \notin B$ , there exist open sets  $U, V \subseteq X$  with  $a \in U$ ,  $B \subseteq V$ .
- $X$  is **T4** or **normal** when  $X$  is T1 and for all disjoint closed sets  $A, B \subseteq X$  there exist disjoint open sets  $U, V \subseteq X$  with  $A \subseteq U$  and  $B \subseteq V$ .

**Definition.** Let  $X$  be a topological space.

- $X$  is **first countable** when for every  $a \in X$ , there exists a countable set  $B_a$  of open sets in  $X$  which contain  $a$  such that for every open set  $W$  in  $X$  with  $a \in W$ , there exists  $U \in B_a$  with  $a \in U \subseteq W$ .
- $X$  is **second countable** when there exists a countable basis for the topology on  $X$ .

*Example.* (i)  $X$  is T1 if and only if every 1-point subset of  $X$  is closed in  $X$

(ii) Every compact Hausdorff space is regular.

(iii) Every second countable regular space is normal.

(iv) Every metric space is normal.

(v) If  $X$  is second countable, then every open cover admits a countable subcover.

(vi) Every second countable space  $X$  contains a countable dense subset.

**1.2 Lemma. (Urysohn)** If  $X$  is normal and  $A, B \subseteq X$  are disjoint and closed, then there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

**1.3 Theorem. (Tietze Extension)** If  $X$  is normal and  $f : A \rightarrow \mathbb{R}$  is continuous for some  $A \subseteq X$  closed, then there exists a continuous map  $F : X \rightarrow \mathbb{R}$  such that  $F|_A = f$  and  $\sup_{a \in A} |f(a)| = \sup_{x \in X} |F(x)|$ .



**1.4 Theorem. (Urysohn's Metrization)** If  $X$  is second countable and regular, then  $X$  is metrizable.

**Definition.** An  $n$ -dimensional topological manifold is a Hausdorff, second countable topological space  $M$  which is **locally homeomorphic** to  $\mathbb{R}^n$ , meaning for every  $p \in M$ , there exists an open set  $U \subseteq M$  with  $p \in U$  and an open set  $V \subseteq \mathbb{R}^n$  and a homeomorphism  $\phi : U \subseteq M \rightarrow V \subseteq \mathbb{R}^n$ . Such a homeomorphism  $\phi$  is called a **(local) coordinate chart** or **chart** on  $M$  at  $p$ . The domain  $U$  of a chart  $\phi : U \subseteq M \rightarrow \phi(U) \subseteq \mathbb{R}^n$  is called a (local) **coordinate neighbourhood** at  $p$ . Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \rightarrow \phi_k(U_k) : k \in K\}$$

where  $K$  is any non-empty set such that  $M = \bigcup_{k \in K} U_k$ . Such a set of charts is called an **atlas** for  $M$ .

**Definition.** Two charts are called  $\phi : U \rightarrow \phi(U)$  and  $\psi : V \rightarrow \psi(V)$  are called **(smoothly) compatible** when either  $U \cap V = \emptyset$  or  $\phi^{-1} \circ \psi$  and  $\psi \circ \phi^{-1}$  are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas  $\mathcal{A}$  on  $M$  can be extended to a unique maximal smooth atlas  $\mathcal{M}$  on  $M$  by adding to  $\mathcal{A}$  every possible homeomorphism  $\psi : U \subseteq M \rightarrow \psi(U) \subseteq \mathbb{R}^n$  which is compatible with all of the existing charts (since if  $\psi$  and  $\chi$  are both compatible with every chart  $\phi \in \mathcal{A}$ , then  $\psi$  and  $\chi$  will be compatible with each other). The maps  $\psi \circ \phi^{-1}$  are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas  $\mathcal{M}$  on  $M$  is called a **smooth structure** on  $M$ .

**Definition.** An  $n$ -dimensional **smooth (or  $C^\infty$ ) manifold** is an  $n$ -dimensional topological manifold with a smooth structure.

**Remark.** A topological manifold can have different smooth structures. For example, take  $\mathcal{A} = \{\phi\}$  where  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is the identity map, and  $\mathcal{B} = \{\psi\}$  where  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism given by  $\psi(x) = x^3$ , since  $\sqrt[3]{x}$  is not smooth at the origin.

What if we tried  $\mathcal{B} = \{\psi\}$  where  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism which is not  $C^\infty$ ? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

**Remark.** We can give a smooth manifold  $M$  an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \rightarrow B(0, 1)$
- $\phi : U \subseteq M \rightarrow (0, 1)^n$
- $\phi : U \subseteq M \rightarrow \mathbb{R}^n$

Note that the maximal atlas  $\mathcal{M}$  is determined from any subset  $\mathcal{A} \subset \mathcal{M}$  such that the domains of the charts in  $\mathcal{A}$  cover  $M$ .

**Definition.** Let  $M$  be an  $m$ -dimensional smooth manifold and  $N$  be an  $n$ -dimensional smooth manifold and let  $f : M \rightarrow N$  be a function. Then we say  $f$  is **smooth** at  $p$  when for some (hence for any) chart  $\phi$  on  $M$  at  $p$  and for some (hence any) chart  $\psi$  on  $N$  at  $f(p)$ , the map  $\psi^{-1} \circ f \circ \phi$  is smooth at  $x = \phi(p)$ , and  $f$  is **smooth** if  $f$  is smooth at every  $p \in M$ . We say that  $f$  is a **diffeomorphism** when  $f$  is invertible and both  $f$  and  $f^{-1}$  are smooth. We say that  $M$  and  $N$  are **diffeomorphic**, and write  $M \cong N$ , when there exists a diffeomorphism  $f : M \rightarrow N$ .

*Remark.* It is conceivable that a topological manifold  $M$  could be both of dimension  $n$  and of dimension  $m$  with  $n \neq m$ . To do this, we would need to have a homeomorphism from an open set in  $\mathbb{R}^n$  to an open set in  $\mathbb{R}^m$ . In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When  $M$  is smooth, it is easy to see that this cannot happen. If  $\psi \circ \phi^{-1}$  and  $\phi \circ \psi^{-1}$  were smooth inverses, then the matrices  $D(\psi \circ \phi^{-1})(\phi(p))$  and  $D(\phi \circ \psi^{-1})(\psi(p))$  would be inverse matrices. But then a product of a matrix in  $M_{m \times n}(\mathbb{R})$  and in  $M_{n \times m}(\mathbb{R})$  cannot be inverses when  $m \neq n$ .

*Remark.* Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.

There are other kinds of manifolds (other than  $C^\infty$  manifolds); for example, one can define  $C^k$  manifolds, or analytic  $C^\omega$  manifold has an atlas in which the transition maps are analytic.

- Example.*
1.  $\mathbb{R}^n$  is a smooth  $n$ -dimensional manifold. It can be given an atlas consisting of 1 chart, the identity map.
  2. Any  $n$ -dimensional vector space over  $\mathbb{R}$  is a smooth  $n$ -dimensional manifold. It can be given an atlas with one chart. If  $\{u_1, \dots, u_n\}$  is a basis for  $V$ , then one can define  $\phi : V \rightarrow \mathbb{R}^n$  by  $\phi(\sum t^i u_i) = (t^1, \dots, t^n) = t \in \mathbb{R}^n$ .
  3. Every open subset of a smooth  $n$ -dimensional manifold is also a smooth  $n$ -dimensional manifold.
  4.  $M_{n \times m}(\mathbb{R})$  is an  $n \cdot m$ -dimensional manifold with pointwise  $\mathbb{R}^{nm}$  structure.
  5.  $\{A \in M_{n \times m}(\mathbb{R}) : \text{rank}(A) = \min\{n, m\}\}$  is a smooth manifold with one chart, since it is an open submanifold of  $M_{n \times m}$ . Suppose  $n > m$ ; then take all  $n \times n$  submatrices which have non-zero determinant (open by continuity of  $\det$ ), and maximal rank means that  $A$  is contained in one of these open subsets.
  6. The disjoint union of countably many  $n$ -dimensional smooth manifolds.
  7. The cartesian product of finitely many smooth manifolds is a smooth manifold. Let  $\dim(M_k) = n_k$ , the  $\dim(M_1 \times \dots \times M_\ell) = \sum_{k=1}^\ell n_k$ . If  $\phi_k : U_k \subseteq M_k \rightarrow \phi_k(U_k) \subseteq \mathbb{R}^{n_k}$  is a chart on  $M_k$ , then  $\chi_k : \prod_{k=1}^\ell U_k \rightarrow \prod_{k=1}^\ell \mathbb{R}^{n_k}$  given by  $\chi_k(p_1, \dots, p_\ell) = (\phi_1(p), \dots, \phi_\ell(p))$  is a chart in  $M_1 \times \dots \times M_\ell$ .
  8. One can show that  $\mathbb{S}^n$  is a smooth  $n$ -dimensional manifold.

*Remark.* For  $A \in M_{n \times m}(\mathbb{R})$ , we denote the entry in the  $k^{\text{th}}$  row and  $\ell^{\text{th}}$  column by  $A_\ell^k$ .

*Example.*  $\mathbb{S}^n$  is an example of an  $n$ -dimensional smooth manifold. It can, for example, be given a smooth atlas which contains  $2(n+1)$  charts as follows. For  $1 \leq k \leq n+1$ , let

$$\begin{aligned} U_k &= \{x \in \mathbb{S}^n : x^k > 0\} \\ \phi_k : U_k &\rightarrow B(0, 1) \subseteq \mathbb{R}^n \\ \phi_k(x) &= (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1}) \\ \phi_k^{-1}(t^1, \dots, t^n) &= \left(t^1, \dots, t^{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} (t^i)^2}, t^k, \dots, t^n\right) \end{aligned}$$

and the corresponding opposite charts for  $x^k < 0$ . Note that  $\mathbb{S}^n$  is a metric space. It has 2 standard metrics: either the one inherited from  $\mathbb{R}^n$ , or the arclength distance  $d_s(u, v) = \cos^{-1}(u \cdot v)$ .

We can also give  $\mathbb{S}^n$  an atlas which only uses 2 charts, by stereographic projection from a north pole and a south pole.

This stereographic projection also shows that the rational points on the sphere are dense in  $\mathbb{S}^n$ , via the map

$$\phi(x) = \alpha\left(\frac{1}{1-x^{n+1}}\right) = \left(\frac{x^1}{1-x^{n+1}}, \dots, \frac{x^n}{1-x^{n+1}}\right)$$

One can also find  $\phi^{-1}$  and verify that they are both rational functions. In particular,  $\phi^{-1}(\mathbb{Q}^n) \subseteq \mathbb{S}^n$  is dense.

*Example.* The projective space  $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$  is commonly defined in at least 3 ways:

$$\begin{aligned}\mathbb{P}^n &= \{1\text{-dimensional subspaces of } \mathbb{R}^{n+1}\} \\ \mathbb{P}^n &= \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times = \{[x] : 0 \neq x \in \mathbb{R}^{n+1}\}, [x] = \{tx : t \in \mathbb{R}^\times\} \\ \mathbb{P}^n &= \mathbb{S}^n / \pm 1\end{aligned}$$

We can give  $\mathbb{P}^n$  a smooth atlas with  $n+1$  charts as follows: for  $1 \leq k \leq n+1$ , set

$$\begin{aligned}U_k &= \{[x] \in \mathbb{P}^n : x^k \neq 0\} \\ \phi_k : U_k &\rightarrow \mathbb{R}^n, \phi_k([x]) = \left(\frac{x^1}{x^k}, \dots, \frac{x^{k-1}}{x^k}, \frac{x^{k+1}}{x^k}, \dots, x^{n+1}x^k\right)\end{aligned}$$

with  $\phi_k^{-1}(t_1, \dots, t^n) = [(t_1, \dots, t^{k-1}, 1, t^k, \dots, t^n)]$ .

### EXAMPLES OF SMOOTH MAPS

- The inclusion  $f : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$  given by  $f(x) = x$
- The quotient map  $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$
- The exponential map  $f : \mathbb{R} \rightarrow \mathbb{S}^1$  given by  $f(t) = e^{i2\pi t}$ , or more generally  $f : \mathbb{R}^n \rightarrow \mathbb{T}^n$  given by  $f(t^1, \dots, t^n) = (e^{2\pi i t^1}, \dots, e^{2\pi i t^n})$
- The determinant map  $f : M_n(\mathbb{R}) \rightarrow \mathbb{R}$  given by  $f(A) = \det(A)$  is smooth
- For  $A \in M_n(\mathbb{R})$ , left and right multiplication by  $A$ , the transpose map, and the inverse map  $f(A) = A^{-1}$  are smooth.

