

Fractal Geometry

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I. Topics in Fractal Geometry

1 DIMENSION THEORY

1.1 CONSTRUCTING MEASURES IN METRIC SPACES

[*TODO: fill in proofs and transfer to measure section*] Let X be a metric space.

Definition. Given $A, B \subseteq X$, say $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Say A, B have **positive separation** if $d(A, B) > 0$.

If A, B are compact and disjoint, then they have positive separation. We say that an outer measure μ^* is a **metric outer measure** if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ when A, B have positive separation.

Example. The Lebesgue outer measure is a metric outer measure. [*TODO: prove*]

1.1 Theorem. μ^* is a metric outer measure if and only if every Borel set is μ^* -measurable (in the sense of Caratheodory).

PROOF [*TODO: prove this (homework), and find a proof of the converse? (may not be true)*] ■

Suppose $\mathcal{A} \subseteq \mathcal{B}$ are both covers of X containing \emptyset and $\mathcal{C} : \mathcal{B} \rightarrow [0, \infty]$ with $\mathcal{C}(\emptyset) = 0$. Let $\mu_{\mathcal{A}}^*$ and $\mu_{\mathcal{B}}^*$ be the corresponding extensions of \mathcal{C} and $\mathcal{C}|_{\mathcal{A}}$. Then by definition, $\mu_{\mathcal{B}}^*(E) \leq \mu_{\mathcal{A}}^*(E)$ for all $E \in \mathcal{P}(X)$.

Let X be a metric space, \mathcal{A} cover X containing \emptyset . Suppose for each $x \in X$ and $\delta > 0$, there exists $A \in \mathcal{A}$ such that $x \in A$ and $\text{diam } A \leq \delta$. Let $\mathcal{C} : \mathcal{A} \rightarrow [0, \infty]$ with $\mathcal{C}(\emptyset) = 0$. Set $\mathcal{A}_{\epsilon} = \{A \in \mathcal{A} : \text{diam}(A) \leq \epsilon\}$, and define μ_{ϵ}^* by extending $\mathcal{C}|_{\mathcal{A}_{\epsilon}}$. In particular, as ϵ decreases, μ_{ϵ}^* increases, and define

$$\mu^*(E) = \sup_{\epsilon} \mu_{\epsilon}^*(E) = \lim_{\epsilon \rightarrow 0} \mu_{\epsilon}^*(E)$$

1.2 Theorem. As defined above, μ^* is a metric outer measure.

PROOF [*TODO: prove this, homework*] ■

Example. The Lebesgue measure arises this way; in fact, the μ_{ϵ}^* are all the same outer measure.

Definition. We say that a collection of subsets \mathcal{C} is a **semi-algebra** if it contains \emptyset , is closed under finite intersections, and complements are finite disjoint unions of sets in \mathcal{C} . We then say that μ is a **measure on a semi-algebra** if $\mu : \mathcal{C} \rightarrow [0, \infty]$ has

- (i) $\mu(\emptyset) = 0$
- (ii) If $E_1, \dots, E_n \in \mathcal{C}$ are disjoint and $\bigcup_{i=1}^n E_i \in \mathcal{C}$, then $\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$.
- (iii) If $\{E_i\}_{i=1}^{\infty} \in \mathcal{C}$ are pairwise disjoint and $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$

An **algebra** is a semi-algebra which is closed under finite unions and complements. Then a **measure on an algebra** is a map μ satisfying the same above constraints.

1.3 Theorem. *A measure μ on a semi-algebra \mathcal{C} has a unique extension to a measure on $\mathcal{A} = \langle \mathcal{C} \rangle$, the algebra generated by \mathcal{C} .*

PROOF It is easy to verify that \mathcal{A} is the set of all finite unions of elements in \mathcal{C} . Thus we extend μ to \mathcal{A} where if $A = \bigcup_{i=1}^n C_i$, set $\mu(A) = \sum_{i=1}^n \mu(C_i)$.

Check: well-defined and a measure ■

We then appeal to Caratheodory extension theorem to get a measure μ (on a σ -algebra) that extends μ from \mathcal{A} .

Let $\Sigma = \{1, \dots, k\}$ be our alphabet and let Σ^* denote the set of all words on Σ . We then associate to Σ^* a heirarchy of subsets of \mathbb{R}^n where to each $\sigma \in \Sigma$, get some subset X_σ ; set $\mathcal{E} = \{X_\sigma : \sigma \in \Sigma^*\}$. We also assume that

$$X_\sigma \supseteq \bigcup_{i=1}^k X_{\sigma i}$$

disjointly. Suppose $\tilde{\mu} : \mathcal{E} \rightarrow [0, \infty]$ has $\mu(X_\sigma) = \sum_{i=1}^k \mu(X_{\sigma i})$. We assume that for every infinite sequence (i_1, i_2, \dots) , with $\sigma_j = (i_1, \dots, i_j)$, $\lim_{j \rightarrow \infty} |X_{\sigma_j}| = 0$ and $\lim_{j \rightarrow \infty} \mu(X_{\sigma_j}) = 0$ uniformly with respect to length.

We set $E_k = \bigcup_{\omega \in \Sigma^k} X_\omega$ and $E = \bigcap_{k=1}^\infty E_k$. Let $\mathcal{C} = \{\emptyset, X_\omega \cap E : \omega \in \Sigma^*\}$. Define $\mu : \mathcal{C} \rightarrow [0, \infty]$ by $\mu(X_\omega \cap E) = \tilde{\mu}(X_\omega)$ and $\mu(\emptyset) = 0$.

1.4 Proposition. *\mathcal{C} is a semialgebra and μ is a measure on a semialgebra.*

PROOF We have closure under finite intersections due to the disjoint / nested property. Moreover,

$$(X_\omega \cap E)^c = \bigcup_{\substack{\sigma \in \Sigma^{|\omega|} \\ \sigma \neq \omega}} X_\sigma \cap E$$

is closed under complementation.

It is left as an exercise to verify that μ is a measure. ■

Thus μ extends to a measure μ on the σ -algebra generated by \mathcal{C} . Moreover, since $\text{diam } X_\omega \rightarrow 0$ as $|\omega| \rightarrow \infty$, μ extends to a metric outer measure and hence the σ -algebra contains the Borel sets.

1.2 HAUSDORFF MEASURE AND DIMENSION

For the remainder of this chapter, if X is a metric space and $U \subseteq X$, we denote $|U| = \text{diam}(U)$.

