

REPLACE

Alex Rutar^{*}
University of Waterloo

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^{*}*arutar@uwaterloo.ca*

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I. REPLACE

1. For $a, b, k \in \mathbb{N}$,

$$\binom{a+b}{k} = \sum_{j=1}^k \binom{a}{j} \cdot \binom{b}{k-j} \quad (0.1)$$

We prove this with a bijection:

$$\mathcal{B}(a+b, k) \rightleftharpoons \bigcup_{j=0}^k \mathcal{B}(a, j) \times \mathcal{B}(b, k-j)$$

given by $S \mapsto (S \cap \{1, \dots, a\}, (S \cap \{a+1, \dots, a+b\})^{(-a)})$ and $(P, Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n, i)$ is the set of i -element subsets of $\{1, 2, \dots, n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity

$$\binom{x+y}{k} = \sum_{j=0}^k \binom{x}{j} \binom{y}{k-j}$$

in $\mathbb{Q}[x, y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i! \binom{x}{i}$, so multiplying our identity by $k!$, we get

$$(x+y)_k = \sum_{j=0}^k \binom{k}{j} (x)_j (y)_{k-j}$$

Compare this with the standard binomial theorem

$$(x+y)^k = \sum_{j=0}^k \binom{k}{j} x^j y^{k-j}$$

These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s, t \geq 1$,

$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^n \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1}$$

Let $\mathcal{M}(m, r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m, r)| = \binom{m+r-1}{r-1}$. Let's define a bijection

$$\mathcal{M}(n, s+t) \rightleftharpoons \bigcup_{k=1}^n \mathcal{M}(k, s) \times \mathcal{M}(n-k, t) \quad (0.2)$$

$\mu = (m_1, \dots, m_{s+t}) \mapsto ((m_1, \dots, m_s), (m_{s+1}, \dots, m_{s+t}))$ and $(v, \theta) \mapsto v\theta$. Note that if f, g are polynomials of degree d and e respectively, then $\sum_{k=0}^n f(k)g(n-k)$ is a polynomial in n of degree $d+e-1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n &= \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n = \sum_{(m_1, \dots, m_{s+t})} z^{m_1 + \dots + m_{s+t}} \\ &= \left(\sum_{m=0}^{\infty} z^m \right)^{s+t} \\ &= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s} \frac{1}{(1-z)^t} \\ &= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell \\ &= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^n \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1} \right) \end{aligned}$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a (1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z = -q$ and $n = -t$?

3. Consider

$$(x_1 + x_2)^n = \sum_{i=0}^n \binom{n}{i} x_1^i x_2^{n-i}$$

and

$$(x_1 + x_2)^n = \sum_{f: N_n \rightarrow \{1,2\}} \prod_{j=1}^n x_{f(j)}$$

More generally, we can consider

$$(x_1 + \dots + x_k)^n = \sum_{f: N_n \rightarrow N_k} \prod_{j \in N_n} x_{f(j)}$$

If we set all $x_1 = \dots = x_k = 1$, then k^n gives the number of functions from N_n to N_k . If we set $x_i = q^i$ for all $i \in N_k$, then we get

$$\left(\frac{q - q^{k+1}}{1 - q} \right)^n = (q + q^2 + \dots + q^k)^n = \sum_{f: N_n \rightarrow N_k} q^{f(1) + \dots + f(n)}$$

Collect all the terms in $(x_1 + \dots + x_k)^n$ that produce the same monomial. Given a multiset μ with $m_1 + \dots + m_k = n$, write $x_1^{m_1} \dots x_k^{m_k} = \underline{x}^\mu$. Then

$$(x_1 + \dots + x_k)^n = \frac{n!}{m_1! \dots m_k!} \underline{x}^\mu = \sum_{\mu \in \mathcal{M}(n, k)} \binom{n}{\mu} \underline{x}^\mu$$

