Fractal Geometry

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Winter 2020[†]

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[†] Last updated: January 9, 2020

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I. Topics in Fractal Geometry

1 Dimension Theory

1.1 The Cantor Set

Define maps $f_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ given by $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$. Let $C_0 = [0, 1]$; given some C_k, define $C_{k+1} = f_1(C_k) \cup f_2(C_k)$; since the f_i are linear, C_k is compact. We thus define $C_{1/3} = \bigcap_{n=0}^{\infty} C_n$, the classical **Cantor set**.

If $x \in C_{1/3}$, then x is an accumulation point: given $\epsilon > 0$, get N so that $3^{-N} < \epsilon$ then and thus some endpoint of C_N disjoint from x is within distance ϵ of x. Thus $C_{1/3}$ is a perfect set and therefore uncountable. Another way to see that the Cantor set is uncountable is to note that $C_{1/3}$ is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ with the product topology (via ternary expansions). Moreover, since $\lambda(C_{1/3}) \leq \lambda(C_n) = \frac{2^n}{3^n}$ for any $n \in \mathbb{N}$ we see that $\lambda(C_{1/3}) = 0$.

More generally, we may define C_r where $r \in (0, 1/2)$ by the above process with the functions $f_1(x) = rx$ and $f_2(x) = rx + 1 - r$. Again, $C_r \cong \{0, 1\}^{\mathbb{N}}$ topologically and $\lambda(C_r) = 0$; but already, we see that our classical analytic perspectives (topological, Lebesgue-measure-theoretic, cardinality) are insufficient to distinguish the C_r for distinct r.

1.2 Box Dimensions

Definition. Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set, and for each $\delta > 0$, let $N_\delta(E)$ be the least number of closed balls of diameter δ. We then define the **upper box dimension** of E

$$\overline{\dim}_B E = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{|\log \delta|}$$

and similarly $\underline{\dim}_B E$ (the **lower box dimension**) with a lim inf in place of lim sup. If $\underline{\dim}_B E = \overline{\dim}_B E$, then we define the **box dimension** to be this shared quantity.

If I is any interval, it is easy to see that $\dim_B I = 1$. [***TODO: include proof of invariance on choice of ball***] Note that if $N_\delta(E) \sim \delta^{-s}$, then $\dim_B E = S$.

Example. Let's show that the box dimension of $C_{1/3}$ exists, and compute it. Given some $\delta > 0$, let n be so that $3^{-n} \leq \delta < 3^{-(n-1)}$. Certainly we can cover $C_{1/3}$ by Cantor intervals of level n, so that $N_\delta(C_{1/3}) \leq 2^n$. Moreover, the endpoints of Cantor inverals of level $n - 1$ are distance at least $3^{-(n-1)} > \delta$ apart. Thus $N_\delta(C_{1/3})$ is at least the number of endpoints of level $n - 1$, i.e. $N_\delta(C_{1/3}) \geq 2^n$. Thus $N_\delta(C_{1/3}) = 2^n$, so that

$$\frac{\log 2}{\log 3} = \frac{\log 2^n}{\log 3^n} \leq \frac{\log N_\delta(C_{1/3})}{|\log \delta|} \leq \frac{\log 2^n}{\log 3^{n-1}} = \frac{n}{n-1} \cdot \frac{\log 2}{\log 3}$$

and, as $\delta \to 0$, $n \to \infty$ so that the $C_{1/3} = \frac{\log 2}{\log 3}$.

More generally, using the same technique, we may compute $C_r = \frac{\log 2}{\log 1/r}$.

However, the box dimension has poor properties: for example, we may verify $\dim_B\{0, 1, 1/2, 1/3, \ldots\} = \frac{1}{2}$. But this is very concerning from a measure theoretic perspective, since this is a countable set with larger "dimension" than some uncountable sets (e.g. C_r for small r).

1.3 Constructing Measures in Metric Spaces

Let X be a metric space.

Definition. Given $A, B \subseteq X$, say $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$. Say A, B have **positive separation** if $d(A,B) > 0$.

If A, B are compact and disjoint, then they have positive separation. We say that an outer measure μ^* is a **metric outer measure** if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ when A, B have positive separation.

Example. The Lebesgue outer measure is a metric outer measure. [***TODO: prove***]

> ***1.1 Theorem.*** *μ^* is a metric outer measure if and only if every Borel set is μ^*−measurable (in the sense of Caratheodory).*

Proof [***TODO: prove this (homework), and find a proof of the converse? (may not be true)***] ■

Suppose $\mathcal{A} \subseteq \mathcal{B}$ are both covers of X containing $\emptyset$ and $\mathcal{C} : \mathcal{B} \to [0,\infty]$ with $\mathcal{C}(\emptyset)$. Let $\mu^*_{\mathcal{A}}$ and $\mu^*_{\mathcal{B}}$ be the corresponding extensions of $\mathcal{C}$ and $\mathcal{C}|_{\mathcal{A}}$. Then by definition, $\mu^*_{\mathcal{B}}(E) \leq \mu^*_{\mathcal{A}}(E)$ for all $E \in \mathcal{P}(X)$.