### PARTITIONS OF UNITY

**1.5 Lemma.** *Every open cover of a manifold has an (at most) countable subcover.*

**PROOF** Let  $\mathcal{S}$  be any open cover of  $M$ , and let  $\mathcal{B}$  be a countable basis for the topology on  $M$ . For each  $p \in M$ , choose  $U_p \in \mathcal{S}$  with  $p \in U_p$ , then choose  $B_p \in \mathcal{B}$  with  $p \in B_p \subseteq U_p$ . Then  $\{B_p : p \in M\} \subseteq \mathcal{B}$  is an open cover of  $M$ , and it is a subset of  $\mathcal{B}$ , so it is (at most) countable; but then  $\{U_p : p \in M\}$  gives an at most countable subcover of  $\mathcal{S}$ . ■

As a result, every manifold has a countable basis  $\mathcal{B}$  such that for each  $B \in \mathcal{B}$ , there is a chart  $\phi : U \rightarrow \phi(U)$  on  $M$  with  $\phi(U) = B(0, 2)$  and  $\phi(B) = B(0, 1)$ .

**1.6 Lemma.** *Let  $M$  be a manifold, and let  $\mathcal{S}$  be any open cover of  $M$ . Then there exists an at most countable open cover  $\mathcal{B}$  of  $M$  such that*

1. *for each  $B \in \mathcal{B}$  there is a chart  $\phi_B : C_B \rightarrow \phi_B(C_B) = B(0, 1)$  with  $B \subseteq C_B \subseteq U_B \subseteq \mathcal{S}$  for some  $U_B \in \mathcal{S}$  and  $\phi_B(B) = B(0, 1)$ .*

2.  $\{C_B : B \in \mathcal{B}\}$  is locally finite, meaning that every point in  $M$  has an open neighbourhood which only intersects with finitely many of the sets  $C_B$ ,  $B \in \mathcal{B}$  (and hence also the sets  $\overline{B}$ ,  $B \in \mathcal{B}$ ).

PROOF Choose a countable set  $\mathcal{V} = \{V_1, V_2, \dots\}$  of regular coordinate balls which cover  $M$  with charts  $\phi_i : W_i \rightarrow \phi_i(W_i) = B(0, 2)$  such that  $V_i = \phi_i^{-1}(B(0, 1))$ . We use the sets  $V_i$  to construct a strongly ascending chain of compact sets  $K_i$  in  $M$  with  $K_i \subseteq H_{i+1}^{-1}$  for each  $i$ , and  $M = \bigcup_{i=1}^{\infty} K_i$  as follows:

- Let  $K_1 = \overline{V_1}$ ; since  $K_1$  is compact, we can choose  $\ell_1 \in \mathbb{N}$  so that  $K_1 \subseteq V_1 \cup \dots \cup V_{\ell_1}$ .
- Then we let  $K_2 = \overline{V_1 \cup \dots \cup V_{\ell_1}}$ . Since  $K_2$  is compact, we can choose  $\ell_2 > \ell_1$  so that  $K_2 \subseteq V_1 \cup \dots \cup V_{\ell_2}$ , and set  $K_3 = \overline{V_1 \cup \dots \cup V_{\ell_2}}$ .

Repeat the above process to obtain  $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq K_3^\circ \subseteq \dots$  with  $\bigcup_{i=1}^k K_i = M$ . For each  $m \in \mathbb{N}$ , note that  $K_{m+1} \setminus K_m^\circ$  is compact and contained in the open set  $K_{m+2} \setminus K_{m-1}$  (with  $K_0 = \emptyset$ ). For each  $p \in K_{m+1} \setminus K_m^\circ$ , choose  $U_p \in \mathcal{S}$  with  $p \in U_p$  and then choose a regular coordinate ball  $B_p$  and a chart  $\phi_p : C_p \subseteq M \rightarrow \phi_p(C_p) = B(0, 2) \subseteq \mathbb{R}^n$  with  $\phi_p(B_p) = B(0, 1)$  and  $C_p \subseteq U_p \cap (K_{m+2}^\circ \setminus K_{m-1})$ . The coordinate balls  $B_p$ ,  $p \in K_{m+1} \setminus K_m^\circ$  cover the compact set  $K_{m+1} \setminus K_m^\circ$ , so we can choose a finite set  $\mathcal{B}_m$  of such regular coordinate balls  $B_p$  so that  $K_{m+1} \setminus K_m^\circ \subseteq \bigcup \mathcal{B}_m \subseteq K_{m+2}^\circ \setminus K_{m-1}$ .

Now, the set  $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$  is a countable set of such regular coordinate balls. Note that for each  $B \in \mathcal{B}$ , we have chart  $\phi_B : C_B \rightarrow \phi_B(C_B) = B(0, 2)$  and the set  $\{C_B : B \in \mathcal{B}\}$  is locally finite since every point in  $M$  is contained in one of the sets  $K_{m+2}^\circ \setminus K_{m-1}$  and each of these sets only intersects with the coordinate balls from the finite sets  $\mathcal{B}_l$  with  $m-2 \leq l \leq m+2$ . ■

**1.7 Theorem. (Partitions of Unity)** Let  $M$  be a smooth manifold, and let  $\mathcal{S}$  be any open cover of  $M$ . There exists a set  $\{\psi_u : u \in \mathcal{S}\}$  of smooth maps  $\psi_u : M \rightarrow \mathbb{R}$  such that

1.  $\psi_u(M) \subseteq [0, 1]$  for each  $u \in \mathcal{S}$
2.  $\text{supp}(\psi_u) \subseteq U$  for each  $U \in \mathcal{S}$
3.  $\{\text{supp}(\psi_u) : u \in \mathcal{S}\}$  is locally finite: every point in  $M$  contains an open neighbourhood which only intersects finitely many of the sets  $\text{supp}(\psi_u)$ ,  $u \in \mathcal{S}$
4.  $\sum_{u \in \mathcal{S}} \psi_u = 1$

Such a set of functions  $\{\psi_u : u \in \mathcal{S}\}$  is called a (smooth) **partition of unity** on  $M$  for  $\mathcal{S}$  (or **subordinate** to  $\mathcal{S}$ ).

PROOF Let  $\mathcal{B}$  be a countable set of regular coordinate balls as in the previous lemma. Recall that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(t) = \begin{cases} e^{1/t} & : t < 0 \\ 0 & : t \geq 0 \end{cases}$$

is smooth, so the function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  given by  $g(x) = f(|x|^2 - 1)$  is smooth with  $g(x) > 0$  for  $|x| < 1$  and  $g(x) = 0$  for  $|x| \geq 1$ . For each  $B \in \mathcal{B}$ , we define a smooth bump function  $\sigma_B : M \rightarrow \mathbb{R}$  by

$$\sigma_B(p) = \begin{cases} g(\phi_B(p)) & : p \in B \\ 0 & : p \notin B \end{cases}$$

where  $\phi_B : C_B \subseteq M \rightarrow \phi_B(C_B) = B(0, 2)$  with  $\phi_B(B) = B(0, 1)$  as in the previous lemma. Note that  $\sigma(B)$  is smooth with  $\sigma_B(p) > 0$  for  $p \in B$  and  $\sigma_B(p) = 0$  for  $p \notin B$ . Now for each  $B \in \mathcal{B}$ ,

define  $\tau'_B : M \rightarrow \mathbb{R}$  by

$$\tau_B = \frac{\sigma_B}{\sum_{c \in \mathcal{B}} \sigma_c}$$

Note that  $\sum_{c \in \mathcal{B}} \sigma_c$  is well-defined by the local finiteness of  $\mathcal{B}$  and  $\sum_{c \in \mathcal{B}} \sigma_c(p) > 0$ . Furthermore, note that  $\tau_B(p) > 0$  for all  $p \in B$ , and  $\tau_B(p) = 0$  for all  $p \notin B$ , and  $\sum_{B \in \mathcal{B}} \tau_B = 1$ . Then define  $\rho_V : M \rightarrow \mathbb{R}$  by  $\rho_V = \sum_{B \in \mathcal{B}_V} \tau_B$ . ■

## 2 IMMERSIONS, EMBEDDING, SUBMANIFOLDS

**2.1 Theorem. (Inverse Function Theorem)** Let  $U \subseteq \mathbb{R}^n$  be open,  $p \in U$ , and  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth and suppose  $Df(p)$  is invertible. Then  $f$  is a local diffeomorphism.

**2.2 Corollary.** Let  $n < m$  and  $U \subseteq \mathbb{R}^n$  be open, and let  $p \in U$ , and  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  be smooth and suppose  $Df(p)$  has rank  $n$ . Then the range of  $f$  is locally equal to the graph of a smooth function. Such a map  $f$  is called a local **immersion** at  $p$ .

**PROOF** Since  $Df(p)$  is an  $m \times n$  matrix of rank  $n$ , some  $n$  rows of  $Df(p)$  form an invertible submatrix. Reorder the variables in  $\mathbb{R}^m$  (if necessary) so that the top  $n$  rows form an invertible matrix. Write elements in  $U \subseteq \mathbb{R}^n$  as  $t$  and write elements of  $\mathbb{R}^m$  as  $(x, y)$ . Also write  $(x, y) = f(t) = (u(t), v(t))$  so

$$Df = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

with  $\frac{\partial u}{\partial t}(p)$  invertible. Then by the inverse function theorem,  $u(t)$  is a local diffeomorphism. Say  $u : U_0 \subseteq U \rightarrow V_0 \subseteq \mathbb{R}^n$  is the diffeomorphism, and let  $g : V_0 \rightarrow U_0$  be its inverse. Then the range of  $f$  is locally equal to the graph of the function  $y = v(g(x)) =: h(x)$ . If  $(x, y) \in \Gamma(f)$  with  $(x, y) = f(t) = (u(t), v(t))$ , then since  $x = u(t)$  we have  $t = g(x)$  so  $y = v(t) = v(g(x)) = h(x)$ . If  $(x, y) \in \Gamma(h)$ , then  $y = h(x) = v(g(x))$  and we can choose  $t = g(x)$  to get  $x = u(t)$  and  $y = v(g(x)) = v(t)$  so that  $(x, y) = (u(t), v(t)) = f(t)$ . ■

**2.3 Theorem. (Implicit Function)** Let  $n < m$ ,  $U \subseteq \mathbb{R}^m$  be open,  $p \in U$ , and  $f : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  be smooth. Suppose  $Df(p)$  has rank  $n$  and let  $q = f(p)$ . Then the level set  $f^{-1}(q)$  is locally equal to a graph of a smooth function.

**2.4 Theorem.** Let  $U \subseteq \mathbb{R}^n$  be open with  $p \in U$ , let  $f : U \rightarrow \mathbb{R}^m$  be smooth with  $f(p) = q$  and suppose that  $Df$  has constant rank  $r$  in  $U$ . Then the level set (or fibre)  $f^{-1}(q)$  is locally equal to the graph of a smooth function (with  $n - r$  independent variables and  $r$  dependent variables).

**PROOF** Since  $Df$  is an  $m \times n$  matrix of rank  $r$ , there is some  $r \times r$  submatrix of  $Df(p)$  which is invertible; without loss of generality, it is the upper left submatrix. Write elements in  $\mathbb{R}^n$  as  $(x, y)$  with  $x \in \mathbb{R}^r$  and  $y \in \mathbb{R}^{n-r}$  and write elements in  $\mathbb{R}^m$  as  $(u, v)$  with  $u \in \mathbb{R}^r$  and  $v \in \mathbb{R}^{m-r}$ , with say  $p = (a, b)$  and  $q = f(p) = (c, d)$ . Then we have  $(u, v) = f(x, y) = (u(x, y), v(x, y))$  so that

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

with  $\frac{\partial u}{\partial x}(p) = \frac{\partial u}{\partial x}(a, b)$  being an invertible  $r \times r$  matrix. Define  $F : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^m$  by  $F(x, y) = (u(x, y), y)$ . Then

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & I \end{pmatrix}$$

so that  $DF(p)$  is invertible. By the IVT,  $F$  is a local diffeomorphism, say  $F : U_0 \subseteq U \subseteq \mathbb{R}^m \rightarrow V_0 \subseteq \mathbb{R}^m$  is a diffeomorphism with  $U_0$  an open rectangular box. Let  $G : V_0 \rightarrow U_0$  denote the smooth inverse of  $F$ . Note that  $G$  is of the form  $G(u, y) = (g(u, y), y)$  for some smooth function  $g : V_0 \rightarrow \mathbb{R}^r$ . We claim that  $f^{-1}(q) = f^{-1}(c, d)$  is locally equal to the graph of  $x = g(c, y)$ . First, note that

$$(u, y) = F(G(u, y)) = F(g(u, y), y) = (u(g(u, y), y), y)$$

so that, in particular,  $u(g(u, y), y) = u$  and so

$$f(G(u, y)) = (u(g(u, y), y), v(g(u, y), y)) = (u, h(u, y))$$

where  $h(u, y) = v(g(u, y), y)$ . Thus

$$Df(x, y) \cdot DG(u, y) = D(f \circ G)(u, y) = \begin{pmatrix} I & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial y} \end{pmatrix}$$

Since  $Df$  has constant rank  $r$  and  $DG$  is invertible, the matrix on the right is of rank  $r$  for all  $(u, v) \in V_0$ . Thus it follows that  $\frac{\partial h}{\partial y} = 0$  for all  $u, b$ , so that  $h(u, y)$  is independent of  $y$  and  $h(u, y) = h(u, b)$  for all  $y$ ; let  $k(u) = h(u, b)$ . Let us calculate  $k(c)$ . We have

$$\begin{aligned} f(a, b) = (c, d) &\implies (u(a, b), v(a, b)) = (c, d) \\ &\implies u(a, b) = c \\ &\implies F(a, b) = (u(a, b), b) = (c, b) \\ &\implies (a, b) = G(c, b) \\ &\implies (c, d) = f(a, b) = f(G(c, b)) = (c, h(c, b)) = (c, k(c)) \\ &\implies k(c) = d \end{aligned}$$

Finally, let us show that  $f^{-1}(c, d)$  is (locally) the graph of  $x = g(c, y)$ . We have

$$\begin{aligned} (x, y) = f^{-1}(c, d) &\implies f(x, y) = (c, d) \\ &\implies u(x, y) = c \text{ and } v(x, y) = d \\ &\implies F(x, y) = (u(x, y), y) = (c, y) \\ &\implies (x, y) = G(c, y) = (g(c, y), y) \\ &\implies x = g(c, y) \end{aligned}$$

We thus have

$$\begin{aligned} x = g(c, y) &\implies G(c, y) = (g(c, y), y) = (x, y) \\ &\implies f(x, y) = f(G(c, y)) = (c, h(c, y)) = (c, k(c)) = (c, d) \end{aligned}$$

as required. ■

**Definition.** When  $N$  and  $M$  are smooth manifolds and  $f : N \rightarrow M$  is a smooth map, we say that  $f$  has **rank**  $r$  at  $p \in N$  when for some (hence for every) chart  $\phi$  on  $N$  at  $p$  and for some (hence every) chart  $\psi$  on  $M$  at  $f(p)$ , the matrix  $D(\psi f \phi^{-1})(\phi(p))$  has rank  $r$ .

