

# Representation Theory of Finite Groups

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# I. Introduction

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Let  $G$  be a finite group of order  $n$ , and write  $G = \{g_1, \dots, g_n\}$ . Fix  $g \in G$ ; then  $gg_i = gg_j$  if and only if  $i = j$ . Thus there exists some  $\sigma_g \in S_n$  such that  $gg_i = g_{\sigma_g(i)}$  for all  $i \in \{1, 2, \dots, n\}$ . In particular,  $\phi : G \rightarrow S_n$  by  $\phi(g) = \sigma_g$  is an embedding (injective group homomorphism). This observation is usually referred to as Cayley's Theorem.

Now let  $V$  be an  $n$ -dimensional complex vector space. We then denote  $\text{GL}(V)$  as the group of invertible linear operators  $T : V \rightarrow V$ . Now define  $\psi : S_n \rightarrow \text{GL}(V)$  by  $\psi(\sigma) = T_\sigma$  where if  $\{b_1, \dots, b_n\}$  is a basis for  $V$  and  $T_\sigma(b_i) = b_{\sigma(i)}$ . This is an injective group homomorphism, so  $\psi \circ \phi : G \rightarrow \text{GL}(V)$  is an embedding of  $G$  into  $\text{GL}(V)$ .

**Definition.** Let  $G$  be a finite group, and  $V$  a finite dimensional  $\mathbb{C}$ -vector space. A **representation** of  $G$  is a group homomorphism  $\rho : G \rightarrow \text{GL}(V)$ . We call  $\dim(V)$  the **degree** of the representation.

In particular, if  $V$  is  $n$ -dimensional, then  $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ .

*Example.* 1. Consider  $\rho : G \rightarrow \text{GL}(\mathbb{C}) \cong \mathbb{C}^\times$  given by  $\rho(g) = 1$  for all  $g \in G$ . This is called the *trivial representation*.

2. Consider  $\rho : S_n \rightarrow \mathbb{C}^\times$  given by  $\rho(\sigma) = \text{sgn}(\sigma)$ , which is called the *sign representation*.

3. The representation of  $G$  afforded by Cayley's theorem is called the *regular representation* of  $G$ . The next example is a good way to understand the regular rep of  $G$ .

4. Consider  $G$ ,  $X = \{x_1, \dots, x_n\}$ , and  $V = \text{Free}(X)$ . Suppose  $G$  acts on  $X$ . Then  $\rho : G \rightarrow \text{GL}(V)$  given by  $\rho(g)(x_i) = gx_i$ . In particular, if we take  $X = G$ , then this is the regular representation of  $G$ .

5. Consider the 4-gon, with vertices labelled  $a, b, c, d$ . Take  $X = \{a, b, c, d\}$  and the regular representation  $\rho : D_4 \rightarrow \text{GL}(V)$ . This action has a geometric notion.

6. Let  $C_n$  be a cyclic group of order  $n$ ; let us define some  $\rho : C_n \rightarrow \text{GL}(V)$ . Say  $\rho(x) = T$  where  $t \in \text{GL}(V)$ ; then this is a representation if and only if  $T^n = I$ .

**Definition.** We say that two representations  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  are **isomorphic** if there exists an isomorphism  $T : V \rightarrow W$  such that for all  $g \in G$ ,

$$T \circ \rho(g) = \tau(g) \circ T$$

Suppose  $\rho : G \rightarrow \text{GL}(V)$  and  $T : V \rightarrow W$  is an isomorphism. Then we can define  $\tau : G \rightarrow \text{GL}(W)$  by  $\tau(g) = T \circ \rho(g) \circ T^{-1}$ ; this  $\rho \cong \tau$ . In other words, the representation is unique up to isomorphism under change of basis.

*Example.* Consider  $G = \{g_1, \dots, g_n\} = \{h_1, \dots, h_n\}$ , and fix  $g \in G$ . Let  $gg_i = g_{\alpha(i)}$  and  $gh_i = h_{\beta(i)}$  where  $\alpha, \beta \in S_n$ . Fix an  $n$ -dimensional vector space  $V$  with basis  $\{b_1, \dots, b_n\}$ . Then two regular representations are given by

$$\rho_1 : G \rightarrow \text{GL}(V), \rho_1(g)(b_i) = b_{\alpha(i)}$$

$$\rho_2 : G \rightarrow \text{GL}(V), \rho_2(g)(b_i) = b_{\beta(i)}$$

Let  $\gamma \in S_n$  be such that  $h_{\gamma(i)} = g_i$ , and define  $T : V \rightarrow V$  by  $T(b_i) = b_{\gamma(i)}$ . Then

$$gg_i = g_{\alpha(i)} = gh_{\gamma(i)} = h_{\beta\gamma(i)} = g_{\gamma^{-1}\beta\gamma(i)}$$

so that  $\alpha = \gamma^{-1}\beta\gamma$ . Thus for each  $b_i$ ,

$$\begin{aligned} T \circ \rho_1(g) \circ T^{-1}(b_i) &= T \circ \rho_1(g)(b_{\gamma^{-1}(i)}) \\ &= T(b_{\alpha\gamma^{-1}(i)})b_{\gamma\alpha\gamma^{-1}(i)} \\ &= b_{\beta(i)} = \rho_2(g)(b_i) \end{aligned}$$

so that  $T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$ .

Note: conjugate elements have the same cycle type.

## SUBREPRESENTATIONS

What should a subrepresentation of  $\rho : G \rightarrow \text{GL}(V)$  mean?

We would like a subspace  $W \leq V$  such that  $\tau : G \rightarrow \text{GL}(W)$  is a representation given by  $\tau(g)(w) = \rho(g)(w)$  for all  $w \in W$ . Moreover, to make this well-defined, we need  $W$  to be  $\rho(g)$ -invariant for every  $g \in G$  ( $\rho(g)(W) \subseteq W$ ).

Suppose  $T : V \rightarrow V$  is a linear operator, and  $W \leq V$  is a  $T$ -invariant subspace; i.e.  $T(W) \subseteq W$ . In particular, the restriction operator  $T_W : W \rightarrow W$  is well-defined.

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. A subspace  $W \subseteq V$  is said to be  **$G$ -stable** if  $W$  is  $\rho(g)$ -invariant for all  $g \in G$ . A **subrepresentation** of  $\rho$  is a representation  $\rho_W : G \rightarrow \text{GL}(W)$  where for all  $g \in G$  and  $w \in W$ ,  $\rho_W(g)(w) = \rho(g)(w)$  where  $W$  is a  $G$ -stable subspace of  $V$ .

*Example.* Suppose  $\rho : G \rightarrow \text{GL}(V)$  be the regular representation. Take  $W = \text{span}\{\sum_{g \in G} v_g\}$ , which is clearly  $G$ -stable, and  $\rho_W : G \rightarrow \text{GL}(W)$  is isomorphic to the trivial representation.

Similarly, let  $\rho : S_n \rightarrow \text{GL}(V)$  be the regular representation,  $W = \text{span}\{\sum_{\sigma \in S_n} \text{sgn}(\sigma)v_\sigma\}$ ; this is isomorphic to the sign representation.

**0.1 Theorem.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation,  $W \leq V$   $G$ -stable. Then there exists a  $G$ -stable subspace  $W'$  such that  $V = W \oplus W'$ .

**PROOF** Take any inner product  $\langle x, y \rangle$  on  $V$ . Then for any  $x, y \in V$ , define

$$\langle x, y \rangle^* = \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle$$

This is also an inner product. Let  $x, y \in V$  and let  $h \in G$ . Then

$$\begin{aligned} \langle \rho(h)(x), \rho(h)(y) \rangle^* &= \sum_{g \in G} \langle \rho(g)\rho(h)(x), \rho(g)\rho(h)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(gh)(x), \rho(gh)(y) \rangle \\ &= \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle \end{aligned}$$

Thus every  $\rho(h)$  is unitary with respect to  $\langle \cdot, \cdot \rangle^*$ . Let  $W \leq V$  be  $G$ -stable, and take  $W' = W^\perp$  with respect to  $\langle \cdot, \cdot \rangle^*$ . Then  $V = W \oplus W'$ . Let's see that  $W^\perp$  is  $G$ -stable. Let  $x \in W^\perp$ ,  $w \in W$ ,

and  $g \in G$ , so that

$$\begin{aligned} \langle \rho(g)(x), w \rangle^* &= \langle x, \rho(g)^*(w)^* \rangle = \langle x, \rho(g)^{-1}(w) \rangle^* \\ &= \langle x, \underbrace{\rho(g^{-1})(w)}_{\in W} \rangle^* \\ &= 0 \end{aligned}$$

and  $\rho(g)(W^\perp) \subseteq W^\perp$  as required.  $\blacksquare$

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation, and  $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$  where each  $W_i$  is  $G$ -stable. For each  $i$ , let  $\rho_i = \rho|_{W_i}$ . For each  $v = \sum w_i \in V$ , we have  $\rho(g)(v) = \sum \rho(g)(w_i) = \sum \rho_i(g)(w_i)$ . In this case, we write

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$$

and call  $\rho$  a direct sum of the  $\rho_i$ 's.

The previous definition is written as an internal direct sum of  $V$ . Externally, given vector spaces  $W_1, \dots, W_k$  and representations  $\rho_i : G \rightarrow \text{GL}(W_i)$ , we can define

$$(\rho_1 \oplus \cdots \oplus \rho_k) : G \rightarrow \text{GL}(W_1 \oplus \cdots \oplus W_k)$$

by  $(\rho_1 \oplus \cdots \oplus \rho_k)(g)(w_1, \dots, w_k) = (\rho_1(g)(w_1), \dots, \rho_k(g)(w_k))$ . If  $\rho_i : G \rightarrow \text{GL}(W_i)$  is a subrepresentation for  $\rho : G \rightarrow \text{GL}(V)$ , we often say “ $W_i$  is a subrepresentation of  $V$ ”.

**Definition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. We say  $\rho$  is **irreducible** if  $V \neq \{0\}$  and the only  $G$ -stable subspaces of  $V$  are  $\{0\}$  and  $V$ .

Clearly,

**0.2 Theorem.** Every representation  $\rho : G \rightarrow \text{GL}(V)$  can be written as a direct sum of irreducible sub-representations.

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(\mathbb{C}^3)$  be the permutation representation with respect to the standard basis  $\{e_1, e_2, e_3\}$ . Consider  $W_1 = \text{span}\{e_1 + e_2 + e_3\}$  and  $W_2 = \text{span}\{e_1 - e_2, e_2 - e_3\}$ . Is  $W_2$  irreducible?

More generally, if  $V = W_1 \oplus \cdots \oplus W_k$  and  $\dim W_i = 1$  and  $\deg(\rho_i) = 1$ ,

$$\rho(gh)(\sum w_i) = \sum \rho_i(gh)(w_i) = \sum \rho_i(g)\rho_i(h)(w_i) = \sum \rho_i(h)\rho_i(g)(w_i)$$

so that  $\rho(gh) = \rho(hg)$ . In our example, this does not happen, since  $\rho(g) \neq I$  when  $g \neq 1$  and  $S_3$  is not abelian.

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(V)$  be the regular representation. Let  $W_1 = \text{span}\{\sum_{\sigma \in S_3} v_\sigma\}$  and  $W_2 = \text{span}\{\sum_{\sigma \in S_3} \text{sgn}(\sigma)v_\sigma\}$ , and Now let's focus on  $W_3$ . A basis for  $W_3$  is given by

$$\begin{aligned} e_1 &= v_e - v_{(123)} & e_2 &= v_e - v_{(123)} \\ e_3 &= v_{(12)} - v_{(13)} & e_4 &= v_{(12)} - v_{(23)} \end{aligned}$$

Recall that  $S_3 = \langle (12), (123) \rangle$ ; suffices to show stability with respect to generators.

$$\begin{aligned} \rho(12) : e_1 &\mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto e_2, e_4 \mapsto e_1 \\ \rho(123) : e_1 &\mapsto e_2 - e_1, e_2 \mapsto -e_1, e_3 \mapsto e_4 - e_3, e_4 \mapsto -e_3 \end{aligned}$$

Let  $U_1 = \text{span}\{e_1 - e_4, e_2 + e_3 - e_1\}$

## 1 TENSOR PRODUCTS

Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be representations. We define the representation  $\rho \otimes \tau : G \rightarrow \text{GL}(V \otimes W)$

$$(\rho \otimes \tau)(g)(v \otimes w) = \rho(g)(v) \otimes \tau(g)(w)$$

## 2 CHARACTER THEORY

We define the character of  $\rho$  by  $\rho : G \rightarrow \mathbb{C}$  as  $\chi(G) = \text{Tr}(\rho(g))$ .

*Remark.* If we choose a basis  $\beta$  for  $V$ , then define  $A(g) = [\rho(g)]_\beta$  and  $\chi(G)$  is given by the sum of the diagonal entries of  $A(g)$ . Furthermore, if  $A, B \in M_n(\mathbb{C})$ , then  $\text{Tr}(AB) = \text{Tr}(BA)$ .

