Harmonic Analysis

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I. Harmonic Analysis

1 Locally Compact Groups

Definition. Let G be a group. A topology τ on G is a **group topology** provided that
- $x \mapsto x^{-1} : G \to G$ is continuous, and
- $(x,y) \mapsto xy : G \times G \to G$ is continuous.

We call (G,τ) a **topological group** where we omit τ when it is clear from context.

Equivalently, we may assert that $(x,y) \mapsto xy^{-1}$ is $\tau \times \tau - \tau$–continuous. Write $L_g(x) = gx$ and $R_g(x) = xg$ to denote the left and right multiplication maps; then it is easy to see that L_g and R_g are homeomorphisms. Similarly, $x \mapsto x^{-1}$ is a homeomorphism.

Definition. We say that a subset $A \subset G$ is **symmetric** if $A^{-1} = A$.

We have the following basic properties:

1.1 Proposition. *Let (G,τ) be a topological group.*
(i) If $\emptyset \neq A \subseteq G$ and U is open, then $AU = \{ay : a \in A, y \in U\}$ and likewise UA are open.
(ii) Given $U \in \tau$ and $x \in U$, then there is a symmetric $V \in \tau$ with $e \in V$ such that $VxV \subseteq U$. In particular, if $e \in U$, then we can find symmetric V so that $V^2 \subseteq U$.
(iii) If H is a subgroup of G, then $\overline{H}$ is also a subgroup.
(iv) An open subgroup is automatically closed.
(v) If $K, L \subseteq G$ are compact, then KL is compact.
(vi) If K is compact and C is closed in G, then KC is closed.

In $(\mathbb{R},+)$, then $\mathbb{Z}+\sqrt{2}\mathbb{Z}$ is not closed, so it is necessary to assume compactness in (vi).

Proof (i) $AU = \bigcup_{a\in A} L_a(U)$ is a union of open sets.

(ii) Consider the continuous map $(y,z) \mapsto yxz$. Since $exe = x \in U$, there is a $\tau \times \tau$–neighbourhood of (e,e) which maps into U have a basic neighbourhood $V_1 \times V_2$. Let $V = V_1 \cap V_2$. Moreover, we may replace V by $V^{-1} \cap V$. to attain symmetry.

(iii) Let $x, y \in \overline{H}$, $U \in \tau$ with $xy \in U$. Then (ii) provides V with $VxyV \subseteq U$. But $Vx \cap H \neq \emptyset$ and $\neq yV \cap H$ so there are $_1 \in Vx \cap H$, $h_2 \in yV \cap H$, and $h_1 h_2 \in VxyV \subseteq U$. Thus $U \cap H \neq \emptyset$. Thus $xy \in \overline{H}$.

To use nets for inverses, if $x \in \overline{H}$, then $x = \lim_\alpha x_\alpha$ where $(x_\alpha)_{\alpha\in A} \subset H$ is a net. Then $x^{-1} = \lim_\alpha x_\alpha^{-1} \in \overline{H}$ as each $x_\alpha^{-1} \in H$.

(iv) If H is an open subgroup, then $H = G \setminus \bigcup_{x \in G\setminus H} xH$ is closed.

(v) $K \times L$ is compact, and hence so is its image under multiplication.

(vi) If $x \in \overline{KC}$, then $x = \lim_\alpha k_\alpha x_\alpha$ where $k_\alpha \in H$ and $x_\alpha \in C$. Since K is compact, we may assume (passing to a subnet if necessary) $k = \lim_\alpha k_\alpha$ exists in K. Then

$$k^{-1}x = \lim_\alpha k_\alpha^{-1} \cdot \lim_\alpha k_\alpha x_\alpha = \lim_\alpha k_\alpha^{-1} k_\alpha x_\alpha = \lim x_\alpha \in C$$

so $x = kk^{-1}x \in KC$. ■

1.1 Homogenous Spaces

Let (G,τ) be a topological group, H a subgroup of G, and ${G}/{H} = \{xH; x\in G\}$. Let $\pi : G \to {G}/{H}$ be given by $\pi(x) = xH$ be the projection map. The **quotient topology** on ${G}/{H}$ is $\tau_{G/H} = \{W \in {G}/{H} : \pi^{-1}(W)\in\tau\}$. Notice that if $U \in \tau\setminus\{\emptyset\}$, then $UH = \pi^{-1}(\pi(U))$ is open, so $\pi : G \to {G}/{H}$ is an open map.

1.2 Proposition. *Let (G,τ), H be as above.*

(i) The map $(x,yH)\mapsto xyH : G\times {G}/{H} \to {G}/{H}$ is $\tau\times\tau_{G/H} - \tau_{G/H}$ continuous and open.

(ii) If H is normal, then $({G}/{H}, \tau_{G/H})$ is a topological group.

(iii) If H is closed, then $\tau_{G/H}$ is Hausdorff.

Proof (i) Let $x,y\in G$, $W\in\tau_{G/H}$ satisfy $xyH = \pi(xy)\in W$. Then $xy\in\pi^{-1}(W)$ and by Proposition 1.1 we have $V\in\tau$ with $e\in V$ such that $VxyV\subseteq\pi^{-1}(W)$. But then $(x,\pi(y))\in Vx\times\pi(yV)\in\tau\times\tau_{G/H}$ and the latter set maps into $\pi(VxyV)\subseteq W$.
Also, if $U\in\tau\times\tau_{G/H}$, then $U = \bigcup_{(x,yH)\in U} V_x\times W_{yH}$ and

$$\pi(U) = \bigcup_{(x,yH)\in U} \pi(V_x\pi^{-1}(W_{yH}))$$

since π is open.

(ii) Recall that $(xH)(yH) = xyH$ is our multiplication operation on ${G}/{H}$ and π is a group homomorphism. Then the following diagram commutes: We have that $\pi\times\mathrm{id}$ is open and $(x,yH)\mapsto xyH$ is open from (i), so the multiplication from ${G}/{H}\times{G}/{H}\to{G}/{H}$ must be open and continuous.

(iii) If $x,y\in G$ with $\pi(x)\neq\pi(y)$, then $e\notin xHy^{-1}$. Now $xHy^{-1} = L_x(R_{y^{-1}}(H))$ so xHy^{-1} is closed. Hence by the last proposition, there is a symmetric open V with $e\in V$ so $V^2\subseteq G\setminus(xHy^{-1})$. But then $e\notin(VxH)(VyH)^{-1} = VxHy^{-1}V$: if we had $e = vxhy^{-1}u$ with $v,u\in V$ and $h\in H$, then $v^{-1}u^{-1} = xhy^{-1}\in V^2\cap(xHy^{-1}) = \emptyset$, a contradiction. Hence $VxH\cap VyH = \emptyset$ so $\pi(Vx)$, $\pi(Vy)$ is a pair of separating neighbourhoods of $\pi(x)$, $\pi(y)$. ∎

1.3 Corollary. *G is Hausdorff if and only if there exists $x\in G$ so that $\{x\}$ is closed.*

Proof In a Hausdorff space, points are closed. Conversely, if $\{x\}$ is closed, $\{e\} = L_{x^{-1}}(\{x\})$ is closed and a normal subgroup. Then $G\cong {G}/{\{e\}}$ is Hausdorff. ∎

If (G,τ) is not Hausdorff, then $\{e\}\subsetneq\overline{\{e\}}$ is the smallest closed subgrup in G. Thus $\overline{\{e\}}\subseteq\bigcap_{x\in G} x\overline{\{e\}}x^{-1}\subseteq\overline{\{x\}}$ so $\overline{\{e\}}$ is normal. In particular, ${G}/{\overline{\{e\}}}$ is Hausdorff.

Definition. A **locally compact group** is a Hausdorff topological group (G,τ) which is locally compact.

(i) If there is any $U\in\tau\setminus\{\emptyset\}$ such that $\overline{U}$ is compact, then for any $x\in U$, we have $e\in x^{-1}U\subseteq L_{x^{-1}}(\overline{U})$ so $\overline{x^{-1}U}$ is compact. If $V\in\tau$ with $e\in V$ and $\overline{V}$ compact, then for any $x\in H$, $x\in xV$ and $\overline{xV}\subseteq L_x(\overline{V})$ and $\overline{xV}$ is compact. In particular, (G,τ) is locally compact if and only if there is some $U\in\tau\setminus\{\emptyset\}$ with $\overline{U}$ compact.

(ii) If (G,τ) is locally cmpact and N is a closed normal subgroup, then $({}^{G}\!/\!{}_{N}, \tau_{G/N})$ is locally compact. Indeed, $U \in \tau \setminus \{e\}$ with $\overline{U}$ compact, then $\overline{\pi(U)} \subseteq \pi(\overline{U})$ is compact.

Example. (i) If G is any group and τ is the discrete topology, then (G,τ_d) is locally compact.

(ii) If $((\mathbb{R},+),\tau_{\|\cdot\|})$ is locally compact.

(iii) If $\{G_i\}_{i\in I}$ is a family of locally compact groups, then $\prod_{i\in I} G_i$ is a locally compact group if and only if all but finitely many (G_i,τ_i) are compact.

(iv) $((\mathbb{R}^n,+),\tau_{\|\cdot\|})$ is a locally compact group

(v) Suppose $\{F_i\}_{i\in I}$ is an infinite family of finite groups (with discrete topologies), then $G = \prod_{i\in I} F_i$ is a compact group. If $F \subset I$ is finite, then $N_F = \{(x_i)_{i\in I} \in G : x_i = e \text{ for } i \in F\}$ is open and a normal subgroup. $\{N_F : F \subset I \text{ finite}\}$ is a base for the topology at e.

(vi) Let $(\Bbbk,\tau)$ be a locally compact field.. Then $\det^{-1}(\Bbbk \setminus \{0\}) = \mathrm{GL}_n(\Bbbk) \subseteq M_n(\Bbbk) \cong \Bbbk^{n^2}$ is an open subset and multiplicative subgroup, and hence locally compact. Notice that multiplication is governed by linear equations, and hence continuous, while inversion is continuous thanks to Cramer's rule.
Here are some common closed subgroups:

$$\overset{-1}{\det}(\{1\}) = \mathrm{SL}_n(\Bbbk)$$
$$O_n(\Bbbk) = \{x \in \mathrm{GL}_n(\Bbbk) : x^{-1} = X^T\}$$

As a special case, consider $U_n = \{x \in \mathrm{GL}_n(\mathbb{C}) : x^* = x^{-1}\}$ is governed by continuous equations, and thus closed in $M_n(\mathbb{C})$. Furthermore, U_n is bounded in $M_n(\mathbb{C})$, and hence compact.

1.2 *p*-adic Numbers

Let p be a prime in $\mathbb{N}$. We will establish product structures and topologies on

$$\mathbb{O}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : a_k \in \{0,1,\ldots,p-1\} \right\} \cong \{0,1,\ldots,p-1\}^{\mathbb{N}}$$
$$\mathbb{Q}_p = \left\{ \sum_{k=N}^{\infty} a_k p^k : N \in \mathbb{Z}, a_k \in \{0,1,\ldots,p-1\} \right\}$$

are topological rings and a topological field respectively. Let $R_p = \prod_{n=0}^{\infty} \mathbb{Z}\big/p^{n+1}\mathbb{Z}$ which is a ring under pointwise operations.

1.4 Lemma. *The map $\rho : R_p \times R_p \to [0,1]$ given by*

$$\rho(x,y) = \sum_{n\in\mathbb{N}_0} \frac{\rho_n(x_n,y_n)}{p^n} \qquad \rho_n(x_n,y_n) = \begin{cases} 1 & : x_n = y_n \\ 0 & : x_n \neq y_n \end{cases}$$

is a metric on R_p which satisfies

- *(additively invariant): $\rho(x+z,y+z) = \rho(x,y)$ for $x,y,z \in R_p$*
- *τ_ρ is the product topology*

Proof Additive invariance is routine. Notice that if $\frac{1}{p^m} \geq \epsilon > \frac{1}{p^{m+1}}$, then the open ϵ–ball around a point x is $\{x_0\} \times \cdots \{x_m\} \times \prod_{n=m+1}^{\infty} \mathbb{Z}\big/p^{n+1}\mathbb{Z}$ is product-open. Conversely, any product-open set is a finite union of such ϵ–balls. ■

1.5 Corollary. *The function $\|x\|_p = \rho(x,0)$ in R_p satisfies*

- $\|x\|_p = 0$ *if and only if* $x = 0$
- $\left\|x+y\right\|_p \leq \|x\|_p + \left\|y\right\|_p$
- $\left\|xy\right\|_p \leq \|x\|_p \left\|y\right\|_p$
- $\|-x\|_p = \|x\|_p$

Hence (R_p, τ_ρ) is a compact topological ring.

Proof The properties follow directly using additive invariance. To see that R_p is a topological ring, if $(x_\alpha),(y_\alpha)$ have $x = \lim x_\alpha$ and $y = \lim y_\alpha$, then, for example,

$$\begin{aligned}\left\|xy - x_\alpha y_\alpha\right\|_p &\leq \left\|xy - x_\alpha y\right\|_p + \left\|x_\alpha y - x_\alpha y_\alpha\right\|_p \\ &\leq \|x - x_\alpha\|_p + \left\|y - y_\alpha\right\|_p\end{aligned}$$

as $\left\|y\right\|_p, \|x_\alpha\|_p \leq 1$. ■

We now view $\mathbb{O}_p$ as a closed subring of R_p. Define $\alpha : \mathcal{O}_p \to R_p$ be given on $a = \sum_{k=0}^{\infty} a_k p^k$ by

$$\alpha(a) = \left(\sum_{k=0}^{n} a_k p^k + p^{n+1}\mathbb{Z}\right)_{n=0}^{\infty}.$$

This map is an injection with range $\alpha(\mathcal{O}_p) = \{(x_n)_{n=0}^{\infty} \in R_p : x_n = \pi_n(x_{n+1}) \text{ for all } n\}$ where $\pi_n : \mathbb{Z}\big/p^{n+2}\mathbb{Z} \to \mathbb{Z}\big/p^{n+1}\mathbb{Z}$ is the canonical quotient map. In fact, this is called an inductive limit with respect to the maps π_n. Hence it is routine to show that

- $\alpha(\mathbb{O}_p)$ is a subring of R_p, and
- $\alpha(\mathbb{O}_p)$ is closed in R_p (just check net limits in product topology)

If $a,b \in \mathbb{O}_p$, define $a+b = \alpha^{-1}(\alpha(a)+\alpha(b))$.

Remark. (i) $1 + \sum_{k=1}^{\infty} 0 \cdot p^k$ is the multiplicative identity in $\mathbb{O}_p$. Then $-1 = \sum_{k=0}^{\infty}(p-1)p^k$.

(ii) If $n \in \mathbb{N}$, we can uniquely write $n = \sum_{k=0}^{m(n)} a_k p^k$ with $a_k \in \{0,\ldots,p-1\}$. Then $n \cdot 1 = \sum_{k=0}^{m(n)} a_k p^k \in \mathbb{O}_p$. In particular, $n \mapsto n \cdot 1 : \mathbb{N} \to \mathbb{O}_p$ is an additive semigroup homomorphism with dense ring. Hence $n \mapsto n \cdot 1 : \mathbb{Z} \to \mathbb{Q}_p$ has dense range. We call $\mathbb{O}_p$ the ***p*–adic integers**.

Let $a = \sum_{k=0}^{\infty} a_k p^k$ in $\mathbb{O}_p$. Let

$$\begin{aligned}\nu_p(a) &= \min\{k \in \mathbb{N}_0 : a_k \neq 0\}, \min\emptyset = -\infty \\ |a|_p &= p^{-\nu_p(a)}, p^{-\infty} = 0\end{aligned}$$

and notice that $|a|_p = \|\alpha(a)\|_p$. However, $|a|_p$ has even nicer properties:

(i) $|a|_p = 0$ if and only if $a = 0$

(ii) $\nu_p(ab) = \nu_p(a) + \nu_p(b)$. Thus $|ab|_p = |a|_p|b|_p$

(iii) $\nu_p(a+b) \geq \min\{\nu_p(a), \nu_p(b)\}$. Thus $|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$

Notice that (i) and (ii) imply that $\mathbb{O}_p$ is an integral domain.

1.6 Proposition. *The multiplicative unit group of $\mathbb{O}_p$ is $\mathbb{O}_p \setminus p\mathbb{O}_p = \{a \in \mathbb{O}_p : |a|_p = 1\}$. Hence $\mathbb{O}_p^\times$ is open and a topological group.*

Proof The second equality is trivial. If $a \in \mathbb{O}_p^\times$, then $|a|_p$, $|a^{-1}|_p \leq 1$ and $1 = |1|_p = |aa^{-1}|_p = |a|_p|a^{=1}|_p$, so $|a|_p = 1$. If $|a|_p = 1$, let

$$x = \alpha(a) = \left(\sum_{k=0}^{n} a_k p^k + p^{n+1}\mathbb{Z}\right)_{n=0}^{\infty} \in R_p.$$

Then $x_n \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ since $p \nmid \sum_{k=0}^n a_k p^k$ in $\mathbb{Z}$. Hence there is $y_n \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ so $x_n y_n = 1 + p^{n+1}\mathbb{Z}$ and thus

$$1 + p^n\mathbb{Z} = \pi_N(1 + p^{n+2}\mathbb{Z}) = \pi_n(x_{n+1}y_{n+1}) = \pi(x_{n+1})\pi(y_{n+1}) = x_n\pi_n(y_{n+1})$$

so that $\pi_n(y_{n+1}) = y_n$. Thus if $y \in \alpha(\mathbb{O}_p)$, i.e. $y = \alpha(b)$ with $ab = \alpha^{-1}(\alpha(a)\alpha(b)) = \alpha^{-1}((1 + p^{n+1}\mathbb{Z})_{n=0}^\infty) = 1$ and $a \in \mathbb{O}_p^\times$.

Since $p\mathbb{O}_p$ is closed, we see that $\mathbb{O}_p^\times$ is open in $\mathbb{O}_p$. Continuity of multiplication follows (ii). Finally, if $a, b \in \mathbb{O}_p$,

$$|a^{-1} - b^{-1}|_p = |a|_p|a^{-1} - b^{-1}|_p|b|_p = |b - a|_p \qquad \blacksquare$$

1.7 Corollary. *Each ideal in $\mathbb{O}_p$ is of the form $p^k\mathbb{O}_p$ for $k \in \mathbb{N}_0$.*

Proof If I is an ideal in $\mathbb{O}_p$, then let $k(I) = \min\{k \in \mathbb{N}_0 : \nu_p(a) = k \text{ for some } a \in I\}$. Then there is $a \in I$ with $\nu_p(a) = k(I)$, so $p^{-k(I)} \in a\mathbb{O}_p^\times \subseteq a\mathbb{O}_p \subseteq I$. Thus $p^{-k(I)}\mathbb{O}_p \subseteq I$. Clearly $I \subseteq p^{-k(I)}\mathbb{O}_p$ as well. $\blacksquare$

We now extend the valuation and norm to $\mathbb{Q}_p$. If $a = \sum_{k\in\mathbb{Z}} a_k p^k \in \mathbb{Q}_p$, let $\nu_p(a) = \min\{k \in \mathbb{Z} : a_k \neq 0\}$ and $|a|_p = p^{-\nu_p(a)}$. Then for $a \in \mathbb{Q}_p \setminus\{0\}$ admits (formal) factorization

$$a = \sum_{k=\nu_p(a)}^{\infty} a_k p^k = p^{\nu_p(a)} \sum_{k=\nu_p(a)}^{\infty} a_k p^{k-\nu_p(a)} = p^{\nu_p(a)} \underbrace{\sum_{k=0}^{\infty} a_{k+\nu_p(a)} p^k}_{:=a' \in \mathbb{O}_p^\times}$$

Thus, if $a, b \in \mathbb{Q}_p \setminus\{0\}$, we define multiplication and addition by $ab = p^{\nu_p(a)+\nu_p(b)}a'b'$ and $a^{-1} = p^{-\nu_p(a)}(a')^{-1}$. Furthermore, assuming $\nu_p(a) \leq \nu_p(b)$, we define addition by

$$a + b = p^{\nu_p(a)}(a' + p^{\nu_p(b)-\nu_p(a)}b')$$

and $0 + a = a$, $0a = 0$. Notice that $|ab|_p = |a|_p|b|_p$, $|1/a|_p = 1/|a|_p$ and if $\nu_p(a) \leq \nu_p(b)$, $|a + b|_p = p^{-\nu_p(a)}|a' + p^{\nu_p(b)-\nu_p(a)}b'|_p \leq |a|_p$ so, generally, $|a + b|_p \leq \max\{|a|_p, |b|_p\}$. Also, if $|a|_p = 0$, then $|a| = 0$. Thus $(\mathbb{Q}_p, |\cdot|_p)$ is a "normed field", and hence a topological field.

Note that

$$\mathbb{O}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\} = \{a \in \mathbb{Q}_p : |a|_p < p\}$$

is a compact open neighbourhood of 0, so $\mathbb{Q}_p$ is locally compact. Moreover, each $p^k\mathbb{Q}_p = \{a \in \mathbb{Q}_p : |a|_p < p^{k-1}\}$ is a closed and open ball about 0.

1.3 Haar Integral and Haar Measure

Let G be a locally compact group. Define for $f : G \to \mathbb{C}$, $x \in G$, $f \cdot x = f \circ L_x$, and $x \cdot f = f \circ R_x$. Notice that $(f, x) \mapsto f \cdot x$, as an adjoint action, is a right (group) action of G on functions. Likewise, $(x, f) \mapsto x \cdot f$ is a left action.

1.8 Proposition. *If $f \in C_c(G)$, then $\lim_{x\to e} \|f \cdot x - f\|_\infty = 0 = \lim_{x\to e} \|x \cdot f - f\|_\infty$.*

Proof Let $\epsilon > 0$, W a symmetric relatively compact neighbourhood of e, and let $K = \overline{W} \operatorname{supp} f$. Given $y \in K$, $x \mapsto |f(xy) - f(y)|$ is continuous, so there is a neighbourhood U_y of e so $|f(xy) - f(y)| < \epsilon/2$ whenever $x \in U_y$. Let V_y be a symmetric neighbourhood of e so that $V_y^2 \subseteq U_y$. Then $K \subseteq \bigcup_{y\in K} V_y y$ so by compactness get some finite subcover indexed by $\{y_1, \dots, y_n\}$. Set $V = \left(\bigcap_{j=1}^n V_{y_j}\right) \cap W$ and note that V is a symmetric relatively compact neighbourhood of e.