Let X be a metric space, $\mathcal{A}$ cover X containing $\emptyset$. Suppose for each $x \in X$ and $\delta > 0$, there exists $A \in \mathcal{A}$ such that $x \in A$ and $A \leq \delta$. Let $\mathcal{C} : \mathcal{A} \to [0,\infty]$ with $\mathcal{C}(\emptyset) = 0$. Set $\mathcal{A}_\epsilon = \{A \in \mathcal{A} : (A) \leq \epsilon\}$, and define μ^*_ϵ by extending $\mathcal{C}|_{\mathcal{A}_\epsilon}$. In particular, as ϵ decreases, μ^*_ϵ increases, and define

$$\mu^*(E) = \sup_\epsilon \mu^*_\epsilon(E) = \lim_{\epsilon \to 0} \mu^*_\epsilon(E)$$

> ***1.2 Theorem.*** *As defined above, μ^* is a metric outer measure.*

Proof [***TODO: prove this, homework***] ■

Example. The Lebesgue measure arises this way; in fact, the μ^*_ϵ are all the same outer measure.

1.4 Hausdorff Measure and Dimension

For the remainder of this chapter, if X is a metric space and $U \subseteq X$, we denote $|U| = (U)$.

Definition. A **δ−cover** of a set $F \subseteq X$ is any countable collection $\{U_n\}_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty U_n \supseteq F$ and $|U_n| \leq \delta$.

Let $\mathcal{A} = \mathcal{P}(X)$, and $\mathcal{A}_\delta = \{A \subseteq X : |A| \leq \delta\}$. For $\delta \geq 0$, put $\mathcal{C}_s(A) = |A|^s$. Then for $s \geq 0$, $\delta > 0$, and $E \subseteq$, we define

$$\begin{aligned} H^s_\delta(E) &= \inf\left\{\sum_{n=1}^\infty |U_n|^s : \{U_n\} \text{ is a } \delta\text{−cover of } E\right\} \\ &= \inf\left\{\sum_{n=1}^\infty \mathcal{C}_s(U_n) : \bigcup_{n=1}^\infty U_n \supseteq E, U_n \in \mathcal{A}_\delta\right\} \end{aligned}$$

This is the outer measure as constructed in **??** with covering family A_δ and function $\mathcal{C}_s$. In particular, as $\delta \to 0$, H^s_δ increases; in particular, by Theorem 1.2, $H^s(E) = \sup_\delta H^s_\delta(E)$ is a metric outer measure. Then apply Caratheodory (**??**) to get the s−dimensional Hausdorff measure, which is a complete Borel measure.

Example. (i) H^0 is the counting measure on any metric space.

(ii) Take $X = \mathbb{R}$ and $s = 1$. Then H^1 is the Lebesgue measure (on Borel sets). To see this, we have

$$\begin{aligned}\lambda(E) &= \inf\left\{\sum_{n=1}^{\infty}|I_n| : \bigcup_{n=1}^{\infty} I_n \supseteq E, |I_n| \le \delta\right\}\\ &\ge H^1_\delta(E)\end{aligned}$$

for any $\delta > 0$; and conversely, take any δ−cover of E, say $\{U_n\}_{n=1}^{\infty}$ and set $I_n = \overline{\operatorname{conv} U_n}$ so $|I_n| = |U_n| \le \delta$. Thus $\sum_{n=1}^{\infty}|U_n| = \sum_{n=1}^{\infty}|I_n| \ge \lambda(E)$ for any such cover, so $\lambda(E) = H^1_\delta(E)$ for any $\delta > 0$. Thus $\lambda(E) = H^1(E)$ for any Borel set E.

(iii) More generally, if $X = \mathbb{R}^n$ and $s = n$, then $\lambda = \pi_n \cdot H^n$ where π_n is the n−dimensional volume of the ball of diameter 1. [***TODO: this is annoying exercise***]

Suppose $s < t$. Then $H^s(E) \ge H^t(E)$.

II. Stochastic Calculus

Definition. Given a measure space $(\Omega, \mathcal{F}, \mathbb{P})$, a measurable function $f : \Omega \to \mathbb{R}$ is called a **random variable**.

