Topics in Graph Theory

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I. Graph Colourings

1 List Colourings

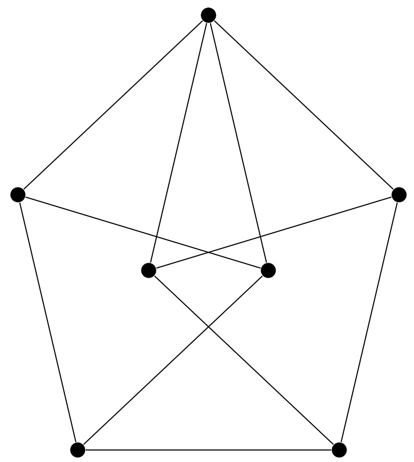
Recall that a colouring of a graph G is an assignment to each $v \in V(G)$ an element $c(v)$ of some set C called "colors" such that if v and v' are neighbours, then $c(v) \neq c(v')$. Then the **chromatic number** $\chi(G)$ is the smallest cardinality $|C|$ such that there exists a colouring of G from C.

There are some basic upper bounds on the chromatic number of a graph:

1. $\chi(G) \leq |V(G)|$, by colouring every vertex distinctly
2. $\chi(G) \leq \Delta(G) + 1$, by randomly colouring the graph based on colours not used on the neighbours

Note that these upper bounds are in fact tight; for example, the complete graph is tight for both, and an odd cycle is tight for (2).

There are some graphs for which the chromatic number is not known: consider the graph given by $V(G) = \mathbb{R}^2$ where vertices are adjacent if they have euclidean distance 1. This graph is not 3–colorable, by taking for example the subgraph



Recently there was a construction showing that the graph is not 4–colourable, and there is an easy upper bound of 7, so that $5 \leq \chi(G) \leq 7$.

We also define the notion of a list colouring:

Definition. A list assignment is an assignment of a set $L(v)$ of colors to each vertex v. Then a graph is k–list-colorable if you can always colour $V(G)$ whenever every vertex has a list of size at least k.

Note that $\chi(G) \leq \chi_\ell(G)$ since asssigning an identical list of size k is a valid list assignmnet and yields a standard coloring. In many cases list colorings can be hard to determine, but in some cases the exact value is known. Consider the complete bipartite graph $K_{k,q}$ where $q \geq k$. We then have the following classification:

1.1 Proposition. *$\chi_\ell(K_{k,q}) \le k$ if and only if $q < k^k$, and $\chi_\ell(K_{k,q}) = k+1$ if and only if $q \ge k^k$.*

Proof Note that $\chi_\ell(K_{k,q}) \le k+1$ always works by taking arbitrary colors on the k−side, and on the q−side, since the lists have size k, there is always a distinct color.

Now $q < k^k$. Try to color the k vertices such that two vertices have the same color. If this works, then for every list of size k on the q−side, there are only $k-1$ disallowed colours, so we may choose a valid color from the corresponding list. Otherwise, every vertex on the k−side has a distinct color; this is forced precisely when all the lists are disjoint. But then since $q < k^k$, there must be some selection of colors from the lists on the k−side such that the set of colors is distinct from every list on the q−side, and we may choose colors from the q−side without issue.

Otherwise if $q \ge k^k$, consider lists given by disjoint sets on the k−side, and then for every possible assignment of colors on the k−side, give a corresponding list for some vertex of the q−side that contains a list with those colors. Since $q \ge k^k$, we will exhaust all possibilities, so there is no valid coloring from those lists. ■

Recall that a planar graph G is one for which there exists an embedding of G into the plane such that each edge is a disjoint curve. Note that it suffices to consider edges which are polygonal curves, which consist of a finite number of straight line segments; in fact we can also do it with straight line segments (requiring that the graph is simple).

1.2 Theorem. (Thomassen) *If G is planar, then $\chi_\ell(G) \le 5$.*

In fact, we prove a stronger statement. We call an "almost-triangulation" a planar drawing in which every face except possibly the infinite face is a triangle. We prove this: let w be a given almost-triangulation with lists of available colour $L(v)$ assigned to every vertex v such that

1. $|L(v)| = 5$ for all vertices that are not on the infinite face,
2. two neighbouring vertices of the infinite face, a and b are colored distinctly,
3. and all other vertices of the infinite face have lists of 3 colours.

Then this almost-triangulation has a proper list colouring with respect to the given lists.

This implies the theorem since any planar drawing can be made an almost-triangulation by adding edges, and 5-element lists can be reduced to lists of the size above.

Proof We consider two cases in an induction proof.

1. There is a "long diagonal" connecting two of the vertices of the infinite face (that is not an edge of the infinite face).
2. There is no long diagonal.

The induction is on the number of vertices. When $n = 1, 2$ it is trivial, and when $n = 3$ it is a 3-cycle and it is certainly fine.

Now for the induction step, we have the two cases.

1. Cut the graph along the long diagonal to get G_1, G_2. Without loss of generality, G_1 is exactly as described in the statement, so it can be properly list coloured from the given lists. Then give the endpoints of the copied long diagonal in G_2 so that the endpoint colours are fixed; and by induction, colour it as well. Since the endpoints have the same colouring, we can put the two coloured graphs back together to obtain a proper list colouring of G.

2. Let $u \in V(G)$ be the neighbour of a on the infinite face different from b. Consider the neighbourhood of u, $N(u) = \{a, w, v_1, v_2, \ldots, v_k\}$ where w is on the infinite face different from a. We have $|L(w)| = 3$ and $|L(v_i)| = 5$ for all $i = 1, \ldots, k$ since there is no long diagonal. Choose two different colours γ and Δ in $L(u) \setminus \{\alpha\}$; they certainly exist since $|L(u)| = 3$. Delete γ and δ from all the lists of vertices in $\{v_1, \ldots, v_k\}$, and then by induction we can list colour $G \setminus \{u\}$ from the modified lists. This can be extended to a list colouring of G since u shares no colour in its list with any $\{v_1, \ldots, v_k\}$, and at least one of δ or γ will not be used in w. ■

n–connected means if you remove any n vertices, the graph remains connected

Take K_4, and have lists with colours $1, 2, 3, 4$ (or any graph which is uniquely 4–colorable). Inscribe a triangle in each face with lists $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$. Always align so that the degree 3 vertex is adjacent to the $1, 2$ and $1, 3$.

1.3 Theorem. (Grötsch) *If G is planar with girth at least 4, then $\chi(G) \leq 3$ and $\chi_\ell \leq 4$.*

If G is planar with n vertices and e edges, then $e \leq 3n - 6$ so that $\delta \leq 5$. If G is planar with n vertices and e edges with girth 4, then $e \leq 2n - 4$ so $\delta \leq 3$. This gives an easy proof of the list colouring value.

1.4 Theorem. *Let G be planar with girth at least 5. Then $\chi_\ell(G) \leq 3$.*

Proof Suppose G is a planar graph with girth at least 5 such that

1. There are at most 6 pre-colored vertices on the outer face which form a path or a cycle (edges need not be on the outer face),
2. there are some vertices with $|L(u)| = 2$ on the outer face boundary, and
3. There are no edges joining vertices with $|L(v)| < 3$ except for those in (1)

We will prove by induction on $|V(G)|$. Assume that G is a minimal counterexample. Then

1. $|V(G)| \leq 3$
2. G is connected
3. Outer face bounded by a cycle
4. No cut vertex in the graph (G is 2-connected); outer cycle has C
5. C has no chord
6. No separating cycle with at most 6 vertices
7. Pre-coloured path/cycle is a non-empty path (can just remove an edge)
8. No path of length 2 inside C except (see paper)
9. No path of length 3 inside C except starting at a list-2-vertex
10. The precolored path P and the outer cycle C has $|V(C)| \leq |V(P)| + 2$.

We will allow some precolored vertices which form a path or cycle with at most 6 vertices (edges can be chords), and some vertices with $|L(u)| = 2$, all on the outer face boundary. Except for edges in this path/cycle, there are no other edges joining vertices with $|L(u)| < 3$. All other vertices have at least 3 available colors. ■

1.5 Theorem. (Grötsch) *If G is planar with girth at least 4, then $\chi(G) \leq 3$.*

Proof If there is no 4-cycle, we are done by the previous theorem. If G contains no 4-cycle, we may simply add a 4-cycle artifically by adding edges.

Note that we may even precolor a 4-cycle or 5-cycle. Then that coloring can be extended to G. Suppose G is a minimal counterexample. First note that there is no separating 4 or 5 cycle: otherwise, one can colour the interior and exterior of the cycle. Thus assume the precolored cycle is on the boundary. If there is another separating 4 or 5 cycle inside. Then colour the outer face by induction, then the inner face.

Let C be a 4-cycle in G, and C is facial. If C is pre-colored, we have a problem: we can assume $C \neq C_0$, for if not, delete an edge in C_0 and refer to the original case. In this case, we may ... ■

1.6 Proposition. *The following are equivalent:*

(i) $\chi(G) \leq 3$

(ii) There exists an orientation of G such that all cycles are balanced modulo 3

(iii) There exists an orientation of G such that all closed walks are balanced modulo 3

Proof $(iii \Rightarrow ii)$ is immediate.

