Fractal Geometry

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Contents

I. Topics in Fractal Geometry

1 Dimension Theory

1.1 Constructing Measures in Metric Spaces

[***TODO: fill in proofs and transfer to measure section***] Let X be a metric space.

Definition. Given $A, B \subseteq X$, say $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$. Say A, B have **positive separation** if $d(A,B) > 0$.

If A, B are compact and disjoint, then they have positive separation. We say that an outer measure μ^* is a **metric outer measure** if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ when A, B have positive separation.

Example. The Lebesgue outer measure is a metric outer measure. [***TODO: prove***]

> ***1.1 Theorem.*** *μ^* is a metric outer measure if and only if every Borel set is μ^*−measurable (in the sense of Caratheodory).*

Proof [***TODO: prove this (homework), and find a proof of the converse? (may not be true)***] ■

Suppose $\mathcal{A} \subseteq \mathcal{B}$ are both covers of X containing $\emptyset$ and $\mathcal{C} : \mathcal{B} \to [0,\infty]$ with $\mathcal{C}(\emptyset)$. Let $\mu^*_{\mathcal{A}}$ and $\mu^*_{\mathcal{B}}$ be the corresponding extensions of $\mathcal{C}$ and $\mathcal{C}|_{\mathcal{A}}$. Then by definition, $\mu^*_{\mathcal{B}}(E) \le \mu^*_{\mathcal{A}}(E)$ for all $E \in \mathcal{P}(X)$.

Let X be a metric space, $\mathcal{A}$ cover X containing $\emptyset$. Suppose for each $x \in X$ and $\delta > 0$, there exists $A \in \mathcal{A}$ such that $x \in A$ and $\operatorname{diam} A \le \delta$. Let $\mathcal{C} : \mathcal{A} \to [0,\infty]$ with $\mathcal{C}(\emptyset) = 0$. Set $\mathcal{A}_\epsilon = \{A \in \mathcal{A} : \operatorname{diam}(A) \le \epsilon\}$, and define μ^*_ϵ by extending $\mathcal{C}|_{\mathcal{A}_\epsilon}$. In particular, as ϵ decreases, μ^*_ϵ increases, and define

$$\mu^*(E) = \sup_\epsilon \mu^*_\epsilon(E) = \lim_{\epsilon \to 0} \mu^*_\epsilon(E)$$

> ***1.2 Theorem.*** *As defined above, μ^* is a metric outer measure.*

Proof [***TODO: prove this, homework***] ■

Example. The Lebesgue measure arises this way; in fact, the μ^*_ϵ are all the same outer measure.

1.2 The Subdivision Method

Definition. We say that a collection of subsets $\mathcal{C}$ is a **semi-algebra** if it contains $\emptyset$, is closed under finite intersections, and complements are finite disoint unions of sets in $\mathcal{C}$. We then say that μ is a **measure on a semi-algebra** if $\mu : \mathcal{C} \to [0,\infty]$ has

(i) $\mu(\emptyset) = 0$

(ii) If $E_1, \ldots, E_n \in \mathcal{C}$ are disjoint and $\bigcup_{i=1}^n E_i \in \mathcal{C}$, then $\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$.

(iii) If $\{E_i\}_{i=1}^\infty \in \mathcal{C}$ are pairwise disjoint and $\bigcup_{i=1}^\infty E_i \in \mathcal{C}$, then $\mu(\bigcup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu(E_i)$.
An **algebra** is a semi-algebra which is closed under finite unions and complements. Then a **measure on an algebra** is a map μ satisfying the same above constraints.

1.3 Theorem. *Let μ be a measure on a semi-algebra $\mathcal{C}$. Then μ has a unique extension to a measure on $\mathcal{A} = \langle \mathcal{C} \rangle$, the algebra generated by $\mathcal{C}$.*

Proof It is easy to verify that $\mathcal{A}$ is the set of all finite unions of elements in $\mathcal{C}$. Thus we extend μ to $\mathcal{A}$ where if $A = \dot{\bigcup}_{i=1}^n C_i$, set $\mu(A) = \sum_{i=1}^n \mu(C_i)$.

[***TODO: prove***] Check: well-defined and a measure ■

Let $\Sigma = \{1, \ldots, k\}$ and let Σ^* denote the set of all words on Σ. We then associate to Σ^* a heirarchy of subsets $\{X_\sigma : \sigma \in \Sigma^*\}$ with $X_\sigma \subseteq \mathbb{R}^n$. Set $\mathcal{E} = \{X_\sigma : \sigma \in \Sigma^*\}$. When we say heirarchy, we mean that for any $\sigma \in \Sigma^*$,

$$X_\sigma \supseteq \dot{\bigcup_{i=1}^k} X_{\sigma i}$$

disjointly. We also assume that for every infinite sequence $(i_1, i_2, \ldots)$, with $\sigma|j = (i_1, \ldots, i_j)$, $\lim_{j\to\infty} |X_{\sigma|j}| = 0$ and $\lim_{j\to\infty} \mu_0(X_{\sigma|j}) = 0$ uniformly with respect to length.

Suppose $\mu_0 : \mathcal{E} \to [0, \infty]$ is any function such that $\mu(X_\sigma) = \sum_{i=1}^k \mu(X_{\sigma i})$. Set $E_k = \bigcup_{\omega \in \Sigma^n} X_\omega$ and $E = \bigcap_{i=k}^\infty E_k$. Let $\mathcal{C} = \{\emptyset\} \cup \{X_\omega \cap E : \omega \in \Sigma^*\}$ and extend μ_0 to a function $\mu : \mathcal{C} \to [0, \infty]$ by the rule $\mu(X_\omega \cap E) = \mu_0(X_\omega)$. We then have the following result.

1.4 Proposition. *In the above construction, $\mathcal{C}$ is a semialgebra and μ is a measure on a semialgebra.*

Proof Closure under finite intersections is immediate since the X_σ are either nested are disjoint. Moreover,

$$(X_\omega \cap E)^c = \bigcup_{\substack{\sigma \in \Sigma^{|\omega|} \\ \sigma \neq \omega}} X_\sigma \cap E$$

is closed under complementation.

Let's first see that μ is a measure on a semi-algebra. We have $\mu(\emptyset) = 0$ by definition. Suppose $\bigcup_{i=1}^n X_{\sigma_i} = X_\tau$ for some $\tau \in \Sigma^*$. Clearly τ is a prefix of each σ_i. Let's prove by induction on $m = \max\{|\sigma_i| - |\tau| : 1 \leq i \leq n\}$ that the formula holds.

If $m = 0$, this is immediate since since the union is over a single element. Otherwise, suppose $m \in \mathbb{N}$ is arbitrary. Let $S = \{i : |\sigma_i| - |\tau| = m\}$ and partition S into classes $S_1, \ldots, S_k$ where σ_i and σ_j are in the same class if they have the same parent. But then for any S_i with common parent τ_i, we must have $\bigcup_{i \in S_i} X_{\sigma_i} \cap E = X_{\tau_i} \cap E$ disjointly, so that $\mu(X_{\tau_i} \cap E) = \sum_{i \in S_i} \mu(X_{\sigma_i} \cap E)$ by assumption on μ_0 above. Let $S_0 = \{1, \ldots, n\} \setminus \bigcup_{i=1}^k S_i$ denote the set of remainind indices. Then $X_\tau = \bigcup_{i \in S_0} X_{\sigma_i} \cup \bigcup_{i=1}^k X_{\tau_i}$ where $|\sigma_i| - |\tau| < m$ by definition of S_0 and $|\tau_i| - |\tau| < m$ since τ_i is a parent of some σ with $|\sigma| - |\tau| = m$. But then apply the induction hypothesis to get

$$\mu(X_\tau) = \sum_{i=1}^k \mu(X_{\tau_i} + \sum_{i \in S_0} X_{\sigma_i} = \sum_{i=1}^k \sum_{j \in S_i} \mu(X_{\sigma_i}) + \sum_{i \in S_0} X_{\sigma_i} = \sum_{i=1}^n \mu(X_{\sigma_i})$$

as required.

Finally, suppose $\bigcup_{i=1}^{\infty} X_{\sigma_i} = X_\tau$ for some $\tau \in \Sigma^*$. It suffices to show that $\mu(X_\tau) \le \sum_{i=1}^{\infty} \mu(X_{\sigma_i}) + \epsilon$ for any $\epsilon > 0$. If $\sum_{i=1}^{\infty} \mu(X_{\sigma_i}) = \infty$, this inequality holds trivially. Otherwise, there exists sume N such that $\sum_{i=N+1}^{\infty} \mu(X_{\sigma_i}) < \epsilon$. Then $\bigcup_{i=1}^{N} X_{\sigma_i} \subseteq X_\tau$. Let $m = \max\{|\sigma_i|\}$, and for any ω with $|\omega| = m$ and $X_\omega \subseteq X_\tau$, either $X_\omega \subseteq X_{\sigma_i}$ for some i or X_ω is disjoint from each X_{σ_i}. Then let $\{X_{\omega_1}, \dots, X_{\omega_m}\}$ be the maximal set of such ω such that X_ω is disjoint from each X_{σ_i} for all $1 \le i \le N$. But now $X_\tau = \bigcup_{i=1}^{N} X_{\sigma_i} \cup \bigcup_{i=1}^{m} X_{\omega_i}$, and apply the property proven earlier to get

$$\mu(X_\tau) \le \sum_{i=1}^{N} \mu(X_{\sigma_i}) \le \sum_{i=1}^{\infty} \mu(X_{\sigma_i}) + \epsilon$$

as required. Thus, μ is in fact a measure on a semi-algebra.

Thus, μ extends to the σ-algebra $\mathcal{M}$ generated by $\mathcal{C}$. It remains to show that $\mathcal{M}$ contains the Borel sets in E. To do this, it suffices to show that the outer measure μ^* is in fact a metric outer measure. Let $F_1, F_2 \subseteq E$ be arbitrary such that $\mathrm{dist}(F_1, F_2) \ge \delta > 0$. We wish to show for any $\epsilon > 0$ that

$$\mu^*(F_1) + \mu^*(F_2) \le \mu^*(F_1 \cup F_2) + \epsilon.$$

Get N such that whenever $|\omega| \ge N$, we have $|X_\omega| < \delta$. Write $E = \bigcup_{\omega \in \Sigma^N} X_\omega$. In particular, since $|X_\omega| < \delta$, we cannot have both $F_1 \cap X_\omega \neq \emptyset$ and $F_2 \cap X_\omega \neq \emptyset$.

Let $\{X_{\sigma_i}\}_{i=1}^{\infty}$ be a cover for $F_1 \cup F_2$ such that $\sum_{i=1}^{\infty} \mu(X_{\sigma_i}) < \mu^*(F_1 \cup F_2) + \epsilon$. By writing $X_{\sigma_i} = \bigcup_{\alpha \in \Sigma^N} X_{\sigma_i \alpha}$ (which does not change the value of the sum and still covers F_1), we may assume that $|X_{\sigma_i}| < \delta$. In particular, there exists a partition $\mathbb{N} = T_1 \cup T_2$ such that for each $i \in T_1$, X_{σ_i} intersects F_1 and not F_2, and similarly for each $i \in T_2$. But then $\{X_{\sigma_i}\}_{i \in T_1}$ is a cover for F_1, and $\{X_{\sigma_i}\}_{i \in T_2}$ is a cover for F_2, so

$$\mu^*(F_1) + \mu^*(F_2) \le \sum_{i \in T_1} \mu(X_{\sigma_i}) + \sum_{i \in T_2} \mu(X_{\sigma_i}) = \sum_{i=1}^{\infty} \mu(X_{\sigma_i}) \le \mu^*(F_1 \cup F_2) + \epsilon$$

as required. Thus μ^* is a metric outer measure, and hence the σ-algebra contains the Borel sets. ∎

1.3 Hausdorff Measure and Dimension

For the remainder of this chapter, if X is a metric space and $U \subseteq X$, we denote $|U| = \mathrm{diam}(U)$.

Definition. A **δ–cover** of a set $F \subseteq X$ is any countable collection $\{U_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} U_n \supseteq F$ and $|U_n| \le \delta$.

Let $\mathcal{A} = \mathcal{P}(X)$, and $\mathcal{A}_\delta = \{A \subseteq X : |A| \le \delta\}$. For $\delta \ge 0$, put $\mathcal{C}_s(A) = |A|^s$. Then for $s \ge 0$, $\delta > 0$, and $E \subseteq$, we define

$$\begin{aligned} H_\delta^s(E) &= \inf\left\{ \sum_{n=1}^{\infty} |U_n|^s : \{U_n\} \text{ is a } \delta\text{–cover of } E \right\} \\ &= \inf\left\{ \sum_{n=1}^{\infty} \mathcal{C}_s(U_n) : \bigcup_{n=1}^{\infty} U_n \supseteq E, U_n \in \mathcal{A}_\delta \right\} \end{aligned}$$

This is the outer measure as constructed in **??** with covering family A_δ and function $\mathcal{C}_s$. In particular, as $\delta \to 0$, H^s_δ increases; in particular, by Theorem 1.2, $H^s(E) = \sup_\delta H^s_\delta(E)$ is a metric outer measure. Then apply Caratheodory (**??**) to get the s–dimensional Hausdorff measure, which is a complete Borel measure.

Example. (i) H^0 is the counting measure on any metric space.

(ii) Take $X = \mathbb{R}$ and $s = 1$. Then H^1 is the Lebesgue measure (on Borel sets). To see this, we have

$$\begin{aligned} \lambda(E) &= \inf\left\{\sum_{n=1}^{\infty} |I_n| : \bigcup_{n=1}^{\infty} I_n \supseteq E, |I_n| \leq \delta\right\} \\ &\geq H^1_\delta(E) \end{aligned}$$

for any $\delta > 0$; and conversely, take any δ–cover of E, say $\{U_n\}_{n=1}^\infty$ and set $I_n = \overline{\operatorname{conv} U_n}$ so $|I_n| = |U_n| \leq \delta$. Thus $\sum_{n=1}^\infty |U_n| = \sum_{n=1}^\infty |I_n| \geq \lambda(E)$ for any such cover, so $\lambda(E) = H^1_\delta(E)$ for any $\delta > 0$. Thus $\lambda(E) = H^1(E)$ for any Borel set E.