If $x \in V$, then for $y \in K$ we have $y \in V_{y_j} y_j$ for some j, i.e. $yy_j^{-1} \in V_{y_j}$, and hence

$$xy = xyy_j^{-1} y_j \in V V_{y_j} y_j \subseteq V_{y_j}^2 y_j \subseteq U_{y_j} y_j.$$

Note that $xy = x' y_j$ for some $x' \in U_{y_j}$. Similarly, since $y_j y^{-1} \in U_{y_j}$, we $y_j = x'' y$ for some $x'' \in U_{y_j}$. Thus by definition of U_{y_j}, we have

$$|f(xy) - f(y)| \le |f(x' y_j) - f(y_j)| + |f(x'' y) - f(y)| < \epsilon.$$

Otherwise, if $y \notin K$, then $Wy \cap \operatorname{supp}(f) = \emptyset$: by contrapositive, if $x \in \operatorname{supp}(f) \cap Wy$, then $yx^{-1} \in W$ so $y \in \overline{W} \operatorname{supp} f = K$. Thus for $x \in V \subseteq W$, we have $f(xy) = 0 = f(y)$, and the desired result follows. ■

1.9 Theorem. (Existence of Haar Integral) *There exists a linear functional $I : C_c(G) \to \mathbb{C}$ satisfying*

- *(positivity): $I(f) > 0$ if $f \in C_c^+(G) = \{g \in C_c(G) \setminus \{0\} : g \ge 0\}$.*
- *(left invariance): $I(f \cdot x) = I(f)$ for $f \in C_c(G)$, $x \in G$.*

Let for $f, \phi \in C_c^+(G)$

$$(f : \phi) = \inf\left\{\sum_{j=1}^n c_j : \text{there are } x_1, \dots, x_n \in G, c_i > 0, n \in \mathbb{N} \text{ s.t. } f \le \sum_{j=1}^n c_j \phi \cdot x_j\right\}$$

Notive that $0 < \frac{\|f\|_\infty}{\|\phi\|_\infty} \le (f : \phi)$. Also, if $U = \{x \in G : \phi(x) > \frac{1}{2}\|\phi\|_\infty\}$, then $\operatorname{supp} f$ is covered by finitely many $x^{-1}U$, $x \in G$, and thus $(f : \phi) < \infty$.

Claim I *For f, g in $C_c^+(G)$, we have*

(i) $(f \cdot x : \phi) = (f : \phi)$ *for x in G*
(ii) $(cf : \phi) = c(f : \phi) = (f : \frac{1}{c}\phi)$ *for $c > 0$*
(iii) $(f + g, \phi) \le (f : \phi) + (g : \phi)$.
(iv) $(f : \phi) \le (f : g)(g : \phi)$

Proof Note that (i) and (ii) are straightforward. To see (iii) and (iv), consider

$$f \le \sum_{j=1}^{n} c_j \phi \cdot x_j \qquad g \le \sum_{j=n+1}^{N} c_j \phi \cdot x \qquad f \le \sum_{k=1}^{m} b_k g \cdot y_k$$

so that $f+g \le \sum_{j=1}^{N} c_j \phi \cdot x_k$ and $(f+g:\phi) \le \sum_{j=1}^{n} c_j + \sum_{j=n+1}^{N} c_j$, giving (iii). To get (iv), note $f \le \sum_{k=1}^{m} b_k \sum_{j=n+1}^{N} c_j \phi \cdot (x_j y_k)$ so $(f:\phi) \le \sum_{k=1}^{m} b_k \sum_{j=k+1}^{N} c_j$, giving (iv).

We wish to "homogonize" the effect of ϕ. Fix $\psi_0 \in C_c^+(G)$ and let $I_\phi(f) = \frac{(f:\phi)}{(\psi_0:\phi)}$. Then $I_\phi : C_c^+(G) \to \mathbb{R}_{\ge 0}$ is
(i') left translation invariant
(ii') $\mathbb{R}_{\ge 0}$–homogenous
(iii') subadditive.
by using the claim above directly. Thus by (iv), $(\psi_0 : \phi) \le (\psi_0 : f)(f:\phi)$ and $(f:\phi) \le (f:\psi_0)(\psi_0:\phi)$, giving
iv' $0 < \frac{1}{(\psi_0:f)} \le I_\phi(f) \le (f:\psi_0)$.

Claim II *If $f, g \in C_c^+(G)$, $\epsilon > 0$, there is a neighbourhood V of e such that*

$$I_\phi(f) + I_\phi(g) < I_\phi(f+g) + \epsilon$$

whenever $\phi \in C_c^+(G)$ with $\operatorname{supp}(\phi) \subseteq V$.

Proof Let $k \in C_c^+(G)$ be so $k|_{\operatorname{supp}(f+g)} = 1$ and let $\delta > 0$. Take $h = f+g+\delta k$ and $f' = \frac{f}{h}$, $g' = \frac{g}{h} \in C_c^+(G)$. Uniform continuity of f', g' provides a neighbourhood v of e such that $|f'(x) - f'(y)| < \delta$, $|g'(x) - g'(y)| < \delta$ if $y^{-1}x \in V$. If $\phi \in C_c^+(G)$, $\operatorname{supp}(\phi) \subseteq V$, and $x_1, \ldots, x_n$ in G, $c_1, \ldots, c_n > 0$ satisfy

$$h \le \sum_{j=1}^{n} c_j \phi_j \cdot x_j^{-1}$$

then for x in G

$$\begin{aligned} f(x) = f'(x)h(x) &\le \sum_{j=1}^{n} f'(x) c_j \phi(x_j^{-1}x) \\ &\le \sum_{j=1}^{n} [f'(x_j) + \delta] c_j \phi(x_j^{-1}x) \end{aligned}$$

by properties of f', g' proven above and $\operatorname{supp}(\phi) \subseteq C$. Likewise,

$$g \le \sum_{j=1}^{n} [g'(x_j) + \delta] c_j \phi \cdot x_j^{-1}.$$

Now $f' + g' = (f+g)/h = \frac{f+g}{f+g+\delta k} \le 1$ so

$$\begin{aligned} (f:\phi) + (g:\phi) &\le \sum_{j=1}^{n} [f'(x_j) + \delta] c_j + \sum_{j=1}^{n} [g'(x_j) + \delta] c_j \\ &\le \sum_{j=1}^{n} [1 + 2\delta] c_j \end{aligned}$$

and $(f:\phi)+(g:\phi)\le(1+2\delta)(h:\phi)$. Divide by $(\psi_0:\phi)$ and (iii') and (iv') above to get

$$I_\phi(f)+I_\phi(g)\le(1+2\delta)I_\phi(h)\le(1+2\delta)[I_\phi(f+h)+\delta I_\phi(k)]$$

Thus with sufficiently small δ, $I_\phi(f)+I_\phi(g)<I_\phi(f+g)+\epsilon$.

We are now in position to complete the proof.

Claim III *Construction of the functional I.*

Proof Inequality (iv') tells us that

$$x_\phi=(I_\phi(f))_{f\in C_c^+(G)}\in\prod_{f\in C_c^+(G)}[\frac{1}{(\psi_0:f)},(f:\psi_0)]=X$$

which, by Tychonoff, is compact. For ϕ,ϕ' in $\Phi=\{\psi\in C_c^+(G):\psi(e)=1\}$, $\phi\le\phi'$ if $\phi\ge\phi'$ pointwise, which is a preorder. Notice that $\phi\phi'\le\phi\wedge\phi'$ (pointwise minimum), so that $(\Phi,\le)$ is directed. Hence $(x_\phi)_{\phi\in\Phi}$ admits a converging subnet $x=\lim_{\mu\in M}x_{\phi_\mu}$ in X.

Write $x=(I(f))_{f\in C_c^+(G)}$, so $I(f)=\lim_{\mu\in M}I_{\phi_\mu}(f)$ for each $f\in C_c^+(G)$. Then it follows that from (i'), (ii'), and (iii') that for f,g in $C_c^+(G)$, we have

$$I(F\cdot x)=I(f)\qquad\qquad I(cf)=cI(f)\qquad\qquad I(f+g)\le I(f)+I(g)$$

for $x\in G$, $c>0$. Moreover, by cofinality, if V is a neighbourhood of e, then $\operatorname{supp}(\phi_\mu)\subseteq V$ for μ sufficiently large in M. Hence given $\epsilon>0$, by Claim II, $I_{\phi_\mu}(f)+I_{\phi_\mu}(g)<I_{\phi_\mu}(f+g)+\epsilon$ for μ sufficiently large in M. Since $\epsilon>0$ as arbitrary, we have $I(f)+I(g)\le I(f+g)$.

Let $I(0)=0$. If $f\in C_c^{\mathbb{R}}(G)$ and $f=f_1-f_2=g_1-g_2$ with $f_1,f_2,g_1,g_2\ge0$, then $h=f_1+g_2=g_1+f_2$ satisfies that $I(h)=I(f_1)+I(g_2)=I(g_1)+I(f_2)$ and hence we may define $I(f)=I(f_1)-I(f_2)$, which do not depend on the choice of f_1,f_2. One may check that $I:C_c^{\mathbb{R}}(G)\to\mathbb{R}$ is $\mathbb{R}-$linear. Finally, if $f\in C_c(G)$, let $I(f)=I(\operatorname{Re}f)+iI(\operatorname{Im}f)$. It is left as an exercise to verify that $I:C_c(G)\to\mathbb{C}$ is $\mathbb{C}-$linear.

Finally, the fact that $I(f\cdot x)=I(f)$ for $f\in C_c(G)$ and $x\in G$ follows for f in $C_c^+(G)$ as above. If $f\in C_c^+(G)$, then (iv') tellus us that $I(f)\ge\frac{1}{(\psi_0:f)}>0$. ■

Remark. (i) In Claim III, $I_\phi(\psi_0)=1$ so $I(\psi_0)=1$.

(ii) If G is discrete, then $\psi_0=1_{\{e\}}=\min\Phi$. Then $I_{\psi_0}(f)=\frac{(f:\psi_0)}{(\psi_0:\psi_0)}=\sum_{x\in G}f(x)$ for $f\in C_c^+(G)$.

(iii) If $G=\mathbb{R}$, let ψ_0 be the linear function which is 0 on $(-\infty,-1/2-\delta)\cup(1/2+\delta,\infty)$, 1 on $(-1/2+\delta,1/2-\delta)$, and continuied linearly on the remainder. Notice that $(\psi_0,\phi_n)\approx n$, so $\frac{(f:\phi_n)}{(\psi_0:\phi_n)}$ is approximately the Riemann-Darboux upper sum.

(iv) Examine $\mathbb{O}_p$, $\psi_0=1_{\mathbb{O}_0}$, $\psi_n=1_{p^n\mathbb{O}_p}$.

1.10 Theorem. (Harr Measure) *Let $\mathcal{B}(G)$ denote the Borel $\sigma-$algebra on G. Then there is a Radon measure $m:\mathcal{B}(G)\to[0,\infty]$ such that*

- *m is left invariant: $m(xE)=m(E)$ for $x\in G$, $E\in\mathcal{B}(G)$*
- *$m(U)>0$ for $U\in\tau\setminus\{\emptyset\}$.*

Proof The Riesz Representation Theorem provides a measure $m:\mathcal{B}(G)\to[0,\infty]$ for which

$$\int_G f\,\mathrm{d}m=I(f)$$

for $f \in C_c(G)$. Notice that

$$\int_G f \cdot x \, \mathrm{d}m = I(f \cdot x) = \int_G f$$

for any $x \in G$, $f \in C_c(G)$. Thus if $U \in \tau$, $\operatorname{supp} f \subseteq U$ if and only if $\operatorname{supp}(f \cdot x) \subseteq x^{-1}U$ for $x \in G$ and $f \in C_c(G)$. Thus

$$\begin{aligned} m(U) &= \sup\{I(f) : f \in C_c(G), 0 \leq f \leq 1 \text{ and } \operatorname{supp}(f) \subseteq U\} \\ &= \sup\{I(f \cdot x) : f \in C_c(G), 0 \leq f \leq 1 \text{ and } \operatorname{supp}(f) \subseteq U\} \\ &= \sup\{I(g) : g \in C_c(G), 0 \leq g \leq 1, \operatorname{supp}(g) \subseteq x^{-1}U \\ &= m(x^{-1}U). \end{aligned}$$

Therefore, for any $E \in \mathcal{B}(G)$, we have

$$\begin{aligned} m(E) &= \inf\{m(U) : E \in U, U \in \tau\} \\ &= \inf\{m(xU) : E \subseteq U, U \in \tau\} \\ &= \inf\{m(xU) : xE \subseteq xU, U \in \tau\} = m(xE). \end{aligned}$$

Finally, if $U \in \tau \setminus \{\emptyset\}$, there is $f \in C_c^+(G)$ with $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subseteq U$, so $m(U) \geq I(f) > 0$. ■

Remark. If $E \in \mathcal{G}(G)$, $m(E) < \infty$, then $m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$. Inner regularity need not hold on infinite measure sets: taking $G = \mathbb{R}_d \times \mathbb{R}$, then $\mathbb{R}_d \times \{0\}$ is closed, and thus Borel. However, $m(E) = \infty$ while $m(K) = 0$ for each compact $K \subset E$.

1.11 Theorem. (Uniqueness of Haar Measure) *Let $m' : \mathcal{B}(G) \to [0, \infty]$ be any Radon measure such that $m(xE) = m'(E)$ for $x \in G$ and $E \in \mathcal{B}(G)$. Then there is $c \geq 0$ such that $m' = cm$.*

Proof Fix a symmetric neighbourhood $W = W^{-1}$ of e with $\overline{W}$ compact. Given $f \in C_c^+(G)$, $\epsilon > 0$, and U a neighbourhood of e such that $\|f - x \cdot f\|_\infty < \epsilon$. Let $V = U \cap W$. Then let $x \in G$, and for any $x' \in G$ with $x'x^{-1} \in V$, we have

$$\begin{aligned} \left| \int_G f(yx) dm'(y) - \int_G f(yx') dm'(y) \right| &\leq \left\| x \cdot f - x' \cdot f \right\|_\infty m(\operatorname{supp}(f)x^{-1} \cup \operatorname{supp}(f)Vx^{-1}) \\ &< \epsilon m(\operatorname{supp}(f)x^{-1} \cup \operatorname{supp}(f)Wx^{-1}) \end{aligned}$$

and hence $x \mapsto \int_G x \cdot f \, dm'$ is continuous at each point in G. Thus

$$D_f(x) = \frac{\int_G x \cdot f \, dm'}{\int_G f \, dm}$$

defines a continuous function on G.

If $f, g \in C_c^+(G)$, then $(x,y) \mapsto f(x)g(y^{-1})$ is non-negative, continuous, Borel measurable, and compactly supported on $G \times G$. Then by left-invariance and Tonelli's theorem,

$$\begin{aligned}\left(\int_G f\,dm\right)\cdot\left(\int_G g(y^{-1})dm'(y)\right) &= \int_G\int_G f(x)g(y^{-1})dm'(y)dm(x)\\ &= \int_G\int_G f(x)g((x^{-1}y)^{-1})dm'(y)dm(x)\\ &= \int_G\int_G f(x)g(y^{-1}x)dm(x)dm'(y)\\ &= \int_G\int_G f(yx)g(x)dm(x)dm'(y)\\ &= \int_G g(y)\left(\int_G f(yx)dm'(y)\right)dm(x)\end{aligned}$$

Thus,

$$\int_G g(y^{-1})dm'(y) = \int_G g(x)D_f(x)dm(x).$$

But if we have any other $f' \in C_c^+(G)$, then we would have

$$\int_G g(x)D_f(x)dm(x) = \int_G g(y^{-1})dm'(y) = \int_G g(x)D_{f'}(x)dm(x)$$

so it follows that $D_f = D_{f'}$ m–a.e. Since D_f, $D_{f'}$ are continuous, we see that $D_f = D_{f'}$ everywhere. In particular, $D_f(e) = D_{f'}(e)$, or that

$$\frac{\int_G f\,dm'}{\int_G f\,dm} = D_f(e) = D_{f'}(e) = \frac{\int_G f'dm'}{\int_G f'dm}$$

Let c denote this common value, so $c\int_G f\,dm = \int_G f\,dm'$ for $f \in C_c^+(G)$. Hence $m' = cm$. ■

Example. (i) If G is discrete, then $C_c(G)$ is composed of functions with finite support, and $m(E)$ is (a multiple of) the counting measure. In the finite case, we normalize by $|G|$.

(ii) If $G = \mathbb{R}^n$, $I(f) = \int_{\mathbb{R}^n} f$ and m is n-dimensional Lebesgue measure.

(iii) Let $G = \mathrm{GL}_n(\mathbb{R})$.

   (a) If $t \in \mathrm{GL}_n(\mathbb{R})$, then for $f \in C_c(\mathbb{R}^n)$,

   $$\int_{\mathbb{R}^n} f \circ t(y)dy = \frac{1}{|\det t|}\int_{\mathbb{R}^n} f(y)dy.$$

   Indeed, show that this holds for an elementary matrix t, and $\mathrm{GL}_n(\mathbb{R})$ is the algebra generated by these elements.

   (b) If $X \in \mathrm{GL}_n(\mathbb{R})$, then $L_X : M_n(\mathbb{R}) \to M_n(R)$ is isomorphic to

   $$\begin{pmatrix} y_1 \\ \vdots \\ y_n\end{pmatrix} \mapsto \begin{pmatrix} xy_1 \\ \vdots \\ xy_n\end{pmatrix} : (\mathbb{R}^n)^n \to (\mathbb{R}^n)^n$$

and hence $\det L_X = \det X^n$. Thus if $f \in C_c(M_n(\mathbb{R}))$, we have that

$$\int_{M_n(\mathbb{R})} f(xy)dy = \int_{M_n(\mathbb{R})} f \circ L_X(y)dy = \frac{1}{|\det X|^n}\int_{M_n(\mathbb{R})} f(y)dy.$$

Now since $\mathrm{GL}_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$, so $C_c(\mathrm{GL}_n(\mathbb{R})) \subset C_c(M_n(\mathbb{R}))$, and we define for $f \in C_c(\mathrm{GL}_n(\mathbb{R}))$

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{R})} f(y)\frac{1}{|\det y|^n}dy.$$

Then for $x \in \mathrm{GL}_n(\mathbb{R})$, we have

$$\begin{aligned} I(f\cdot x) &= \int_{\mathrm{GL}_n(\mathbb{R})} f(xy)\frac{1}{|\det xy|^n}\cdot|\det x|^n dy \\ &= \int_{\mathrm{GL}_n(\mathbb{R})} f(y)\frac{1}{|\det y|^n}\cdot\frac{|\det x|^n}{|\det x|^n}dy = I(f) \end{aligned}$$

If $M_{\mathrm{GL}_n(\mathbb{R})}$ is the measure associated with I, then with m the Lebesgue measure on $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we have

$$\frac{dm_{\mathrm{GL}_n(\mathbb{R})}}{dm}(y) = \frac{1}{|\det y|^n}$$

(c) If we take $\mathbb{R}^\times \cong \mathrm{GL}_1(\mathbb{R})$, then

$$I(f) = \int_{\mathbb{R}^\times} f(y)\frac{dy}{|y|}$$

(d) Consider $\mathbb{C}^\times = \mathbb{C}\setminus\{0\} \subseteq \mathbb{C} \cong \mathbb{R}^2$. Then $[L_{x+iy}]_{(1,i)} = \begin{pmatrix} x & -y \\ y & x\end{pmatrix}$ so that $\det L_{x+iy} = |x+iy|^2$. Thus we get an integral on $G = \mathbb{C}^\times$ by

$$I(f) = \int_{\mathbb{C}^\times} f(z)\frac{dz}{|z|^2}$$

(e) On $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{R})$, we likewise find Haar integral

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{C})} f(y)\frac{1}{|\det y|^{2n}}dy.$$

(iv) Suppose G admits an open (hence closed) subgroup H. If m is a Haar measure on G, then $m_H = m|_{\mathcal{B}(H)}$ is a Haar measure on H. Let T be a transversal for left cosets of H (A of C), so $G = \bigcup_{t\in T} tH$. If $U \subset G$ is open with $m(U) < \infty$, then

$$\begin{aligned} \{t \in T : U \cap tH \neq \emptyset\} &= \{t \in T : M(U \cap tH) > 0\} \\ &= \bigcup_{n=1}^{\infty}\{t \in T : m(U\cap tH) < \frac{1}{n}\} \end{aligned}$$

is countable, so if $E \in \mathcal{B}(G)$, $m(E) < \infty$, $E \subseteq \bigcup_{j=1}^\infty t_jH$ and then

$$\begin{aligned} m(E) &= \sum_{j=1}^\infty m(E\cap t_jH) = \sum_{j=1}^\infty m(t_j^{-1}(E\cap t_jH)) \\ &= \sum_{j=1}^\infty m_H((t_j^{-1}E)\cap H) \end{aligned}$$

(v) Suppose H is a closed, non-open subgroup of G. We wish to see that for compact $K \subseteq H$, $m(K) = 0$.
 (a) Given open U with $K \subseteq U$, then there is open V with $e \in V$ so $VK \subseteq U$. Indeed, for $x \in K$, find open U_x with $e \in U_x$, so $U_x x \subseteq U$. Find open V_x, $e \in V_x$, so $V_x^2 \subseteq U_x$, then $K \subseteq \bigcup_{j=1}^n V_{x_j} x_j$ where $x_1, \dots, x_j \in K$. Let $V = \bigcap_{j=1}^n V_{x_k}$. If $x \in K$, $x \in V_{x_j} x_j$ for soe j so $Vx \subseteq VV_{x_j} x_j \subseteq V_{x_j}^2 x_j \subseteq U_{x_j} x_j \subseteq U$, i.e. $VK = \bigcup_{x \in K} V_x \subseteq U$.
 (b) Suppose we had compact $K \subseteq H$ such that $m(K) > 0$. We may find open U so $K \subseteq U$ and $m(U) < 2m(K)$ (by outer regularity). Take V as above. Since H is non-open, there is $x \in V \setminus H$. Then
  - $K \cap xK = \emptyset$ as $K \subseteq H$, while
  - $K \cup xK \subseteq U$.

 . Thus $2m(K) = m(K \cup xK) \leq m(U) < 2m(K)$, a contradiction.
 Thus, a closed non-open subgroup H of G is always m-locally null. Hence, if G is σ-compact, then closed non-open H are m-null. However, if $G = \mathbb{R} \times \mathbb{R}_d$, $H = \{0\} \times \mathbb{R}_d$ is closed, m-locally null, but not m-null.