To see $(ii \Rightarrow iii)$, we can simply take the orientation from (ii). If a closed walk is not a cycle, it has a repeated vertex, and we can verify that the walk is balanced on each component.

For $(i \Rightarrow ii)$, we must simply orient the edges such that $0- > 1$, $1- > 2$, $2- > 3$

For $(iii \Rightarrow i)$, colour some vertex 0. Then for any other vertex, take a path connecting the vertices and walk along the path by adding one for every forward traversal, and subtract one for each backwards traversal, modulo 3. If there are multiple paths, then the multiple paths would form a walk which is balanced modulo 3, so the lengths must be the same. ■

Definition. A **cut** in a graph. Partition the vertex set into two pieces. Then a cut is the set of edges between the two vertex sets. A **minimal cut** is a cut containing no other cuts.

Note that a cut is minimal if and only if each side of the cut is connected. If G is planar, then the dual graph is formed as follows: each face becomes a vertex, and the vertices are joined by an edge if the corresponding faces are adjacent. The number of edges is unchanged, and the number of vertices and faces swaps.

Given an orientation on the original graph, we can pass the orientation to the dual graph by setting the orientation anticlockwise relative to the intersection. Let $E \subseteq E(G)$. Then E is a minimal cut in G if and only if E^* is a cycle, and E is a cycle in G if and only if E^* is a minimal cut in G^*.

Assume G is planar. If G is 4-edge-connected, then each cut has at least 4 edges, G^* has girth at least 4, then $\chi(G^*) \leq 3$, then the following equivalent things hold:

(i) G^* has an orientation so that all cycles are balanced modulo 3

(ii) Ghas an orientation such that all cuts are balanced modulo 3

(iii) G has an orientation such that $d^+(v) \equiv d^-(v)$ modulo 3

1.7 Conjecture. (Tutte) *If G is 4-edge-connected, then there exists an orientation on G such that all degrees are balanced modulo 3.*

Currently proven for 6-edge-connected. If G is 4-edge connected, then there exists an orientation on G and a flow 1 or 2 on each edge such that at each vertex the inflow equals the outflow. This is equivalent to the conjecture by reversing the orientation for all edges which have flow 2, or by simply placing flow 1 on every edge in the graph.

In fact, one can remove the modular condition. Assume each edge has a flow 1 or 2 or 3 or 4, and assume that each inflow is equivalen to the out flow modulo 5.

1.8 Proposition. *If G is planar and 4-edge-onnected, then there exists an orientation such that G is balanced modulo 3.*

1.9 Proposition. *If G is cubic and 3-edge-connected, there exists an orientation which is balanced modulo 3 if and only if G is bipartite.*

Does there exists an orientation on G such that G is balanced modulo k? Or such that each vertex v has out degree $p(v)$ modulo k?

If the second holds for every p and k is odd, then the first holds. Let v be a vertex with degree $d(v)$; we want that $d^+(v) \cong d^-(v)$, in other words that $2d^+(v) \cong d(v) \pmod{k}$, s

$$\frac{k-1}{2} \cdot 2d^+(v) \cong \frac{k-1}{2} d(v) \Rightarrow d^+(v) \cong \frac{-(k-1)}{2} d(v)$$

Suppose $k = 2$. Here's a necessary condition: then $|E(G)| = \sum_{v \in V(G)} d^+(v) \cong \sum_{v \in V(G)} p(v)$, modulo 2. In fact, if G is connected and $\sum_{v \in V(G)} p(v) \cong |E(G)$, then such an orientation exists. Do do this, fix any orientation. If there is a vertex which does not satisfy the requirements, by parity, there must be some other vertex which does not satisfy the requirements. Take a path connecting the vertices and flip all the edges, repeating until the graph is balanced.

1.10 Conjecture. (Jaeger) *If G is 1000-edge-connected, then there exists an orientation on G balanced modulo 3.*

This has been proven in the affirmative for 8-edge-connected, then 6-edge-connected. It is enough to prove this for 5.

1.11 Conjecture. (Jaeger) *If G is* $(2k-2)$*–edge-connected, then there exists an orientation on G that is balanced modulo k if k is odd.*

It has been shown that if there is a $(2k^2 + 2)$–edge-connected graph, then there exists an orientation on G with any out degrees modulo k, also true for k is even. If G is $(3k-3)$–edge connected, then the same holds, but only for k odd.

Suppose G is 4-edge-connected: then there exists an orientation of G balanced modulo 4. This is equivalent to the 3-flow conjecture. Given an orientation balanced modulo 3, by a previous exercise, we can also balance each vertex modulo k for any k.

If $k = 5$, the statement says that G is 8-edge-connected implies G is balanced modulo 5.

Let G be 2-edge-connected, then there exists an orientation on G with flow values on $\{1, 2, 3, 4\}$ such that the inflow and the outflow are equal for all $v \in V(G)$. It suffices to verify this for cubic 3-connected graphs. Note that for cubic graphs, the edge and vertex connectivity are the same. k connected means there are k internally vertex disjoint paths, and k–edge-connected means there are k internally edge disjoint paths.

Example. Assume the 5-flow-conjecture holds for G cubic 3-connected. Then prove that it holds for G 2-edge-connected. There's a couple cases: if there is a vertex of degree 2 with edges going to the same vertex, simply add the same flow value going in and out. If there is a vertex of degree 2 with edges going to distinct edges, simply merge the edges, apply induction, and then apply the flow assigned to that edge to both pieces.

If there is a vertex with degree large, remove two of the edges so as not to create a bridge, and apply the same argument. What happens if we have all vertex of degree 3? We need to deal with the case where G is 2-edge connected. Isolate the pair of edges e_1

and e_2. First close the loops, and then multiply the flows or perhaps re-orient so that the edges agree.

Now assume G is cubic and 3-edge-connected. Then take the graph and replace every edge by 3 edges to get some G' that is 9-edge-connected. Therefore, by the result above (Jaeger with $k = 5$), it has an orientation that is balanced modulo 5. Then replace each triple of edges with the oriented net sum of the number of edges.

Example. K_8 is 7-edge-connected nad has no orientation balanced modulo 5.

Let's consider factors modulo k. A d-factor is a spanning subgraph of G such that every vertex of the subgraph has degree d.

> ***1.12 Theorem.*** *Let G be bipartite with bipartition $V(G) = A \cup B$ with $V(G) = \{v_1, \dots, v_n\}$. For every v_i, let d_i be a natural number. We want a spanning subgraph of $H \subseteq G$ such that $d_H(v_i) \cong d_i \pmod{k}$ where k is odd. Then H exists if G is $(3k-3)$–edge connected and $\sum_{v_i \in A} d_i \cong \sum_{v_i \in B} d_i \pmod{k}$.*

Proof Apply the $(3k-3)$ result, and assign the function $p(v_i) = d_i$ for $v_i \in A$ and $p(v_j) = d(v_j) - d_j$ for $v_j \in B$. Certainly $\sum_{v_i \in V(G)} p_i \cong |E(G)|$ modulo k by the modular summation condition on the d_i. Then we simply take all $A- > B$ edges. ■

Recall that if G is 9-edge-connected, then there exists an orientation on G balanced modulo 5. We've shown that if G is 9-edge-connected, then Tutte's 5-flow theorem follows. Jaeger conjectured that this in fact holds for 8-edge-connected graphs.

Example. Which K_n have an orientation balanced modulo 5? If n is odd, this always works, since then all the vertex degrees have even degree, and we can simply use an eulerian tour. Now K_8 is 7-edge connected, and does not have an orientation balanced modulo 5.

We can write $7 = 7 + 0 = 6 + 1 = 5 + 2 = 4 + 3$; and if K_8 is balanced modulo 5, then all $d^+(v), d^-(v) \in \{1, 6\}$. If such an orientation exists, we must have 4 with out degree 6, and 4 with out degree 1 by counting flows. But then on the $d^+ = 1$ side, the sum of the out degrees is 4, but it must be at least 6 (by counting internal vertices).

We can do K_{10}: partition into copies of K_5, make each balanced modulo 5, and then add all edges from one side to the other with the same orientation. We can also generalize this, by adding two vertices.

Let G be bipartite with $N(G) = A \cup B$. Set $V(G) = \{v_1, \dots, v_n\}$, with $d_1, d_2, \dots, d_n \in \mathbb{N}$. Then we want to find $G \supseteq H$ such that $d_H(v_i) \equiv d_i \pmod{k}$ where $\sum_{v_i \in A} d_i = \sum_{v_i \in B} d_i$. This is always doable if G is $(3k-3)$–edge connected (for k odd), else G is $(2k^2+k)$–edge-connected that k is even.

Let G be a graph and partition G into sides A and B such that the number edges between them is maximal. Let H be the graph induced by the maximum cut edges, so that H is bipartite. Then the following properties hold:

(i) $d_H(v) \geq \frac{1}{2} d_G(v)$.
(ii) $|E(H)| \geq \frac{1}{2}|E(G)|$
(iii) If G is $(2k-1)$–edge connected, then H is k–edge-connected.