(iii) More generally, if $X = \mathbb{R}^n$ and $s = n$, then $\lambda = \pi_n \cdot H^n$ where π_n is the n–dimensional volume of the ball of diameter 1.
We will verify that $H^n \leq m$ where m is n–dimensional Lebesgue measure on $\mathbb{R}^n$; the general result is harder and left as an exercise. To see this, we have

$$\begin{aligned} m(E) &= \inf\left\{\sum_{i=1}^{\infty} \operatorname{vol}(C_i) : C_i \text{ cube}, \bigcup_{i=1}^{\infty} C_i \supseteq E, \text{sides } \leq \frac{1}{\sqrt{n}}\delta\right\} \\ &= \inf\left\{\sum_{i=1}^{\infty}\left(\frac{1}{\sqrt{n}}\right)^n |C_i|^n : \{C_i\} - \delta\text{-cover of cubes of } E\right\} \\ &\geq c_n \inf\left\{\sum_{i=1}^{\infty} |c_i|^n : \text{all } \delta\text{–covers of } E = c_n H^n_\delta(E)\right\} \end{aligned}$$

where $c_n = (1/\sqrt{n})^n \leq 1$.

(iv) If $s < t$, then $H^s(E) \geq H^t(E)$.

Suppose $s < t$. Clearly $H^s(E) \geq H^t(E)$, but we can in fact make stronger statements. Suppose we have some U_i where $|U_i| \leq \delta$, and

$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^s |U_i|^{t-s} \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$

so that

$$H^t_\delta(E) \leq \delta^{t-s} \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i=1}^\infty \ \delta\text{–cover of } E\right\} = \delta^{t-s} H^s_\delta(E).$$

In particular, as $\delta \to 0$, $H^t_\delta(E) \to H^t(E)$ and $H^s_\delta(E) \to H^s(E)$ and $\delta^{t-s} \to 0$ since $s < t$. Thus if $H^s(E) \neq \infty$, then $H^t(E) = 0$ for all $t > s$. Similarly, if $H^t(E) > 0$, then $H^s(E) = \infty$ for all $s < t$. As a result, there exists some unique number $S_0 := \dim_H(E) \geq 0$ such that for all $s < S_0$, $H^s(E) = \infty$, and for all $t > S_0$, $H^t(E) = 0$. We call this value the **Hausdorff dimension** of E. Note that $H^{S_0}(E) \in [0, \infty]$ and all choices are possible.

Example. (i) Since $1 = m([0,1]) = H^1([0,1])$, $\dim_H[0,1] = 1$

(ii) $\dim_H \mathbb{R} = 1$ but $m(\mathbb{R}) = H^1(\mathbb{R}) = \infty$.
(iii) It is possible to have $S_0 = 1$ but $m(E) = 0$.
(iv) There is a Cantor-like set with Hausdorff-dimension 0.
(v) If E is countable and $s > 0$, $H^s_\delta(E) \leq \sum_{x\in E} |\{x\}|^s = 0$. In particular, there exist compact countable sets, and in this case, $\dim_H C = 0$ while $H^0(C) = \infty$.

Here are some basic properties of Hausdorff dimension.

1.5 Proposition. (Properties of Hausdorff Dimension) *(i) If $A \subseteq B$, then $\dim_H A \leq \dim_H B$.*
(ii) If $F \subseteq \mathbb{R}^n$, then $\dim_H F \leq n$.
(iii) If $U \subset \mathbb{R}^n$ is open, then $\dim_H U = n$.
(iv) If $F = \bigcup_{i=1}^\infty F_i$, then $\dim_H(F) = \sup_{i\in\mathbb{N}} \dim_H F_i$.

PROOF (i) If $H^s(B) = 0$, then $H^s(A) = 0$ by monotonicity of measures so $\dim_H A \leq \dim_H B$.
(ii) First consider the unit cube $I^n \subset \mathbb{R}^n$. Then

$$H^s_{\sqrt{n}\delta}(I^n) \leq \left(\frac{2}{\delta}\right)^n (\sqrt{n}\delta)^s = 2^n \sqrt{n}^n \delta^{s-n}$$

so if $s > n$, then $\delta^{s-n} \to 0$ as $\delta \to 0$. Thus for all $s > n$, $H^s(I^n) = \lim_{\delta\to 0} H^s_{\sqrt{n}\delta}(I^n) = 0$ so that $\dim_H(I^n) \leq n$. Moreover, $\mathbb{R}^n$ is the countable union of unit cubes, so that $H^s(\mathbb{R}^n) = 0$ and $\dim_H(\mathbb{R}^n) \leq n$. Then appeal to (i).
(iii) Cubes have positive Hausdorff n–measure.
(iv) If $s > \sup\{\dim_H F_i\}$, then $H^s(F_i) = 0$ for all i and by subadditivity $H^s(F) = 0$. Thus $s \geq \dim_H F$. By monotonicity, $\dim_H F \geq \dim_H F_j$ for all j. ■

Suppose $X = \mathbb{R}^n$, $E \subseteq \mathbb{R}^n$, $\lambda > 0$. Set $\lambda E = \{\lambda e : e \in E\}$: then $H^s(\lambda E) = \lambda^s H^s(E)$ since there is a bijection between δ–covers and $\lambda\delta$–covers.

Definition. Let X, Y be metric spaces. A function $f : X \to Y$ is called **Lipschitz** if there exists C such that $d(f(x), f(y)) \leq Cd(x,y)$.

Certainly if f is Lipschitz, then f is uniformly continuous. Functions $f : \mathbb{R} \to \mathbb{R}$ with bounded derivative are Lipschitz by the mean value theorem.

Definition. A function $f : X \to Y$ is **Hölder continuous** with exponent α if there exists c such that $d(f(x), f(y)) \leq cd(x,y)^\alpha$.

Example. (i) If $\alpha = 1$, then f is Lipschitz, and if $\alpha = 0$, then f is bounded.
(ii) If $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha > 0$, then f is constant (by considering derivatives). Thus the most interesting cases occur for $0 < \alpha \leq 1$.

1.6 Proposition. *If $f : X \to Y$ is Hölder continuous with exponent α. Then $H^{s/\alpha}(f(E)) \leq cH^s(E)$ for some constant c.*

PROOF If $\{U_i\}$ are a δ-cover of E, then $\{f(U_i)\}$ cover $f(E)$. Then $\operatorname{diam} f(U_i) = \sup\{d(f(x), f(y)) : x, y \in U_i\} \leq c \sup\{d(x,y)^\alpha : x, y \in U_i\} = C \cdot (\operatorname{diam} U_i)^\alpha$. Thus if $\{U_i\}$ is a δ–cover of E, then $\{f(U_i)\}$ is a $c\delta^\alpha$–cover of $f(E)$. Passing through the definition, we get $H^{s/\alpha} \leq c^{s/\alpha} H^s(E)$. ■

We then have the easy corollaries

1.7 Corollary. $\dim_H f(X) \leq \frac{1}{\alpha} \dim_H X$.

1.8 Corollary. *If f is an isometry, then $H^s(f(X)) = H^s(X)$.*

1.9 Corollary. *If $f : X \to Y$ are bi-Lipschitz, then $\dim_H X = \dim_H Y$.*

Example. Let C denote the Cantor set. Let's show that $\frac{1}{2} \leq H^s(C) \leq 1$ for $s = \frac{\log 2}{\log 3}$. In particular, this implies that $\dim_H C = \frac{\log 2}{\log 3}$.

Let $\delta = 3^{-n}$ and cover C with a δ–covering with generation n Cantor intervals. Then $H^s_\delta(C) \leq \sum_{I \in C_n} |I|^s = 2^n 3^{-ns} = 1$ by choice of s. Thus $\lim_{\delta \to 0} H^s_\delta(C) = \lim_{n \to \infty} H^s_{3^{-n}}(C) \leq 1$.

For the lower bound, take any δ–cover $\{U_i\}$ of C. Without loss of generality, we may assume that the U_i are open intervals. Since C is compact, get some finite subcover $U_1, \ldots, U_N$. For each i, get $k_i \in \mathbb{N}$ so that $3^{-(k_i+1)} \leq |U_i| < 3^{-k_i}$; set $k = \max\{k_1, \ldots, k_N\}$. Since U_i intersects at most 1 interval in C_{k_i}, U_i intersects at most 2^{k-k_i} intervals of C_k. Thus $2^k \leq \sum_{i=1}^N 2^{k-k_i}$ where $2^{k-k_i} = 2^k 3^{-sk_i} = 2^k 3^{-s(k_i+1)} \leq 2^k |U_i|^s 3^s$. Thus

$$2^k \leq \sum_{i=1}^N 2^k |U_i|^s 3^s$$

so $\frac{1}{2} = 3^{-s} \leq \sum_{i=1}^N |U_i|^s \leq \sum_{i=1}^\infty |U_i|^s$ so $H^s_\delta(C) \geq \frac{1}{2}$ so $H^s(C) \geq \frac{1}{2}$.

1.10 Proposition. *Let (X,d) be a metric space. If $\dim_H X < 1$, then X is totally disconnected.*

Proof Let $x \in X$ and define $f : X \to [0, \infty)$ by $f(z) = d(z,x)$. Then f is Lipschitz with constant 1 so $\dim_H f(X) \leq \dim_H X < 1$ so $m(f(X)) = 0$. Then if $y \neq x$, $d(y,x) = f(y) > 0$ while $f(x) = 0$. In particular, $(0, f(y)) \not\subset f(X)$ so there exists $0 < r < f(y)$ such that $r \notin f(X)$. Then $U_1 = \{z \in X : f(z) < r\}$ and $U_2 = \{z \in X : f(z) > r\}$ are disconnecting sets for X separating x and y.

1.4 Box Dimensions

Definition. Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set, and for each $\delta > 0$, let $N_\delta(E)$ be the least number of closed balls of diameter δ. We then define the **upper box dimension** of E

$$\overline{\dim}_B E = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{|\log \delta|}$$

and similarly $\underline{\dim}_B E$ (the **lower box dimension**) with a liminf in place of limsup. If $\underline{\dim}_B E = \overline{\dim}_B E$, then we define the **box dimension** to be this shared quantity.

If I is any interval, it is easy to see that $\dim_B I = 1$. Note that if $N_\delta(E) \sim \delta^{-s}$, then $\dim_B E = S$.

Example. Let's show that the box dimension of $C_{1/3}$ exists, and compute it. Given some $\delta > 0$, let n be so that $3^{-n} \leq \delta < 3^{-(n-1)}$. Certainly we can cover $C_{1/3}$ by Cantor intervals of level n, so that $N_\delta(C_{1/3}) \leq 2^n$. Moreover, the endpoints of Cantor inverals of level $n-1$ are distance at least $3^{-(n-1)} > \delta$ apart. Thus $N_\delta(C_{1/3})$ is at least the number of endpoints of level $n-1$, i.e. $N_\delta(C_{1/3}) \geq 2^n$. Thus $N_\delta(C_{1/3}) = 2^n$, so that

$$\frac{\log 2}{\log 3} = \frac{\log 2^n}{\log 3^n} \leq \frac{\log N_\delta(C_{1/3})}{|\log \delta|} \leq \frac{\log 2^n}{\log 3^{n-1}} = \frac{n}{n-1} \cdot \frac{\log 2}{\log 3}$$

and, as $\delta \to 0$, $n \to \infty$ so that the $\dim_B C_{1/3} = \frac{\log 2}{\log 3}$.

More generally, using the same technique, we may compute $\dim_B C_r = \frac{\log 2}{\log 1/r}$.

However, the box dimension has poor properties: for example, we may verify $\dim_B\{0, 1, 1/2, 1/3, \ldots\} = \frac{1}{2}$. In particular, the box dimension does not have countable stability (the box dimension of any singleton is 0). But this is very concerning from a measure theoretic perspective, since this is a countable set with larger "dimension" than some uncountable sets (e.g. C_r for small r).

1.11 Theorem. *The value of the various box dimensions are equal for all following definitions of $N_\delta(E)$:*

1. *least number of open balls of radius δ that cover E*
2. *least number of cubes of side length δ*
3. *the number of δ–mesh cubes that intersect E: $[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$ for $(m_1, \ldots, m_n) \in \mathbb{Z}^n$.*
4. *the largest number of disjoint closed balls of radius δ with centers in E.*

Proof Throughout, from the logarithms in the definition, it suffices to bound $N_\delta^{(i)}(E)$ with respect to $N_\delta(E)$ up to some constant factor either with respect to δ or with respect to N_δ.

1. Exercise.
2. Exercise.
3. In general, the diameter of a δ–cube in $\mathbb{R}^n$ is $\sqrt{n}\delta$. Let $N_\delta^{(3)}(E)$ denote the number of δ–mesh cubes intersecting E. Then the cubes which intersect E cover E and these have diameter $\sqrt{n}\delta$, so $N_{\sqrt{n}\delta}(E) \le N_\delta^{(3)}(E)$.
   Conversely, any set with diameter at most δ is contained in at most 3^n δ–mesh cubes. Thus $N_\delta^{(3)}(E) \le 3^n N_\delta(E)$.
4. Let $N_\delta^{(4)}$ denote the largest number of disjoint balls of radius δ centred in E. Say $B_1, \ldots, B_{N_\delta^{(4)}(F)}$ are such balls. If $x \in F$, then $d(x, B_i) \le \delta$ for some i, else $B(x, \delta)$ would be disjoint from all B_i, contradicting maximality. Thus the balls $B_1^1, \ldots, B^1_{N_\delta^{(4)}(E)}$ cover E and have diameter 4δ, so $N_{4\delta}(E) \le N_\delta^{(4)}(E)$.
   Conversely, let $U_1, \ldots, U_{N_\delta(E)}$ be any collection of sets of diameter at most δ that cover E. Let $B_1, \ldots, B_m$ be any disjoint balls with radius δ and centres $x_i \in E$. Since the U_j cover E, each $x_i \in U_{j(i)}$ for some $j(i)$ so $U_{j(i)} \subseteq B_i$ and $U_{j(i)} \cap B_k = \emptyset$ for $k \ne i$. Thus $N_\delta(E) \ge N_\delta^{(4)}(E)$. ∎

Note that, in the box dimension computation, it suffices to verify along a sequence of $(\delta_k)_{k=1}^\infty \to 0$ such that $\delta_{k+1} \ge c \cdot \delta_k$ for some $c > 0$ (i.e. not faster than exponentially).