(vi) The measure on $(\mathbb{Q}_p, +)$ is determined by $(\mathbb{O}_p, +)$. Likewise, the measure $\mathrm{GL}_n(\mathbb{Q}_p)$is determined by $\mathrm{GL}_n(\mathbb{O}_p) = \{a \in M_n(\mathbb{O}_p) : \det a \in \mathbb{O}_p^\times\}$

(vii) On $\mathrm{GL}_n(\mathbb{O}_p)$, we have Haar integral

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{Q}_p)} f(y) \frac{1}{|\det y|_p^n} dy$$

(viii) G is compact if and only if $m(G) < \infty$. The forward is clear since m is Radon. If G is not compact, let U be a open neighbourhood of e so $\overline{U}$ is compact, so $0 < m(U) < \infty$. For any compact set K, $KU \subseteq K\overline{U}$ is compact, hence $KU \subsetneq G$. Inductively find $(x_n)_{n=1}^\infty \subset G$ so $x_{n+1} \notin \{x_1, \dots, x_n\}U$. Notice that $x_j V \cap x_k V = \emptyset$ for $j \neq k$ for V a neighbourhood of e with $V = V^{-1}, V^2 \subset U$. Then $m(G) \geq nm(V)$ for any $n \in \mathbb{N}$, so $m(G) = \infty$.

Let G be a locally compact group equipped with Haar measure m. Then

$$L^1(G) = \{f : G \to \mathbb{C} : f \text{ measurable}, \|f\|_1 = \int_G |f| dm < \infty\} / \sim_m \quad \textit{a.e.}$$

This is a Banach space. Recall that by definition of the Lebesgue integral

$$S^1(G) = \mathrm{span}\{\chi_E : E \in \mathcal{B}(G), m(E) < \infty\} / \sim_m \quad \textit{a.e.}$$

If $0 < m(E) < \infty$, then, given $\epsilon > 0$, there are compact $K \subseteq E$ and open $U \supseteq E$. Hence Urysohn's lemma provides $f \in C_c^+(G)$ such that $f|_L = 1$, $\mathrm{supp}(f) \subseteq U$, and $0 \leq f \leq 1$. Hence $\|f - 1_E\|_1 < \epsilon$. Note that if $f, g \in C_c(G)$, then $f = g$ m a.e. if and only if $f = h$. Thus $C_c(G) \subseteq L^1(G)$ is dense.

1.4 The Modular Function

If $x \in G$, define $m_x : \mathcal{B}(G) \to [0, \infty]$ by $m_x(E) = m(Ex)$. Then since R_x is a homeomorphism, one may verify that
- m_x is left invariant,
- $m_x(U) = m(Ux) > 0$ if U is non-empty and open, and

- $m_x(K) = m(Kx) < \infty$ if K is compact.

Hence by uniqueness of Haar measure there exists some function $\Delta : G \to \mathbb{R}^\times$ such that $m_x = \Delta(x)m$. In fact, Δ is a group homomorphism. To see this, if $E \in \mathcal{B}(G)$ with $0 < m(E) < \infty$ and $x, y \in G$, then

$$\Delta(xy)m(E) = m(Exy) = \Delta(y)m(Ex) = \Delta(x)\Delta(y)(E).$$

Denote by $x \cdot f$ the function $x \cdot f(y) = f(yx)$. We then have the following result:

1.12 Proposition. *(i) For any $f \in L^1(G)$ and $x \in G$, $x \cdot f \in L^1(G)$ with*

$$\int_G f \,\mathrm{d}m = \Delta(x) \int_G x \cdot f \,\mathrm{d}m.$$

(ii) Δ is a continuous function.

Proof (i) Let $E \in \mathcal{B}(G)$ with $m(E) < \infty$. Then

$$\Delta(x) \int \mathbf{1}_E \,\mathrm{d}m = \Delta(x)m(E) = m(Ex) = \int \mathbf{1}_{Ex} \,\mathrm{d}m = \int_G x^{-1} \cdot \mathbf{1}_E \,\mathrm{d}m$$

since $\mathbf{1}_{Ex} = x^{-1} \cdot \mathbf{1}_E$. Thus replacing x by x^{-1}, we have

$$\int \mathbf{1}_E \,\mathrm{d}m = \Delta(x) \int x \cdot \mathbf{1}_E \,\mathrm{d}m$$

so that the desired result holds for characteristic functions. Then by density of simple functions in L^1 and applying dominated convergence, the result holds for any $f \in L^1$.

(ii) Let $f \in C_c^+(G)$, $\epsilon > 0$, and $V = V^{-1}$ be a relatively compact neighbourhood of e so $\|x \cdot f - f\|_\infty < \epsilon$ for any $x \in V$. Then for $x \in V$, applying (i) above,

$$\begin{aligned}|\Delta(x) - 1| &= \frac{|\Delta(x) \int f \,\mathrm{d}m - \int_G f \,\mathrm{d}m|}{\int f \,\mathrm{d}m} \\ &\leq \frac{1}{\int f \,\mathrm{d}m} \int |x^{-1} \cdot f - f| \,\mathrm{d}m < \epsilon \frac{m(\mathrm{supp}(f)V)}{\int f \,\mathrm{d}m}.\end{aligned}$$

where $\mathrm{supp}(x^{-1} \cdot f - f) \subseteq \mathrm{supp}(f)V$ and $\mathrm{supp}(f)V$ has compact closure so that $m(\mathrm{supp}(f)V) < \infty$. Moreover, as $\epsilon \to 0$, we may arrange for V to be decreasing, yielding continuity of Δ at e. Now if $x, y \in G$ are arbitrary, we have

$$|\Delta(xy) - \Delta(y)| = |\Delta(x) - 1|\Delta(y)$$

which shows that Δ is continuous at y. ∎

1.13 Proposition. *(i) The integral $f \mapsto \int_G f(x) \frac{1}{\Delta(x)} \,\mathrm{d}x$ on $C_c(G)$ is right invariant.*

(ii) For $f \in L^1(G)$,

$$\int_G f(x^{-1}) \frac{1}{\Delta(x)} \,\mathrm{d}x = \int_G f(x) \,\mathrm{d}x$$

Proof (i) If $y \in G$ and $f \in C_c(G)$, then

$$\int_G y \cdot f(x) \frac{1}{\Delta(x)} \, \mathrm{d}x = \int_G f(xy) \frac{1}{\Delta(x)} \, \mathrm{d}x = \int_G f(xy) \frac{1}{\Delta(xy)} \Delta(y) \, \mathrm{d}x$$
$$= \int_G f(x) \frac{1}{\Delta(x)} \, \mathrm{d}x$$

(ii) If $f \in C_c^+(G)$, then for any $y \in G$,

$$\int_G f \cdot y(x^{-1}) \frac{1}{\Delta(x)} \, \mathrm{d}x = \int_G f((xy^{-1})^{-1}) \frac{1}{\Delta(x)} \, \mathrm{d}x$$
$$= \int_G f(x^{-1}) \frac{1}{\Delta(x)} \, \mathrm{d}x$$

by the proof above. Hence by uniqueness of left Haar integral, there is $c > 0$ so that

$$\int_G f(x^{-1}) \frac{1}{\Delta(x)} \, \mathrm{d}x = c \int_G f(x) \, \mathrm{d}x$$

for $f \in C_c(G)$. Notice, by continuity of $f \mapsto \int_G f \, \mathrm{d}m$ on $L^1(G)$, this holds for $f \in L^1(G)$. Now, if $c \neq 1$, there is a relatively compact neighbourhood $U = U^{-1}$ of e such that $|\Delta(x) - 1| < \frac{1}{2}|c - 1|$ for $x \in U$. Then we have

$$\begin{aligned} 0 &= \left| \int_G \mathbf{1}_U(x^{-1}) \frac{1}{\Delta(x)} \, \mathrm{d}x - c \int_G \mathbf{1}_U(x) \, \mathrm{d}x \right| \\ &= \left| \int_U \left(\frac{1}{\Delta(x)} - c \right) \mathrm{d}x \right| \\ &= \left| \int_U \left(\frac{1}{\Delta(x)} - 1 + 1 - c \right) \mathrm{d}x \right| \\ &\geq m(U) \left| |1 - c| - \frac{1}{2}|c - 1| \right| = \frac{m(U)}{2} |1 - c| > 0 \end{aligned}$$

which is a contradiction. ■

For $f \in L^1(G)$, $x \in G$, we let

$$\begin{aligned} x * f(y) &= f(x^{-1}y) \\ f * x(y) &= f(yx^{-1}) \frac{1}{\Delta(x)} \\ f^*(x) &= \overline{f(x^{-1})} \frac{1}{\Delta(x)} \end{aligned}$$

Notice that $\|f\|_1 = \|x * f\|_1 = \|f * x\|_1 = \|f^*\|_1$. Moreover,
- $x' * (x * f) = (x'x) * f$ and $(f * x) * x' = f * (xx')$
- $f^{**} = f$.

- $(x * f)^* = f^* * x^{-1}$. Indeed, for m-a.e. y, we have

$$\begin{aligned}(x * f)^*(y) &= \overline{[x * f](y^{-1})}\frac{1}{\Delta(y)}\\ &= \overline{f(x^{-1}y^{-1})}\frac{1}{\Delta(y)}\\ &= \overline{f((yx)^{-1})}\frac{1}{\Delta(yx)}\frac{1}{\Delta(x^{-1})}\\ &= f^* * x^{-1}(y)\end{aligned}$$

1.14 Proposition. *For $f \in L^1(G)$, $\lim_{x\to e}\|x * f - f\|_1 = 0 = \lim_{x\to e}\|f * x - f\|_1$.*

Proof First, consider $g \in C_c(G)$ and $\epsilon > 0$, and let $V = V^{-1}$ be a relatively compact neighbourhood of e so $\|g \cdot x - g\|_\infty < \epsilon$ for $x \in V$. Then

$$\|x * g - g\|_1 = \int_G |g \cdot x^{-1}\,\mathrm{d}m \leq \left\|g \cdot x^{-1} - g\right\|_\infty m(V\operatorname{supp}(g)) < \epsilon m(V\operatorname{supp}(g))$$

so $\lim_{x\to e}\|x * g - g\|_1 = 0$. If $f \in L^1(G)$, $\epsilon > 0$, find $g \in C_c(G)$ such that $\|f - g\|_1 < \epsilon$. Then

$$\begin{aligned}\|x * f - f\|_1 &\leq \|x * f - x * g\|_1 + \|x * g - g\|_1 + \|g - f\|_1\\ &< 2\epsilon + \|x * g - g\|_1\end{aligned}$$

where $\|x * g - g\|_1 \to 0$ as $x \to e$. Since $\epsilon > 0$ was arbitrary, we are done.

Now, for f, x as above,

$$\|f_x - f\|_1 = \|(f * x - f)^*\|_1 = \left\|x^{-1} * f^* - f^*\right\|_1$$

where $x^{-1} \to e$ as $x \to e$. ■

1.15 Theorem. (Weil Integral Formula) *Let N be a closed normal subgroup of G.*

(i) If $f \in C_c(G)$, then $x \mapsto \int_N f(xn)dn : G \to \mathbb{C}$ is constant on cosets and hence defines a function $T_N f : G/N \to \mathbb{C}$. Furthermore, $T_N(C_c^+(G)) \subseteq C_c^+(G/N)$ and the operator $T_N : C_c(G) \to C_c(G/N)$ is linear and covariant:

$$(T_N f)\cdot(yN) = T_N(f \cdot y)$$

for $f \in C_c(G)$ and $y \in G$.

(ii) The functional on $C_c(G)$ given by $f \mapsto \int_{G/N} T_n f(xN)\,\mathrm{d}xN$ is hence a Haar integral on G, so we may write

$$\int_{G/N}\int_N f(xn)\,\mathrm{d}n\,\mathrm{d}xN = \int_G f(x)\,\mathrm{d}x$$

Proof (ii) is a direct consequence of (i); let's see the proof of (i).

The N-invariance of the first function is evident. Let $f \in C_c(G)$. We inspect the continuity of $T_N f$ at x in G. Given $\epsilon > 0$, let $V = V^{-1}$ be a real compact neighbourhood of e, so $\left\|f \cdot y - f\right\|_\infty < \epsilon$ for $y \in V$. Let $g \in C_c^+(G)$ satisfy that $0 \le g \le 1$ and $g|_{Vx^{-1}\operatorname{supp}(f)} = 1$. For $y \in V$, $yN = q_N(y) \in q_N(V)$ so

$$\begin{aligned}|T_N f(yxN) - T_N f(xN)| &\le \int_N |f(yx) - f(xn)| g(n)\,\mathrm{d}n \\ &< \epsilon m_N(\operatorname{supp}(g) \cap N)\end{aligned}$$

Notice as $\epsilon \to 0$, we may shrink V and hence $\operatorname{supp}(g)$. Hence $T_N f$ is continuous at xN. Now, $\operatorname{supp}(T_N f) \subseteq q_N(\operatorname{supp} f)$ so $T_N f \in C_c(G/N)$.

If $g \in C_c^+(G)$, $x \in G$ is such that $f(x) > 0$, let U be a neigbourhood of e, $f(xy) > \frac{1}{2} f(x)$ for $y \in U$. Then

$$T_N f(xN) = \int_N f(xn)\,\mathrm{d}n \ge \frac{1}{2} f(x) m_N(U \cap N) > 0$$

■

1.16 Corollary. *If N is closed and normal in G, Then $\Delta_G|_N = \Delta_N$.*

Proof Let $n' \in N$ and $f \in C_c^+(G)$. Then

$$\begin{aligned}\int_G n' \cdot f(x)\,\mathrm{d}x &= \int_{G/N} \int_N f(xnn')\,\mathrm{d}n\,\mathrm{d}xN \\ &= \int_{G/N} \frac{1}{\Delta_N(n')} \int_N f(xn)\,\mathrm{d}n\,\mathrm{d}xN = \frac{1}{\Delta_N(n')} \int_G f(x)\,\mathrm{d}x\end{aligned}$$

so that $\Delta_G(n') = \Delta_N(n')$. ■

Definition. We say that G is **unimodular** if $\Delta = 1$ on G.

1.17 Proposition. *G is unimodular in any of the following cases:*

(i) G is abelian, discrete, or compact

(ii) G is perfect: $G = \overline{[G, G]}$

(iii) $G/Z(G)$ is unimodular.

(iv) There is a closed, unimodular normal subgroup N such that G/N is compact.

Proof (i) This is (nearly) obvious for G abelian or discrete. If G is compact, then $\log \circ \Delta : G \to (\mathbb{R}, +)$ is a continuous homomorphism whose range is a compact subgroup.

(ii) Any commutator $[x, y] \in xyx^{-1}y^{-1} \in \ker \Delta$.

(iii) Let $Z = Z(G)$. If $y \in G$ and $f \in C_c(G)$, we have by Weyl's integral formula

$$\begin{aligned}\int_G y \cdot f(x)dx &= \int_{G/Z}\int_Z f(xz)\,\mathrm{d}z\,\mathrm{d}xZ \\ &= \int_{G/Z}\int_Z f(xyz)\,\mathrm{d}z\,\mathrm{d}xZ \\ &= \int_{G/Z} T_Z f(xyZ)\,\mathrm{d}xZ \\ &= \int_{G/Z} T_Z f(xZyZ)dxZ \\ &= \int_{G/Z} T_Z f(xZ)dxZ = \int_G f(x)dx\end{aligned}$$

(iv) We have $\Delta_G|_N = \Delta_N = 1$ by assumption, i.e. $N \in \ker\Delta_G$, so Δ_G induces a homomorphism $\overline{\Delta} : G/N \to (0,\infty)$ where $\Delta_G = \overline{\Delta} \circ \pi_N$. Verify that $\overline{\Delta}$ is continuous, so $\log\circ\overline{\Delta} : G/N \to \mathbb{R}^+$ is a continuous homomorphism, whose range is a closed subgroup. It follows that $\Delta_G = 1$ on G. ∎

Example. Here are some examples of unimodular groups.

(i) Let $\Bbbk$ be a locally compact field with $|\Bbbk| > 3$. Then $\mathrm{SL}_n(\Bbbk)$ is perfect.

Let $\{E_{ij}\}_{i,j=1}^n$ be a matrix unit for $M_n(\Bbbk)$, so $E_{ij}E_{k\ell} = \delta_{jk}E_{il}$.

If $\lambda \in \Bbbk$, i,j,k distinct (i.e. $n \geq 3$), then

$$[e+\lambda E_{ik}, e+E_{kj}] = (e+\lambda E_{ik})(e+E_{kj})(e-\lambda E_{ik})(e-E_{kj}) = e+\lambda E_{ij}.$$

If $n = 2$, then

$$\left[\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}, \begin{pmatrix}1 & \beta \\ 0 & 1\end{pmatrix}\right] = e+\lambda E_{12}$$

where $\lambda = (1-\alpha^2)\beta$.

If $S = \langle e+\lambda E_{ij} : \lambda \in \Bbbk, i \neq j\rangle$. Using only elemnentary operations induced by multiplying by elements of S on the left, and element a of $\mathrm{SL}_n(\Bbbk)$ satisfies (see pic)

By an evident induction, we see that there are $s_1, s_2 \in S$ so $s_1 s a s_2 = e$. Thus $a = s^{-1}s_1^{-1}s_2^{-1} \in S$.

(ii) Let $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. Consider $G = \mathrm{GL}_n(\Bbbk)$. Notice that $Z = Z(G) = \Bbbk^\times e$. From (i), $\mathrm{SL}_n(\Bbbk)$ is perfect.

Let $H = Z \cdot \mathrm{SL}_n(\Bbbk)$, which is closed (check!) and $H/Z \cong \mathrm{SL}_n(\Bbbk)/Z \cap \mathrm{SL}_n(\Bbbk)$ is perfect, being the quotient of a perfect group, hence unimodular.

If $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{R}$ and n is odd, $H = G$. Else if $\Bbbk = \mathbb{R}$ and n is even, then $H = \mathrm{GL}_n(\mathbb{R})_e = \det^{-1}((0,\infty))$ (connnected component of e) and $G = \mathrm{GL}_n(\mathbb{R})_e \,\dot\cup\, (-e)\mathrm{GL}_n(\mathbb{R})_e$, so $G/H \cong \{-1,1\}$ is compact.

(iii) $E(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$. Since $N = \mathbb{R}^n \rtimes \{e\}$ is closed, normal, and abelian, and $G/N = \mathrm{SO}(n)$ is compact.

(iv) Consider
$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$
then
$$Z(\mathbb{H}) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$
has $\mathbb{H}/Z \cong \mathbb{R}^2$.

Remark. In A1, a "Braconnier" modular function $\delta : \mathrm{Aut}(G) \to (0, \infty)$ is defined.

(i) If $\gamma : G \to \mathrm{Aut}(G)$ has $\gamma(x)(y) = xyx^{-1}$, so γ is a homomorphism. Then $\delta(\gamma(x)) = \frac{1}{\Delta(x)}$.

(ii) If G is compact and $\alpha \in \mathrm{Aut}(G)$, then $\alpha(G) = G$ so $1 = m(G) = m(\alpha(G))$ and $\delta(\alpha) = 1$.

(iii) If G is discrete and $\alpha \in \mathrm{Aut}(G)$, then for any non-empty finite $F \subsetneq G$, we have $|F| = |\alpha(F)|$, and it follows that $\delta(\alpha) = 1$.

(iv) If G is unimodular and H an open subgroup of G, then H is unimodular However, there is some subtlety here:
$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$
is closed in $\mathrm{SL}_2(\mathbb{R})$ and $H \cong \mathbb{R} \rtimes (0, \infty)$ is not unimodular. Moreover,
$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}, a > 0 \right\}$$
is open, normal and abelian in $G = \left\{ \begin{pmatrix} 2^n & r \\ 0 & 1 \end{pmatrix} n \in \mathbb{Z}, r \in \mathbb{R} \right\}$ and G is closed in $\mathrm{GL}_2(\mathbb{R})$.

1.5 The Convolution Algebra of Measures

Let G be a locally compact group. Let
$$M(G) = \{\mu : \mathcal{B}(G) \to \mathbb{C} : \mu \text{ Radon measure}\} = \operatorname{span} M_+(G)$$
$$M_+(G) = \{\mu : \mathcal{B}(G) \to [0, \infty) | \mu \text{ Radon}]$$

If $\mu \in M_+(G)$ with $\mu(G) < \infty$ so μ is finite.

Recall the Hahn-Jordan Decomposition: each μ in $M(G)$ admits a decomposition $\mu = \sum_{k=0}^{3} i^k \mu_k$ where each $\mu_i \in M_+(G)$, $\mu_0 \perp \mu_2$, and $\mu_1 \perp \mu_3$. Any measures satisfying this decomposition are unique.

Definition. If $\mu \in M(G)$, we define the $|\mu| : \mathcal{B}(G) \to [0, \infty)$ by
$$|\mu|(E) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(E_k)| : E = \dot{\bigcup_{k=1}^{\infty}} E_k, E_k \in \mathcal{B}(G) \right\}$$
and $|\mu| = M_+(G)$. If $\mu = \sum_{k=0}^{3} i^k \mu_k$ as in Hahn-Jordan, then $|\mu_0 - \mu_2| = \mu_0 + \mu_2$ and $|\mu_1 - \mu_3| = \mu_1 + \mu_3$. Furthermore,
$$|\mu| \leq |\mu_0 - \mu_2| + |\mu_1 - \mu_3| \text{ and } |\mu_0 - \mu_2||\mu_1 - \mu_3| \leq |\mu|$$

1.18 Theorem. ***(Riesz-Markov Duality)*** *Let $C_0(G) = \overline{C_c(G)} \subseteq C_b(G)$ with the uniform topology. Then $C_0(G)^* \cong M(G)$ through the map $\mu \mapsto \langle \mu, \cdot \rangle$ where $\langle \mu, f \rangle = \int_G f \,\mathrm{d}\mu$. Moreover, $\left\|\langle \mu, \cdot \rangle\right\|_{op} = |\mu|(G)$.*

Remark. Let $\mathcal{B}^\infty(G) = \{f : G \to \mathbb{C} : f \text{ bounded and Borel measurable}\}$, which is a Banach space under the uniform norm. Note that $\mathcal{B}^\infty(G) = \overline{\operatorname{span}}\{1_E : E \in \mathcal{B}(G)\}$. We have

$$\left|\int_G f \,\mathrm{d}\mu\right| \leq \int_G |f| d|\mu| \leq \|f\|_\infty \left\|\mu\right\|_1$$

If $\mu \in M(G)$ and $\epsilon > 0$, then inner regularity provides compact $K \subseteq G$ such that $|\mu|(K) > |\mu(G) - \epsilon$. Hence $|\mu|(G \setminus K) < \epsilon$. Then $\left\|\mu - \mu_k\right\|_1 = \left\|\mu_{G\setminus K}\right\|_1 = |\mu|(G \setminus K) < \epsilon$.

1.19 Lemma. ***(Continuous Fubini)*** *Let X, Y be locally compact spaces, $\mu \in M(X)$, $\nu \in M(Y)$. Then there is a measure $\mu \times \nu \in M(X \times Y)$ such that*

$$\int_{X\times Y} f \,\mathrm{d}(\mu \times \nu) + \int_Y \int_X f \,\mathrm{d}\mu \,\mathrm{d}\nu = \int_X \int_Y f \,\mathrm{d}\nu \,\mathrm{d}\mu$$

for any $f \in C_b(X)$.