How to see this? If $v \in B \subseteq V(G)$ is a given vertex, then the number of edges in the cut must be at least as large as the number of internal edges from v on side A (or we could swap v to the other side and get a better cut). This shows (i) and (ii). To show (iii), suppose H is not k–edge-connected ... (see paper).

1.13 Theorem. *Let $k \in \mathbb{N}$, and G a $(6k-7)$–edge connected connected graph with k odd. Let $V(G) = \{v_1, \ldots, v_n\}$ and $d_1, \ldots, d_n \in \mathbb{N}$ given. We wish to find $H \subseteq G$ such that $d_H(v_i) \cong d_i$ (mod k). This can be done if for every partition $V(G) = A \cup B$, $\sum_{v_i} \in Ad_i \cong \sum_{v_i \in B} d_i$ (mod k).*

Proof By the previous arguments, there exists some $H' \subseteq G$ with H' bipartite and $(3k-3)$–edge connectivity, and apply the previous result with $H' \supseteq H$ satisfying the result. ■

1.14 Conjecture. *If G is simple and 4–regular, then G contains a 3-regular subgraph.*

If $|E(G)| > 2|V(G)|$, then $G \supseteq H$ all vertices degree equivalent to 0 modulo 3 where H has nonempty edge set.

Suppose G is 4–regular, perhaps with multiple edges. Then G with an extra edge contains a 3-regular subgraph.

Now consider the previous theorem where all $d_i = k$, and we work modulo $2k$. If G is $[2(2(2k)^2 + 2k) - 1]$–edge-connected with $|V(G)|$ even, then $G \supseteq H$ has all degrees congruent to k modulo $2k$.

2 Group Valued Flows

Find an orientation on G such that G is almost balanced. There exists some small $\epsilon > 0$ such that $E(A,B) \le (1+\epsilon)E(B,A)$ ($E(A,B)$ is number of oriented edges from A to B) and $E(B,A) \le (1+\epsilon)E(A,B)$. Given an abelian group Γ and $F \subseteq \Gamma$, and G is a graph. We want an F–flow in G, in other words that each edge e gets some $g \in G$ such that the sum of the in flow is equal to the sum of the out flow.

If G is 6-edge-connected, then G has a $\{1,2\}$–flow. If G is 2-edge connected, then G has a $\{1,2,3,4,5\}$–flow.

Let $f(F,\Gamma)$ denote the smallest k such that every k–edge connected has an F–flow. For example, if $F = \{1,3\} \subseteq \mathbb{Z}$, this is not always possible. But we do have

2.1 Theorem. *$f(F,\Gamma)$ exists if and only if the odd sum condition holds.*

The odd sum condition is the statement that it is possible to have a sum of an even number of elements and a sum of an odd number of elements have equal value.

Recall:

- 5-flow conjecture: if G if 2-edge-connected, then there exists a flow with values $1,2,3,4$
- 3-flow-conjecture: if G is 4-edge-connected, then there exists a flow with values $1,2$
- $(2+\epsilon)$-flow-conjecture: if G is $\alpha(\epsilon)$-edge-connected, then there exists a flow with values $[1, 1+\epsilon]$

To have a flow with only 1, graph must be Eulerian.

If G is 3-edge-connected, the $(2+\epsilon)$ flow need not exist (for example, with a 3-strut: two components with 3 edges joining them).

Let $c(x,y)$ denote the number of edge-disjoint paths from x to y (the **local edge connectivity**).

2.2 Theorem. (Mader Lifting) *Let G be a graph and v a vertex with neighbours. Fix a neighbour of v such that $d(v) \geq 4$, w, w' and remove the edges $\{v, w\}$ and $\{v, w'\}$ and add the edge $\{w, w'\}$. This is called a lift.*

The lifting can be chosen such that all $c(x, y)$ remains the same for all $x \neq y$, $x \neq v$, $y \neq v$.

Proof TODO ■

Let Γ be an abelian group and $F \subseteq \Gamma$. Then a $F-$flow is an assignment of $g \in F$ to each $v \in V(G)$ such that the sum of the ingoing edges is equal to the sum of the outgoing edges.

2.3 Theorem. *Suppose F satisfies the odd-sum condition; in other words, there exists $a_i, b_i \in F$ so that $a_1 + \cdots + a_{2p} = b_1 + \cdots + b_{2q+1}$. Then there exists a function $f(F, \Gamma) \leq 3k - 1$ where $k = 2p + 2q + 1$ such that every $f(F, \Gamma)-$edge-connected graph has an $F-$flow.*

Proof By induction on $|E(G)|$. First suppose $|V(G)| = 2$. If there are an even number of vertices, choose half the orientations in either direction and take the same element of F on all edges. If $|E(G)|$ is odd, then there are at least $3k - 1$ edges, so take k and use the odd-sum identity and use the even trick as before.

Select a vertex v. Suppose v has a neighbour w such that there are multiple edges. Then repeatedly lift pairs of edges until there is only one pair of vertices left, which will have an even number of edges between them. Then color $G \setminus \{v\}$ inductively, and add back the pairs (if $\deg(v)$ is even). For the induction, use Mader's trick (1) if there is a vertex with even degree.

If there is a vertex with odd degree, apply Mader's trick to two neighbours. But then the induction only fails if there is a vertex with degree precisely $3k$. Fix an orientation of G such that all vertices have out-degree 0, modulo k. Then there are three cases: the out degree is 0, k, $2k$, or $3k$.

We can reduce this to the case where every vertex has in-degree k or out-degree k. But then G is bipartite (sorting by in or out degree).

Let G be bipartite and $k-$regular: then $G = M_1 \cup M_2 \cup \cdots \cup M_k$ where each M_i is a perfect matching (by Hall's theorem repeatedly). But then write $\{1, \ldots, k\} = \{a_1, \ldots, a_{2q}, -b_1, \ldots, -b_{2q+1}\}$ and apply that flow to each edge in M_k, and we are done. ■

2.4 Theorem. (Seymour) *If G is $2-$edge-connected, then G has a $(\Gamma \setminus \{0\})$-flow for $|\Gamma| \geq 6$.*

If $|\Gamma| \geq 3$, then take $a + a = 2a$ or if $a + a = 0$ or all a, then $a + b = c$. If G is $8-$edge-connected, then G has a $\Gamma-$flow using only $a, 2a$.

To see the $(2 + \epsilon)$ conjecture, take $\Gamma = \mathbb{R}$ and $F = \{1, 1 + 1/k\}$. Then

$$\underbrace{1 + 1 + \cdots + 1}_{k+1} = \underbrace{(1 + 1/k) + \cdots + (1 + 1/k)}_{k}$$

If G is $6k-$edge-connected, then G has a $(1, 1 + 1/k)-$flow, and the conjecture follows with $f(\epsilon) = 6/\epsilon$.

Almost balanced orientation

Take a $1, 1 + 1/k$ flow, and ignore the flows values (keep only the orientation). Take an arbitrary cut A, B, and let $E(A, B)$ denote the number of edges from side A to side B. Then

$$|E(A, B)| \leq |A \to B \text{ flow}| = |B \to A \text{ flow}| \leq \left(1 + \frac{1}{k}\right)|E(B, A)|$$

and likewise in reverse; thus, it gives an almost-balanced orientation.

Identify $\mathbb{R}^2 = \mathbb{C}$, and consider the group $R_3 = \{z : z^3 = 1\}$.

2.5 Theorem. *G has a $\{1,2\}$–flow if and only if G has an R_3–flow.*

Proof Assume that G has a $\{1,2\}$–flow; we prove the claim by induction on the number of edges. Fix a vertex v. If v has an incoming flow of α and an outgoing flow of α, then we lift the two edges and use induction. The only other case is that all incoming edges have flow 2 and all outgoing edges have flow 1. Then there are twice as many outgoing edges as incoming edges, so we may separate the vertex into multiple vertices such that each incoming edge is 2 and the pair of outgoing edges is 1. Then the edges of flow 2 form a perfect matching, and the edges of flow 1 form a 2-factor (which is a disjoint collection of cycles), where the edges in the cycle alternate in direction. Then we give flow $1 \in R_3$ to the edges with flow 2, and alternate labels $e^{\pm\frac{2\pi i}{3}}$ on the cycle. Re-identifying vertices preserves the incoming and outgoing flow, so we are done.

Conversely, in the case when an incoming and outgoing flow are the same, we use the same argument as above. Arguing by sign, this forces such a vertex to either have all incoming or all outgoing edges. Then each of 1, $e^{2\pi i/3}$, and $e^{2\pi i/3}$ must occur the same number of times. But then we can separate the graph into a cubic bipartite graph, which therefore has a $1,2$–flow. ∎

2.6 Theorem. *If G is a cubic graph, then the following are equivlent:*

(i) G has a $\{1,2\}$–flow
(ii) G has a R_3–flow
(iii) G has an S^1–flow
(iv) G is bipartite

Assume G is cubic and has an S^1–flow. Re-orienting edges and changing sign (which preserves a S^1–element), we can guarantee that the flow only uses the 3 roots of unity.