4. How can we interpret

$$P_n(q) = \prod_{i=1}^n (1 + q + q^2 + \cdots + q^{i-1})$$

In general, if we set $q = 1$, we see that $P_n(1) = n!$. We might hope that there is some weight function on permutations $w : \mathcal{S}_n \rightarrow \mathbb{N}$ such that $P_n(q) = \sum_{\sigma \in \mathcal{S}_n} q^{w(\sigma)}$. Recall the bijection $I_n : \mathcal{S}_n \rightarrow \mathcal{Q}_n$ from chapter 1. Let's find some weight function $v : \mathcal{Q}_n \rightarrow \mathbb{N}$ such that $\sum_{\rho \in \mathcal{Q}_n} x^{v(\rho)} = P_n(q)$, then “pull back” the definition of $v : \mathcal{Q}_n \rightarrow \mathbb{N}$ to get a definition for $\omega : \mathcal{S}_n \rightarrow \mathbb{N}$. Note that $\sum_{h \in \mathbb{N}_r} q^{h-1} = 1 + q + \cdots + q^{r-1}$. Thus

$$\sum_{\rho=(h_1, \dots, h_n) \in \mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^n (1 + q + \cdots + q^{i-1}) = P_n(q)$$

so we can define $v(\rho) = |\rho| - n$ and $\sum_{\rho \in \mathcal{Q}_n} q^{|\rho| - n} = P_n(q)$. We also have

$$\sum_{\rho \in \mathcal{Q}_n} q^{(h_1-1)+\cdots+(h_n-1)} = (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2}) \cdots (1 + q)(1)$$

For notation, define $[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q}$. Then $[m]_q! = [m]_q [m-1]_q \cdots [1]_q$.

	1	q	q ²	q ³	q ⁴
$q[3]_q$	0	1	1	1	
$[2]_q[3]_q$	1	2	2	1	
$-q[2]_q[3]_q$	0	-1	-2	-2	-1
$q^2[2]_q[3]_q$	0	0	1	2	2
$[6]_q$	1	1	1	1	1

so that $[6]_q = (1 - q + q^2)[2]_q[3]_q$. An **inversion** in $\sigma = a_1 \dots a_n \in \mathcal{S}_n$ is a pair (i, j) of indices $1 \leq i < j \leq n$ with $a_i > a_j$. Define $\text{Inv}(\sigma)$ as the set of inversions of σ , and $\text{inv}(\sigma) = |\text{Inv}(\sigma)|$. Notice that if $\sigma = a_1 \dots a_n \mapsto \rho = (h_1, \dots, h_n)$, then for each $1 \leq i \leq n$, $h_i - 1$ is the number of inversions of σ with i in the first coordinate. Recall

$$\begin{aligned} \mathcal{S}_n &\rightleftharpoons \mathcal{B}(n, k) \times \mathcal{S}_k \times \mathcal{S}_{n-k} \\ \sigma = a_1 \dots a_n &\leftrightarrow (A, \beta, \gamma) \\ \text{inv}(\sigma) &= w(A) + \text{inv}(\beta) + \text{inv}(\gamma) \end{aligned}$$

Assuming such a weight function $w(A)$ exists, then

$$\begin{aligned} [n]_q! &= \sum_{\sigma \in \mathcal{S}_n} q^{\text{inv}(\sigma)} = \sum_{(A, \beta, \gamma)} q^{w(A) + \text{inv}(\beta) + \text{inv}(\gamma)} \\ &= [k]_q! \cdot [n-k]_q! \cdot \sum_{A \in \mathcal{B}(n, k)} q^{w(A)} \end{aligned}$$

so that

$$\sum_{A \in \mathcal{B}(n, k)} q^{w(A)} = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!} = \left[\begin{matrix} n \\ k \end{matrix} \right]_q$$

$$\sum_{S \in \mathcal{B}(n,k)} q^{\text{sum}(S)} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

0.1 Theorem. Let V be an n -dimensional vector space over a finite field \mathbb{F}_q . Then for $0 \leq k \leq n$, the number of k -dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

0.2 Lemma. Let $L : V \rightarrow W$ be a linear transformation that is surjective. Then $\dim V = \dim W + \dim(\ker L)$. So if this is over a finite field \mathbb{F}_q , every $w \in W$ is the image of exactly $q^{\dim(\ker L)}$ vectors $v \in V$.

For every $w \in W$, is the image of exactly q^k vectors in V . The number of ordered bases of V is $q^{\binom{n}{2}}(q-1)^n[n]_q!$.