Definition. A δ -**cover** of a set $F \subseteq X$ is any countable collection $\{U_n\}_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty U_n \supseteq F$ and $|U_n| \leq \delta$.

Let $\mathcal{A} = \mathcal{P}(X)$, and $\mathcal{A}_\delta = \{A \subseteq X : |A| \leq \delta\}$. For $\delta \geq 0$, put $\mathcal{C}_s(A) = |A|^s$. Then for $s \geq 0$, $\delta > 0$, and $E \subseteq X$, we define

$$\begin{aligned} H_\delta^s(E) &= \inf \left\{ \sum_{n=1}^{\infty} |U_n|^s : \{U_n\} \text{ is a } \delta\text{-cover of } E \right\} \\ &= \inf \left\{ \sum_{n=1}^{\infty} \mathcal{C}_s(U_n) : \bigcup_{n=1}^{\infty} U_n \supseteq E, U_n \in \mathcal{A}_\delta \right\} \end{aligned}$$

This is the outer measure as constructed in ?? with covering family \mathcal{A}_δ and function \mathcal{C}_s . In particular, as $\delta \rightarrow 0$, H_δ^s increases; in particular, by [Theorem 1.2](#), $H^s(E) = \sup_\delta H_\delta^s(E)$ is a metric outer measure. Then apply Caratheodory (??) to get the s -dimensional Hausdorff measure, which is a complete Borel measure.

Example. (i) H^0 is the counting measure on any metric space.

(ii) Take $X = \mathbb{R}$ and $s = 1$. Then H^1 is the Lebesgue measure (on Borel sets). To see this, we have

$$\begin{aligned} \lambda(E) &= \inf \left\{ \sum_{n=1}^{\infty} |I_n| : \bigcup_{n=1}^{\infty} I_n \supseteq E, |I_n| \leq \delta \right\} \\ &\geq H_\delta^1(E) \end{aligned}$$

for any $\delta > 0$; and conversely, take any δ -cover of E , say $\{U_n\}_{n=1}^{\infty}$ and set $I_n = \overline{\text{conv } U_n}$ so $|I_n| = |U_n| \leq \delta$. Thus $\sum_{n=1}^{\infty} |U_n| = \sum_{n=1}^{\infty} |I_n| \geq \lambda(E)$ for any such cover, so $\lambda(E) = H_\delta^1(E)$ for any $\delta > 0$. Thus $\lambda(E) = H^1(E)$ for any Borel set E .

(iii) More generally, if $X = \mathbb{R}^n$ and $s = n$, then $\lambda = \pi_n \cdot H^n$ where π_n is the n -dimensional volume of the ball of diameter 1.

We will verify that $H^n \leq m$ where m is n -dimensional Lebesgue measure on \mathbb{R}^n ; the general result is harder and left as an exercise. To see this, we have

$$\begin{aligned} m(E) &= \inf \left\{ \sum_{i=1}^{\infty} \text{vol}(C_i) : C_i \text{ cube}, \bigcup_{i=1}^{\infty} C_i \supseteq E, \text{sides} \leq \frac{1}{\sqrt{n}} \delta \right\} \\ &= \inf \left\{ \sum_{i=1}^{\infty} \left(\frac{1}{\sqrt{n}} \right)^n |C_i|^n : \{C_i\} \text{ } \delta\text{-cover of cubes of } E \right\} \\ &\geq c_n \inf \left\{ \sum_{i=1}^{\infty} |c_i|^n : \text{all } \delta\text{-covers of } E = c_n H_\delta^n(E) \right\} \end{aligned}$$

where $c_n = (1/\sqrt{n})^n \leq 1$.

(iv) If $s < t$, then $H^s(E) \geq H^t(E)$.

Suppose $s < t$. Clearly $H^s(E) \geq H^t(E)$, but we can in fact make stronger statements. Suppose we have some U_i where $|U_i| \leq \delta$, and

$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^s |U_i|^{t-s} \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$

so that

$$H_\delta^t(E) \leq \delta^{t-s} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i=1}^{\infty} \text{ } \delta\text{-cover of } E \right\} = \delta^{t-s} H_\delta^s(E).$$

In particular, as $\delta \rightarrow 0$, $H_\delta^t(E) \rightarrow H^t(E)$ and $H_\delta^s(E) \rightarrow H^s(E)$ and $\delta^{t-s} \rightarrow 0$ since $s < t$. Thus if $H^s(E) \neq \infty$, then $H^t(E) = 0$ for all $t > s$. Similarly, if $H^t(E) > 0$, then $H^s(E) = \infty$ for all $s < t$. As a result, there exists some unique number $S_0 := \dim_H(E) \geq 0$ such that for all $s < S_0$, $H^s(E) = \infty$, and for all $t > S_0$, $H^t(E) = 0$. We call this value the **Hausdorff dimension** of E . Note that $H^{S_0}(E) \in [0, \infty]$ and all choices are possible.

Example. (i) Since $1 = m([0, 1]) = H^1([0, 1])$, $\dim_H[0, 1] = 1$

(ii) $\dim_H \mathbb{R} = 1$ but $m(\mathbb{R}) = H^1(\mathbb{R}) = \infty$.

(iii) It is possible to have $S_0 = 1$ but $m(E) = 0$.

(iv) There is a Cantor-like set with Hausdorff-dimension 0.

(v) If E is countable and $s > 0$, $H_\delta^s(E) \leq \sum_{x \in E} |\{x\}|^s = 0$. In particular, there exist compact countable sets, and in this case, $\dim_H C = 0$ while $H^0(C) = \infty$.

Here are some basic properties of Hausdorff dimension.

1.5 Proposition. (Properties of Hausdorff Dimension) (i) If $A \subseteq B$, then $\dim_H A \leq \dim_H B$.

(ii) If $F \subseteq \mathbb{R}^n$, then $\dim_H F \leq n$.

(iii) If $U \subset \mathbb{R}^n$ is open, then $\dim_H U = n$.

(iv) If $F = \bigcup_{i=1}^\infty F_i$, then $\dim_H(F) = \sup_{i \in \mathbb{N}} \dim_H F_i$.

PROOF (i) If $H^s(B) = 0$, then $H^s(A) = 0$ by monotonicity of measures so $\dim_H A \leq \dim_H B$.