**2.5 Corollary.** Let  $N$  and  $M$  be smooth manifolds, with  $p \in N$ . Let  $f : N \rightarrow M$  be smooth with  $f(p) = q \in M$ . Suppose  $f$  has constant rank  $r$  in an open neighbourhood of  $p$ . Then there exists a chart  $\phi$  on  $N$  at  $p$  and a chart  $\psi$  on  $M$  at  $q = f(p)$  such that  $\phi(p) = 0$  and  $\psi(q) = 0$  and

$$(\psi \circ f \circ \phi^{-1})(x^1, \dots, x^r, \dots, x^n) = (x^1, \dots, x^r, 0, \dots, 0)$$

where  $n = \dim(N)$  and  $m = \dim(M)$ .

**PROOF** Choose any chart  $\phi_0$  on  $N$  at  $p$  and any chart  $\psi_0$  on  $M$  at  $q$  with  $\phi_0(p) = 0$  and  $\psi_0(q) = 0$ . Then  $D(\psi_0 f \phi_0^{-1})$  has constant rank  $r$  near 0. Let  $\phi_1$  and  $\psi_1$  be linear permutation maps so that the upper left  $r \times r$  submatrix of  $D(\psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1})(0)$ . Say  $f_1 = \psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1}$ . Let  $F, G, f_1$  be as in the proof of the rank theorem (for the function  $f_1$ ). Let us verify that for the charts  $\phi = F \phi_1 \phi_0$  and  $\psi = H \psi_1 \psi_0$  where  $H(u, v) = (u, v - k(u))$  we have  $(\psi f \phi^{-1})(u, y) = (u, 0)$ . ■

**2.6 Corollary.** When  $f : M \rightarrow N$  is a smooth map of smooth manifolds with constant rank  $r$  in  $M$ , for  $q \in \text{im } f$ , the level set (fibre)  $f^{-1}(q)$  can be given charts (obtained from canonical charts) to make it a smooth  $(\dim M - r)$ -dimensional manifold.

**Definition.** Let  $N$  and  $M$  be smooth manifolds (of dimensions  $m$  and  $n$ ). A smooth map  $f : N \rightarrow M$  is called a (smooth) **immersion** when  $f$  has rank  $n$  in  $N$ . An **immersed submanifold** of  $M$  is the image of an immersion  $f : N \rightarrow M$  or the image of an injective immersion  $f : N \rightarrow M$ .

Note that when  $f : N \rightarrow M$  is injective, we can give the image  $f(N)$  a smooth atlas which makes  $f : N \rightarrow f(N)$  a diffeomorphism. When we do this, the resulting topology on  $f(N) \subseteq M$  does not necessarily agree with the subspace topology of  $M$ .

**Definition.** An **embedded submanifold** of  $M$  is a subset  $N \subseteq M$  which is a smooth manifold such that the inclusion map  $f : N \rightarrow M$  (given by  $f(p) = p$ ) is an immersion such that the topology in the previous remark agrees with the subspace topology.

When  $f : M \rightarrow N$  is a smooth map of smooth manifolds of constant rank  $r$  and  $q \in \text{im } f$ , the level set  $f^{-1}(q)$  is an embedded submanifold of  $M$ .

**Remark.** When  $N \subseteq M$  is an embedded submanifold,

- If  $f : M \rightarrow K$  is smooth, then the restriction  $f : N \rightarrow K$  is smooth
- If  $f : K \rightarrow M$  is smooth and  $f(K) \subseteq N$ , then  $f : K \rightarrow N$  is smooth

**Example.**  $\text{SL}_n(\mathbb{R})$  is a smooth manifold. Recall that  $\text{GL}_n(\mathbb{R})$  is a smooth  $n^2$ -dimensional manifold, since it is open in the  $n^2$ -dimensional vector space  $M_n(\mathbb{R})$ . We have  $\text{SL}_n(\mathbb{R}) = f^{-1}(\{1\})$  where  $f$  is the determinant evaluation map. Then for fixed  $\ell$ ,  $\det X = \sum_{i=1}^n (-1)^{i+\ell} X_{\ell}^i \deg X_{(\ell)}^{(i)}$ , where  $X_{(\ell)}^{(i)}$  is the matrix obtained from  $X$  by removing row  $i$  and column  $\ell$ . We have

$$Df = \left( \mathbb{P} f x_1^1, \dots, \frac{\partial f}{\partial x_n^n} \right) \in M_{1 \times n^2}(\mathbb{R})$$

with  $\frac{\partial f}{\partial x_{\ell}^{\ell}} = (-1)^{k+\ell} \det X_{(\ell)}^{(k)}$ , so that  $Df = 0$  if and only if  $\det X = 0$ . Thus  $f$  has constant rank 1, so  $\text{SL}_n(\mathbb{R}) = f^{-1}(1)$  is an embedded submanifold of  $M_n(\mathbb{R})$  of dimension.

### 3 TANGENT VECTORS

**Definition.** A vector  $u$  in  $\mathbb{R}^n$  at a point  $a \in \mathbb{R}^n$  is an ordered pair  $(a, u)$ .

**Definition.** Let  $M$  be a smooth manifold and let  $p \in M$ . A **tangent vector** on  $M$  at  $p$  is a set of vectors  $X = \{\phi_*x : \phi \text{ is a chart on } M \text{ at } p\}$ , where  $\phi_*x$  is a vector in  $\mathbb{R}^n$  at the point  $x = \phi(p)$  such that when  $\phi$  and  $\psi$  are two charts on  $M$  at  $p$ , we have  $\psi_*X = D(\psi\phi^{-1})(\phi(p))\phi_*X$ .

The set of all tangent vectors on  $M$  at  $p$  is denoted by  $T_pM$ . Note that  $T_pM$  is an  $n$ -dimensional vector space. When  $I \subseteq \mathbb{R}$  is an open interval,  $s \in I$ , and  $\alpha : I \subseteq \mathbb{R} \rightarrow M$  is a smooth map with  $\alpha(s) = p$ , we define  $\alpha'(s)$  to be the tangent vector  $\alpha'(s) \in T_pM$  given by  $\phi_*\alpha'(s) = \beta'(s)$  where  $\beta(t) = \phi(\alpha(t))$ . Note that, by the chain rule, we do have  $\phi_*\alpha'(s) = D(\psi\phi^{-1})\phi_*\alpha'(s)$ .

When  $\phi$  is a chart on  $M$  at  $p$ , we often write

$$x = x(p) = \phi(p) = (\phi^1(p), \dots, \phi^n(p)) = (x^1(p), \dots, x^n(p))$$

(so each  $x^k = \phi^k$  is a function  $x^k, \phi^k : U \subseteq M \rightarrow \mathbb{R}$ ). When  $\psi$  is another chart and we write  $y = \psi(p)$ , we often write  $y = y(x) = (\psi\phi^{-1})(x) = (y^1(x), \dots, y^n(x))$  and we write

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

With this notation, if  $u = \phi_*X$  and  $v = \psi_*X$ , then  $v = D(\psi\phi^{-1})u = \frac{\partial y}{\partial x}u$ , so  $V^k = \sum_{i=1}^n \frac{\partial y^k}{\partial x^i} u^i$ .

**Definition.** Let  $f : M \rightarrow N$  be a smooth map of smooth manifolds with  $p \in M$ . We define the **induced map** or the **pushforward**  $f_*$  or the **differential**  $df$  to be the map  $f_* = df : T_pM \rightarrow T_{f(p)}N$  given as follows. Given  $X \in T_pM$ , choose  $\alpha : (-\epsilon, \epsilon) \rightarrow M$  smooth with  $\alpha(0) = p$ ,  $\alpha'(0) = X$ , when we let  $\beta(t) = f(\alpha(t))$  and define  $df(x) = f_*(x) = \beta'(0)$ . Given a chart  $\phi$  on  $M$  at  $p$  and  $\psi$  on  $N$  at  $f(p)$ , if  $u = \phi_*X$  and  $V = \psi_*(f_*X)$ , then verify that  $v = D(\psi f \phi^{-1})(\phi(p))u$ .

1. When  $\phi$  is a chart on  $M$  at  $p$  and  $\psi$  is a chart on  $N$  at  $f(p)$ ,  $\psi_*f_*X = D(\psi f \phi^{-1})_{\phi(p)}\phi_*X$
2. The map  $df = f_*$  is linear
3. If  $g : L \rightarrow M$  and  $f : M \rightarrow N$  are smooth, then  $(f \circ g)_* = f_* \circ g_*$ .
4. When  $\iota : M \rightarrow M$  is the identity map,  $d\iota : T_pM \rightarrow T_pM$  is the identity map
5. If  $f : M \rightarrow N$  is a diffeomorphism, then  $f_* : T_pM \rightarrow T_pM$  is an isomorphism.
6. For  $f : M \rightarrow N$  smooth,  $f$  is of rank  $r$  at  $p$  if and only if  $f_*$  is of rank  $r$  at  $p$ .

When  $U \subseteq \mathbb{R}^n$  is open,  $U$  is a manifold with atlas  $\{\emptyset\}$  where  $\phi$  is the identity map. In this case, we identify  $X \in T_pU$  with  $\phi_*X \in \mathbb{R}^n$ . With this convention,  $\phi_*X$  is equal to  $\phi_*X$  where the second  $\phi_*$  is the pushforward. When  $N \subseteq M$  is a submanifold (immersed or embedded), the inclusion map  $\iota : N \rightarrow M$  is an injective immersion. Thus, the map  $\iota_* : T_pN \rightarrow T_pM$ . In this situation, we identify  $T_pN$  with the subspace  $\iota_*(T_pN) \subseteq T_pM$ .

Let  $X$  be the vector on  $\mathbb{S}^2$  at  $p$  with  $\phi_*X = (1, 0)$ . Let  $\iota : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  be the inclusion map. We have  $\phi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$  with  $u = \phi_*X = (1, 0)$ . Then  $\iota_*X = D(\iota\phi^{-1})_{\phi(p)}\phi_*X$  where  $\psi$  is the identity on  $\mathbb{R}^3$ .

#### TANGENT VECTORS AS DIFFERENTIAL OPERATORS

Recall that a vector  $u \in \mathbb{R}^n$  at a point  $a \in \mathbb{R}^n$  acts as a differential operator on smooth maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by directional derivative. Choose any smooth map  $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  with



$\alpha(0) = a$  and  $\alpha'(0) = u$ , and define  $u(f) = u_a(f) = D_u f(a) = \beta'(0)$  where  $\beta(t) = f(\alpha(t))$ . Since  $\beta(t) = f(\alpha(t))$ , we have  $\beta'(t) = Df(\alpha(t)) \cdot \alpha'(t)$  so

$$\begin{aligned} u(f) &= D_u f(a) = \beta'(0) = Df(a) \cdot u \\ &= \left( \frac{\partial f}{\partial x^1}(a), \dots, \frac{\partial f}{\partial x^n}(a) \right) \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) \cdot u^i \end{aligned}$$

or as a differential operator,  $u = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}$ .

**Definition.** When  $M$  is a smooth manifold,  $p \in M$ , and  $X \in T_p M$ ,  $X$  acts as a differential operator on a smooth function  $f : M \rightarrow \mathbb{R}$  as follows: choose a smooth map  $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow M$  with  $\alpha(0) = p$  and  $\alpha'(0) = X$ , and define  $X(f) = X_p(f) = \beta'(0)$  where  $\beta(t) = f(\alpha(t))$ .