The remark implies a number of facts:

- (i)  $\rho \cong \tau$ , then  $\text{Tr}(\rho(g)) = \text{Tr}(\tau(g))$ .
- (ii)  $\text{Tr}(T)$  is the sum of eigenvalues of  $T$
- (iii)  $\chi(1) = \dim(V)$ .

**2.1 Proposition.** *For every  $g \in G$  the eigenvalues of  $\rho(g)$  have modulus 1. In particular,  $\chi(g^{-1}) = \overline{\chi(g)}$ .*

**PROOF** Set  $n = |G|$ ; then  $\rho(g)^n = \rho(g^n) = I$  so that  $\lambda^n - 1 = 0$  for any eigenvalue  $\lambda$ , so  $|\lambda| = 1$ . Furthermore,

$$\overline{\chi(g)} = \overline{\sum \lambda_i} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \chi(g^{-1})$$

proving the second component. ■

**2.2 Proposition.** *Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$ . Then  $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$  and  $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$ .*

**PROOF** Let  $\beta_1 = \{v_1, \dots, v_n\}$  be a basis for  $V$  and  $\beta_2 = \{w_1, \dots, w_m\}$  a basis for  $W$ .

Then a basis for  $V \oplus W$  is given by  $\beta = \{(v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)\}$ . In particular,

$$[(\rho \oplus \tau)(g)]_\beta = \begin{pmatrix} [\rho(g)]_{\beta_1} & \\ & [\tau(g)]_{\beta_2} \end{pmatrix}$$

and the trace result follows.

A basis for  $V \otimes W$  is given by  $\gamma = \{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$  in lexicographic order. Fix  $g \in G$ , and set  $A = [\rho(g)]_{\beta_1}$ ,  $B = [\tau(g)]_{\beta_2}$ . Fix  $v_i \otimes w_j \in \gamma$ . Then

$$\begin{aligned} (\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1i}v_1 + \dots + a_{ni}v_n) \otimes (b_{1j}w_1 + \dots + b_{mj}w_m) \\ &= \dots + a_{ii}b_{jj} \cdot (v_i \otimes w_j) + \dots \\ &= \text{Tr}([\rho \otimes \tau](g))_\delta = \sum_{i,j} a_{ii}b_{jj} = \text{Tr}(A)\text{Tr}(B) = \chi_\rho(g) \cdot \chi_\tau(g) \end{aligned} \quad \blacksquare$$



*Example.* Suppose  $\rho : S_n \rightarrow \text{GL}(\mathbb{C}^n)$  is the permutation representation with respect to  $\{e_1, \dots, e_n\}$ . Then  $\chi(\sigma) = |\{e_i : \rho(\sigma)(e_i) = e_i\}| = |\text{Fix}(\sigma)|$ , which is the number of indices  $i$  fixed by  $\sigma$ . Since  $S_n$  acts transitively on  $\{1, \dots, n\}$ , there is exactly 1 orbit, so by Burnside's lemma,

$$n! = |S_n| = \sum_{\sigma \in S_n} \chi(\sigma)$$

*Example.* Let  $\rho : G \rightarrow \text{GL}(V)$  be the regular representation. Note that if  $g \neq 1$ , then for all  $h \in G$ ,  $gh \neq h$ . In particular, this means that  $\chi(g) = 0$  if  $g \neq 1$ , and  $\chi(1) = |G|$  (the dimension of  $V$ ).

*Example.* Let  $\rho : S_3 \rightarrow \text{GL}(V)$  be the regular representation. Recall that  $V = W_1 \oplus W_2 \oplus U_1 \oplus U_2$  where  $W_1$  is the trivial representation,  $W_2$  is the sign representation, and  $U_1, U_2$  are isomorphic. Let  $S_3 = \langle (12), (123) \rangle$ ; then we have

$x_1$	1	1
$x_2$	-1	1
$x_3$	$a$	$b$
$x_4$	$a$	$b$

In particular,  $\chi(12) = 1 - 1 + 2a = 0$  and  $\chi(123) = 1 + 1 + 2b = 0$ , so  $b = -1$ .

*Example.* Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. In particular,  $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)$  so that  $\text{Tr} \rho(ghg^{-1}) = \text{Tr} \rho(h)$  so  $\chi(ghg^{-1}) = \chi(h)$ ; in other words, that characters are constant on conjugacy classes.

**2.3 Lemma. (Schur)** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible representations, and suppose  $T : V \rightarrow W$  is linear such that for all  $g \in G$ ,  $\tau(g) \circ T = T \circ \rho(g)$ . Then either  $T = 0$  or  $T$  is an isomorphism and  $\rho \cong \tau$ . Moreover, if  $V = W$  and  $\rho = \tau$ , then  $T$  is a scalar multiple of the identity.

**PROOF** Assume  $T \neq 0$ .

Let's first see that  $T$  is injective, and let  $v \in \ker(T)$ . Then for any  $g \in G$ ,  $T(\rho(g)(v)) = \tau(g)(T(v)) = 0$ , so  $\rho(g)(v) \in \ker(T)$ . Thus  $\ker(T)$  is  $G$ -stable (with respect to  $\rho$ ). Since  $\rho$  is irreducible and  $T \neq 0$ ,  $\ker(T) = \{0\}$ .

We also have that  $T$  is surjective. Let  $v \in \text{Im}(T)$  and say  $v = T(X)$  with  $x \in V$ . Then for  $g \in G$ ,  $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \text{Im}(T)$  so  $\text{Im}(T)$  is  $G$ -stable, and again by irreducibility of  $\tau$ ,  $\text{Im}(T) = W$ . Thus  $T$  is an isomorphism.

Now let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $T$  and consider  $T' = T - \lambda I$ . Now, note that for  $g \in G$ ,  $\rho(g)T' = T'\rho(g)$ , but  $T'$  has non-trivial kernel, so in fact  $T' = 0$ . ■

**2.4 Corollary.** Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible, and  $T : V \rightarrow W$  linear. Consider

$$T' = \frac{1}{|G|} \sum_{g \in G} \tau(g)^{-1} T \rho(g)$$

Then

- (i) If  $T' \neq 0$ , then  $\rho \cong \tau$  via  $T'$ .
- (ii) If  $V = W$ ,  $\rho = \tau$ , then  $T' = \text{Tr}(T)/\dim(V) \cdot I$ .

PROOF Clearly  $T' : V \rightarrow W$  is linear, and for any  $h \in G$ ,

$$\begin{aligned}\tau(h)T' &= \tau(h) \frac{1}{|H|} \sum_{g \in G} \tau(g^{-1})T\rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(hg^{-1})T\rho(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1})T(\rho(g)) \\ &= \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1})T\rho(g)\rho(h) \\ &= T'\rho(h)\end{aligned}$$

If  $V = W$  and  $\rho = T$ , then  $\text{Tr}(T') = \frac{1}{|G|} \text{Tr}(T) \cdot |G| = \text{Tr}(T) = \alpha \dim(V)$ , so  $\alpha = \text{Tr}(T)/\dim(V)$ . ■

Let  $\rho : G \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$  be irreducible representations, and  $T : V \rightarrow W$  linear. Let  $\beta$  be a basis for  $V$  and  $\gamma$  a basis for  $W$ . Then for  $g \in G$ , let  $[\rho(g)]_\beta = (a_{ij}(g))$ ,  $[\tau(g)]_\gamma = (b_{kl}(g))$ ,  $[T]_\beta^\gamma = (x_{ki})$ , and  $[T']_\beta^\gamma = (x'_{ki})$ .

By matrix multiplication,  $x'_{ki} = \frac{1}{|G|} \sum_g \sum_{j,l} b_{kl}(g^{-1})x_{lj}a_{ji}(g)$ . If  $\rho \not\cong \tau$ , then  $T' = 0$ , so by viewing the RHS as a polynomial in the  $x_{ij}$ , we have

$$\frac{1}{|G|} \sum_g b_{kl}(g^{-1})a_{ji}(g) = 0$$

But now if  $\rho = \tau$ , then  $T' = \lambda I$  where  $\lambda = \text{Tr}(T)/\dim(V)$  so that

$$\frac{1}{|G|} \sum_g \sum_{j,l} a_{kl}(g^{-1})x_{lj}a_{ji}(g) = \lambda \delta_{ki} = \frac{1}{\dim(V)} \sum_{j,l} \delta_{ki} \delta_{jl} x_{lj}$$

Then by equating coefficients of  $x_{lj}$ , we have

$$\frac{1}{|G|} \sum_g a_{kl}(g^{-1})a_{ji}(g) = \frac{1}{\dim(V)} \delta_{ki} \delta_{jl}$$

*Remark.* If  $G$  is a finite group, then consider the vector space of all functions  $\phi : G \rightarrow \mathbb{C}$ . For any  $\phi, \psi$  in this vector space,  $\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_g \phi(g) \overline{\psi(g)}$  defines an inner product. Then if  $\chi_1, \chi_2$  are characters of  $G$ , then

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \overline{\chi_2(g)}$$

We thus have:

**2.5 Theorem.** If  $\chi$  is a character of an irreducible representation, then  $\langle \chi, \chi \rangle = 1$ , and if  $\chi_1$  and  $\chi_2$  correspond to non-isomorphic representations, then  $\langle \chi_1, \chi_2 \rangle = 0$ .

PROOF Say  $[\rho(g)]_\beta = (a_{ij}(g))$  where  $\rho$  is an irreducible representation with character  $\chi$ . Then

$$\begin{aligned} \langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_g \chi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_g \chi(g^{-1}) \chi(g) \\ &= \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g^{-1}) a_{jj}(g) = \sum_{i,j} \left( \frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{jj}(g) \right) \\ &= \sum_{i,j} \left( \frac{1}{|G|} \sum_g a_{ii}(g^{-1}) a_{ii}(g) \right) \\ &= \sum_i \frac{1}{\dim(V)} = 1 \end{aligned}$$

To see the second part,

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \chi_2(g^{-1}) = \frac{1}{|G|} \sum_g \sum_{i,j} a_{ii}(g) a_{jj}(g^{-1}) = \sum_{i,j} 0 = 0 \quad \blacksquare$$

If  $\chi$  is a character corresponding to an irreducible representation, we say  $\chi$  is irreducible. If  $\rho$  and  $\tau$  are isomorphic representations, we say  $\chi_\rho$  and  $\chi_\tau$  are isomorphic (in fact  $\chi_\rho = \chi_\tau$ ).

**2.6 Corollary.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation with character  $\chi$ . Say  $V = W_1 \oplus \cdots \oplus W_k$  is an irreducible decomposition of  $V$ . If  $\tau : G \rightarrow \text{GL}(W)$  is an irreducible representations with character  $\phi$ , then the number of  $W_i$  isomorphic to  $W$  (i.e.  $\rho_i \cong \tau$ ) is  $\langle \chi, \phi \rangle$ .

PROOF Write  $\chi = n_1 \chi_1 + \cdots + n_l \chi_l$ , where the  $\chi_i$  are pairwise non-isomorphic. Then  $\langle \chi, \chi_i \rangle = n_i$ . ■

Let  $\tau : G \rightarrow \text{GL}(V)$  be irreducible, and let  $\tau$  have character  $\phi$ . Then

$$\langle \chi, \phi \rangle = \sum_{i=1}^k \langle \chi_i, \phi \rangle$$

Now,  $\langle \chi_i, \phi \rangle = 1$  if and only if  $\rho_i \cong \tau$ , so that  $\langle \chi, \phi \rangle$  counts the number of times in which  $\tau$  appears in the irreducible decomposition of  $\rho$ .

**2.7 Corollary.** If two representations of  $G$  have the same character, then they are isomorphic.

PROOF They have the same irreducible decomposition. ■

**2.8 Corollary.** If  $\rho : G \rightarrow \text{GL}(V)$  is a representation and  $\chi$  is a character, then  $\langle \chi, \chi \rangle \in \mathbb{N}$  and  $\langle \chi, \chi \rangle = 1$  if and only if  $\chi$  is irreducible.

PROOF If  $\chi_1, \dots, \chi_k$  are irreducible, write  $\chi = n_1 \chi_1 + \cdots + n_k \chi_k$  so that  $\langle \chi, \chi \rangle = n_1^2 + \cdots + n_k^2 \in \mathbb{N}$ . ■

**2.9 Proposition.** Every irreducible representation of  $G$  occurs as a subgroup for the regular representation of  $G$ , with multiplicity equal to its degree.

PROOF Let  $\chi$  be an irreducible character of  $G$ . Then

$$\langle \chi, \chi_{\text{reg}} \rangle = \frac{1}{|G|} \sum_g \chi(g) \overline{\chi_{\text{reg}}(g)} = \frac{1}{|G|} \chi(1) \overline{\chi_{\text{reg}}(1)} = \frac{1}{|G|} \deg(\chi) \quad \blacksquare$$

**2.10 Corollary.** Let  $\chi_1, \dots, \chi_k$  be the distinct irreducible characters of  $G$ , with  $\deg(\chi_i) = n_i$ . Then  $\sum n_i^2 = |G|$  for  $g \neq 1$ ,  $\sum_{i=1}^k n_i \chi_i(g) = 0$

PROOF Recall that  $\chi_{\text{reg}} = n_1 \chi_1 + \dots + n_k \chi_k$ . Then  $\chi_{\text{reg}}(1) = |G| = n_1^2 + \dots + n_k^2$ , and evaluation at  $g \neq 1$  gives the desired result.  $\blacksquare$

**Definition.** Let  $G$  be a group. A function  $f : G \rightarrow \mathbb{C}$  is called a class function if  $f$  is constant on each conjugacy class, i.e. for all  $a, b \in G$ ,  $f(bab^{-1}) = f(a)$ .