(ii) First consider the unit cube $I^n \subset \mathbb{R}^n$. Then

$$H_{\sqrt{n}\delta}^s(I^n) \leq \left(\frac{2}{\delta}\right)^n (\sqrt{n}\delta)^s = 2^n \sqrt{n}^n \delta^{s-n}$$

so if $s > n$, then $\delta^{s-n} \rightarrow 0$ as $\delta \rightarrow 0$. Thus for all $s > n$, $H^s(I^n) = \lim_{\delta \rightarrow 0} H_{\sqrt{n}\delta}^s(I^n) = 0$ so that $\dim_H(I^n) \leq n$. Moreover, \mathbb{R}^n is the countable union of unit cubes, so that $H^s(\mathbb{R}^n) = 0$ and $\dim_H(\mathbb{R}^n) \leq n$. Then appeal to (i).

(iii) Cubes have positive Hausdorff n -measure.

(iv) If $s > \sup\{\dim_H F_i\}$, then $H^s(F_i) = 0$ for all i and by subadditivity $H^s(F) = 0$. Thus $s \geq \dim_H F$. By monotonicity, $\dim_H F \geq \dim_H F_j$ for all j . ■

Suppose $X = \mathbb{R}^n$, $E \subseteq \mathbb{R}^n$, $\lambda > 0$. Set $\lambda E = \{\lambda e : e \in E\}$: then $H^s(\lambda E) = \lambda^s H^s(E)$ since there is a bijection between δ -covers and $\lambda\delta$ -covers.

Definition. Let X, Y be metric spaces. A function $f : X \rightarrow Y$ is called **Lipschitz** if there exists C such that $d(f(x), f(y)) \leq Cd(x, y)$.

Certainly if f is Lipschitz, then f is uniformly continuous. Functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with bounded derivative are Lipschitz by the mean value theorem.

Definition. A function $f : X \rightarrow Y$ is **Hölder continuous** with exponent α if there exists c such that $d(f(x), f(y)) \leq cd(x, y)^\alpha$.

Example. (i) If $\alpha = 1$, then f is Lipschitz, and if $\alpha = 0$, then f is bounded.

(ii) If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\alpha > 0$, then f is constant (by considering derivatives). Thus the most interesting cases occur for $0 < \alpha \leq 1$.

1.6 Proposition. If $f : X \rightarrow Y$ is Hölder continuous with exponent α . Then $H^{s/\alpha}(f(E)) \leq cH^s(E)$ for some constant c .

PROOF If $\{U_i\}$ are a δ -cover of E , then $\{f(U_i)\}$ cover $f(E)$. Then $\text{diam } f(U_i) = \sup\{d(f(x), f(y)) : x, y \in U_i\} \leq c \sup\{d(x, y)^\alpha : x, y \in U_i\} = C \cdot (\text{diam } U_i)^\alpha$. Thus if $\{U_i\}$ is a δ -cover of E , then $\{f(U_i)\}$ is a $c\delta^\alpha$ -cover of $f(E)$. Passing through the definition, we get $H^{s/\alpha} \leq c^{s/\alpha} H^s(E)$. ■

We then have the easy corollaries

1.7 Corollary. $\dim_H f(X) \leq \frac{1}{\alpha} \dim_H X$.

1.8 Corollary. If f is an isometry, then $H^s(f(X)) = H^s(X)$.

1.9 Corollary. If $f : X \rightarrow Y$ are bi-Lipschitz, then $\dim_H X = \dim_H Y$.

Example. Let C denote the Cantor set. Let's show that $\frac{1}{2} \leq H^s(C) \leq 1$ for $s = \frac{\log 2}{\log 3}$. In particular, this implies that $\dim_H C = \frac{\log 2}{\log 3}$.

Let $\delta = 3^{-n}$ and cover C with a δ -covering with generation n Cantor intervals. Then $H_\delta^s(C) \leq \sum_{I \in C_n} |I|^s = 2^n 3^{-ns} = 1$ by choice of s . Thus $\lim_{\delta \rightarrow 0} H_\delta^s(C) = \lim_{n \rightarrow \infty} H_{3^{-n}}^s(C) \leq 1$.

For the lower bound, take any δ -cover $\{U_i\}$ of C . Without loss of generality, we may assume that the U_i are open intervals. Since C is compact, get some finite subcover U_1, \dots, U_N . For each i , get $k_i \in \mathbb{N}$ so that $3^{-(k_i+1)} \leq |U_i| < 3^{-k_i}$; set $k = \max\{k_1, \dots, k_N\}$. Since U_i intersects at most 1 interval in C_{k_i} , U_i intersects at most 2^{k-k_i} intervals of C_k . Thus $2^k \leq \sum_{i=1}^N 2^{k-k_i}$ where $2^{k-k_i} = 2^k 3^{-sk_i} = 2^k 3^{-s(k_i+1)} \leq 2^k |U_i|^s 3^s$. Thus

$$2^k \leq \sum_{i=1}^N 2^k |U_i|^s 3^s$$

so $\frac{1}{2} = 3^{-s} \leq \sum_{i=1}^N |U_i|^s \leq \sum_{i=1}^\infty |U_i|^s$ so $H_\delta^s(C) \geq \frac{1}{2}$ so $H^s(C) \geq \frac{1}{2}$.

1.10 Proposition. Let (X, d) be a metric space. If $\dim_H X < 1$, then X is totally disconnected.

PROOF Let $x \in X$ and define $f : X \rightarrow [0, \infty)$ by $f(z) = d(z, x)$. Then f is Lipschitz with constant 1 so $\dim_H f(X) \leq \dim_H X < 1$ so $m(f(X)) = 0$. Then if $y \neq x$, $d(y, x) = f(y) > 0$ while $f(x) = 0$. In particular, $(0, f(y)) \not\subset f(X)$ so there exists $0 < r < f(y)$ such that $r \notin f(X)$. Then $U_1 = \{z \in X : f(z) < r\}$ and $U_2 = \{z \in X : f(z) > r\}$ are disconnecting sets for X separating x and y .

1.3 BOX DIMENSIONS

Definition. Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set, and for each $\delta > 0$, let $N_\delta(E)$ be the least number of closed balls of diameter δ . We then define the **upper box dimension** of E

$$\overline{\dim}_B E = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{|\log \delta|}$$

and similarly $\underline{\dim}_B E$ (the **lower box dimension**) with a \liminf in place of \limsup . If $\underline{\dim}_B E = \overline{\dim}_B E$, then we define the **box dimension** to be this shared quantity.

If I is any interval, it is easy to see that $\dim_B I = 1$. Note that if $N_\delta(E) \sim \delta^{-s}$, then $\dim_B E = S$.