When  $\phi$  is a chart on  $M$  at  $p$ , then

$$\begin{aligned} X(f) &= (\phi_* X)(f \circ \phi^{-1}) = D_{\phi_* X}(f \circ \phi^{-1})(\phi(p)) \\ &= D(f \circ \phi^{-1})(\phi(p)) \cdot (\phi_* X) \\ &= \sum_{i=1}^n \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p)) \cdot u^i \end{aligned}$$

where  $u = \phi_* X \in \mathbb{R}^n$ . So when  $u = \phi_* X \in \mathbb{R}^n$ ,  $X$  acts as the differential operator  $X = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i} \Big|_p$  where  $\frac{\partial}{\partial x^i} \Big|_p(f) = \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p))$ . With this notation,

$$T_p M = \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

If  $\phi$  and  $\psi$  are two charts at  $p$  on  $M$ , then  $T_p M$  has two representations as differential operators. Let us determine how  $\frac{\partial}{\partial x^k}$  and  $\frac{\partial}{\partial y^\ell}$  are related. When  $X \in T_p M$ ,  $u = \phi_* X \in \mathbb{R}^n$  and  $v = \psi_* X \in \mathbb{R}^n$ , we have  $V = D(\psi \circ \phi^{-1})(\phi(p)) \cdot u = \left( \frac{\partial y}{\partial x} \right)(\phi(p)) \cdot u$ . When  $u = \frac{\partial}{\partial x^j}$ ,

$$v = \left( \frac{\partial y}{\partial x} \right) \cdot e_k$$

so  $v^\ell = \left( \frac{\partial y}{\partial x} \right)_k^\ell = \frac{\partial y^\ell}{\partial x^k}$  so that

$$v = \sum_{i=1}^n \frac{\partial y^i}{\partial x^k} \frac{\partial}{\partial y^i}$$

**Definition.** A **derivation** on  $M$  at  $p$  is a linear map  $L : C^\infty(M) \rightarrow \mathbb{R}$  or  $L : C_p^\infty(M) \rightarrow \mathbb{R}$  where  $C_p^\infty(M)$  is the vector space of **germs** of smooth functions on  $M$  at  $p$ , which satisfies the product rule at  $p$ :

$$L(fg) = L(f) \cdot g(p) + f(p) \cdot L(g)$$

Every  $X \in T_p M$  gives a derivation on  $M$  at  $p$ . Moreover, it can be shown that every derivation on  $M$  at  $p$  is of this form. Thus allows us to give an alternate definition for  $T_p M$  as the space of derivations on  $M$  at  $p$ .

**Definition.** Let  $TM$  be the disjoint union of all the tangent spaces. A **vector field** on  $M$  is a function  $X : M \rightarrow TM$  such that  $X(p) \in T_p M$ .

Given a chart  $\phi : U \rightarrow \phi(U)$  on  $M$ , the restriction of  $X$  to  $U$  determines and is determined by the vector field  $\phi_* X$  on  $\phi(U) \subseteq \mathbb{R}^n$  by  $(\phi_* X)(\phi(p)) = \phi_*(X(p))$ , or  $(\phi_* X)(x) = \phi_*(X(\phi^{-1}(x))) \in \mathbb{R}^n$ . We say that  $X$  is **smooth** at  $p$  when for some chart  $\phi$  on  $M$  at  $p$ , the vector field  $\phi_* X$  is smooth at  $\phi(p)$ . When  $X$  is a smooth vector field on  $M$ ,  $X$  acts as a differential operator  $X : C^\infty(M) \rightarrow C^\infty(M)$  by  $X(f)(p) = X_p(f)$ .

The space of smooth vector fields on  $M$  is  $\Gamma(M, TM) = \Gamma(TM) = \mathcal{X}(M)$ .

### THE PUSHFORWARD OR DIFFERENTIAL

If  $X$  is a smooth vector field on a smooth manifold  $N$  and  $f : N \rightarrow M$  is a smooth map, for each point  $p \in N$ , we have the linear map  $df = f_* : T_p N \rightarrow T_{f(p)} M$ .

Note that  $f_*$  does not in general give a map  $f_* : \Gamma(TN) \rightarrow \Gamma(TM)$ , if  $f$  is not surjective, or  $f$  is not injective with  $p, q \in N$  with  $p \neq q$  and  $f(p) = f(q)$  and  $f_* X_p \neq f_* X_q$ .

If  $f : N \rightarrow M$  is a diffeomorphism, then  $f_*$  does give a well-defined bijective map  $f_* : \Gamma(TN) \rightarrow \Gamma(TM)$ . If  $f$  is an injective immersion, then  $f : N \rightarrow f(N)$  is a diffeomorphism.

### THE LIE BRACKET OF VECTOR FIELDS

**Definition.** When  $X$  and  $Y$  are two smooth vector fields on  $M$ , we define the **Lie bracket** of  $X$  and  $Y$ , denoted by  $[X, Y](f)$ , by  $[X, Y]f = X(Y(f)) - Y(X(f))$  for all  $f \in C^\infty(M)$ .

Note that  $[X, Y]$  satisfies the product rule since

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X(f \cdot Y(g) + g \cdot Y(f)) - Y(f \cdot X(g) + g \cdot X(f)) \\ &= f \cdot X(Y(g)) + X(f) \cdot Y(g) + g \cdot X(Y(f)) + X(g) \cdot Y(f) - Y(f \cdot X(g)) - Y(g \cdot X(f)) \\ &= g[X, Y](f) + g[X, Y](g) \end{aligned}$$

Given a chart  $\phi : U \rightarrow \phi(U)$  on  $M$  at  $p$ , we can calculate a formula for the Lie bracket: say  $u = \phi_* X$  and  $v = \phi_* Y$  ( $u(x) = \phi_*(X_{\phi^{-1}(x)})$ ,  $v(x) = \phi_*(Y_{\phi^{-1}(x)})$ ). Then for  $f \in C^\infty(M)$ ,

$$\begin{aligned} [X, Y]_p(f) &= X_p(Y(f)) - Y_p(X(f)) \\ &= \sum_i u^i \frac{\partial}{\partial x^i} \left( \sum_j v^j \frac{\partial g}{\partial x^j} \right) - \sum_i v^i \frac{\partial}{\partial x^i} \left( \sum_j u^j \frac{\partial g}{\partial x^j} \right) \\ &= \sum_{i,j} \left( u^i \frac{\partial v^j}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} + u^i v^j \frac{\partial^2 g}{\partial x^i \partial x^j} - v^i \frac{\partial u^j}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} - v^i u^j \frac{\partial^2 g}{\partial x^i \partial x^j} \right) \\ &= \sum_{i,j} \left( \frac{\partial v^j}{\partial x^i} \cdot u^i - \frac{\partial u^j}{\partial x^i} \cdot v^i \right) \frac{\partial g}{\partial x^j} \end{aligned}$$

Thus  $[X, Y]_p$  is a vector in  $T_p M$ . It is the vector given by  $w^j = \sum_i \left( \frac{\partial v^j}{\partial x^i} u^i - \frac{\partial u^j}{\partial x^i} v^i \right)$  and  $w = \sum_j w^j \frac{\partial}{\partial x^j} = Dv \cdot u - Du \cdot v$ .

## INTEGRAL CURVES AND FLOWS

Given a smooth vector field  $X$  on a smooth manifold  $M$ , and given  $p \in M$ , the existence and uniqueness theorem for (systems) of ODEs guarantees that there is a unique smooth map (or curve)  $\alpha_p : I_p \subseteq \mathbb{R} \rightarrow M$  where  $I$  is the (unique) maximal open interval  $\alpha$  and  $\alpha(0) = p$  and  $\alpha'(t) = X_{\alpha(t)}$ . A stronger version of the existence and uniqueness theorem also guarantees that  $\alpha_p(t)$  varies smoothly with  $p$  to give a unique smooth map  $\theta : U \subseteq M \times \mathbb{R} \rightarrow M$  where  $U$  is the (unique) maximal open connected domain given by  $\theta(p, t) = \alpha_p(t)$ .

*Example.* (i) Find a vector field which is a parabola at each point.

(ii) Find a smooth vector field so that the solution curves have vertical asymptote.

When a vector field  $X$  on a 2 dimensional manifold  $M$ , we define the **index** of  $X$  at  $p$  as follows. Choose a chart  $\phi : C \rightarrow \phi(C) = B(0, 2)$  on  $M$  at  $p$ . Thus  $U = \phi_*X$  is a vector field in  $\mathbb{R}^2$  with no zeros in  $B(0, 2)$  except at 0.

When we restrict  $u$  to the circle  $\mathbb{S}^1$  and we define the index of  $X$  at  $p$  to be the winding number of this map  $u : \mathbb{S}^1 \rightarrow \mathbb{C} \setminus \{0\}$ . When a vector field on  $X$  has finitely many isolated zeros, the index of  $X$  is the sum of the indices at the zeros of  $X$ .

**3.1 Theorem.** When  $X$  is a smooth vector field with isolated zeros on a **compact** 2-dimensional manifold  $M$ ,  $\text{Ind } X = \chi(M)$ , the Euler characteristic of  $M$ .

## 4 LIE GROUPS

**Definition.** A Lie group  $G$  is both a smooth manifold and a group such that the group operations  $\mu : G \times G \rightarrow G$  and inversion  $\nu : G \rightarrow G$  are smooth maps.

*Example.*  $O_n(\mathbb{R}) = \{A \in \text{GL}_n(\mathbb{R}) : A^T A = I\}$  is a Lie group. Define  $F : \text{GL}_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$  by  $F(X) = X^T X$ . Thus  $O_n(\mathbb{R}) = F^{-1}(I)$ . When  $n = 2$ ,  $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$  we have  $F(X) = \begin{pmatrix} x^2 + y^2 & xz + yw \\ xz + yw & z^2 + w^2 \end{pmatrix}$  so that

$$DF = \begin{pmatrix} 2x & 2y & 0 & 0 \\ z & w & x & y \\ z & w & x & y \\ 0 & 0 & 2z & 2w \end{pmatrix}$$

In general, for  $A \in \text{GL}_n(\mathbb{R})$ ,  $F(R_A(X)) = F(XA) = A^T X^T X A = L_{A^T} R_A F(X)$ . Thus by the chain rule,  $DF(XA) \cdot DR_A(X) = DL_{A^T}(X^T X A) \cdot DR_A(X^T X) \cdot DF(X)$ , so we can identify  $T_p \text{GL}_n(\mathbb{R})$  or  $T_p M_n(\mathbb{R})$  with the vectr space  $M_n(\mathbb{R})$ . Note that  $L_{A^T}$  and  $R_A$  are diffeomorphisms of  $\text{GL}_n(\mathbb{R})$ , so  $DL_{A^T}$  and  $DR_A$  are invertible. Thus  $\text{rank } DF(XA) = \text{rank } DF(X)$ . In particular, taking  $X = I$ ,  $\text{rank } DF(A) = \text{rank } DF(I)$ , so  $F$  has constant rank. Let us calculate  $\text{rank } DF : T_I \text{GL}_n(\mathbb{R}) \rightarrow T_I M_n(\mathbb{R})$ . Let  $A \in T_I \text{GL}_n(\mathbb{R})$ , so  $A \in M_n(\mathbb{R})$ , and let  $\alpha(t) = I + tA$  so that  $\alpha(0) = I$  and  $\alpha'(0) = A$ . Then  $DF(I) \cdot A = \beta'(0)$  where  $\beta(t) = F(\alpha(t)) = (I + tA)^T (I + t(A + A^T) + t^2 A^T A)$ . Then  $\beta'(t) = A + A^T + 2tA^T A$ , so  $\beta'(0) = A + A^T$  so that  $DF(I) \cdot A = A + A^T$ . The range of  $DF$  at  $I$  is the set of matrices  $B$  of the form  $B = A + A^T$  for some matrix  $A \in M_n(\mathbb{R})$ , or equivalently, the set of symmetric matrices in  $M_n(\mathbb{R})$ . Thus the dimension of the range of  $DF$  is  $(n^2 + n)/2$ , so  $F$  has constant rank  $r = (n^2 + n)/2$  and thus  $\dim O_n(\mathbb{R}) = n^2 - r = \frac{n^2 - n}{2}$ .

Thus by the constant rank theorem,  $O_n(\mathbb{R})$  is a regular embedded submanifold of  $\text{GL}_n(\mathbb{R})$ . In fact,  $T_I O_n(\mathbb{R})$  can be identified with  $\ker DF(I) \subseteq T_I \text{GL}_n(\mathbb{R}) = M_n(\mathbb{R})$ , which is

$\{A \in M_n(\mathbb{R}^n) : A^T + A = 0\}$ . One can do the same for  $U_n(\mathbb{C}) = \{A \in GL(\mathbb{C}) : A^*A = I\}$  and  $A^* = \overline{A}^T$ .

**Definition.** When  $f : M \rightarrow M$  is a diffeomorphism and  $X \in \Gamma(M, TM)$ , we say that  $X$  is **invariant** under  $f$  when  $f_*X = X$  (where  $f_*(X_p) = X_{f(p)}$  for all  $p \in M$ ). When  $G$  is a Lie group and  $X \in \Gamma(G, TG)$ , we say that  $X$  is **left-invariant** when  $X$  is invariant under the left multiplication map  $\ell_a : G \rightarrow G$  where  $\ell_a(p) = ap$  for all  $a \in G$ .