**2.11 Proposition.** Let  $\rho : G \rightarrow \text{GL}(V)$  be a representation. Then

$$\rho_f = \sum_g f(g) \rho(g)$$

is a linear operator on  $V$ . If  $\rho$  is irreducible of degree  $n$ , then  $\rho_f = \lambda I$ , where  $\lambda = \frac{|G|}{n} \langle f, \bar{\chi} \rangle$  where  $\chi$  is the character of  $\rho$ .

PROOF Note that

$$\begin{aligned} \rho_f \circ \rho(h) &= \sum_g f(g) \rho(g) \rho(h) = \sum_g f(g) \rho(gh) \\ &= \sum_g f(hgh^{-1}) \rho(hg) \\ &= \sum_g f(g) \rho(h) \rho(g) = \rho(h) \circ \rho_f \end{aligned}$$

so by Schur,  $\rho_f = \lambda I$  where  $\lambda = \text{Tr}(\rho_f)/n$ . However,  $\text{Tr}(\rho_f) = \text{Tr}(\sum_g f(g) \rho(g)) = \sum_g f(g) \chi(g) = |G| \langle f, \bar{\chi} \rangle$ .  $\blacksquare$

Recall that

- $\langle \chi, \chi \rangle = 1$  if and only if  $\chi$  is irreducible
- If  $\chi_\rho$  and  $\chi_\tau$  are irreducible then  $\langle \chi_\rho, \chi_\tau \rangle = 0$  if  $\rho \not\cong \tau$ , and 1 otherwise.
- If  $\chi'$  is an irreducible subrepresentation of  $\chi$ , then  $\langle \chi, \chi' \rangle$  is the multiplicity of  $\chi'$  in  $\chi$ .
- $|G| = n_1^2 + \dots + n_k^2$  where  $n_i$  is the multiplicity of  $\chi_i$  as an irreducible subrepresentation of the regular representation.
- Every irreducible character is a character of some subrepresentation of the regular rep?
- ... every irreducible representation is a subrepresentation of the regular rep?

and

$$\rho_f = \sum_g f(g)\rho(g) = \lambda I$$

where  $\lambda = |G|/\dim(V) \cdot \langle f, \bar{\chi} \rangle$ .

**2.12 Proposition.** *Let  $G$  be a group. The irreducible characters of  $G$  form an orthonormal basis for the vector space of class functions on  $G$ .*

**PROOF** Let  $\beta = \{\chi_1, \dots, \chi_k\}$  be the irreducible characters of  $G$ . We know that  $\beta$  is orthonormal, and hence linearly independent. Let  $W = \text{span}(\beta)$ . To show  $W = V$  where  $V$  is the space of class functions, we prove that  $W^\perp = \{0\}$ . Let  $f \in W^\perp$ , and suppose  $\rho : G \rightarrow \text{GL}(V)$  is irreducible. By A2,  $\bar{\chi}_1, \dots, \bar{\chi}_k$  are all irreducible characters of  $G$ . Thus  $\rho_f = 0$ . By considering irreducible decompositions,  $\rho_f = 0$  for all representations  $\rho : G \rightarrow \text{GL}(V)$ . In particular, when  $\rho$  is the regular representation,

$$0 = \rho_f(v_1) = \sum_g f(g)\rho(g)(v_1) = \sum_g f(g)v_g$$

so by independence of  $\{v_g : g \in G\}$ ,  $f(g) = 0$  for all  $g \in G$ . ■

**2.13 Corollary.** *The number of irreducible characters of  $G$  is equal to the number of conjugacy classes of  $G$ .*

**PROOF** Let  $C_1, \dots, C_k$  be the conjugacy classes. Then a basis for  $V_{\text{class}} = \{\phi_1, \dots, \phi_k\}$  where each  $\phi_i$  is the indicator for  $C_i$ . Since bases must have the same size, the result follows. ■

**2.14 Proposition.** *Let  $G$  be a group,  $g \in G$ , and  $O_g$  the conjugacy class of  $g$ . Let  $\chi_1, \dots, \chi_k$  be the irreducible characters of  $G$ . Then*

1.  $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$
2. If  $h$  is not conjugate to  $g$ , then  $\sum_{i=1}^k \chi_i(g)\overline{\chi_i(h)} = 0$ .

**PROOF** Define  $\phi : G \rightarrow \mathbb{C}$  where  $\phi(x)$  is the indicator function for  $O_g$ . Write  $\phi = \sum_{i=1}^k \lambda_i \chi_i$  where

$$\lambda_i = \langle \phi, \chi_i \rangle = \frac{1}{|G|} \sum_x \phi(x)\overline{\chi_i(x)} = \frac{|O_g|\overline{\chi_i(g)}}{|G|}$$

Therefore,

$$\phi(x) = \frac{|O_g|}{|G|} \sum_{i=1}^k \overline{\chi_i(g)} \chi_i(x)$$

Then the result follows by evaluating  $\phi$  at  $g$  and  $h$ . ■

*Example.* Let's compute the character table of  $S_3$ . There are 2 degree 1 representations, and 3 irreducible characters since there are three conjugacy classes (cycle types). In particular,  $|S_3| = 6 = 1^2 + 1^2 + n_3^2$ , so  $n_3 = 2$ .

	$\epsilon$ $(12)$ $(123)$
$(\text{triv})\chi_1$	$1$ $1$ $1$
$(\text{sgn})\chi_2$	$1$ $-1$ $1$
$\chi_3$	$2$ $a$ $b$

Note that the columns must be orthogonal, so by the previous proposition, we have  $a = 0$  and  $b = -1$ .

Let  $\chi_1, \dots, \chi_k$  be the irreducible characters of  $G$ . Then  $\sum_{g \in G} \chi_i(g) = |G|$  and  $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|O_g|$ .

Let  $G$  be abelian. By A1,  $G$  has  $|G|$  representations of degree 1, and  $[G : [G, G]] = |G|$ . Since  $G$  has  $|G|$  conjugacy classes, these are all of the irreducible representations of  $G$ . Suppose  $G$  is a group whose irreducible representations are all degree one. Since  $n_1^2 + \dots + n_k^2 = |G|$ , then  $k = |G|$ .

**2.15 Proposition.** *Let  $H$  be an abelian subgroup of  $G$ . Then any irreducible representation of  $G$  has degree at most  $[G : H]$ .*

PROOF Let  $\rho : G \rightarrow \text{GL}(V)$  be an irreducible representation of  $G$ . Consider the restriction  $\tilde{\rho} : H \rightarrow \text{GL}(V)$ . Let  $W \leq V$  be an irreducible subrepresentation of  $\tilde{\rho}$ . Since  $H$  is abelian,  $\dim W = 1$ . Suppose  $W = \text{span}\{x\}$ , and let  $W' = \{\rho(g)(x) : g \in G\}$  so that  $W'$  is  $G$ -stable, and in fact  $W' = V$  since  $\rho$  is irreducible.

Take  $g \in G$  and  $h \in H$ , so  $\rho(gh) = \rho(g)\rho(h)(x) = \rho(g)(\alpha x) = \alpha\rho(g)(x)$ . Say  $g_1, \dots, g_m$  are coset representatives of  $H$  in  $G$ . Then  $V = W' = \text{span}\{\rho(g_i)(x) : 1 \leq i \leq m\}$ , then  $\dim(V) \leq m = [G : H]$ . ■

*Example.* Consider  $D_4$ . Then the number of degree 1 representations is  $[D_4 : \langle r^2 \rangle] = 4$ . Since there are 5 conjugacy classes, we know that there are 5 irreducible representations, so that  $n_5^2 = 8$ . Let's make the character table:

$D_4$	1	$r$	$r^2$	$s$	$rs$
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	1	1	1	-1	-1
$\chi_4$	1	-1	1	-1	1
$\chi_5$	2	$a$	$b$	$c$	$d$

But then by column orthogonality, we have  $a = 0$ ,  $b = -2$ ,  $c = 0$ ,  $d = 0$ .

*Example.* Consider  $S_4$ . Then  $[S_4 : A_4] = 2$  so there are two degree 1 representations (the trivial and the sign), and the conjugacy classes are given by 1, (12), (12)(34), (123), (1234), so there are 5 irreducible representations. Since  $24^2 = 1^2 + 1^2 + n_3^2 + n_4^2 + n_5^2$ , we have  $22 = n_3^2 + n_4^2 + n_5^2$ , which forces  $n_3 = 2$  and  $n_4 = n_5 = 3$ . Now we have

$D_4$	1	(12)	(12)(34)	(123)	(1234)
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	2	1	1	-1	-1
$\chi_4$	3	-1	1	-1	1
$\chi_5$	3	$a$	$b$	$c$	$d$

Note that  $K = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$  and  $H = \{1, (12), (13), (123), (132), (23)\}$ , so  $S_4 = KH$ . Let  $\rho$  be an irreducible representation of  $H$  of degree 2:

$S_3$	1	(12)	(123)
$\alpha_1$	1	1	1
$\alpha_2$	1	-1	1
$\alpha_3$	2	0	-1

Then  $\rho : S_4 \rightarrow \text{GL}(V)$  by  $\rho(kh) := \rho(h)$  is an irreducible representation of  $S_4$  since  $K \trianglelefteq S_4$ .

### 3 INDUCED REPRESENTATIONS

Given a subgroup  $H \leq G$  and a representation  $\rho : H \rightarrow \text{GL}(V)$ , construct a representation of  $G$ . Let  $H \leq G$  and  $\rho : H \rightarrow \text{GL}(V)$  a representation. Say the cosets of  $H$  in  $G$  are  $g_1H, \dots, g_mH$ . For each  $i$ , let  $g_iV = \{g_iv : v \in V\}$  be an isomorphic copy of  $G$ , and let  $W = \bigoplus_{i=1}^m g_iV$  so that every  $w \in W$  can be uniquely written as  $w = g_1v_1 + \dots + g_mv_m$ , where  $m = [G : H]$ . Fix  $g \in G$ ; then there exists  $\pi \in S_m$  such that for every  $i$ ,  $gg_i = g_{\pi(i)}h_i$ ,  $h_i \in H$ . We then define  $\text{Ind}_H^G(\rho) : G \rightarrow \text{GL}(W)$  by

$$\text{Ind}_H^G(\rho)(g)\left(\sum g_iw_i\right) = \sum g_{\pi(i)}\rho(h_i)v_i$$

*Example.* Let  $\{1\} \leq G$  and suppose  $\rho : \{1\} \rightarrow \text{GL}(\mathbb{C})$  is the trivial representation. Then  $G = \{g_1, \dots, g_n\}$ . Then for  $g \in G$ ,  $gg_i1 \in G$  and

$$\text{Ind}(\rho)(s)\left(\sum_{i=1}^n g_i\alpha_i\right) = \sum g_i\rho(1)(\alpha_i) = \sum g_i\alpha_i$$

so that  $\text{Ind}(\rho)$  is isomorphic to the regular representation.

*Example.* Consider  $\langle r \rangle \leq D_n$ , and let  $\rho : \langle r \rangle \rightarrow \text{GL}(\mathbb{C})$  be given by  $\rho(r)(1) = \zeta_n$ . Let the coset representatives be given by  $\epsilon$  and  $s$ .

- (i) Let  $r \in D_n$  so  $r\epsilon = \epsilon r$  and  $rs = sr^{n-1}$ . Fix  $W = \epsilon\mathbb{C} \oplus s\mathbb{C}$ . Then  $\text{Ind}(\rho) : D_n \rightarrow \text{GL}(W)$  is given by  $\text{Ind}(\rho)(r)(\epsilon\alpha_1 + s\alpha_2) = \epsilon\zeta_n\alpha_1 + 1 + s\zeta_n^{n-1}\alpha_2$ .
- (ii) Let  $s \in D_n$ . Then  $s\epsilon = \epsilon s$  and  $ss = \epsilon\epsilon$ . Then  $\text{Ind}(\rho)(s)(\epsilon\alpha_1 + s\alpha_2) = s\rho(\epsilon)(\alpha_1) + \epsilon\rho(\epsilon)(\alpha_2) = s\alpha_1 + \epsilon\alpha_2$ .

Take the basis  $\beta = \{\epsilon, s\}$  for  $W$ , so we have

$$[\text{Ind}(\rho)(r)]_\beta = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{n-1} \end{pmatrix} \quad [\text{Ind}(\rho)(s)]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

### 4 NON-COMMUTATIVE MODULE THEORY

Let  $R$  be a ring with unity and  $(M, +)$  an abelian group. We can equip  $\text{End}(M)$  with a ring structure given by  $(f + g)(x) = f(x) + g(x)$  and  $fg(x) = f(g(x))$ .

**Definition.** A (left)  $R$ -module is an abelian group  $(M, +)$  equipped with a unitary ring homomorphism  $\alpha : R \rightarrow \text{End}(M)$ .