Definition. A **stochastic process** $X = \{X_t\}_{t \in T}$ is a collection of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Typically $t \in \mathbb{Z}^+$ or $t \in \mathbb{R}^+$ (including 0); t is a discrete or continuous time parameter. Given some $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is called a **realization** or **path** of this process. We will regard $\{X_t\}_{t \geq 0}$ as a random element in some path space, equipped with a proper $\sigma-$algebra and probability.

Consider $X_t(\omega)$ as a function $X : [0, \infty) \times \Omega \to \mathbb{R}$ equipped with the product $\sigma-$algebra.

Definition. The **distribution** of a stochastic process is the collection of all its finite-dimensional distributions.

Two processes X and Y can be "the same" in different senses:

Definition. Two process $X = \{X_t\}_{t \geq 0}$ and $Y = \{Y_t\}_{t \geq 0}$ are called **distinguishable** if almost all their sample paths agree; in other words, $\mathbb{P}(X_t = Y_t, 0 \leq t < \infty) = 1$. We say that Y is a **modification** of X if for each $t \geq 0$ we have $\mathbb{P}(X_t = Y_t) = 1$. Finally, X and Y are said to have the **same distribution** if all the finite dimensional distributions agree. In other words, if for all $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, we have $(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (Y_{t_1}, \ldots, Y_{t_n})$.

Example. Let X be a continuous stochastic process and N a Poisson point process on $[0, \infty)$. Then define

$$Y_t := \begin{cases} X_t & : t \notin N \\ X_t + 1 & : t \in N \end{cases}$$

Thus $\mathbb{P}(X_t = Y_t) = 1$ for all t, so X is a modification of Y. However, $\mathbb{P}(X_t = Y_t, t \geq 0) = 0$, so that X and Y are not indistinguishable.

A filtration formalizes the idea of "information acquired over time".

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **filtration** is a non-decreasing family $\{\mathcal{F}_t\}_{t \geq 0}$ of sub-σ-algebras of $\mathcal{F}$ so that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. We write $F_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$.

Let $\{X_t\}_{t \geq 0}$ be a stochastic process. The filtration generated by $\{X_t\}_{t \geq 0}$ is $\{\sigma(X_s : 0 \leq s \leq t)\}_{t \geq 0}$, in other words $\mathcal{F}_t$ is the smallest $\sigma-$algbra which makes X_s measurable for all $s \in [0, t]$.

Definition. A stochastic process $\{X_t\}_{t \geq 0}$ is called **adapted** to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if X_t is $\mathcal{F}_t-$measurable for every $t \geq 0$.

The filtration generated by $\{X_t\}_{t \geq 0}$ is the smallest filtration which makes $(X_t)_{t \geq 0}$ adapted.

A filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is said to satisfy the "usual condition" if

1. It is right-continuous: $\lim_{s \to t^+} := \bigcap_{s > t} \mathcal{F}_s = \mathcal{F}_t$
2. $\mathcal{F}_0$ contains all the $\mathbb{P}-$null events in $\mathcal{F}$.

2 Martingale Theory

Consider a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in S})$ where $S = \mathbb{N}$ or $S = \mathbb{R}^+$.

Definition. A **random time** T is called a stopping time if $\{T \le t\} \in \mathcal{F}_t$ ("we know it happens when it happens").

Example. (i) Constants are trivial stopping times.

(ii) Last hit a constant before N is not a stopping time

2.1 Proposition. *If S, T are stopping times, $T \vee S$, $T \wedge S$, $T + S$ are stopping times.*

Proof That $T \wedge S$ and $T \vee S$ are stopping times are trivial. For $T + S$, $\{T + S > t\} = \{T = 0, S > t\} \cup \{0 < T \le t, T + S > t\} \cup \{T > t\}$. It suffices to prove that

$$\{0 < T \le t < T + S > t\} = \bigcup_{\substack{r\in\mathbb{Q}^+ \\ 0<r<t}} \{r < T \le t, S > t - r\}.$$

If there exists r with $r < T \le t$, then $S > t = r$ and $S + T > r + (t - r) = t$, so $\supseteq$ holds. Conversely, if $0 < T \le t$ and $T + S \ge t$¡ then there exists $r \in \mathbb{Q}$ such that $r < T$ and $r + S > t$. Hence $r < T \le t$ and $S > t - r$. ■

Definition. The σ−algebra generated by a stopping time T is the collection of all the events A for which $A \cap \{T \le t\} \in \mathcal{F}_t$ for every $t \ge 0$. This is the "information you collect until the stopping time".

Exercise: show that the collection given in the definition above is actually a σ−algebra.

We write $X_{T\wedge t}$ is a random variable evaluated at time $T \wedge t$ (or T); in other words, $(X_{T\wedge t})(\omega) = X_{T\wedge t}(\omega)$. Then $\{T_{T\wedge t}\}_{t\ge 0}$, or X^T, is a stochastic process stopped at time t.