Note that  $(\ell_a)_*(X) = X$  for all  $a \in G$ .

On the other hand, if we define a vector field  $X$  on  $G$  by the formula  $X_a = (\ell_a)_*A$  where  $A \in T_eG$ , then  $X$  is left invariant since for all  $a, b \in G$ ,

$$(\ell_a)_*X_b = ((\ell_a)_* \circ (\ell_b)_*)(X_e) = X_{ab}$$

**Definition.** A **Lie algebra** is a vector space  $V$  with an alternating bilinear map  $[\cdot, \cdot] : V \times V \rightarrow V$  which satisfies the Jacobi identity  $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ .

*Example.*  $M_n(\mathbb{R})$  is a Lie algebra using  $[A, B] = AB - BA$ , as one can verify directly. More generally, when  $V$  is a vector space,  $\text{End } V$  is a Lie algebra with Lie bracket  $[A, B] = AB - BA$ . For example, when  $M$  is a smooth manifold,  $X(M) = \Gamma(M, TM)$  is a vector space with Lie bracket  $[X, Y](f) = X(Y(f)) - Y(X(f))$ .

Given  $A \in T_eG$ , there is a unique left invariant vector field  $X$  on  $G$  with  $X_e = A$ , and  $X$  is given by  $X_p = (\ell_p)_*A$ . By the assignment if  $X$  and  $Y$  are left-invariant vector fields on a Lie group  $G$ , then  $[X, Y]$  is left invariant since  $(\ell_a)_*[X, Y] = [(\ell_a)_*X, (\ell_a)_*Y] = [X, Y]$ .

**Definition.** For a Lie group  $G$ , the **Lie algebra** of  $G$ , denoted by  $\mathfrak{g}$ , is the Lie algebra of left-invariant vector fields on  $G$ .

Equivalently, we may define  $\mathfrak{g} = T_eG$  with the corresponding Lie algebra given by  $[A, B] = [X, Y]_e$ , where  $A, B \in T_eG = \mathfrak{g}$ , and  $X, Y$  are the left invariant vector fields on  $G$  with  $X_e = A$  and  $Y_e = B$ .

**Definition.** A **Lie subgroup** of a Lie group  $G$  is a subgroup  $H \subseteq G$  that is also an immersed (or embedded) submanifold.

Let  $G$  be a Lie subgroup of  $GL_n(\mathbb{R})$ . We identify  $T_pGL_n(\mathbb{R})$  with  $M_n(\mathbb{R})$ , and we identify  $T_pG$  with a subspace of  $M_n(\mathbb{R})$ .

*Example.* 1. Given  $A \in T_I G \subseteq M_n(\mathbb{R})$ , find a formula for  $U_p = U(P)$ , where  $P \in G \subseteq M_n(\mathbb{R})$  and  $U$  is the left-invariant vector field on  $G$  with  $U_I = A$ .

We have  $U_p = (L_p)_*A$ , where  $L_p : G \rightarrow G$  is given by  $L_p(X) = PX$ . Note that  $L_p$  is the restriction of the map  $L_p : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ . This map  $L_p$  is linear, so  $DL_p$  is equal to  $L_p$  as a linear map on  $M_n(\mathbb{R})$ . Thus we have  $U_p = (P_p)_*(A) = (DL_p)A = PA$ .

2. Given  $A, B \in \mathfrak{G} = T_I G \subseteq M_n(\mathbb{R})$ , let  $U$  and  $V$  be given by  $U(P) = PA$  and  $V(P) = PB$ . Note that  $U = R_A$ ,  $V = R_B$ , so  $DU = R_A$  and  $DV = R_B$  as linear maps on  $M_n(\mathbb{R})$ , and we have

$$[A, B] = [U, V]_I = DV(I)U(I) - DU(I)V(I) = R_B(A) - R_A(B) = AB - BA$$

3. Let  $A \in \mathfrak{G} = T_I(G) \subseteq M_n(\mathbb{R})$ , let  $U(P) = PA$ . We need to find the integral curve  $\alpha : I \subseteq \mathbb{R} \rightarrow G$  with  $\alpha(0) = I$ . Then we want  $\alpha'(t) = U(\alpha(t)) = \alpha(t)A$  for all  $t$ . The solution to this DE is given by  $\alpha(t) = e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \dots$  so that  $\alpha'(t) = (e^{tA})A$ . As a consequence of the above formula, note that  $\mathfrak{g} = \{A \in M_n(\mathbb{R}) : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$ .

Thus formula allows us to give an explicit description of the Lie algebras of many Lie subgroups of  $GL_n(\mathbb{R})$ .

Given  $A \in M_n(\mathbb{R})$ ,  $\det e^A = e^{\text{tr} A}$ . By Schur's Theorem or the Jordan Normal Form, there is a matrix  $P \in GL_n(\mathbb{C})$  so that  $P^{-1}AP = T$  where  $T$  is upper triangular, so that

$$\det e^A = \det(Pe^T P^{-1}) = \det e^T = e^{\text{tr} A}$$

Recall when  $G$  is a Lie subgroup of  $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$  and if  $J = T_I G \subseteq T_I GL_n(\mathbb{R})$ , the left invariant vector field  $U$  on  $G$  with  $U(I) = A \in J$  is given by  $U(P) = PA$ . The Lie bracket on  $J$  is given by  $[A, B] = AB - BA$ , and the integral curve of  $U(P) = PA$  is given by  $\alpha : \mathbb{R} \rightarrow G$  where  $\alpha(t) = e^{tA}$ , and hence

$$J = \{A \in M_n(\mathbb{R}) : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$$

For example, the Lie algebra of  $SL_n(\mathbb{R})$  is

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : e^{tA} \in SL_n(\mathbb{R}) \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \det e^{tA} = 1 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : e^{\text{tr} tA} = 1 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \text{tr}(tA) = 0 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \text{tr}(A) = 0\} \end{aligned}$$

The Lie algebra of  $O_n(\mathbb{R})$  is

$$\begin{aligned} \mathfrak{o}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : e^{tA} \in O_n(\mathbb{R}) \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : (e^{tA})^T (e^{tA}) = I \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : (e^{tA^T})(e^{tA}) = I \forall t\} \end{aligned}$$

If  $(e^{tA^T})(e^{tA}) = I$  for all  $t \in \mathbb{R}$ , then  $\frac{d}{dt}(e^{tA^T})(e^{tA}) = \frac{d}{dt}I$  so that

$$(e^{tA^T} A^T (e^{tA}) + (e^{tA^T})(e^{tA}) \cdot A = 0$$

and taking  $t = 0$  gives  $A^T + A = 0$ . Then  $A^T = -A$  so  $tA^T = -tA$  so  $e^{tA^T} = e^{-tA} = (e^{tA})^{-1}$  for all  $t$ , so  $e^{tA^T} \cdot e^{tA} = I$  for all  $t$ . Thus  $\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$ .

Table of Lie algebras:

$G$	$\mathfrak{g}$
$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$GL_n^+(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A > 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$	$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \text{tr} A = 0\}$
$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T A = I\}$	$\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$
$SO_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T A = I, \det A = 1\}$	$\mathfrak{so}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$
$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A \neq 0\}$	$\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$
$SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A = 1\}$	$\mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \text{tr} A = 0\}$
$O_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : A^T A = I\}$	$\mathfrak{o}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A + A^T = 0\}$
$SO_n(\mathbb{C}) = \{A \in SL_n(\mathbb{C}) : A^T A = I, \det A = 1\}$	$\mathfrak{so}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \text{tr} A = 0, A + A^T = 0\}$
$U_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : A^* A = I\}$	$\mathfrak{u}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A^* + A = 0\}$
$SU_n(\mathbb{C}) = \{A \in SL_n(\mathbb{C}) : A^* A = I\}$	$\mathfrak{su}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \text{tr} A = 0, A^* + A = 0\}$

## 5 SMOOTH $k$ -FORMS

Suppose  $\alpha : I \subseteq \mathbb{R} \rightarrow U \subseteq \mathbb{R}^3$  and let  $f : U \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ , then the length of  $\alpha$  is

$$\int_C dL = \int_\alpha dL = \int_{t \in I} |\alpha'(t)| dt$$

and

$$\int_C f dL = \int_\alpha f dL = \int_{t \in I} f(\alpha(t)) |\alpha'(t)| dt$$

Given  $\sigma : R \subseteq \mathbb{R}^2 \rightarrow U \subseteq \mathbb{R}^3$ ,  $f : U \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ , the area of  $\text{im } \sigma$  is given by

$$\begin{aligned} \sigma(s, t) &= (x(s, t), y(s, t), z(s, t)) \\ D\sigma &= \begin{pmatrix} \frac{\partial}{\partial s} x(s, t) & \frac{\partial}{\partial t} x(s, t) \\ \frac{\partial}{\partial s} y(s, t) & \frac{\partial}{\partial t} y(s, t) \\ \frac{\partial}{\partial s} z(s, t) & \frac{\partial}{\partial t} z(s, t) \end{pmatrix} \end{aligned}$$

and denote  $\sigma_s, \sigma_t$  as the respective columns, so

$$A = \int_S dA = \int_\sigma dA = \iint_{(s,t) \in R} |\sigma_s(s, t) \times \sigma_t(s, t)| ds dt$$

and

$$\int_S f dA = \int_\sigma f dA = \iint_{(s,t) \in R} f(\sigma(s, t)) |\sigma_s \times \sigma_t| ds dt$$

For  $\alpha : I \subseteq \mathbb{R} \rightarrow U \subseteq \mathbb{R}^3$ ,  $F : U \rightarrow \mathbb{R}^3$ , say  $F = (P, Q, R)$ , then

$$\begin{aligned} W &= \int_C F \cdot T dL = \int_\alpha F \cdot T dL \\ &= \int_{t \in I} F(\alpha(t)) \cdot \frac{\alpha'(t)}{|\alpha'(t)|} |\alpha'(t)| dt \\ &= \int_{t \in I} (P(\alpha(t))x'(t) + Q(\alpha(t))y'(t) + R(\alpha(t))z'(t)) dt \\ &= \int_\alpha P dx + Q dy + R dz \end{aligned}$$

**Definition.** A **smooth  $k$ -form** in  $U \subseteq \mathbb{R}^n$  is an expression of the form  $a(x) = \sum_I a_I(x) dx^I$  where the sum is taken over multi-indices  $I = (i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq n$  and each  $a_I : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth map and  $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$ .

For a smooth map  $\sigma : R \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^n$  and  $s = \text{im } \sigma$  and for  $a(x) = \sum_I a_I(x) dx^I$ , we define

$$\int_S a = \int_\sigma a := \sum_I \int_R a_I(\sigma(t)) \det \left( \frac{\partial x^I}{\partial t} \right) dt^{i_1} \dots dt^{i_k}$$

where

$$\frac{\partial x^I}{\partial x} = \begin{pmatrix} \frac{\partial x^{i_1}}{\partial t^1} & \dots & \frac{\partial x^{i_1}}{\partial t^k} \\ \vdots & & \vdots \\ \frac{\partial x^{i_k}}{\partial t^1} & \dots & \frac{\partial x^{i_k}}{\partial t^k} \end{pmatrix}$$

For  $a(x) = \sum_I a_I(x) dx^I$ , we define  $da = \sum_I \sum_j \frac{\partial a_I}{\partial x_j} dx^j \wedge dx^I$ , using the rule  $dx^j \wedge dx^i = -dx^i \wedge dx^j$ . With this notation, Gauss' Theorem and Stoke's Theorem become

$$\int_S d\alpha = \int_{\partial S} \alpha$$

where  $S = \text{im } \sigma$ ,  $\sigma : R \subseteq \mathbb{R}^{k+1} \rightarrow \mathbb{R}^n$ ,  $\alpha = \sum a_I dx^I$  is a  $k$ -form, and  $d\alpha$  is a  $(k+1)$ -form.

### THE EXTERIOR ALGEBRA

If  $V$  is a vector space with basis  $\{u_1, \dots, u_n\}$ , then the dual space  $V^* = \{\text{linear maps } g : V \rightarrow \mathbb{R}\}$  has dual basis  $\{f^1, \dots, f^k\}$  where each  $f^k : V \rightarrow \mathbb{R}$  and  $f^k(u_\ell) = \delta_\ell^k$ .

We have a canonical evaluation map  $E : V \rightarrow V^{**}$  given by  $E(v)(g) = g(v)$ , which is an isomorphism.

The space  $\Lambda^k V = \{\text{alternating } k\text{-linear maps } L : V^* \times \dots \times V^* \rightarrow \mathbb{R}\}$  has a basis

$$\{U_i = U_{i_1} \wedge \dots \wedge U_{i_k} : I \text{ is an increasing multi-index}\}$$

where for  $v^i \in V$  and  $g^i \in V^*$ ,

$$(v_1 \wedge \dots \wedge v_k)(g^1, \dots, g^k) = \det \begin{pmatrix} g^1(v_1) & \dots & g^1(v_k) \\ \vdots & & \vdots \\ g^k(v_1) & \dots & g^k(v_k) \end{pmatrix}$$

Also  $\Lambda^k V^*$  has basis given similarly.