Example. Let's show that the box dimension of $C_{1/3}$ exists, and compute it. Given some $\delta > 0$, let n be so that $3^{-n} \leq \delta < 3^{-(n-1)}$. Certainly we can cover $C_{1/3}$ by Cantor intervals of level n , so that $N_\delta(C_{1/3}) \leq 2^n$. Moreover, the endpoints of Cantor intervals of level $n-1$ are distance at least $3^{-(n-1)} > \delta$ apart. Thus $N_\delta(C_{1/3})$ is at least the number of endpoints of level $n-1$, i.e. $N_\delta(C_{1/3}) \geq 2^n$. Thus $N_\delta(C_{1/3}) = 2^n$, so that

$$\frac{\log 2}{\log 3} = \frac{\log 2^n}{\log 3^n} \leq \frac{\log N_\delta(C_{1/3})}{|\log \delta|} \leq \frac{\log 2^n}{\log 3^{n-1}} = \frac{n}{n-1} \cdot \frac{\log 2}{\log 3}$$

and, as $\delta \rightarrow 0$, $n \rightarrow \infty$ so that the $\dim_B C_{1/3} = \frac{\log 2}{\log 3}$.

More generally, using the same technique, we may compute $\dim_B C_r = \frac{\log 2}{\log 1/r}$.

However, the box dimension has poor properties: for example, we may verify $\dim_B \{0, 1, 1/2, 1/3, \dots\} = \frac{1}{2}$. In particular, the box dimension does not have countable stability (the box dimension of any singleton is 0). But this is very concerning from a measure theoretic perspective, since this is a countable set with larger "dimension" than some uncountable sets (e.g. C_r for small r).

1.11 Theorem. *The value of the various box dimensions are equal for all following definitions of $N_\delta(E)$:*

1. *least number of open balls of radius δ that cover E*
2. *least number of cubes of side length δ*
3. *the number of δ -mesh cubes that intersect E : $[m_1\delta, (m_1+1)\delta] \times \dots \times [m_n\delta, (m_n+1)\delta]$ for $(m_1, \dots, m_n) \in \mathbb{Z}^n$.*
4. *the largest number of disjoint closed balls of radius δ with centers in E .*

PROOF Throughout, from the logarithms in the definition, it suffices to bound $N_\delta^{(i)}(E)$ with respect to $N_\delta(E)$ up to some constant factor either with respect to δ or with respect to N_δ .

1. Exercise.
2. Exercise.
3. In general, the diameter of a δ -cube in \mathbb{R}^n is $\sqrt{n}\delta$. Let $N_\delta^{(3)}(E)$ denote the number of δ -mesh cubes intersecting E . Then the cubes which intersect E cover E and these have diameter $\sqrt{n}\delta$, so $N_{\sqrt{n}\delta}(E) \leq N_\delta^{(3)}(E)$.

Conversely, any set with diameter at most δ is contained in at most 3^n δ -mesh cubes. Thus $N_\delta^{(3)}(E) \leq 3^n N_\delta(E)$.

4. Let $N_\delta^{(4)}$ denote the largest number of disjoint balls of radius δ centred in E . Say $B_1, \dots, B_{N_\delta^{(4)}(E)}$ are such balls. If $x \in E$, then $d(x, B_i) \leq \delta$ for some i , else $B(x, \delta)$ would be disjoint from all B_i , contradicting maximality. Thus the balls $B_1^1, \dots, B_{N_\delta^{(4)}(E)}^1$ cover

E and have diameter 4δ , so $N_{4\delta}(E) \leq N_\delta^{(4)}(E)$.

Conversely, let $U_1, \dots, U_{N_\delta(E)}$ be any collection of sets of diameter at most δ that cover E . Let B_1, \dots, B_m be any disjoint balls with radius δ and centres $x_i \in E$. Since the U_j cover E , each $x_i \in U_{j(i)}$ for some $j(i)$ so $U_{j(i)} \subseteq B_i$ and $U_{j(i)} \cap B_k = \emptyset$ for $k \neq i$.

Thus $N_\delta(E) \geq N_\delta^{(4)}(E)$. ■

Note that, in the box dimension computation, it suffices to verify along a sequence of $(\delta_k)_{k=1}^\infty \rightarrow 0$ such that $\delta_{k+1} \geq c \cdot \delta_k$ for some $c > 0$ (i.e. not faster than exponentially).

1.12 Proposition. $\dim_H(E) \leq \underline{\dim}_B(E)$.

PROOF Suppose we cover E by $N_\delta(E)$ sets of diameter at most δ . Then $\inf\{\sum |U_i|^s : \{U_i\} \delta\text{-cover of } E\} \leq \delta^s N_\delta(E)$ so that $H_\delta^s(E) \leq \delta^s N_\delta(E)$. Suppose $s < \dim_H E$, so $H^s(E) > \lambda$ for some $\lambda > 0$. Then $\delta^s N_\delta(E) \geq \lambda$ so that $\frac{\log N_\delta(E)}{-\log \delta} \geq s + \frac{\log \lambda}{-\log \delta}$. Then as $\delta \rightarrow 0$, $\liminf \frac{\log N_\delta(E)}{-\log \delta} \geq s$. Thus $\underline{\dim}_B E \geq \dim_H E$. ■

1.13 Proposition. (Properties of Box Dimension) (i) $\underline{\dim}_B E = \underline{\dim}_B \overline{E}$ and $\overline{\dim}_B E = \overline{\dim}_B \overline{E}$
 (ii) $\underline{\dim}_B E = n$ if E is dense in an open set in \mathbb{R}^n .
 (iii) $\underline{\dim}_B(E \cup F) = \max(\underline{\dim}_B E, \underline{\dim}_B F)$. However, $\underline{\dim}_B E \cup \underline{\dim}_B F \geq \max\{\underline{\dim}_B E, \underline{\dim}_B F\}$ and the inequality can hold strictly.
 (iv) Box dimension is Lipschitz invariant.

1.14 Theorem. (Mass Distribution Principle) Let μ be a finite Borel measure on F with $\mu(F) > 0$. Suppose there exists $c > 0$ and $\delta_0 > 0$ such that whenever $|U| \leq \delta_0$, $\mu(U) \leq c|U|^s$. Then $H^s(F) \geq \frac{\mu(F)}{c} > 0$.