Proof Let $\mathcal{A} = \operatorname{span}\{\phi \times \psi : \phi \in C_0(X), \psi \in C_0(Y)\}$ where $\phi \times \psi(x,y) = \phi(x)\psi(y)$. Then $\overline{\mathcal{A}} = C_0(X \times Y)$ in the uniform topology. We define for $f \in \mathcal{A}$

$$J(f) = \int_X \int_Y f \,\mathrm{d}\nu \,\mathrm{d}\mu \int_Y \int_X f \,\mathrm{d}\mu \,\mathrm{d}\nu$$

so that J is linear on $\mathcal{A}$ and

$$|J(F)| \leq \|f\|_\infty |\mu|(X)|\nu|(Y)$$

so J is bounded. Thus J is uniformly continuous and hence extens uniquely to $C_0(X \times Y)$ as a bounded linear functional with $\|J\| \leq \left\|\mu\right\|_1 \|\nu\|_1$. By Riesz-Markov, there is $\mu \times \nu \in M(X \times Y)$ such that $J(f) = \int_{X\times Y} f \,\mathrm{d}(\mu \times \nu)$. Uniform limits are pointwise limits, so LDCT tells us that we have Fubini for $f \in C_0(X \times Y)$.

By inner regularity, find $(K_n)_{n=1}^\infty$ and $(L_n)_{n=1}^\infty$ so that $\lim_{n\to\infty} \left\|\mu - \mu_{K_n}\right\|_1 = 0 = \lim_{n\to\infty} \left\|\nu - \nu_{L_n}\right\|_1$. For each n, let $f_n \in C_c(X \times Y)$ be such that $f|_{K_n \times L_n} = f_n|_{K_n \times L_n}$ (Urysohn). We also notice that $(\mu \times \nu)_{K_n \times L_n} = \mu_{K_n} \times \nu_{L_n}$. Then check that $\lim_{n\to\infty} \left\|(\mu \times \nu)_{K_n \times L_n} - \mu \times \nu\right\|_1 = 0$. Then for $f \in C_b(X \times Y)$,

$$\begin{aligned}
\int_{X\times Y} f \,\mathrm{d}(\mu \times \nu) &= \lim_{n\to\infty} \int_{X\times Y} f \,\mathrm{d}(\mu \times \nu)_{K_n \times L_n} \\
&= \lim_{n\to\infty} \int_{X\times Y} f_n \,\mathrm{d}(\mu \times \nu)_{K_n \times L_n} \\
&= \lim_{n\to\infty} \int_X \int_Y f_n \,\mathrm{d}\nu_{L_n} \,\mathrm{d}\mu_{K_n} \\
&= \lim_{n\to\infty} \int_X \int_Y f \,\mathrm{d}\nu_{L_m} \,\mathrm{d}\mu_{K_n} = \int_X \int_Y f \,\mathrm{d}\nu \,\mathrm{d}\mu
\end{aligned}$$

■

1.20 Theorem. *Given μ, ν in $M(G)$, there is a unique measure $\mu * \nu$ in $M(G)$ such that*

$$\int_G f \,\mathrm{d}\mu * \nu = \int_G \int_G f(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

for $f \in C_0(G)$. The map $(\mu, \nu) \mapsto \mu * \nu$ is bilinear, associative, and satisfies $\left\|\mu * \nu\right\|_1 \leq \left\|\mu\right\|_1 \|\nu\|_1$. Hence $(M(G), *)$ is a Banach algebra.

Claim I *Given $\mu \in M(G)$ and $f \in C_0(G)$, define $f \cdot \mu, \mu \cdot f : G \to \mathbb{C}$ by*

$$f \cdot \mu(x) = \langle \mu, x \cdot f \rangle = \int_G f(yx)\,\mathrm{d}\mu(y)$$

$$\mu \cdot f(x) = \langle \mu, f \cdot x \rangle = \int_G f(xy)\,\mathrm{d}\mu(y)$$

Notice that $\mu \cdot f(e) = f \cdot \mu(e)$. Then $f \cdot \mu$, $\mu \cdot f$ in $C_0(G)$.

Proof Indeed, let us check continuity of $\mu \cdot f$. Let $\epsilon > 0$ and V be a neighbourhood of e such that $|f(x) - f(x')| < \epsilon$ for $x'x^{-1} \in V$ (uniform continuity, $\overline{C_c(G)} = C_0(G)$, uniform limit of uniformly cts is uniformly cts). Then

$$|\mu \cdot f(x) - \mu \cdot f(x')| \leq \int_G |f(xy) - f(x'y)|\,\mathrm{d}|\mu|(y) \leq \epsilon|\mu|(G)$$

is uniformly continuous, hence continuous. Notice that $|\mu \cdot f(x)| \leq \int_G |f(xy)|\,\mathrm{d}|\mu|(Y) \leq \|f\|_\infty \left\|\mu\right\|_1$. Now, given $\epsilon > 0$, let $K \subseteq G$ be compact so $\left\|\mu - \mu_K\right\|_1 < \epsilon$ and $f' \in C_c(G)$ be so $\|f - f'\|_\infty < \epsilon$. Then

$$\begin{aligned}
\left\|\mu \cdot f - \mu_K \cdot f'\right\|_\infty &\leq \left\|\mu \cdot f - \mu_k \cdot f\right\|_\infty + \left\|\mu_K \cdot f - \mu_K \cdot f'\right\|_\infty \\
&\leq \left\|\mu - \mu_K\right\|_1 \|f\|_\infty + \left\|\mu_K\right\|_1 \left\|f - f'\right\|_\infty \\
&< (\left\|\mu\right\|_1 + \|f\|_\infty)\epsilon
\end{aligned}$$

and $\operatorname{supp}(\mu_K \cdot f') \subseteq \operatorname{supp}(f')K^{-1}$, so $u_k \cdot f \in C_c(G)$. Thus $\mu \cdot f \in C_0(G)$. Similarly, $f \cdot \mu \in C_0(G)$. It is evident that $(\mu, f) \mapsto \mu \cdot f$ and $(f, \mu) \mapsto f \cdot \mu$ are bilinear.

Claim II *If $\mu, \nu \in M(G)$, then $\mu \cdot (f \cdot \nu) = (\mu \cdot f) \cdot \nu$ and $\langle \mu, f \cdot \mu \rangle = \langle \nu, \mu \cdot f \rangle$.*

Proof Use Fubini for continuous functions: for x in G,

$$\begin{aligned}
\mu \cdot (f \cdot \nu)(x) = \int_G (f \cdot \nu)(xy)\,\mathrm{d}\mu(y) &= \int_G \int_G f(zxy)\,\mathrm{d}\nu(z)\,\mathrm{d}\mu(y) \\
&= \int_G \int_G f(zxy)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(z) \\
&= \int_G (\mu \cdot f)(zx)\,\mathrm{d}\nu(z) = (\mu \cdot f) \cdot \nu(x)
\end{aligned}$$

Claim III *For μ, ν in $M(G)$, define for f in $C_0(G)$*

$$\langle \mu * \nu, f \rangle = \langle \mu, \nu \cdot f \rangle = \langle \nu, f \cdot \mu \rangle$$

Then $\mu * \nu$ is unique and satisfies the required properties.

Proof Uniqueness follows by Riesz-Markov (...?)
Moreover, $(\mu,\nu)\mapsto \mu*\nu$ is evidently bilinear and

$$|\langle \mu*\nu, f\rangle| = |\langle \mu, \nu\cdot f\rangle| \le \left\|\mu\right\|_1 \|\nu\cdot f\|_\infty \le \left\|\mu\right\|_1 \|\nu\|_1 \|f\|_\infty .$$

This shows that

- $f\mapsto \langle \mu\times\nu, f\rangle$ is bounded, and hence $\mu*\nu$ is an element of $M(G)$ (by Riesz-Markov)
- $\left\|\mu*\nu\right\|_1 \le \left\|\mu\right\|_1 \|\nu\|_1$

It remains to see associativity. If $\rho\in M(G)$, then for $f\in C_0(G)$,

$$\begin{aligned}\langle \mu*(\nu*\rho)\rangle &= \langle \nu*\rho, f\cdot\mu\rangle = \langle \nu, \rho\cdot(f\cdot\mu)\rangle\\ &= \langle \mu*\nu, \rho\cdot f\rangle\end{aligned}$$

■

Remark. (i) If $\mu\in M(G)$, let $R_\mu, L_\mu : C_0(G)\to C_0(G)$ be given by $L_\mu f = \mu\cdot f$, $R_\mu f = f\cdot\mu$. Then for $\nu\in M(G)$, $\mu*\nu = R_\mu^*(\nu) = L_\nu^*(\mu)$ which shows $\nu\mapsto\mu*\nu$ or $\nu\mapsto\nu*\mu$ are each w^*-w^*-continuous. Note that $(\mu,\nu)\mapsto\mu*\nu$ may not be w^*-w^*-continuous.
(ii) Let for x in G $\delta_x(E)$ be the point mass measure at x. Then $\langle\delta_x, f\rangle = f(x)$ for $f\in C_0(G)$. Then $\delta_x*\delta_y = \delta_{xy}$.
(iii) If $\mu,\nu\in M_+(G)$, then $\mu*\nu\in M_+(G)$.

1.6 Atomic-Continuous and Lebesgue Decompositions

Let $\mu\in M(G)$ and set

$$A(\mu) = \{x\in G : |\mu|(\{x\})>0\} = \bigcup_{n=1}\infty \left\{x\in G : |\mu(\{x\})>\frac{1}{n}\right\}$$

so $A(\mu)$ is countable. For any $x\in G$, $|\mu|(\{x\}) = |\mu(\{x\})|$ by definition of $|\mu|$ and hence

$$\sum_{x\in A(\mu)} |\mu(\{x\})| = \sum_{x\in A(\mu)} |\mu|(\{x\}) = |\mu|(A(\mu)) < \infty.$$

We then define the **atomic** or **discrete** part of μ by

$$\mu_d = \sum_{x\in A(\mu)} \mu(\{x\})\delta_x \in M(G)$$

and the **continuous part** by

$$\mu_c = \mu - \mu_d d$$

Then $\mu_c\perp\mu_d$ so

$$\left\|\mu\right\|_1 = |\mu|(G) = |\mu_c|(G) + |\mu_d|(G) = \left\|\mu_c\right\|_1 + \left\|\mu_d\right\|_1$$

The set $M_d(G) = \overline{\operatorname{span}}\{\delta_x : x\in G\}$ is a subspace of $M(G)$, and $M_d(G)\cong\ell^1(G)$ isometrically. Thus $M_c(G) = \operatorname{im} P_c$ is a closed subspace.

If G is discrete, then $|\mu_c|(G)=0$ so $M(G) = M_d(G)\cong\ell^1(G)$.

If G is not discrete, then $\{e\}$ is a closed, non-open subgroup, so for $x\in G$, $m(\{x\})=0$ where m is the Haar measure. Hence the measures absolutely continuous with respect to m satisfy $M_d(G)\subseteq M_c(G)$. We can employ the Lebesgue decomposition to write $\mu_c = \mu_a + \mu_{cs}$

where $\mu_a \ll m$ and $\mu_a \perp \mu_{cs}$. Moreover, the Radon-Nikodym derivative has $\frac{d\mu}{dm} \in L^1(G)$. To conclude,

$$\begin{aligned} M(G) &= M_c(G) \oplus_1 M_d(G) \\ &= M_a(G) \oplus_1 M_{cs}(G) \oplus_1 M_d(G) \\ &\cong L^1(G) \oplus_1 M_{cs}(G) \oplus_1 \ell^1(G) \end{aligned}$$

Certainly $\ell^1(G)$ is a subalgebra. We will show that the remaining components are also subalgebras.

Remark. Given $\mu, \nu \in M(G)$, we formed a **Radon product** $\mu \times \nu$ which satisfies

$$\int_G \int_G f(x,y) \, d\mu(x) \, d\mu(y) = \int_{G\times G} f \, d(\mu \times \nu) = \int_G \int_G f(x,y) \, d\nu(y) \, d\mu(x)$$

for $f \in C_0(G)$. This extends to $f \in C_b(G \times G)$. Similarly,

$$\begin{aligned} \int_G f \, d(\mu * \nu) &= \int_G \int_G f(xy) \, d\mu(x) \, d\nu(y) = \int_G \int_G f(xy) \, d\nu(y) \, d\mu(x) \\ &= \int_{G\times G} f \circ p \, d(\mu \times \nu) \end{aligned}$$

where $p : G \times G \to G$ is the product.

If $E \in \mathcal{B}(G)$, then $\pi^{-1}(E) \in \mathcal{B}(G \times G)$. Indeed, since p is continuous, $p^{-1}(U) \in \tau_G \times \tau_G \subseteq \mathcal{B}(G \times G)$ for U open, so the result follows.

1.21 Theorem. *If $\mu, \nu \in M(G)$, $E \in \mathcal{B}(G)$, then*

$$\mu * \nu(E) = (\mu \times \nu) \circ p^{-1}(E)$$

for $E \in \mathcal{B}(G)$.

Proof First note that

$$(\mu \times \nu)(p^{-1}(E)) = \int_{G\times G} \mathbf{1}_{\pi^{-1}(E)} \, d(\mu \times \nu) = \int_{G\times G} \mathbf{1}_E \circ p \, d(\mu \times \nu).$$

Now given Jordan decomposition $\mu = \sum_{k=0}^3 i^k \mu_k$ and $\nu = \sum_{j=0}^3 i^j \nu_j$, we have

$$\mu * \nu = \sum_{k,j=0}^3 i^{k+j} \mu_k * \nu_k$$

so it suffices to show the result for $\mu, \nu \in M_+(G)$.

First let $K \subseteq G$ be compact and $\epsilon > 0$. Find open U so $K \subseteq U$ and $\mu * \nu(U \setminus K) < \epsilon$, and by Urysohn find $f \in C_c(G, [0,1])$ such that $f|_K = 1$ and $\text{supp}(f) \subseteq U$. Then

$$\begin{aligned} (\mu \times \nu)(p^{-1}(K)) &= \int_{G\times G} \mathbf{1}_K \circ p \, d(\mu \times \nu) \\ &\leq \int_{G\times G} f \circ p \, d(\mu \times \nu) = \int_G f \, d(\mu \times \nu) \\ &\leq \int_G \mathbf{1}_U \, d(\mu \times \nu) = \mu * \nu(U) < \mu * \nu(K) + \epsilon \end{aligned}$$

so that $(\mu\times\nu)\circ p^{-1}(K)\le \mu*\mu(K)$.

Now let N be $\mu\times\nu$-null. If $K\subseteq p^{-1}(N)$ is compact, then

$$\begin{aligned}(\mu\times\nu)(K)&\le(\mu\times\nu)(p^{-1}(p(K)))\\&\le\mu*\nu(\pi(K))\le\mu*\nu(N))=0\end{aligned}$$

so that $(\mu\times\nu)(\pi^{-1}(N))=\sup\{\mu\times\nu(K):K\subseteq N,K \text{ compact}\}=0$.

Now let $U\subseteq G$. For each $n\in\mathbb{N}$, find compact $K_n\subseteq U$ so $\mu*\nu(U)<\mu*\nu(K_n)+1/n$ and find $f_n\in C_c(G,[0,1])$ such that $f_n|_{K_n}=1$, $\mathrm{supp}(f_n)\subseteq U$, and $g_n=\max\{f_1,\dots,f_n\}$. Set $F=\bigcup_{n=1}^{\infty}K_n$ so $\mu\times\nu(U\setminus F)=0$. Thus $g_n\to\mathbf{1}_U$ as $n\to\infty$ on $G\setminus(U\setminus F)$ and hence, by above, $g_n\circ p\to\mathbf{1}_U\circ p=\mathbf{1}_{p^{-1}(U)}$ $\mu\times\nu-$ a.e.. Hence by monotone convergence,

$$\begin{aligned}(\mu\times\nu)(p^{-1}(U))&=\int_{G\times G}\mathbf{1}_U\circ p\,\mathrm{d}(\mu\times\nu)=\lim_{n\to\infty}\int_{G\times G}g_n\circ p\,\mathrm{d}(\mu\times\nu)=\lim_{n\to\infty}\int_G g_n\,\mathrm{d}(\mu*\nu)\\&=\int_G\mathbf{1}_U\,\mathrm{d}(\mu*\nu)=\mu*\nu(U)\end{aligned}$$

Finally, let $E\in\mathcal{B}(G)$ be any Borel set. Find open $U_n\supseteq E$ such that $\mu*\nu(U_n)<\mu*\nu(E)+1/n$. Let $V_n=\bigcap_{k=1}^{n}U_k$, so $\mathbf{1}_{V_n}\to\mathbf{1}_E$ (non-increasing) on $G\setminus(\bigcap_{n=1}^{\infty}U_n\setminus E)$, i.e. $\mu*\nu-$ a.e.. Hence, again by above, $\mathbf{1}_{V_n}\circ p\to\mathbf{1}_E\circ p$ $\mu\times\nu-$ a.e.. Thus by LDCT (since our measures are finite),

$$\begin{aligned}\mu\times\nu(\pi^{-1}(E))&=\int_{G\times G}\mathbf{1}_E\circ\pi\,\mathrm{d}(\mu\times\nu)=\lim_{n\to\infty}\int_{G\times G}\mathbf{1}_{V_n}\circ p\,\mathrm{d}(\mu\times\nu)\\&=\lim_{n\to\infty}\int_G\mathbf{1}_{V_n}\,\mathrm{d}(\mu*\nu)=\int_G\mathbf{1}_E\,\mathrm{d}(\mu*\nu)=\mu*\nu(E)\end{aligned}$$ ∎

Remark. If U,g_n are as in the above proof, then

$$\begin{aligned}\mu*\nu(U)&=\int_G\mathbf{1}_U\,\mathrm{d}(\mu*\nu)=\lim_{n\to\infty}\int_G g_n\,\mathrm{d}(\mu*\nu)\\&=\lim_{n\to\infty}\int_G\int_G g_n(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\\&\le\int_G\int_G\mathbf{1}_U(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\end{aligned}$$

Then for $E\in\mathcal{B}(G)$,

$$\mu*\nu(E)\le\int_G\int_G\mathbf{1}_E(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)=\int_G\mu(Ey^{-1})\,\mathrm{d}\mu(y).$$

1.22 Corollary. *Let $\mu,\nu\in M_+(G)$. If $N\in\mathcal{B}(G)$ has that Ny^{-1} is ν-null for $y\in G$, then N is $\mu\times\nu$-null.*

Proof Use the remark above. ∎

1.23 Theorem. *Each of $M_a(G)$ and $M_c(G)$ is a (two-sided) ideal in $M(G)$.*

Proof If $N\in\mathcal{B}(G)$ is

- m-null, then so are Ny^{-1}, $x^{-1}N$ for $y,x\in G$.

- a singleyon, $N = \{z\}$, then $\{z\}y^{-1} = \{zy^{-1}$ and $x^{-1}\{z\} = \{x^{-1}z\}$

Thus if $\nu \in M_a(G)$ or $\nu \in M_c(G)$, then the same is true of $\mu * \nu$ and $\nu * \mu$ for $\mu \in M(G)$. ■

Remark. If $\nu \in M + a(G)$, then if $f = \frac{d\nu}{dm}$ satisfies $\int_G h\, d\nu = \int_G hf\, dm$ for $h \in C_0(G)$. What can we learn about $\frac{d(\mu*\nu)}{dm}$, $\frac{d(\nu*\mu)}{dm}$?

1.24 Theorem. *Let X be a locally compact space, $\mathcal{L}$ a Banach space, and let*

$$C_b(X,\mathcal{L}) = \left\{ f : X \to \mathcal{L} : F \text{ continuous}, \|f\|_\infty = \sup_{x\in X} \|f(x)\| < \infty \right\}.$$

Then there is a bilinear map

$$(f,\mu) \mapsto \int_X f\, d\mu : C_b(X,\mathcal{L}) \times M(X) \to \mathcal{L}$$

such that

- $\left\|\int_X f\, d\mu\right\| \leq \|f\|_\infty \left\|\mu\right\|_1$ *and*
- $T\left(\int_X f\, d\mu\right) = \int_X T \circ f\, d\mu$ *for* $T \in \mathcal{B}(\mathcal{L},\mathcal{L}')$.

Proof Let

$$\mathcal{S} = \mathcal{S}(X,\mathcal{L}) = \operatorname{span}\{\mathbf{1}_E \xi : E \in \mathcal{B}(X), \xi \in \mathcal{L}\}.$$

If $\Phi \in \mathcal{L}$, then it admits a standard form $\Phi = \sum_{j=1}^n \mathbf{1}_{E_j}\xi_j$ where for $i \neq j$, $E_i \cap E_j = \emptyset$ and $\xi_i \neq \xi_j$. Note that $\|\Phi\|_\infty = \sup_{x\in X} \|\Phi(x)\| = \max_{j=1,\ldots,n} \|\xi_i\|$. Then for Φ in standard form,

$$(\Phi,\mu) \mapsto \int_X \Phi\, d\mu := \sum_{j=1}^n \mu(E_j)\xi_j$$

from $\mathcal{S} \times M(X \to \mathcal{L}$ is bilinear and

$$\left\|\int_X \Phi\, d\mu\right\| \leq \sum_{j=1}^n |\mu(E_j)| \left\|\xi_j\right\| \leq \left\|\mu\right\|_1 \|\Phi\|_\infty.$$

Now let $\mathcal{S}^\infty$ denote the uniform closure of $\mathcal{S}$ and the bilinear pairing extends isometrically.

Now assume X is compact. Then for $F \in C(X,\mathcal{L})$, $F(X)$ is totally bounded in $\mathcal{L}$ so for $\epsilon > 0$, $F(X) \subseteq \bigcup_{j=1}^n B(\xi_j,\epsilon)$. Let $E_1 = F^{-1}(\mathcal{B}(\xi,\epsilon))$, $E_{i+1} = F^1(B(\xi_k,\epsilon)) \setminus \bigcup_{j=1}^{k-1} F^{-1}(B(\xi_j,\epsilon))$. Then $\phi_\epsilon = \sum_{j=1}^n \mathbf{1}_{E_j}\xi_j$ satisfies $\|\Phi_\epsilon - F\|_\infty < \epsilon$. Thus $C(X,\mathcal{L}) \subseteq S^\infty(X,\mathcal{L})$. Thus we may define $\int_X F\, d\mu$ for $\mu \in M(X)$ and $F \in C(X,\mathcal{L})$.

