

PMATH 465

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I. Fundamentals of Manifolds

1 INTRODUCTION TO TOPOLOGY

BASIC CONSTRUCTIONS

Definition. A **topology** on a set X is a set τ of subsets of X such that

- (i) $\emptyset \in \tau$ and $X \in \tau$
- (ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$.

The sets $U \in \tau$ are called the **open sets** in X , and sets of the form $X \setminus U$ for some open set U are called the **closed sets** in X .

Definition. When X is a topological space and $A \subseteq X$, the **interior** of A (denoted A°) is the union of all open sets contained in A . Similarly, we define the **closure** of A (denoted \bar{A}) as the intersection of all closed sets containing A . Then the **boundary** of A , denoted by ∂A , is the set $\partial A = \bar{A} \setminus A^\circ$.

Example. Let X be any set. The **discrete topology** on X is the topology $\tau = \mathcal{P}(X)$, and the **trivial topology** on X is the topology $\tau = \{\emptyset, X\}$.

Definition. A **basis** for a topology on a set X is a set \mathcal{B} of subsets of X

- (i) $\bigcup_{B \in \mathcal{B}} B = X$
- (ii) for all $a \in X$ and $U, V \in \mathcal{B}$ such that $a \in U \cap V$, then there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When \mathcal{B} is a basis for a topology on X , the topology on X **generated** by \mathcal{B} is the set τ of subsets of X such that for $W \subseteq X$, $W \in \tau$ if and only if for all $a \in W$, there exists $U \in \mathcal{B}$ such that $a \in U \subseteq W$.

Note that τ , as above, is a topology on X since

- (i) $\emptyset \in \tau$ vacuously and $X \in \tau$ obviously.
- (ii) If $A_k \in \tau$ for all $k \in K$ (where K is any set of indices), then given $a \in \bigcup_{k \in K} A_k$, we can choose $\ell \in K$ so that $a \in A_\ell$. Then since $A_\ell \in \tau$, we can choose $U_\ell \in \mathcal{B}$ so that $a \in U_\ell \subseteq A_\ell$. Thus $a \in U_\ell \subseteq A_\ell \subseteq \bigcup_{k \in K} A_k$.
- (iii) By induction, it suffices to prove that if $A, B \in \tau$, then $A \cap B \in \tau$. Suppose $A, B \in \tau$, and let $a \in A \cap B$. Since $A \in \tau$, we can choose $U \in \mathcal{B}$ so that $a \in U \subseteq A$. Since $B \in \tau$, we can choose $V \in \mathcal{B}$ so that $a \in V \subseteq B$. Then we have $a \in U \cap V$. Since \mathcal{B} is a basis, we can choose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$, so $a \in W \subseteq U \cap V \subseteq A \cap B$.

Note that when τ is the topology on X generated by the basis \mathcal{B} , for $A \subseteq X$, $A \in \tau$ if and only if there exists some $S \subseteq \mathcal{B}$ such that $A = \bigcup_{s \in S} s$. In this sense, the topology τ on X generated by the basis \mathcal{B} is the coarsest topology which contains \mathcal{B} .

Definition. (Subspace Topology) When Y is a topological space and $X \subseteq Y$ is a subset of Y , we define the **subspace topology** on X to be the topology for which a set $U \subseteq X$ is open if and only if $U = X \cap V$ for some open set V .

If \mathcal{C} is a basis for the topology on Y , then $\mathcal{B} = \{X \cap V \mid V \in \mathcal{C}\}$ is a basis for the subspace topology on X .

Definition. (Disjoint Union Topology) If X and Y are topological spaces with $X \cap Y = \emptyset$, then the **disjoint union topology** on $X \cup Y$ is the topology in which a subset $U \subseteq X \cup Y$ is open in $X \cup Y$ if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y .

Definition. (Product Topology) If X and Y are topological spaces, the **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{C}, V \in \mathcal{D} \}$$

where \mathcal{C} and \mathcal{D} are bases for the topologies on X, Y respectively.

Definition. (Infinite Product Topology) We define the infinite product to be

$$\prod_{k \in K} \left\{ f : K \rightarrow \bigcup_{k \in K} X_k \mid f(k) \in X_k \text{ for all } k \in K \right\}$$

There are two standard topologies on X . The first is the **box topology**,

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid U_k \text{ is open in } X_k \right\}$$

and the **product topology**

$$\mathcal{B} = \left\{ \prod_{k \in K} U_k \mid \begin{array}{l} U_k \text{ is open in } X_k \\ U_k = X_k \text{ for all but finitely many indices } k \end{array} \right\}$$

Example. (Metric Topology) \mathbb{R}^n has a standard **inner product**, and for $u, v \in \mathbb{R}^n$, $\langle u, v \rangle = u \cdot v = V^T u = \sum_{i=1}^n u_i v_i$. This gives the standard norm on \mathbb{R}^n for $u \in \mathbb{R}^n$, $\|u\| = \sqrt{\langle u, u \rangle}$. This gives the standard metric on \mathbb{R}^n : for $a, b \in \mathbb{R}^n$, $d(a, b) = \|b - a\|$.

Given a metric on a set Y , we obtain (by restriction) an induced metric on any subset $X \subseteq Y$. Given a metric space X , we define the **metric topology** on X to be the topology which is generated by the set of open balls

$$B(a, r) = \{ x \in X \mid d(a, x) < r \}$$

where $x \in X, r > 0$.