However, the implication (iii) implies (ii) does not work in general. If G is planar, then G has an R_3–flow if and only if G^* has chromatic number at most 3 (this is Grotsch's theorem). Let's show that G has an S^1–flow if and only if a homomorphic image of $G^* \subseteq U$ where U is a unit distance graph.

Assume G has an S^1–flow, and let e be an edge from x to y with flow $g \in S^1$. Then on the dual graph, we rotate the orientation counterclockwise and keep the same flow. Fix some vertex x^* and assert x^* is at $(0,0)$. Now take v^*, and a flow from x^* to v^*, and place v^* at $g_1 + g_2 - g_3 + g_4$. This is well-defined since the cycles in the dual graph are balanced. The reverse construction certainly works as well.

Now consider some dual graph G^*. If $G^* \subseteq U$, then G has an S^1–flow. However if $\chi(G^*) \geq 4$, then G has no R_3–flow.

... TODO: draw this dual graph explicitly (from one of the first classes), give the S^1 flow. Also: connection to list colouring.

Recall that $R_k = \{z \in \mathbb{C} : z^k = 1\}$.

- If there exists an orintation of G balanced modulo 3, then G has an R_3–flow.
- G has an orientation balanced modulo 3 if and only if G has a $\{1,2\}$–flow in $\mathbb{Z}$ or $\mathbb{Z}_3$.
- Conjecture of Kamal Jain: $f(S^1, \mathbb{R}^2) = 4$, $f(S^2, \mathbb{R}^3) = 2$.
- Tutte 3-flow implies conjecture of Jain

If there exists a balanced orientation modulo k, then G has an R_k–flow.

Proof Induction. The base case is straightforward (two vetices with a number of edges). Thus take a vertex v with some incoming edge and outgoing edge. Then one can lift the edge, and use induction ... ∎

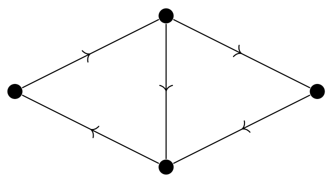
2.7 Proposition. *If G has an R_5–flow, then there exists an orientation balanced modulo 5.*

Proof Assume G has an R_5–flow. As before, assume we have a vertex v with an incoming and outgoing edge with the same flow; then we lift those edges and replace the flow and use induction. Thus we assume that all vertices v have all outgoing edges with flow distinct from all incoming flow... ∎

Let T denote the set of all vectors in $\mathbb{R}^3$ with one 0 and two ± 1.

2.8 Theorem. *G has a T–flow if and only if $G = H_1 \cup H_2 \cup H_3$ such that every edge is covered twice and every vertex of H_i has even degree. Furthermore, if G is cubic, then the following are equivalent:*

(i) *G has a T–flow*
(ii) *$G = M_1 \cup M_2 \cup M_3$ where each M_i is a perfect matching.*
(iii) *G is 3-edge-colourable*
(iv) *G is class 1 (Vizing theorem, +1 case)*



Proof

Given $H_1 \cup H_2 \cup H_3$, to each edge, assign (a_1, a_2, a_3) where

$$a_i = \begin{cases} 1 & : e \in H_i \text{ with the same orientation} \\ -1 & : e \in H_i \text{ with the opposite orientation} \\ 0 & : e \notin H_i \end{cases}$$

clearly this has exactly two ± 1 and that the edge sums work.

Conversely, let

$$\begin{aligned} H_1 &= \{e : \text{first coordinate is } \pm 1\} \\ H_1 &= \{e : \text{second coordinate is } \pm 1\} \\ H_1 &= \{e : \text{third coordinate is } \pm 1\} \end{aligned}$$

Now suppose G is cubic. If G is a union of perfect matchings, take $M_1 \cup M_2$, $M_1 \cup M_3$, $M_2 \cup M_3$. Conversely, take $M_i = E(G) \setminus E(H_i)$. ∎

Note that this also shows that if we have a graph covering, then we have an S^2–flow (scale T–flow by $\sqrt{2}$).

2.9 Theorem. *Suppose G is $(3k-1)$–edge-connected. Then G is covered by k even graphs such that every edge is covered precisely $k-1$ times.*

Proof Let $\mathbb{Z}_2^k = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, and let F be the set of all vectors with precisely one 0. Suppose G has an F–flow; then take H_i composed of the edges in $E(G)$ where the i–th coordinate is 1. ■

2.10 Theorem. *Every 2-edge-connected planar graph G has a T–flow.*

Proof Consider G, so that G^* is 4-colorable. Assign every color to the vertex of the tetrahedron, and place the vertices on these corners. Then given some edge $(x^*, y^*) \in E(G^*)$, give the edge the flow equal to the vector corresponding to the edge of the tetrahedron.■

Recall:

2.11 Conjecture. *If G is 4-edge-connected, it contains an R_3-flow.*

This conjecture is equivalent to Tutte's 3-flow conjecture.

2.12 Conjecture. *If G is 8-edge-connected (or maybe 9-edge-connected), it contains an R_5-flow.*

This conjecture implies Tutte's 5-flow conjecture. Is there some kind of converse?

To prove the 5-flow conjecture, it is sufficient to consider cubic graphs which are 3-connected.

2.13 Proposition. *If G has a $(\Gamma \setminus \{0\})$–flow (NZ Γ–flow) and if $|\Gamma'| \geq |\Gamma|$, then G has a NZ Γ'–flow.*

Let $f(G,\Gamma)$ denote the number of NZ Γ–flows (allowing G to have loops and multiple edges). Fix a non-loop edge e, and consider the graphs G/e (edge contraction) and $G - e$. We can then generate a bijection ... TODO: type But then $f(G/e,\Gamma) = f(G,\Gamma) + f(G-e,\Gamma)$. Futhermore, for the base case, let G be a graph with 1 vertex and q loops. But then $\Gamma(G,\Gamma) = (|\Gamma| - 1)^q$.

Now if G has a Γ–flow, then G has a NZ $\mathbb{Z}_n$–flow, so G has a $\{1, 2, \ldots, n-1\}$–flow, so G has a NZ $Z_{n'}$-flow for any $n' \geq n$.

What happens if Γ' is infinite? If it contains an infinite copy of $\mathbb{Z}$, we are certainly done; otherwise, G' finite groups of arbitrary large size, and we stop when we have a subgroup of size at least $|\Gamma|$.

2.14 Proposition. *If G is 4-edge-connected, then G contains two edge-disjoint spanning trees.*

If G is 3-edge-connected, this is not true (except for small base cases). For example, if G must have at least $2n - 2$ edges, while if G is cubic, then G has $3n/2$ edges.

Proof To do this, we use Mader's lifting theorem. ■

2.15 Corollary. *More generally, if G is $2k$–edge-connected, then G contains k edge-disjoint spanning trees.*

Proof We prove this by induction on $|E(G)|$. Delete e if possible such that $G - e$ is $2k$–edge-connected, and use induction. Thus assume no such edge exists. Using the proposition below, there exists some vertex v with degree $2k$. Then apply Mader's lifting theorem to remove the vertex, and get k loops which is $2k$–edge-connected. By induction, get k edge-disjoint spanning trees. If each edge from the lift has a distinct tree, then we are done. Otherwise, suppose some pair of edges in the lift has the same tree. ■

2.16 Proposition. *Consider the case when G is minimally k–edge-connected. Then $\delta(G) = k$.*

Proof For any edge e, there is a cut (A,B) containing e of size k. In particular, choose (A,B) such that one size of the cut B has minimal size. If $|B| = 1$, then the result follows. Otherwise, $|B| > 1$ so B must have some internal edge e' Now take another cut containing e' of minimal size. ... see paper ■

2.17 Proposition. *Let $A \subseteq V(T)$ such that $|A|$ is even. Then there exists some pairing of A and edge disjoint paths betwen those elements.*

Proof By induction. If there is some leaf v not in A, remove v and apply induction. Otherwise, every leaf is in A, so select an arbitrary one, and look at its neighbour in T. If its neighbour is not in A, remove v, and label its neighbour and apply induction. If its neighbour is in A, connect the two and apply induction to the rest of G. ■

2.18 Corollary. *Let G be 4 edge connected, so G contains T_1, T_2 edge-disjoint spanning trees. Then $G - E(T_1)$ is connected, and for each odd degree vertex, add a path in T_1 joining odd degree vertex. Then this graph is even and connected (hence Eulerian).*

2.19 Corollary. *If G is 4-edge-connected, then $G = H_1 \cup H_2$ where each H_i is even. Then G has a nowhere zero 4-flow.*

We say $H \subseteq G$ is an r–factor in G if $V(H) = V(G)$ and H is R–regular. Suppose $r = p \cdot q$ and G is r–regular. In this case, is G a union of a p q–factors?