Finally, let $\mu \in M(X)$ and let $(K_n)_{n=1}^\infty$ be a sequence of compact sets such that $|\mu|(X \setminus K_n) < 1/n$, so $\left\|\mu - \mu_{K_n}\right\|_1 \to 0$ as $\to \infty$. Then let

$$\xi_n = \int_{K_n} F\, d\mu = \int_K F\, d\mu_{K_n}$$

for any $K \supseteq K_n$. Then $\{\xi_n\}_{n=1}^\infty$ is Cauchy since $\|\xi_n - \xi_m\| = \left\|\int_K F\, d(\mu_{K_n} - \mu_{K_m})\right\| \leq \|F\|_\infty \left\|\mu_{K_n} - \mu_{K_m}\right\|$. Let $\int_X F\, d\mu = \lim_{n\to\infty} \xi_n$, which is independent of the choice of K_n.

Now if $T \in \mathcal{B}(\mathcal{L},\mathcal{L}')$, first apply T to Φ in $S(X,\mathcal{L})$, and then by approimations to Ψ in $S^\infty(X,\mathcal{L})$, and then extend using the construction above. ■

1.25 Theorem. *Let G be a locally compact group, $\mathcal{L}$ a Banach space, and suppose there is an action*

$$(x,\xi) \mapsto x \cdot \xi : G \times \mathcal{L} \to \mathcal{L}$$

such that

- *$x \mapsto x \cdot \xi$ is continuous for each ξ*
- *$\xi \mapsto x \cdot \xi$ is linear for each x*
- *there is $C > 0$ such that $\|x \cdot \xi\| \le C \|\xi_1\|$ for $x \in G$, $\xi \in \mathcal{L}$.*

Then there is a bilinear map $(\mu,\xi) \mapsto \mu \cdot \xi : M(G) \times \mathcal{L} \to \mathcal{L}$ such that $\left\|\mu \cdot \xi\right\| \le C \left\|\mu\right\|_1 \|\xi\|$ and $(\mu * \nu) \cdot \xi = \mu \cdot (\nu \cdot \xi)$ for $\mu, \nu \in M_G$ and $\xi \in \mathcal{L}$.

Proof We let

$$\mu \cdot \xi = \int_G x \circ \xi \, \mathrm{d}\mu(x).$$

The bilinear and boundedness is clear. To check associaticity, let $w \in \mathcal{L}^*$ and check that $w\Big((\mu * \nu) \cdot \xi\Big) = w(\mu \cdot (\nu \cdot \xi))$ by Fubini for continuous integrands. ∎

Definition. A **Banach G-module** is an action $G \times X \to X$ which is

- linear in X
- multiplicative and continuous in G
- uniformly bounded in G:$\|x \cdot \xi\| \le C \|\xi\|$.

In addition, we say that the G-module is **non-degenerate** of $e \cdot \xi = \xi$.

In this context, the prior theorem essentially states that a Banach G-module is a Banach $M(G)$-module.

Remark. A **representation** of G on a Banach space X is a homomorphism $\pi : G \to \mathcal{B}(X)$ ($\mathcal{B}(X)$ is the set of bounded linear operators on X) such that $\pi(e) = I$. We typically also assume

- **boundedness**: $\sup_{x \in G} \|\pi(x)\| \le C < \infty$
- **strong operator continuity**: $x \mapsto \pi(x)\xi$ is continuous for any $\xi \in X$.

1.26 Corollary. *If $\pi : G \to \mathcal{B}(X)$ is a (bounded, SOT) representation of G, then π induces a bounded homomorphism $\pi_M : M(G) \to \mathcal{B}(X)$ such that $\pi_M(\mu)\xi = \int_G \pi(x)\xi \, \mathrm{d}\mu(x)$ with $\pi_M(\delta_x) = \pi(x)$.*

Example. Recall that if $f \in L^1(G)$ and $x \in G$, we have actions $G \times L^1(G) \to L^1(G)$ by $(x,f) \mapsto x * f$, where $x * f(y) = f(x^{-1}y)$; and similarly $f * x(y) = f(yx^{-1})/\Delta(x)$. These make $L^1(G)$ both a left and right isometric G-module. Hence the last theorem provides us with a Banach $M(G)$-module structure

$$\mu * f = \int_G x * f \, \mathrm{d}\mu(x)$$

$$f * \mu = \int_G f * x \, \mathrm{d}\mu(x)$$

1.27 Lemma.

- *$L^1 \cap C_0(G)$ with norm $\|\cdot\|_1 + \|\cdot\|_\infty$ is a Banach space, dense in $L^1(G)$, and which is a left Banach G-module (in $L^1(G)$).*

- *The space*
$$L^1 \cap C^\delta(G) = \{f \in C(G) : \|f\|_1 < \infty, \|f/\Delta\|_\infty < \infty\}$$
is a Banach space with $\|\cdot\|_1 + \|\cdot/\Delta\|_\infty$, *which is dense in* $L^1(G)$ *and a right Banach G-module.*

Proof TODO. ■

1.28 Theorem. *Let* $\nu \in M_a(G)$ *with* $f = \frac{\mathrm{d}\nu}{\mathrm{d}m} \in L^1(G)$.

(i) For $\mu \in M(G)$, $\frac{\mathrm{d}(\mu*\nu)}{\mathrm{d}m} = \mu * f$ *and* $\frac{\mathrm{d}(\nu*\mu)}{\mathrm{d}m} = f * \mu$

(ii) If, further, $\mu \in M_a(G)$ *with* $g = \frac{\mathrm{d}\mu}{\mathrm{d}m}$, *then* $\frac{\mathrm{d}(\mu*\nu)}{\mathrm{d}m} = g * f$ *where*

$$g * f = \int_G g(x) x * f \, dx = \int_G f(x) g * x \, \mathrm{d}x$$

Proof Note that $C_c(G) \subseteq (L^1 \cap C_0(G)) \cap (L^1 \cap C^\Delta(G))$. Let $(f_n)_{n=1}^\infty \subset C_c(G)$ such that $\lim_{n\to\infty} \|f - f_n\|_1 = 0$, and then for $g \in C_c(G)$

$$\begin{aligned}
\int_G h \, \mathrm{d}(\nu * \mu) &= \int_G \int_G h(xy) f(x) \, \mathrm{d}x \, \mathrm{d}\mu(y) \\
&= \int_G \int_G h(x) f(xy^{-1}) \frac{1}{\Delta(y)} \, \mathrm{d}x \, \mathrm{d}\mu(y) \\
&= \lim_{n\to\infty} \int_G \int_G h(x) f_n(xy^{-1}) \frac{1}{\Delta(y)} \, \mathrm{d}x \, \mathrm{d}\mu(y) \\
&= \lim_{n\to\infty} \int_G h(x) \int_G f_n(xy^{-1}) \frac{1}{\Delta(y)} \, \mathrm{d}\mu(y) \, \mathrm{d}x \\
&= \lim_{n\to\infty} \int_G h(x) f_n * \mu(x) \, \mathrm{d}x && 7 f_n \in C_c(G) \subseteq L^1 \cap C^\delta(G) \\
&= \int_G h(x) f * \mu(x) \, \mathrm{d}x && \left\| f_n * \mu - f * \mu \right\|_1 \le \|f_n - f\|_1 \left\| \mu \right.
\end{aligned}$$

so $\frac{\mathrm{d}(\nu*\mu)}{\mathrm{d}m} = f * \mu$. The left case is similar, and (ii) is similar. ■

Note that we may write for m-a.e. x,

$$f * g(x) = \int_G f(y) g(y^{-1}x) \, \mathrm{d}y = \int_G f(xy^{-1}) g(y) \frac{1}{\Delta(y)} \, \mathrm{d}y$$

Remark. In the finite group setting, representations of G are in correspondence with submodules of $\mathbb{C}[G]$. A natural question to ask is: when does $M(G)$ replace $\mathbb{C}[G]$?

If $Q = M(G)/M_c(G) \cong \ell^1(G)$, then

$$x \cdot (\mu + M_c(G)) = \delta_x * \mu + M_c(G)$$

so for $\alpha \in \ell^1(G)$,

$$x \cdot \left(\sum_{y\in G} \alpha(y) \delta_y \right) = \sum_{y \in G} \alpha(y) \delta_{xy} = \sum_{y\in G} \alpha(x^{-1}y) \delta_y.$$

This is a bounded homomorphism of G into $\mathcal{B}(\ell^1(G)) \cong \mathcal{B}(Q)$ which is not strong operator continuous if G is not discrete.

A summability kernel in $L^1(G)$ is a net (f_α) introduced in A2. We will show that

- contractive summability kernals alwats exist: $\|f_\alpha\|_1 \leq 1$.
- If (f_α) is a summability kernel, then $\lim_\alpha f_\alpha * f = f = \lim_\alpha f * f_\alpha$ in L^1-norm, for f in $L^1(G)$.

Definition. Let X be a Banach space. Then it is a **Banach $L^1(G)$-module** if there is a bilinear action $(f,\xi) \mapsto f \cdot \xi : L^1(G) \times X \to X$ such that for $f, g \in L^1(G)$, $\xi \in X$,

- $f \cdot (g \cdot \xi) = (f * g) \cdot \xi$
- $\|f \cdot \xi\| \leq c\|f\|_1 \|\xi\|$ for some $c > 0$.

Further, this is **non-denenerate** if $X_0 = \operatorname{span}\{f \cdot \xi : f \in L^1(G) < \xi \in X\}$ is dense in X.

1.29 Theorem. *If X is a non-degenerate Banach $L^1(G)$-module, then it is a Banach G-module with*

$$\int_G f(x)x \cdot \xi \, \mathrm{d}x = f \cdot \xi$$

for $f \in L^1(G)$ and $\xi \in X$.

Proof Let (f_α) be a contractive summability kernel in $L^1(G)$. We define for x in G, $\xi_0 = \sum_{j=1}^n f_j \cdot \xi_j$ in X_0,

$$x \cdot \xi_0 = \sum_{j=1}^n (x * f_j) \cdot \xi_j.$$

Notice that if $0 = \sum_{j=1}^n f_j \cdot \xi_h$, then bilinearity of X as an $L^1(G)$-module provides

$$0 = (x * f_\alpha) \cdot 0 = \sum_{j=1}^n (x * f_\alpha * f_j) \cdot \xi_j \to \sum_{j=1}^n (x * f_j) \cdot \xi_j.$$

Thus this map is well-defined.

If $\xi_0 = \sum_{j=1}^n f_j \cdot \xi_j \in X_0$ and $x \in G$, then

$$\begin{aligned} \|x \cdot \xi_0\| &= \left\| \lim_\alpha \sum_{j=1}^n (x * f_\alpha * f_j) \cdot \xi_j \right\| \\ &= \lim_\alpha \|(x * f_\alpha) \cdot \xi_0\| \\ &\leq \lim_\alpha C \|x * f_\alpha\|_1 \|\xi_0\| = C\|\xi_0\| \end{aligned}$$

Hence, define $\pi_0(x) \in \mathcal{L}(X_0)$, $\pi_0(x)\xi_0 = x \cdot \xi_0$ for $x \in G$, $\xi_0 \in X_0$ satisfies that $\{\pi_0(x) : x \in G\}$ is a group of operators bounded by C¡ and hence $\pi_0(x)$ extends uniquely to a linear bounded operator $\pi(x)$ on X. Thus $\{\pi(x) : x \in G\}$ is a group in $\mathcal{B}(X)$, bounded by C. For $x \in G$ and $\xi \in X$, define $x \cdot \xi = \pi(x)\xi$.

We wish to see strong operator continuity. Let $\epsilon > 0$. If $\xi \in X$, there is $\xi_0 = \sum_{j=1}^n f_j \cdot \xi_j \in X_0$ with $\|\xi - \xi_0\| < \epsilon$. Then

$$\begin{aligned} \limsup_\alpha \|f_\alpha \cdot \xi - \xi\| &\leq \limsup_\alpha \left[\underbrace{\|f_\alpha \cdot \xi - f_\alpha \cdot \xi_0\|}_{\leq \|f_\alpha\|_1 \|\xi - \xi_0\|} + \|f_\alpha \cdot \xi_0 - \xi_0\| + \|\xi_0 - \xi\| \right] \\ &\leq (C+1)\epsilon \end{aligned}$$

so that $\lim_\alpha f_\alpha * \xi = \xi$. Now let α be so $\|f_\alpha \cdot \xi - \xi\| < \epsilon$ and, for $x_0 \in G$,

$$\begin{aligned}\limsup_{x\to x_0}\|x\cdot\xi - x_0\cdot\xi\| &\le \limsup_{x\to x_0}\left[\underbrace{\|x\cdot\xi-(x*f_\alpha)\|}_{\le C\|\xi-f_\alpha\cdot\xi\|}+\underbrace{\|(x*f_\alpha)\cdot\xi-(x_0*f_\alpha)\cdot\xi\|}_{\le C\|x*f_\alpha-x_0*f_\alpha\|\|\xi\|}+\underbrace{\|(x_0*f)\cdot\xi-x_0\cdot\xi\|}_{\le C\|f_\alpha\cdot\xi-\xi\|}\right]\\ &\le 2C\epsilon\end{aligned}$$

■

1.7 Unitary Representations

Let $\mathcal{H}$ be a hilbert space, and $\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(H) : U^*U = UU^* = I\}$ denote the unitary group. This is a topological group with respect to the operator norms. On $\mathcal{B}(\mathcal{H})$, we define two coarser topologies:

- *(strong operator)*: The initial topology

$$\tau_{\text{so}} = \sigma\Big(\mathcal{B}(\mathcal{H}), \{T \mapsto T\xi : \mathcal{B}(\mathcal{H}) \to (\mathcal{H}, \tau_{\|\cdot\|}\}_{\xi\in\mathcal{H}}\Big).$$

- *(weak operator)*: The initial topology

$$\begin{aligned}\tau_{\text{wo}} &= \sigma\Big(\mathcal{B}(\mathcal{H}), \{T \mapsto \langle T\xi, \eta\rangle\}_{\xi,\eta\in\mathcal{H}} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}\Big)\\ &= \sigma\Big(\mathcal{B}(\mathcal{H}), \{T \mapsto T\xi : \mathcal{B}(\mathcal{H}) \to (\mathcal{H}, w)\}_{\xi\in\mathcal{H}}\Big)\end{aligned}$$

Notice that $\tau_{\text{wo}} \subseteq \tau_{\text{so}}$ on $\mathcal{B}(\mathcal{H})$. That is, strong operator convergence implies weak operator convergence (in nets).

Example. Let $B(\mathcal{B}(\mathcal{H})) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \le 1\}$, which is a semigroup in $\mathcal{B}(\mathcal{H})$. Then set

- *(unilateral shift)*: $S \in \mathcal{B}(\ell^2(\mathbb{N}))$, $S\delta_n = \delta_{n+1}$, so $S^*\delta_n = \delta_{n-1}$ if $n > 1$, and $S^*\delta_0 = 0$ of $n = 1$.
- *(bilateral shift)*: $U \in \mathcal{B}(\ell^2(\mathbb{Z}))$, $U\delta_n = \delta_{n+1}$ so $U^*\delta_n = \delta_{n-1}$

We now have

(a) $\tau_{\text{wo}} \subsetneq \tau_{\text{so}}$ on $B(\mathcal{B}(\mathcal{H}))$, if $\dim\mathcal{H} = \infty$, since $S^n \to 0$ in τ_{wo}, while S^n does not converge to anything in the strong operator topology.

(b) $T \mapsto T^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is wo-wo continuous, but not so-so continuous if $\dim\mathcal{H} = \infty$. For example, $(S^n)^* = (S^*)^n \to 0$ but S^n des not converge to 0

(c) If $T_0 \in \mathcal{B}(\mathcal{H})$, $T \mapsto TT_0$, $T \mapsto T_0T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are each wo-wo continuous. However, if $\dim\mathcal{H} = \infty$, then $(T, T') \mapsto TT'$ is not $\tau_{\text{wo}} - \tau_{\text{wo}} - \tau_{\text{wo}}$ continuous. Note that $U^n, (U^*)^n \to 0$ as $n \to \infty$, but $U^nU^{*n} = I$.

1.30 Proposition. *(i) $(S,T) \mapsto ST : B(\mathcal{B}(\mathcal{H}) \times B(\mathcal{B}(\mathcal{H}) \to B(\mathcal{B}(\mathcal{H}))$ is $\tau_{\text{so}} \times \tau_{\text{so}} - \tau_{\text{so}}$-continuous.*

(ii) $\tau_{\text{so}}|_{\mathcal{U}(\mathcal{H})} = \tau_{\text{wo}}|_{\mathcal{U}(\mathcal{H})}$
Hence $(\mathcal{U}(\mathcal{H}), \tau_{\text{wo}}) = (\mathcal{U}(\mathcal{H}), \tau_{\text{so}})$ is a topological group.

Proof (i) Let $T_\alpha \to T$ and $S_\alpha \to T$ strong operator in $B(\mathcal{B}(\mathcal{H})$. Then for $\xi \in \mathcal{H}$,

$$0 \le \|S_\alpha T_\alpha\xi - ST\xi\| \le \underbrace{\|S_\alpha T_\alpha\xi - S_\alpha T\xi\|}_{\|T_\alpha\xi - T\xi\|} + \|S_\alpha T\xi - ST\xi\| \to 0$$

(ii) Let $U_\alpha \to U$ in $\mathcal{U}(\mathcal{H})$. Then for $\xi \in \mathcal{H}$,

$$\begin{aligned}\|U_\alpha\xi - U\xi\|^2 &= \langle U_\alpha\xi - U\xi, U_\alpha\xi - U\xi\rangle \\ &= 2\|\xi\| \qquad\qquad 2 - 2\,\mathrm{Re}\langle U_\alpha\xi, U\xi\rangle \\ &\to 2\|\xi\|^2 - 2\,\mathrm{Re}\langle U\xi, U\xi\rangle = 0\end{aligned}$$

so that $U_\alpha \to U$ strong operator in $\mathcal{U}(\mathcal{H})$.

We thus have that $(U,V) \mapsto UV : \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ is strong operator continuous, and hence weak operator continuous by (ii). In (b) above, we remarked that $U \mapsto U^{-1} = U^*$ is wo-wo continuous. ■

Remark. If $\dim \mathcal{H} = \infty$, then $(\mathcal{U}(\mathcal{H}), \tau_{\mathrm{wo}})$ is not locally compact.

1.31 Proposition. *$\mathcal{U}(\mathcal{H})$ is the largest subgroup in $B(\mathcal{B}(\mathcal{H}))$.*

Proof If $U, U^{-1} \in B(\mathcal{B}(\mathcal{H}))$, then for $\xi \in \mathcal{H}$,

$$\|\xi\| = \|U^*U\xi\| \le \|U\xi\| \le \|\xi\|$$

so that $\|U\xi\| = \|\|\|$. Hence $\langle \xi, \xi\rangle = \|\xi\|^* = \|U\xi\|^*2 = \langle U^*U\xi, \xi\rangle$. Now for $\xi, \eta \in \mathcal{H}$, the polarization identity gives

$$\begin{aligned}4\langle \xi, \eta\rangle &= \sum_{k=0}^{3} i^k \langle \xi + i^k\eta, \xi + i^k\eta\rangle \\ &= \sum_{k=0}^{3} i^k \langle U^*U(\xi + i^k\eta), \xi + i^k\eta\rangle \\ &= 4\langle U^*U\xi, \eta\rangle\end{aligned}$$

so $U^*U = I$. Then $U^* = U^*UU^{-1} = U^{-1}$. ■

Definition. Let G be a locally compact group. A **unitary representation** is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ which is τ_G-wo continuous.

Example. Consider $\lambda : G \to \mathcal{U}(L^2(G))$ given by $\lambda(x)f(y) = f(x^{-1}y)$ m-a.e., $f \in L^2(G)$.

If G is not a discrete group, then $\lambda : G \to (\mathcal{U}(\mathcal{H}), \tau_{\|\cdot\|})$ is not continuous. However, $\lambda : G \to (\mathcal{U}(\mathcal{H}), \tau_{\mathrm{so}})$ is continuous (proof just like for translation on $L^1(G)$).

1.32 Theorem. *There is a bijective correspondence between any two of*

(i) unitary representations of G

(ii) contractive representations of G on Hilbert spaces

(iii) non-degenerate bounded $*$-homomorphisms from $L^1(G)$ to $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space

(iv) non-degenerate bounded contractive homomorphisms from $L^1(G)$ to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is Hilbert.

Proof $(i \leftrightarrow ii)$ Last proposition

$(ii \leftrightarrow iv)$ Coincidence of Banach G-modules with Banach $L^1(G)$-modules, $(C = 1)$, $\pi : G \to B(\mathcal{B}(\mathcal{H}))$ is a continuous homomorphism with $\pi(e) = I$, then $\pi_1(f)\xi = \int_G f(x)\pi(x)\xi \, dx$.

If $\sigma : L^1(G) \to \mathcal{B}(\mathcal{H})$ homomorphism, $\|\sigma\| \leq 1$, then define $\pi(x) = \lim_\alpha \sigma(x * f_\alpha)$ (weak operator) where (f_α) is a contractive summability kernel in $L^1(G)$ (A2Q1).

$(i \leftrightarrow iii)$ Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then for $f \in L^1(G)$, $\xi, \eta \in \mathcal{H}$,

$$\begin{aligned}\langle \pi_1(f)^* \xi, \eta\rangle &= \langle \xi, \pi_1(f)\eta\rangle \\ &= \int_G \langle \xi, \pi(x)\eta\rangle \overline{f(x)}\,\mathrm{d}x \\ &= \int_G \langle \pi(x^{-1})\xi, \eta\rangle \overline{f(x)}\,\mathrm{d}x \\ &= \int_G \langle \pi(x)\xi, \eta\rangle \overline{f(x^{-1})}\frac{1}{\Delta(x)}\,\mathrm{d}x = \langle \pi_1(f^*)\xi, \eta\rangle\end{aligned}$$

so $\pi_1(f)^* = \pi_1(f^*$.

Conversely, if $\sigma : L^1(G) \to \mathcal{B}(\mathcal{H})$ is a bounded $*$-homomorphism, then with π as before, we have for x in G

$$\pi(x)^* = \mathrm{wo} - \lim_\alpha \sigma(x * f_\alpha)^{-1} = \mathrm{wo} - \lim_\alpha (f_\alpha^* * x^{-1}) = \pi(x^{-1})$$

(check last step!). ∎

Definition. Let G be a group. A function $u : G \to \mathbb{C}$ is **of positive type** (or positive definite) if for any $x_1, \ldots, x_n \in G$, $[u(x_i^{-1}x_j)]$ is a positive semidefinite matrix. If G is a locally compact group, let $\mathcal{B}^+(G) = \{u : G \to \mathbb{C} : u \text{ continuous positive type}\}$.