MAPS ON TOPOLOGICAL SPACES

Definition. When X and Y are topological spaces and $f : X \rightarrow Y$, we say that f is **continuous** when it has the property that $f^{-1}(V)$ is open in X for every open set V in Y . We say that $f : X \rightarrow Y$ is a **homeomorphism** when f is bijective and both f and f^{-1} are continuous. Then X, Y are **homeomorphic** if there exists a homeomorphism $f : X \rightarrow Y$.

1.1 Theorem. (Glueing Lemma) Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a function. Suppose either

(i) $X = \bigcup_{k \in K} A_k$ where each A_k is open in X , or

(ii) $X = \bigcup_{k=1}^n A_k$ where each A_k is closed in X

and each restriction map $f_k : A_k \rightarrow Y$ is continuous, then f is continuous.

PROOF Exercise. ■

Definition. A topological space X is **compact** when it has the property that for every set \mathcal{S} of open subsets of X with $X = \bigcup_{U \in \mathcal{S}} U$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{S}$ such that $X = \bigcup_{F \in \mathcal{F}} F$.

Note that when $X \subseteq Y$ is a subspace, X is compact if and only if X has the property that for every set \mathcal{T} with $X \subseteq \bigcup_{T \in \mathcal{T}} T$, there exists a finite subset $\mathcal{G} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup_{G \in \mathcal{G}} G$.

Definition. A topological space X is **connected** when there do not exist non-empty disjoint open sets $U, V \subseteq X$ such that $X = U \cup V$.

Note that if Y is a metric space and $X \subseteq Y$ is a subspace, then X is connected if and only if there do not exist open sets $U, V \subseteq Y$ such that

$$X \cap U \neq \emptyset, X \cap V \neq \emptyset, U \cap V = \emptyset, \text{ and } X \subseteq U \cup V$$

Definition. A topological space X is called **path connected** when it has the property that for all $a, b \in X$, there exists a continuous map $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = a$ and $\alpha(1) = b$.

It is easy to see that if X is path connected, then X is connected.

Definition. Let X be a topological space. If we define a relation \sim on X by taking $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a \in A$ and $b \in A$.

It is clear that this is an equivalence relation. Note that when X is a topological space, its connected components are connected, and each connected subspace of X is contained in one of its connected components.

Definition. Let X be a topological space. Define a relation \approx on X by $a \approx b$ if and only if there exists a continuous map $\alpha : [0, 1] \rightarrow X$ with $\alpha(0) = a$ and $\alpha(1) = b$. Such a map α is called a **continuous path**.

One can show that if X is **locally path connected** (which means that X has a basis for its topology which consists of path connected sets), then the path components of X are equal to the connected components of X , and that these components are open.

QUOTIENT TOPOLOGY

Definition. (Quotient Topology) Let X be a topological space and let \sim be an equivalence relation on X . The set of equivalence classes is denoted X/\sim , and X/\sim is called the **quotient** of X by \sim . The map $\pi : X \rightarrow X/\sim$ given by $\pi(a) = [a]$ is called the **natural projection map** or **quotient map**. We define the **quotient topology** on X/\sim by stipulating that for $W \subseteq X/\sim$, W is open in X/\sim if and only if $\pi^{-1}(W)$ is open in X .

When a group G acts on a topological space X , we define an equivalence relation \sim on X by $a \sim b$ if and only if $b = g \cdot a$ for some $g \in G$. The equivalence classes are orbits. In this context, we also write X/\sim as X/G .

When X, Y are any topological spaces and $\pi : X \rightarrow Y$ is surjective, we can define an equivalence relation \sim on X by $a \sim b$ if and only if $\pi(a) = \pi(b)$. We then have a natural bijection from Y to X/\sim in which $y \in Y$ corresponds to the fibre $\pi^{-1}(y) \in X/\sim$.

If Y has the topology such that for $W \subseteq Y$, W is open in Y if and only if $\pi^{-1}(W)$ is open in X . In this case, we also use the terminology “quotient map” for π .

Remark. Let X be a topological space and let \sim be an equivalence relation on X . Let Y be any set. If $f : X \rightarrow Y$ is constant on the equivalence classes, then f induces a well-defined map $\bar{f} : X/\sim \rightarrow Y$ given by define $\bar{f}([a]) = f(a)$.

Example. Define an equivalence class on $[0, 1] \subseteq \mathbb{R}$ by $s \sim t$ if and only if $s = t$ or $\{s, t\} = \{0, 1\}$. Then $[0, 1]/\sim \cong \mathbb{S}^1$. Define $f : [0, 1] \rightarrow \mathbb{S}^1$ by $f(t) = e^{i2\pi t}$. Note that $f(0) = f(1)$, so f induces a continuous map $\bar{f} : [0, 1]/\sim \rightarrow \mathbb{S}^1$. The inverse map can be constructed as follows. We define $g : \mathbb{S}^1 \rightarrow [0, 1]/\sim$ by

$$g(x, y) = \begin{cases} \left[\frac{1}{2\pi} \cos^{-1} x \right] & : y \geq 0 \\ \left[1 - \frac{1}{2\pi} \cos^{-1} x \right] & : y \leq 0 \end{cases}$$

Then g is continuous by the Glueing lemma.