PROOF Let $\{U_i\}$ be a δ -cover of F with $\delta \leq \delta_0$. Then $\mu(F) \leq \mu(\bigcup_{i=1}^\infty U_i) \leq \sum_{i=1}^\infty \mu(U_i) \leq c \sum_{i=1}^\infty |U_i|^s$. Thus $\inf\{\sum_{i=1}^\infty |U_i|^s : \{U_i\} \delta\text{-cover of } F\} \geq \frac{\mu(F)}{c}$ and let $\delta \rightarrow 0$. ■

Example. Let $C(r)$ denote the Cantor set with contraction ratio r . Define $\mu(I_\omega \cap C) = r^{|\omega|}$, and extend to the uniform r -Cantor measure. We now apply the mass distribution principle. Let U be arbitrary with $r^{k+1} \leq |U| < r^k$. Then U cannot intersect 3 level k intervals (or U would have diameter greater than r^k). Thus $\mu(U) = \mu(U \cap C) \leq c\mu(I_\omega) = 3^s \dots$ So $\dim_G(C_r) = \frac{\log 2}{|\log r|}$.

1.15 Proposition. Suppose μ is a finite Borel measure on \mathbb{R}^n and $F \subseteq \mathbb{R}^n$ is Borel. Let $0 < c < \infty$.

- (i) If $\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^s} \leq c$ for all $x \in F$, then $H^s(F) \geq \frac{\mu(F)}{c}$
- (ii) If $\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^s} \geq c$ for all $x \in F$, then $H^s(F) \leq \frac{10^s}{c} \mu(\mathbb{R}^n) < \infty$.

PROOF (i) Fix $\epsilon > 0$. For each $\delta > 0$, let $F_\delta = \{x \in X : \mu(B(x, r)) \leq (c + \epsilon)r^s\}$. By hypothesis, $F \subseteq \bigcup_{\delta > 0} F_\delta$; moreover, for $\delta_1 < \delta_2$, $F_{\delta_1} \supseteq F_{\delta_2}$.

Fix some δ and take a δ -cover $\{U_i\}_{i=1}^\infty$ of F . This is also a δ -cover for F_δ . If $x \in F_\delta$, $\mu(B(x, |U_i|)) \leq (c + \epsilon)|U_i|^s$ as $|U_i| \leq \delta$. Since $U_i \subseteq B(x_i, |U_i|)$ for any $x_i \in U_i$, if $U_i \cap F_\delta \neq \emptyset$, then $x_i \in U_i \cap F_\delta$, so that $\mu(U_i) \leq \mu(B(x_i, |U_i|)) \leq (c + \epsilon)|U_i|^s$. Thus

$$\begin{aligned} F_\delta &= \bigcup_{U_i \cap F_\delta \neq \emptyset} (U_i \cap F_\delta) \\ &\leq \sum_{i=1}^\infty (c + \epsilon)|U_i|^s \end{aligned}$$

so that $\mu(F_\delta) \leq (c + \epsilon)H_\delta^s(F)$. But $F_\delta \rightarrow F$ and $H_\delta^s(F) \rightarrow H^s(F)$, so that $\mu(F) \leq (c + \epsilon)H^s(F)$, where $\epsilon > 0$ is arbitrary.

- (ii) Fix $\epsilon > 0$ and $\delta > 0$. Let $\mathcal{B} = \{B(x, r) : x \in F, 0 < r \leq \delta, \mu(B(x, r)) \geq (x - \epsilon)r^s\}$. By assumption, $F \subseteq \bigcup_{B \in \mathcal{B}} B$. Use the Vitali covering lemma, so there exists disjoint balls $B_1, B_2, \dots \in \mathcal{B}$ such that B'_i is the ball with the same centre and 5 times the radius, then $\bigcup_{i=1}^{\infty} B'_i \supseteq F$. Since $\text{diam } B(x, r) = 2r$, $|B'_i| \leq 10r \leq 10\delta$ so the $\{B'_i\}_{i=1}^{\infty}$ are a 10δ -cover of F . Thus

$$\begin{aligned} H_{10\delta}^s(F) &\leq \sum_{i=1}^{\infty} |B'_i|^s = \sum_{i=1}^{\infty} |B_i|^s 5^s \\ &= \sum_{i=1}^{\infty} (2r_i)^s 5^s \\ &\leq 10^s \sum_{i=1}^{\infty} \frac{\mu(B_i)}{c - \epsilon} \\ &= \frac{10^s}{c - \epsilon} \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \frac{10^s}{c - \epsilon} \mu(\mathbb{R}^n) \end{aligned}$$

and taking $\delta \rightarrow 0$ and noting that $\epsilon > 0$ is arbitrary, we have $H^s(F) \geq \frac{10^s \mu(\mathbb{R}^n)}{c}$. \blacksquare

1.16 Proposition. Suppose F is Borel and $0 < H^s(F) < \infty$. Then there exists c and a compact $E \subseteq F$ such that $H^s(E) > 0$ and $H^s(B(x, r) \cap E) \leq cr^s$ for all $x \in E$ and $r > 0$.

PROOF Let

$$F_1 = \left\{x : \limsup_{r \rightarrow 0} \frac{H^s(F \cap B(x, r))}{r^s} > 10^{s+1}\right\}$$

and apply (b) above with $\mu = H^s|_F$ so that

$$H^s(F_1) \leq \frac{10^s}{10^{s+1}} \mu(\mathbb{R}^n) = \frac{1}{10} H^s(F).$$

In particular, $H^s(F \setminus F_1) \geq \frac{9}{10} H^s(F) > 0$. For all $x \in F \setminus F_1$, there exists $r_0(x)$ such that for all $r \leq r_0$, then

$$\frac{H^s(F \cap B(x, r))}{r^s} \leq 10 \cdot 10^{s+1} = 10^{s+2}.$$

Let

$$E_n = \left\{x \in F \setminus F_1 : \frac{H^s(F \cap B(x, r))}{r^s} \leq 10^{s+2} \text{ for all } r \leq \frac{1}{n}\right\}$$

so that $\bigcup_{n=1}^{\infty} E_n = F \setminus F_1$. By continuity of measure, $H^s(E_n) \rightarrow H^s(F \setminus F_1) > 0$ so there exists N such that $H^s(E_N) > 0$. Since H^s is inner regular (TODO prove), get $E \subseteq E_N$ compact such that $H^s(E) > 0$. Then if $x \in E$, $x \in E_N$ so $H^s(E \cap B(x, r)) \leq H^s(F \cap B(x, r)) \leq 10^{s+2} r^s$ if $r \leq 1/N$. For any r , $H^s(E \cap B(x, r)) \leq H^s(F) = C_0$. If $r > 1/N$, then $C_0 \leq C_0 N^s r^s$. Take $c = \max\{10^{s+2}, C_0 N^s\}$. \blacksquare

Remark. The assumption $H^s(F) < \infty$ can be removed when F is closed.