0.3 Theorem. Let V be an n -dimensional vector space over a finite field \mathbb{F}_q . For $0 \leq k \leq n$, the number of k -dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

PROOF Let $\text{OB}(V)$ be the set of ordered bases of V , and let $G(V, k)$ be the set of k -dimensional subspaces of V . Define a function

$$\text{OB}(V) \rightarrow \bigcup_{U \in G(V, k)} (\{U\} \times \text{OB}(U) \times \text{OB}(V/U))$$

as follows. Given (v_1, \dots, v_n) an ordered basis of V , let $U = \text{span}_{\mathbb{F}_q}\{v_1, \dots, v_k\}$. Then $(v_1, \dots, v_k) \in \text{OB}(U)$ and $(v_{k+1} + U, \dots, v_n + U) \in \text{OB}(V/U)$. Consider the map $L : V \rightarrow V/U$ given by $L(v) = v + U$, so that every $v + U$ in V/U is the image of q^k vectors in V . Thus $(v_{k+1} + U, \dots, v_n + U)$ is the image of $q^{k(n-k)}$ sequences (z_{k+1}, \dots, z_n) of vectors in V . Thus the function $(v_1, \dots, v_n) \mapsto (U, (v_1, \dots, v_k), (v_{k+1} + U, \dots, v_n + U))$ is surjective and hits everything on the RHS $q^{k(n-k)}$ times. But then counting both sides,

$$\begin{aligned} q^{\binom{n}{2}}(q-1)^n[n]_q! &= \sum_{U \in G(V, k)} 1 \cdot q^{\binom{k}{2}}(q-1)^k[k]_q! \cdot q^{\binom{n-k}{2}}(q-1)^{n-k}[n-k]_q! \cdot q^{k(n-k)} \\ q^{\binom{n}{2}}[n]_q! &= |G(V, k)| \cdot [k]_q! \cdot [n-k]_q! q^{\binom{k}{2} + \binom{n-k}{2} + k(n-k)} \\ [n]_q! &= |G(V, k)| \cdot [k]_q! \cdot [n-k]_q! \end{aligned}$$

giving our desired result. ■

A **set partition** π of a set V is a collection of subsets $\pi = \{B_1, \dots, B_k\}$ of V such that

- Each B_i is not empty
- $B_i \cap B_j = \emptyset$ if $i \neq j$
- $B_1 \cup \dots \cup B_k = V$

Let $\Pi(n, k)$ be the set of set partitions of N_n with k blocks, and set $S(n, k) = |\Pi(n, k)|$. Certainly $S(0, 0) = 1$ for the empty set partition. If $n \geq 1$, then $S(n, 0) = 0$, $S(n, n) = 1$, and $S(n, 1) = 1$. We can also define a recurrence relation. Let $\Pi'(n, k)$ be those $\pi \in \Pi(n, k)$ in which $\{n\}$ is a block, and $\Pi''(n, k)$ is the set of π in which n is in a block of size at least 2. Note that $\Pi'(n, k) \rightleftharpoons \Pi(n-1, k-1)$ by removing or adding the independent element.

Furthermore, the function which removes the element n from a block in $\Pi''(n, k)$ is a surjective function onto $\Pi(n-1, k)$ which hits every element of $\Pi(n-1, k)$ k times. Thus combining these observations, $S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$. Thus we can compute

$S(n, k)$	0	1	2	3	4	5	6
0	1	X	X	X	X	X	X
1	0	1	X	X	X	X	X
2	0	1	1	X	X	X	X
3	0	1	3	1	X	X	X
4	0	1	7	6	1	X	X
5	0	1	15	25	10	1	X
6	0	1	31		1		

From homework 2, we have that

$$x^n = \sum_{k=0}^n k! S(n, k) \binom{n}{k}$$

Invert this using Binomial Inversion.

0.4 Theorem. (Binomial Inversion) Let a_0, a_1, \dots be a sequence.

PROOF For $h \in \mathbb{N}$, let $b_h = \sum_{i=0}^h \binom{h}{i} a_i$. Let $A(t) = \sum_{i=0}^{\infty} a_i t^i$ and $B(t) = \sum_{h=0}^{\infty} b_h t^h$. Then

$$\begin{aligned} B(t) &= \sum_{h=0}^{\infty} t^h \sum_{i=0}^h \binom{h}{i} a_i \\ &= \sum_{i=0}^{\infty} a_i t^i \sum_{h=i}^{\infty} \binom{h}{h-i} t^{h-i} \\ &= \sum_{i=0}^{\infty} a_i t^i \sum_{j=0}^{\infty} \binom{i+j}{j} t^j = \sum_{i=0}^{\infty} \frac{a_i t^i}{(1-t)^{i+1}} \\ &= \frac{1}{1-t} \sum_{i=0}^{\infty} a_i \left(\frac{t}{1-t} \right)^i = \frac{1}{1-t} A\left(\frac{t}{1-t} \right) \end{aligned}$$