2.20 Proposition. *Next time, we will show that G is 6–edge-connected and 9-regular implies that G is a union of 3 3-factors, and similarly for 4-edge-connected (if the 3-flow conjecture is true).*

2.21 Proposition. *Let G be a planar 2–connected graph. Then the following are equivalent:*
- *(i) If G is cubic, then $G = M_1 \cup M_2 \cup M_3$ where M_i are 1-factors.*
- *(ii) G is 4-colourable.*
- *(iii) If G is 9-regular, then $G = F_1 \cup F_2 \cup F_3$ where F_i are 3-regular.*

Proof $(i \Rightarrow ii)$ Assume the 4 color theorem. Color the faces with colours from K_4 (Klein 4-group), then for any edge in G, colour it equal to the sum of adjacent elements on the faces. But then this is a valid 3-coloring, since the sum of distinct elements of K_4 is 0 precisely when the elements are the same, and no two edges of the same face can be coloured the same.

$(ii \Rightarrow i)$ Conversely, suppose $G = M_1 \cup M_2 \cup M_3$. Consider $M_1 \cup M_2$, which is a 2-factor and hence a collection of disjoint cycles. This G gets a face coloring using colors 0 and 1, with $C_1(p)$. Repeat with $M_1 \cup M_3$ and get $C_2(p)$. Then set $C(p) = (C_1(p), C_2(p))$.

$(iii \Rightarrow ii)$ Let G be planar cubic 2-edge-connected. Take each vertex v with edges e_1, e_2, e_3 and expand the vertex into three vertices each with 4 edges connecting them, and apply (iii) to this graph. Then precisely one of e_1, e_2, e_3 is contained in some distinct 3-factor F_i. Color that edge i. ■

2.22 Proposition. *Let G be planar and 2-edge-connected. Then the edges of G can be colored with $\{1,2,3\}$ and $c_1(v), c_2(v), c_3(v)$ (the color count for each vertex v) are all equal or two are equal and the third differs by ± 2.*

Proof Assume not. Consider the following conditions:

(i) The number of blocks (piece with no cut vertex) is minimum

(ii) The number of vertex degrees not equivalent to 0 mod 3 is minimal

(iii) The sum of the vertex degrees greater than 3 is minimal

(iv) Number of edges is minimal

Let G_0 be 2-connected and minimal with respect to the above conitions. Let's determine some properties of G_0.

1. If there are blocks, separate and apply induction, permuting the colors in one of the components of necessary. Thus the number of blocks is 1
2. Let v be a vertex with degree not 0 modulo 3. If v has degree 7, separate the vertex into a pair v_1, v_2 (degrees 2 and 5) and with an additional edge between v_1, v_2. Then the number of blocks is unchanged, and the number of not 0 modulo 3 degrees goes down, so we may apply induction. If v has degree 5, separate the vertex into a pair v_1, v_2 (degrees 1 and 4), with 2 additional edges, and repeat the same process. Every possible degree falls into one of these degrees. Thus, every vertex has degree equivalent to 0 modulo 3.
3. A least one vertex has degree at least 6.
4. G is 3-edge connected; if not, get a bridge with 2 edges, do the standard swap.
5. For all x of degree at least 3, there exists y such that $\{x, y\}$ is a 2-vertex-cut. Suppose some vertex x has degree at least $n \geq 6$. Then separate into a vertex x' with degree 3 and x'' with degree $n-3$. Suppose the number of blocks is now 2. Then there exists some vertex y such that every x' to x'' path contains y.
6. Every vertex has at least 3 neighbours. ■

2.23 Proposition. *We have the following increasing sequence of dependencies:*

1. *If G is 4-edge-connected.*
2. *G contains two edge-disjoint spanning trees.*
3. *$G \supseteq H_1 \cup H_2$ where H_i are eulerian*
4. *$G = H_1 \cup H_2 \cup H_3$ such H_i is even and every edge is covered twice.*
5. *There is a cycle double cover (collection of cycles)*
6. *$L(G)$ is Hamiltonian*
7. *G has an S^2–flow.*

Proof We show 3 implies 4. Put $E(H_3)$ as the symmetric difference of $E(H_1)$ and $E(H_2)$. ■

If G is 40-regular, can G be factorized into 5 8–factors? Yes: if G is $2k$–regular, can decompose into k 2–factors (take an Euler walk and split the vertices into 2, with all the out weight or all the in-weight).

2.24 Theorem. *Let $r = k \cdot q$ where q is odd.*

(i) Suppose k is odd, G is r–regular, and $(3k-3)$–edge-connected. Then we can factor G into k q–factors.

(ii) If k is even, the same holds with $(2k^2+k)$–edge-connected and $|V(G)|$ is even.

Proof Recall if k is odd and G is $(3k-3)$–connected, then G has an orientation with prescribed out-degree modulo k (given degree constraints, which are satisfied when $|V(G)|$ is even). Moreover, if G has odd edge connectivity at least $3k-2$, then G has an orientation balanced modulo k.

Fix a vertex v with in-degree α and out-degree β. Then $\alpha \cong \beta \pmod{k}$ and $\alpha+\beta \cong 0 \pmod{k}$, so $\alpha \cong 0 \pmod{k}$ and $\beta \cong 0 \pmod{k}$. Thus we can split each vertex at α and β ... ■

If G is a 9–reglar graph with odd edge connectivity at least 7, then $G = F_1 \cup F_2 \cup F_3$ where the F_i are 3–factors. If Tutte's 3-flow holds, we may take 5 instead.

2.25 Proposition. *Let $n \equiv 3 \pmod{6}$. Then for any $s_1, \dots, s_k$ with $s_i \geq 2$ and $\sum_{i=1}^k s_i = n$, G can be factorized into k factors $M_1, \dots, M_k$ where each M_i is s_i–regular.*

Proof Let $o_1, \dots, o_\ell$ and $e_1, \dots, e_m$ enumerate s_i with o_i odd and e_i even. Get G as a union of $r/3$ 3–factors, and identify ... ■

Let G have $\Delta(G) \leq r$, with $r = r_1 + \cdots + r_m$ and $r_i \geq 2$ satisfying the same conditions as the previous proposition. Then there is a decomposition $G = G_1 \cup \cdots \cup G_m$ where $\Delta(G_i) \leq r_i$.

2.26 Theorem. *If G is 2-edge-connected, then G has a flow with values $\{1, 2, \dots, 7\}$.*

Proof Since G is 2-edge-connected, let's first see that $G = H_1 \cup H_2 \cup H_3$ where each H_i is even. By induction.

(i) If a vertex has degree 2, flatten the vertex and apply induction.

(ii) If G has a vertex with degree at least 4, lift some pair of vertices and apply induction.

(iii) If G is cubic and 2–connected, fold the bridge (standard technique) and merge the pieces by induction

(iv) Finally, if G is cubic and 3-connected, double every edge to get a 6-edge-connected, and thus contains three 3 edge disjoint spanning trees T_1, T_2, T_3. Return the trees to G, and set $H_1 = G - E(T_1)$, and for each edge e, there is some tree not containing e.

Then on each H_i, pick an orientation on the cycles and assign values ± 1 on H_1, ± 2 on H_2, and ± 4 on H_3. Flip the orientations on edges with negative values. ■

Definition. Let G be an undirected graph and D a directed graph.

- G is k–connected if for all $x, y \in V(G)$, there are k internally vertex disjoint paths between x and y.
- G is k–edge-connected if there are k edge disjoint paths between any x and y.
- D is strongly k–connected if there are k internally disjoint directed paths.
- D is strongly k–edge-connected (or k-arc-connected) if there are k internally edge disjoint directed paths.

2.27 Theorem. (Robbin) *Let G be a graph. Then G has an orientation that makes it strongly connected if and only if G is 2-edge-connected.*

Proof Since G is 2-edge-connected, begin with a cycle and orient the edges in the same direction. If we are not done, there exists some edge adjacent to a vertex already in some oriented cycle. This is called an ear decomposition. ■

2.28 Theorem. *Let G be a mixed graph: some edges have orientations, while other edges do not. Then G has no bridge and no directed cut (a cut with all edges with the same orientation) if and only if G can be extended to a strongly oriented G.*

Proof Let's see that every edge is in a consistent cycle. Suppose e is an edge from x to y, and let B be the set of vertices which can be reached with a directed path from y. If e is oriented from x to y with no consistent cycle, then it means that every other path connecting x and y must have the same orientation across some cut, a contradiction. If e is not directed, the only case we need to treat is when the A to B cut containing e has all edges oriented from A to B. But then since e is not a bridge, there exists some distinct edge e' with orientation from A to B, which is contained in a consistent cycle, and this cycle ust contained e. ■

2.29 Theorem. (Nash-Williams) *Let G be a graph. Then G has an orientation that makes it k–arc-conected if and only if G is $2k$–edge-connected.*

Proof Delete edges until G is minimally $2k$–edge-connected. Consider a vertex x with degree precisely $2k$. By mader's lifting theorem, we can lift all the pairs of edges so that G remains $2k$–edge-connected, and possibly leaving some x_0 with multiple edges. Apply induction on the lifted graph to get an orientation on G'. Then maintain the orientations on the edges to v. The only case that to verify that x and y are k–arc-connected for any y that is not a neigbour of x.