1.4 POTENTIAL-THEORETIC METHODS

Definition. For $s \geq 0$, the s -potential at x due to μ is

$$\phi_s(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{\|x - y\|^s}$$

and the s -energy of μ

$$I_s(\mu) = \int_{\mathbb{R}^n} \phi_s d\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{d\mu(x)d\mu(y)}{\|x - y\|^s}$$

Example. (i) If $s = 0$, then $\phi_0(x) = \mu(\mathbb{R}^n)$ and $I_0(\mu) = \mu(\mathbb{R}^n)^2 < \infty$.

(ii) If $s > 0$ and $\mu = \delta_0$, then $I_s(\delta_0) = \phi_s(0) = \infty$

(iii) If $n = 1$ and $\mu = m|_{[0,1]}$, $s < 1$. Then $I_s(\mu) = \int_0^1 \int_0^1 \frac{dx dy}{|x - y|^s} < \infty$.

1.17 Theorem. Let F be a closed set, $s > 0$.

- (i) If there exists a finite, non-zero measure μ supported on F such that $I_s(\mu) < \infty$, then $H^s(F) = \infty$ implies that $\dim_H F \geq s$.
- (ii) If $H^s(F) > 0$, then there exists a finite non-zero measure μ on F such that $I_t(\mu) < \infty$ for all $t < s$.

PROOF (i) Suppose $I_s(\mu) < \infty$ for μ a finite measure on F . We will show that $\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^s} = 0$ for μ a.e. $x \in F$. Assuming this, then $H^s(F) \geq \frac{\mu(F \setminus N)}{\epsilon}$ for some μ -null N , but this holds for any $\epsilon > 0$, so $H^s(F) = \infty$.

Let $F_1 = \{x \in F : \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^s} > 0\}$. We want to show that $\mu(F_1) = 0$. We first show that $\phi_s(\mu) = \infty$ on F_1 . If $x \in F_1$, then there exists $\epsilon > 0$ and $\{r_i\}_{i=1}^\infty$ converging to 0 such that $\mu(B(x, r_i)) \geq \epsilon r_i^s$. Since $I_s(\mu) < \infty$ for some $s > 0$, μ is not atomic so by downward continuity of measure, $\mu(B(x, q)) \rightarrow \mu(\{x\}) = 0$ as $q \rightarrow 0$. Thus get q_i such that $\mu(B(x, q_i)) < \frac{\epsilon}{2} r_i^s$. Let $A_i = B(x, r_i) \setminus B(x, q_i)$, so that $\mu(A_i) \geq \frac{\epsilon}{2} r_i^s$. Relabelling the r_i if necessary, we may assume that $r_{i+1} < q_i$ so that the annuli are disjoint and nested. In particular,

$$\begin{aligned} \phi_s(x) &= \int_{\mathbb{R}^n} \frac{d\mu(y)}{\|x - y\|^s} \\ &\geq \sum_{i=1}^\infty \int_{A_i} \frac{d\mu(y)}{\|x - y\|^s} \\ &\geq \sum_{i=1}^\infty \frac{1}{\max_{y \in A_i} \|x - y\|^s} \mu(A_i) \\ &\geq \sum_{i=1}^\infty \frac{1}{r_i^s} \mu(A_i) \geq \sum_{i=1}^\infty \frac{1}{r_i^s} \cdot \frac{\epsilon}{2} r_i^s = \infty \end{aligned}$$

But now,

$$\infty > I_s(\mu) = \int_{\mathbb{R}^n} \phi_s d\mu \geq \int_{F_1} \phi_s d\mu$$

so if $\phi_s = +\infty$ on F_1 , then $\mu(F_1) = 0$.

- (ii) Suppose $H^s(F) > 0$. By the previous proposition, there exists sompact $E \subseteq F$ with $0 < H^s(E) < \infty$ and $H^s(E \cap B(x, r)) \leq cr^s$ for all $x \in E$ and $r > 0$. Put $\mu = H^s|_E$. Then $\mu(B(x, r)) \leq cr^s$ for all $x \in E$. For $x \in E$,

$$\phi_i(x) = \int_{\|x-y\| \leq 1} \frac{d\mu(y)}{\|x-y\|^t} + \int_{\|x-y\| > 1} \frac{d\mu(y)}{\|x-y\|^t}.$$

Certainly the second integral is finite independent of x . The first integral is finite since

$$\begin{aligned} \int_{\|x-y\| \leq 1} \frac{d\mu(y)}{\|x-y\|^t} &= \sum_{k=0}^{\infty} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-(k+1)})} \frac{d\mu(y)}{\|x-y\|^t} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{2^{-(k+1)t}} \mu(B(x, 2^{-k})) \\ &\leq \sum_{k=0}^{\infty} \frac{c}{2^{-(k+1)t}} \cdot 2^{-ks} < \infty \end{aligned}$$

since $s > t$. Again, this bound does not depend on x . Thus ϕ_t is a bounded function on E , so that $I_t(\mu) < \infty$. ■

“can’t have both the measure and it’s fourier transform small”

Suppose f is integrable on \mathbb{R}^n or $\mu \in M(\mathbb{R}^n)$ is a complex measure. We then define the **fourier transform**

$$\begin{aligned} \hat{f}(z) &= \int_{\mathbb{R}^n} f(x) e^{-ix \cdot z} dm(x) \\ \hat{\mu}(z) &= \int_{\mathbb{R}^n} e^{-ix \cdot z} d\mu(x) \end{aligned}$$

If $f, g \in L^1$, then $f * g \in L^1$ by

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}^n} f(x-y)g(y) dy \\ f * \mu(x) &= \int_{\mathbb{R}^n} f(x-y) d\mu(y) \end{aligned}$$

By Fubini, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ and $\|f * \mu\| \leq \|f\|_1 \|\mu\|_{M(\mathbb{R}^n)}$. One reason for doing this is that L^1 is not closed under pointwise multiplication. Importantly, we have

$$\begin{aligned} (f * g)^\wedge(z) &= \hat{f}(z) \hat{g}(z) \\ (f * \mu)^\wedge(z) &= \hat{f}(z) \hat{\mu}(z) \end{aligned}$$

in other words that the fourier transform converts convolution to multiplication.