Let $z = t/(1-t)$, so that $t = z/(1+z)$. Thus

$$B\left(\frac{z}{1+z} \right) = (1+z)A(z)$$

so that

$$\begin{aligned} \sum_{i=0}^{\infty} a_i z^i &= \frac{1}{1+z} B\left(\frac{z}{1+z} \right) = \sum_{h=0}^{\infty} b_h \frac{z^h}{(1+z)^{h+1}} \\ &= \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\infty} \binom{j+h}{h} z^h (-z)^j \\ &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} b_j \end{aligned}$$

Thus for all $m \in \mathbb{N}$, $a_m = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} b_j$. ■

In particular, applying this to Stirling numbers of the second kind, for all $n \in \mathbb{N}$ in $\mathbb{R}[x]$, we have

$$x^n = \sum_{k=0}^n S(n, k) \binom{x}{k} k!$$

Let $b_i = i^n$ for $i = 0, 1, 2, \dots$. If $k > n$ or $k > i$, then $S(n, k) \binom{i}{k} = 0$; thus,

$$\begin{aligned} i^n &= \sum_{k=0}^n S(n, k) \binom{i}{k} k! = \sum_{k=0}^{\min(n, i)} S(n, k) \binom{i}{k} k! = \sum_{k=0}^i S(n, k) \binom{i}{k} k! \\ &= \sum_{k=0}^i \binom{i}{k} a_k \end{aligned}$$

where $a_k = k! S(n, k)$ for all $k \in \mathbb{N}$. But then apply binomial inversion to get

$$a_k = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} b_j$$

Suppose $m^n = \sum_{k=0}^n S(n, k) \binom{m}{k} k!$. Then $[m]_q^n = \sum_{k=0}^{\infty} S[n, k]_q \left[\begin{matrix} m \\ k \end{matrix} \right]_q [k]_q!$, where $S[n, k]_q = \sum_{\pi \in \Pi(n, k)} q^{w(\pi)}$. Is there some function $w : \Pi(n, k) \rightarrow \mathbb{N}$ that makes this work?

B.5 For \mathcal{S} a set of BTs, let \mathcal{R} be the trees in \mathcal{S} with a red root and \mathcal{B} be the trees in \mathcal{S} with a blue root, so $\mathcal{S} = \mathcal{R} \cup \mathcal{B}$ disjointly. Let $r(T)$ count the number of red nodes, and $b(T)$ count the number of blue nodes, and let $S(x, y) = \sum_{T \in \mathcal{S}} x^{r(T)} y^{b(T)}$. In particular, $S(t, t) = \sum_{T \in \mathcal{S}} t^{n(T)}$ where $n(T) = r(T) + b(T)$ is the total number of nodes.

We have bijections

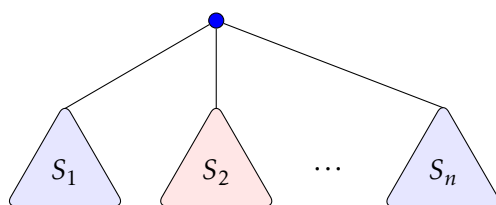
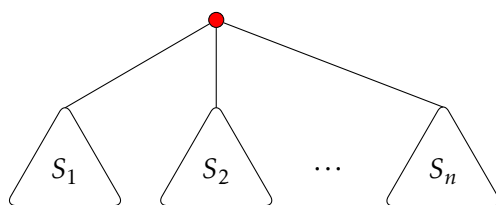
$$\begin{aligned} \mathcal{R} &\simeq \{\bullet\} \times \bigcup_{k=0}^{\infty} \mathcal{S}^k \\ \mathcal{B} &\simeq \{\bullet\} \times ((\epsilon \cup \mathcal{R})(\mathcal{B}\mathcal{R})^*(\epsilon \cup \mathcal{B})) \\ \mathcal{S} &\simeq \mathcal{R} \cup \mathcal{B} \end{aligned}$$

so that

$$\begin{aligned} S &= R + B \\ R &= \frac{x}{1 - S} \\ B &= y(1 + R) \frac{1}{1 - BR} (1 + B) \end{aligned}$$