In particular, the same proof shows that \mathbb{R}/\mathbb{Z} is homeomorphic to \mathbb{S}^1 .

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ can be defined in several ways. \mathbb{P}^n is the set of all 1-dimensional vector subspaces of \mathbb{R}^{n+1} , or $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$, or $\mathbb{P}^n = \mathbb{S}^n / \pm 1$ where $\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$.

Let us show that $\mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times$ is homeomorphic to $\mathbb{S}^n / \pm 1$. Define $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{S}^n$ by $f(x) = x/|x|$, and $g = \pi \circ f$. Then g is given by $g(x) = \{\pm x/|x|\}$. Note that for $t \in \mathbb{R}^\times$,

$$g(tx) = \left[\frac{t}{|t|} \cdot \frac{x}{|x|} \right] = \left[\frac{x}{|x|} \right]$$

since $t/|t| = \pm 1$. Thus g induces a continuous map \bar{g} on the quotient. We construct the inverse map in a similar way.

Definition. Let X be a topological space. Then

- X is **T1** when for all $a, b \in X$ there exists an open set U in X with $a \in U$ and $b \notin U$
- X is **T2** or **Hausdorff** when for all $a, b \in X$, there exist disjoint open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$
- X is **T3** or **regular** when X is T1 and for every $a \in X$ and every closed set $B \subseteq X$ with $a \notin B$, there exist open sets $U, V \subseteq X$ with $a \in U$, $B \subseteq V$.
- X is **T4** or **normal** when X is T1 and for all disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

Definition. Let X be a topological space.

- X is **first countable** when for every $a \in X$, there exists a countable set B_a of open sets in X which contain a such that for every open set W in X with $a \in W$, there exists $U \in B_a$ with $a \in U \subseteq W$.
- X is **second countable** when there exists a countable basis for the topology on X .

Example. (i) X is T1 if and only if every 1-point subset of X is closed in X

(ii) Every compact Hausdorff space is regular.

(iii) Every second countable regular space is normal.

(iv) Every metric space is normal.

(v) If X is second countable, then every open cover admits a countable subcover.

(vi) Every second countable space X contains a countable dense subset.

1.2 Lemma. (Urysohn) If X is normal and $A, B \subseteq X$ are disjoint and closed, then there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

1.3 Theorem. (Tietze Extension) If X is normal and $f : A \rightarrow \mathbb{R}$ is continuous for some $A \subseteq X$ closed, then there exists a continuous map $F : X \rightarrow \mathbb{R}$ such that $F|_A = f$ and $\sup_{a \in A} |f(a)| = \sup_{x \in X} |F(x)|$.

1.4 Theorem. (Urysohn's Metrization) If X is second countable and regular, then X is metrizable.

Definition. An n -dimensional topological manifold is a Hausdorff, second countable topological space M which is **locally homeomorphic** to \mathbb{R}^n , meaning for every $p \in M$, there exists an open set $U \subseteq M$ with $p \in U$ and an open set $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \subseteq M \rightarrow V \subseteq \mathbb{R}^n$. Such a homeomorphism ϕ is called a **(local) coordinate chart** or **chart** on M at p . The domain U of a chart $\phi : U \subseteq M \rightarrow \phi(U) \subseteq \mathbb{R}^n$ is called a **(local) coordinate neighbourhood** at p . Note that we can choose a set of charts

$$\mathcal{A} = \{\phi_k : U_k \subseteq M \rightarrow \phi_k(U_k) : k \in K\}$$

where K is any non-empty set such that $M = \bigcup_{k \in K} U_k$. Such a set of charts is called an **atlas** for M .

Definition. Two charts are called $\phi : U \rightarrow \phi(U)$ and $\psi : V \rightarrow \psi(V)$ are called **(smoothly) compatible** when either $U \cap V = \emptyset$ or $\phi^{-1} \circ \psi$ and $\psi \circ \phi^{-1}$ are smooth (meaning partial derivatives of all orders exist). We say that an atlas is **smooth** if every pair of charts is compatible.

Note that a smooth atlas \mathcal{A} on M can be extended to a unique maximal smooth atlas \mathcal{M} on M by adding to \mathcal{A} every possible homeomorphism $\psi : U \subseteq M \rightarrow \psi(U) \subseteq \mathbb{R}^n$ which is compatible with all of the existing charts (since if ψ and χ are both compatible with every chart $\phi \in \mathcal{A}$, then ψ and χ will be compatible with each other). The maps $\psi \circ \phi^{-1}$ are called **transition maps** or **change of coordinate maps**. A maximal smooth atlas \mathcal{M} on M is called a **smooth structure** on M .

Definition. An n -dimensional **smooth (or C^∞) manifold** is an n -dimensional topological manifold with a smooth structure.

Remark. A topological manifold can have different smooth structures. For example, take $\mathcal{A} = \{\phi\}$ where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is the identity map, and $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism given by $\psi(x) = x^3$, since $\sqrt[3]{x}$ is not smooth at the origin.

What if we tried $\mathcal{B} = \{\psi\}$ where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism which is not C^∞ ? This is trivially a smooth atlas.

Typically, a manifold is given with a standard smooth structure.

Remark. We can give a smooth manifold M an (at most countable) atlas of charts all of which are of one of the forms

- $\phi : U \subseteq M \rightarrow B(0, 1)$
- $\phi : U \subseteq M \rightarrow (0, 1)^n$
- $\phi : U \subseteq M \rightarrow \mathbb{R}^n$

Note that the maximal atlas \mathcal{M} is determined from any subset $\mathcal{A} \subset \mathcal{M}$ such that the domains of the charts in \mathcal{A} cover M .