Let A be a component containins x and B a component containing y, and suppose for contradiction A and B define a cut with less than k directed edges in some direction. If there is another vertex z in A, we have a contradiction by induction on the lifted G. But then if $x \in A$ is the only vertex, there are the same number of edges going out and in. ■

2.30 Conjecture. *If G is $f(k)$–connected, then there exists an orientation on G which is strongly k-connected.*

This is open for $k = 3$.

If G is cubic 3–connected, then there is no orientation which makes G strongly 2–connected.

2.31 Conjecture. (Frank) *There exists an orientation on G so it is strongly k–connected if and only if for all $S \subseteq V(G)$ with $|S| < k$, $G \setminus S$ is $2(k - |S|)$–edge-connected.*

The forward implication is trivial, but the reverse implciation is false for each $k \geq 2$. However, it does hold for $k = 2$.

Proof If G is cubic 3-connected, then G has a 1-factor M in G. Double the edges contained in M. Then $G \setminus M$ is composed of disjoint cycles. Give those any orientation, and the resulting orientation on G is strongly 2–connected.

Now let $v \in V(G)$ arbitrary and fix a direced cut in $G \setminus v$. Then this cut must have size at least 2.

2.32 Proposition. *Let G be 4-regular 4-edge-connected. Then there exists an orientation on G so that it is strongly 2-connected.*

Proof By induction. Let v be a vertex incident with 2 other vertices with 2 double edges. ∎

2.33 Theorem. *G has a strongly connected orientation if and only if G is 4-edge-connected and for any vertex v, $G \setminus v$ is 2-edge-connected.*

Let G be a graph with $|V(G)| \geq 3$ satisfying

1. G is 2-connected
2. G is 3-edge-connected
3. G is almost 4-edge-connected (exists some y_0 with a 3-edge-cut)
4. $G \setminus v$ is almost 2–edge-connected (only way to make a bridge is to delete the two vertices joining an edge).

Then there exists an orientation on G (denoted by D) such that D is strong and for all v, $D \setminus v$ is almost strong.

If G is 2–edge-connected, then G has a S^6–flow..

3 Infinite Graphs

3.1 Lemma. (König Infinity) *If G is infinite, locally finite, and connected, then there exists a path of infinite length.*

Proof Fix some vertex v_1. Every vertex $v \in G$ can be reached from v_1, and G is infinite, there must be some child v_2 of v_1 contained in infinitely many simple paths originating at v_1. Repeat this process (axiom of dependent choice) to construct the desired path. ∎

3.2 Theorem. (Erdös) *Let G be a countable graph. G is planar if and only if every finite $G' \subseteq G$ is planar.*

Proof List the vertices $\{v_i\}_{i=1}^{\infty}$ and let G_k be the subgraph induced by $\{v_1, \dots, v_k\}$. Let P be the graph with vertex set the set of all plane drawings of all G_k (up to homeomorphism). We say p_1 and p_2 are connected if $p_1 \in G_k$, $p_2 \in G_{k+1}$, and p_2 restricted to G_k is homeomorphic to p_1. Then P is an infinite locally finite graph, so by König's infinity lemma, P has an infinite path $T = (p_1, p_2, \dots)$ where we may take each p_k to be a planar embedding of G_k. ∎

Let $\mathcal{F}$ be a collection of finite forbidden k–colored subsets. Then a k–coloring of A' is $\mathcal{F}$–admissible if it does not contained an element in $\mathcal{F}$.

3.3 Theorem. (De Bruijn-Erdös) *A has a $\mathcal{F}$−admissible colouring if and only if every finite $A' \subseteq A$ has an $\mathcal{F}$−admissible colouring.*

Proof By Zorn, extend $\mathcal{F}$ to some maximal $\mathcal{F}_0$ such that every finite subset $A' \subseteq A$ has an $\mathcal{F}_0$−admissible colouring. By maximality of $\mathcal{F}_0$, there is exactly one vertex-colour pair $(\{x\}, i)$ that is not an element of $\mathcal{F}_0$. Repeat this for every vertex; verify that this is a valid colouring. ■

3.4 Theorem. (Tutte) *Let G be locally finite. Then G has a 1-factor if and only if the number of odd components of $G - S$ is at most $|S|$.*

Proof Let G be a graph, $V_0 \subseteq V(G)$ and all x in V_0 of finite degree. Then G has a matching covering V_0 if and only if for all $S \subseteq V(G)$ (S finite), the number of odd finite components consisting of vertices in V_0 is less than $|S|$. Colour the edges $0, 1$, and let $\mathcal{F}$ forbid a vertex having a pair of adjacent 1's, and any vertex $v \in V_0$ does not have all adjacent 0. Let $E_f \subseteq E(G)$ be a finite set of edges. Let V_1 be the set of vertices v in V_0 such that all edges adjacent to v are in E_f. Consider the graph G consisting of V_1, $N(V_1)$, and a complete graph K_m, and include every edge from $N(V_1)$ and K_m. ... ■

3.5 Theorem. *G is a union of k–forests if and only if for any finite $A \subseteq V(G)$, $|E(G[A])| \leq k(|A| - 1)$.*

There exists a connected graph G and vertices x, y so that for any path P from x to y, $G - E(P)$ is disconnected. Let G_0 be k−connected and $x_1, x_2, \dots, x_k \in V(G)$ far apart. Now fix a copy of G_0. Then for every path $P = (e_1, \dots, e_k)$ of length k in G_0, take a new copy of G_0 and add G_0 by placing each x_k on each e_k. Repeat this construction on each new G_0. Suppose we delete $k - 1$ vertices...

3.6 Theorem. *G is k−edge-connected*

3.7 Theorem. *If G is locally inite and connected, $A_0 \subseteq V(G)$ is finite. Then G "looks like a finite graph" in the sense that $V(G) \setminus A_0$ can be partitioned into a finite number of sets, each of which is a **singleton** or a **boundary-linked set**.*

A boundary linked set is an infinite set with some outgoing edges, where for each outgoing edge, there is a distinct one-way infinite path, all belonging to the same **end** We say $P \sim P'$ if and only if P and P' are not separated by a finite cut, and the ends are the equivalence classes of $\sim$. It is straightforward to see that $\sim$ is an equivalence relation. If two paths are in the same end, then there are infinitely many disjoint paths connecting them. In a sense, the ends of a graph are the number of ways to go off to infinity in some disjoint way.

Now let B be infinite, with finite boundary of size k'. We will always choose k' to be minimal.

Proof Label the outgoing edges of A_0 by $e_1, \dots, e_k$, with neighbourhood A_1. We may assume $k > 1$, for if $k = 1$, this statement is immediately true. Now let B be a component with finite boundary of size k'. Choose k' minimal, where $k' \leq j$. We prove this by induction on $k - k'$.

First suppose $k - k' = 0$. Pick a $1 - \infty$ path P_0, and separate G into components. Let A_2 be the boundary of the infinite connected component G_2 containing P_0, so the boundary has size k'. Since $k' = k$, we may apply Menger's theorem (infinite case) to get $P_{1,1}, \dots, P_{1,k}$

edge disjoint $A_0 - A_2$ paths. Then remove A_2 from G_2, and let A_3 be the boundary of a connected component of G_2 containing P_0. As before, there must be at least k' edges, so we may choose $P_{2,1}, P_{2,2}, \dots, P_{2,k}$. Always do this with minimal total length. Continue this to get an infinite family of paths rowwise edge disjoint. Then one can construct k disjoint 1-way infinite paths. We must ensure that all paths are in the same end.

Let H be a component with a distinct end, and suppose (for example) P_3 has edges in H. Consider $P_{i,1}, \dots, P_{i,k}$, which uses the boundary of H in only finitely many ways. Thus $P_{i,1}, \dots, P_{i,k}$ use H in only finitely many ways, for if not, then they would extend arbitrarily far into H, contradicting minimality.

Now suppose $k' < k$. Contract the maximal components (under inclusion) A' with boundary of size k' into a vertex a'. Do that for all infinite components with boundary size k' to get a set of components A', contracted to vertices a'. Now apply induction to the resulting graph to get some connecting paths. If some path goes through some vertex a', when we expand to A', applying the base case to A', there are k' edge-disjoint paths in A' which allows us to extend the edges to a' into finite paths through A'. This is doable since the incoming edges need not be paired with the outgoing edges in a fixed order (taking minimal connections). ∎

3.8 Theorem. (Nash-Williams) *Let G be finite and $2k$–edge-connected. Then G has an orientation such that G is k–arc-connected.*

We can extend Nash-Williams to the infinite case:

3.9 Theorem. *Let G be possibly infinite and $8k$–edge-connected. Then G has an orientation such that G is k–arc-connected.*

Let G be 1000-connected. Then there exists $H \subseteq G$ such that H is finite and 2-connected. Moreover, there exists some $H \subseteq G$ such that H is a subdivision of a 3-connected graph.