1.12 Proposition. $\dim_H(E) \le \underline{\dim}_B(E)$.

Proof Suppose we cover E by $N_\delta(E)$ sets of diameter at most δ. Then $\inf\{\sum |U_i|^s : \{U_i\}\delta\text{–cover of } E\} \le \delta^s N_\delta(E)$ so that $H_\delta^s(E) \le \delta^s N_\delta(E)$. Suppose $s < \dim_H E$, so $H^s(E) > \lambda$ for some $\lambda > 0$. Then $\delta^s N_\delta(E) \ge \lambda$ so that $\frac{\log N_\delta(E)}{-\log \delta} \ge s + \frac{\log \lambda}{-\log \delta}$. Then as $\delta \to 0$, $\liminf \frac{\log N_\delta(E)}{-\log \delta} \ge s$. Thus $\underline{\dim}_B E \ge \dim_H E$. ∎

1.13 Proposition. (Properties of Box Dimension) *(i)* $\underline{\dim}_B E = \underline{\dim}_B \overline{E}$ *and* $\overline{\dim}_B E = \overline{\dim}_B \overline{E}$

(ii) $\underline{\dim}_B E = n$ *if* E *is dense in an open set in* $\mathbb{R}^n$.

(iii) $\overline{\dim}_B(E \cup F) = \max(\overline{\dim}_B E, \overline{\dim}_B F)$. *However,* $\underline{\dim}_B E \cup \underline{\dim}_B F \geq \max\{\underline{\dim}_B E, \underline{\dim}_B F\}$ *and the inequality can hold strictly.*

(iv) *Box dimension is Lipschitz invariant.*

1.14 Theorem. (Mass Distribution Principle) *Let* μ *be a finite Borel measure on* F *with* $\mu(F) > 0$. *Suppose there exists* $c > 0$ *and* $\delta_0 > 0$ *such that whenever* $|U| \leq \delta_0$, $\mu(U) \leq c|U|^s$. *Then* $H^s(F) \geq \frac{\mu(F)}{c} > 0$.

Proof Let $\{U_i\}$ be a δ-cover of F with $\delta \leq \delta_0$. Then $\mu(F) \leq \mu(\bigcup_{i=1}^\infty U_i) \leq \sum_{i=1}^\infty \mu(U_i) \leq c\sum_{i=1}^\infty |U_i|^s$. Thus $\inf\{\sum_{i=1}^\infty |U_i|^s : \{U_i\}\delta\text{-cover of } F\} \geq \frac{\mu(F)}{c}$ and let $\delta \to 0$. ■

Example. Let $C(r)$ denote the Cantor set with contraction ratio r. Define $\mu(I_\omega \cap C) = r^{|\omega|}$, and extend to the uniform r-Cantor measure. We now apply the mass distribution principle. Let U be arbitrary with $r^{k+1} \leq |U| < r^k$. Then U cannot intersect 3 level k intervals (or U would have diameter greater than r^k). Thus $\mu(U) = \mu(U \cap C) \leq c\mu(I_\omega) = 3^s$... So $\dim_G(C_r) = \frac{\log 2}{|\log r|}$.

1.15 Proposition. *Suppose* μ *is a finite Borel measure on* $\mathbb{R}^n$ *and* $F \subseteq \mathbb{R}^n$ *is Borel. Let* $0 < c < \infty$.

(i) *If* $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \leq c$ *for all* $x \in F$, *then* $H^s(F) \geq \frac{\mu(F)}{c}$

(ii) *If* $\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s} \geq c$ *for all* $x \in F$, *then* $\mathcal{P}^s(E) \leq \frac{2^s\mu(F)}{c}$.

(iii) *If* $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \geq c$ *for all* $x \in F$, *then* $H^s(F) \leq \frac{10^s}{c}\mu(\mathbb{R}^n) < \infty$.

(iv) *If* $\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s} \leq c$ *for all* $x \in F$, *then* $\mathcal{P}^s(E) \geq \frac{10^s\mu(F)}{c}$.

Proof (i) Fix $\epsilon > 0$. For each $\delta > 0$, let

$$F_\delta = \{x \in X : \mu(B(x,r)) \leq (c+\epsilon)r^s \text{ for all } 0 < r \leq \delta\}.$$

By hypothesis, $F \subseteq \bigcup_{\delta>0} F_\delta$; moreover, for $\delta_1 < \delta_2$, $F_{\delta_1} \supseteq F_{\delta_2}$.
Fix some δ and take a δ-cover $\{U_i\}_{i=1}^\infty$ of $F \supseteq F_\delta$. If $x \in F_\delta$, since $|U_i| \leq \delta$, $\mu(B(x,|U_i|)) \leq (c+\epsilon)|U_i|^s$. Moreover, since $U_i \subseteq B(x_i,|U_i|)$ for any $x_i \in U_i$, if $U_i \cap F_\delta \neq \emptyset$, take any $x_i \in U_i \cap F_\delta$ and $\mu(U_i) \leq \mu(B(x_i,|U_i|)) \leq (c+\epsilon)|U_i|^s$. Thus

$$\mu(F_\delta) \leq \sum_{i:U_i\cap F_\delta\neq\emptyset} \mu(U_i) \leq \sum_{i=1}^\infty (c+\epsilon)|U_i|^s$$

so that $\mu(F_\delta) \leq (c+\epsilon)\mathcal{H}^s_\delta(F)$. Taking limits, we have $\mu(F) \leq (c+\epsilon)\mathcal{H}^s(F)$; but $\epsilon > 0$ is arbitrary, so we are done.

(ii) For each $\delta > 0$, let

$$F_\delta = \{x \in X : \mu(B(x,r)) \geq (c-\epsilon)r^s \text{ for all } 0 < r \leq \delta\}.$$

By hypothesis, $F \subseteq \bigcup_{\delta>0} F_\delta$; moreover, for $\delta_1 < \delta_2$, $F_{\delta_1} \supseteq F_{\delta_2}$.

We first show that for any $\delta_0 \le \delta$, $\mu(F) \ge \frac{(c-\epsilon)}{2^s}\mathcal{P}^s_{\delta_0}(F_\delta)$. Fix a δ_0-packing of F_δ, say $\{B_i\}_{i=1}^\infty$ where the $B_i = B(x_i, r_i)$ are disjoint, $r_i \le \delta_0$, and $x_i \in F_\delta$. Then since the B_i are disjoint, we have

$$\mu(F) \ge \mu(F_\delta) \ge \sum_{i=1}^\infty \mu(B_i) \ge \sum_{i=1}^\infty (c-\epsilon)\frac{|B_i|^s}{2^s};$$

but this holds for any δ_0-packing, so taking the supremum yields the inequality. In particular, we have as $\delta_0 \to 0$, $\mu(F) \ge \frac{(c-\epsilon)}{2^s}\mathcal{P}^s_0(F_\delta) \ge \frac{(c-\epsilon)}{2^s}\mathcal{P}^s(F_\delta)$. But this holds for any F_δ, and since $\mathcal{P}^s$ is indeed a measure, we have $\mu(F) \ge \frac{(c-\epsilon)}{2^s}\mathcal{P}^s(F)$ as required.

(iii) Fix $\epsilon > 0$ and $\delta > 0$. Let $\mathcal{B} = \{B(x,r) : x \in F, 0 < r \le \delta, \mu(B(x,r)) \ge (c-\epsilon)r^s\}$. By assumption, $F \subseteq \bigcup_{B\in\mathcal{B}} B$. Use the Vitali covering lemma, so there exists disjoint balls $B_1, B_2, \ldots \in \mathcal{B}$ such that B_i' is the ball with the same centre and 5 times the radius, then $\bigcup_{i=1}^\infty B_i' \supseteq F$. Since $\operatorname{diam} B(x,r) = 2r$, $|B_i'| \le 10r \le 10\delta$ so the $\{B_i'\}_{i=1}^\infty$ are a 10δ-cover of F. Thus

$$\begin{aligned} H^s_{10\delta}(F) &\le \sum_{i=1}^\infty |B_i'|^s = \sum_{i=1}^\infty |B_i|^s 5^s \\ &= \sum_{i=1}^\infty (2r_i)^s 5^s \\ &\le 10^s \sum_{i=1}^\infty \frac{\mu(B_i)}{c-\epsilon} \\ &= \frac{10^s}{c-\epsilon}\mu\left(\bigcup_{i=1}^\infty B_i\right) \le \frac{10^s}{c-\epsilon}\mu(\mathbb{R}^n) \end{aligned}$$

and taking $\delta \to 0$ and noting that $\epsilon > 0$ is arbitrary, we have $H^s(F) \ge \frac{10^s\mu(\mathbb{R}^n)}{c}$.

(iv) Let $\{F_i\}_{i=1}^\infty$ be any cover of F. Since $\mathcal{P}_0(F_i') \le \mathcal{P}_0(F_i)$ when $F_i' \subseteq F_i$, we may assume $F_i \subseteq F$. It is enough to show that $\sum_{i=1}^\infty \mathcal{P}^s_0(F_i) \ge \frac{10^s}{c+\epsilon}\mu(F)$ for any fixed $\epsilon > 0$.
Let $\delta > 0$ and let $\mathcal{B} = \{B(x,r) : x \in F_i, 0 < r \le \delta, \mu(B(x,r)) \le (c+\epsilon)r^s\}$ and let $\mathcal{C} = \{B(x, r/5) : B(x,r) \in \mathcal{B}\}$. By assumption, $F_i \subseteq \bigcup_{B\in\mathcal{C}} C$. By the Vitali covering theorem, there exists disjoint balls $\{B_i\}_{i=1}^\infty \subset \mathcal{C}$ with $B_i = B(x_i, r_i)$, such that $\bigcup_{i=1}^\infty B(x_i, 5r_i) \supseteq F_i$. Note that $B(x_i, 5r_i) \in \mathcal{B}$, so that

$$\mu(F_i) \le \sum_{i=1}^\infty \mu(B(x_i, 5r_i)) \le \sum_{i=1}^\infty (c+\epsilon)10^s|B_i|^s$$

where the B_i are disjoint with radius at most $\delta/5$ and thus $\frac{10^{-s}}{c+\epsilon}\mu(F_i) \le \mathcal{P}^s_{\delta/5}(F_i)$. Then taking the limit as δ goes to zero gives $\frac{10^{-s}}{c+\epsilon}\mu(F_i) \le \mathcal{P}^s_0(F_i)$. But then

$$\frac{10^s}{c+\epsilon}\mu(F) \le \sum_{i=1}^\infty \frac{10^s}{c+\epsilon}\mu(F_i) \le \sum_{i=1}^\infty \mathcal{P}^s_0(F_i)$$

but as above, the F_i are an arbitrary cover for F, and $\epsilon > 0$ was arbitrary, so that $\frac{10^s}{c}\mu(F) \le \mathcal{P}^s(F)$. ■

1.16 Proposition. *Suppose F is Borel and $0 < H^s(F) < \infty$. Then there exists c and a compact $E \subseteq F$ such that $H^s(E) > 0$ and $H^s(B(x,r) \cap E) \le cr^s$ for all $x \in E$ and $r > 0$.*

Proof Let
$$F_1 = \left\{x : \limsup_{r \to 0} \frac{H^s(F \cap B(x,r))}{r^s} > 10^{s+1}\right\}$$
and apply (b) above with $\mu = H^s|_F$ so that
$$H^s(F_1) \le \frac{10^s}{10^{s+1}} \mu(\mathbb{R}^n) = \frac{1}{10} H^s(F).$$
In particular, $H^s(F \setminus F_1) \ge \frac{9}{10} H^s(F) > 0$. For all $x \in F \setminus F_1$, there exists $r_0(x)$ such that for all $r \le r_0$, then
$$\frac{H^s(F \cap B(x,r))}{r^s} \le 10 \cdot 10^{s+1} = 10^{s+2}.$$
Let
$$E_n = \left\{x \in F \setminus F_1 : \frac{H^s(F \cap B(x,r))}{r^s} \le 10^{s+2} \text{ for all } r \le \frac{1}{n}\right\}$$
so that $\bigcup_{n=1}^{\infty} E_n = F \setminus F_1$. By continuity of measure, $H^s(E_n) \to H^s(F \setminus F_1) > 0$ so there exists N such that $H^s(E_N) > 0$. Since H^s is inner regular (TODO prove), get $E \subseteq E_N$ compact such that $H^s(E) > 0$. Then if $x \in E$, $x \in E_N$ so $H^s(E \cap B(x,r)) \le H^s(F \cap B(x,r)) \le 10^{s+2} r^s$ if $r \le 1/N$. For any r, $H^s(E \cap B(x,r)) \le H^s(F) = C_0$. If $r > 1/N$, then $C_0 \le C_0 N^s r^s$. Take $c = \max\{10^{s+2}, C_0 N^s\}$. ■

Remark. The assumption $H^s(F) < \infty$ can be removed when F is closed.

1.5 Potential-Theoretic Methods

Definition. For $s \ge 0$, the **s-potential at x due to μ** is
$$\phi_s(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{\|x-y\|^s}$$
and the **s-energy of μ**
$$I_s(\mu) = \int_{\mathbb{R}^n} \phi_s \, d\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{d\mu(x) d\mu(y)}{\|x-y\|^s}$$

Example. (i) If $s = 0$, then $\phi_0(x) = \mu(\mathbb{R}^n)$ and $I_0(\mu) = \mu(\mathbb{R}^n)^s < \infty$.
(ii) If $s > 0$ and $\mu = \delta_0$, then $I_s(\delta_0) = \phi_s(0) = \infty$
(iii) If $n = 1$ and $\mu = m|_{[0,1]}$, $s < 1$. Then $I_s(\mu) = \int_0^1 \int_0^1 \frac{dx dy}{|x-y|^s} < \infty$.

1.17 Theorem. *Let F be a closed set, $s > 0$.*
(i) If there exists a finite, non-zero measure μ supported on F such that $I_s(\mu) < \infty$, then $H^s(F) = \infty$ implies that $\dim_H F \ge s$.
(ii) If $H^s(F) > 0$, then there exists a finite non-zero measure μ on F such that $I_t(\mu) < \infty$ for all $t < s$.

Proof (i) Suppose $I_s(\mu) < \infty$ for μ a finite measure on F. We will show that $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} = 0$ for μ a.e. $x \in F$. Assuming this, then $H^s(F) \geq \frac{\mu(F\setminus N)}{\epsilon}$ for some μ-null N, but this holds for any $\epsilon > 0$, so $H^s(F) = \infty$.