Definition. Let M be an m -dimensional smooth manifold and N be an n -dimensional smooth manifold and let $f : M \rightarrow N$ be a function. Then we say f is **smooth** at p when for some (hence for any) chart ϕ on M at p and for some (hence any) chart ψ on N at $f(p)$, the map $\psi^{-1} \circ f \circ \phi$ is smooth at $x = \phi(p)$, and f is **smooth** if f is smooth at every $p \in M$. We say that f is a **diffeomorphism** when f is invertible and both f and f^{-1} are smooth. We say that M and N are **diffeomorphic**, and write $M \cong N$, when there exists a diffeomorphism $f : M \rightarrow N$.

Remark. It is conceivable that a topological manifold M could be both of dimension n and of dimension m with $n \neq m$. To do this, we would need to have a homeomorphism from an open set in \mathbb{R}^n to an open set in \mathbb{R}^m . In fact, this cannot happen by invariance of domain, proven using tools from algebraic topology.

When M is smooth, it is easy to see that this cannot happen. If $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ were smooth inverses, then the matrices $D(\psi \circ \phi^{-1})(\phi(p))$ and $D(\phi \circ \psi^{-1})(\psi(p))$ would be inverse matrices. But then a product of a matrix in $M_{m \times n}(\mathbb{R})$ and in $M_{n \times m}(\mathbb{R})$ cannot be inverses when $m \neq n$.

Remark. Manifolds are sometimes constructed using quotient constructions. These quotients can be given by polygons with pairs of edges identified up to orientation.

There are other kinds of manifolds (other than C^∞ manifolds); for example, one can define C^k manifolds, or analytic C^ω manifold has an atlas in which the transition maps are analytic.

- Example.*
1. \mathbb{R}^n is a smooth n -dimensional manifold. It can be given an atlas consisting of 1 chart, the identity map.
 2. Any n -dimensional vector space over \mathbb{R} is a smooth n -dimensional manifold. It can be given an atlas with one chart. If $\{u_1, \dots, u_n\}$ is a basis for V , then one can define $\phi : V \rightarrow \mathbb{R}^n$ by $\phi(\sum t^i u_i) = (t^1, \dots, t^n) = t \in \mathbb{R}^n$.
 3. Every open subset of a smooth n -dimensional manifold is also a smooth n -dimensional manifold.
 4. $M_{n \times m}(\mathbb{R})$ is an $n \cdot m$ -dimensional manifold with pointwise \mathbb{R}^{nm} structure.
 5. $\{A \in M_{n \times m}(\mathbb{R}) : \text{rank}(A) = \min\{n, m\}\}$ is a smooth manifold with one chart, since it is an open submanifold of $M_{n \times m}$. Suppose $n > m$; then take all $n \times n$ submatrices which have non-zero determinant (open by continuity of \det), and maximal rank means that A is contained in one of these open subsets.
 6. The disjoint union of countably many n -dimensional smooth manifolds.
 7. The cartesian product of finitely many smooth manifolds is a smooth manifold. Let $\dim(M_k) = n_k$, the $\dim(M_1 \times \dots \times M_\ell) = \sum_{k=1}^\ell n_k$. If $\phi_k : U_k \subseteq M_k \rightarrow \phi_k(U_k) \subseteq \mathbb{R}^{n_k}$ is a chart on M_k , then $\chi_k : \prod_{k=1}^\ell U_k \rightarrow \prod_{k=1}^\ell \mathbb{R}^{n_k}$ given by $\chi_k(p_1, \dots, p_\ell) = (\phi_1(p), \dots, \phi_\ell(p))$ is a chart in $M_1 \times \dots \times M_\ell$.
 8. One can show that \mathbb{S}^n is a smooth n -dimensional manifold.

Remark. For $A \in M_{n \times m}(\mathbb{R})$, we denote the entry in the k^{th} row and ℓ^{th} column by A_ℓ^k .

Example. \mathbb{S}^n is an example of an n -dimensional smooth manifold. It can, for example, be given a smooth atlas which contains $2(n+1)$ charts as follows. For $1 \leq k \leq n+1$, let

$$\begin{aligned} U_k &= \{x \in \mathbb{S}^n : x^k > 0\} \\ \phi_k : U_k &\rightarrow B(0, 1) \subseteq \mathbb{R}^n \\ \phi_k(x) &= (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^{n+1}) \\ \phi_k^{-1}(t^1, \dots, t^n) &= \left(t^1, \dots, t^{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} (t^i)^2}, t^k, \dots, t^n\right) \end{aligned}$$

and the corresponding opposite charts for $x^k < 0$. Note that \mathbb{S}^n is a metric space. It has 2 standard metrics: either the one inherited from \mathbb{R}^n , or the arclength distance $d_s(u, v) = \cos^{-1}(u \cdot v)$.

We can also give \mathbb{S}^n an atlas which only uses 2 charts, by stereographic projection from a north pole and a south pole.