This map  $\alpha$  defines a multiplication between elements of  $r$  and  $m$  given by  $rm = \alpha(r)(m)$ .

*Example.* (i) If  $F$  is a field, a  $F$ -module is a  $F$ -vector space.

(ii)  $M$  is a  $\mathbb{Z}$ -module if and only if  $M$  is an abelian group.

(iii)  $R$  is an  $R$ -module (left multiplication)

(iv) If  $I$  is a left ideal of  $R$ , then  $I$  is a left  $R$ -module.

(v)  $R = M_n(F)$ , and  $V = F^n$ . Then  $V$  is an  $R$ -module.

(vi) Let  $R$  be a ring and  $I$  a left ideal of  $R$ . Then  $R/I = \{a+I : a \in R\}$ , so  $R/I$  is an  $R$ -module with  $r(a+I) = ra+I$ .

Let  $M$  be an  $R$ -module. We say a subgroup  $(N, +)$  of  $(M, +)$  is an  $R$ -submodule of  $M$  if  $N$  is  $\alpha(r)$ -invariant for each  $r \in R$ .

**Definition.** Let  $G$  be a finite group and  $F$  a field. We define the group algebra  $F[G] = \{\alpha_1 g_1 + \cdots + \alpha_n g_n : \alpha_i \in F\}$  equipped with  $G$ -pointwise addition and multiplication  $ag_i \cdot bg_j = (ab)g_i g_j$ , extended by distributivity.

*Example.* Let  $M$  be a  $\mathbb{C}[G]$ -module. Then  $M$  is also a  $\mathbb{C}$ -vector space, and  $\rho : G \rightarrow \text{GL}(M)$  given by  $\rho(g)(m) = gm$  is a representation.

*Example.* If  $\rho : G \rightarrow \text{GL}(V)$  be a representation, the  $\rho$  induces a  $\mathbb{C}[G]$ -multiplication on  $V$ , making  $V$  a  $\mathbb{C}[G]$ -module. Moreover, if  $N \leq M$  is a submodule, then it is  $\rho(cg)$ -invariant for any  $cg \in \mathbb{C}[G]$  if and only if  $N$  as a subspace of  $M$  is  $G$ -stable.

To be precise, we have  $cg \cdot v = \rho(g)(cv)$ . In fact, there is an isomorphism of categories from representations of  $G$  and  $\mathbb{C}[G]$ -modules.

**Definition.** Let  $N, M$  be  $R$ -modules. We say  $\psi : N \rightarrow M$  is a (module) homomorphism if  $\phi$  commutes with the structures on  $N$  and  $M$ .

If  $\phi : N \rightarrow M$  is a homomorphism where  $N, M$  are  $\mathbb{C}[G]$ -modules, with multiplication maps  $\rho$  and  $\tau$ . Then  $\phi \circ \rho = \tau \circ \phi$ , in other words that it is an intertwining map. Note that  $\rho : G \rightarrow \text{GL}(V)$  is faithful if only if the unique zero map on  $v$  is 0.

**Definition.** Let  $M$  be an  $R$ -module. The **annihilator**  $\text{Ann}(M) = \{r \in R : rm = 0\}$ . Then  $M$  is **faithful** if  $\text{Ann}(M) = (0)$ .

**4.1 Proposition.** Let  $M$  be an  $R$ -module. Then  $\text{Ann}(M)$  is a (2-sided) ideal of  $R$ . Moreover,  $M$  is a faithful  $R/\text{Ann}(M)$ -module.

**Definition.** An  $R$ -module  $M$  is **irreducible** if  $M \neq (0)$  and the only submodules of  $M$  are  $(0)$  and  $M$ .

Recall that a division ring is a unital ring such that every non-zero element is invertible.

**4.2 Theorem. (Schur)** Let  $M$  be an irreducible  $R$ -module. Then  $\text{End}_R(M)$  is a division ring.

**4.3 Theorem.** Let  $M, N$  be  $R$ -modules and let  $\psi : M \rightarrow N$  be a module homomorphism. Then  $M/\ker \psi \cong \psi(M) \leq N$ .

**4.4 Proposition.** Let  $M$  is an irreducible  $R$ -module, then  $M \cong R/I$ , where  $I$  is a maximal left ideal. Conversely, if  $I$  is a maximal left ideal, then  $R/I$  is irreducible.

**PROOF** Let  $M$  be an irreducible  $R$ -module and fix  $0 \neq m \in M$ , and define  $\phi : R \rightarrow M$  by  $\phi(r) = rm$ , so  $\phi$  is a homomorphism and  $R/\ker \phi \cong \phi(R) = M$  by irreducibility. But then  $I$  is maximal since  $R/I \cong M$  is simple. ■

**Definition.** Let  $R$  be a ring. Then the **Jacobson radical** of  $R$  is  $J(R) = \bigcap_{\text{irred left } M} \text{Ann}(M)$ .

**Definition.** A left ideal  $I$  of  $R$  is called **left quasiregular** if for all  $a \in I$ ,  $R(1 + a) = R$ .

**4.5 Theorem.** If  $R$  is a ring, then the following are equivalent:

- (i)  $a \in J(R)$ .
- (ii)  $Ra$  is left quasiregular
- (iii)  $a \in \bigcap_{I \leq R \text{ maximal}} I$ .



**PROOF** ( $i \Rightarrow ii$ ) Let  $a \in J(R)$  and for contradiction assume for some  $x \in R$   $R(1 + xa) \neq R$ . Thus there exists a maximal left ideal  $I$  such that  $R(1 + xa) \subseteq I$ , so that  $R/I$  is an irreducible  $R$ -module. Thus  $a(R/I) = (0)$ , so that  $a(\overline{1}) = \overline{a} = \overline{0}$ , so  $xa \in I$  and  $1 \in I$ , a contradiction.

( $ii \Rightarrow iii$ ) Assume  $Ra$  is left quasiregular. Assume there exists some maximal left ideal  $I$  with  $a \notin I$ . Since  $R/I$  is irreducible,  $I + Ra/I \leq R/I$  is a non-zero ideal. By irreducibility,  $I + Ra/I = R/I$ , so there exists  $x \in R$  so that  $\overline{xa} = \overline{-1}$ , so  $1 + xa \in I$  is left-invertible, so  $I = R$ , a contradiction.

( $iii \Rightarrow i$ ) Let  $A = \bigcap_{I \text{ left max}} I$ . Suppose there exists an irreducible module  $M$  so that  $AM \neq (0)$ . Then there exists  $0 \neq m \in M$  such that  $Am \neq (0)$ . Note that  $am$  is a left  $R$ -submodule of  $M$ , so there exists  $a \in A$  so that  $am = -m$ . Thus  $(1 + a)m = 0$ , so if  $(1 + a)$  is in a maximal left ideal, then  $1 + a - a$  is as well. Thus  $(1 + a)$  is left-invertible, so  $m = 0$ , a contradiction. ■

*Remark.*

$$J(R) = \bigcap_{M \text{ irreducible}} \text{Ann}(M) = \bigcap_{\text{left max}} I = \sum_{\text{left quasi-reg}} Ra$$

Let  $a \in J(R)$ ,  $x \in R$ , and suppose  $R(1 + ax) \neq R$ , so  $R(1 + ax) \subseteq I$  where  $I$  is left maximal. Thus  $R/I$  is irreducible, so  $a(x + I) = \overline{0}$ , so  $ax \in I$ , so  $1 \in I$ .

If  $a \in J(R)$ , then  $1 + a$  is invertible so get  $b \in R$  so that  $b(1 + a) = -a$ . Then since  $a + b + ba = 0$ , so  $b \in J(R)$ . By the same argument, get  $c \in J(R)$  with  $c(1 + b) = -b$ . But then subtracting, manipulating, we get  $cb = ba$  so that  $a + b = b + c$  and in fact  $a = c$ . Thus  $(1 + a)b = b + ab = b + cb = -a$ . Thus  $(1 + a)b = -a$ , so  $(1 + a)R = R$ . Thus  $J(R) = \{x : xr \text{ is right quasiregular}\}$ .

**Definition.** A ring is **semiprimitive** if  $J(R) = (0)$ .

Recall that

$$J(R) = \bigcap_{\text{left max}} I = \bigcap_{\text{irred left}} \text{Ann}(M) = \bigcap_{\text{left quasi-ref}} \{Ra : \forall x, R(1 + xa) = R\}$$

*Example.* 1.  $J(\mathbb{Z}) = \bigcap_{p \text{ prime}} \langle p \rangle$

2.  $J(F[[x]]) = \langle x \rangle$

3.  $J(\mathbb{Z}_{12}) = \langle 2 \rangle \cap \langle 3 \rangle = \langle 6 \rangle$

**Definition.** Let  $R$  be a ring. We say  $a \in R$  is **nilpotent** if there exists  $n = n(a) \in \mathbb{N}$  such that  $a^n = 0$ . An ideal (left, right, both) is **nil** if every element is nilpotent. An ideal  $I$  (left, right, both) is **nilpotent** if there exists some  $n \in \mathbb{N}$  such that  $I^n = (0)$ .

**4.6 Proposition.** Every nil left ideal of  $R$  is contained in  $J(R)$ .

**PROOF** It suffices to show that for every nil element  $a$  that  $(1 + a)$  is invertible. Indeed, since  $a^n = 0$  for some  $n$ ,  $(1 - a + a^2 - \dots + (-1)^{n-1} a^{n-1})(1 + a) = 1$ . ■

**4.7 Proposition.**  $J(R/J(R)) = (0)$ , in other words,  $R/J(R)$  is semiprimitive.

**PROOF**

$$J(R/J(R)) = \bigcap_{\substack{I \subseteq R \\ J(R) \subseteq I}} I/J(R) = \bigcap_{\substack{I \subseteq R \\ \text{left max}}} I/J(R) = J(R)/J(R) = (0) \quad \blacksquare$$

**Definition.** A ring  $R$  is **(left) Artinian** if whenever  $I_1 \supseteq I_2 \supseteq \cdots$  is a descending chain of left ideals, then there exists  $N \in \mathbb{N}$  such that  $I_k = I_N$  for all  $k \geq N$ .

*Example.* (i)  $\mathbb{Z}$  is not Artinian.

(ii) If  $R$  Artinian, then  $M_n(R)$  is Artinian. If  $I$  is an ideal of  $M_n(R)$ , then  $I = M_n(I')$  where  $I'$  is an ideal of  $R$ .

(iii) Division rings are artinian

(iv) Suppose  $R$  is an  $F$ -algebra, where  $F$  is a field (isomorphic copy of  $F$  contained in the center of  $R$ ). If  $\dim_F R < \infty$ , then  $R$  is Artinian

(v) If  $F$  is a field and  $G$  is a finite group, then  $F[G]$  is Artinian since  $\dim F[G] = |G| < \infty$

**4.8 Proposition.** *If  $R$  is Artinian, then  $J(R)$  is nilpotent.*

**PROOF** Consider  $J(R) \supseteq J(R)^2 \supseteq \cdots$ . Thus there exists  $N$  such that  $J(R)^k = J(R)^N$  for all  $k \geq N$ . Let  $I = J(R)^N$ ; let's see that  $I = (0)$ . Suppose  $I \neq (0)$ . Let  $A$  be a minimal left ideal of  $R$  such that  $IA \neq (0)$ . Let  $a \in A$  so that  $Ia \neq (0)$ , so  $Ia$  is a left ideal and  $I(Ia) = I^2a = Ia$ . Thus by minimality,  $A = Ia$  so there is some  $x \in I$  such that  $a = xa$ . Thus  $(1 - x)a = 0$  so  $a = 0$ , a contradiction. ■

**4.9 Theorem. (Maschke)** *Let  $G$  be a finite group. If  $F$  is a field such that  $\text{char}(F) = 0$  or  $\text{char}(F) = p$  does not divide  $|G|$ , then  $F[G]$  is semiprimitive and Artinian (and hence semisimple, by the assignment).*

**PROOF** Since  $\dim_F F[G] < \infty$ ,  $F[G]$  is Artinian. For contradiction, suppose  $I$  is a nonzer nil ideal of  $R$ . Take  $0 \neq x \in I$ , so  $x = \sum a_g g$  where  $a_h \neq 0$  for some  $h \in G$ . By multiplying by  $h^{-1}$ , we may assume  $a_1 \neq 0$ . For each  $a \in F[G]$ , define  $T_a : F[G] \rightarrow F[G]$  by  $T_a(v) = av$ , so  $T_a$  is a  $F$ -linear operator. Note that  $T_x = \sum a_g T_g$  so that  $\text{Tr}(T_x) = \sum a_g \text{Tr}(T_g)$ , so  $x$  is not nilpotent, a contradiction. ■

## ARTIN-WEDDERBURN THEORY

**Definition.** A ring  $R$  is **primitive** if it has a faithful, irreducible module.

Note that primitive rings are semiprimitive.

*Example.* If  $D$  is a division ring, then  $M_n(D)$  is primitive. In particular,  $D^n$  is faithful and irreducible

Let  $R$  be primitive and commutative. Then if  $M$  is faithful and irreducible,  $M \cong R/I$  where  $I$  is a maximal ideal so  $R$  is a field.