Now consider $g_s(t) = \|t\|^{-s}$. Then

$$\phi_s(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{\|x-y\|^s} = \int_{\mathbb{R}^n} g_s(x-y) d\mu(y) = g_s * \mu(x)$$

It is known that $\hat{g}_s(z) = c(n, s) \|z\|^{s-n}$ for $0 < s < n$. In particular, $\hat{\phi}_s(z) = \hat{g}_s(z) \hat{\mu}(z) = c(n, s) \|z\|^{s-n} \hat{\mu}(z)$.

1.18 Theorem. (Parseval) We have

$$\int f \cdot \bar{g} \, dx = (2\pi)^n \int \hat{f} \cdot \bar{\hat{g}} \, dz$$

for $f, g \in L^2$ and thus $\int |f|^2 = (2\pi)^n \int |\hat{f}|^2$. When g is “nice”,

$$\int g(x) \, d\mu(x) = (2\pi)^n \int \hat{g}(z) \overline{\hat{\mu}(z)} \, dz$$

In particular (with some technicalities ...)

$$\begin{aligned} I_s(\mu) &= \int \phi_s(x) \, d\mu(x) = c_n \int \hat{\phi}_s(z) \overline{\hat{\mu}(z)} \, dz \\ &= c'_n \int \|z\|^{s-n} |\hat{\mu}(z)|^2 \, dz \end{aligned}$$

Example. If $|\hat{\mu}(z)| \leq C \|z\|^{-t/z}$, then $\dim_H \text{supp } \mu \geq t$.

PROOF We have since $\hat{\mu}(z)$ is bounded that

$$\begin{aligned} I_s(\mu) &= c \int \|z\|^{s-n} |\hat{\mu}(z)|^2 \, dz \\ &= c \left(\int_{\|z\| \leq 1} \|z\|^{s-n} |\hat{\mu}(z)|^2 \, dz + \int_{\|z\| > 1} \|z\|^{s-n} |\hat{\mu}(z)|^2 \, dz \right) \\ &\leq c \left(\int_{\|z\| \leq 1} C_0 \|z\|^{s-n} \, dz + \int_{\|z\| \geq 1} \|z\|^{s-n} \|z\|^{-t} \, dz \right) \\ &= c \left(c_1 \int_0^1 r^{s-n} r^{n-1} \, dr + \int_1^\infty t^{s-t-1} \, dt \right) < \infty \end{aligned}$$

as $s < t$. Thus $I_s(\mu) < \infty$ for any $0 < s < t$, and apply the energy theorem. ■

1.5 PROJECTIONS OF FRACTALS

Let $F \subset \mathbb{R}^2$ be a region and consider the (orthogonal) projection onto some line through the origin. Write $\text{proj}_\theta(f)$ to denote the projection onto the line L_θ . Note that $d(\text{proj}_\theta(x), \text{proj}_\theta(y)) \leq d_{\mathbb{R}^2}(x, y)$ so proj_θ is Lipschitz and $\dim_H \text{proj}_\theta F \leq \min\{1, \dim_H F\}$.

If L is a line segment, then for all values of θ (except for 2), then the projection has maximal dimension.

1.19 Theorem. Let $F \subseteq \mathbb{R}^2$ be closed.

- (i) If $\dim_H F \leq 1$, then $\dim_H \text{proj}_\theta F = \dim_H F$ for a.e. θ .
- (ii) If $\dim_H F > 1$, then $m(\text{proj}_\theta F) > 0$ for a.e. θ .

PROOF (i) Choose $0 < s < \dim_H F$, so $H^s(F) > 0$. Thus there exists some μ on F such that $I_s(\mu) < \infty$. Write $x.\theta$ to denote the projection of x onto the line L_θ . Then define μ_θ on $\text{proj}_\theta F$ by

$$\int_{-\infty}^{\infty} f(t) \, d\mu_\theta(t) = \int f(x.\theta) \, d\mu(x)$$

for all $f \in C_c(\mathbb{R})$ (Radon-Markov). Note that $\mu_\theta(S) = \mu(\text{proj}_\theta^{-1}(S))$. We will show that $\int_0^\pi I_s(\mu_\theta) d\theta < \infty$, so that $I_s(\mu_\theta) < \infty$ for a.e. θ and we will be done.

We have since $|x.\theta - y.\theta| = \|x - y\| \cos(\theta - (x - y))$.

$$\begin{aligned} \int_0^\pi I_s(\mu_\theta) d\theta &= \int_0^\pi \int_F \int_F \frac{d\mu(x) d\mu(y)}{|x.\theta - y.\theta|^s} \\ &= \int_0^\pi \int_F \int_F \frac{d\mu(x) d\mu(y)}{\|x - y\|^s |\cos(\theta - (x - y))|^s} \\ &= \int_F \int_F \left(\int_0^\pi \frac{d\theta}{|\cos(\theta - (x - y))|^s} \right) \frac{d\mu(x) d\mu(y)}{\|x - y\|^s} \\ &= \int_{F \times F} \left(\int_0^\pi \frac{d\theta}{|\cos \theta|^s} \right) \frac{d\mu(x) d\mu(y)}{\|x - y\|^s} \end{aligned}$$

Note that $\int_0^\pi \frac{d\theta}{|\cos \theta|^s} < \infty$, but the remaining term is just the s -energy of μ , which is finite.

- (ii) Assume $\dim_H F > 1$, so there exists some $t > 1$ such that $H^t(F) > 0$. Get μ on F such that $I_1(\mu) < \infty$. Define μ_θ as above. We will show that μ_θ is absolutely continuous with density in L^2 for almost every θ . Then $f_\theta \neq 0$ in L^2 since $\mu_\theta \neq 0$ so that $m\{x : f_\theta(x) \neq 0\} > 0$ where $\{x : f_\theta(x) \neq 0\} \subseteq \text{supp } \mu_\theta$.