This stereographic projection also shows that the rational points on the sphere are dense in \mathbb{S}^n , via the map

$$\phi(x) = \alpha\left(\frac{1}{1-x^{n+1}}\right) = \left(\frac{x^1}{1-x^{n+1}}, \dots, \frac{x^n}{1-x^{n+1}}\right)$$

One can also find ϕ^{-1} and verify that they are both rational functions. In particular, $\phi^{-1}(\mathbb{Q}^n) \subseteq \mathbb{S}^n$ is dense.

Example. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ is commonly defined in at least 3 ways:

$$\begin{aligned}\mathbb{P}^n &= \{1\text{-dimensional subspaces of } \mathbb{R}^{n+1}\} \\ \mathbb{P}^n &= \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^\times = \{[x] : 0 \neq x \in \mathbb{R}^{n+1}\}, [x] = \{tx : t \in \mathbb{R}^\times\} \\ \mathbb{P}^n &= \mathbb{S}^n / \pm 1\end{aligned}$$

We can give \mathbb{P}^n a smooth atlas with $n+1$ charts as follows: for $1 \leq k \leq n+1$, set

$$\begin{aligned}U_k &= \{[x] \in \mathbb{P}^n : x^k \neq 0\} \\ \phi_k : U_k &\rightarrow \mathbb{R}^n, \phi_k([x]) = \left(\frac{x^1}{x^k}, \dots, \frac{x^{k-1}}{x^k}, \frac{x^{k+1}}{x^k}, \dots, x^{n+1}x^k\right)\end{aligned}$$

with $\phi_k^{-1}(t_1, \dots, t^n) = [(t_1, \dots, t^{k-1}, 1, t^k, \dots, t^n)]$.

EXAMPLES OF SMOOTH MAPS

- The inclusion $f : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ given by $f(x) = x$
- The quotient map $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$
- The exponential map $f : \mathbb{R} \rightarrow \mathbb{S}^1$ given by $f(t) = e^{i2\pi t}$, or more generally $f : \mathbb{R}^n \rightarrow \mathbb{T}^n$ given by $f(t^1, \dots, t^n) = (e^{2\pi i t^1}, \dots, e^{2\pi i t^n})$
- The determinant map $f : M_n(\mathbb{R}) \rightarrow \mathbb{R}$ given by $f(A) = \det(A)$ is smooth
- For $A \in M_n(\mathbb{R})$, left and right multiplication by A , the transpose map, and the inverse map $f(A) = A^{-1}$ are smooth.

PARTITIONS OF UNITY

1.5 Lemma. *Every open cover of a manifold has an (at most) countable subcover.*

PROOF Let \mathcal{S} be any open cover of M , and let \mathcal{B} be a countable basis for the topology on M . For each $p \in M$, choose $U_p \in \mathcal{S}$ with $p \in U_p$, then choose $B_p \in \mathcal{B}$ with $p \in B_p \subseteq U_p$. Then $\{B_p : p \in M\} \subseteq \mathcal{B}$ is an open cover of M , and it is a subset of \mathcal{B} , so it is (at most) countable; but then $\{U_p : p \in M\}$ gives an at most countable subcover of \mathcal{S} . ■

As a result, every manifold has a countable basis \mathcal{B} such that for each $B \in \mathcal{B}$, there is a chart $\phi : U \rightarrow \phi(U)$ on M with $\phi(U) = B(0, 2)$ and $\phi(B) = B(0, 1)$.

1.6 Lemma. *Let M be a manifold, and let \mathcal{S} be any open cover of M . Then there exists an at most countable open cover \mathcal{B} of M such that*

1. *for each $B \in \mathcal{B}$ there is a chart $\phi_B : C_B \rightarrow \phi_B(C_B) = B(0, 1)$ with $B \subseteq C_B \subseteq U_B \subseteq \mathcal{S}$ for some $U_B \in \mathcal{S}$ and $\phi_B(B) = B(0, 1)$.*

2. $\{C_B : B \in \mathcal{B}\}$ is locally finite, meaning that every point in M has an open neighbourhood which only intersects with finitely many of the sets C_B , $B \in \mathcal{B}$ (and hence also the sets \overline{B} , $B \in \mathcal{B}$).

PROOF Choose a countable set $\mathcal{V} = \{V_1, V_2, \dots\}$ of regular coordinate balls which cover M with charts $\phi_i : W_i \rightarrow \phi_i(W_i) = B(0, 2)$ such that $V_i = \phi_i^{-1}(B(0, 1))$. We use the sets V_i to construct a strongly ascending chain of compact sets K_i in M with $K_i \subseteq H_{i+1}^{-1}$ for each i , and $M = \bigcup_{i=1}^{\infty} K_i$ as follows:

- Let $K_1 = \overline{V_1}$; since K_1 is compact, we can choose $\ell_1 \in \mathbb{N}$ so that $K_1 \subseteq V_1 \cup \dots \cup V_{\ell_1}$.
- Then we let $K_2 = \overline{V_1 \cup \dots \cup V_{\ell_1}}$. Since K_2 is compact, we can choose $\ell_2 > \ell_1$ so that $K_2 \subseteq V_1 \cup \dots \cup V_{\ell_2}$, and set $K_3 = \overline{V_1 \cup \dots \cup V_{\ell_2}}$.