$\mathbb{R}^n$  has standard basis  $\{e_1, \dots, e_n\}$ , which we can consider as differential operators  $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ . The dual basis for  $(\mathbb{R}^n)^*$  is denoted by  $\{dx^1, \dots, dx^n\}$  where  $dx^k(\frac{\partial}{\partial x^\ell}) = \delta_\ell^k$ . So for example,

$$(dx^{i_1} \wedge \dots \wedge dx^{i_k}) \left( \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \begin{cases} (-1)^\sigma & : J = \sigma(I) \\ 0 & : \text{otherwise} \end{cases}$$

A smooth  $k$ -form on  $U \subseteq \mathbb{R}^n$  is a smooth map  $\alpha : U \subseteq \mathbb{R}^n \rightarrow \Lambda^k(\mathbb{R}^n)^*$ . Note that  $dx^I : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$  has

$$\begin{aligned} dx^I(u_1, \dots, u_k) &= dx^I \left( \sum_{j_1=1}^n u_1^{j_1} \frac{\partial}{\partial x^{j_1}}, \dots, \sum_{j_k=1}^n u_k^{j_k} \frac{\partial}{\partial x^{j_k}} \right) \\ &= \sum_{\text{all } J} u_1^{j_1} \dots u_k^{j_k} \underbrace{dx^I \left( \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right)}_{(-1)^\sigma \text{ if } J=\sigma(I); 0 \text{ otherwise}} \\ &= \sum_{\sigma \in S_n} (-1)^\sigma u_1^{i_{\sigma(1)}} \dots u_k^{i_{\sigma(k)}} \\ &= \det(A^I) \end{aligned}$$

where  $A^I$  consists of the rows  $i_1, \dots, i_k$  of the matrix  $A = (u_1, \dots, u_k)$ .

### **$k$ -FORMS AT A POINT ON A MANIFOLD**

Let  $M$  be a smooth manifold and fix a point  $p \in M$ . Given a chart  $\phi$  on  $M$  at  $p$ ,  $T_p M$  has a basis  $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ . We denote the dual basis for  $T_p^* M = (T_p M)^*$  by  $\{dx^1, \dots, dx^n\}$ . Then  $\Lambda^k T_p^* M$  has basis  $\{dx^I : I \text{ increasing}\}$  (it is  $\binom{n}{k}$  dimensional). If  $X_j = \sum_{i=1}^n u_j^i \frac{\partial}{\partial x^i} \in T_p M$ , then

$$dx^I(X_1, \dots, X_k) = \det \begin{pmatrix} u_1^{i_1} & \cdots & u_k^{i_1} \\ \vdots & & \vdots \\ u_1^{i_k} & \cdots & u_k^{i_k} \end{pmatrix}$$

An element  $\alpha \in \Lambda^k T_p^* M$  can be written uniquely as  $\alpha = \sum_{I \text{ increasing}} a_I dx^I$  with  $a_I \in \mathbb{R}$ , and  $\alpha$  is called a  $k$ -form on  $M$  at  $p$ .

### **Change of Coordinates**

Suppose that  $\phi$  and  $\psi$  are two charts on  $M$  at  $p$ , so that  $T_p M$  has bases  $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$  and  $\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\}$ , and  $T_p^* M$  has dual bases  $\{dx^1, \dots, dx^n\}$  and  $\{dy^1, \dots, dy^n\}$  with corresponding bases for  $\Lambda^k T_p^* M$ . Let  $\alpha \in \Lambda^k T_p^* M$ . Say  $\alpha = \sum_I a_I dx^I = \sum_J b_J dy^J$ . If  $X = \sum_i u^i \frac{\partial}{\partial x^i} = \sum_j v^j \frac{\partial}{\partial y^j}$ , then  $v = D(\psi \phi^{-1})$  is

$$v^j = \sum_i \frac{\partial y^j}{\partial x^i} u^i$$

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

where  $y(x) = \psi \circ \phi^{-1}(x)$ . Then for each increasing multi-index  $I$ ,

$$\begin{aligned} a_I &= \alpha \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \\ &= \left( \sum_J b_J dy^J \right) \left( \sum_{\ell_1} \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \frac{\partial}{\partial y^{\ell_1}}, \dots, \sum_{\ell_k} \frac{\partial y^{\ell_k}}{\partial x^{i_k}} \frac{\partial}{\partial y^{\ell_k}} \right) \\ &= \sum_{\text{incr } J} \sum_{\text{all } L} b_J \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\ell_k}}{\partial x^{i_k}} dy^L \left( \frac{\partial}{\partial y^{\ell_1}}, \dots, \frac{\partial}{\partial y^{\ell_k}} \right) \\ &= \sum_{\text{incr } J} \sum_{\sigma \in S_n} (-1)^\sigma b_J \frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_{\sigma(k)}}}{\partial x^{i_k}} \\ &= \sum_{\text{incr } J} b_J \det \left( \frac{\partial y^J}{\partial x^I} \right) \end{aligned}$$

where

$$\frac{\partial y^J}{\partial x^I} = \begin{pmatrix} \frac{\partial y^{j_1}}{\partial x^{i_1}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{i_1}} \\ \vdots & & \vdots \\ \frac{\partial y^{j_1}}{\partial x^{i_k}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{i_k}} \end{pmatrix}$$



When  $\alpha = \sum_I a_I dx^I = \sum_J b_J dy^J$ ,

$$a_I = \sum_J b_J \det \left( \frac{\partial y^J}{\partial x^I} \right)$$

For an increasing multi-index  $L$ , taking  $b_K = 1$  and  $b_J = 0$  for  $J \neq L$ , we obtain

$$dy^L = \sum_I a_I dx^I, a_I = 1 \cdot \det \left( \frac{\partial y^L}{\partial x^I} \right)$$

so

$$dy^L = \sum_I \det \left( \frac{\partial y^L}{\partial x^I} \right) dx^I$$

### THE WEDGE PRODUCT OR EXTERIOR PRODUCT

When  $\phi$  is a chart on  $M$  at  $p$  and  $\alpha = \sum_I a_I dx^I \in \Lambda^k T_p^* M$  and  $\beta = \sum_J b_J dx^J \in \Lambda^\ell T_p^* M$ , we would like to define  $\alpha \wedge \beta \in \Lambda^{k+\ell} T_p^* M$  by  $\alpha \wedge \beta = \sum_{I,J} a_I b_J dx^I \wedge dx^J$  (where we can use the rule  $dx^i \wedge dx^j = -dx^j \wedge dx^i$  to put the multi-index in increasing order). We need to make sure that the definition does not depend on the choice of the chart  $\phi$ . Note that for vectors  $X_1, \dots, X_{k+\ell} \in T_p M$ , given (in the chart  $\phi$ ) by  $X_j = \sum_{i=1}^n u_j^i \frac{\partial}{\partial x^i}$  we have

$$\begin{aligned} (dx^I \wedge dx^J)(X_1, \dots, X_{k+\ell}) &= \det \begin{pmatrix} u_1^{i_1} & \cdots & u_{k+\ell}^{i_1} \\ \vdots & & \vdots \\ u_1^{i_k} & \cdots & u_{k+\ell}^{i_k} \\ u_1^{j_1} & \cdots & u_{k+\ell}^{j_1} \\ \vdots & & \vdots \\ u_1^{j_\ell} & \cdots & u_{k+\ell}^{j_\ell} \end{pmatrix} \\ &= \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma u^{i_1}_{\sigma(1)} \cdots u^{i_k}_{\sigma(k)} u^{j_1}_{\sigma(k+1)} \cdots u^{j_\ell}_{\sigma(k+\ell)} \\ &= \sum_{\tau} \sum_{\mu} \sum_{\nu} (-1)^\tau (-1)^\mu (-1)^\nu u^{i_1}_{\mu(\tau(1))} \cdots u^{i_k}_{\mu(\tau(k))} u^{j_1}_{\nu(\tau(k+1))} \cdots u^{j_\ell}_{\nu(\tau(k+\ell))} \end{aligned}$$

where the sums are over  $\tau$  a permutation of  $\{1, \dots, k+\ell\}$  so that  $\tau(1) < \cdots < \tau(k)$  and  $\tau(k+1) < \cdots < \tau(k+\ell)$ ,  $\mu$  is a permutation of  $\{\tau(1), \dots, \tau(k)\}$  and  $\nu$  is a permutation of  $\{\tau(k+1), \dots, \tau(k+\ell)\}$ , so that

$$\begin{aligned} &= \sum_{\tau} (-1)^\tau \det \begin{pmatrix} u_{\tau(1)}^{i_1} & \cdots & u_{\tau(1)}^{i_k} \\ \vdots & & \vdots \\ u_{\tau(k)}^{i_1} & \cdots & u_{\tau(k)}^{i_k} \end{pmatrix} \det \begin{pmatrix} u_{\tau(k+1)}^{j_1} & \cdots & u_{\tau(k+1)}^{j_\ell} \\ \vdots & & \vdots \\ u_{\tau(k+\ell)}^{j_1} & \cdots & u_{\tau(k+\ell)}^{j_\ell} \end{pmatrix} \\ &= \sum_{\tau} (-1)^\tau dx^I(X_{\tau(1)}, \dots, X_{\tau(k)}) dx^J(X_{\tau(k+1)}, \dots, X_{\tau(k+\ell)}) \end{aligned}$$

Thus for  $\alpha = \sum_I a_I dx^I$ ,  $\beta = \sum_J b_J dx^J$ ,  $\gamma = \sum_{I,J} a_I b_J dx^I \wedge dx^J$  we have

$$\gamma(X_1, \dots, X_{k+\ell}) = \sum_{\tau \in T_{k,\ell}} (-1)^\tau \alpha(X_{\tau(1)}, \dots, X_{\tau(k)}) \cdot \beta(X_{\tau(k+1)}, \dots, X_{\tau(k+\ell)})$$

where  $T_{k,l}$  is the set of permutations  $\tau$  of  $\{1, \dots, k + \ell\}$  such that  $\tau(1) < \dots < \tau(k)$  and  $\tau(k+1) < \dots < \tau(k+\ell)$ .

**Definition.** When  $f : M \rightarrow N$  is a smooth map of smooth manifolds and  $p \in M$ , we define the **pullback**

$$f^* = f^*(p) : \Lambda^k T_{f(p)}^* N \rightarrow \Lambda^k T_p^* M$$

by  $f^*(\beta)(X_1, \dots, X_k) = \beta(f_* X_1, \dots, f_* X_k)$  where  $\beta \in \Lambda^k T_{f(p)}^* N$  and each  $X_j \in T_p M$  so that  $f_* X_j \in T_{f(p)} N$ .

Let  $M$  be a chart on  $M$  at  $p$  and  $\psi$  a chart on  $N$  at  $f(p)$ . Let  $\beta = \sum b_J dx^J$ , write  $X_j = \sum_i u_j^i \frac{\partial}{\partial x^i}$  and say  $\alpha = f^* \beta = \sum_I a_I dx^I$ .

$$\begin{aligned} a_I &= \alpha \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \\ &= \beta \left( f_* \frac{\partial}{\partial x^{i_1}}, \dots, f_* \frac{\partial}{\partial x^{i_k}} \right) \\ &= \left( \sum_J b_J dy^J \right) \left( \sum_{\ell_1} \frac{\partial y^{\ell_1}}{\partial x^{i_1}}, \dots, \sum_{\ell_k} \frac{\partial y^{\ell_k}}{\partial x^{i_k}} \right) \\ &= \sum_J \sum_{\text{incr all } L} B_J \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \dots \frac{\partial y^{\ell_k}}{\partial x^{i_k}} dy^J \left( \frac{\partial}{\partial y^{\ell_1}}, \frac{\partial}{\partial y^{\ell_k}} \right) \\ &= \sum_J \sum_{\sigma \in S_k} (-1)^\sigma b_J \frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}} \dots \frac{\partial y^{j_{\sigma(k)}}}{\partial x^{i_k}} \\ &= \sum_J b_J \det \left( \frac{\partial y^J}{\partial x^I} \right) \end{aligned}$$

where  $y = y(x) = (\psi f \phi^{-1})(x)$ .

**Definition.** A  $k$ -form at each point  $p$  on a smooth manifold  $M$  is given by a map  $\alpha : M \rightarrow \bigcup_{p \in M} \Lambda^k T_p^* M$ , where  $\alpha(p) \in \Lambda^k T_p^* M$  for all  $p \in M$ . We say that such a map  $\alpha$  is **smooth** at  $p \in M$  when for some (hence for every) chart  $\phi : U \subseteq M \rightarrow \phi(U) \subseteq \mathbb{R}^m$  on  $M$  at  $p$ , when we write the restriction of  $\alpha$  to  $U$  as  $\alpha(p) = \sum_I \alpha_I(p) dx^I$ , the coefficient functions  $\alpha_I : U \subseteq M \rightarrow \mathbb{R}$  are smooth. Such a map  $\alpha : M \rightarrow \bigcup_{p \in M} \Lambda^k T_p^* M$  is called smooth (on  $M$ ) when it is smooth at every point  $p \in M$ .