Substituting R and B using the first two equations, we get

$$S - \frac{x}{1 - S} = \frac{y \left(1 + \frac{x}{1 - S}\right) \left(1 + S - \frac{x}{1 - S}\right)}{1 - \frac{x}{1 - S} \left(S - \frac{x}{1 - S}\right)}$$



II. Power Series Identities

- (i) $\frac{1}{(1-z)^h} = \sum_{k=0}^{\infty} \binom{k+h-1}{h-1} z^k$
- (ii) Let a_0, a_1, \dots be a sequence, and $b_h = \sum_{i=0}^h \binom{h}{i} a_i$. Then $a_m = \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} b_i$.
- (iii) General Binomial Series. For $k \in \mathbb{N}$, let $\binom{y}{k} = \frac{y(y-1)\dots(y-k+1)}{k!} \in \mathbb{Q}[y]$. Then we define

$$(1+x)^y = \sum_{k=0}^{\infty} \binom{y}{k} x^k$$

which is a power series in x . Each coefficient of $[x^n]$ is in $\mathbb{Q}[y]$. Then by Vandermone convolution,

$$\begin{aligned} (1+x)^y (1+x)^z &= \sum_{i=0}^{\infty} \binom{y}{i} x^i \sum_{j=0}^{\infty} \binom{z}{j} x^j \\ &= \sum_{n=0}^{\infty} x^n \left(\sum_{i=0}^n \binom{y}{i} \binom{z}{n-i} \right) \\ &= \sum_{n=0}^{\infty} \binom{y+z}{n} x^n = (1+x)^{y+z} \end{aligned}$$

Furthermore, if $y = -p < 0$ is an integer, then

$$\begin{aligned} (1+x)^{-p} &= \sum_{k=0}^{\infty} \binom{-p}{k} x^k \\ &= \sum_{k=0}^{\infty} \binom{k+p-1}{p-1} x^k \end{aligned}$$

For $\alpha \in \mathbb{C}$, $f(x) = (1+x)^\alpha$ is analytic for $|x| < 1$. In particular, by Taylor's theorem,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} c_k x^k$$

where $c_k = \frac{1}{k!} \frac{d^k}{dx^k} (1+x)^\alpha \big|_{x=0}$.

Consider the class \mathcal{Q} of (unrooted) trees in which every vertex has odd degree. We identify $\mathcal{Q}^\bullet \equiv \chi * \xi_{\text{odd}}[\mathcal{N}]$ for some class \mathcal{N} describing the components of $T \setminus \{v\}$. A structure in \mathcal{N} is a rooted tree in which every vertex has an even number of children. Moreover, $\mathcal{N} \equiv \chi * \xi_{\text{even}}[\mathcal{N}]$. Note that the exponential generating function for ξ_{odd} is $\sum_{j=0}^{\infty} \frac{e^{2j+1}}{(2j+1)!}$, and similarly for the even components. This give

$$\begin{aligned} \mathcal{Q}^\bullet &= x \cdot E_{\text{odd}}(N(x)) = x \cdot \sinh(N(x)) \\ N(x) &= x \cdot E_{\text{even}}(N(x)) = x \cdot \cosh(N(x)) \end{aligned}$$

Now apply LIFT with $K = \mathbb{Q}$, $G(u) = \cosh(u)$, and $F(u) = \sinh(u)$, so $F'(u) = \cosh(u)$. Now for $n \geq 2$,