1.33 Theorem. (Gelfand-Naimark) *Let G be a locally compact group. Then $u \in B^+(G)$ if and only if there is a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ and $\xi \in \mathcal{H}$ such that $u(x) = \langle \pi(x)\xi, \xi\rangle$.*

Proof We will use that G is a topological group, not necessarily locally compact.

$(\Longleftarrow)$ If $u = \langle \pi(\cdot)\xi, \xi\rangle$, then for x_i in G and λ_i in $\mathbb{C}$, we have

$$\sum_{i=1}^n \sum_{j=1}^n \overline{\lambda_j}\lambda_i u(x_i^{-1}x_j) = \langle \sum_{j=1}^n \lambda_i \pi(x_j)\xi, \sum_{i=1}^n \lambda_i \pi(x_i)\xi\rangle \geq 0$$

$(\Longrightarrow)$ Let $\mathbb{C}[G] = \left\{\sum_{i=1}^n \alpha_x x, \alpha_x \in \mathbb{C}, x \in G, \alpha_x \neq 0 \text{ for finitely many } x\right\}$ denote the free $\mathbb{C}$-vector space over G. Define $\Lambda : G \to \mathcal{L}(\mathbb{C}[G])$ by

$$\Lambda(x)\sum_{y\in G}\alpha_y y = \sum_{y\in G}\alpha_y (xy) = \sum_{y\in G}\alpha_{x^{-1}y}y.$$

Then $\Lambda(xx') = \Lambda(x)\Lambda(x')$ for x, x' in G, and $\Lambda(e) = I$.

On $\mathbb{C}[G] \times \mathbb{C}[G]$, define

$$\left[\sum_{x\in G}\alpha_x x, \sum_{y\in G}\beta_y y\right]_y = \sum_{x\in G}\sum_{y\in G}\alpha_x\overline{\beta_y}u(y^{-1}x).$$

Notice that $[\cdot,\cdot]_u$ is positie and Hermitian, since for x, y in G,

$$\begin{pmatrix} u(e) & u(y^{-1}x) \\ u(x^{-1}y) & u(e)\end{pmatrix}$$

is positive semidefinite, hence Hermitian, so $u(x^{-1}y) = u(y^{-1}x)$. Hence $[\cdot,\cdot]_u$ has Cauchy schwarz inequality $|[\alpha,\beta]_u| \le [\alpha,\alpha]_u^{1/2}[\beta,\beta]_u^{1/2}$. Hence $\mathcal{K}_u = \{\alpha \in \mathbb{C}[G] : [\alpha,\alpha]_n = 0\}$ is a subspace of $\mathbb{C}[G]$.

Note that for $x \in G$, $\alpha,\beta \in \mathbb{C}[G]$,

$$[\Lambda(x)\alpha, \Lambda(x)\beta]_u = \sum_{y\in G}\sum_{z\in G} \alpha_y\overline{\beta_z}u((xz)^{-1}xy) = [\alpha,\beta]_y.$$

In particular, $\Lambda(x)\mathcal{K}_u \subseteq \mathcal{K}_u$ for each $x \in G$. Hence we may define $\pi_0 : G \to \mathcal{K}(\mathcal{H}_0)$ where $\mathcal{H}_0 = \mathbb{C}[G]\big/\mathcal{K}_u$ and $\pi_0(x)(\alpha + \mathcal{K}_u) = \Lambda(x)\alpha + \mathcal{K}_u$ is well-defined. Furthermore, $\pi_0(xx') = \pi_0(x)\pi_0(x')$ for $x,x' \in G$, and $\pi_0(e) = I$. Define on $\mathcal{H}_0 \times \mathcal{H}_0$

$$\langle \alpha + \mathcal{K}_u, \beta + \mathcal{K}_u\rangle_u = [\alpha,\beta]_u$$

which is an inner product on $\mathcal{H}_0$. We note from above that each $\pi_0(x)$ is unitary on $\mathcal{H}_0$: $\pi_0(x^{-1}) = \pi_0(x)$ and

$$\langle \pi_0(x)(\alpha+\mathcal{K}_u), \pi_0(x)(\beta+\mathcal{K}_u)\rangle_u = [\Lambda(x)\alpha,\Lambda(x)\beta]_u = [\alpha,\beta]_u = \langle \alpha+\mathcal{K}_u, \beta+\mathcal{K}_u\rangle$$

We let $\mathcal{H} = \overline{\mathcal{H}_0}$ be the completion with respect to $\|\xi\| = \langle \xi,\xi\rangle_u^{1/2}$, so $\mathcal{H}$ is a Hilbert space. Each element of the group of operators $\{\pi_0(x) : x \in G\}$ extends to a unitary on $\mathcal{H}$, so we get a group of unitaries $\{\pi(x) : x \in G\}$. Notive that for $x \in G$,

$$\langle \pi(x)(e + \mathcal{K}_u), e + \mathcal{K}_u\rangle = [x,e]_u = u(x)$$

so we let $\xi = e + \mathcal{K}_u$. Notice then, that

$$|u(x)| = |\langle \pi(x)\xi,\xi\rangle| \le \|\pi(x)\xi\|\|\xi\| \le \|\xi\|^2 = u(e)$$

so u is bounded. If $\alpha,\beta \in \mathbb{C}[G]$,

$$\langle \pi(x)(\alpha+\mathcal{K}_u), \beta+\mathcal{K}_u\rangle = \sum_{y\in G}\sum_{z\in G}\alpha_y\overline{\beta_z}u(z^{-1}xy)$$

so $x \mapsto \langle \pi(x)(\alpha+\mathcal{K}_u),\beta+\mathcal{K}_u\rangle$ is continuous.

If $\xi,\eta \in \mathcal{H}$, $\epsilon > 0$, find $\alpha,\beta \in \mathbb{C}[G]$ so $\|(\alpha+\mathcal{K}_u) - \xi\| \le \epsilon$ and $\left\|(\beta+\mathcal{K}_u) - \eta\right\| < \epsilon$ and then

$$\begin{aligned}|\langle \pi(x)\xi,\eta\rangle - \langle \pi(x)(\alpha+\mathcal{K}_u),\beta+\mathcal{K}_u\rangle| &\le |\langle \pi(x)(\xi - (\alpha+\mathcal{K}_u)),\eta\rangle| + |\langle \pi(x)(\alpha+\mathcal{K}_u), \eta - (\beta+\mathcal{K}_u)\rangle| \\ &\le \epsilon\left\|\eta\right\| + (\|\xi\|+\epsilon)\epsilon\end{aligned}$$

so by taking limits of bounded continuous functions, we see that $\pi : G \to (\mathcal{U})\mathcal{H}, \tau_{\mathrm{wo}})$ is continuous. ∎

1.34 Proposition. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation and $K \subseteq H$ a closed subspace. Let $P = P_K$ denote the orthogonal projection onto K. Then $\pi(G)K \subseteq K$ (K is π-invariant) if and only if $P\pi(x) = \pi(x) \circ P$ for any $x \in G$.*

Proof We have ∎

Definition. We say that a unitary representation π is **irreducible** if it admits no closed invariant subspaces.

1.35 Lemma. *Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation. Then π is irreducible if and only if*

$$\pi(G)' = \{T \in \mathcal{B}() : T\pi(x) = \pi(x)T \text{ for all } x \in G\} = \mathbb{C}U.$$

Proof ($\Longrightarrow$) Let K be π-invariant, $\{0\} \subsetneq K \subsetneq K$. Then $P_K \in \pi(G)' \setminus \mathbb{C}I$.
($\Longleftarrow$) Let $T \in \pi(G)'$. Then for x in G,

$$T^*\pi(x) = (\pi(x^{-1}T)^* = (T\pi(x^{-1}))^* = \pi(x)T^* \qquad \blacksquare$$

so that $T^* \in \pi(G)'$. Thus if $T \in \pi(G)'\backslash\mathbb{C}I$, at least one of $\operatorname{Re} T = \frac{1}{2}(T+T^*)$ or $\operatorname{im} T = \frac{1}{2i}(T-T^*)$ is not in $\mathbb{C}I$. Thus there is $S = S^* \in \pi(G)' \setminus \mathbb{C}I$. Since normal operators with singleton spectrum are always always multiples of the identity, we must have $|\sigma(H)| \geq 2$.

Let U be a non-empty non-dense open set in $\sigma(S)$ and find $f \in C(\sigma(S))$ such that $f|_U = 0$. If $g \in C(\sigma(S))$ is non-zero with $\operatorname{supp}(g) \subseteq U$, then$f(H)g(H) = fg(H) = 0$, so $\operatorname{im} g(H) \subseteq \ker f(H) \neq \{0\}$. THen $\ker f(H)$ is π-invariant: if $x \in G$ ahd $\xi \in \ker f(H)$ is arbitrary, then $f(H)\pi(x)\xi = \pi(x)f(H)\xi = 0$ so $\pi(x)\xi \in \ker f(H)$.

1.36 Corollary. *If G is abelian, then every irreducible unitary representation is one-dimensional.*

Proof If π is an irreducible representation, then $\pi(G) \subseteq \pi(G)' = \mathbb{C}I$. Hence for $x \in G$, there is $\sigma(x) \in \mathbb{C}$ such that $\pi(x) = \sigma(x)I$. Notice that for also $y \in G$,

$$\sigma(xy)I = \pi(xy)I = \pi(x)\pi(y)I = \sigma(x)\sigma(y)I$$

and

$$\overline{\sigma(x)} = (\sigma(x)I)^* = \pi(x)^* = \pi(x^{-1}) = \sigma(x^{-1})I$$

so $\sigma(x) \in \pi$. $\blacksquare$

2 Gelfand Theory

Definition. We say that $\mathcal{A}$ is a **commutative Banach algebra** if $\mathcal{A}$ is a Banach space with a commuting associative product such that $\|ab\| \leq \|a\|\|b\|$.

Example.
(i) Let X be a locally compact Hausdorff space, $\mathcal{A} = C_0(X)$. This is unital if and only if X is compact.
(ii) Consider $L^1(G)$ where G is an abelian locally compact group. This is unital if and only if G is discrete. The reverse is clear; forwardly, if f acts as a unit, i.e. $f * g = g$ for all $g \in L^1(G)$, let $(f_\alpha)_\alpha$ be a summability kernel so $f = \lim_\alpha f * f_\alpha = \lim_\alpha f_\alpha$. Thus if $h \in C_0(G)$, then

$$\int_G hf \,\mathrm{d}m = \lim_\alpha \int_G hf_\alpha \,\mathrm{d}m = h(e) = \int_G h\delta_e$$

and hence $m_f = \delta_e$. Thus $M_a(G) \cap M_d(G) \neq \emptyset$, so G is discrete.

(iii) Let S be an abelian semigroup, and define a convolution product on $\ell^1(S)$ by

$$\left(\sum_{s\in S}\alpha(s)\delta_s\right)*\left(\sum_{t\in S}\beta(t)\delta_t\right)=\sum_{u\in G}\left(\sum_{\substack{(s,t)\in S\times S\\ st=u}}\alpha(s)\beta(t)\right)\delta_u.$$

If S is unital, then $\ell^1(S)$ is unital. Otherwise, consider $S=\{s_1,\dots,s_n\}$ by $s_is_j=s_i$ if $i\neq j$, and 0 otherwise. Then the unit has norm n (?)

(iv) If G is an abelian locally compact group, $M(G)$ is a commutative Banach algebra.

(v) Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and let $\mathcal{A}(\mathcal{D})=\{f\in C(\overline{\mathbb{D}}):f\text{ holomorphic}\}$ ([***TODO: related to semigroup algebra*** $\ell^1(\{0\}\cup\mathbb{N})$ ***from analytic functions***].)

Definition. The **(Gelfand) spectrum** of a commutative Banach algebra $\mathcal{A}$ is$\hat{}$ A=$\{\chi:\mathcal{A}\to\mathbb{C}:\chi$ linear and multiplicative$\}\setminus\{0\}$. We refer to χ as **characters**.

2.1 Proposition. *Let $\mathcal{A}$ be a unital commutative Banach algebra. Then for $\chi\in\hat{\mathcal{A}}$,*

(i) $\chi(1_{\mathcal{A}})=1$

(ii) If $a\in\mathcal{A}^\times$, then $\chi(a)\neq 0$

(iii) $|\chi(a)|\le\|a\|$, *i.e.* $\chi\in\mathcal{A}^*$, $\|\chi\|\le 1$.

Proof (i) There is a so that $\chi(a)\neq 0$, so $\chi(1_{\mathcal{A}})\chi(a)=\chi(a)$ so $\chi(1_{\mathcal{A}})=1$.

(ii) $\chi(a)\chi(a^{-1})=1$

(iii) If $\lambda\in\mathbb{C}$, then $|\lambda|>\|a\|$ for some $a\in\mathcal{A}$, then $\left\|\frac{1}{\lambda}a\right\|<1$ so

$$(\lambda 1_{\mathcal{A}}-1)^{-1}=[\lambda(1_{\mathcal{A}}-\frac{1}{\lambda}a)]^{-1}=\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{1}{\lambda^n}a^n.$$

Thus $\lambda-\chi(a)=\chi(\lambda 1_{\mathcal{A}}-a)\neq 0$. But $\chi(\chi(a)1_{\mathcal{A}}-a)=0$ so we cannot have $|\chi(a)|\ge\|a\|$. ■

2.2 Corollary. *If $\mathcal{A}$ is unital, then $\hat{\mathcal{A}}$ is w^*-compact in $\mathcal{A}$.*

Proof Since $\hat{\mathcal{A}}\subseteq B(\mathcal{A}^\times)$, it suffices to show that $\hat{\mathcal{A}}$ is w^*-closed.
If $\chi\in\overline{\hat{\mathcal{A}}}^{w^*}$, say $\chi=\lim_\alpha\chi_\alpha$ for $\chi_\alpha\in\hat{\mathcal{A}}$. Then for $a,b\in\mathcal{A}$,

$$\chi(ab)=\lim_\alpha\chi_\alpha(ab)=\lim_\alpha\chi_\alpha(a)\chi_\alpha(b)=\chi(a)\chi(b)$$
$$\chi(1_{\mathcal{A}})=\lim_\alpha{}_\alpha(1_{\mathcal{A}})=1$$

so $\chi\neq 0$. ■

2.3 Lemma. *Let $\mathcal{A}$ be a unital commutative Banach algebra, and $\mathcal{I}\subsetneq\mathcal{A}$ an ideal. Then*

(i) $\mathcal{I}\cap\mathcal{A}^\times=\emptyset$,

(ii) $\overline{\mathcal{I}}\subsetneq\mathcal{A}$ *is an ideal, and*

(iii) $\mathcal{I}$ is contained in a proper maximal ideal M

(iv) if $\mathcal{I}$ is maximal, it is closed.

Proof (i) Standard.

(ii) If $\|b\| < 1$ in $\mathcal{A}$, then $1_{\mathcal{A}} - b \in \mathcal{A}^{\times}$ with $\left(1^{-1}_{\mathcal{A}-b}\right) = \sum_{n=0}^{\infty} b^n$. Thus $\mathcal{I} \cap \left(1_{\mathcal{A}} + B^0(\mathcal{A})\right) = \emptyset$. Thus $\overline{I} \cap (1_{\mathcal{A}} + B^0(\mathcal{A})) = \emptyset$ too. If $a \in \overline{\mathcal{I}}$, then $a = \lim_{n\to\infty} a_n$ with each $a_n \in \mathcal{I}$. If $b \in \mathcal{A}$, then $ba = \lim_{n\to\infty} ba_n$ with each $ba_n \in \mathcal{I}$.

(iii) Standard.

(iv) $\mathcal{I} \subseteq \overline{\mathcal{I}} \subsetneq \mathcal{A}$ ∎

2.4 Theorem. *If $\mathcal{A}$ is a unital Banach algebra, then the "spectrum" $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \in \mathcal{A} \setminus \mathcal{A}^{\times}\}$ is a non-empty compact subset of $\mathbb{C}$.*

Proof Just like for $\mathcal{B}(X)$. ∎

2.5 Theorem. (Mazur) *If $\mathcal{A}^{\times} = \mathcal{A} \setminus \{0\}$, then $\mathcal{A} \cong \mathbb{C}$.*

Proof If there were $a \in \mathcal{A} \setminus \mathbb{C}1_{\mathcal{A}}$, then $\lambda 1_{\mathcal{A}} - a \neq 0$ for all $\lambda \in \mathbb{C}$, so $\lambda 1_{\mathcal{A}} - a \in \mathcal{A}^*$. This contradicts the last theorem. ∎

2.6 Theorem. (Maximal Ideals) *Let $\mathcal{A}$ be a unital commutative Banach algebra. Then $\{\ker\chi : \chi \in \hat{\mathcal{A}}\}$ is the collection of distinct maximal ideals.*

Proof Each $\mathcal{A}/\ker\chi \cong \chi(\mathcal{A}) = \mathbb{C}$, so $\ker\chi$ is maximal. If $\ker\chi = \ker\chi'$, then $\chi(a)1_{\mathcal{A}} - a \in \ker\chi = \ker\chi'$ so $0\chi'(\chi(a)1_{\mathcal{A}} - a) = \chi(a) - \chi'(a)$ for $a \in \mathcal{A}$.

Conversely, if $\mathcal{M}$ is a maximal idea in $\mathcal{A}$, then $\mathcal{A}/\mathcal{M}$ is a field. By Mazur's theorem, $\mathcal{A}/\mathcal{M} \cong \mathbb{C}$, so the projection map is a character. ∎

2.7 Corollary. *(i) $\mathcal{A} \setminus \mathcal{A}^{\times} = \bigcup_{\chi \in \hat{\mathcal{A}}} \ker\chi$*

(ii) $\sup_{\chi\in\hat{\mathcal{A}}} |\chi(a)| = \lim_{n\to\infty} \|a^n\|^{1/n}$

Proof (i) We already saw $\mathcal{A}^{\times} \subseteq \mathcal{A} \setminus \bigcup_{\chi\in\hat{\mathcal{A}}} \ker\chi$. If $a \in \mathcal{A} \setminus \mathcal{A}^{\times}$, then $a\mathcal{A}$ is a proper ideal contained in a maximal ideal $\ker\chi$.

(ii) If $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, then $\lambda \in \sigma(a)$ if and only if $\lambda = \chi(a)$. Then apply the spectral radius formula. ∎

3 Abelian Locally Compact Groups

Let G be an abelian locally compact group. Then $M(G)$ and $L^1(G)$ are commutative. Let

$$\widehat{G} = \{\sigma : G \to \mathbb{T} | \sigma \text{ continuous group homomorphism}\}$$

Then Schur's lemma tells us that $\widehat{G}$ is the set of all irreducible unitary representations of G, as $U(\mathbb{C}) = \mathbb{T}$.

3.1 Theorem. *Let G be an abelian locally compact group.*

(i) $\widehat{L^1(G)} \cong \widehat{G}$, where each element of $\widehat{L^1(G)}$ is given by $f \mapsto \langle f, \sigma\rangle = \int_G f\sigma\, dm$.

(ii) $\widehat{G} \cup \{0\}$ is w^*-compact in $L^\infty(G) \cong L^1(G)^*$, and thus $\widehat{G}$ is locally compact

(iii) $(\widehat{G}, w^*)$ is a locally compact group under pointwise operations ($\sigma^{-1} = \overline{\sigma}$).

Proof (i) If $\sigma \in \widehat{G}$, so $\sigma : G \to \pi = U(\mathbb{C})$, then $\chi = \sigma_1 : L^1(G) \to \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$ is a non-degenerate, and hence non-zero multiplicative functional, i.e. in $\widehat{L^1(G)}$. Also, if $\sigma \neq \tau$, then $\sigma_1 \neq \tau_1$, since there exists f such that $\int_G f\sigma \neq \int_G f\tau$.

Now, let $\chi \in \widehat{L^1(G)}$. If G is discrete, then $L^1(G) \cong \ell^1(G)$ is unital, and hence we have $\|\chi\| \le 1$. If G is not discrete, we let $\mathcal{A} = L^1(G) \oplus \mathbb{C}\delta_e \subset M(G)$. Then define $\tilde{\chi} : \mathcal{A} \to \mathbb{C}$ by $\tilde{\chi}(f + \alpha\delta_e) = \chi(f) + \alpha$. It is straightforward that $\tilde{\chi} \in \widehat{\mathcal{A}}$, so $\|\chi\| \le \|\tilde{\chi}\| \le 1$ from a proposition last class. Thus $\chi : L^1(G) \to \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$ is a contractive representation, and hence self-adjoint, and hence there is $\sigma : G \to U(\mathbb{C}) \cong \pi$ such that $\chi = \sigma_1$.

(ii) It suffices to show that $\widehat{G} \cup \{0\}$ is w^*-closed in $L^\infty(G)$. Let σ be an element of the w^*-closure of $\widehat{G}$, so $\sigma = w^* - \lim_\alpha \sigma_\alpha$ for each $\sigma_\alpha \in \widehat{G}$. Then for $f, g \in L^1(G)$,

$$\langle f * g, \sigma\rangle = \lim_\alpha \langle f * g, \sigma_\alpha\rangle = \lim_\alpha \langle f, \sigma_\alpha\rangle\langle g, \sigma_\alpha\rangle = \langle f, \sigma\rangle\langle g, \sigma\rangle$$

so $\sigma_1 \in \widehat{L^1(G)} \cup \{0\}$, i.e. $\sigma \in \widehat{G} \cup \{0\}$.

Since the w^*-topology is Hausdorff, there is a neighbourhood U of X and V of) so that $U \cap V \neq \emptyset$. Hence $0 \in \overline{U}^{w^*}$, so U is a relatively compact neighbourhood of X by the spectral radius formula.

(iii) Let $\pi_m : L^\infty(G) \to \mathcal{B}(L^2(G))$ be given by $\pi_m(\phi)f = \phi f$. Then for $f, g \in L^2(G)$, then $f\overline{g} \in L^1(G)$ by Cauchy-Schwarz and

$$\langle \pi_m(\phi)f, g\rangle = \int_G \phi f \overline{g}\,\mathrm{d}m = \langle \phi, f\overline{g}\rangle.$$

On the other hand, if $f \in L^1(G)$, $f = \operatorname{sgn} f|f|^{1/2}|f|^{1/2}$ with

$$\langle \phi, f\rangle = \int_G \phi f\,\mathrm{d}m = \langle \pi_m(\phi)\operatorname{sgn} f|f|^{1/2}, |f|^{1/2}\rangle.$$

Hence $\phi_\alpha \to \phi$ w^* in $L^\infty(G)$ if and only if $\pi_m(\phi_\alpha) \to \pi_m(\phi)$ in the weak operator topology, i.e. π_m is a $w^* - w.o.$ homeomorphism.

Then $\pi_m(\widehat{G}) \subseteq \mathcal{U}(L^2(G))$ so $\widehat{G}$ embeds homeomorphically into a topological group, and is thus a topological group. ■

3.2 Proposition. *(i) If G is discrete, then $\widehat{G}$ is compact.*

(ii) If G is compact, then $\widehat{G}$ is discrete.