Let $F_1 = \{x \in F : \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} > 0\}$. We want to show that $\mu(F_1) = 0$. We first show that $\phi_s(\mu) = \infty$ on F_1. If $x \in F_1$, then there exists $\epsilon > 0$ and $\{r_i\}_{i=1}^{\infty}$ converging to 0 such that $(B(x, r_i)) \geq \epsilon r_i^s$. Since $I_s(\mu) < \infty$ for some $s > 0$, μ is not atomic so by downward continuity of meaure, $\mu(B(x,q)) \to \mu(\{x\}) = 0$ as $q \to 0$. Thus get q_i such that $\mu(B(x,q_i)) < \frac{\epsilon}{2} r_i^s$. Let $A_i = B(x, r_i) \setminus B(x, q_i)$, so that $\mu(A_i) \geq \frac{\epsilon}{2} r_i^s$. Relabelling the r_i if necessary, we may assume that $r_{i+1} < q_i$ so that the annuli are disjoint and nested. In particular,

$$
\begin{aligned}
\phi_s(x) &= \int_{\mathbb{R}^n} \frac{d\mu(y)}{\|x-y\|^s} \\
&\geq \sum_{i=1}^{\infty} \int_{A_i} \frac{d\mu(y)}{\|x-y\|^s} \\
&\geq \sum_{i=1}^{\infty} \frac{1}{\max_{y\in A_i} \|x-y\|^s} \mu(A_i) \\
&\geq \sum_{i=1}^{\infty} \frac{1}{r_i^s} \mu(A_i) \geq \sum_{i=1}^{\infty} \frac{1}{r_i^s} \cdot \frac{\epsilon}{2} r_i^s = \infty
\end{aligned}
$$

But now,

$$
\infty > I_s(\mu) = \int_{\mathbb{R}^n} \phi_s d\mu \geq \int_{F_1} \phi_s d\mu
$$

so if $\phi_s = +\infty$ on F_1, then $\mu(F_1) = 0$.

(ii) Suppose $H^s(F) > 0$. By the previous proposition, there exists sompact $E \subseteq F$ with $0 < H^s(E) < \infty$ and $H^s(E \cap B(x,r)) \leq cr^s$ for all $x \in E$ and $r > 0$. Put $\mu = H^s|_E$. Then $\mu(B(x,r)) \leq cr^s$ for all $x \in E$. For $x \in E$,

$$
\phi_i(x) = \int_{\|x-y\|\leq 1} \frac{d\mu(y)}{\|x-y^t\|} + \int_{\|x-y\|>1} \frac{d\mu(y)}{\|x-y\|^t}.
$$

Certainly the second integral is finite independent of x. The first integral is finite since

$$
\begin{aligned}
\int_{\|x-y\|\leq 1} \frac{d\mu(y)}{\|x-y^t\|} &= \sum_{k=0}^{\infty} \int_{B(x,2^{-k})\setminus B(x,2^{-(k+1)})} \frac{d\mu(y)}{\|x-y\|^t} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{2^{-(k+1)t}} \mu(B(x, 2^{-k})) \\
&\leq \sum_{k=0}^{\infty} \frac{c}{2^{-(k+1)t}} \cdot 2^{-ks} < \infty
\end{aligned}
$$

since $s > t$. Again, this bound does not depend on x. Thus ϕ_t is a bounded function on E, so that $I_t(\mu) < \infty$. ∎

"can't have both the measure and it's fourier transform small"

Suppose f is integrable on $\mathbb{R}^n$ or $\mu \in M(\mathbb{R}^n)$ is a complex measure. We then define the **fourier transform**

$$\hat{f}(z) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot z} \, \mathrm{d}m(x)$$

$$\hat{\mu}(z) = \int_{\mathbb{R}^n} e^{-ix\cdot z} \, \mathrm{d}\mu(x)$$

If $f, g \in L^1$, then $f * g \in L^1$ by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \, \mathrm{d}y$$
$$f * \mu(x) = \int_{\mathbb{R}^n} f(x-y) \, \mathrm{d}\mu(y)$$

By Fubini, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ and $\|f * \mu\| \leq \|f\|_1 \|\mu\|_{M(\mathbb{R}^n)}$. One reason for doing this is that L^1 is not closed under pointwise multiplication. Importantly, we have

$$(f * g)\hat{}(z) = \hat{f}(z)\hat{g}(z)$$
$$(f * \mu)\hat{}(z) = \hat{f}(z)\hat{\mu}(z)$$

in other words that the fourier transform converts convolution to multiplication.

Now consider $g_s(t) = \|t\|^{-s}$. Then

$$\phi_s(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(y)}{\|x-y\|^s} = \int_{\mathbb{R}^n} g_s(x-y) \, \mathrm{d}\mu(y) = g_s * \mu(x)$$

It is known that $\hat{g}_s(z) = c(n,s) \|z\|^{s-n}$ for $0 < s < n$. In particular, $\hat{\phi}_s(z) = \hat{g}_s(z)\hat{\mu}(z) = c(n,s) \|z\|^{s-n} \hat{\mu}(z)$.

1.18 Theorem. (Parseval) *We have*

$$\int f \cdot \overline{g} \, \mathrm{d}x = (2\pi)^n \int \hat{f} \cdot \overline{\hat{g}} \, \mathrm{d}z$$

for $f, g \in L^2$ and thus $\int |f|^2 = (2\pi)^n \int |\hat{f}|^2$. When g is "nice",

$$\int g(x) \, \mathrm{d}\mu(x) = (2\pi)^n \int \hat{g}(z) \overline{\hat{\mu}(z)} \, \mathrm{d}z$$

In particular (with some technicalities ...)

$$I_s(\mu) = \int \phi_s(x) \, \mathrm{d}\mu(x) = c_n \int \hat{\phi}_s(z) \overline{\hat{\mu}(z)} \, \mathrm{d}z$$
$$= c_n' \int \|z\|^{s-n} |\hat{\mu}(z)|^2 \, \mathrm{d}z$$

Example. If $|\hat{\mu}(z)| \leq C \|z\|^{-t/z}$, then $\dim_H \operatorname{supp} \mu \geq t$.

Proof We have since $\hat{\mu}(z)$ is bounded that

$$\begin{aligned}
I_s(\mu) &= c\int \|z\|^{s-n}\,|\hat{\mu}(z)|^2\,\mathrm{d}z \\
&= c\left(\int_{\|z\|\le 1}\|z\|^{s-n}\,|\hat{\mu}(z)|^2\,\mathrm{d}z + \int_{\|z\|>1}\|z\|^{s-n}\,|\hat{\mu}(z)|^2\,\mathrm{d}z\right) \\
&\le c\left(\int_{\|z\|\le 1} C_0\,\|z\|^{s-n}\,\mathrm{d}z + \int_{\|z\|\ge 1}\|z\|^{s-n}\,\|z\|^{-t}\,\mathrm{d}z\right) \\
&= c\left(c_1\int_0^1 r^{s-n}r^{n-1}\,\mathrm{d}r + \int_1^\infty t^{s-t-1}\,\mathrm{d}r\right) < \infty
\end{aligned}$$

as $s < t$. Thus $I_s(\mu) < \infty$ for any $0 < s < t$, and apply the energy theorem. ■

1.6 Projections of Fractals

Let $F \subset \mathbb{R}^2$ be a region and consider the (orthogonal) projection onto some line through the origin. Write $\mathrm{proj}_\theta(f)$ to denote the projection onto the line L_θ. Note that $d(\mathrm{proj}_\theta(x), \mathrm{proj}_\theta(y)) \le d_{\mathbb{R}^2}(x,y)$ so proj_θ is Lipschitz and $\dim_H \mathrm{proj}_\theta F \le \min\{1, \dim_H F\}$.

If L is a line segment, then for all values of θ (except for 2), then the projection has maximal dimension.

1.19 Theorem. *Let $F \subseteq \mathbb{R}^2$ be closed.*

(i) If $\dim_H F \le 1$, then $\dim_H \mathrm{proj}_\theta F = \dim_H F$ for a.e. θ.

(ii) If $\dim_H F > 1$, then $m(\mathrm{proj}_\theta F) > 0$ for a.e. θ.

Proof (i) Choose $0 < s < \dim_H F$, so $H^s(F) > 0$. Thus there exists some μ on F such that $I_s(\mu) < \infty$. Write $x.\theta$ to denote the projection of x onto the line L_θ. Then define μ_θ on $\mathrm{proj}_\theta F$ by

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}\mu_\theta(t) = \int f(x.\theta)\,\mathrm{d}\mu(x)$$

for all $f \in C_c(\mathbb{R})$ (Radon-Markov). Note that $\mu_\theta(S) = \mu(\mathrm{proj}_\theta^{-1}(S))$. We will show that $\int_0^\pi I_s(\mu_\theta)\,\mathrm{d}\theta < \infty$, so that $I_s(\mu_\theta) < \infty$ for a.e. θ and we will be done.

We have since $|x.\theta - y.\theta| = \|x-y\|\cos(\theta - (x-y))$.

$$\begin{aligned}
\int_0^\pi I_s(\mu_\theta)\,\mathrm{d}\theta &= \int_0^\pi\int_F\int_F \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x.\theta - y.\theta|^s} \\
&= \int_0^\pi\int_F\int_F \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{\|x-y\|^s\,|\cos(\theta-(x-y))|^s} \\
&= \int_F\int_F\left(\int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\theta-(x-y))|^s}\right)\frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{\|x-y\|^s} \\
&= \int_{F\times F}\left(\int_0^\pi \frac{\mathrm{d}\theta}{|\cos\theta|^s}\right)\frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{\|x-y\|^s}
\end{aligned}$$

Note that $\int_0^\pi \frac{\mathrm{d}\theta}{|\cos\theta|^s} < \infty$, but the remaining term is just the s–energy of μ, which is finite.

(ii) Assume $\dim_H F > 1$, so there exists some $t > 1$ such that $H^t(F) > 0$. Get μ on F such that $I_1(\mu) < \infty$. Define μ_θ as above. We will show that μ_θ is absolutely continuous with density in L^2 for almost every θ. Then $f_\theta \neq 0$ in L^2 since $\mu_\theta \neq 0$ so that $m\{x : f_\theta(x) \neq 0\} > 0$ where $\{x : f_\theta(x) \neq 0\} \subseteq \operatorname{supp} \mu_\theta$.
Recall that $f \in L^2$ if and only if $\hat{f} \in L^2$. We have

$$\begin{aligned} |\hat{\mu_\theta}(z)|^2 &= \int e^{-ivz} \, \mathrm{d}\mu_\theta(v) \overline{\int e^{-izw} \, \mathrm{d}\mu_\theta(w)} \\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{-iz(v-w)} \, \mathrm{d}\mu_\theta(v) \, \mathrm{d}\mu_\theta(w) \\ &= \int_{F\times F} e^{-iz(x-y).\theta} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \end{aligned}$$

so that

$$\begin{aligned} |\hat{\mu_\theta}(z)|^2 + |\hat{\mu_{\theta+\pi}}(z)|^2 &= \int_{F\times F} \left(e^{-iz(x-y).\theta} + e^{-iz(x-y).(-\theta)}\right) \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= 2 \int_{F\times F} \cos(z(x-y).\theta) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \end{aligned}$$

First note that

$$\begin{aligned} \int_0^{2\pi} |\hat{\mu_\theta}(z)|^2 \, \mathrm{d}\theta &= \int_0^{\pi} |\hat{\mu_\theta}(z)|^2 + |\hat{\mu_{\theta+\pi}}(z)|^2 \, \mathrm{d}\theta \\ &= 2 \int_0^{\pi} \int_f \int_F \cos(z(x-y).\theta) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \, \mathrm{d}\theta \\ &= 2 \int_0^{\pi} \int_f \int_F \cos\!\left(z \left\|x-y\right\| \cos(\theta - (x-y))\right) \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \, \mathrm{d}\theta \\ &= \int_F \int_F \left(\int_0^{2\pi} \cos\!\left(z \left\|x-y\right\| \cos(\theta)\right) \mathrm{d}\theta \right) \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= 2\pi \int_F \int_F J_0(z \left\|x-y\right\|) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y). \end{aligned}$$

We now have (concealing some technicalities in verifying the application of Fubini)

$$\begin{aligned} \int_0^{2\pi} \int_{-\infty}^{\infty} |\hat{\mu_\theta}(z)|^2 \, \mathrm{d}z \, \mathrm{d}\theta < \infty &= \int_{-\infty}^{\infty} \int_0^{2\pi} |\hat{\mu_\theta}(z)|^2 \, \mathrm{d}z \, \mathrm{d}\theta < \infty \\ &= 2\pi \int_{-\infty}^{\infty} \int_F \int_F J_0(z \left\|x-y\right\|) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= 2\pi \int_F \int_F \left(\int_{-\infty}^{\infty} J_0(z \left\|x-y\right\|) \, \mathrm{d}z \right) \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= 2\pi \int_F \int_F \left(\int_{-\infty}^{\infty} J_0(w) \, \mathrm{d}w \right) \frac{\mathrm{d}\mu(x) \, \mathrm{d}\mu(y)}{\left\|x-y\right\|} < \infty \end{aligned}$$

by the integral of the Bessel function and the fact that $I_1(\mu) < \infty$. ■

Bessel function: $J_0(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \cos(u \cos\theta) \, \mathrm{d}\theta$.

2 Iterated Function Systems

2.1 Invariant Sets and Measures

Let X be a complete metric space and $F_1,\dots,F_m$ a family of contractions from X to X (i.e. functions with $0 < r_i < 1$ with $d(F_i(x),F_i(y)) \le r_i d(x,y)$). Then there exists $E \subseteq X$ with E compact such that $E = \bigcup_{i=1}^n F_i(E)$.

Let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X. For $A \subseteq X$, let $A_r = \{y \in X : d(a,y) < r \text{ for some } a \in A\}$. We then define the **Hausdorff metric** on $\mathcal{K}(X)$ as follows:

$$D(A,B) = \inf\{r > 0 : A \subseteq B_r, B \subseteq A_r\}$$

2.1 Proposition. *D, as defined above, is in fact a metric and when X is complete, $\mathcal{K}(X)$ is also complete.*

Proof We verify the properties for D to be a metric:

(i) Suppose $D(A,B) = 0$. Then get a sequence a_n in A converging to any $b \in B$, i.e. $b \in \overline{A} = A$ and $B \subseteq A$. Similarly, $B \subseteq A$.

(ii) $D(A,B) = D(B,A)$ is clear

(iii) Fix $A,B,C \in \mathcal{K}(X)$, $d_1 = D(A,C)$, $d_2 = D(C,B)$. Fix $\epsilon > 0$ and let $a \in A$ be arbitrary. Get $c \in C$ so that $D(a,c) < d_1 + \epsilon/2$. Then get $b \in B$ so that $D(c,b) < d_2 + \epsilon/2$. Thus $d(a,b) < d_1 + d_2 + \epsilon$ so $A \subseteq B_{d_1+d_2+\epsilon}$ for all $\epsilon > 0$. Similarly, $B \subseteq A_{d_1+d_2+\epsilon}$. Thus $D(A,B) \le d_1 + d_2$.