$$\begin{aligned}
 |\mathcal{Q}_n| &= \frac{1}{n} |\mathcal{Q}_n^\bullet| = \frac{1}{n} \cdot n! [x^n] Q^\bullet(x) \\
 &= (n-1)! [x^n] x \sinh(N(x)) \\
 &= (n-1)! [x^{n-1}] \sinh(N(x)) \\
 &= (n-1)! \cdot \frac{1}{n-1} [u^{n-2}] F'(u) G(u)^{n-1} \\
 &= (n-2)! \cdot [u^{n-2}] \cosh(u)^n \\
 &= (n-2)! [u^{n-2}] \left(\frac{e^u + e^{-u}}{2} \right)^n \\
 &= \frac{(n-2)!}{2^n} [u^n] \sum_{j=0}^n \binom{n}{j} (e^u)^j (e^{-u})^{n-j} \\
 &= \frac{(n-2)!}{2^n} [u^{n-2}] \sum_{j=0}^n \binom{n}{j} e^{(2j-n)u} \\
 &= \frac{(n-2)!}{2^n} \sum_{j=0}^n \binom{n}{j} [u^{n-2}] \sum_{i=0}^{\infty} \frac{((2j-n)u)^i}{i!} \\
 &= \frac{(n-2)!}{2^n} \sum_{j=0}^n \binom{n}{j} \frac{(2j-n)^{n-2}}{(n-2)!} \\
 &= \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (2j-n)^{n-2}
 \end{aligned}$$

If n is odd, then this summation is zero, as expected.

ENDOFUNCTIONS

An **endofunction** is any function $\phi : X \rightarrow X$. If $|X| = n$, then there are n^n endofunctions $\phi : X \rightarrow X$. Call this class \mathcal{F} . We can define the **functional directed graph** of $\phi : X \rightarrow X$ with vertices X and directed edges $v \rightarrow \phi(v)$ for $v \in X$. When we say ϕ is connected, we mean the underlying undirected graph is connected. Call this class \mathcal{C} . Certainly $\mathcal{F} \equiv \xi[\mathcal{C}]$.

What is the expected number of components among all n^n endofunctions on $\{1, 2, \dots, n\}$? Certainly $F(x) = \exp(C(x))$ for the EGFs $F(x)$ and $C(x)$ for \mathcal{F} and \mathcal{C} respectively. Let $c(\phi)$ be the number of connected components of $\phi \in \mathcal{F}_X$. Then

$$F(x, y) = \sum_{n=0}^{\infty} \left(\sum_{\phi \in \mathcal{F}_n} y^{c(\phi)} \right) \frac{x^n}{n!}$$

Recall $F(x) = \sum_{k=0}^{\infty} \frac{C(x)^k}{k!}$, where $C(x)^k$ is the generating function for a graph with k connected components. Thus

$$F(x, y) = \sum_{k=0}^{\infty} \frac{(C(x)y)^k}{k!} = \exp(yC(x))$$

Let's determine the structure of a connected endofunction $\phi \in \mathfrak{C}_X$. By following arrows, the graph must contain a directed cycle; in fact, this directed cycle must be unique. The same argument allows use to decompose the graph into a set of components, one for each vertex in the directed cycle. But then each component is in fact a rooted tree. We can thus identify $\mathfrak{C} \equiv \mathcal{C}[\mathcal{R}]$ where $\mathcal{R} = \mathcal{T}^\bullet$ is the class of rooted trees and \mathcal{C} is the class of cyclic permutations. Passing to EGFs, we have

$$\begin{aligned} F(x, y) &= \exp(yC(x)) \\ C(x) &= \log\left(\frac{1}{1-R(x)}\right) \\ R(x) &= x \exp(R(x)) \end{aligned}$$

Thus,

$$F(x, y) = \exp\left(\log\left(\left(\frac{1}{1-R(x)}\right)^y\right)\right) = \left(\frac{1}{1-R(x)}\right)^y$$

Apply LIFT with $R(x) = x \exp(R(x))$, $G(u) = \exp(u)$, $F(u) = \frac{1}{(1-u)^y}$. Then $F'(u) = uF(u) = \frac{y}{(1-u)^{y+1}}$. Thus the total number of components among all n^n $\phi \in \mathcal{F}_n$ is

$$\begin{aligned} n![x^n]yF(x, y)|_{y=1} &= n!y \frac{1}{n} [u^{n-1}] \frac{y}{(1-u)^{y+1}} \exp(u)^n \Big|_{y=1} \\ &= (n-1)! [u^{n-1}] \exp(nu) \left[\frac{(1-u)^{y+1} - y(y+1)(1-u)^y}{(1-u)^{2y+2}} \right]_{y=1} \\ &= (n-1)! [u^{n-1}] \exp(nu) \left[\frac{(1-u)^2 - 2(1-u)}{(1-u)^4} \right] \end{aligned}$$