Recall that $f \in L^2$ if and only if $\hat{f} \in L^2$. We have

$$\begin{aligned} |\hat{\mu}_\theta(z)|^2 &= \int e^{-ivz} d\mu_\theta(v) \overline{\int e^{-izw} d\mu_\theta(w)} \\ &= \int_{\mathbb{R} \times \mathbb{R}} e^{-iz(v-w)} d\mu_\theta(v) d\mu_\theta(w) \\ &= \int_{F \times F} e^{-iz(x-y).\theta} d\mu(x) d\mu(y) \end{aligned}$$

so that

$$\begin{aligned} |\hat{\mu}_\theta(z)|^2 + |\hat{\mu}_{\theta+\pi}(z)|^2 &= \int_{F \times F} (e^{-iz(x-y).\theta} + e^{-iz(x-y).(-\theta)}) d\mu(x) d\mu(y) \\ &= 2 \int_{F \times F} \cos(z(x-y).\theta) d\mu(x) d\mu(y) \end{aligned}$$

First note that

$$\begin{aligned} \int_0^{2\pi} |\hat{\mu}_\theta(z)|^2 d\theta &= \int_0^\pi |\hat{\mu}_\theta(z)|^2 + |\hat{\mu}_{\theta+\pi}(z)|^2 d\theta \\ &= 2 \int_0^\pi \int_f \int_F \cos(z(x-y).\theta) d\mu(x) d\mu(y) d\theta \\ &= 2 \int_0^\pi \int_f \int_F \cos(z\|x-y\| \cos(\theta - (x-y))) d\mu(x) d\mu(y) d\theta \\ &= \int_F \int_F \left(\int_0^{2\pi} \cos(z\|x-y\| \cos(\theta)) d\theta \right) d\mu(x) d\mu(y) \\ &= 2\pi \int_F \int_F J_0(z\|x-y\|) d\mu(x) d\mu(y). \end{aligned}$$

We now have (concealing some technicalities in verifying the application of Fubini)

$$\begin{aligned}
 \int_0^{2\pi} \int_{-\infty}^{\infty} |\hat{\mu}_\theta(z)|^2 dz d\theta &< \infty = \int_{-\infty}^{\infty} \int_0^{2\pi} |\hat{\mu}_\theta(z)|^2 dz d\theta < \infty \\
 &= 2\pi \int_{-\infty}^{\infty} \int_F \int_F J_0(z\|x-y\|) d\mu(x) d\mu(y) \\
 &= 2\pi \int_F \int_F \left(\int_{-\infty}^{\infty} J_0(z\|x-y\|) dz \right) d\mu(x) d\mu(y) \\
 &= 2\pi \int_F \int_F \left(\int_{-\infty}^{\infty} J_0(w) dw \right) \frac{d\mu(x) d\mu(y)}{\|x-y\|} < \infty
 \end{aligned}$$

by the integral of the Bessel function and the fact that $I_1(\mu) < \infty$. ■

Bessel function: $J_0(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \cos(u \cos \theta) d\theta$.

2 ITERATED FUNCTION SYSTEMS

Let X be a complete metric space and F_1, \dots, F_m a family of contractions from X to X (i.e. functions with $0 < r_i < 1$ with $d(F_i(x), F_i(y)) \leq r_i d(x, y)$). Then there exists $E \subseteq X$ with E compact such that $E = \bigcup_{i=1}^m F_i(E)$.

Let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X . For $A \subseteq X$, let $A_r = \{y \in X : d(a, y) < r \text{ for some } a \in A\}$. We then define the **Hausdorff metric** on $\mathcal{K}(X)$ as follows:

$$D(A, B) = \inf\{r > 0 : A \subseteq B_r, B \subseteq A_r\}$$

2.1 Proposition. *D , as defined above, is in fact a metric and when X is complete, $\mathcal{K}(X)$ is also complete.*

PROOF We verify the properties for D to be a metric:

- (i) Suppose $D(A, B) = 0$. Then get a sequence a_n in A converging to any $b \in B$, i.e. $b \in \bar{A} = A$ and $B \subseteq A$. Similarly, $B \subseteq A$.
- (ii) $D(A, B) = D(B, A)$ is clear
- (iii) Fix $A, B, C \in \mathcal{K}(X)$, $d_1 = D(A, C)$, $d_2 = D(C, B)$. Fix $\epsilon > 0$ and let $a \in A$ be arbitrary. Get $c \in C$ so that $D(a, c) < d_1 + \epsilon/2$. Then get $b \in B$ so that $D(c, b) < d_2 + \epsilon/2$. Thus $d(a, b) < d_1 + d_2 + \epsilon$ so $A \subseteq B_{d_1+d_2+\epsilon}$ for all $\epsilon > 0$. Similarly, $B \subseteq A_{d_1+d_2+\epsilon}$. Thus $D(A, B) \leq d_1 + d_2$.

Completeness is left as an exercise. ■

2.2 Theorem. *Let $\{F_1, \dots, F_m\}$ be an IFS on X . Then there exists a unique compact set $E \subseteq X$ such that $E = \bigcup_{i=1}^m F_i(E)$.*

PROOF Define $F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ by $F(A) = \bigcup_{i=1}^m F_i(A)$. Let $r = \max\{r_1, \dots, r_m\} < 1$. We will show that $D(F(A), F(B)) \leq rD(A, B)$. Set $d = D(A, B)$; it suffices to show that $F_i(A) \subseteq (F_i(B))_{r(d+\epsilon)}$ for any $\epsilon > 0$. Indeed, take $a \in A$, so there exists $b \in B$ so that $d(a, b) \leq d + \epsilon$. Then $d(F_i(a), F_i(b)) \leq r(d + \epsilon)$.

Then F is a contraction map on $\mathcal{K}(X)$, so that $F^{(k)}(A) \rightarrow E$ for some unique E . ■

If $F_i(A) \subseteq A$, then $E = \bigcap_{k=0}^{\infty} F^{(k)}(A)$.

2.3 Lemma. *If $(A_k)_{k=1}^{\infty} \subset \mathcal{K}(X)$ with $A_1 \supseteq A_2 \supseteq \dots$, then $A_k \rightarrow \bigcap_{i=1}^{\infty} A_i$.*

PROOF Let $A_0 = \bigcap_{k=1}^{\infty} A_k$. We want to prove that $D(A_{k_1}, A_0) \rightarrow 0$. Certainly $A_0 \subseteq A_k$. Conversely, we must check that for any $r > 0$, there exists n_r such that $A_k \subseteq (A_0)_r$. Note that $(A_0)_r$ is an open set. Then $\{(A_0)_r, A_n^c : n \in \mathbb{N}\}$ is an open cover for A_1 . Hence there exists a finite subcover $(A_0)_r, A_{n_1}^c, \dots, A_{n_N}^c$. Thus for any $k > \max\{n_1, \dots, n_N\}$, $A_k \subseteq (A_0)_r$, as required. ■