3.10 Proposition. *For all k, there exists a k–connected planar graph.*

3.11 Proposition. *Let G be $4k$–edge-connected and $A_0 \subseteq V(G)$ finite. Then G has an immersion of a finite $2k$–edge-connected Eulerian graph with vertex set A_0. Contract the boundary-linked components to get G', which is $4k$–edge connected. Get $2k$–edge-disjoint spanning trees $T_1, \dots, T_{2k}$, and get $H_1 \subseteq T_1 \cup T_2$ Eulerian, etc. Take $G'' = H_1 \cup \dots \cup H_k$. Since each H_i is Eulerian, every cut is even, so that G'' is $2k$–edge-connected.*

Now apply Mader's lifting theorem on G'' on all vertices outside A_0. Suppose we lift some e_1 and e_2 connected to some a contracted from A. Then get 1-way disjoint paths from e_1 and e_2, which we may reconnect within G...

This may not work. Given some vertex a contracted from A, define the lifting graph $L(G'', a)$ defined on $e_1, \dots, e_q$. Two of e_i, e_j are neighbours if the lifting of e_i, e_j preserves $2k$–edge-connectivity. By Mader's lifting theorem, $H_1 = L(G'', a)$ has an edge. Now define the pseudo-lifting graph $H_2 = PL(G, A)$ defined on $e_1, \dots, e_q$, where two of e_i, e_j are neighbours if P_i and P_j are joined by a path P edge-disjoint from $P_1, P_2, \dots, P_q$. It suffices to show that H_1 and H_2 share an edge.

We first show that H_2 is connected. Equivalently, partition $H_2 = A \cup B$; let's show there is a path from A to B. But this follows by taking a minimal path. Now, it suffices to show H_1 has a spanning complete bipartite subgraph. We prove this by induction. Take the case when $d(v) = 4$. If the lifting graph is complete, then we are fine. Otherwise, lift some pair *of vertices, and there is a $2k-2$-cut, and arguing by counting, we get a complete bipartite subgraph.*

Now let v have $d(v) \geq 6$ and even. Suppose for contradiction $\overline{H}_1$ is connected, so there exists x, y which are not neighbours such that $G-x-y$ is connected unless G is complete or a cycle. We know G cannot be complete since H_1 is non-empty. Let's also show H_1 is not a cycle.

Algorithm O_1: selected an undirected cycle and directs it in an arbitrary direction. This algorithm terminates if and only if the initial graph was Eulerian. Moreover, every cut in this direct graph will be balanced, so the result edge connectivity is 1/2 the original.

Algorithm O_2: suppose there are $2k-1$ mixed paths connecting vertices x and y. Then identify x and y.

Let's show that repeatedly applying O_1 and O_2 results in some oriented G'. Suppose this stops with some G' with $n' \geq 2$ and $e' \geq (2k-1)n'$. The the number of undirected edges is at most $n'-1$ (or we would have a cycle). Thus the number of directed edges is at least $(2k-1)n'-n'+1 = (2k-2)n'+1$. Thus H has n vertices and $e \geq (k-1)(n-1)+1$ edges.

Partition G into components K_1, K_2 connected by $k-1$ edges and edge set e_1, e_2. Then

$$\begin{aligned}
e_1 &\leq (k-1)(n_1-1) \\
e_2 &\leq (k-1)(n_2-1) \\
(k-1)(n-1)+1 \leq e &\leq (k-1)(n_1+n_2-2)+k-1 = (k-1)(n-2)+k-1 = (k-1)(n-1)
\end{aligned}$$

a contradiction.

Now follow the O_2' algorithm backwards. Let's show that we never get a cut with fewer than k directed edges. Suppose such a cut existed after separating some x and y. Then necessarily x and y are on opposite sides of the cut. Since the directed edges of G form an Eulerian subgraph (we only add them in cycles), the number of edges in each direction must be the same.

3.12 Theorem. *Suppose G is $8k$–edge-connected. Then there exists an orientation on G so that it is k–arc-connected.*

Proof Assume first G is locally finite, so that $E(G) = \{e_1, e_2, \ldots, \}$. Let A_0' be composed of the vertices on e_1, x and y. Then join x and y with $2k$ oriented edges. Then $A_0 = e_1 \cup e_2$.

There exists an immersion of a $4k$–edge-connected graph on A_0. Use $O_1' + O_2'$ on A_0 so that it is k–arc-connected.

Now let G be countable. Fix some vertex v such that $G-v$ is connected. Then split v into vertices of finite degree such that G has the same edge connectivity.

Recall that if T is a tree and $A \subseteq V(T)$, then A or $A \setminus \{a\}$ can be paired in T by edge disjoint paths. We will prove this later for infinite graphs.

Now apply this, and since $G-v$ is connected, we may apply this result to pair all the neighbours of v. But now we may inductively group the edges of v into arbitrarily large groups consisting of distinct members. ■

3.13 Theorem. *Let $A \subseteq V(T)$, then T has an A–pairing or a near-A-pairing.*

Proof We first show the finite case. Fix some root vertex $v \in T$. Then there exists an A–path $P[a_1, a_2]$ such that $A \setminus \{a_1, a_2\}$ is contained in some component of $T - E(P)$ which contains the root v. To see this, simply choose a_1, a_2 so that the component containing r contains the maximum number of vertices in A.

In general, given a path $P \subseteq T$, $E(T) \setminus E(P)$ is a disjoint union of trees. Write $V(T) = \{v_1, v_2, \ldots\}$.

Step 1: suppose there exists some v_i, v_j such that all vertices of $A \setminus \{v_i, v_j\}$ are in the same component of $T - E(P_1)$. Repeat as long as possible using Zorn. To make this precise, let $\mathcal{P}$ be a path-system pairing some vertices in A such that all vertices in $A - \{\text{ends of } \mathcal{P}\}$. Order the set of path-systems by inclusion; then some chain $\mathcal{C}$ has an upper bound given by taking unions, say, $\mathcal{P}^*$. We must verify that the vertices not in A lie in the same component. Take some pair of vertices u, v in A not covered by the maximal $\mathcal{P}^*$. If there is some path Q connecting them, then Q would be in $\mathcal{P}^*$. If we are left with finitely many unpaired vertices in A after step 1, we apply the finite proof with possibly 1 vertex leftover.

Step 2: Let v_i be the smallest unpaired vertex after step 1. Suppose we can pair v_i with some v_j, and consider the path Q connecting v_i and v_j. Then the remaining tree is divided into components such that for each component, there is either 1 vertex of A or infinitely many vertices of A. If there are finitely many vertices in A, then take a minimal path within that component, which would be a valid path in step 1, a contradiction. Suppose there is some component which has 1 vertex u, and take the component with minimal distance from v_i. Then connect v_i with u with some path P. If the component of $T - P$ containing v_j has finitely many components, we argue as before. Otherwise, v_j is alone; but then we may pair v_j and u, contradicting the property from step 1.

We now have the following property: for any minimal unpaired v_i and some $a \in A$, take a path P from v_i to a such that each component has either no vertices in A or infinitely many vertices in A. We must verify that the infinite components have the property from step 1. Suppose not; fix some component C, take some path P, and apply step 1. Suppose there is some vertex z in C which is unpaired. Then we can reconstruct the pairing such that, without loss of generality, z has a path to Q disjoint from the pairings. But then if there are multiple components, we simply pair these extra vertices using paths through P. ∎

3.14 Conjecture. *Let G be 2–edge-connected. Then G has a collection of cycles covering each edge between 1 and two times.*

For finite graphs, this is equivalent to having a collection of cycles covering each edge exactly twice. We know we can do this covering each edge at most 3 times. We proved this by showing that we could cover any such graph using 3 even graphs.

3.15 Proposition. *Let G be 2–edge-connected (and possibly infinite). Then G has a collection of cycles covering each edge between 1 and $\aleph_0$ times.*

3.16 Theorem. *Let G be 2–edge-connected and fix $v_0 \in V(G)$. Then G has a collection $\mathcal{C}_n$ of cycles such that*

- *(i) $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots$ such that $\mathcal{C}_n$ covers each edge at most finitely many times and $G_n = \bigcup_{C \in \mathcal{C}_n} C$ is an induced subgraph.*
- *(ii) G_n contains all vertices of distance at most n from v_0.*
- *(iii) If we add an external matching M to the vertex-boundary of G_n, then $G_n + M$ has a collection of cycles $\mathcal{C}_M$ covering each edge only finitely often and each edge in M at least once.*

Proof Take $\mathcal{C}_0 = \{C_0\}$, where C_0 is a shortest cycle containing v_0. Then (i), (ii), and (iii) hold trivially.

Now suppose we have constructed $\mathcal{C}_{n-1}$, and let B be the boundary of G_{n-1}. Let H be the set of neighbours of G_{n-1}. We then construct the set of $H-$paths. First given some $v \in H$ neighbour to some $N(v) \subseteq G_{n-1}$, take as many $w_1 - v - w_2$ pairs of edges. Remove these edges, and take a spanning tree T containing the vertices in H which have unpaired edges, let $A = N(G_{n-1})$, and apply the previous proposition to get edge-disjoint paths in T connecting the vertices of A. This is the set of $H-$paths.