For each $j \geq 1$, let $M_j \subseteq \mathbb{N}$ be a set of **allowed multiplicities** (for parts of size j)

$$\lambda \mapsto \underline{m}(\lambda) = \langle m_1, m_2, m_3, \dots \rangle$$

We require that only finitely many $j \geq 0$ have $0 \notin m_j$. Consider $\mathcal{Z} \subseteq \mathcal{Y}$ given by

$$\mathcal{Z} = \{\lambda \in \mathcal{Y} : m_j(\lambda) \in M_j \text{ for all } j \geq 1\}$$

0.5 Theorem.

$$\Phi_{\mathcal{Z}}(x, y) = \sum_{\lambda \in \mathcal{Z}} x^{n(\lambda)} y^{k(\lambda)} = \prod_{j=1}^{\infty} \left(\sum_{m \in M_j} x^j y^m \right)$$

PROOF Let $\hat{\mathcal{Z}} = \{\underline{m}(\lambda) : \lambda \in \mathcal{Z}\}$. For $\ell \geq 1$, let $\mathcal{M}(\ell) = \{\rho \in \mathcal{M} : r_j = 0 \text{ if } j > \ell\}$. Then $\bigcup_{\ell=1}^{\infty} \mathcal{M}(\ell) = \mathcal{M}$. Consider partitions in $\hat{\mathcal{Z}} \cap \mathcal{M}(\ell)$ when ℓ is bigger than the greatest index i such that $0 \notin M_i$. Then $\lambda \in \hat{\mathcal{Z}} \cap \mathcal{M}(\ell)$ if and only if $\underline{m}(\lambda) = \langle m_1, m_2, \dots \rangle$ with $m_j \in M_j$ and

$m_j = 0$ if $j > \ell$. Thus $\hat{\mathcal{Z}} \cap \mathcal{M}(\ell) \simeq M_1 \times \cdots \times M_\ell$. Thus

$$\begin{aligned} \sum_{\substack{\lambda \in \mathcal{Z} \\ \underline{m}(\lambda) \in \mathcal{M}(\ell)}} x^{n(\lambda)} y^{k(\lambda)} &= \sum_{\rho \in M_1 \times \cdots \times M_\ell} x^{r_1 + 2r_2 + \cdots + \ell r_\ell} y^{r_1 + \cdots + r_\ell} \\ &= \sum_{r \in M_1} x^r y^r \sum_{r \in M_2} x^{2r} y^r \cdots \sum_{j \in M_\ell} x^{\ell r} y^r \\ &= \prod_{j=1}^{\ell} \left(\sum_{r \in M_j} x^{jr} y^r \right) \end{aligned}$$

Since $\mathcal{M}(1) \subseteq \mathcal{M}(2) \subseteq \cdots \subseteq \mathcal{M}$ and $\bigcup_{\ell=1}^{\infty} \mathcal{M}(\ell) = \mathcal{M}$, by taking limits,

$$\begin{aligned} \Phi_{\mathcal{Z}}(x, y) &= \lim_{\ell \rightarrow \infty} \Phi_{\hat{\mathcal{Z}} \cap \mathcal{M}(\ell)}(x, y) \\ &= \lim_{\ell \rightarrow \infty} \prod_{j=1}^{\ell} \left(\sum_{r \in M_j} x^{jr} y^r \right) \\ &= \prod_{j=1}^{\infty} \left(\sum_{r \in M_j} x^{jr} y^r \right) \end{aligned} \quad \blacksquare$$

Example. Partitions with distinct parts \mathcal{D} . Then $M_j = \{0, 1\}$ for all $j \geq 1$. Then $\phi_{\mathcal{D}}(x, y) = \prod_{j=1}^{\infty} (1 + x^j y)$

Example. Partitions with only odd parts \mathcal{O} . Then $M_j = \{0\}$ if $j = 2i$ is even, and $M_j = \mathbb{N}$ if $j = 2i + 1$ is odd. Then

$$\Phi_{\mathcal{O}}(x, y) = \prod_{i=0}^{\infty} \left(\sum_{r \in \mathbb{N}} x^{(2i+1)r} y^r \right) = \prod_{i=0}^{\infty} \frac{1}{1 - x^{2i+1} y}$$