Repeat the above process to obtain $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq K_3^\circ \subseteq \dots$ with $\bigcup_{i=1}^k K_i = M$. For each $m \in \mathbb{N}$, note that $K_{m+1} \setminus K_m^\circ$ is compact and contained in the open set $K_{m+2} \setminus K_{m-1}$ (with $K_0 = \emptyset$). For each $p \in K_{m+1} \setminus K_m^\circ$, choose $U_p \in \mathcal{S}$ with $p \in U_p$ and then choose a regular coordinate ball B_p and a chart $\phi_p : C_p \subseteq M \rightarrow \phi_p(C_p) = B(0, 2) \subseteq \mathbb{R}^n$ with $\phi_p(B_p) = B(0, 1)$ and $C_p \subseteq U_p \cap (K_{m+2}^\circ \setminus K_{m-1})$. The coordinate balls B_p , $p \in K_{m+1} \setminus K_m^\circ$ cover the compact set $K_{m+1} \setminus K_m^\circ$, so we can choose a finite set \mathcal{B}_m of such regular coordinate balls B_p so that $K_{m+1} \setminus K_m^\circ \subseteq \bigcup \mathcal{B}_m \subseteq K_{m+2}^\circ \setminus K_{m-1}$.

Now, the set $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ is a countable set of such regular coordinate balls. Note that for each $B \in \mathcal{B}$, we have chart $\phi_B : C_B \rightarrow \phi_B(C_B) = B(0, 2)$ and the set $\{C_B : B \in \mathcal{B}\}$ is locally finite since every point in M is contained in one of the sets $K_{m+2}^\circ \setminus K_{m-1}$ and each of these sets only intersects with the coordinate balls from the finite sets \mathcal{B}_l with $m-2 \leq l \leq m+2$. ■

1.7 Theorem. (Partitions of Unity) Let M be a smooth manifold, and let \mathcal{S} be any open cover of M . There exists a set $\{\psi_u : u \in \mathcal{S}\}$ of smooth maps $\psi_u : M \rightarrow \mathbb{R}$ such that

1. $\psi_u(M) \subseteq [0, 1]$ for each $u \in \mathcal{S}$
2. $\text{supp}(\psi_u) \subseteq U$ for each $U \in \mathcal{S}$
3. $\{\text{supp}(\psi_u) : u \in \mathcal{S}\}$ is locally finite: every point in M contains an open neighbourhood which only intersects finitely many of the sets $\text{supp}(\psi_u)$, $u \in \mathcal{S}$
4. $\sum_{u \in \mathcal{S}} \psi_u = 1$

Such a set of functions $\{\psi_u : u \in \mathcal{S}\}$ is called a (smooth) **partition of unity** on M for \mathcal{S} (or **subordinate** to \mathcal{S}).

PROOF Let \mathcal{B} be a countable set of regular coordinate balls as in the previous lemma. Recall that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{1/t} & : t < 0 \\ 0 & : t \geq 0 \end{cases}$$

is smooth, so the function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $g(x) = f(|x|^2 - 1)$ is smooth with $g(x) > 0$ for $|x| < 1$ and $g(x) = 0$ for $|x| \geq 1$. For each $B \in \mathcal{B}$, we define a smooth bump function $\sigma_B : M \rightarrow \mathbb{R}$ by

$$\sigma_B(p) = \begin{cases} g(\phi_B(p)) & : p \in B \\ 0 & : p \notin B \end{cases}$$

where $\phi_B : C_B \subseteq M \rightarrow \phi_B(C_B) = B(0, 2)$ with $\phi_B(B) = B(0, 1)$ as in the previous lemma. Note that $\sigma(B)$ is smooth with $\sigma_B(p) > 0$ for $p \in B$ and $\sigma_B(p) = 0$ for $p \notin B$. Now for each $B \in \mathcal{B}$,

define $\tau'_B : M \rightarrow \mathbb{R}$ by

$$\tau_B = \frac{\sigma_B}{\sum_{c \in \mathcal{B}} \sigma_c}$$

Note that $\sum_{c \in \mathcal{B}} \sigma_c$ is well-defined by the local finiteness of \mathcal{B} and $\sum_{c \in \mathcal{B}} \sigma_c(p) > 0$. Furthermore, note that $\tau_B(p) > 0$ for all $p \in B$, and $\tau_B(p) = 0$ for all $p \notin B$, and $\sum_{B \in \mathcal{B}} \tau_B = 1$. Then define $\rho_V : M \rightarrow \mathbb{R}$ by $\rho_V = \sum_{B \in \mathcal{B}_V} \tau_B$. ■

2 IMMERSIONS, EMBEDDING, SUBMANIFOLDS

2.1 Theorem. (Inverse Function Theorem) Let $U \subseteq \mathbb{R}^n$ be open, $p \in U$, and $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be smooth and suppose $Df(p)$ is invertible. Then f is a local diffeomorphism.

2.2 Corollary. Let $n < m$ and $U \subseteq \mathbb{R}^n$ be open, and let $p \in U$, and $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ be smooth and suppose $Df(p)$ has rank n . Then the range of f is locally equal to the graph of a smooth function. Such a map f is called a local **immersion** at p .