Completeness is left as an exercise. ■

2.2 Theorem. *Let $\{F_1,\dots,F_m\}$ be an IFS on X. Then there exists a unique compact set $E \subseteq X$ such that $E = \bigcup_{i=1}^m F_i(E)$.*

Proof Define $F : \mathcal{K}(X) \to \mathcal{K}(X)$ by $F(A) = \bigcup_{i=1}^m F_i(A)$. Let $r = \max\{r_1,\dots,r_m\} < 1$. We will show that $D(F(A),F(B)) \le rD(A,B)$. Set $d = D(A,B)$; it suffices to show that $F_i(A) \subseteq (F_i(B))_{r(d+\epsilon)}$ for any $\epsilon > 0$. Indeed, take $a \in A$, so there exists $b \in B$ so that $d(a,b) \le d + \epsilon$. Then $d(F_i(a),F_i(b)) \le r(d+\epsilon)$.

Then F is a contraction map on $\mathcal{K}(X)$, so that $F^{(k)}(A) \to E$ for some unique E. ■

If $F_i(A) \subseteq A$, then $E = \bigcap_{k=0}^\infty F^{(k)}(A)$.

2.3 Lemma. *If $(A_k)_{k=1}^\infty \subset \mathcal{K}(X)$ with $A_1 \supseteq A_2 \supseteq \cdots$, then $A_k \to \bigcap_{i=1}^\infty A_i$.*

Proof Let $A_0 = \bigcap_{k=1}^\infty A_k$. We want to prove that $D(A_{k_1},A_0) \to 0$. Certainly $A_0 \subseteq A_k$. Conversely, we must check that for any $r > 0$, there exists n_r such that $A_k \subseteq (A_0)_r$. Note that $(A_0)_r$ is an open set. Then $\{(A_0)_r, A_n^c : n \in \mathbb{N}\}$ is an open cover for A_1. Hence there exists a finite subcover $(A_0)_r, A_{n_1}^c,\dots,A_{n_N}^c$. Thus for any $k > \max\{n_1,\dots,n_N\}$, $A_k \subseteq (A_0)_r$, as required. ■

2.4 Theorem. *Let $X \subseteq \mathbb{R}^n$ be compact and let $\{F_i\}_{i=1}^m$ be an IFS on X with attractor E. Assume we are given probabilities $\{p_i\}_{i=1}^m$ such that $\sum_{i=1}^m p_i = 1$. Then there exists a unique Borel probability measure μ such that*

$$\mu(A) = \sum_{i=1}^m p_i \mu(F_i^{-1}(A))$$

for all Borel sets A. Moreover,

(i) $\int g \, \mathrm{d}\mu = \sum_{i=1}^{m} p_i \int g(F_i(x)) \, \mathrm{d}\mu(x)$

(ii) $\mathrm{supp}(\mu) = E$

(iii) If the IFS satisfies the strong separation condition, then $\mu(E_\sigma) = p_\sigma$.

Remark. In the case of an IFS of similarities, μ is called a **self-similar measure**.

Proof Let $M_1(X)$ denote the set of all Borel probability measures on X. Define a metric on $M(X)$ by

$$d(\mu, \nu) = \sup\left\{\left|\int g \, \mathrm{d}\mu - \int g \, \mathrm{d}\nu\right| : |g(x) - g(y)| \leq \left\|x - y\right\|\right\}.$$

Step 1: verify that this in fact a metric which makes $M(X)$ a complete metric space. [***TODO: Falconer Techniques Proposition 1.9***]

Step 2: Define $H : M(X) \to M(X)$ where $H(\nu) = H_\nu$ is the measure that satisfies

$$H_\nu(A) = \sum_{i=1}^{m} p_i \nu(F_i^{-1}(A))$$

for all A Borel. Verify that H_ν is a Borel probability measure. We have

$$\begin{aligned} H_\nu(A) = \int \mathbf{1}A \, \mathrm{d}H_\nu &= \sum_{i=1}^{m} p_i \int \mathbf{1}F_i^{-1}(A) \, \mathrm{d}\nu \\ &= \sum_{i=1}^{m} p_i \int \mathbf{1}A(F_i(x)) \, \mathrm{d}\nu(x) \end{aligned}$$

and extending by density of simple functions in L^1, we have

$$\int g \, \mathrm{d}H_\nu = \sum_{i=1}^{m} p_i \int g(F_i(x)) \, \mathrm{d}\nu(x)$$

Step 3: Check that H_ν is a contraction. We have

$$\begin{aligned} d(H_\mu, H_\nu) &= \sup\left\{\left|\int_g \mathrm{d}H_\mu - \int_G \mathrm{d}H_\nu\right| : \mathrm{Lip}(g) \leq 1\right\} \\ &= \sup_{\mathrm{Lip}(g) \leq 1} \left|\sum_{i=1}^{m} \left(\int g(F_i(x)) \, \mathrm{d}\mu(x) - \int g(F_i(x)) \, \mathrm{d}\nu(x)\right)\right| \\ &\leq \sup_{\mathrm{Lip}(g) \leq 1} \sum_{i=1}^{m} p_i r_i \left|\int r_i^{-1} g(F_i(x)) \, \mathrm{d}(\mu - \nu)(x)\right| \end{aligned}$$

where r_i is the contraction factor of F_i. Moreover, notice that

$$\begin{aligned} \left|r_i^{-1} g(F_i(x)) - r_i^{-1} g(F_i(y))\right| &\leq r_i^{-1} \left\|F_i(x) - F_i(y)\right\| \\ &\leq \left\|x - y\right\| \end{aligned}$$

so that $r_i^{-1} g \circ F_i$ is Lipschitz with constant at most 1. Thus

$$d(\mu,\nu) \geq \left| \int r_i^{-1} g \circ F_i \, \mathrm{d}(\mu-\nu)(x) \right|$$

so that

$$d(H_\mu, H_\nu) \leq \sum_{i=1}^m p_i r_i d(\mu,\nu) \leq \max\{r_i : i = 1,\dots,m\} d(\mu,\nu)$$

and thus H is in fact a contraction map.

Step 4: By the Banach contraction mapping principle, there exists a unique fixed point $\mu \in M_1(X)$. But then

$$\mu(A) = H(\mu)(A) = \sum_{i=1}^m p_i \mu(F_i^{-1}(A))$$

for any Borel A.

It remains to show the properties.

(i) Set $S = \operatorname{supp}(\mu)$. Then $1 = \mu(S) = \sum_{i=1}^m p_i \mu(F_i^{-1}(S))$ which forces $\mu(F_i^{-1}(S)) = 1$. Thus $F_i^{-1}(S) \supseteq S$ since they are of full measure, so $S \supseteq F_i(S)$. If $\mu(A) > 0$, then $\sum_{i=1}^m p_i \mu(F_i^{-1}(A)) > 0$, so there exists i such that $F_i^{-1}(A) \cap S \neq \emptyset$. Thus $A \cap F_i(S) \neq \emptyset$. But $S \setminus \left(\bigcup_{i=1}^m F_i(S) \right) \cap F_j(S) = \emptyset$ for all j, so that $\mu(S \setminus \bigcup_{i=1}^m F_i(S)) = 0$ and thus $\mu(S) = 1$. Thus $S = \bigcup_{i=1}^m F_i(S)$ so that $S = E$.

(ii) Assume the SSC. Then

$$\begin{aligned}
\mu(E_\sigma) &= \sum_{i=1}^m p_i \mu(F_i^{-1}(E_\sigma)) \\
&\geq p_{\sigma_1} \mu(E_{\sigma_2 \dots \sigma_k}) \\
&= p_{\sigma_1} \left(\sum_{i=1}^m p_i \mu(F_i^{-1}(E_{\sigma_2 \dots \sigma_k})) \right) \\
&\geq \dots \geq p_\sigma
\end{aligned}$$

On the other hand, since $E = \bigcup_{\sigma \in \Sigma^k} E_\sigma$ disjointly,

$$\begin{aligned}
1 = \mu(E) &= \sum_{\sigma \in \Sigma^k} \mu(E_\sigma) \\
&\geq \sum_{\sigma \in \Sigma^k} p_\sigma = \left(\sum_{i=1}^m p_i \right)^k = 1
\end{aligned}$$

■

Definition. If the attractor E of an IFS $\{F_1,\dots,F_m\}$ has the property that the sets $F_i(E)$ are disjoint, we say E satisfies the **strong separation condition**. We say that the IFS satisfies the **open set condition** if there exists a non-empty bounded open V such that $\dot{\bigcup}_{i=1}^m F_i(U) \subseteq U$.

The strong separation condition implies the open set condition by taking, say, $U = \{x : d(x,E) < \epsilon\}$ where $\epsilon = \frac{1}{2} \min_{i \neq j}(d(F_i(E), F_j(E))) > 0$.

2.2 Dimensional Properties of the Attractor

2.5 Theorem. *Let F be the attractor of the IFS $\{F_i\}_{i=1}^m$ with contraction factors $\{r_1,\dots,r_m\}$. If the IFS satisfies the SSC, then $\dim_H E = s$ where $\sum_{i=1}^m r_i^s = 1$. Moreover, $0 < H^s(E) < \infty$.*

Proof Write $A_\sigma = F_\sigma(A)$ for each $\sigma \in \Sigma^* = \{1,\dots,m\}^*$. Fix $\delta > 0$ and pick k such that $r^k|E| < \delta$. Then the sets $\{E_\sigma : \sigma \in \Sigma^k\}$ is a δ-cover of E. Then

$$\begin{aligned} H_\delta^s(E) &\le \sum_{\sigma\in\Sigma^k} |E_\sigma|^s = \left(\sum_{\sigma\in\Sigma^k} r_\sigma^s\right)|E|^s \\ &= \left(\sum_{i=1}^m r_j^s\right)^k |E|^s = |E|^s \end{aligned}$$

so that $H^s(E) \le |E|^s < \infty$.

To get a lower bound, intending to use the mass distribution principle, we will construct a measure μ on E such that $\mu(U) \le c|U|^s$ for all open E. Define a measure μ on E by the rule $\mu(E_\sigma) = r_\sigma^s$. Using the subdivision method, one may verify that this is in fact a measure. But then $E_\sigma = \bigcup_{j=1}^m E_{\sigma j}$, so

$$\sum_j \mu(E_{\sigma j}) = \sum_j (r_{\sigma j})^s = r_\sigma^s \sum_j r_j^s = r_\sigma^s = \mu(E_\sigma).$$

Now consider $B(x,r)$ where $x \in E$. Let $r < d = \min_{i\neq j} d(F_i(E), F_j(E)) > 0$, and get $k \in \mathbb{N}$ such that $r_\sigma \cdot d \le r < r_{\sigma^-} d$ for $\sigma \in \Sigma^k$. Suppose $\sigma \neq \sigma'$ with $\sigma, \sigma' \in \Sigma^k$, and let j be maximal such that $\sigma|j = \sigma'|j$. Then

$$d(F_{\sigma|j} \circ F_{\sigma'_{j+1}}(E), F_{\sigma|j} \circ F_{\sigma_{j+1}}(E)) = r_{\sigma|j} \cdot d \ge r_{\sigma|k-1} \cdot d > r$$

so that $d(E_{\sigma'}, E_\sigma) > r$. If $y \in B(x,r) \cap E$, then $y \in E_\sigma$ so $B(x,r) \cap E \subseteq E_\sigma$. Thus $\mu(B(x,r) \cap E) \le \mu(E_\sigma) = r_\sigma^s \le \frac{r^s}{d^s} = c(\operatorname{diam} B(x,r))^s$.

But given any U such that $U \cap E \neq \emptyset$, we may take $U \subset B(x, |U|)$ for any choice of $x \in E \cap U$. ∎

2.6 Theorem. *Suppose E is a compact, non-empty subset of X and let $a, r_0 > 0$. Suppose for all closed balls B with centre in E and radius $r < r_0$, there exists a contraction map $g : E \to E \cap B$ such that $d(g(x), g(y)) \ge ar \cdot d(x,y)$ for all $x, y \in E$. Then if $s = \dim_H E$, then $H^s(E) \le 4^s a^{-s} < \infty$ and $\underline{\dim}_B(E) = \overline{\dim}_B(E) = s$.*

Example. Let E denote the Cantor set under the IFS $\{S_1, S_2\}$, and let B be the Cantor interval C_σ. Then $\operatorname{diam}(B) = r_\sigma$, and $g : E \to E \cap B$ is the map S_σ. Then $d(g(x), g(y)) = r_\sigma d(x,y)$.

Proof Let $N_r(E)$ denote the maximum number of disjoint closed balls of radius r with centers in E. Assume for contradiction there exists $r < \min\{a^{-1}, r_0\}$ with $N_r(E) > a^{-s} r^{-s}$.

Get some $r > s$ such that $N_r(E) > a^{-t} r^{-t}$, so we may get m disjoint closed balls $B_1, \dots, B_m$ with centres in E of radius r, and each of them gives rise to a map $g_i : E \to E \cap B_i$ such that $d(g_i(x), g_i(y)) \ge ard(x,y)$ for all x, y in E. Set $d_0 = \min_{i\neq j} d(B_i \cap E, B_j \cap E) > 0$. But then

$$\begin{aligned} d(g_{i_1} \circ \cdots \circ g_{i_k}(x), g_{j_1} \circ \cdots \circ g_{j_k}(y)) &\ge (ar)^{q-1} d(g_{i_q} \circ \cdots g_{i_k}(x), g_{j_q} \circ g_{j_k}(y)) \\ &\ge (ar)^{q-1} d_0 \ge (ar)^k d_0 > 0. \end{aligned}$$

On the other hand, $\operatorname{diam} E_\sigma \leq (\text{max contraction factor})^k |E|$ which converges to 0 as k goes to infinity.

Intending to use the mass distribution principle, define a measure on μ by $\mu(E_{i_1 \dots i_k}) = m^{-k}$ using the subdivision method. Take $U \cap E \neq \emptyset$ and $\operatorname{diam} U < d_0$. Pick k such that $(ar)^{k+1} d_0 \leq |U| < (ar)^k d_0$. Then

$$\mu(U) \leq \mu(E_{i_1 \dots i_k}) = m^{-k} \leq (ar)^{rk} \leq |U|^t \frac{(ar)^t}{d_0}$$

and by the mass distribution principle, $\dim_H(E) \geq t > s$, a contradiction.