Another way to think about smooth  $k$ -forms is as follows. Consider  $\alpha : M \rightarrow \bigcup_{p \in M} \Lambda^k T_p^* M$  with  $\alpha(p) \in \Lambda^k T_p^* M$  for all  $p \in M$ . Let  $\Lambda^k T^* M = \bigcup_{p \in M} \Lambda^k T_p^* M$  and define the **projection map**  $\pi : \Lambda^k T^* M \rightarrow M$  by  $\pi(\alpha_p) = p$ , when  $\alpha_p \in \Lambda^k T_p^* M$ . We give  $\Lambda^k T^* M$  the structure of a smooth vector bundle of rank  $\binom{n}{k}$  on  $M$  as follows. For each chart  $\phi : U \rightarrow \phi(U)$  on  $M$ , we define a chart

$$\Phi : \pi^{-1}(U) = \bigcup_{p \in U} \Lambda^k T_p^* M \rightarrow \phi(U) \times \Lambda^k(\mathbb{R}^n)^* \equiv \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$

by  $\Phi(\alpha_p) = (\phi(p), \sum_I \alpha_I(p) dx^I)$ , where the restriction of  $\alpha$  to  $U$  is given by  $\alpha(p) = \sum_I \alpha_I(p) dx^I$ .

With this definition, a smooth  $k$ -form on  $M$  is a smooth map  $\alpha : M \rightarrow \Lambda^k T^* M$  such that  $\pi(\alpha(p)) = p$ . We denote the space of all  $k$ -forms on  $M$  by  $\Omega^k(M)$  or  $\Gamma(M, \Lambda^k T^* M)$  (sections) or  $\Gamma(\Lambda^k T^* M)$ .

When  $U \subseteq \mathbb{R}^n$  is open and  $p \in U$ , we often identify

$$\begin{aligned} T_p U &= T_p \mathbb{R}^n = \mathbb{R}^n = M_{n \times 1}(\mathbb{R}) \\ T_p^* U &= T_p^* \mathbb{R}^n = \mathbb{R}^n = M_{1 \times n}(\mathbb{R}) \\ \Lambda^k T_p^* U &= \Lambda^k T_p^* \mathbb{R}^n = \Lambda^k(\mathbb{R}^n)^* = \text{span}\{dx^I : I \text{ incr}\} \\ TU &= \bigcup_{p \in U} T_p U = \bigcup_{p \in U} \mathbb{R}^n = U \times \mathbb{R}^n \\ T^* U &= U \times (\mathbb{R}^n)^* \\ \Lambda^k T^* U &= U \Lambda^k(\mathbb{R}^n)^* \\ \mathfrak{X}(U) &= \Gamma(TU) = C^\infty(U, \mathbb{R}^n) \\ \Omega^k(U) &= \Gamma(\Lambda^k T^* U) = \Gamma(U \times \Lambda^k(\mathbb{R}^n)^*) = C^\infty(U, \Lambda^k(\mathbb{R}^n)^*) \end{aligned}$$

where we identify  $\Gamma(TU) = C^\infty(U, \mathbb{R}^n)$  by  $X : U \rightarrow U \times \mathbb{R}^n$  and  $X(p) = (p, u(p))$  with  $u : U \rightarrow \mathbb{R}^n$ .

When  $U \subseteq M$  is open and  $p \in U$ ,  $T_p U = T_p M$ ,  $T_p^* U = T_p^* M$ ,  $\Lambda^k T_p^* U = \Lambda^k T_p^* M$ . When  $\phi : U \rightarrow \phi(U)$  is a chart on  $M$  and if  $X \in \mathfrak{X}(M) = \Gamma(TM)$  is given locally by  $X(p) = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x^i}$  then

$$(\phi_* X)(x) = \sum_{i=1}^n u^i(x) \frac{\partial}{\partial x^i}$$

for  $x \in \phi(U)$  where  $u^i(x) = X^i(\phi^{-1}(x))$ ,  $X^i(p) = u^i(\phi(p))$ . If  $\alpha \in \Omega^k(M) = \Gamma(\Lambda^k T^* M)$  and the restriction of  $\alpha$  to  $U$  is given by

$$\alpha(p) = \sum_{I \text{ incr}} \alpha_I(p) dx^I$$

then

$$(\phi^{-1})^*(\alpha)(x) = \sum_{I \text{ incr}} a_I(x) dx^I$$

where

$$a_I(x) = \alpha_I(\phi^{-1}(x)) \quad \alpha_I(p) = a_I(\phi(p))$$

Note that when  $f \in C^\infty(M)$ , we have  $df = f_* \in \Omega^1(M)$ . Indeed, we have  $f : M \rightarrow \mathbb{R}$ ,  $df(p) = f_*(p) : T_p M \rightarrow T_p \mathbb{R} = \mathbb{R}$ , so  $df(p) = f_*(p) \in T_p^* M$  so that  $df = f_* : M \rightarrow T^* M$  with  $df(p) = f_*(p) \in T_p^* M$  for all  $p \in M$ .

Locally,  $df = f_*$  is given by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

To be precise, if  $\phi : U \rightarrow \phi(U)$  is a chart on  $M$  then the restriction of  $f$  to  $U$  is given by

$$df(p) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i$$

so that

$$(\phi^{-1})^*(df)(x) = \sum_{i=1}^n \frac{\partial f \circ \phi^{-1}}{\partial x^i} dx^i$$

**Definition.** When  $M$  is a smooth manifold, we define a smooth 0-form on  $M$  to be a smooth function  $f : M \rightarrow \mathbb{R}$ .

We define

$$\begin{aligned}\Omega^0(M) &= C^\infty(M) = C^\infty(M, \mathbb{R}) \\ \Lambda^0 T_p^* M &= \mathbb{R} \\ \Lambda^0 T^* M &= \bigcup_{p \in M} \mathbb{R} = M \times \mathbb{R}\end{aligned}$$

so that  $\Omega^0(M) = \Gamma(\Lambda^0 T^* \mathbb{R}) = C^\infty(M)$ .

Now, we want to define  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  by the local definition

$$d\left(\sum_I a_I x^*\right) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

To be precise, given  $\alpha \in \Omega^k(M)$ , if  $\phi : U \rightarrow \phi(U)$  is a chart and the restriction of  $\alpha$  to  $U$  is given by  $\alpha(p) = \sum \alpha_I(p) dx^I$ , so

$$(\phi^{-1})^*(\alpha)(x) = \sum_I a_I(x) dx^I$$

then we want  $d\alpha$  restricted to  $U$  to be given by

$$d\alpha(p) = \sum d\alpha_I(p) \wedge dx^I$$

that is

$$(\phi^{-1})(d\alpha)(x) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

However, it is not immediate that this definition does not depend on the chosen chart  $\phi$ .

**5.1 Theorem. (Exterior Derivative)** Let  $M$  be a smooth manifold. Then there exists a unique  $\mathbb{R}$ -linear map  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  for all  $k \geq 0$  such that

1.  $d : \Omega^0(M) \rightarrow \Omega^1(M)$  is given by  $df = f_*$  (as previously defined)
2. If  $\alpha \in \Omega^k(M)$  and  $\beta \in \Omega^\ell(M)$ ,  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
3.  $d^2 = 0$  (where  $d^2 = d \circ d$ ).

Such a map  $d$  is given locally by  $d\left(\sum_I a_I dx^I\right) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$ .

**PROOF** We claim that if such a map  $d$  exists, then  $d$  is determined locally: for  $\alpha, \beta \in \Omega^k(M)$ , if  $\alpha(p) = \beta(p)$  for all  $p \in U$  where  $U \subseteq \mathbb{R}^n$  is open, then  $d\alpha(p) = d\beta(p)$  for all  $p \in U$ . Suppose  $\alpha = \beta$  in  $U$ . Let  $\gamma = \beta - \alpha$ , so  $\gamma = 0$  in  $U$ . Let  $p \in U$ .

Choose a smooth bump function  $S : M \rightarrow \mathbb{R}$  with  $S = 1$  in a neighbourhood of  $p$  and  $\text{supp}(S) \subseteq U$ . Then  $s\gamma = 0 \in \Omega^k(M)$ , so that

$$\begin{aligned}0 &= d(s\gamma) \\ &= ds \wedge \gamma + s \wedge d\gamma \\ &= 0 \wedge \gamma + 1 \wedge d\gamma\end{aligned}$$

in a neighbourhood of  $p$ . Thus  $d\gamma = 0$  in a neighbourhood of  $p$ , so that  $d\gamma(p) = 0$ . In particular,  $p \in U$  was arbitrary, so  $\gamma = 0$  in  $U$ .

We now claim that if such a map  $d$  exists, then  $d$  must be given locally by the formula  $d(\sum_I a_I dx^I) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$ . To be precise, if  $\alpha \in \Omega^k(M)$  and  $\phi : U \rightarrow \phi(U)$  is a chart, and if the restriction of  $\alpha$  to  $U$  is given by  $\alpha(p) = \sum_I \alpha_I(p) dx^I$  so that  $(\phi^{-1})^*(\alpha)(x) = \sum_I a_I(x) dx^I$  then  $d\alpha$ , restricted to  $U$ , is given by

$$(d\alpha)(p) = \sum_I d\alpha_I(p) dx^I$$

so that

$$(\phi^{-1})^*(d\alpha)(x) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j}(x) dx^j \wedge dx^I$$

Suppose  $d$  exists. Let  $\alpha \in \Omega^k(M)$  and  $\phi : U \rightarrow \phi(U)$  be a chart on  $M$ . Say the restriction of  $\alpha$  to  $U$  is given by  $\alpha(p) = \sum_I \alpha_I(p) dx^I$  for all  $p \in U$ . Fix  $p \in U$ . Use a smooth step function which is equal to 1 in an open neighbourhood of  $p$  and supported in  $U$  to construct smooth functions  $\beta_I : M \rightarrow \mathbb{R}$  and  $\gamma^i : M \rightarrow \mathbb{R}$  such that  $\beta_I = \alpha_I$  and  $\gamma^i = \phi^i$  in an open neighbourhood of  $p$  (where  $\phi : U \rightarrow \phi(U)$  is given by  $\phi = (\phi^1, \dots, \phi^n)$ ). Define  $\beta \in \Omega^k(M)$  by  $\beta = \sum_I \beta_I d\gamma^I$ . Then  $\beta = \alpha$  in an open neighbourhood of  $p$ . Then in this neighbourhood  $d\alpha = d\beta = d(\sum_I \beta_I d\gamma^I) = \sum_I d\beta_I \wedge d\gamma^I$  by linearity and rules (2) and (3). Also, by our knowledge of  $d : \Omega^0(M) \rightarrow \Omega^1(M)$  in the open neighbourhood, we have  $d\beta_I = d\alpha_I$  and  $d\gamma^i = d\phi^i = dx^i$ .

On the other hand, given a chart  $\phi : U \rightarrow \phi(U)$ , we can define  $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$  by the local formula

$$d(\sum_I \alpha_I dx^I) = \sum_I d\alpha_I \wedge dx^I$$

and then  $d$  is linear and satisfies the required properties since

$$\begin{aligned} d\left(\sum_I a_I dx^I \wedge \sum_J b_J dx^J\right) &= d\left(\sum_{I,J} a_I b_J dx^I \wedge dx^J\right) \\ &= \sum_{I,J} \sum_j \left(\frac{\partial a_I}{\partial x^j} b_J + a_I \frac{\partial b_J}{\partial x^i}\right) dx^j \wedge dx^I \wedge dx^J \\ &= \sum_{I,J} \sum_j \left(\frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I\right) \wedge b_J dx^J + (-1)^k (a_I dx^I) \wedge \left(\frac{\partial b_J}{\partial x^j} dx^j \wedge dx^I\right) \end{aligned}$$

and

$$\begin{aligned} d^2(\sum_I a_I dx^I) &= d\left(\sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I\right) \\ &= \sum_I \sum_{i,j} \frac{\partial^2 a_I}{\partial x^i \partial x^j} dx^i \wedge dx^j \wedge dx^I \\ &= 0 \end{aligned}$$

Finally, note that we can define  $d : \Omega^k \rightarrow \Omega^{k+1}$  globally by using the local formula in the domains of all charts, and the definitions agree on intersections of charts. ■

As an exercise, we have the following formulas:

- (i) For  $\alpha \in \Omega^k(M)$ ,  $\beta \in \Omega^\ell(M)$ ,  $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$ .
- (ii) For  $f : L \rightarrow M$ ,  $g : M \rightarrow N$ ,  $(g \circ f)^* = f^* \circ g^*$ . Also, if  $f : M \rightarrow M$  is the identity, then  $f^* : \Omega^k(M) \rightarrow \Omega^k(M)$  is the identity.
- (iii)  $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$
- (iv)  $d \circ f^* = f^* \circ d$
- (v) For  $\alpha \in \Omega^k(M)$  and each  $X_i \in \mathfrak{X}(M)$ , then

$$\begin{aligned} d\alpha(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\alpha(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \end{aligned}$$

In particular, when  $\alpha = f \in \Omega^0(M)$ ,  $df(X) = X(f)$  and when  $\alpha \in \Omega^1$ ,

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

## 6 INTEGRATION

If  $\alpha(x)dx^1 \wedge \dots \wedge dx^n$  is a smooth  $n$ -form on an open set  $U \subseteq \mathbb{R}^n$  with compact support in  $U$ , we define

$$\begin{aligned} \int_U \alpha &= \int_U a(x) dx^1 \wedge \dots \wedge dx^n \\ &= \int_U a(x) dx^1 dx^2 \dots dx^n \\ &= \int_{\mathbb{R}^n} a(x) dx^1 \dots dx^n \end{aligned}$$

where we extend  $a(x)$  by zero to all of  $\mathbb{R}^n$ .