PROOF Since $Df(p)$ is an $m \times n$ matrix of rank n , some n rows of $Df(p)$ form an invertible submatrix. Reorder the variables in \mathbb{R}^m (if necessary) so that the top n rows form an invertible matrix. Write elements in $U \subseteq \mathbb{R}^n$ as t and write elements of \mathbb{R}^m as (x, y) . Also write $(x, y) = f(t) = (u(t), v(t))$ so

$$Df = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

with $\frac{\partial u}{\partial t}(p)$ invertible. Then by the inverse function theorem, $u(t)$ is a local diffeomorphism. Say $u : U_0 \subseteq U \rightarrow V_0 \subseteq \mathbb{R}^n$ is the diffeomorphism, and let $g : V_0 \rightarrow U_0$ be its inverse. Then the range of f is locally equal to the graph of the function $y = v(g(x)) =: h(x)$. If $(x, y) \in \Gamma(f)$ with $(x, y) = f(t) = (u(t), v(t))$, then since $x = u(t)$ we have $t = g(x)$ so $y = v(t) = v(g(x)) = h(x)$. If $(x, y) \in \Gamma(h)$, then $y = h(x) = v(g(x))$ and we can choose $t = g(x)$ to get $x = u(t)$ and $y = v(g(x)) = v(t)$ so that $(x, y) = (u(t), v(t)) = f(t)$. ■

2.3 Theorem. (Implicit Function) Let $n < m$, $U \subseteq \mathbb{R}^m$ be open, $p \in U$, and $f : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ be smooth. Suppose $Df(p)$ has rank n and let $q = f(p)$. Then the level set $f^{-1}(q)$ is locally equal to a graph of a smooth function.

2.4 Theorem. Let $U \subseteq \mathbb{R}^n$ be open with $p \in U$, let $f : U \rightarrow \mathbb{R}^m$ be smooth with $f(p) = q$ and suppose that Df has constant rank r in U . Then the level set (or fibre) $f^{-1}(q)$ is locally equal to the graph of a smooth function (with $n - r$ independent variables and r dependent variables).

PROOF Since Df is an $m \times n$ matrix of rank r , there is some $r \times r$ submatrix of $Df(p)$ which is invertible; without loss of generality, it is the upper left submatrix. Write elements in \mathbb{R}^n as (x, y) with $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^{n-r}$ and write elements in \mathbb{R}^m as (u, v) with $u \in \mathbb{R}^r$ and $v \in \mathbb{R}^{m-r}$, with say $p = (a, b)$ and $q = f(p) = (c, d)$. Then we have $(u, v) = f(x, y) = (u(x, y), v(x, y))$ so that

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

with $\frac{\partial u}{\partial x}(p) = \frac{\partial u}{\partial x}(a, b)$ being an invertible $r \times r$ matrix. Define $F : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $F(x, y) = (u(x, y), y)$. Then

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & I \end{pmatrix}$$

so that $DF(p)$ is invertible. By the IVT, F is a local diffeomorphism, say $F : U_0 \subseteq U \subseteq \mathbb{R}^m \rightarrow V_0 \subseteq \mathbb{R}^m$ is a diffeomorphism with U_0 an open rectangular box. Let $G : V_0 \rightarrow U_0$ denote the smooth inverse of F . Note that G is of the form $G(u, y) = (g(u, y), y)$ for some smooth function $g : V_0 \rightarrow \mathbb{R}^r$. We claim that $f^{-1}(q) = f^{-1}(c, d)$ is locally equal to the graph of $x = g(c, y)$. First, note that

$$(u, y) = F(G(u, y)) = F(g(u, y), y) = (u(g(u, y), y), y)$$

so that, in particular, $u(g(u, y), y) = u$ and so

$$f(G(u, y)) = (u(g(u, y), y), v(g(u, y), y)) = (u, h(u, y))$$

where $h(u, y) = v(g(u, y), y)$. Thus

$$Df(x, y) \cdot DG(u, y) = D(f \circ G)(u, y) = \begin{pmatrix} I & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial y} \end{pmatrix}$$

Since Df has constant rank r and DG is invertible, the matrix on the right is of rank r for all $(u, v) \in V_0$. Thus it follows that $\frac{\partial h}{\partial y} = 0$ for all u, b , so that $h(u, y)$ is independent of y and $h(u, y) = h(u, b)$ for all y ; let $k(u) = h(u, b)$. Let us calculate $k(c)$. We have

$$\begin{aligned} f(a, b) = (c, d) &\implies (u(a, b), v(a, b)) = (c, d) \\ &\implies u(a, b) = c \\ &\implies F(a, b) = (u(a, b), b) = (c, b) \\ &\implies (a, b) = G(c, b) \\ &\implies (c, d) = f(a, b) = f(G(c, b)) = (c, h(c, b)) = (c, k(c)) \\ &\implies k(c) = d \end{aligned}$$

Finally, let us show that $f^{-1}(c, d)$ is (locally) the graph of $x = g(c, y)$. We have

$$\begin{aligned} (x, y) = f^{-1}(c, d) &\implies f(x, y) = (c, d) \\ &\implies u(x, y) = c \text{ and } v(x, y) = d \\ &\implies F(x, y) = (u(x, y), y) = (c, y) \\ &\implies (x, y) = G(c, y) = (g(c, y), y) \\ &\implies x = g(c, y) \end{aligned}$$

We thus have

$$\begin{aligned} x = g(c, y) &\implies G(c, y) = (g(c, y), y) = (x, y) \\ &\implies f(x, y) = f(G(c, y)) = (c, h(c, y)) = (c, k(c)) = (c, d) \end{aligned}$$

as required. ■

Definition. When N and M are smooth manifolds and $f : N \rightarrow M$ is a smooth map, we say that f has **rank r** at $p \in N$ when for some (hence for every) chart ϕ on N at p and for some (hence every) chart ψ on M at $f(p)$, the matrix $D(\psi f \phi^{-1})(\phi(p))$ has rank r .