Therefore $N - r(E) \leq a^{-s} r^{-s}$ for all small r. We may now compute

$$\overline{\dim}_B E = \limsup_{r \to 0} \frac{\log N_r(E)}{|\log r|} \leq \limsup_{r \to 0} \frac{\log a^{-s} r^{-s}}{\log r^{-1}} = s$$

so that $\overline{\dim}_B E \geq \underline{\dim}_B E \geq \dim_H E = s$. In particular, $\mathcal{H}^s_{2r}(E)$ is bounded above by the sum of the covering balls of radius $2r$, so $\mathcal{H}^s_{2r}(E) \leq 4^s a^{-s}$. ∎

2.7 Corollary. *Let E be the attractor of similarities $\{F_i\}_{i=1}^m$. If $s = \dim_H E$, then $\mathcal{H}^s(E) < \infty$ and $\dim_B E = s$.*

Proof We need to produce continuous $g : E \to E \cap B$ for any ball B with radius r centred at $x \in E$. For $x \in E$ with $r < |E|$, there exists some infinite sequence $(i_1, i_2, \dots)$ representing x. Choose k so that $r_{i_1} \cdots r_{i_k} |E| \leq r < r_{i_1} \cdots r_{i_{k-1}} |E|$. In particular,

$$r \cdot r_{\min} < r_{i_1} \cdots r_{i_k} |E|$$

so that $E_{i_1 \dots i_k} \subseteq B(x, r)$. Now define $g : E \to E \cap B(x,r)$ by $g = F_{i_1} \circ \cdots \circ F_{i_k}$ has image contained in $E \cap B(x,r)$, and

$$d(g(x), g(y)) = r_{i_1} \cdots r_{i_k} d(x,y) \geq r \cdot r_{\min} |E|^{-1} d(x,y).$$

Take $a = r_{\min} |E|^{-1}$ and apply the previous theorem. ∎

In fact, more is true in the strong separation case. Given $0 < r < |E|$, let $\Lambda_r = \{\sigma \in \Sigma^k : r_\sigma \leq r < r_{\sigma^-}\}$. Given $x \in E$, let $\Lambda_r(x) = \{\sigma \in \Lambda_r : B(x,r) \cap F_\sigma(E) \neq \emptyset\}$. Choose some $\sigma \in \Lambda_r(x)$ with maximal length. Pick some index i such that if $\lambda \in \Lambda_r(x)$, then $\lambda = (\sigma_1 \dots, \sigma_i, \lambda_{i+1}, \dots, \Lambda_N)$. But then

$$\begin{aligned} 2r \geq d(F_\sigma(E), F_\lambda(E)) &= r_{\sigma_1} \cdots r_{\sigma_k} d(F_{\sigma_{i+1}} \circ \cdots \circ F_{\sigma_L}(E), F_{\lambda_{i+1}} \circ \cdots \circ F_{\lambda_N}) \\ &\geq r_{\sigma_1} \cdots r_{\sigma_i} d_0 \end{aligned}$$

so that $2r \geq r_{\sigma_1} \cdots r_{\sigma_i} d_0$. But then combining the above inequalities, we have

$$r_{\sigma_1} \cdots r_{\sigma_i} \cdot r_{\sigma_{i+1}} \cdots r_{\sigma_{L-1}} > r \geq r_{\sigma_1} \cdots r_{\sigma_i} \frac{d_0}{2}$$

so there exists some C such that $L - i \leq C$. Thus $|\Lambda_r(x)| \leq m^C$ is a universal constant.

Definition. We say that the IFS has the **weak separation condition** if there exists C suth that $|\Lambda_r(x)| \leq C$.

2.8 Corollary. *If E is a self-similar set from an IFS that has the WSC, then $\mathcal{H}^s(E) > 0$ for $s = \dim_H(E)$.*

Proof It is enough to check the setup of the assignment question. Let $N \subseteq E$ with $|N| = r$, $x \in E$. Then $B(x,r) \supseteq N$. Check that $E = \bigcup_{\sigma\in\Lambda_r} F_\sigma(E)$, so

$$B(x,r) \cap E \subseteq \bigcup_{\sigma\in\Lambda_r(x)} F_\sigma(E) = \bigcup_{j=1}^{m} N_j.$$

Let $m = \max_{r,x} \Lambda_r(x)$. Let $g_j = F_\sigma^{-1} : F_\sigma(E) \to E$, so that

$$\begin{aligned} d(g_j(z), g_j(y)) &= d(F_\sigma^{-1}(z), F_\sigma^{-1}(y)) = r_\sigma^{-1} d(z,y) \\ &= r_\sigma^{-1} d(z,y) \geq r^{-1} d(z,y) \\ &= |N|^{-1} d(z,y) \end{aligned}$$

for all z, y in E. By the homework, we have $\mathcal{H}^s(E) \geq m^{-1} > 0$. ■

2.9 Proposition. *If E is the self-similar set from an IFS satisfying the weak separation condition, then there exists $a, b > 0$ such that*

$$ar^s \leq \mathcal{H}^s(E \cap B(x,r)) \leq br^s$$

for all $r < |E|$ and $x \in E$.

Proof Without loss of generality $|E| = 1$. Fix x, r and pick $\sigma \in \Lambda_r(x)$ such that $x \in F_\sigma(E)$ and $|F_\sigma(E)| \leq r_\sigma \leq r$. Thus $F_\sigma(E) \subseteq B(x,r) \cap E$. Thus

$$\mathcal{H}^s(B(x,r)\cap E) \geq \mathcal{H}^s(F_\sigma(E)) = r_\sigma^s \mathcal{H}^s(E) \geq r^s (r_{\min})^s \mathcal{H}^s(E)$$

so that $\mathcal{H}^s(B(x,r)\cap E) \leq \sum_{\sigma\in\Lambda_r(x)} r^s \mathcal{H}^s(E) \leq C\mathcal{H}^s(E) r^s$. ■

2.3 Assouad Dimensions

In some sense, the upper and lower assouad dimensions are a measurement of the smallest and largest local dimension of a set E. We define the **upper assouad dimension**

$$\dim_A E = \inf\left\{\alpha : \exists C_1, C_2 > 0 \text{ s.t. } \forall 0 < r < R \leq C_1 \sup_{x\in E} N_r(B(x,R)\cap E) \leq C_2 \left(\frac{R}{r}\right)^\alpha\right\}$$

and the **lower assouad dimension**

$$\dim_L E = \sup\left\{\alpha : \exists C_1, C_2 > 0 \text{ s.t. } \forall 0 < r < R \leq C_1 \inf_{x\in E} N_r(B(x,R)\cap E) \geq C_2 \left(\frac{R}{r}\right)^\alpha\right\}.$$

If $E \subseteq \mathbb{R}^n$ is bounded, then $\dim_A E \leq n$. To see this, fix $x \in E$ and look at the n-dimensional cube $Y(x,R)$ centred at x with sides of length R. Then $N_r(Y(x,R)\cap E) \leq (2R/r)^n$. Take $B(x,R) \subseteq Y(x,2R)$ so $N_r(B(x,R)\cap E) \leq 4^n (R/r)^n$.

$\mathbb{R}^n$ has a property called **doubling**, which means there exists a constant M such that $N_{R/2}(B(x,R)) \leq M$. In fact, $\dim_A E < \infty$ if and only if E is doubling.

Get M so that E is doubling, and show that α such that $M^\alpha = 2$ works. The other direction is easier.

2.10 Proposition. *(i)* $\dim_A E \geq \overline{\dim}_B E$.
(ii) $\dim_L E \leq \underline{\dim}_B E$

Proof If $\dim_A E = \infty$ we are done. Thus assume $d = \dim_A E$. Fix $\epsilon > 0$ and get C_1, C_2 such that $N_r(B(x,R)) \leq C_2(R/r)^{d+\epsilon}$. Cover E by finitely many balls of radius C_1 centred at points of E, say $B_1, \ldots, B_m$. Then

$$N_r(E) \leq \sum_{j=1}^{m} N_r(B_j) \leq mC_2\left(\frac{C_1}{r}\right)^{d+\epsilon}$$

so that

$$\limsup_{r\to 0} \frac{\log N_r(E)}{|\log r|} \leq \limsup_{r\to 0} \frac{\log mC_2C_1^{d+\epsilon} - (\log r)(d+\epsilon)}{-\log r} = d+\epsilon$$

and thus $\overline{\dim}_B E \leq d + \epsilon$ for any $\epsilon > 0$.
(ii) is an exercise. ■

Remark. It is also known that $\dim_L E \leq \dim_H E$, but this is more difficult to prove.

If E has an isolated point, then $\dim_L E = 0$. In fact,

2.11 Proposition. $\dim_L E > 0$ *if and only if E is* ***uniformly perfect****, which means there exists $c > 0$ such that $(B(z,R) \setminus B(z,cR)) \cap E \neq \emptyset$ whenever $B(z,R) \cap E \neq \emptyset$.*

Example. Consider the Cantor set $C((r_j)_{j=1}^{\infty})$. Recall that

$$\overline{\dim}_B C((r_j)_{j=1}^{\infty}) = \limsup_{n\to\infty} \frac{\log 2}{\log(r_1 \cdots r_n)^{1/n}}$$
$$\dim_H C((r_j)_{j=1}^{\infty}) = \underline{\dim}_B C((r_j)_{j=1}^{\infty}) = \liminf_{n\to\infty} \frac{\log 2}{\log(r_1 \cdots r_n)^{1/n}}$$

Moreover, one can show that

$$\dim_A C((r_j)_{j=1}^{\infty}) = \limsup_{k\to\infty}\left(\sup_n \frac{\log 2}{\log(r_{n+1}\cdots r_{n+k})^{1/k}}\right)$$
$$\dim_L C((r_j)_{j=1}^{\infty}) = \liminf_{k\to\infty}\left(\inf_n \frac{\log 2}{\log(r_{n+1}\cdots r_{n+k})^{1/k}}\right)$$

In fact, given any $0 \leq A < B < C < D \leq 1$, it can be arranged so that $\dim_L(C) = A$, $\underline{\dim}_B = B$, $\overline{\dim}_B = C$, and $\dim_A(C) = D$.

2.12 Theorem. *If E is a self-similar set satisfying the SSC, then* $\dim_L E = \dim_A E$.

Proof We say previously that E has the following property. Let $a, r_0 > 0$. Then for any U such that $U \cap E \neq \emptyset$ and $|U| \leq r_0$, there exists a map $f : E \cap U \to E$ with $a|U|^{-1} d(x,y) \leq d(f(x), f(y)))$ for all $x, y \in E \cap U$. Similarly, for any closed ball B with centre in E and radius $r \leq r_0$, there exists a map $g : E \to E \cap B$ such that $ard(x,y) \leq d(g(x), g(y))$ for all $x, y \in E \cap B$.

We know that for all $\epsilon > 0$, there exists C_ϵ such that $\frac{1}{C_\epsilon} \leq N_r(E) \leq C_\epsilon r^{-s-\epsilon}$ for sufficiently small r. Fix $0 < r < R$ and $x \in E$. Then consider $B(x,R) \cap E$. and get $f : B(x,R) \cap E \to E$ with $a|U|^{-1} d(x,y) \leq d(f(x), f(y))$. Consider $f(B(x,R) \cap E)$. Suppose $\{U_i\}$ are a $ar/(2R)$-cover. Then $\{f^{-1}(U_i)\}$ cover $B(x,R) \cap E$ and $\operatorname{diam} f^{-1}(U_i)$. Let $x, y \in f^{-1}(U_i)$, so that $f(x), f(y) \in U_i$ and $a|U|^{-1}(d(x,y)) \leq \operatorname{diam} U_i$. Thus $d(x,y) \leq \frac{|U_i|}{a|U|^{-1}} \leq r$.

Thus

$$\begin{aligned} N_r(B(x,R) \cap E) &\leq N_{ar/(2R)}(f(B(x,R) \cap E)) \leq N_{ar/(2R)}(E) \\ &\leq C_\epsilon \left(\frac{ar}{2R}\right)^{-s-\epsilon} \\ &= C'_\epsilon \left(\frac{R}{r}\right)^{s+\epsilon} \end{aligned}$$

so that $\dim_A E \leq s + \epsilon$. But $\dim_A E \geq \overline{\dim}_B E = s$, so $\dim_A E = s$. For $\dim_L E = s$, use 2. ■

Example. Let $X = \{1/n : n \in \mathbb{N}\} \cup 0$. Then $\dim_L X = 0$, $\dim_B X = 1/2$, and $\dim_A X = 1$.

If E is a self-similar set in $\mathbb{R}$ that fails the WSC, then $\dim_A E = 1$.

2.13 Theorem. *Let $X = \{x_j\}_{j=1}^\infty \cup \{0\} \subset \mathbb{R}$ where $\sum_{i=1}^\infty x_j < \infty$, $\{x_j\}$ is decreasing, and $\{x_j - x_{j+1}\}_j$ is decreasing. Then*

- *if $\{x_j\}$ is **lacunary** (there exists $\lambda > 0$ such that $x_j/x_{j+1} \geq \lambda$), then $\dim_A X = 0$, and*
- *$\dim_A X = 1$ otherwise.*

Proof Let $a_j = x_j - x_{j+1}$ so a_j is a decreasing sequence, and $x_j = \sum_{i=j}^\infty a_i$. Note that there exists $\epsilon > 0$ such that $a_j \geq \epsilon \sum_{j+1}^\infty a_i = \epsilon x_{j+1}$ if and only if $x_j \geq (1+\epsilon) x_{j+1}$ if and only if (x_j) is lacunary with $\lambda = 1 + \epsilon$.

First suppose (x_j) not lacunary. Then for each $N_0 \in \mathbb{N}$, there exist infinitely many k such that $a_k / x_{k+1} < 1/N_0$. Given such k, choose $N \in \mathbb{N}$ such that

$$\frac{1}{N+1} \leq \frac{a_k}{x_{k+1}} < \frac{1}{N}$$

Let $R = x_{k+1}$, $r = a_k$ so that $R/r \leq N+1$ and $\frac{x_{k+1}}{N+1} \leq a_k = r < R/N < R$. Look at $B(0,R) \cap X = \{x_j\}_{j=k+1}^\infty \cup \{0\}$. Then the intervals $\{[x_{k+1} - (s+1)r, x_{k+1} - sr]\}_{s=0}^{N-1}$ are each contained in $[0, x_{k+1}]$, and $x_{k+1} - Nr \geq 0$ as $x_{k+1}/N > r$. Since $r = a_k \geq a_i$, each interval contains some x_j, so $N_r(B(0,R) \cap X) \geq N/2$ and $R/r = x_{k+1}/a_k > N$.