**Definition.** A ring  $R$  is **simple** if  $R \neq (0)$  and  $R$  has no proper non-zero two-sided ideals. For example,  $M_n(D)$  is simple. If  $J \leq M_n(D)$  is an ideal, then  $J = M_n(I)$  for some ideal  $I$  of  $D$ .

*Remark.* If  $R$  is irreducible, then  $R$  is simple. However, the converse does not hold since  $M_2(\mathbb{R})$  is simple but  $I = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$  is a left ideal.

**4.10 Proposition.** *Every simple ring is primitive.*

**PROOF** Let  $R$  be simple and  $I$  be a maximal left ideal of  $R$  so that  $M := R/I$  is irreducible. Since  $\text{Ann}(M)$  is an ideal of  $R$  and  $\text{Ann}(M) \neq R$ ,  $\text{Ann}(M) = (0)$ . ■

For the remainder of this section,  $R$  is primitive,  $M$  is faithful and irreducible, and  $D = \text{End}_R(M)$  is a division ring. We give  $M$  the structure of a  $D$ -module by  $\phi \cdot m = \phi(m)$ .

**Definition.** We say  $R$  **acts densely** on  $M$  if for all  $D$ -linearly independent  $v_1, \dots, v_n \in M$  and all  $w_1, \dots, w_n \in M$ , there exists  $r \in R$  such that  $rv_i = w_i$  for  $i = 1, 2, \dots, n$ .

*Remark.* Suppose  $\dim_D M < \infty$  and  $R$  acts densely on  $M$ . If  $\{v_1, \dots, v_n\}$  is a  $D$ -basis, then for all  $w_1, \dots, w_n$ , there exists  $r \in R$  so that  $rv_i = w_i$ . Thus  $R \cong \{T : M \rightarrow M : D\text{-linear}\} \cong M_n(D)$ .

**4.11 Lemma.** *If for every finite dimensional  $D$ -subspace  $V$  of  $M$  and every  $m \in M \setminus V$  there exists  $r \in R$  such that  $rV = (0)$  but  $rm \neq 0$ , then  $R$  acts densely on  $M$ .*

**PROOF** Let  $v_1, \dots, v_n$  be  $D$ -linearly independent in  $M$  and suppose  $w_1, \dots, w_n$  are in  $M$ . For each  $i$ , let  $V_i = \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ . By assumption, since  $v_i \notin V_i$ , there exists  $t_i \in R$  so that  $t_i V_i = (0)$  but  $t_i v_i \neq 0$ . Observe that  $Rt_i v_i = M$  since  $M$  is irreducible, so get  $r_i \in R$  such that  $r_i t_i V_i = (0)$  and  $r_i t_i v_i = w_i$ . Let  $t = r_1 t_1 + \dots + r_n t_n$ , so  $tv_i = w_i$ . ■

**4.12 Theorem. (Jacobson Density)** *Let  $R$  be primitive and  $M$  a faithful irreducible  $R$ -module. Then  $R$  acts densely on  $M$ .*

**PROOF** Let  $V$  be a finite dimensional  $D$ -subspace of  $M$ , and let  $m \in M \setminus V$ . We proceed by induction on  $\dim V$ . If  $\dim V = 0$ ,  $V = (0)$ , and take  $r = 1$ . Proceeding inductively, suppose  $\dim V > 0$  and  $0 \neq w \in V$  with  $V = V_0 \oplus \text{span}\{w\}$ , where  $\dim V_0 = \dim V - 1$ . Let  $A(V_0) = \{x \in R : xV_0 = (0)\}$ . By induction, for every  $y \in V_0$ , there exists  $r \in A(V_0)$  such that  $ry \neq 0$ . Note that  $A(V_0)$  is a left ideal: since  $w \notin V_0$ ,  $A(V_0)w \neq (0)$  so  $A(V_0)w = M$  by irreducibility. Consider  $\tau : M \rightarrow M$  given by  $\tau(aw) = am$ , where  $a \in A(V_0)$ . This is well-defined for if  $aw = a'w$ , then  $(a - a')w = 0$  so  $(a - a')V = 0$  (since  $V = V_0 \oplus \text{span}_D\{w\}$ ). For contradiction, assume that if  $r \in R$  and  $rV = (0)$ , then  $rm = 0$ . Thus  $(a - a')m = 0$  so  $am = a'm$  and  $\tau(a2) = \tau(a'w)$  and  $\tau$  is well-defined. Notice that  $\tau \in \text{End}_R(M) = D$ . For all  $a \in A(V_0)$ ,  $am = \tau(aw) = a\tau(w)$  so  $a(m - \tau(w)) = 0$ . Thus by the inductive hypothesis,  $M - \tau(w) \in V_0$ , so  $m \in V_0 \oplus \text{span}_D(w) = V$ . ■

**4.13 Proposition.** *If  $R$  is primitive and (left) Artinian, then  $R \cong M_n(D)$  where  $D \cong \text{End}_R(M)$ .*

**PROOF** We first show that  $\dim_D(M) < \infty$ . Suppose  $\{v_1, v_2, \dots\}$  is infinite and  $D$ -linearly independent. For each  $m$ , let  $I_m = \{r \in R : rv_i = 0 \text{ for } i = 1, \dots, m\}$ , so that  $I_1 \supseteq I_2 \supseteq \dots$ . By the JDT,  $R$  acts densely on  $M$ . In particular, for every  $m > 1$ , there exists  $r \in R$  so that  $rv_1 = \dots = rv_{m-1} = 0$  but  $rv_m = v_m \neq 0$ , so  $r \in I_{m-1} \setminus I_m$ . Thus  $I_1 \supsetneq I_2 \supsetneq \dots$ , contradicting Artinianity.

Consider the map  $\phi : R \rightarrow \text{End}_D(M) \cong M_n(D)$  by  $\phi(r) = (v_i \mapsto rv_i)$ . Then by the homework, this is a ring isomorphism. ■

In particular, on A4, we prove that every semiprimitive Artinian ring is a finite direct sum of primitive Artinian rings. We thus have

**4.14 Theorem. (Artin-Wedderburn)** *Every semiprimitive Artinian (i.e. semisimple) ring is isomorphic to a finite direct sum of matrix rings over division rings, i.e.  $R \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_k}(D_k)$*

Note that the  $D_i$  and the  $n_i$  are unique up to reordering.

**4.15 Corollary.** *Every commutative semisimple ring is isomorphic to a finite direct sum of fields.*

Let  $R$  be primitive  $F$ -algebra where  $F$  is a field. Let  $M$  be a faithful, irreducible  $R$ -module, and  $D = \text{End}_R(M)$ . For  $\alpha \in F$ , consider  $\phi_\alpha : M \rightarrow M$  given by  $\phi_\alpha(m) = \alpha m$  since  $F \subseteq Z(R)$ ,  $\phi_\alpha \in D$ .

Now define  $\psi : F \rightarrow D$  by  $\psi(\alpha) = \phi_\alpha$ , which is an injective homomorphism. Furthermore, for each  $\psi \in D$ ,  $\psi(\phi_\alpha(m)) = \phi(\alpha m) = \phi_\alpha(\phi(m))$  so  $\phi(\phi_\alpha(m)) = \phi(\alpha m) = \alpha \phi(m) = \phi_\alpha(\phi(m))$  so  $\phi \circ \phi_\alpha = \phi_\alpha \circ \phi$ , so  $D$  is an  $F$ -algebra.

**4.16 Lemma.** *Suppose  $F = \bar{F}$ . If  $D$  is a division  $F$ -algebra which is algebraic over  $F$ , then  $D = F$ .*

PROOF Let  $a \in D$ , and let  $p(x) \in F[x]$  with  $p(a) = 0$ . Then  $p(x) = \prod_i (x - \lambda_i)$  with  $\lambda_i \in F$ . However,  $p(a) = \prod_i (a - \lambda_i)$  since  $F \subseteq Z(D)$ . Since  $D$  is a division ring,  $(a - \lambda_i) = 0$  so that  $a = \lambda_i \in F$ . ■

*Remark.* Suppose  $D$  is a division  $F$ -algebra. If  $\dim_F(D) < \infty$ , then  $D$  is algebraic over  $F$ .

**4.17 Theorem.** *Let  $F = \bar{F}$ . If  $R$  is a finite dimensional semisimple  $F$ -algebra, then  $R \cong M_{n_1}(F) \oplus \cdots \oplus M_{n_k}(F)$ .*

PROOF Write  $R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$ , so that  $\dim_F(D_i) < \infty$ . Thus since each  $D_i$  is an  $F$ -algebra with finite dimension, each  $D_i = F$ . ■

We thus have

**4.18 Theorem.** *If  $F = \bar{F}$ ,  $G$  a finite group, and  $\text{char } F = 0$  or  $\text{char } F \nmid |G|$ , then  $F[G]$  is semisimple and thus  $F[G] \cong M_{n_1}(F) \oplus \cdots \oplus M_{n_k}(F)$ .*

*Remark.* Suppose  $F = \mathbb{C}$ . Then  $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ . Taking  $\dim_{\mathbb{C}} : |G| = n_1^2 + \cdots + n_k^2$ .

**4.19 Lemma.** *Let  $R$  be semisimple, so that  $R = M_1 \oplus \cdots \oplus M_k$  where  $M_i$  are irreducible.*

- (i) *If  $M$  is an irreducible  $R$ -module, then  $M \cong M_i$  for some  $i$ .*
- (ii) *If  $R \cong N_1 \oplus \cdots \oplus N_m$  is another irreducible decomposition, then  $m = n$  and up to reordering  $M_i \cong N_i$ .*

PROOF (i) Let  $M \cong R/I$  where  $I$  is left maximal. Then  $\phi_i : M_i \rightarrow R \rightarrow R/I \cong M$ , so either  $\phi_i = 0$  or  $\phi_i$  is an isomorphism. Suppose  $\phi_i = 0$  for all  $i$ . Then  $\phi = \sum \phi_i$ , so  $\phi : R \rightarrow R/I$  as  $\phi(1) = 0$ , so  $1 \in I$ , a contradiction.

(ii) The maximal submodules of  $R$  are precisely  $P_i := \bigoplus_{j \neq i} M_j$ . ■

Let  $D$  be a division ring,  $R = M_n(D)$  semisimple, so  $R = M_1 \oplus \cdots \oplus M_n$  where each  $M_i$  is the ideal composition of column  $i$  of  $D^n$ . Then  $R \cong D^n \oplus \cdots \oplus D^n$ . Since  $R$  is semisimple, Artin-Wedderburn implies that  $R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$  as rings, so that

$$R \cong D_1^{n_1} \oplus \cdots \oplus D_1^{n_1} \oplus \cdots \oplus D_k^{n_k} \oplus \cdots \oplus D_k^{n_k}$$

and in fact

$$\begin{aligned}\mathbb{C}[G] &\cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \\ &\cong \underbrace{\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_1}}_{n_1 \text{ times}} \oplus \cdots \oplus \underbrace{\mathbb{C}^{n_k} \oplus \cdots \oplus \mathbb{C}^{n_k}}_{n_k \text{ times}}\end{aligned}$$

Let  $M$  be an irreducible  $\mathbb{C}[G]$ -module. By the lemma,  $M \cong \mathbb{C}^{n_i}$  for some  $i$ . The degree of the associated representation is  $\dim_{\mathbb{C}} M = n_i$ , and whenever  $M$  occurs in  $\mathbb{C}[G]$  (regular representation)  $n_i$  times. Moreover,  $k$  is the number of conjugacy classes of  $G$ .

Exercise: if  $C$  is a conjugacy class and  $z_C = \sum_{g \in C} g \in \mathbb{C}[G]$ ,  $\{z_C : C \text{ conj class}\}$  forms a basis for  $Z(\mathbb{C}[G])$  (use Artin-Wedderburn).

*Example.* (i) In  $\mathbb{C}[S_3]$ , we have  $\mathbb{C}[S_3] \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ .

(ii) If  $G$  is abelian with  $|G| = n$ , then  $\mathbb{C}[G] \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}$   $n$  times.

(iii) If  $G, H$  are abelian, then  $\mathbb{C}[G] \cong \mathbb{C}[H]$  if and only if  $|G| = |H|$ .

**4.20 Theorem.** Say  $R \subseteq S$  and  $a \in S$ . Then the following are equivalent:

1.  $a$  is integral over  $R$ .
2.  $R[a]$  is a finitely generated  $R$ -module.
3. There exists a subring  $R \subseteq T \subseteq S$  such that  $a \in T$  and  $T$  is a finitely generated  $R$ -module.

## 5 FACTS ABOUT NON-COMMUTATIVE MODULES

General structures on modules:

**Definition.** A **(left)  $R$ -module** is an abelian group  $(M, +)$  equipped with a unitary ring homomorphism  $\alpha : A \rightarrow \text{End}(M)$ . If  $N, M$  be  $R$ -modules, then a group homomorphism  $\psi : N \rightarrow M$  is a **(module) homomorphism** if  $\psi(rm) = r\psi(m)$  for any  $r \in R$ . The kernel and image of  $\psi$  are submodules of  $N$  and  $M$  respectively. The **annihilator**  $\text{Ann}(M) = \{r \in R : rm = 0\}$ . Then  $M$  is **faithful** if  $\text{Ann}(M) = (0)$ .