Note that if  $f : U \rightarrow V$  is a diffeomorphism of open sets  $U, V$  in  $\mathbb{R}^n$  which preserves orientation, then for a smooth  $n$ -form  $\beta$  with compact support in  $V$ ,

$$\int_U f^* \beta = \int_V \beta$$

(if  $\beta(y) = b(y)dy^1 \wedge \dots \wedge dy^n$  then

$$(f^* \beta)(x) = b(f(x)) \det(Df) dx^1 \wedge \dots \wedge dx^n$$

so that

$$\begin{aligned} \int_U f^* \beta &= \int_U b(f(x)) |\det Df(x)| dx^1 \dots dx^n \\ &= \int_V b(y) dy^1 \dots dy^n = \int_V \beta \end{aligned}$$

Let  $M$  be a smooth oriented  $n$ -dimensional manifold and  $\omega$  a smooth  $n$ -form on  $M$  with compact support. We define the integral as follows. Choose finitely many positively

oriented charts  $\phi_i : U_i \subseteq M \rightarrow \phi_i(U_i) \subseteq \mathbb{R}^n$  such that  $\text{supp}(\omega) \subseteq \bigcup_{i=1}^m U_i =: U$ . Let  $U_0 = M \setminus \text{supp}(\omega)$  so that  $M = \bigcup_{i=0}^m U_i$ . Choose a partition of unity  $\{\rho_i : 0 \leq i \leq m\}$  subordinate to the cover  $\{U_i : 0 \leq i \leq m\}$  and note that

$$\omega = \sum_{i=0}^m \rho_i \omega = \sum_{i=0}^m \rho_i \omega$$

Then we define

$$\int_M \omega = \sum_{i=0}^m \int_M \rho_i \omega$$

where

$$\int_M \rho_i \omega = \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\rho_i \omega) = \int_{\mathbb{R}^n} (\phi_i)^{-1}(\rho_i \omega)$$

**6.1 Proposition.** *Integration, as define here, does not depend on the choice of charts or the choice of the partition of unity  $\rho_i$ .*

**PROOF** Let  $\psi_i : V_i \rightarrow \psi_i(V_i)$  be positively oriented charts  $\text{supp}(\omega) \subseteq V \subseteq \bigcup_{j=1}^\ell V_j$  and add  $V_0 = M \setminus \text{supp}(\omega)$ . Let  $\sigma_j$  be a subordinate partition of unity. Then for each  $i$  and  $j$ , we can restrict  $\phi_i : U_i \rightarrow \phi_i(U_i)$  and  $\phi_j : V_j \rightarrow \psi_j(V_j)$  to get new charts

$$\begin{aligned} \phi_{i,j} : U_i \cap V_j &\rightarrow \phi_{i,j}(U_i \cap V_j) \\ \psi_{i,j} : U_i \cap V_j &\rightarrow \psi_{i,j}(U_i \cap V_j) \end{aligned}$$

and a subordinate partition of unity  $\{\rho_i \sigma_j : 0 \leq i \leq m, 0 \leq j \leq \ell\}$  subordinate to  $U_i \cap V_j$ . Then

$$\begin{aligned} \sum_j \int_{\psi_j(V_j)} (\psi_j^{-1})^* \sigma_j \omega &= \sum_j \int_{\psi_j(V_j)} (\psi_j^{-1})^* \left( \sum_i \rho_i \cdot \sigma_j \omega \right) \\ &= \sum_{i,j} \int_{\psi_j(V_j)} (\psi_j^{-1})^* \rho_i \sigma_j \omega \\ &= \sum_{i,j} \int_{\psi_j(U_i \cap V_j)} (\psi_{i,j}^{-1})^* \rho_i \sigma_j \omega \\ &= \sum_{i,j} \int_{\psi_{i,j}(U_i \cap V_j)} (\psi_{i,j}^{-1})^* \rho_i \sigma_j \omega \\ &= \sum_{i,j} \int_{\phi_{i,j}(U_i \cap V_j)} (\phi_{i,j}^{-1})^* \rho_i \sigma_j \omega \\ &= \sum_i \int_{\phi_i(U_i \cap V)} (\phi_i^{-1})^* \rho_i \sum_j \sigma_j \omega \\ &= \sum_i \int_{\phi_i(U_i)} (\phi_i^{-1})^* \rho_i \omega \end{aligned}$$

■

**6.2 Proposition.** *If  $M$  and  $N$  are oriented  $n$ -dimensional manifolds and if  $f : N \rightarrow M$  is a positively oriented diffeomorphism, then for a smooth  $n$ -form on  $M$  with compact support, we have*

$$\int_N f^* \omega = \int_M \omega$$

When  $f : N \rightarrow M$  is a smooth immersion of smooth oriented manifolds of dimensions  $n = \dim N$  and  $M = \dim M$  and  $\alpha \in \Omega^n(M)$ , we often write

$$\int_{f(N)} \omega = \int_N f^* \omega.$$

When  $N \subseteq M$  and the inclusion map  $\iota : N \hookrightarrow M$  is an immersion, we write  $\int_N \alpha = \int_N \iota^* \alpha$ .

*Example.* When  $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\beta \in \Omega^n(\mathbb{R}^m)$  is given by  $\beta = \sum_J b_J(y) dy^J$ , the pullback  $\alpha = f^* \beta$  is given by

$$\alpha = \sum_J b_J(f(x)) \det \left( \frac{\partial y^J}{\partial x} \right)$$

and

$$\int_{f(D)} \sum_J b_J(f(x)) \det \frac{\partial y^J}{\partial x} dx^1 \dots dx^n$$

**6.3 Theorem.** *Let  $M$  be a smooth orientable manifold and suppose  $M$  has a finite positively oriented atlas with charts  $\phi_i : U_i \subseteq M \rightarrow \phi_i(U_i) \subseteq \mathbb{R}^n$  for  $0 \leq i \leq m$  such that  $\phi_j \setminus (U_j) \setminus \phi_j(U_i \cap U_0)$  has measure zero for all  $j \neq 0$ . Then*

$$\int_M \alpha = \int_{U_0} \alpha|_{U_0} = \int_{\phi_0(U_0)} (\phi_0^{-1})^* \alpha$$

**PROOF** Let  $\rho_k : 0 \leq k \leq m$  be a partition of unity subordinate to the cover  $\{U_k : 0 \leq k \leq m\}$ . Then

$$\begin{aligned} \int_M \alpha &= \sum_{k=0}^m \int_{\phi_k(U_k)} (\phi_k^{-1})^* \phi_k \alpha \\ &= \sum_{k=0}^m \int_{\phi_k(U_k \cap U_0)} (\phi_k^{-1})^* (\phi_k \alpha) \\ &= \sum_{k=0}^m \int_{\phi_k(U_k \cap U_0)} (\phi_{k,0}^{-1})^* (\rho_k \alpha)|_{U_k \cap U_0} \\ &= \int_{U_0} \alpha|_{U_0} \\ &= \int_{\phi_0(U_0)} (\phi_0^{-1})^* \alpha \end{aligned}$$

where  $\phi_{k,0}$  is the restriction of  $\phi_k$  to  $U_k \cap U_0$ . ■



*Example.* Let  $\alpha = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \in \Omega^2(\mathbb{R}^3)$ . Find  $\int_{\mathbb{S}^2} \alpha = \int_{\mathbb{S}^2} \iota^* \alpha$ , where  $\iota: \mathbb{S}^2 \rightarrow \mathbb{R}^3$  is the inclusion map.

Using spherical coordinates chart given by  $(x, y, z) = \phi^{-1}(\psi, \theta) = (\sin \psi \cos \theta, \sin \psi \sin \theta, \cos \psi)$  for  $\psi \in (0, \pi)$  and  $\theta \in (0, 2\pi)$ . We have

$$D\phi^{-1} = \begin{pmatrix} \cos \psi \cos \theta & -\sin \psi \sin \theta \\ \cos \psi \sin \theta & \sin \psi \cos \theta \\ -\sin \psi & 0 \end{pmatrix}$$

Then

$$\begin{aligned} \int_{\mathbb{S}^2} \alpha &= \int_{(0, \pi) \times (0, 2\pi)} (\phi^{-1})^* \alpha \\ &= \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} \sin \psi \cos \theta \det \begin{pmatrix} \cos \psi \sin \theta & \sin \psi \cos \theta \\ -\sin \psi & 0 \end{pmatrix} - \sin \psi \cos \theta \det \begin{pmatrix} \cos \psi \cos \theta & -\sin \psi \sin \theta \\ -\sin \psi & 0 \end{pmatrix} + \cos \psi d\psi d\theta \\ &= \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} (\sin^3 \psi \cos^2 \theta + \sin^3 \psi \sin^2 \theta + \sin \psi \cos^2 \psi) d\psi d\theta \\ &= \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} \sin \psi d\psi d\theta \\ &= 4\pi \end{aligned}$$

If  $\alpha = x \wedge dz + \dots$ , then

$$\begin{aligned} dy &= \frac{\partial y}{\partial \psi} d\psi + \frac{\partial y}{\partial \theta} d\theta = \sin \theta d\psi + \cos \psi \sin \theta d\theta \\ dz &= \frac{\partial z}{\partial \psi} d\psi + \frac{\partial z}{\partial \theta} d\theta = -\sin \psi d\psi \end{aligned}$$

For the sphere, let  $\sigma(\psi, \theta) = \phi^{-1}(\psi, \theta)$ . Then

$$\begin{aligned} A &= \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} |\sigma_{\psi} \times \sigma_{\theta}| d\psi d\theta \\ &= \iint \sqrt{\sin^4 \psi \cos^2 \theta + \sin^4 \psi \sin^2 \theta + \sin^2 \psi \cos^2 \psi} d\psi d\theta \\ &= \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} \sin \psi d\psi d\theta \end{aligned}$$

*Example.* Let  $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$  be given in spherical coordinates by  $f(\psi, \theta) = (\psi, 2\theta)$  (this really means  $(\phi f \psi^{-1})(\psi, \theta) = (\psi, 2\theta)$ ). Find  $\int_{\mathbb{S}^2} f^* \alpha$  where  $\alpha$  is as above.

$$\begin{aligned}
\int_{\mathbb{S}^2} f^* \alpha &= \int_{(0,\pi) \times (0,2\pi)} (\phi^{-1})^* f^* \iota^* \alpha \\
&= \int_{(0,\pi) \times (0,2\pi)} (\phi^{-1})^* f^* \phi^* (\phi^{-1})^* \alpha \\
&= \int_{(0,\pi) \times (0,2\pi)} (\phi f \phi^{-1})^* (\phi^{-1})^* \alpha \\
&= \int_{\psi=0}^{\pi} \int_{\theta=0}^{2\pi} \det \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \sin \psi d\psi d\theta \\
&= 8\pi
\end{aligned}$$

In fact, for any smooth map  $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ ,  $\frac{1}{4\pi} \int f^* \alpha \in \mathbb{Z}$ . We define the **degree** of  $f$  to be this value.

## 7 MANIFOLDS WITH BOUNDARY

**Definition.** An  $n$ -dimensional **topological manifold with boundary** is a Hausdorff, second-countable topological space  $M$  such that for every point  $p \in M$  has an open neighbourhood which is homeomorphic either to an open set in  $\mathbb{R}^n$  or to an open set in  $\mathbb{H}^n$ . Such a manifold with boundary can be given an atlas of charts  $\phi_k : U_k \rightarrow \mathbb{H}^n$ ,  $k \in K$ , such that for each  $k \in K$ , either  $\phi_k(U_k) \subseteq \mathbb{R}^n$  or  $\phi_k(U_k) \subseteq \mathbb{H}^n$  with  $\phi_k(U_k) \cap \partial\mathbb{H}^n \neq \emptyset$ . We say that  $M$  is **smooth** when it has such an atlas for which the transition maps

$$\phi_\ell \circ \phi_k^{-1} : \phi_k(U_k \cap U_\ell) \rightarrow \phi_\ell(U_k \cap U_\ell)$$

are all **smooth** where for  $A \subseteq \mathbb{R}^n$ , we say  $g : A \rightarrow \mathbb{R}^m$  is smooth when  $g$  extends to a smooth map  $g : U \rightarrow \mathbb{R}^m$  where  $U$  is open with  $A \subseteq U$ . A **boundary point** of such a smooth manifold  $M$  with boundary is a point  $p \in M$  for which there is a chart  $\phi$  which sends  $p$  to  $\partial\mathbb{H}^n$ .