Otherwise (x_j) is lacunary. Then there exists $\epsilon > 0$ such that $a_j \geq \epsilon \sum_{j+1}^\infty a_j = \epsilon x_{j+1}$. Choose $0 < r < R$, $x \in X$, and look at $B(x,R)$. If $x \leq R$, then $B(x,R) \cap X = [0, x+R] \cap X = \{x_j\}_{j=k}^\infty \cup \{0\}$. Choose minimal k such that $x_k \leq x + R$. If $r \leq a_k$, pick i such that $a_i < r \leq a_{i-1}$. Then $r > a_i \geq \epsilon x_{i+1}$. Thus $[0, x_{i+1}] \cap X$ can be covered by $1/\epsilon$ intervals of length r. Thus $N_r(B(x,R) \cap X) \leq 1/\epsilon + i - k + 1 = C + i - k$.

Compare with R/r. Here, $2R \geq x_k$ since $[0, x_k] \subset B(x,R)$ and

$$R \geq \frac{x_k}{2} \geq \frac{\lambda}{2} x_{k+1} \geq \cdots \geq \frac{\lambda^{i-k-1}}{2} x_{i-1}$$

and $r \leq a_{i-1} = x_{i-1} - x_i \leq x_{i-1}$ so that $R/r \geq C_1 \lambda^{i-k}$ since $\lambda > 1$. Thus

$$N_r(B(x,R) \cap X) \leq C + i - k \leq C'_\delta \lambda^{(i-k)\delta} \leq C'_0 \left(\frac{R}{r}\right)^\delta$$

Otherwise, $r > a_k \geq \epsilon x_{k+1}$, then $[0, x_{k+1}] \cap X$ is covered by $\frac{1}{\epsilon}$ intervals of length r. Thus $B(x,R) \cap X = [0, x_{k+1}] \cap X \cup \{x_k\}$, so that

$$N_r(B(x,r) \cap X) \leq \frac{1}{\epsilon} + 1 \leq C_\delta \left(\frac{R}{r}\right)^\delta$$

for any $\delta > 0$.

If $x > R$, then $B(x,R) \cap X = \{x_i\}_{i=J}^k$ where

$$2R \geq x_k - x_J \geq \lambda^{k-J} x_J - x_J = (\lambda^{k-J} - 1)x_J.$$

Take $r < R$, so that $r \geq a_k \geq \epsilon x_{k+1}$ and $B(x,R) \cap X = [0, x_{k+1}] \cap X \cup \{x_k\}$, where $x_{k+1} < r/\epsilon$. Thus $N_r(B(x,R) \cap X) \leq 1/\epsilon + 1 = c_1$. Since $R/r > 1$,

$$N_r(B(x,R)) \leq C_\delta \left(\frac{R}{r}\right)^\delta$$

for all $\delta > 0$. Otherwise, $r \leq a_J$ and $N_r(B(x,R) \cap X) \leq J - k + 1$, and $R/r \geq (\lambda^{J-k} - 1)x_J/a_J \geq c\lambda^{J-k}$. Finally, if $a_J < r < a_k$, pick i such that $a_J \leq a_i \leq r < a_{i-1} \leq a_j$ and

$$N_r(B(x,R) \cap X) \leq N_r([0, x_{i+1}] \cap X \cup \{x_j\}_{j=k}^i) \leq \frac{1}{\epsilon} + i - k + 1$$

and $R/r \geq c\lambda^{i-k}$ so $\dim_A X = 0$. ■

3 Sizes of Measures

We consider the space $M_+(\mathbb{R}^n)$, which is the set of finite, regular Borel measures on $\mathbb{R}^n$ equipped with the convolution product. How can we compuare sizes of measures? We might say $\dim_H(\operatorname{supp}\mu)$.

Example. Consider the measure $\mu = \sum_{n=1}^\infty \frac{1}{2^n} \delta_{q_n}$ where $\{q_n\}$ is an enumeration of $\mathbb{Q} \cap [0,1]$. Then $\dim_H(\operatorname{supp}\mu) = 1$, which is misleading since μ is singular with respect to Lebesgue measure.

Definition. We define the **Hausdorff dimension of a measure** $\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}$.

However, this value can be misrepresentative of the measure since it assigns a global value.

Definition. We define the **upper local dimension** of μ at x by

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

and similarly for the **lower local dimension**. If these two values coincide, we call this the **local dimension** of μ at x.

Example.

1. Suppose $\mu = m|_{[0,1]^n}$. Then $\mu(B(x,r)) \sim r^n$, so $\dim_{\mathrm{loc}} m(x) = 1$
2. If $\mu = \delta_0$, then $\dim_{\mathrm{loc}} \mu(0) = 0$.
3. Let E be the self-similar set satisfying the WSC, and let $s = \dim_H E$. Then if $\mu = H^s|_E$, we saw $H^s(B(x,R) \cap E) \sim r^s$ where $0 < H^s(E) < \infty$. Then $\dim_{\mathrm{loc}} \mu(x) = s$ for all $x \in E$.

4. If $x \notin \operatorname{supp}\mu$, then $\dim_{\mathrm{loc}}\mu(x) = +\infty$.
5. Let μ be the uniform Cantor measure on $C(1/3)$,=. If $x \in C_n$, then $\mu(B(x, 3^{-n})) = \mu(C_n) = 2^{-n}$, so one may compute that the local dimension of μ at x to be $\log(2)/\log(3)$.

Example. If μ is the weight $(p, 1-p)$ Cantor measure, given $x \in C(1/3)$ with symbolic representation $(x_i)_i$, one may show that $\dim_{\mathrm{loc}}\mu(x) = \lim_{n\to\infty} \frac{p_{x_1}\cdots p_{x_n}}{n\log 1/3}$. In fact, one may show that the set of local dimensions is the interval $[\log p/\log(1/3), \log(1-p)/\log(1/3)]$ when $p > 1-p$.

Recall if $\mu(E) > 0$ and $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \leq c$, then for any $x \in E$, $\dim_H E \geq s$. Similarly, if $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \geq c$, then for any $x \in E$, $\dim_H E \leq s$. In particular, when $E = \operatorname{supp}\mu$, if $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ for each $x \in E$, then for any $\epsilon > 0$, there exists $r_\epsilon > 0$ so that for all $r \leq r\epsilon$,

$$\frac{|\log \mu(B(x,r))|}{|\log r|} \geq s - \epsilon$$

so that $|\log\mu(B(x,r))| \geq (s-\epsilon)|\log r|$, so $\mu(B(x,r)) \leq r^{s-\epsilon}$. Thus $\limsup_{r\to 0} \frac{\log\mu(B(x,r))}{r^{s-\epsilon}} \leq 1$ for all $x \in E$, so $\dim_H E \geq s$. In fact, it suffices to show this for μ a.e. x. Recall

3.1 Proposition. *If $H^s(E) > 0$, then there exists c and $F \subseteq E$ compact with $0 < H^s < \infty$ and $H^s(F \cap B(x,r)) \leq cr^s$ for all $x \in F$ and $r > 0$.*

3.2 Corollary. *If $\dim_H E > s$, then there exists a measure μ with $0 < \mu < \infty$ and $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ for all $x \in E$.*

Proof Since $\dim_H E > s$, $H^s(E) > 0$. Get F from the proposition and take $\mu = H^s|_F$. Then

$$\frac{\log\mu(B(x,r))}{\log r} = \frac{\log H^s(B(x,r)\cap E)}{\log r} \geq \frac{\log cr^s}{\log r}.$$

Thus $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ for all $x \in E$. ∎

3.3 Corollary. $\dim_H E = \sup\{s : \exists\mu$ *with* $0 < \mu(E) < \infty, \underline{\dim}_{\mathrm{loc}}\mu(x) \geq s\forall\mu$ *a.e.* $x \in E\}$.

Recall that $\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}$.

3.4 Proposition. $\dim_H \mu = \sup\{x : \underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ *for* μ *a.e.* $x\}$.

Proof Let $d = \dim_H \mu$ and D denote the value of the RHS. Suppose $d < D$, and get $d < s < D$. Then $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ for all $x \in E_s$. Let $\mu(E) > 0$, so $\mu(E \cap E_s) > 0$ and $\dim_H E \geq s$, a contradiction. Now suppose $D < s < d$, so $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ does not occur for a.e. x. Thus $\underline{\dim}_{\mathrm{loc}}\mu(x) \leq s$ for all $x \in F_s$ where $\mu(F_s) > 0$, and $\dim_H F_s \leq s < d$, a contradiction. ∎

Example. If μ is the uniform Cantor measure, then $\dim_{\mathrm{loc}}\mu = \frac{\log 2}{\log 3}$ for all $x \in C(1/3)$. Thus $\dim_H \mu = \frac{\log 2}{\log 3}$. In particular, if $\mu(E) > 0$, then $\dim_H E = \frac{\log 2}{\log 3}$.

Example. Consider the IFS $F_1(x) = x/2$ and $F_2(x) = x/2 + 1/2$. Let $p_1 = p$ and $p_2 = 1-p$, and μ_p the associated Hausdorff measure. However, since the images of $[0,1]$ are not positively separated, it is more challenging to compute $\mu(B(x, 2^{-n}))$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Let

$$s(p) = \frac{-(p\log p + (1-p)\log(1-p))}{\log 2}.$$

Note that $s(1/2) = 1$ and as $s(p)$ increases as p increases from 1/2.

3.5 Theorem. *Let $0 < p \le 1/2$ and μ_p the described measure. Then $\dim_H \mu_p = s(p) = \dim_{\mathrm{loc}} \mu_p(x)$ for μ_p-a.e. x.*

Proof Define $X_k(x) = 1$ if $x_k = 0$ and 0 if $x_k = 1$, there x_k is the kth digit in the binary expansion of x. These are i.i.d. random variables (with respect to any μ_p), so by the Strong Law of Large Numbers,

$$\frac{1}{n}\sum_{k=1}^{n} X_k(x) \xrightarrow{n\to\infty} \mathbb{E}[X_1] = \mu_p[0,1/2] = p.$$

Thus for μ_p-a.e. x, $\lim S_n^{(0)}(x)/n \to p$ where $S_n^{(0)}(x) = \#\{j \le n : x_j = 0\}$. Similarly, $\lim S_n^{(1)}(x)/n \to 1-p$.

Let $K_p = \{x \in [0,1] : \lim S_n^{(0)}(x)/n = p\}$, so $\mu_p(K_p) = 1$. Note that $K_p = \{x \in [0,1] : \lim S_n^{(1)}(x)/n = 1-p\}$. Given $x \in [0,1]$ where $x = (x_j)_j$, $I_n(x) = I_{x_1 x_2 \dots x_n}$, $\mu_p(I_{x_1\dots x_n}) = p^{S_n^{(0)}(x)}(1-p)^{S_n^{(1)}(x)}$. Thus

$$\frac{\log \mu_p(I_{x_1\dots x_n})}{\log 2^{-n}} = \frac{S_n^{(0)}(x)\log p + S_n^{(1)}(x)\log(1-p)}{n\log 1/2} \to \frac{-(p\log p + (1-p)\log(1-p))}{\log 2} = s(p)$$

Given some $B(x,r)$ arbitrary, pick minimal k such that $I_k \subseteq B(x,r)$, where $2^{-(k-1)} = m(I_{k-1}) \ge r$ and $(k+1)\log 2 \ge |\log r| \ge (k-1)\log 2$, so that

$$\frac{|\log \mu_p(B(x,r))|}{|\log r|} \le \frac{|\log \mu_p(I_k(x))|}{(k-1)\log 2} \to s(p)$$

as $k \to \infty$ for μ_p-a.e. x.

3.6 Lemma. *If* $\limsup \frac{\mu(I_k(x))}{|I_k(x)|^t} \le c$ *for all $x \in F$ with $\mu(F) > 0$, then $\dim_H F \ge t$.*

Assuming this, we note that $\frac{\log \mu_p(I_k(x))}{\log |I_k(x)|} \to s(p)$, so for any $\epsilon > 0$ and $k \ge k_\epsilon$, $\mu_p(I_k(x)) \le |I_k(x)|^{s(p)-\epsilon}$ and $\limsup \frac{\mu_p(I_k(x))}{|I_k(x)|^{s(p)-\epsilon}} \le 1$. Thus $\dim_H K_p \ge s(p) - \epsilon$ for any $\epsilon > 0$, so $\dim_H K_p \ge s(p)$. This forces $\dim_H \mu_p \ge s(p)$.

Let's see a proof of the lemma. Our goal is to show that $H^t(F) > 0$ implies that $\dim_H F \ge t$. Let $\epsilon > 0$ and set $F_\delta = \{x \in F : \frac{\mu(I_k(x))}{|I_k(x)|^t} \le (c+\epsilon)\}$ for all k such that $2^{-k} \le \delta$.

Since F_δ increases to F as $\delta \to 0$. Let $\{U_i\}$ be a $\delta/4$-cover of F and choose $x_i \in U_i \cap F_\delta$ and let $B_i = B(x_i, |U_i|) \supseteq U_i$. Choose the minimal integer k_i such that $I_{k_i}(x_i) \subseteq B_i$. Consider $B_i \setminus I_{k_i-1}(x_i)$, and suppose there exists $y_i \in F_\delta \cap (B_i \setminus I_{k_i-1}(x_i))$. Let $I_{k_i-1}(y_i)$ be the diadic unit interval of level $k_i - 1$ containing y. Then $I_{k_i-1}(x_i) \cup I_{k_i-1}(y_i) \supseteq B_i \cap F_\delta$, and $I_{k_i}(x_i) \subseteq B_i$. Then $2^{-k_i} \le 2|U_i| \le 2\frac{\delta}{4}$, so $2^{-(k_i-1)} \le 4|U_i| \le \delta$. Thus

$$\mu(U_i \cap F_\delta) \le \mu(B_i \cap F_\delta) \le \mu(I_{k-1}(x_i) \cup I_{k_i-1}(y_i)) \le 2(2^{-(k_i-1)t})(c+\epsilon$$

so that

$$\mu(F_\delta) \le \sum \mu(F_\delta \cap U_i) \le c' \sum 2^{-k_i t} \le c'' \sum |U_i|^t.$$

Thus for ay cover (U_i) of F, $\mu(F_\delta) \le c'' H^t_{\delta/4}(F)$ so $0 < \mu(F)$. Thus $H^t(F) > 0$. ∎

In fact, this also shows that $\dim_H K_p = s(p)$. Put $F_p = \bigcup_{q \le p} K_q$, so $\mu_p(F_p) = 1$. Then

$$\begin{aligned}\frac{1}{k}\log\left(\frac{\mu_p(I_k(x))}{2^{-kt}}\right) &= \frac{1}{k}\left(\log\mu_p(I_k(x)) + lt\log 2\right)\\ &= \frac{1}{k}\left(S_k^{(0)}\log p + s_k^{(1)}(x)\log(1-p)\right) + t\log 2\end{aligned}$$

If $x \in F_p$, then $x \in K_q$ for some $q \le p$, so as $k \to \infty$, $q\log p + (1-q)\log(1-p) + t\log 2 \ge -s(p)\log 2 + t\log 2$. Fix $\epsilon > 0$. For large k,

$$\frac{1}{k}\log\left(\frac{\mu_p(I_k(x))}{2^{-kt}}\right) \ge (t - s(p) - \epsilon)\log 2$$

In particular,

$$\frac{\mu_p(I_k(x))}{|I_k(x)|^t} \ge 2^{k(t-s(p)-\epsilon)} \to \infty$$

for $t > s(p) + \epsilon$. Take any $x \in F_p$, and get minimal k such that $B(x,r) \supseteq I_k(x)$.