Proof (i) We saw that $L^1(G) \cong \ell^1(G)$ is unital with $\widehat{G} \cong \widehat{\ell^1(G)}$.

(ii) If $\sigma \in \widehat{G} \setminus \{1\}$, we normalize $m(G) = 1$. Given $y \in G$ so $\sigma(y) \neq 1$, we have

$$\int_G \sigma(x)\,\mathrm{d}x =_G \sigma(yx)\,\mathrm{d}x = \sigma(y)\int_G \sigma(x)\,\mathrm{d}x$$

so that $\langle \sigma, 1\rangle = \int_G \sigma(x)\,\mathrm{d}x = 0$. Also, $\langle 1, 1\rangle = \int_G 1\,\mathrm{d}x = 1$. Hence

$$\{\tau \in \widehat{G} : \langle \tau, 1\rangle - \langle 1, 1\rangle < 1/2\} = \{1\}$$

is w^*-open. ■

Example. 1. Let $G = \mathbb{Z}$. If $\sigma \in \hat{\mathbb{Z}}$, let $z = \sigma(1)$, then $z^n = \sigma(n)$ for any $n \in \mathbb{Z}$. Write $\sigma = \sigma_z$. Then $z \mapsto \sigma_z : \pi \to \widehat{\mathbb{Z}}$ is a bijection. If $f = \sum_{n\in\mathbb{Z}} f(n)\delta_n \in \ell^1(\mathbb{Z})$, then $\langle f, \sigma_z\rangle = \sum_{n\in\mathbb{Z}} f(n)z^n$, so $z \mapsto \langle f, \sigma_z\rangle$ is continuous, so $z \mapsto \sigma_z : \mathbb{T} \to \widehat{Z}$ is continuous, and thus a homeomorphism.

2. Let $G = \mathbb{R}$ and $\sigma \in \widehat{R}$, so $\sigma(0) = 1$. Hence there is $y_0 > 0$ such that $\int_0^y \sigma(x)\,\mathrm{d}x \neq 0$ for $y \in [-y_0, y_0]$. For such y,

$$0 \neq \int_0^y \sigma(x)\,\mathrm{d}x = \int_y^{2y} \sigma(y+x)\,\mathrm{d}x = \sigma(y)\int_y^{2y} \sigma(x)\,\mathrm{d}x$$

so that

$$\sigma(y) = \frac{\int_0^y \sigma(x)\,\mathrm{d}x}{\int_y^{2y} \sigma(x)\,\mathrm{d}x}.$$

Thus the fundamental theorem of calculus shows that σ is differentiable in a neighbourhood of 0. For $x \in \mathbb{R}$, we have $\sigma'(x) = \sigma(x)\sigma'(0)$. Let $f(x) = e^{-\sigma'(0)x}\sigma(x)$ so $f(0) = 1$, $f'(x) = 0$, so $f(x) = 1$ by the mean value theorem and in fact $\sigma(x) = e^{\sigma'(0)x}$. Since $\sigma(\mathbb{R}) \subseteq \mathbb{T}$, we have that $\sigma'(0) = it \in i\mathbb{R}$. Write $\sigma = \sigma_t$, $\sigma_t(x) = e^{ixt}$.

The map $t \mapsto \sigma_t : \mathbb{R} \to \widehat{R}$ is an injective homomorphism and surjective, from above. If $t_n \to t_0$ in $\mathbb{R}$, $\sigma_{t_n} \to \sigma_{t_0}$ pointwise so by LDCT,

$$\langle f, \sigma_{tn}\rangle = \int_{\mathbb{R}} f(x)e^{it_nx}\,\mathrm{d}x \xrightarrow{n\to\infty} \int_{\mathbb{R}} f(x)e^{it_0x}\,\mathrm{d}x\langle f, \sigma_{t_0}\rangle$$

so $t \mapsto \sigma_t$ from $\mathbb{R} \to \widehat{R}$ is continuous.

To see that the map is open, consider a w^*-open neighbourhood of $1 = \sigma_0$: for $0 < \epsilon < 1 - 2/\pi$,

$$\begin{aligned} U_\epsilon &= \left\{t \in \mathbb{R} : \left|\langle \mathbf{1}_{[-1,1]}, \sigma_t\rangle - \langle \mathbf{1}_{[-1,1]}, \sigma_0\rangle\right| < \epsilon\right\} \\ &= \left\{t \in \mathbb{R} : \left|\int_{-1}^{1} (e^{itx} - 1)\,\mathrm{d}x\right| < \epsilon\right\} \\ &= \left\{t \in \mathbb{R} : 2\left|\frac{\sin(t)}{t} - 1\right\langle \epsilon|\right\} = (-\delta, \delta) \end{aligned}$$

for some $0 < \delta < \pi/2$. Thus $t \mapsto \sigma_t : R \to \widehat{\mathbb{R}}$ is open at 1.

3. Let $G = \mathbb{T}$. Let $\sigma_1 : \mathbb{R} \to \mathbb{T}$ be given by $\sigma_1(x) = e^{ix}$, which is continuous, surjective, and has $\ker\sigma_1 = 2\pi\mathbb{Z}$. If $\tau \in \widehat{\mathbb{T}}$, then $\tau \circ \sigma \in \widehat{\mathbb{R}}$, so $\tau \circ \sigma_1(x) = e^{itx}$ for all $x \in \mathbb{R}$. Also, $\ker\tau \supseteq \ker\sigma_1 = 2\pi\mathbb{Z}$, and hence $t \in \mathbb{Z}$. Write $t = n$, and for $z = \sigma_1(x) \in \pi$,

$$\tau(z) = \tau \circ \sigma_1(x) = e^{inx} = z^n.$$

Write $\tau = \sigma_n$ for $n \in \mathbb{Z}$. We have that $n \mapsto \sigma_n$ is a bijection. It is a homeomorphism as both $\mathbb{Z}$ and $\widehat{\mathbb{T}}$ are discrete.

Definition. (Fourier Transform) If $f \in L^1(G)$, define $\hat{f} : \widehat{G} \to \mathbb{C}$ by

$$\hat{f}(\sigma) = \int_G f(x)\overline{\sigma(x)}\,\mathrm{d}x = \langle f, \overline{\sigma}\rangle.$$

3.3 Proposition. (Riemann-Lebesgue, Gelfand) *The map $f \mapsto \hat{f} : L^1(G) \to C_0(\widehat{G})$ is an injective homomorphism with*

(i) $\|\hat{f}\|_\infty = \lim_{n\to\infty} \|f^{*n}\|_1^{1/n} \le \|f\|_1$

(ii) $\mathcal{A}(\widehat{G}) = \{\hat{f} : f \in L^1(G)\}$ *is dense in* $C_0(\widehat{G})$.

Proof [***TODO: show injective***]

Since $\sigma \mapsto \overline{\sigma}$ is continuous on $\widehat{G}$, $\hat{f}$ is continuous. Letting $\hat{f}(0) = 0$, we see for $\epsilon > 0$, that $g_\epsilon = \max\{|\hat{f}| - \epsilon 1\}$ is supported on

$$(\widehat{G} \cup \{0\}) \setminus \{\sigma \in \widehat{G} \cup \{0\} : |\hat{f}(\sigma)| = |\langle f, \overline{\sigma}\rangle| < \epsilon\}$$

and hence compactly supported with $\||\hat{f}| - g_\epsilon\|_\infty \le \epsilon$ so it follows that $\hat{f} \in C_0(\widehat{G})$.

The formula in (i) is the spectra radius formula. We note that $\mathcal{A}$ is an

- algebra: $\langle f * g, \overline{\sigma}\rangle = \langle f, \overline{\sigma}\rangle\langle g, \overline{\sigma}\rangle$
- conjugate closed: $\widehat{f^*}(\sigma) = \overline{\sigma}_1(f^*) = \overline{\overline{\sigma}_1(f)} = \overline{\hat{f}(\sigma)}$
- point separating: $\sigma \ne \tau$ implies $\overline{\sigma} \ne \overline{\tau}$ so $\langle f, \overline{\sigma}\rangle \ne \langle f, \overline{\tau}\rangle$ for some $f \in L^1(G)$.
- separates points from 0: $L^1(G)^* \cong L^\infty(G) \supseteq \hat{G}$. ■

Hence by Stone-Weierstrass, $\mathcal{A}(\widehat{G})$ is dense in $C_0(\widehat{G})$.

3.4 Theorem. *Let G be an abelian locally compact group. There is a bijective correspondence between*

1. *unitary representations $\pi : G \to \mathcal{U}(\mathcal{H})$, and*
2. *non-degenerate contractive homomorphisms $\Pi : C_0(\widehat{G}) \to \mathcal{B}(\mathcal{H})$ such that $\Pi(\overline{f}) = \Pi(f)^*$*

In the context of this theorem, we will commonly identifty $\pi : G \to \mathcal{U}(\mathcal{H})$ with integrated forms $\pi_1 : L^1(G) \to \mathcal{B}(\mathcal{H})$ and $\pi_0 : C_0(\widehat{G}) \to \mathcal{B}(\mathcal{H})$.

Proof $(i \Rightarrow ii)$ Recall that $\pi_1 : L61(G) \to \mathcal{B}(\mathcal{H})$ satisfies that π_1 is contractive and $\pi_1(f^*) = \pi_1(f)^*$. Notice that $\pi_1(f)^*\pi_1(f) = \pi_1(f^* * f) = \pi_1(f * f^*) = \pi(1(f)\pi_1(f)^*$, so $\pi_1(f)$ is normal. Hence

$$\|\pi_1(f)\| = \|\pi_1(f)^{2n}\|^{1/(2n)} \le \|f^{*(2n)}\|_1^{1/(2n)} \to \|\hat{f}\|_\infty.$$

Thus we may uniquely define linear continuous $\pi_0 : C_0(\widehat{G}) \to \mathcal{B}(\mathcal{H})$ such that $\pi_0(\hat{f}) = \pi_1(f)$. Notice that $\pi_0(\overline{\hat{f}}) = \pi_0(\hat{f^*}) = \pi_1(f^*) = \pi_1(f)^* = \pi_0(\hat{f})^*$ and $\pi_0(\overline{\phi}) = \pi_0(\phi)^*$ for $\phi \in C_0(\widehat{G})$.

$(ii \Rightarrow i)$ We let $\Pi_1 : L^1(G) \to \mathcal{B}(\mathcal{H})$ be given by $\Pi_1(f) = \pi(\hat{f})$, so Π_1 is a homomorphism $(\widehat{f * g} = \hat{f}\hat{g})$ with $\|\Pi_1(f)\| \le \|\hat{f}\|_\infty \le \|f\|_1$, and $\Pi_1(f^*) = \pi(\overline{\hat{f}}) = \pi(\hat{f})^* = \Pi_1(f)^*$. The density of $A(\widehat{G})$ gives that Π_1 is non-degenerate. Hence there is $\pi : G \to \mathcal{U}(\mathcal{H})$ such that $\pi_1 = \Pi_1$.■

Example. Let $\mu \in M_+(\widehat{G}) \setminus \{0\}$ and define $\pi_0^\mu : C_0(\widehat{G}) \to \mathcal{B}(L^2(\widehat{G}, \mu))$ be given by

$$\pi_0^\mu(\phi)h = \phi(h), \phi \in C_0(\widehat{G}), h \in L^2(\widehat{G}, \mu).$$

Clearly, π_0^μ is a contractive homomorphism with $\pi_0^\mu(\overline{\phi}) = \pi_0^\mu(\phi)^*$. Let

$$C_0^+(\widehat{G}) = \{\phi \in C_0^+(\widehat{G}) : \phi(1) = 1 = \|\phi\|_\infty\}$$

be directed by pointwise comparison. Then by inner regularity, $\lim \pi_0^\mu(\phi)h = h$: given $\epsilon > 0$, let $K \subseteq \widehat{G}$ be compact so $\int_{\widehat{G}\setminus K} |h|^2 < \epsilon^2$, eventually $\phi|_K = 1$, so

$$\left(\int_{\widehat{G}} |\phi h - h|^2\right)^{1/2} = \left(\int_{\widehat{G}} (\phi^2 - 1)|h|^2\right)^{1/2} \leq \left(\int_K |h|^2\right)^{1/2} < \epsilon.$$

Hence π_0^μ induces $\pi^\mu : G \to \mathcal{U}(L^2(G,\mu))$ such that $\pi^\mu(x)\pi_0^\mu(\hat{f}) = \pi_0^\mu(\widehat{x * f})$. Now,

$$\widehat{x * f}(\sigma) = \int_G f(x^{-1}y)\overline{\sigma(y)}\,\mathrm{d}y = \int_G f(y)\overline{\sigma(xy)} = \overline{\sigma(x)}\hat{f}(\sigma).$$

Let $\hat{x} : \widehat{G} \to \mathbb{T}$ be given by $\hat{x}(\sigma) = \overline{\sigma(x)}$, so $\widehat{x * f} = \hat{x}\hat{f}$. Thus

$$\pi^\mu(x)\pi_0^\mu(\hat{f})h = \pi_0^\mu(\hat{x}\hat{f})h = \hat{x}\hat{f}h$$

and since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$, and since $\lim \pi_0^\mu(\phi)h = h$, we have

$$\pi^\mu(x)h = \hat{x}h = \overline{\sigma(x)}h(\sigma).$$

Alternatively, we may denote $\pi^\mu = \int_{\widehat{G}}^{\oplus} \overline{\sigma}\,\mathrm{d}\mu(\sigma)$.

Since μ is finite, $j : C_0(\widehat{G}) + \mathbb{C}1 \to L^2(\widehat{G},\mu)$ is a bounded linear map, putting functions into their μ-a.e. equiv classes. Since μ is Radin, $j(C_0(\widehat{G}))$ is dense in $L^2(\widehat{G},\mu)$. Furthermore, $\pi_0^\mu(C_0(\widehat{G}))j(1) = j(C_0(\widehat{G}))$ is dense in $L^2(\widehat{G},\mu)$, i.e. $j(1)$ is a **cyclic vector** for π_0^μ.

3.5 Theorem. (Stone) *Let G be an abelian locally compact group. Let $\pi : G \to \mathcal{U}(\mathcal{H})$ which admits a cyclic vector ξ, i.e. $\overline{\operatorname{span}}\pi(g)\xi) = \mathcal{H}$. Then there is $\mu \in M_+(\widehat{G})$ and a unitary $U : L^2(\widehat{G},\mu) \to \mathcal{H}$ such that $U\pi^\mu(x) = \pi(x)U$ for x in G and $Uj(1) = \xi$.*

Proof Let $\mu \in M(\widehat{G}) \cong C_0(\widehat{G})^*$ be given by

$$\int_{\widehat{G}} \phi\,\mathrm{d}\mu = \langle \pi_0(\phi)\xi|\xi\rangle.$$

Notice if $\phi \geq 0$, then

$$\int_{\widehat{G}} \phi\,\mathrm{d}\mu = \langle \pi_0(\overline{\phi^{1/2}}\phi^{1/2})\xi|\xi\rangle = \left\|\pi_0(\phi^{1/2})\xi\right\|^2 \geq 0$$

so $\mu \in M_+(\widehat{G})$.

Define $U_0 : J(C_0(\widehat{G})) \to \pi_0(C_0(\widehat{G}))\xi$ by $U_0 j(\phi) = \pi_0(\phi)\xi$ and note that

$$\left\|\pi_0(\phi)\xi\right\|^2 = \langle \pi_0(|\phi|^2)\xi|\xi\rangle = \int_{\widehat{G}} |\phi|^2\,\mathrm{d}\mu = \left\|j(\phi)\right\|_2^2.$$

Now $\overline{\operatorname{span}}\pi(G)\xi = \mathcal{H}$ so $\overline{\pi_1(L^1(G))\xi} = \mathcal{H}$. To see this, write $h = \sum_{j=1}^n \alpha_j \pi(x_j)\xi$, where (f_α) is a summability kernel in $L^1(G)$, then

$$\pi_1(f_\alpha)h = \sum_{j=1}^n \alpha_j \pi_1(f_\alpha * x_j)\xi \to \sum_{j=1}^n \alpha_j \pi(x_j)\xi.$$

Thus, $\overline{\pi_0(C_0(\widehat{G}))\xi} = \mathcal{H}$, i.e. ξ is cyclic for $|pi_0$. Thus $U_0 j(C_0(\widehat{G})) = \pi_0(C_0(\widehat{G}))\xi$ is dense in $\mathcal{H}$. Hence U_0 uniquely extends to a surjective isometry, i.e. unitary, $U : L^2(\widehat{G},\mu) \to \mathcal{G}$. Now for $\phi \in C_0(\widehat{G})$, $x \in G$,

$$\pi(x)Uj(\phi) = \pi(x)\pi_0(\phi)\xi = \pi_0(\hat{x}\phi)\xi = Uj(\hat{x}\phi) = U\pi^\mu(x)j(\phi)$$

so $\pi(x)U = U\pi^\mu(x)$ on $\mathcal{H}$. Let $(f_\alpha)_\alpha$ be a continuous summability kernel for $L^1(G)$, then for $f \in L^1(G)$, $\hat{f}_\alpha \hat{f} = \widehat{f_\alpha * f} \to \hat{f}$ in $\|\cdot\|_\infty$ and since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$, $\lim_\alpha \widehat{f_\alpha}\phi = \phi$ uniformly in $C_0(\widehat{G})$ for ϕ in $C_0(\widehat{G})$. Hence $j(\widehat{f_\alpha})j(\phi) = j(\hat{f}_\alpha\phi) \to j(\phi)$ in $L^2(\widehat{G},\phi)$ and by density of $j(C_0(\widehat{G}))$ in $L^2(\widehat{G},\mu)$. $j(\hat{f}_\alpha)h \to h$, $h \in L^2(\widehat{G},\mu)$.

In particular, $j(\hat{f}_\alpha 1) \to j(1)$. Thus

$$Uj(1) = \lim_\alpha Uj(\hat{f}_\alpha) = \lim_\alpha \pi_0(\hat{f}_\alpha)\xi = \lim_\alpha \pi_1(f_\alpha)\xi = \xi \qquad \blacksquare$$

Reverse transform: if $\mu \in M(\widehat{G})$. Define $\mu^\vee : G \to \mathbb{C}$ by $\mu^\vee(x) = \int_{\widehat{G}} \sigma(x)\,\mathrm{d}\mu(\sigma)$.

3.6 Lemma. *The map $(x,\sigma) \mapsto \sigma(x) : G \times \widehat{G} \to \mathbb{T}$ is continuous.*

Proof Fix $\sigma \in \widehat{G}$, $x \in G$, and let $f \in L^1(G)$ be so $\widehat{f}(\sigma) = 1$. We have for $y \in G$ and $\tau \in \widehat{G}$

$$\begin{aligned} |\sigma(x) - \tau(y)| &\le |f(\sigma)\overline{\sigma(x)} - \hat{f}(\tau)\overline{\tau(y)}| + |\hat{f}(\tau)\overline{\tau(y)} - \hat{f}(\sigma)\overline{\tau(y)}| \\ &= |\widehat{f * x}(\sigma) - \widehat{f * y}(\tau)| + |\hat{f}(\tau) - \hat{f}(\sigma)| \\ &\le |\widehat{f * x}(\sigma) - \widehat{f * x}(\tau)| + |\widehat{f * x}(\tau) - \widehat{f * y}(\tau)| + |\hat{f}(\tau) - \hat{f}(\sigma)| \\ &\le \left\| f * x - f * y \right\|_1 \end{aligned}$$

which converges to 0. $\blacksquare$

3.7 Theorem. (Bochner) *$B^+(G) = \{\check{\mu} : \mu \in M_+(\widehat{G})\}$. Hence $B(G) = \{\check{\mu} : \mu \in M(\widehat{G})\}$ is an algebra of (uniformly) continuous functions on G.*

Proof Note that $\mu \in B^+(G)$ if and only if $\tilde{\mu} \in B^+(G)$ where $\tilde{\mu}(x) = \mu(x^{-1})$ for $x \in G$. If $\mu \in M_+(\widehat{G})$, then for $x \in G$,

$$\check{\mu}(x^{-1}) = \int_{\widehat{G}} \sigma(x^{-1})\,\mathrm{d}\mu(\sigma) = \int_{\widehat{G}} \hat{x}\,\mathrm{d}\mu = \langle \pi(x)^M j(1) | j(1) \rangle$$

where $\pi^\mu : G \to U(L^2(\widehat{G},\mu))$ is given by $\pi^\mu(x)f = \hat{x}f$, so that $\check{\mu} \in B^+(G)$.

If $\mu \in B^+(G)$, then Gelfand Naimark provides a representation $\pi : G \to \mathcal{U}(H)$ and $\xi \in H$ (which, by const. is cyclic) such that $u = \langle \pi(\cdot)\xi|\xi \rangle$.

Let $\mu \in M_+(\widehat{G})$ and $U : L^2(\widehat{G},\mu) \to H$ so $U\pi^M(\cdot) = \pi(\cdot)U$ and $Uj(1) = \xi$. Then for $x \in G$,

$$\begin{aligned} u(x) &= \langle \pi(x)\xi|\xi \rangle = \langle \pi(x)Uj(x)|Uj(1) \rangle \\ &= \langle U\pi^\mu(x)j(1)|Uj(1) \rangle = \langle \pi^\mu(x)j(1)|j(1) \rangle \\ &= \int_G \hat{x}1\overline{1}\,\mathrm{d}\mu = \int_G \overline{\sigma(x)}\,\mathrm{d}\mu = \check{\mu}(x^{-1}) \end{aligned}$$

i.e. $\tilde{u} = \check{\mu}$.

We saw

3.8 Proposition. *The map $\mu \mapsto \check{\mu} : M(\widehat{G}) \to B(G)$ is injective.*

Proof If $f \in L^1(G)$, then for $\mu \in M(\widehat{G})$,

$$\begin{aligned}\int_{\widehat{G}} \hat{f}\,\mathrm{d}\mu &= \int_{\widehat{G}}\int_G f(x)\overline{\sigma(x)}\,\mathrm{d}x\,\mathrm{d}\mu(\sigma)\\ &= \int_G f(x)\int_{\widehat{G}} \sigma(x^{-1})\,\mathrm{d}\mu(\sigma)\,\mathrm{d}x\\ &= \int_G f(x)\check{\mu}(x^{-1})\,\mathrm{d}x.\end{aligned}$$

Hence if $\check{\mu} = 0$, then $\mu = 0$ since $A(\widehat{G}) = \{\hat{f} : f \in L^(G)\}$ is dense in $C_0(\widehat{G})$. Since $\mu \mapsto \check{\mu}$ is linear, we are done. ■

Example. (i) Let $\lambda : G \to U(L^2(G))$ be given by $\lambda(x)f(y) = f(x^{-1}y)$. Just like to the fact that $x \mapsto x * f$ is continuous, λ is continuous.