Annihilators:

**Definition.** Let  $R$  be a ring. We say  $a \in R$  is **nilpotent** if there exists  $n = n(a) \in \mathbb{N}$  such that  $a^n = 0$ . An ideal (left, right, both) is **nil** if every element is nilpotent. An ideal  $I$  (left, right, both) is **nilpotent** if there exists some  $n \in \mathbb{N}$  such that  $I^n = (0)$ .

Key example:

**Definition.** Let  $G$  be a finite group and  $F$  a field. We define the **group algebra**  $F[G] = \{\alpha_1 g_1 + \cdots + \alpha_n g_n : \alpha_i \in F\}$  equipped with  $G$ -pointwise addition and multiplication  $ag_i \cdot bg_j = (ab)g_i g_j$ , extended by distributivity.

*Example.* Let  $M$  be a  $\mathbb{C}[G]$ -module. Then  $M$  is also a  $\mathbb{C}$ -vector space, and  $\rho : G \rightarrow \text{GL}(M)$  given by  $\rho(g)(m) = gm$  is a representation. If  $\rho : G \rightarrow \text{GL}(V)$  be a representation, the  $\rho$  induces a  $\mathbb{C}[G]$ -multiplication on  $V$ , making  $V$  a  $\mathbb{C}[G]$ -module. Moreover, if  $N \leq M$  is a submodule, then it is  $\rho(cg)$ -invariant for any  $cg \in \mathbb{C}[G]$  if and only if  $N$  as a subspace of  $M$  is  $G$ -stable. To be precise, we have  $cg \cdot v = \rho(g)(cv)$ . In fact, there is an isomorphism of categories from representations of  $G$  and  $\mathbb{C}[G]$ -modules.

Basic results on modules:

**5.1 Proposition.** Let  $M$  be an  $R$ -module. Then  $\text{Ann}(M)$  is a (2-sided) ideal of  $R$ . Moreover,  $M$  is a faithful  $R/\text{Ann}(M)$ -module.

**5.2 Theorem. (First Isomorphism)** Let  $M, N$  be  $R$ -modules and let  $\psi : M \rightarrow N$  be a module homomorphism. Then  $M/\ker \psi \cong \psi(M) \leq N$ .

Types of modules:

**Definition.** Let  $M$  be an  $R$ -module.

- $M$  is **irreducible** if  $M \neq (0)$  and the only submodules of  $M$  are  $(0)$  and  $M$ .

Types of ideals:

**Definition.** Let  $R$  be a ring.

- A left ideal  $I$  of  $R$  is called **left quasiregular** if for all  $a \in I$ ,  $R(1 + a) = R$ .
- The **Jacobson radical** of  $R$  is  $J(R) = \bigcap_{\text{irred left } M} \text{Ann}(M)$ .

Types of rings:

**Definition.** Let  $R$  be a ring.

- $R$  **semiprimitive** if  $J(R) = (0)$ .
- $R$  is **(left) Artinian** if whenever  $I_1 \supseteq I_2 \supseteq \dots$  is a descending chain of left ideals, then there exists  $N \in \mathbb{N}$  such that  $I_k = I_N$  for all  $k \geq N$ .

Relationships:

**5.3 Proposition.** The following hold:

- Every nil left ideal of  $R$  is contained in  $J(R)$ .
- $R/J(R)$  is semiprimitive.
- If  $R$  is Artinian, then  $J(R)$  is nilpotent.
- $M$  is an irreducible  $R$ -module if and only if then  $M \cong R/I$  as  $R$ -modules, where  $I$  is a maximal left ideal of  $R$ .

**5.4 Theorem. (Schur)** Let  $M$  be an irreducible  $R$ -module. Then  $\text{End}_R(M)$  is a division ring.

**5.5 Theorem.** If  $R$  is a ring, then the following are equivalent:

- (i)  $a \in J(R)$ .
- (ii)  $Ra$  is left quasiregular
- (iii)  $a \in \bigcap_{I \leq R \text{ maximal}} I$ .

Let  $G$  be a finite group with irreducible characters  $\chi_i$  and corresponding representations  $\rho_i$ , and conjugacy classes  $C_i$ .

**5.6 Proposition.** For  $i = 1, \dots, k$ , define  $\omega_i : \{C_1, \dots, C_k\} \rightarrow \mathbb{C}$  by

$$\omega_i(C_j) = \frac{|C_j|\chi_i(g)}{\chi_i(1)}$$

where  $g \in C_j$ . Then  $\omega_i(C_j)$  is an algebraic integer.

**PROOF** Let  $h \in G$ , so that

$$\sum_{g \in C_j} \rho_i(g) = \sum_{g \in G} \rho(h)\rho(g)\rho(h^{-1}) = \rho_i(h) \left( \sum_{g \in C_j} \rho_i(g) \right) \rho_i(h)^{-1}$$

so by Schur's lemma,  $\sum_{g \in C_j} \rho_i(g) = \alpha I$ . Taking traces,  $\sum_{g \in C_j} \text{Tr}(\rho_i(g)) = \alpha \chi_i(1)$ , so  $|C_j| \chi_i(g) = \alpha \chi_i(1)$ .

Now fix  $g \in C_s$ , and define  $a_{ij}(s) = |\{(g_i, g_j) \in C_i \times C_j : g_i g_j = g\}| \in \mathbb{Z}$ . One can verify that the definition does not depend on the choice of  $g$ . Now by the above observation,

$$\begin{aligned} (w_t(c_i)w_t(c_j))I &= \left( \sum_{g_i \in C_i} \rho_t(g_i) \right) \left( \sum_{g_j \in C_j} \rho_t(g_j) \right) \\ &= \sum_{g_i, g_j} \rho_t(g_i g_j) = \sum_{s=1}^k \sum_{g \in C_s} a_{ij}(s) \rho_t(g) \\ &= \sum_{s=1}^k a_{ij}(s) \sum_{g \in C_s} \rho_t(g) \\ &= \sum_{s=1}^k a_{ij}(s) \omega_t(C_s) I \end{aligned}$$

again by the above claim. Thus the finitely generated  $\mathbb{Z}$ -module generated by  $1, w_t(C_1), \dots, w_t(C_k)$  is a subring of  $\mathbb{C}$ . ■

**5.7 Theorem.**  $\chi_i |G|$  for  $i = 1, \dots, k$ , i.e. the degree of an irreducible representation divides  $|G|$ .

PROOF Using the same notation as above,

$$\begin{aligned} \frac{|G|}{\chi_i(1)} &= \frac{|G|}{\chi_i(1)} \langle \chi_i, \chi_i \rangle \\ &= \frac{|G|}{\chi_i(1)} \cdot \frac{1}{|G|} \sum_{g \in G} |\chi_i(g)|^2 \\ &= \frac{1}{\chi_i(1)} \sum_{j=1}^k |C_j| \cdot |\chi_j(g_j)|^2 \\ &= \sum_{j=1}^k \frac{|C_j| \chi_i(g_j)}{\chi_i(1)} \overline{\chi_i(g_j)} \\ &= \sum_{j=1}^k w_i(C_j) \overline{\chi_i(g_j)} \end{aligned}$$

is a finite sum of products of algebraic integers, and hence an algebraic integer. Thus  $|G|/\chi_i(1)$  is an algebraic integer and a rational number, hence an integer. ■

Representations of modules, artin-wedderburn theory E.g. take a group ring and write it as a product of matrix rings over  $\mathbb{C}$ .

proof from class (ring/module theory)

question on induced representations, computational (e.g. think about example from class)

Be comfortable with  $D_4$ , in general, representations of  $D_n$ .

### FROBENIUS RECIPROCITY

Let  $M$  be a  $\mathbb{C}[G]$  module, where  $\dim_{\mathbb{C}} M < \infty$ . Since  $\mathbb{C}[G]$  is semisimple,  $M$  is semisimple (finite direct sum), and write  $M = M_1 \oplus \cdots \oplus M_k$  where each  $M_i$  is irreducible. Let  $N$  be an irreducible  $\mathbb{C}[G]$ -module, so  $\dim_{\mathbb{C}} N < \infty$ . Consider  $\text{Hom}_{\mathbb{C}[G]}(M, N)$ , which is a  $\mathbb{C}$ -vector space. As vector spaces,  $\text{Hom}_{\mathbb{C}[G]}(M, N) \cong \bigoplus_{i=1}^k \text{Hom}(M_i, N)$ . By Schur,  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(M_i, N)$  is 0 if  $M_i$  is not congruent to  $N$ , and 1 if  $M_i \cong N$  (map is a  $\mathbb{C}$ -multiple of the identity). Thus the multiplicity of  $N$  in  $M$  (i.e. the number of  $i$  such that  $M_i \cong N$ ) is  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(M, N)$ . Say  $\rho \sim M$ ,  $\tau \sim N$  (irreducible), then  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(M, N) = \langle \chi_{\rho}, \chi_{\tau} \rangle$ .

Now suppose  $H \leq G$  and  $M$  is a  $\mathbb{C}[H]$ -module, with  $N$  a  $\mathbb{C}[G]$ -module. Let  $M^G := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} M$ , and let  $\iota : M \rightarrow M^G$  by  $\iota(m) = 1 \otimes m$ . For  $f \in \text{Hom}_{\mathbb{C}[G]}(M, N)$ , there exists a unique  $T_f \in \text{Hom}_{\mathbb{C}[G]}(M^G, N)$  such that  $f = T_f \circ \iota$ .

**5.8 Theorem. (Frobenius Reciprocity)** *The map  $\phi : \text{Hom}_{\mathbb{C}[H]}(M, N) \rightarrow \text{Hom}_{\mathbb{C}[G]}(M^G, N)$  given by  $\phi(f) = T_f$  is an isomorphism of  $\mathbb{C}$ -vector spaces.*

**PROOF** Let  $f_1, f_2 \in \text{Hom}_{\mathbb{C}[H]}(M, N)$ , so  $(T_{f_1} + T_{f_2}) \circ \iota = T_{f_1} \circ \iota + T_{f_2} \circ \iota = f_1 + f_2$ . Thus  $T_{f_1} + T_{f_2} = T_{f_1+f_2}$  by uniqueness; similarly,  $T_{\alpha f_1} = \alpha T_{f_1}$ . Thus  $\phi$  is linear.

Suppose  $T_{f_1} = T_{f_2}$ , so  $T_{f_1} \circ \iota = T_{f_2} \circ \iota$  so  $f_1 = f_2$  and we have injectivity. To see surjectivity, let  $F \in \text{Hom}_{\mathbb{C}[G]}(M^G, N)$ , and let  $f = F \circ \iota \in \text{Hom}_{\mathbb{C}[H]}(M, N)$ . Then  $T_f = F$  by uniqueness. ■

Let  $M$  be irreducible as a  $\mathbb{C}[H]$ -module, and  $N$  irreducible as a  $\mathbb{C}[G]$ -module. Let  $\rho : H \rightarrow \text{GL}(M)$  and  $\tau : G \rightarrow \text{GL}(N)$ . Denote the restriction of  $\tau$  to  $H$  by  $\text{Res}_G^H(\tau)$ . By Frobenius reciprocity,  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[H]}(M, N) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(M^G, N)$  and equivalently

$$\langle \chi_{\rho}, \chi_{\text{Res}(\tau)} \rangle_H = \langle \chi_{\text{Ind}(\rho)}, \chi_{\tau} \rangle_G$$

We write  $\text{Res}(\chi_{\tau}) = \chi_{\text{Res}(\tau)}$  and  $\text{Ind}(\chi_{\rho}) = \chi_{\text{Ind}(\rho)}$ . In other words, the number of times  $\rho$  appears in the restriction of  $\tau$  is equal to the number of times  $\tau$  appears in the induced representation of  $\rho$ .

Let  $V, W$  be  $\mathbb{C}[G]$ -modules, and  $\langle V, W \rangle := \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(V, W)$ . In particular, if  $H \leq G$ ,  $V$  is a  $\mathbb{C}[H]$ -module, then  $\text{Ind}_H^G(V) := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ . Then by Frobenius reciprocity, we have the following adjointness relationship:  $\text{Hom}_{\mathbb{C}[G]}(V, \text{Res}_G^H(W)) \cong \text{Hom}_{\mathbb{C}[G]}(\text{Ind}_H^G(V), W)$ .

**5.9 Lemma.** *Let  $V, W$  be  $\mathbb{C}[G]$ -modules, with  $V \sim \chi_{\rho}$  and  $W \sim \chi_{\tau}$  are characters. Then  $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(V, W) = \langle \chi_{\rho}, \chi_{\tau} \rangle$ .*

**PROOF** Write  $W = W_1 \oplus \cdots \oplus W_n$  where the  $W_i$  are irreducible. Then  $\text{Hom}_{\mathbb{C}[G]}(V, W) \cong \bigoplus_{i=1}^n \text{Hom}_{\mathbb{C}[G]}(V, W_i)$ . Taking dimensions, we have

$$\begin{aligned} \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(V, W) &= \sum_{i=1}^n \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[G]}(V, W_i) = \sum_{i=1}^n \langle \chi_{\rho}, \chi_{\tau_i} \rangle \\ &= \langle \chi_{\rho}, \chi_{\tau} \rangle \end{aligned} \quad \blacksquare$$

*Remark.*  $\text{Hom}_{\mathbb{C}[H]}(V, \text{Res}(W)) \cong \text{Hom}_{\mathbb{C}[G]}(\text{Ind}(V), W)$ . In particular, if  $\rho : H \rightarrow \text{GL}(V)$  and  $\tau : G \rightarrow \text{GL}(W)$ ,  $\langle \chi_{\rho}, \text{Res}(\chi_{\tau}) \rangle_H = \langle \text{Ind}(\chi_{\rho}), \chi_{\tau} \rangle$ .