Definition. Say that a measure μ has **exact lower dimension** s if $\underline{\dim}_{\text{loc}}\mu(x) = s$ for μ a.e. x.

Definition. We say that μ is **invariant** under $f : X \to X$ if whenever A is μ-measurable, then $f^{-1}(A)$ is also μ-measurable and $\mu(A) = \mu(f^{-1}(A))$. We say that μ is **ergodic under f** if whenever A is measurable and $f^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(A^c) = 0$.

Example. Let μ be a self-similar measure from an IFS $\{F_1, \dots, F_m\}$ satisfying the strong separation condition. Define f by $f(x) = F_i^{-1}(x)$ if $x \in F_i(E)$. One can verify that μ is invariant and ergodic under f.

> ***3.7 Theorem. (Mean Ergodic)*** *Let $f : X \to X$, μ a finite measure on X which is invariant and ergodic under f. Let $G \in L^1(\mu)$. Then $\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} G(f^{(j)}(g)) = \frac{1}{\mu(X)}\int_X G(z)\,d\mu(z)$.*

> ***3.8 Theorem.*** *Let $f : X \to X$ be Lipschitz and μ a finite Borel measure that is invariant and ergodic under f. Then μ has exact upper and lower dimensions.*

Proof Want to understand $\mu(B(x,r))$. Set $G(y) = \mathbf{1}B(x,r)(y) \in L^1(\mu)$. By the ergodic theorem,

$$\frac{1}{\mu(X)}\int_X \mathbf{1}B(x,r)(z)\,d\mu(z) = \frac{\mu(B(x,r))}{\mu(X)} = \lim \frac{1}{k}\sum_0^{k-1}\mathbf{1}B(x,r)(f^{(j)}(y))$$

for a.e. g. Assume $|f(x) - f(y)| \le c|x-y|$ for all $x, y \in X$. Then

$$|f(x) - f(f^{(j)}(y))| \le c|x - f^{(j)}(y)|.$$

Thus if $f^{(j)} \in B(x,r)$, then $f^{(j+1)}(y) \in B(f(x), cr)$ so that $\mathbf{1}B(x,r)(f^{(j)}(y)) \le \mathbf{1}B(f(x),cr)(f^{(j+1)}(y))$. Take $G_1 = \mathbf{1}B(f(x), cr)$. Then

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k-1}\mathbf{1}B(f(x),cr)(f^{(j)}(f(x))) = \frac{1}{\mu(X)}\int_X G_1 = \frac{\mu(B(f(x),cr))}{\mu(X)}$$

for a.e. y by f invariance. Thus $\mu(B(x,r)) \leq \mu(B(f(x),cr))$. In particular, $\underline{\dim}_{\mathrm{loc}}\mu(f(x)) \leq \underline{\dim}_{\mathrm{loc}}\mu(x)$. Since μ is f-invariant, for any Borel function Φ, $\int \Phi(f)\,\mathrm{d}\mu = \int \Phi\,\mathrm{d}\mu$ so that

$$\int \underline{\dim}_{\mathrm{loc}}\mu(f(x))\,\mathrm{d}\mu(x) = \int \underline{\dim}_{\mathrm{loc}}\mu(x)\,\mathrm{d}\mu(x)$$

so that $\underline{\dim}_{\mathrm{loc}}\mu(f(x)) = \underline{\dim}_{\mathrm{loc}}\mu(x)$ for μ a.e. x. Let $A_k^{(j)} = \{x \in X : \underline{\dim}_{\mathrm{loc}}\mu(x) \in [j/2^k,(j+1)/2^k)\}$ so that the $A_k^{(j)}$ are f-invariant. Then $\mu(A_k^{(j)}) = 0$ or $\mu((A_k^{(j)}) = \mu(X) < \infty$. Then $X = \bigcup_{j=-\infty}^{\infty} A_k^{(j)}$, so that $\mu(x) = \sum_j \mu(A_k^{(j)})$ for all k. Thus there exists a unique k such that $\mu(A_k^{(j)}) = \mu(x)$, so that $\underline{\dim}_{\mathrm{loc}}\mu(x) = \bigcap_{k=1}^{\infty} A_k^{(j(x)}$. ■

3.9 Corollary. *Self-similar measures satisfying the SSC are exact.*

4 Multifractal Analysis

In general, we are interested in the sets $E_\alpha = \{x : \dim_{\mathrm{loc}}\mu(x) = \alpha\} \subseteq \operatorname{supp}\mu$. We are interested in $\dim_H E_\alpha = f_H(\alpha)$. Clearly $0 \leq f_H(\alpha) \leq \dim_H \operatorname{supp}\mu$. We have already seen if $\dim_{\mathrm{loc}}\mu(x) \leq \alpha$ for all $x \in F$, then $\dim_H F \leq \alpha$; thus, $f_H(\alpha) \leq \alpha$.

4.1 Coarse Theory

Let $N_r(\alpha)$ denote the number of r-mesh cubes A with $\mu(A) > r^\alpha$. The coarse multifractal spectrum of μ is given by

$$f_c(\alpha) = \lim_{\epsilon\to 0}\left(\lim_{r\to 0}\frac{\log^+(N_r(\alpha+\epsilon) - N_r(\alpha-\epsilon))}{-\log r}\right)$$

where $\log^+(x) = \max\{0,\log x\}$ and $f_c(\alpha) \geq 0$. For small $r,\epsilon > 0$, we have

$$r^{-(f_c(\alpha)-n)} \leq N_r(\alpha+\epsilon) - N_r(\alpha-\epsilon) \leq r^{-(f_c(\alpha)+n)}$$

which means roughly that the number of r-mesh cubes with measure $\approx r^\alpha$ is about $r^{-f_c(\alpha)}$. We write $\underline{f}_c$ and $\overline{f}_c$ with respect to the limit infimum / supremum.

4.1 Proposition. *$f_H(\alpha) \leq \underline{f}_c(\alpha)$.*

Proof We do this in $\mathbb{R}$. Assume $f_H(\alpha) > 0$. Let $0 < \epsilon < f_H(\alpha)$, $t = f_H(\alpha) - \epsilon$, so that $H^t(E_\alpha) = \infty$. Observe that $\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} = \alpha$ for all $x \in E_\alpha$. In particular, there exists r_ϵ such that for all $r \leq r_\epsilon$

$$(\alpha-\epsilon)|\log k_2 r| \leq |\log\mu(B(x,r))| \leq (\alpha+\epsilon)|\log(k_1 r)$$

so that $(k_2 r)^{-\alpha-\epsilon} \geq \mu(B(x,r)) \geq (k_1 r)^{\alpha+\epsilon}$, for fixed k_1, k_2. In particular, take $k_1 = 3$ and $k_2 = 1/2$ so $3r^{\alpha+\epsilon} \leq \mu(B(x,r)) \leq 2^{\epsilon-\alpha} r^{\alpha-\epsilon}$ for sufficiently small r. Let

$$F_n = \left\{x \in E_\alpha : 3r^{\alpha+\epsilon} \leq \mu(B(x,r)) \leq 2^{\epsilon-\alpha} r^{\alpha-\epsilon} \text{ for all } r \leq \frac{1}{n}\right\}$$

so that $F_n \to E_\alpha$. Thus there exists N such that $H^t(F_N) > 1$. For δ sufficiently small, say $\delta \leq 1/(2N)$, $H_\delta^t(F_N) \geq 1$. Think about the r-mesh intervals that intersect F_N. For some fixed $x \in F_N$ and let A be the r-mesh interval containing x, with adjacent intervals A_R and A_L. Then $A \subseteq B(x,r) \subseteq A_R \cup A \cup A_L \subseteq B(x, r2)$ so that

$$3r^{\alpha+\epsilon} \leq \mu(B(x,r)) \leq \mu(A \cup A_R \cup A_L) \leq \mu(B(x,2r)) \leq r^{\alpha-\epsilon}.$$

Then there exists some $A_0 \in \{A, A_R, A_L\}$ such that $r^{\alpha+\epsilon} \leq \mu(A_0) \leq r^{\alpha-\epsilon}$. Thus the number of mesh intervals with μ-measure in $[r^{\alpha+\epsilon}, r^{\alpha-\epsilon}] \geq 1/3r^{-(f_H(\alpha)-\epsilon)}$. In particular,

$$\frac{\log^+(N_r(\alpha+\epsilon)-N_r(\alpha-\epsilon))}{-\log r} \geq \frac{\log(\frac{1}{3}r^{-(f_H(\alpha)-\epsilon)})}{-\log r} \to f_H(\alpha)-\epsilon$$

and let $\epsilon \to 0$. ■

4.2 Legendre Transform

Let $\beta : \mathbb{R} \to \mathbb{R}$ be convex. We define $f(\alpha) = \int_{q\in\mathbb{R}}(\beta(q)+\alpha q)$; we are interested in the case as $\beta \to 0$. When β is differentiable, $\beta'(q)+\alpha = 0$ if and only if $\alpha = -\beta'(q)$. This q gives the global minimum. Thus $f(\alpha) = \beta(q_\alpha)+\alpha q_\alpha$ where $\alpha = -\beta'(q_\alpha)$. Equivalently, $f(\alpha)$ is the y-intercept to the curve β at q_α.

Let $\mathcal{Q}$ denote the set of r-mesh cubes A with $\mu(A) > 0$, and let $M_r(q) = \sum_{A\in\mathcal{Q}} \mu(A)^q$. If $q \geq 0$ and $\mu(A) \geq r^\alpha$, then $\mu(A)^q \geq r^{\alpha q}$ so

$$M_r(q) \geq \begin{cases} r^{\alpha q} N_r(\alpha) & : q \geq 0 \\ r^{\alpha q} \cdot (\#r-\text{mesh cubes with } \mu(\cdot) \leq r^\alpha & : q < 0 \end{cases}.$$

Put $\underline{\beta}(q) = \liminf_r \frac{\log M_r(q)}{-\log r}$.

4.2 Proposition. $f_C(\alpha) \leq \inf_q\{\beta(q)+\alpha q\}$.

Proof First suppose $q \geq 0$ and fix $n > 0$. Then

$$\begin{aligned} M_r(q) &\geq r^{(\alpha+\epsilon)q} N_r(\alpha+\epsilon) \geq r^{(\alpha+\epsilon)q}(N_r(\alpha+\epsilon)-N_r(\alpha-\epsilon)) \\ &\geq r^{(\alpha+\epsilon)q} r^{-(f_c(\alpha)-n)} = r^{(\alpha+\epsilon)q-f_c(\alpha)+n}. \end{aligned}$$

In particular,

$$\begin{aligned} \frac{\log M_r(q)}{-\log r} &\geq \frac{\big((\alpha+\epsilon)q-\underline{f}_c(\alpha)+n\big)\log r}{-\log r} \\ &= -(\alpha+\epsilon)q+\underline{f}_c(\alpha)-n \end{aligned}$$

so that $\underline{\beta}(q) \geq -(\alpha+\epsilon)q+\underline{f}_c(\alpha)-n$. Thus $\underline{f}_c(\alpha) \leq \underline{\beta}(q)+\alpha q$.

For $q < 0$,

$$\begin{aligned} M_r(q) &\geq r^{q(\alpha-\epsilon)}(N_r(\alpha+\epsilon)-N_r(\alpha-\epsilon)) \\ &\geq r^{q(\alpha-\epsilon)} r^{-(\underline{f}_c(\alpha)-n)} \end{aligned}$$

and the argument proceeds as before. ■

4.3 Fine Theory

Let E_α be as before. Suppose we are given similarities $\{F_i\}$, probabilities $\{p_i\}$, and contraction factors $\{r_i\}$ as i ranges over Λ. We also assume that this IFS satisfies the strong separation condition, i.e. its invariant compact set K satisfies $K = \dot{\bigcup}_{i\in\Lambda} F_i(K)$. Define $\beta(q)$ by $\sum_{i\in\Lambda} p_i^q r_i^{\beta(q)} = 1$. By the implicit function theorem, β is well-defined and differentiable of all orders. Let $q \to -\infty$, so $\beta(q) \to +\infty$, and as $q \to +\infty$, $\beta(q) \to -\infty$. As well, β is a strictly decreasing function. Differentiating, we have

$$\begin{aligned} 0 &= \sum \log p_i p_i^q r_i^{\beta(q)} + p_i^q r_i^{\beta(q)} \log r_i beta'(q) \\ &= \sum p_i^q r_i^{\beta(q)} (\log p_i + \beta'(q) \log r_i) \end{aligned}$$

and differentiating again,

$$0 = \sum p_i^q r_i^{\beta(q)} (\log p_i + \beta'(q)\log r_i)^2 + p_i^q r_i^{\beta(q)} \beta''(q) \log r_i$$

so that

$$\beta''(q) \sum p_i^q r_i^{\beta(q)} (-\log r_i) = \sum p_i^q r_i^{\beta(q)} (\cdot)^2$$

so that $\beta''(q) > 0$ and β is convex. If there exists i such that $\log p_i + \beta'(q)\log r_i \neq 0$, then $\beta''(q) > 0$ for all q. This happens if $\frac{\log p_i}{\log r_i}$ is not constant over i, so we assume we are in this case.