Then λ is called the **left regular representation**. If $f \in L^2(G)$, $\langle \lambda(\cdot)f|f\rangle \in B^+(G)$. We note

$$\langle \lambda(x)f|f\rangle = \int_G f(x^{-1}y)\overline{f(y)}\,\mathrm{d}y = \int_G f(y)\overline{f(xy)}\,\mathrm{d}y.$$

If $f \in L^1 \cap L^2(G)$, then since G is unimodular,

$$\langle \lambda(x)f|f\rangle = \int_G f(y)\overline{f(xy)}\,\mathrm{d}y = \int_G f(y)f^*(y^{-1}x^{-1})\,\mathrm{d}y = f^* * f(x^{-1})$$

and $f^* * f$ is (a.e. equal to) a continuous function on G.

(ii) If $\phi \in C_0^+(G)$¡ then $U_\phi = \{x \in G : \phi(x) > 0\}$ so $\overline{U_\phi} = \mathrm{supp}(\phi)$. Then $\phi^* * \phi(x) = \int_G \phi(y)\phi(xy)\,\mathrm{d}y$ has $\phi^*\phi(x) > 0$ if $xU_\phi \cap U_\phi \neq \emptyset$, i.e. $x \in U_\phi U_\phi^{-1}$. Furthermore, $\mathrm{supp}(\phi^* * \phi) = \overline{U_\phi U_\phi^{-1}} = \mathrm{supp}(\phi)\mathrm{supp}(\phi)^{-1}$.

3.9 Theorem. (Inversion) *Let $B^1(G) = B \cap L^1(G)$.*
(i) $f \in B^1(G)$ implies that $\hat{f} \in L^1(\widehat{G})$
(ii) With suitable normalization of Haar measure on G, $\widehat{G}$, we have for $f \in B^1(G)$ that $f(x) = \int_{\widehat{G}} \hat{f}(\sigma)\sigma(x)\,\mathrm{d}\sigma$, i.e. $\hat{f} \in L^1(\widehat{G}) \cong M_a(\widehat{G})$, so $(\hat{f})^{\check{=}f}$.

Proof If $h \in L^1(G)$ and $f = \check{\mu} \in B^1(G)$, then

$$\int_{\widehat{G}} \hat{h}\,\mathrm{d}\mu = \int_G h(x)\check{\mu}(x^{-1})\,\mathrm{d}x.$$

Now if also $g = \check{\nu} \in B^1(G)$, applying Fubini,

$$\begin{aligned}\int_{\widehat{G}} \hat{h}(\check{\nu})^{\hat{}\mathrm{d}\mu} = \int_{\widehat{G}} \widehat{h * \check{\nu}}\,\mathrm{d}\mu &= \int_G h * \check{\nu}(x)\check{\mu}(x^{-1})\,\mathrm{d}x\\ &= \int_G\int_G h(y)\check{\nu}(y^{-1}x)\check{\mu}(x^{-1})\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_G\int_G h(xy)\check{\nu}(y^{-1})\end{aligned}$$

Since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$ (and $C_c(G)$ is dense in $L^1(G)$), we find that $(\check{\nu})\hat{}\,\mathrm{d}\mu = (\check{\mu})\hat{}\,\mathrm{d}\nu$.

We now build the Haar functional on $C_0(\widehat{G})$. Fix $\psi \in C_c(\widehat{G})$. For each $\sigma \in \operatorname{supp}(\psi)$, find $\mu \in C_c(G) \subseteq L^1(G)$ so $\hat{u}(\sigma) \neq 0$. Hence $\widehat{u^* * u} = \hat{u}^* \hat{u} = |\hat{u}| > 0$ in a neighbourhood of σ. Hence we can find $u_1, \ldots, u_n$ in $C_c(G)$ so

(a) $g = \sum_{j=1}^n u_j^* * u_j \in B^+(C_c(G)) \subseteq B^1(G)$, so $g = \check{\nu}$ for some $\nu \in M_+(\widehat{G})$ by Bochner's theorem.

(b) $\operatorname{supp} \psi \subseteq U_{\hat{g}} = \{\sigma \in \widehat{G} : \hat{g}(\sigma) > 0\}$.

Now let
$$J(\psi) = \int_{\widehat{G}} \frac{\psi}{(\check{\nu})\hat{}}\,\mathrm{d}\nu.$$

If also $f = \check{\mu} \in B^1(G)$ which satisfies (a),(b) above, then

$$\begin{aligned} J(\psi) &= \int_{\widehat{G}} \frac{\psi}{(\check{\nu})\hat{}(\check{\mu})\hat{}}(\check{\mu})\hat{}\,\mathrm{d}\nu \\ &= \int_{\widehat{G}} \frac{\psi}{(\check{\nu})\hat{}(\check{\mu})\hat{}}(\check{\nu})\hat{}\,\mathrm{d}\nu \\ &= \int_{\widehat{G}} \frac{\psi}{(\check{\mu})\hat{}}\,\mathrm{d}\nu \end{aligned}$$

which shows a certain independence of the definition of J from $g = \check{\nu}$.

If $\phi, \psi \in C_c(\widehat{G})$, then using $g = \check{\nu} \in B^1(G)$, so (a), (b) satisfied for both ϕ, ψ, linearity of J is clear. Also, if $g = \hat{\nu} \in B^+ \cap L^1(G) \setminus \{0\}$, then there is $\psi \in C_c^+(\widehat{G})$ so

$$0 < \int_{\widehat{G}} \psi\,\mathrm{d}\nu = \int_{\widehat{G}} \frac{\psi}{(\check{\mu})\hat{}}(\check{\mu})\hat{}\,\mathrm{d}\nu = \int_{\widehat{G}} \frac{\psi}{(\check{\mu})\hat{}}(\check{\nu})\hat{}\,\mathrm{d}\nu$$

and hence $J \neq 0$. Also, initial choice of $g = \hat{\nu}$, given $\psi \in C_c^+(G)$ shows that $J \geq 0$. Now let $\psi \in C_c(\widehat{G})$, $\tau \in \widehat{G}$. Find $g = \hat{\nu} \in B^+ \cap L^1(G)$, so $U_{\widehat{g}} \supseteq \operatorname{supp}(\psi) \cup \operatorname{supp}(\psi\tau)$. Then

Next, if $f = \check{\mu} \in B^+ \cap L^1(G)$, we have for $\psi \in C_c(\widehat{G})$, then a computation similar to above shows
$$\int_{\widehat{G}} \psi\,\mathrm{d}\mu = J(\psi(\check{\mu})\hat{} = J(\psi(\check{\mu})\hat{}) = \int_{\widehat{G}} \psi(\sigma)(\check{\mu})\hat{}(\sigma)\,\mathrm{d}\sigma$$

i.e. $\mu \in M_a(\widehat{G}) \cong L^1(\widehat{G})$, i.e. $\hat{f} = (\check{\mu})\hat{} \in L^1(\widehat{G})$. Since $B^1(G) = \operatorname{span} B^+ \cap L^1(G)$, this gives (i)

Finally, we show (ii). From analogy to (**), if $f = \check{\mu} \in B^1(G)$, then

$$f(x) = \hat{\mu}(x) = \int_{\widehat{G}} \sigma(x)\,\mathrm{d}\mu(x) = \int_{\widehat{G}} \sigma(x)(\check{\mu})\hat{}(\sigma)\,\mathrm{d}\sigma = \int_{\widehat{G}} \sigma(x)\hat{f}(\sigma)\,\mathrm{d}\sigma = (\hat{f})\check{}. \qquad \blacksquare$$

1. If G is compact and $m_G(1) = 1$. Then $\hat{1}(\sigma) = 1$ if $\sigma = 1$, and 0 if $\sigma \neq 1$ for $\sigma \in \widehat{G}$, i.e $\hat{1} = \mathbf{1}_{\{1\}}$ on G. Then by inversion $1 = 1(e) = \int_G \hat{1}(\sigma)\sigma(e)\,\mathrm{d}\sigma = m_{\widehat{G}}(\{1\})$, i.e. $m_{\widehat{G}}$ is counting measure.

2. Suppose G is discrete. Then $\mathbf{1}_{\{e\}} = \mathbf{1}_{\{e\}}^* * \mathbf{1}_{\{e\}} \in B^1(G)$. Then
$$\widehat{\mathbf{1}_{\{e\}}}(\sigma) = \sum_{x \in G} \overline{\sigma(x)} \mathbf{1}_{\{e\}}(x) = 1_{\widehat{G}}(\sigma).$$
Again, the inversion theorem gives
$$m_{\widehat{G}}(\widehat{G}) = \int_{\widehat{G}} \mathbf{1}_{\widehat{G}} \,\mathrm{d}m_{\widehat{G}} = \int_{\widehat{G}} \hat{\mathbf{1}_{\{e\}}}(\sigma)\,\mathrm{d}\sigma = \int_{\widehat{G}} \hat{\mathbf{1}_{\{e\}}}(\sigma)\sigma(e)\,\mathrm{d}\sigma = 1.$$
3. Let $m_{\mathbb{R}}$ be the standard normalization. We let $\alpha, \beta > 0$ be so $\alpha m_{\mathbb{R}}$, $\beta m_{\hat{\mathbb{R}}} \cong \beta m_{\mathbb{R}}$ satisfy the inversion theorem. If $s \in \mathbb{R}$,
$$\int_{\mathbb{R}} e^{-|x|} e^{-isx} \alpha \,\mathrm{d}x = 2\alpha \int_0^\infty e^{-|x|} \cos(sx)\,\mathrm{d}x = \frac{2\alpha}{1+s^2}$$
so $s \mapsto (2\alpha)/(1+s^2)$ is of positive type, as $e^{-|x|}\,\mathrm{d}x$ is positive (Bochner). Inversion gives $e^{-|x|} = 2\alpha \int_{\hat{\mathbb{R}}} \frac{e^{isx}}{1+s^2} B \,\mathrm{d}s$ for $x \in \mathbb{R}$. If $x = 0$, we have $1 = 2\alpha\beta \int_{hat\mathbb{R}} \frac{\mathrm{d}s}{1+s^2} = 2\beta$. Typical normalizations give $\alpha = 1$, $\beta = 1/2\pi$ or $\alpha = \beta = \frac{1}{\sqrt{2\pi}}$.

Given a relatively compact symmetric neighbourhood $V = V^{-1}$ of e in G, let $h_V = \frac{1}{m(V)} \mathbf{1}_V^* * \mathbf{1}_V$. Note that
$$h_V(x) = \frac{1}{m(V)} \int_G \mathbf{1}_V(y^{-1}) \mathbf{1}_V(y^{-1}x)\,\mathrm{d}y = \frac{1}{m(V)} \int_G \mathbf{1}_V(y) \mathbf{1}_{xV}(y)\,\mathrm{d}y = \frac{m(V \cap xV)}{m(V)}.$$
Then
- $h_V \in B^1(G)$,
- $\operatorname{supp}(h_V) \subseteq \overline{V}^2$
- $h_V(e) = 1$
- $h_V \geq 0$, $\int_G h_V \,\mathrm{d}m = m(V)$ (Fubini)

so that $\left(\frac{1}{m(V)} h_V\right)_{V \in N_e}$ of such neighbourhoods of e, preordered by reverse inclusion, is a summability kernel.

We saw that $\lim_{V \to \{e\}} \left\| \frac{1}{m(V)} \widehat{h_V} \phi - \phi \right\|_2 = 0$ for $\phi \in L^2(G)$. Note that
$$B^1(G) = L^1 \cap B(G) \subseteq L^1 \cap L^2(G).$$
If $f \in B^1(G)$, then $\int_G |f|^2 \,\mathrm{d}m \leq \|f\|_\infty \|f\|_1$.

3.10 Theorem. (Plancherel) *With normalization from the inversion theorem, there is a unique unitary $U : L^2(G) \to L^2(\widehat{G})$ such that $Uf = \hat{f}$ if $f \in L^1 \cap L^2(G)$.*

Proof If $f \in L^1 \cap L^2(G)$, then $f^* * f \in B^1(G)$, so the inversion theorem holds. Then
$$\begin{aligned}\int_G |f|^2 \,\mathrm{d}m_G &= \int_G f^*(x^{-1}) f(x)\,\mathrm{d}x = \int_G f^*(x) f(x^{-1}e)\,\mathrm{d}x \\ &= f^* * f(e) = \int_{\widehat{G}} \widehat{f^* * f}(\sigma)\sigma(e)\,\mathrm{d}\sigma \\ &= \int_{\widehat{G}} |\hat{f}|^2 \,\mathrm{d}m_{\widehat{G}}.\end{aligned}$$

Hence $U_0 : L^1 \cap L^2(G) \to L^2(G)$ by $U_0 f = \hat{f}$ is a linear isometry, whose domain is dense in $L^2(G)$, and thus extens uniquely to a linear isometry on $L^2(G)$.

It remains to show that $\operatorname{im} G = L^2(\widehat{G})$. Let $\phi \in (\operatorname{im} U)^\perp \subseteq L^2(\widehat{G})$. If $f \in L^1 \cap L2(G)$, $x \in G$, $x * f \in L^1 \cap L^2(G)$, and then

$$\begin{aligned} 0 &= \int_{\widehat{G}} \overline{\phi} \widehat{x * f} \, \mathrm{d}m_{\widehat{G}} = \int_{\widehat{G}} \overline{\phi} \hat{x} \hat{f} \, \mathrm{d}m_{\widehat{G}} \\ &= \int_{\widehat{G}} \overline{\phi} \hat{f} \hat{x} \mathrm{d}m_{\widehat{G}} = (\overline{\phi} \hat{f})^{\check{}(x^{-1})} \end{aligned}$$

so that $\overline{\phi}(\hat{f}) \in L^1(G)$, by injectivity of the reverse transform, ie $\overline{\phi}\hat{f} = 0$ $m_{\widehat{G}}$-a.e.

Hence $\overline{\phi} h_V = 0$ $m_{\widehat{G}}$–a.e., hence 0 in $L^2(\widehat{G})$. But $\left\| \frac{1}{m(V)} h_V \overline{\phi} - \overline{\phi} \right\|_2 \to 0$, so $\overline{\phi} = 0$. Thus $(\operatorname{im} U)^\perp = \{0\}$ so $\operatorname{im} U = L^2(\widehat{G})$. ■

3.11 Lemma. *(i) If $\phi, \psi \in C_c(\widehat{G})$, then $\phi * \psi = \hat{f}$ for some $f \in B^1(G)$.*
(ii) $A^1(\widehat{G}) = \{\hat{f} : f \in B^1(G) \subseteq L^1 \cap C_0(\widehat{G})\}$ is dense in $L^1(G)$.

Proof (i) As $\phi \in C_c(\widehat{G}) \subseteq L^1 \cap L^2(G)$, so $\check{\phi}$ and $U^* \phi$ make sense. Then for $f \in C_c(G) \subseteq L^1(G) \cap L^2(G)$, we have

$$\begin{aligned} \langle U f | \phi \rangle &= \int_{\widehat{G}} \int_G f(x) \overline{\sigma(x)} \, \mathrm{d}x \overline{\phi(\sigma)} \, \mathrm{d}\sigma \\ &= \int_G f(x) \overline{\int_{\widehat{G}} \phi(\sigma) \sigma(x) \, \mathrm{d}\sigma} \, \mathrm{d}x \\ &= \langle f | \check{\phi} \rangle \end{aligned}$$

Hence $\check{\phi} = U^* \phi$, so $\check{\psi} = U^* \psi$ as well. Let

$$f = (\phi * \psi)\check{} = \check{\phi}\check{\psi} \in L^1 \cap B(G) = B^1(G)$$

so the inversion theorem tells us that $(\hat{f})\check{} = f = (\phi * \psi)\check{}$ and injectivity of reverse transform tells us that $\hat{f} = \phi * \psi$.

(ii) By (i), it suffices to see that $C_c(\widehat{G}) * C_c(\widehat{G})$ is dense in $L^1(\widehat{G})$. Let $(k_\alpha)\alpha$ be a contractive summability kernel for $L^1(\widehat{G})$. Given $f \in L^1(G)$, $\epsilon > 0$, let α be so $\|k_\alpha * f - f\|_1 < \epsilon$. We let $\phi \psi \in C_c(G)$ satisfy $\left\| k_\alpha - \psi \right\|_1 < \epsilon$, so $\left\| f_\psi \right\|_1 < \epsilon$. One then sees that $\left\| \psi * \psi - f \right\|_1 < \epsilon(\|f\|_1 + \epsilon) + 2\epsilon$. ■

3.12 Theorem. (Pontryagin Duality) *The map $x \mapsto \hat{x} : G \to \widehat{\widehat{G}}$ is a bijective homeomorphism.*

Proof Let $\Gamma = \{\hat{x} : x \in G\} \subseteq \widehat{\widehat{G}}$.

We first show that $x \mapsto \hat{x} : G \to \Gamma$ is injective. Notice that $x \mapsto \hat{x}$ is a homomorphism. Let V be a symmetric relatively compact neighbourhood of e in G, and h_V as above. By the inversion theorem, for x in G,

$$h_V(x) = \int_{\widehat{G}} \widehat{h_V}(\sigma) \sigma(x) \, \mathrm{d}\sigma = \int_{\widehat{G}} \hat{h} \overline{\hat{x}} \, \mathrm{d}m_{\widehat{G}}.$$

If $x \neq e$, find V as above so $x \notin V^2$, hence $0 = h_V(x) = \int_{\widehat{G}} \hat{h}\overline{\hat{x}}\,\mathrm{d}m_{\widehat{G}}$, while $1 = h_V(e) = \int_{\widehat{G}} \hat{h}1_{\widehat{G}}\,\mathrm{d}m_{\widehat{G}}$. Thus $\hat{x} \neq \hat{e}$, i.e. $\ker(x \mapsto \hat{x}) = \{e\}$.

We now verify that $x \mapsto \hat{x} : G \to \Gamma$ is a homeomorphism. Let (x_α) be a net in G, $x_0 \in G$. Consider the convergences

(i) $x_\alpha \to x_0$ in G

(ii) $f(x_\alpha) \to f(x_0)$ for $f \in B^1(G)$

(iii) $\hat{x}_\alpha \to \hat{x}_0$ in $\Gamma \subseteq \widehat{\widehat{G}}$

$(i \Rightarrow ii)$ This follows since $B^1(G) \subseteq B(G) \subseteq C_b(G)$.

$(ii \Rightarrow i)$ Given a relatively compact symmetric neighbourhood V of e, let h_V be as above. Then $x_0 * h_V \in B^1(G)$. If (ii) holds, then

$$x_0 * h_V(x_\alpha) \to x_0 * h_V(x_0) = h_V(x_0^{-1}x_0) = h_V(e) = 1$$

so $x_0^{-1}x_\alpha \to e$ in G, i.e. $x_\alpha \to x_0$ in G, so (i) holds.

$(ii \Leftrightarrow iii)$ In $\widehat{\widehat{G}}$, the topology is relativized $w^* = \sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$. Let $\tau = \sigma(L^\infty(\widehat{G}), A^1(\widehat{G}))$, so $\tau \subseteq w^*$ but remains Hausdorff as $\overline{A^1(\widehat{G})}^{\|\cdot\|_1} = L^1(\widehat{G})$. Since $\widehat{\widehat{G}} \subseteq B(L^\infty(\widehat{G}))$ is a subspace of a compact space, we must have $w^*|_{\widehat{\widehat{G}}} = \tau|_{\widehat{\widehat{G}}}$. Now the inversion theorem gives for $f \in B^1(G)$, i.e. $\hat{f} \in A^1(\widehat{G})$, for $x \in G$

$$f(x) = \int_{\widehat{G}} \hat{f}(\sigma)\sigma(x)\,\mathrm{d}\sigma = \int_{\widehat{G}} \hat{f}\overline{\hat{x}}\,\mathrm{d}m_{\widehat{G}}$$

and hence they are equivalent.

By A1Q1, $\Gamma \cong G$ being a locally compact group, is closed in $\widehat{\widehat{G}}$.

Finally, we must show that Γ is dense in $\widehat{\widehat{G}}$. If not, there is some $\chi \in \widehat{\widehat{G}}$ and neighbourhood U of $1_{\widehat{G}}$ in $\widehat{\widehat{G}}$ such that $U^2\chi \cap \Gamma = \emptyset$. Hence if $\phi, \psi \in C_c^+(\widehat{\widehat{G}})$ with $\operatorname{supp}\phi \subseteq U$, $\operatorname{supp}\psi \subseteq U\chi$, then $\operatorname{supp}(\phi * \psi) \subseteq U^2\chi$. Hence $\phi * \psi \neq 0$, while $\phi * \psi|_\Gamma = 0$. By the last lemma, with $\widehat{G}$ playing the role of G, provides $f \in B^1(\widehat{G})$ so $\phi * \psi = \hat{f} : \widehat{\widehat{G}} \to \mathbb{C}$. Thus by the inversion theorem applied to $\widehat{G}$, for x in G, i.e. $\hat{x} \in \Gamma$,

$$\begin{aligned} 0 = \phi * \psi(\hat{x}) = \hat{f}(\hat{x}) &= \int_{\widehat{G}} f(sigma)\overline{(\hat{x}(\sigma)}\,\mathrm{d}\sigma \\ &= \int_G f(\sigma)\sigma(x)\,\mathrm{d}\sigma = \hat{f}(x) \end{aligned}$$

and hence $f = 0$ by injectivity of reverse transform. This contradiction shows that $\Gamma = \widehat{\widehat{G}}$.■

3.13 Corollary. *$f \mapsto \hat{f} : L^1(G) \to A(\widehat{G}) \subseteq C_0(\widehat{G})$ is injective.*

Proof If $f \in L^1(G)$,

$$\hat{f}(\sigma) = \int_G f(x)\overline{\sigma(x)}\,\mathrm{d}x = \int_G f(x)\hat{x}(\sigma)\,\mathrm{d}x = \int_{\widehat{\widehat{G}}} f(\chi)\chi(\sigma)\,\mathrm{d}\chi = \hat{f}(\sigma)$$

where $f \mapsto \check{f} : L^1(\widehat{\widehat{G}}) \to B(\widehat{G})$ is injective. ■

Remark. If $\mu \in M(G)$, define its **Fourier-Stieltjes transform**:

$$\hat{\mu} : \widehat{G} \to \mathbb{C}, \hat{\mu}(\sigma) = \int_G \overline{\sigma(x)}\, \mathrm{d}\mu(x).$$

Let $B(\hat{G}) = \{\hat{\mu} : \mu \in M(G)\}$ is an algebra of uniformly continuous on $\hat{G}$.

4 Compact Groups

5 Introduction to Amenability Theory