Now consider all $H-$paths for all H, and by Zorn add maximally many disjoint cycles within the set of $H-$paths to $\mathcal{C}_n$. The remaining edges in $H-$paths form a forest F. Pair the edges of F; then, extending paths maximally, we get a matching in the boundary of G_{n-1} so we may apply property (iii) to get a collection of cycles covering $\mathcal{G}_{n-1}$ and the matchings.

Now we must show that we can extend $\mathcal{C}_n$ so that G_n is an induced subgraph (there may be many "chord edges"). Take maximally many edge-disjoint cycles consisting of chord edges and add them to $\mathcal{C}_n$. What remains of the edges is a forest, which we can decompose into two pseudo-matchings (matchings with paths, not edges). Extend these matchings to (possibly overlapping cycles, but only finitely mant times) reach the neighbourhod of G_{n-1}. Apply the same argument as before, take maximally many edge-disjoint cycles by Zorn, apply condition (iii) to the remaining forest of path pairings, and add these paths back.

Note that the same proof technique shows that condition (iii) holds in G_n. ■

Let A, B be components separated by a non-empty cut D. We say that $E \subseteq E(G)$ if G has a component G' such that $G' - E$ is disconnected. If D is a cut, then D is separating; the converse is not true (e.g. all the edges in a triangle). Note that every separating set contains a cut. Thus, D is a minimal cut if and only if D is a minimal separating set. Every cut can be decomposed into a disjoint union of minimal cuts. But then every separating set contains a cut, which contains a minimal cut, which is a minimal separating set.

Definition. Let $H \subseteq G$. Then H is **cut-faithful** if every finite minimal cut in H is a cut in G.

We get some properties of cut-faithful subgraphs:

- minimal cuts in H are also minimal cuts in G.
- every finite cut in H is a cut in G, since a union of edge disjoint cuts is a cut. To see this, a cut has the local property that it has even intersection with every cycle.
- If H intersects a minimal finite cut D in G, then H contains the entire cut, essentially by definition.

3.17 Theorem. (Laviolette) *Suppose $G = \bigcup_{i \in I} G_i$ (edge disjoint union) where each G_i is countable, connected, and cut faithful.*

Proof We can assume that G is 2-edge-connected. If G has bridges, take H to be the set of all bridges, and then H is cut-faithful and $G \setminus H$ is a disjoint union of 2-edge-connected components, so we may argue on each component separately.

By the theorem from last time, G can be covered by cycles such that every edge is covered at most countable times. Let $\{C_\alpha\}_{\alpha \in A}$ be the set of cycles. Define a graph structure on A, where α, β are connected by an edge if C_α and C_β share an edge. Let $\{A_i : i \in I(1,1)\}$

be the components of A, and set $G_{i,1,1} = \bigcup_{\alpha \in A_i} C_\alpha$. Here are some basic properties of the $G_{i,1,1}$:

- Each $G_{i,1,1}$ is 2−edge-connected since it is a union of overlapping cycles.
- $G_{i,1,1}$ and $G_{j,1,1}$ are edge disjoint for $i \neq j$.
- Each $G_{i,1,1}$ is countable, since every vertex of A_i must have countable degree: by construction, each edge is covered by at most countably many of the cycles C_α, and each cycle has only finitely many cycles with overlapping edge. Thus since A_i is connected, $G_{i,1,1}$ is countable.

Now for each $G_{i,1,1}$, enumerate all the finite minimal cuts contained in $G_{i,1,1}$ as $\{D_{1,i,1}, D_{2,i,1}, \ldots\}$. Then for each minimal cut $D_{k,i,1}$ in $G_{i,1,1}$ which is not a cut in G, join the edges on each side of the cut, and add a bridge between. But since $D_{k,i,1}$ is not a cut, G' is still 2-connected. Now $\mathcal{C}$ be a collection of cycles covering G' such that each edge is covered at least once and at most countably many times. We can then return this to a collection of cycles in C with the same property in G since each $G_{i,1,1}$ is connected. Now if there is some minimal cut $D_{k,i,1}$ in $G_{i,1,1}$ that is not a cut in G, there is some path P contained in some cycle $C \in \mathcal{C}$ from the two sides of $D_{k,i,1}$ which is not contained entirely in $G_{i,1,1}$. Then recursively add the graphs $G_{j,1,1}$ which contain the path P.

Thus get a collection $\{G_{i,2,1} : i \in I(2,1)\}$ such that each minimal cut $D_{1,i,1}$ is now extended to a cut in G. Repeat this countably many times to get a collection of graphs $\{G_{i,1,2} : i \in I(1,2)\}$ for which each $D_{n,i,1}$ is no longer a violating cut for cut faithfulness. But now repeat this countably many times to get some collection of graphs $\{G_i : i \in I\}$. We must show that the G_i are indeed cut faithful.

Take some minimal cut $E \subset G_i$: we must show that E is a cut in G. Write $G_i = A \cup B$, A', B' neighbours to E. Take a tree T_A connecting all the vertices A' in A and T_B similarly. Then there exists some $j \in I(1,m)$ so that G_j contains the finitely many edges in E, T_A, T_B, so that E must be a cut in G. ■

3.18 Proposition. (Nash-Williams) *A graph can be edge-decomposeed into cycles if and only if G has no odd cut.*

Proof If G is finite or countable, this problem is straightforward. In the uncountable case, apply Laviolette. ■

3.19 Proposition. *If G is $8k$−edge-connected, then there exists an orientation so that G is k−arc-connected.*

Proof We did the countable case earlier: Laviolette's theorem extends this to the uncountable case. ■

3.20 Proposition. *Let G be finite. If $\chi(G) > k$, then there exists $G \supseteq H$ such that $\chi(H) > k$ and $\lambda(H) \geq k$ (λ is edge connectivity).*

Proof Fix some minimal cut $E \subseteq G$ of size $k-1$. We must show that one side of the cut has chromatic number greater than k. Assume not, and k−color each side. Create a bipartite graph $K_{k,k}$ labelled with colours, and an edge if there is an edge joining such colours. Then if there is conflict, there are $p(k-p+1) \geq k$ edges in the complement $K_{k,k}$, so we may apply Hall's theorem. ■

Let $G_1 = G$, and $G_{i+1} = G_i - D_i$, and $G_i = \bigcap_{j<i} G_j$ if i is a limit ordinal. Thus we construct a generalized sequence of cut deletions G_ω for any ordinal ω. We then write $G \to H$ by applying this cut deletion procedure. Conversely, we may construct $G \leftarrow H$ by finite cut additions in the same way. Here's a theorem we will not prove:

3.21 Theorem. *If $G \to H$, then $G \leftarrow H$.*

Note that if $G \leftarrow H$, it is not necessarily true that $G \to H$: one can construct the countable complete graph by cut additions, which has no finite cut.

3.22 Theorem. *Let G be a graph with uncountable chromatic number. Then there exists some $H \subseteq G$ such that $\lambda(H) = \infty$ and $\chi(H)$ uncountable.*

Proof We argue by transfinite induction by applying finite cut deletion.

Suppose by cut deletion we end up with some collection of graph components. We first show that one fo these has $\chi > \aleph_0$ and $\lambda \geq \lambda_0$. Suppose for contradiction we can colour one of these components with colours in $\mathbb{N}$. Let's show that by cut addition we can maintain the countable number. Let $V(G) = \{v_i : i \in I\}$ where I is well-ordered. Enumerate the parts H_i, and focus on some H_i. Fix some vertex v in B, where v has colour $c_i(v)$ and distinguished vertex k_i. Then whenever we have parts H_i, H_j, we keep the colours in i and permute the colours in j if $k_i \leq k_j$. Then define $c(v) = \lim c_i(v)$, where the limit exists since $k_i \leq k_j$ whenever $i > j$. ■

We say a graph has colouring number $(G) = \omega$ where ω is the smallest cardinality such that there exists a well-ordering on $V(G)$ so that all the edges from v to some smaller vertex w is strictly less than ω. Suppose we combine components A, B with $k(A) < k(B)$. Then we reorder the ordering so that $V(B)$ is appended to $V(G)$ under the ordering, and $V(A)$ is unchanged. This only happens finitely many times, so that this reordering is well-defined for each vertex. But then the colouring number is unchanged, so that (G) is preserved under finite cut deletion.

3.23 Theorem. (Hakimi) *If G is a graph, then there exists an orientation on G so that every vertex has outdegree strictly greater than k if and only if there exists a subset $H \subseteq G$ such that $|E(H)| > k|V(H)|$.*

This also holds for k finite and G infinite, where we require $H \subseteq G$ is finite.

3.24 Theorem. *For any orientation on G there exists a vertex of out-degree $\aleph_0$ implie there is some $G \supseteq H$ uncountable and of infinite edge-connectivity.*

Proof Apply finite cut deletion to get a set of disconnected components. Certainly each has infinite edge-connectivity: let's show that some component is uncountable. Put orientations on each component, and then when we combine A with B with $k(A) < k(B)$, re-orient the edges from A to B to face A. Then there are only finitely many re-orientations, contradicting the assumption. ■