*Example.* Consider  $H = S_3 \leq S_4 = G$ , and  $\rho : H \rightarrow \text{GL}(\mathbb{C}^2)$  irreducible of degree 2. Recall

$S_3$	$\epsilon$	(12)	(123)
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

and

$S_4$	$\epsilon$	(12)	(12)(34)	(123)	(1234)
$\phi_1$	1	1	1	1	1
$\phi_2$	1	-1	1	1	-1
$\phi_3$	2	0	*	-1	*
$\phi_4$	3	1	*	0	*
$\phi_5$	3	-1	*	0	*

But now apply Frobenius so  $\langle \text{Ind}(\chi_3), \phi_1 \rangle = \langle \chi_3, \text{Res}(\phi_1) \rangle = \langle \chi_3, \chi_1 \rangle = 0$ . Similarly,  $\langle \text{Ind}(\chi_3), \phi_2 \rangle = \langle \chi_3, \chi_2 \rangle = 0$  and  $\langle \text{Ind}(\chi_3), \phi_3 \rangle = \langle \chi_3, \chi_3 \rangle = 1$ . Now,  $\langle \text{Ind}(\chi_3), \chi_4 \rangle = \langle \chi_3, \text{Res} \phi_4 \rangle = \langle \chi_3, \chi_1 + \chi_3 \rangle = 1$  and finally,  $\langle \text{Ind}(\chi_3), \phi_5 \rangle = \langle \chi_3, \text{Res}(\phi_5) \rangle = \langle \chi_3, \chi_2 + \chi_3 \rangle = 1$ . Thus  $\text{Ind}(\chi_\rho) = \phi_3 + \phi_4 + \phi_5$ .

Suppose  $\text{Ind}_H^G(\rho)$  is irreducible. Then necessarily  $\rho$  is irreducible, for if  $W \leq V$  is a submodule, then  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$  is a  $\mathbb{C}[G]$ -submodule of  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ .

Let  $H \leq G$ , and for  $g \in G$ , let  $H_g = gHg^{-1} \cap H \leq H$ . Let  $\rho : H \rightarrow \text{GL}(V)$  be a representation: then we obtain two representations of  $H_g$ :

1.  $\text{Res}_g(\rho) = \text{Res}_{H_g}^{H_g}(\rho)$
2.  $\rho^g : H_g \rightarrow \text{GL}(V)$  by  $\rho^g(ghg^{-1}) = \rho(h)$

**Definition.** If  $\rho_1$  and  $\rho_2$  are reps of  $G$ , we say  $\rho_1$  and  $\rho_2$  are **disjoint** if  $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 0$ .

**5.10 Theorem. (Mackey)** Let  $H \leq G$ ,  $\rho : H \rightarrow \text{GL}(V)$ . Then  $\text{Ind}_H^G(\rho)$  is irreducible if and only if

1.  $\rho$  is irreducible, and
2. For all  $g \in G - H$ ,  $\rho^g$  and  $\text{Res}_g(\rho)$  are disjoint.

**5.11 Corollary.** Let  $H \trianglelefteq G$ ,  $\rho : H \rightarrow \text{GL}(V)$ . Then  $\text{Ind}_H^G(\rho)$  is irreducible if and only if  $\rho$  is irreducible and for all  $g \in G \setminus H$ ,  $\rho^g$  is disjoint from  $\rho$ .

To approach this theorem, we need some additional group theory.

**Definition.** Let  $H \leq G$  and  $g \in G$ . The **double coset** of  $H$  containing  $G$  is  $HgH = \{h_1gh_2 : h_i \in H\}$ .

Consider  $H \times H$  acting on  $G$  by  $(h_1, h_2) \cdot g = h_1gh_2^{-1}$ , so the double cosets are exactly the orbits of this action. Thus,

1. Two double cosets of  $H$  in  $G$  are equal or disjoint
2. The distinct double cosets of  $H$  in  $G$  partition  $G$

*Example.* Take  $H = \{\epsilon, (12)\} \leq S_3$ . Then  $H(123)H = \{(13), (23), (123), (132)\}$  and  $H\epsilon H = H\{\epsilon, (12)\}$ .

Now let  $H \trianglelefteq G$  with cosets  $g_1H, \dots, g_mH$ . Consider  $Hg_1H, \dots, Hg_mH$ . Then  $g_iH \subseteq Hg_iH$  by normality, so  $Hg_iH = g_iH$ .

**5.12 Proposition.** *Let  $H \leq G$ ,  $\rho : G \rightarrow \text{GL}(V)$ , and  $S$  a set of double coset representatives. Then*

$$\text{Res}_G^H \text{Ind}_H^G(\rho) \cong \bigoplus_{s \in S} \text{Ind}_{H_s}^H(\rho^s)$$

**PROOF** We must construct an isomorphism as  $\mathbb{C}[H]$ -modules. For each  $s \in S$ , let  $W(s) = \text{span}\{x \otimes v : x \in HsH, v \in V\}$ , so that  $W(s)$  is a  $\mathbb{C}[H]$ -submodule of  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$  since double cosets are closed under  $H$ -multiplication. Since the double cosets partition  $G$ ,  $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \bigoplus_{s \in S} W(s)$  as  $\mathbb{C}[G]$ -modules.

It remains to show for  $s \in S$  that  $W(s) \cong \mathbb{C}[H] \otimes_{\mathbb{C}[H_2]} V$  as  $\mathbb{C}[H]$ -module. Consider  $f : V \rightarrow W(s)$  given by  $f(v) = s \otimes v$ . Certainly  $f$  is additive. Now,

$$\begin{aligned} f(shs^{-1} \cdot v) &= f(hv) = s \otimes hv = sh \otimes v \\ &= (shs^{-1})s \otimes v \\ &= shs^{-1}f(v) \end{aligned}$$

so that  $f$  is a  $\mathbb{C}[H_s]$ -module homomorphism. Thus by the universal property, there exists a  $\mathbb{C}[H]$ -module homomorphism  $F : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \rightarrow W(s)$  such that  $F \circ i = f$ . Once can show that  $F$  is the desired isomorphism.  $\blacksquare$

We can now prove Mackey's theorem

**PROOF** We have

$$\begin{aligned} \langle \text{Ind}_H^G(\chi_\rho), \text{Ind}_H^G(\chi_\rho) \rangle &= \langle \chi_\rho, \text{Res}_G^H \text{Ind}_H^G(\chi_\rho) \rangle \\ &= \left\langle \chi_\rho, \sum_{s \in S} \text{Ind}_{H_s}^H(\chi_{\rho^s}) \right\rangle \\ &= \sum_{s \in S} \langle \chi_\rho, \text{Ind}_{H_s}^H(\chi_{\rho^s}) \rangle \\ &= \sum_{s \in S} \underbrace{\langle \text{Res}_H^{H_s}(\chi_\rho), \chi_{\rho^s} \rangle}_{d_s} \\ &= d_1 + \sum_{s \in S, s \neq 1} d_s \\ &= \underbrace{\langle \chi_\rho, \chi_\rho \rangle}_{\geq 1} + \sum_{s \neq 1} d_s \end{aligned}$$

where  $d_s = 0$  precisely when  $\langle \text{Res}_s(\chi_\rho), \chi_{\rho^s} \rangle = 0$ , so the result follows.  $\blacksquare$

**5.13 Corollary.** *Let  $H \trianglelefteq G$  and  $\rho : H \rightarrow \text{GL}(V)$ . Then  $\text{Ind}(\rho)$  is irreducible if and only if  $\rho$  is irreducible and  $\rho \neq \rho^g$  for all  $g \in G \setminus H$ .*

*Example.* Take  $G = D_5$  and  $H = \langle r \rangle \trianglelefteq G$ . Then  $\rho : H \rightarrow \mathbb{C}^\times$  with  $\rho(r) = \zeta_5$ . Let's show that  $\text{Ind}_H^G(\rho)$  is irreducible. Since  $\rho$  is degree 1, it is irreducible. Moreover, since  $H \trianglelefteq G$  and  $\deg(\rho) = 1$ , we must show that  $\rho \neq \rho^g$  for all  $g \in G \setminus H$ . Indeed,  $\rho^{sr^i}(r) = \rho((sr^i)^{-1}rsr^i) = \rho(r^{-1}) = \zeta_5^4$ .

## 6 REPRESENTATION THEORY OF $S_n$

**Definition.** A **partition**  $\lambda$  of  $n \in \mathbb{N}$  is a sequence  $\lambda_1 \geq \dots \geq \lambda_k$  such that  $\sum_{i=1}^k \lambda_i = n$ . We write  $\lambda \vdash n$ . A **Young Tableau** of shape  $\lambda$  is obtained by taking the corresponding Young diagram and filling in the boxes with  $1, 2, \dots, n$  bijectively.

*Remark.* If  $T$  is a  $\lambda$ -Tableau and  $\lambda \vdash n$ , then  $S_n$  acts on  $T$  by  $(\sigma T)(i, j) = \sigma(T(i, j))$ . We say that  $\lambda$ -Tableaux  $T_1$  and  $T_2$  are row-equivlent if their rows contain the same entries.

Let  $T_1, \dots, T_{n!}$  be the  $\lambda$ -Tableaux where  $\lambda \vdash n$  is fixed. Take  $V = \text{span}_{\mathbb{C}}\{T_1, \dots, T_k\}$  which is a  $\mathbb{C}[S_n]$ -module via the previous group action. This is not irreducible:  $W_1 = \text{span}_{\mathbb{C}}\{t_1 + \dots + T_k\}$ .

Let  $[T]$  be the set of row-equivalence classes, say  $[T_1], \dots, [T_\ell]$ . Then  $W_2 = \text{span}_{\mathbb{C}}\{\sum_{T \in [T_1]} T, \dots, \sum_{T \in [T_\ell]} T\}$ .

**Definition.** The row equivalence classes  $[T]$  are called  $\lambda$ -**tabloids**.

Note that  $S_n$  acts on the set of  $\lambda$ -tabloids by  $\sigma \cdot [T] = [\sigma T]$ . Say  $[T_1], \dots, [T_k]$  are all distinct  $\lambda$ -tabloids. We call the  $\mathbb{C}[S_n]$ -module  $M^\lambda = \text{span}_{\mathbb{C}}\{[T_1], \dots, [T_k]\}$  the **permutation module** associated to  $\lambda$ .

Take  $\lambda = (n)$ . Then  $M^\lambda = \text{span}_{\mathbb{C}}\{\}$ .

*Remark.* Let  $\lambda \vdash n$ ,  $M^\lambda$ , with  $[T] \in M^\lambda$ . Let  $[U] \in M^\lambda$ . Then, there exists  $\sigma \in S_n$  such that  $\sigma T = U$ , so  $\sigma[T] = [U]$ . Thus  $M^\lambda = \mathbb{C}[G][T]$  for any  $\lambda$ -tabloid  $[T]$ .

### SPECHT MODULE

**Definition.** Let  $\lambda, T$  a  $\lambda$ -tableau. Then the **row-stabilizer**

$$R_T = \{\sigma \in S_n : \text{for all } i, j, (\sigma T)(i, j), T(i, j) \text{ are in the same row}\}$$

and likewise the **column-stabilizer**. These are subgroups of  $S_n$ .

*Remark.*  $\sigma \in R_T$  if and only if  $\sigma[T] = [\sigma T] = [T]$ .

Let  $H \subseteq S_n$ . Then  $H^- = \sum_{\sigma \in H} \text{sgn}(\sigma) \sigma \in \mathbb{C}[S_n]$ . Define  $K_T = C_T^-$ .

**Definition.** Let  $T$  be a  $\lambda$ -tableau. The **polytabloid** associated to  $T$  is  $e_T = K_T[T] \in M^\lambda$ .

*Example.* Consider  $T = \begin{array}{|c|c|c|} \hline 4 & 1 & 3 \\ \hline 2 & 5 & \\ \hline \end{array}$ . Then  $C_T = \{(15), (24), (15)(24), \epsilon\}$ ,  $K_T = \epsilon + (15)(24) - (15) - (24)$ ,  $e_T = [T] + [(15)(24)T] - [(15)T] - [(24)T]$ .

