Fractal Geometry

Alex Rutar*
University of Waterloo

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*arutar@uwaterloo.ca
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I. Stochastic Calculus

Definition. Given a measure space $(\Omega, \mathcal{F}, \mathbb{P})$, a measurable function $f : \Omega \to \mathbb{R}$ is called a **random variable**.

Definition. A **stochastic process** $X = \{X_t\}_{t\in T}$ is a collection of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Typically $t \in \mathbb{Z}^+$ or $t \in \mathbb{R}^+$ (including 0); t is a discrete or continuous time parameter. Given some $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is called a **realization** or **path** of this process. We will regard $\{X_t\}_{t\geq 0}$ as a random element in some path space, equipped with a proper $\sigma-$algebra and probability.

Consider $X_t(\omega)$ as a function $X : [0,\infty) \times \Omega \to \mathbb{R}$ equipped with the product $\sigma-$algebra.

Definition. The **distribution** of a stochastic process is the collection of all its finite-dimensional distributions.

Two processes X and Y can be "the same" in different senses:

Definition. Two process $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$ are called **distinguishable** if almost all their sample paths agree; in other words, $\mathbb{P}(X_t = Y_t, 0 \leq t < \infty) = 1$. We say that Y is a **modification** of X if for each $t \geq 0$ we have $\mathbb{P}(X_t = Y_t) = 1$. Finally, X and Y are said to have the **same distribution** if all the finite dimensional distributions agree. In other words, if for all $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, we have $(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (Y_{t_1}, \ldots, Y_{t_n})$.

Example. Let X be a continuous stochastic process and N a Poisson point process on $[0,\infty)$. Then define

$$Y_t := \begin{cases} X_t & : t \notin N \\ X_t + 1 & : t \in N \end{cases}$$

Thus $\mathbb{P}(X_t = Y_t) = 1$ for all t, so X is a modification of Y. However, $\mathbb{P}(X_t = Y_t, t \geq 0) = 0$, so that X and Y are not indistinguishable.

A filtration formalizes the idea of "information acquired over time".

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **filtration** is a non-decreasing family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ so that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. We write $F_\infty = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$.

Let $\{X_t\}_{t\geq 0}$ be a stochastic process. The filtration generated by $\{X_t\}_{t\geq 0}$ is $\{\sigma(X_s : 0 \leq s \leq t)\}_{t\geq 0}$, in other words $\mathcal{F}_t$ is the smallest $\sigma-$algbra which makes X_s measurable for all $s \in [0,t]$.

Definition. A stochastic process $\{X_t\}_{t\geq 0}$ is called **adapted** to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if X_t is $\mathcal{F}_t-$measurable for every $t \geq 0$.

The filtration generated by $\{X_t\}_{t\geq 0}$ is the smallest filtration which makes $(X_t)_{t\geq 0}$ adapted.

A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is said to satisfy the "usual condition" if

1. It is right-continuous: $\lim_{s\to t^+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$
2. $\mathcal{F}_0$ contains all the $\mathbb{P}-$null events in $\mathcal{F}$.

1 Martingale Theory

Consider a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in S})$ where $S = \mathbb{N}$ or $S = \mathbb{R}^+$.

Definition. A **random time** T is called a stopping time if $\{T \le t\} \in \mathcal{F}_t$ ("we know it happens when it happens").

Example. (i) Constants are trivial stopping times.

(ii) Last hit a constant before N is not a stopping time

1.1 Proposition. *If S, T are stopping times, $T \vee S$, $T \wedge S$, $T + S$ are stopping times.*

Proof That $T \wedge S$ and $T \vee S$ are stopping times are trivial. For $T + S$, $\{T + S > t\} = \{T = 0, S > t\} \cup \{0 < T \le t, T + S > t\} \cup \{T > t\}$. It suffices to prove that

$$\{0 < T \le t < T + S > t\} = \bigcup_{\substack{r\in\mathbb{Q}^+\\0<r<t}} \{r < T \le t, S > t - r\}.$$

If there exists r with $r < T \le t$, then $S > t = r$ and $S + T > r + (t - r) = t$, so $\supseteq$ holds. Conversely, if $0 < T \le t$ and $T + S \ge t$¡ then there exists $r \in \mathbb{Q}$ such that $r < T$ and $r + S > t$. Hence $r < T \le t$ and $S > t - r$. ∎

Definition. The σ−algebra generated by a stopping time T is the collection of all the events A for which $A \cap \{T \le t\} \in \mathcal{F}_t$ for every $t \ge 0$. This is the "information you collect until the stopping time".

Exercise: show that the collection given in the definition above is actually a σ−algebra.

We write $X_{T\wedge t}$ is a random variable evaluated at time $T \wedge t$ (or T); in other words, $(X_{T\wedge t})(\omega) = X_{T\wedge t}(\omega)$. Then $\{T_{T\wedge t}\}_{t\ge 0}$, or X^T, is a stochastic process stopped at time t.

Definition.