Set $y = 1$ in $\Phi_{\mathcal{D}}(x, y)$ and $\Phi_{\mathcal{O}}(x, y)$. Then

$$\begin{aligned} \Phi_{\mathcal{D}}(x, 1) &= \prod_{j=1}^{\infty} (1 + x^j) = \prod_{j=1}^{\infty} \frac{(1 + x^j)(1 - x^j)}{(1 - x^j)} \\ &= \prod_{j=1}^{\infty} \frac{(1 - x^{2j})}{(1 - x^j)} \\ &= \prod_{i=0}^{\infty} \frac{1}{1 - x^{2i+1}} = \Phi_{\mathcal{O}}(x, 1) \end{aligned}$$

We thus have a bijection $\mathcal{D} \simeq \mathcal{O}$ where if $\lambda \leftrightarrow \mu$, then $n(\lambda) = n(\mu)$ (but in general, lengths are not preserved).

Example. Let \mathcal{A} denote the set of partitions in which each part occurs 0, 1, 4, or 5 times. Then

$$\Phi_{\mathcal{A}}(x, 1) = \prod_{j=1}^{\infty} (1 + x^j + x^{4j} + x^{5j})$$

Let \mathcal{B} denote the set of partitions with no parts congruent to 2 mod 4, and parts divisible by 4 are distinct. Then

$$M_j = \begin{cases} \mathbb{N} & : j \text{ odd} \\ \{0\} & : j \equiv 2 \pmod{4} \\ \{0, 1\} & : 4 \mid j \end{cases}$$

So

$$\Phi_{\mathcal{B}}(x) = \prod_{i=1}^{\infty} \left(\frac{1}{1 - x^{2i-1}} \right) (1 + x^{4i})$$

Now,

$$\begin{aligned} \Phi_{\mathcal{A}}(x) &= \prod_{j=1}^{\infty} (1 + x^j + x^{4j} + x^{5j}) \\ &= \prod_{j=1}^{\infty} (1 + x^j)(1 + x^{4j}) \\ &= \prod_{j=1}^{\infty} (1 + x^{4j}) \frac{(1 + x^j)(1 - x^j)}{(1 - x^j)} \\ &= \prod_{j=1}^{\infty} (1 + x^{4j}) \left(\frac{1 - x^{2j}}{1 - x^j} \right) \\ &= \prod_{j=1}^{\infty} \frac{(1 + x^{4j})}{1 - x^{2j-1}} = \Phi_{\mathcal{B}}(x) \end{aligned}$$

Example. Let $m, n \in \mathbb{Z}$. Count the number of points $(a, b, c) \in \mathbb{Z}^3$ such that $a^2 + 2b^2 + c^2 = m$ and $a^2 + a - 2b + c^2 = n$.

For any $r \geq 0$, the set $\{(a, b, c) \in \mathbb{R}^3 : a^2 + 2b^2 + c^2 = r\}$ is compact and contains only finitely many points with integer coordinates. Call this number $f(m, n)$. We have

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} f(m, n) s^m t^n &= \sum_{(a,b,c) \in \mathbb{Z}^3} s^{a^2+2b^2+c^2} t^{a^2+a-2b+c^2} \\ &= \left(\sum_{a \in \mathbb{Z}} s^{a^2} t^{a^2+a} \right) \left(\sum_{b \in \mathbb{Z}} s^{2b^2} t^{-2b} \right) \left(\sum_{c \in \mathbb{Z}} s^{c^2} t^{c^2} \right) \\ &= \vartheta(st, t) \vartheta(s^2, t^{-2}) \vartheta(st, 1) \\ &= \prod_{i=1}^{\infty} (\dots) \end{aligned}$$

where $\vartheta(x, y) = \sum_{h=-\infty}^{\infty} x^{h^2} y^h$ is the Jacobi theta function, and we can get the product expression using JTPF.

Final Exam, about 1/3 of the exam on each of these topics:

- Exponential generating functions
- Integer partitions
- Everything else (LIFT, formal power series)

For example, let $\mathcal{H}_{a,b}$ denote the set of integer partitions λ such that $\lambda_1 \leq a$ and $k(\lambda) \leq b$. Then we have a bijection $\mathcal{H}_{a,b} \xleftrightarrow{\sim} \mathcal{L}(a,b)$ taking the bounding path for λ . Note that $n(\lambda) = \text{area}(P)$, so $\sum_{\lambda \in \mathcal{H}_{a,b}} q^{n(\lambda)} = a + bb$.