**6.1 Lemma.** Let  $\lambda \vdash n$ ,  $T$  a  $\lambda$ -tableau. Let  $\pi \in S_n$ . Then

1.  $R_{\pi T} = \pi R_T \pi^{-1}$
2.  $C_{\pi T} = \pi C_T \pi^{-1}$
3.  $K_{\pi T} = \pi K_T \pi^{-1}$
4.  $e_{\pi T} = \pi e_T$

PROOF 1.

$$\begin{aligned} \sigma \in R_{\pi T} &\iff \sigma[\pi T] = [\pi T] \\ &\iff \sigma \pi[T] = \pi[T] \\ &\iff \pi^{-1} \sigma \pi[T] = [T] \\ &\iff \pi^{-1} \sigma \pi \in R_T \\ &\iff \sigma \in \pi R_T \pi^{-1} \end{aligned}$$

2. Identical to (1)
- 3.

$$\begin{aligned}
 K_{\pi T} &= C_{\pi T}^{-1} = \sum_{\sigma \in C_{\pi T}} \text{sgn}(\sigma) \sigma \\
 &= \sum_{\sigma \in C_T} \pi^{-1} \text{sgn}(\sigma) \sigma = \sum_{\sigma \in C_T} \text{sgn}(\pi \sigma \pi^{-1}) \pi \sigma \pi^{-1} \\
 &= \pi \left( \sum_{\sigma \in C_T} \text{sgn}(\sigma) \sigma \right) \pi^{-1} \\
 &= \pi K_T \pi^{-1}
 \end{aligned}$$

$$4. e_{\pi T} = K_{\pi T}[\pi T] = \pi K_T \pi^{-1} \pi[T] = \pi K_T[T] = \pi e_T \quad \blacksquare$$

*Remark.*  $\text{span}_{\mathbb{C}}\{e_T : T \text{ } \lambda\text{-tableau}\}$  is a  $\mathbb{C}[S_n]$ -submodule.

**Definition.** Let  $\lambda \vdash n$ . The submodule  $S^\lambda = \text{span}_{\mathbb{C}}\{e_T : T \text{ } \lambda\text{-tableau}\}$  is called the **Specht module** associated to  $\lambda$ .

*Example.* Let  $\lambda = (n)$  and  $T = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \end{bmatrix}$ . Then  $M^\lambda = \text{span}_{\mathbb{C}}\{[T]\}$ ,  $C_T = \{\epsilon\}$ ,  $K_T = 1$ . Then  $E_T = K_T[T] = [T]$ , so if  $U$  is a  $\lambda$ -tableau,  $\pi T = U$  so  $\pi e_T = e_{\pi T} e_U$  and  $S^\lambda = M^\lambda$ .

*Example.*  $\lambda = (1, 1, \dots, 1) \vdash n$ , and  $T$  a  $\lambda$ -tableau. Then  $C_T = S_n$ , so  $K_T = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma$ . Then if  $\pi \in S_n$ ,

$$\begin{aligned}
 e_{\pi T} &= \pi e_T = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \pi \sigma [T] \\
 &= \sum_{\tau \in S_n} \text{sgn}(\pi^{-1} \tau) \tau [T] \\
 &= \text{sgn}(\pi) e_T
 \end{aligned}$$

*Example.* Consider  $\lambda = (n-1, 1) \vdash n$ , and  $T = \begin{bmatrix} i & * & * & * \\ j & & & \end{bmatrix}$ . Set  $v_j := [T]$ . Then  $C_T = \{\epsilon, (ij)\}$ ,  $K_T = \epsilon - (ij)$ ,

$$e_T = (\epsilon - (ij))[T] = (\epsilon - (ij))v_j = v_j - v_i$$

so that  $S^\lambda = \text{span}_{\mathbb{C}}\{v_i - v_j : 1 \leq i < j \leq n\}$ , which has basis  $\{v_1 - v_2, v_1 - v_3, \dots, v_1 - v_n\}$  and  $\dim_{\mathbb{C}} S^\lambda = n - 1$ .

**6.2 Lemma.** Let  $H \leq S_n$ .

1. If  $\pi \in H$ , then  $\pi H^- = H^- \pi = \text{sgn}(\pi) H^-$
2. If  $u, v \in M^\lambda$ , then  $\langle H^- u, v \rangle = \langle u, H^- v \rangle$ .
3. If the transposition  $(bc) \in H$ , then  $H^- = x(\epsilon - (bc))$  for some  $x \in \mathbb{C}[S_n]$ .
4. Let  $\lambda \vdash n$  and  $T$  is a  $\lambda$ -tableau. If  $b, c$  are in the same row of  $T$  and  $(bc) \in H$ , then  $H^-[T] = 0$ .

**PROOF** 1. Direct computation

2.  $\delta_{[T],[U]} = \delta_{[\pi T],[\pi U]}$ , and moreover  $\langle [T], [U] \rangle = \langle \pi[T], \pi[U] \rangle$ , so that

$$\begin{aligned} \langle H^-u, v \rangle &= \sum_{\pi \in H} \langle \text{sgn}(\pi)\pi u, v \rangle \\ &= \sum_{\pi \in H} \langle \pi u, \text{sgn}(\pi)v \rangle \\ &= \sum_{\pi \in H} \langle u, \text{sgn}(\pi^{-1})\pi^{-1}v \rangle \\ &= \langle u, H^-v \rangle \end{aligned}$$

3. Let  $(bc) \in H$  so that  $K = \{\epsilon, (bc)\} \leq H$ . Say  $S$  is a set of coset reps of  $K$  in  $H$ . Then

$$\left( \sum_{\tau \in S} \text{sgn}(\tau)\tau \right) (\epsilon - (bc)) = H^-$$

4. Since  $(bc) \in H$ ,  $H^- = x(\epsilon - (bc))$ . Then  $H^-[T] = x(\epsilon - (bc))[T] = x([T] - [(bc)T]) = 0$  since  $[(bc)T] = [T]$  because  $b, c$  are in the same row. ■

**Definition.** Let  $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$  and  $\mu = (\mu_1, \dots, \mu_m) \vdash n$ , we say that  $\lambda$  **dominates**  $\mu$ , or  $\mu \leq \lambda$ , if for all  $i \geq 1$ ,  $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ , where  $\lambda_i = 0$  for  $i > k$  and  $\mu_i = 0$  for  $i > m$ .

**6.3 Lemma.** Let  $T$  be a  $\lambda$ -tableau,  $U$  and  $\mu$ -tableau,  $\lambda, \mu \vdash n$ . If the elements of a row of  $S$  are all in different columns of  $T$ , then  $\lambda \geq \mu$ .

**PROOF** For every  $i$ , we can sort the entries of  $S$  so that they occur in the first  $i$  rows of  $T$ . Then  $\lambda_1 + \dots + \lambda_i$  is the number of entries of  $T$  in the first  $i$  rows, which is greater than or equal to the number of entries of  $S$  in the first  $i$  rows, which is  $\mu_1 + \dots + \mu_i$ . ■

**6.4 Corollary.** Let  $T$  be a  $\lambda$ -tableau,  $U$  and  $\mu$ -tableau,  $\lambda, \mu \vdash n$ . Suppose  $K_T[S] \neq 0$ .

1.  $\lambda \geq \mu$
2. If  $\lambda = \mu$ , then  $K_T[S] = \pm e_T$ .

**PROOF** 1. Suppose  $K_T[S] \neq 0$ . Suppose  $b, c$  are entries in the same row of  $S$ . For contradiction, suppose  $b, c$  are in the same column of  $T$ , i.e.  $(bc) \in C_T$ . Then by (3) in the lemma,  $K_T = C_T^- = x(\epsilon - (bc))$  for some  $x \in \mathbb{C}[S_n]$ . Thus  $K_T[S] = 0$ , a contradiction, so by the previous lemma, the result follows.

2. By the argument of the previous lemma,  $[S] = \pi[T]$  where  $\pi \in C_T$ . Thus  $K_T[S] = K_T\pi[T] = \text{sgn}(\pi)K_T = \pm e_T$ . ■

**6.5 Corollary.** Let  $u \in M^\lambda$ ,  $T$  a  $\lambda$ -tableau. Then  $K_T u = f e_T$  for some  $f \in \mathbb{C}$ .

**PROOF**  $u = \sum c_i [S_i]$  where  $S_i$  are  $\lambda$ -tableau. Then  $K_T u = \sum c_i K_T [S_i]$  and apply the previous lemma. ■

**6.6 Theorem.** Let  $u$  be a submodule of  $M^\lambda$ . Then  $S^\lambda \subseteq U$  or  $U \subseteq (S^\lambda)^\perp$ . In particular,  $S^\lambda$  is irreducible.

**PROOF** Let  $U$  be a submodule of  $M^\lambda$  and let  $u \in U$ . Say  $T$  is a  $\lambda$ -tableau. Then  $K_T U = f e_T$  for some  $f \in \mathbb{C}$ .

1. Suppose there exists  $U \in U$  and  $\lambda$ -tableau  $T$  such that  $f \neq 0$ . Thus  $e_T = f^{-1} K_T u \in U$ . If  $S$  is a  $\lambda$ -tableau, then  $S = \pi T$ ,  $\pi \in S_n$ . Moreover,  $e_S = e_{\pi T} = \pi e_T \in U$  so  $S^\lambda \subseteq U$ .
2. Suppose  $K_T u = 0$ . Let  $u \in U$ ,  $T$  a  $\lambda$ -tableau be arbitrary. Then

$$\langle u, e_T \rangle = \langle u, K_T[T] \rangle = \langle K_T u, [T] \rangle = 0 \quad \blacksquare$$

**6.7 Proposition.** Suppose  $0 \neq \theta \in \text{Hom}_{\mathbb{C}[S_n]}(S^\lambda, M^\mu)$ . Then  $\lambda \mu$ . If  $\lambda = \mu$ , then  $\theta$  is multiplication by a scalar.

**PROOF** It suffices to find a  $\lambda$ -tableau  $T$  and a  $\mu$ -tableau  $S$  such that  $K_T[S] \neq 0$ . Observe that  $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$  as vector spaces. Moreover, for  $x \in (S^\lambda)^\perp$ ,  $T$  a  $\lambda$ -tableau, and  $\pi \in S_n$ ,

$$\langle \pi x, e_T \rangle = \langle \pi \pi^{-1} x, \pi^{-1} e_T \rangle = \langle x, e_{\pi^{-1} T} \rangle = 0$$

so that  $(S^\lambda)^\perp$  is a submodule. We can extend  $\theta$  to  $\theta \in \text{Hom}_{\mathbb{C}[S_n]}(M^\lambda, M^\mu)$  by setting  $\theta((S^\lambda)^\perp) = 0$ . Since  $\theta \neq 0$ , there exists a  $\lambda$ -tableau  $T$  such that  $\theta(e_T) \neq 0$ . Say  $\theta(e_T) = \sum c_i [S_i]$  where each  $S_i$  has shape  $\mu$ . Now,

$$\begin{aligned} 0 \neq \theta(e_T) &= \theta(K_T[T]) = K_T \theta([T]) \\ &= \sum c_i K_T[S_i] \end{aligned}$$

so there exists some  $i$  such that  $K_T[S_i] \neq 0$ . Thus  $\lambda \succeq \mu$ .

Now suppose  $\lambda = \mu$ . Then

$$\theta(e_T) = K_T \theta([T]) = \sum c_i K_T[S_i] = \sum c_i f_i e_T = \alpha e_T$$

where  $\alpha = \sum c_i f_i$ . Thus for  $\pi \in S_n$ ,  $\theta(e_{\pi T}) = \theta(\pi e_T) = \pi \theta(e_T) = \pi \alpha e_T = \alpha e_{\pi T}$ .  $\blacksquare$

**6.8 Theorem.** The Specht modules are all the irreducible  $\mathbb{C}[S_n]$ -modules up to isomorphism.

**PROOF** It suffices to show that  $S^\lambda \not\cong S^\mu$  for  $\mu \neq \lambda$ . Suppose  $S^\lambda \cong S^\mu$ . The isomorphism between  $S^\lambda$  and  $S^\mu$  gives nonzero  $\theta_1 \in \text{Hom}(S^\lambda, M^\mu)$  and  $\theta_2 \in \text{Hom}(S^\mu, M^\lambda)$ . Thus  $\lambda \succeq \mu$  and  $\mu \succeq \lambda$ , so  $\lambda = \mu$ .  $\blacksquare$

**Definition.** A young tableau is **standard** if its entries are increasing along rows and down columns.

**6.9 Theorem.** A basis for  $S^\lambda$  over  $\mathbb{C}$  is  $\{e_T : T \text{ standard } \lambda\text{-tableau}\}$ .

**Definition.** Let  $T$  be a Young diagram. The **hook length** of the  $i, j$  entry of  $T$  is  $h_{ij}$ , which is the sum of all the blocks to the right and below  $ij$  plus 1.

**6.10 Theorem. (Hook-length Formula)** The number of standard  $\lambda$ -tableau is  $\frac{n!}{\prod_{i,j} h_{ij}}$ .

In particular,  $\dim_{\mathbb{C}} S^\lambda = \frac{n!}{\prod_{i,j} h_{ij}}$ .

*Example.* Take  $\lambda = (3, 2) \vdash 5$ . Then  $\dim S^\lambda = \frac{5!}{4 \cdot 3 \cdot 2} = 5$ .

Final Exam

1. True/False - 10 qs
2. Assignment questions (questions 2 through 5)
3. Frobenius reciprocity / Mackey irreducibility