2.5 Corollary. Let N and M be smooth manifolds, with $p \in N$. Let $f : N \rightarrow M$ be smooth with $f(p) = q \in M$. Suppose f has constant rank r in an open neighbourhood of p . Then there exists a chart ϕ on N at p and a chart ψ on M at $q = f(p)$ such that $\phi(p) = 0$ and $\psi(q) = 0$ and

$$(\psi \circ f \circ \phi^{-1})(x^1, \dots, x^r, \dots, x^n) = (x^1, \dots, x^r, 0, \dots, 0)$$

where $n = \dim(N)$ and $m = \dim(M)$.

PROOF Choose any chart ϕ_0 on N at p and any chart ψ_0 on M at q with $\phi_0(p) = 0$ and $\psi_0(q) = 0$. Then $D(\psi_0 f \phi_0^{-1})$ has constant rank r near 0. Let ϕ_1 and ψ_1 be linear permutation maps so that the upper left $r \times r$ submatrix of $D(\psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1})(0)$. Say $f_1 = \psi_1 \psi_0 f \phi_0^{-1} \phi_1^{-1}$. Let F, G, f_1 be as in the proof of the rank theorem (for the function f_1). Let us verify that for the charts $\phi = F \phi_1 \phi_0$ and $\psi = H \psi_1 \psi_0$ where $H(u, v) = (u, v - k(u))$ we have $(\psi f \phi^{-1})(u, y) = (u, 0)$. ■

2.6 Corollary. When $f : M \rightarrow N$ is a smooth map of smooth manifolds with constant rank r in M , for $q \in \text{im } f$, the level set (fibre) $f^{-1}(q)$ can be given charts (obtained from canonical charts) to make it a smooth $(\dim M - r)$ -dimensional manifold.

Definition. Let N and M be smooth manifolds (of dimensions m and n). A smooth map $f : N \rightarrow M$ is called a (smooth) **immersion** when f has rank n in N . An **immersed submanifold** of M is the image of an immersion $f : N \rightarrow M$ or the image of an injective immersion $f : N \rightarrow M$.

Note that when $f : N \rightarrow M$ is injective, we can give the image $f(N)$ a smooth atlas which makes $f : N \rightarrow f(N)$ a diffeomorphism. When we do this, the resulting topology on $f(N) \subseteq M$ does not necessarily agree with the subspace topology of M .

Definition. An **embedded submanifold** of M is a subset $N \subseteq M$ which is a smooth manifold such that the inclusion map $f : N \rightarrow M$ (given by $f(p) = p$) is an immersion such that the topology in the previous remark agrees with the subspace topology.

When $f : M \rightarrow N$ is a smooth map of smooth manifolds of constant rank r and $q \in \text{im } f$, the level set $f^{-1}(q)$ is an embedded submanifold of M .

Remark. When $N \subseteq M$ is an embedded submanifold,

- If $f : M \rightarrow K$ is smooth, then the restriction $f : N \rightarrow K$ is smooth
- If $f : K \rightarrow M$ is smooth and $f(K) \subseteq N$, then $f : K \rightarrow N$ is smooth

Example. $\text{SL}_n(\mathbb{R})$ is a smooth manifold. Recall that $\text{GL}_n(\mathbb{R})$ is a smooth n^2 -dimensional manifold, since it is open in the n^2 -dimensional vector space $M_n(\mathbb{R})$. We have $\text{SL}_n(\mathbb{R}) = f^{-1}(\{1\})$ where f is the determinant evaluation map. Then for fixed ℓ , $\det X = \sum_{i=1}^n (-1)^{i+\ell} X_{\ell}^i \deg X_{(\ell)}^{(i)}$, where $X_{(\ell)}^{(i)}$ is the matrix obtained from X by removing row i and column ℓ . We have

$$Df = \left(\mathbb{P} f x_1^1, \dots, \frac{\partial f}{\partial x_n^n} \right) \in M_{1 \times n^2}(\mathbb{R})$$

with $\frac{\partial f}{\partial x_{\ell}^{\ell}} = (-1)^{k+\ell} \det X_{(\ell)}^{(k)}$, so that $Df = 0$ if and only if $\det X = 0$. Thus f has constant rank 1, so $\text{SL}_n(\mathbb{R}) = f^{-1}(1)$ is an embedded submanifold of $M_n(\mathbb{R})$ of dimension.

3 TANGENT VECTORS

Definition. A vector u in \mathbb{R}^n at a point $a \in \mathbb{R}^n$ is an ordered pair (a, u) .

Definition. Let M be a smooth manifold and let $p \in M$. A **tangent vector** on M at p is a set of vectors $X = \{\phi_*x : \phi \text{ is a chart on } M \text{ at } p\}$, where ϕ_*x is a vector in \mathbb{R}^n at the point $x = \phi(p)$ such that when ϕ and ψ are two charts on M at p , we have $\psi_*X = D(\psi\phi^{-1})(\phi(p))\phi_*X$.

The set of all tangent vectors on M at p is denoted by T_pM . Note that T_pM is an n -dimensional vector space. When $I \subseteq \mathbb{R}$ is an open interval, $s \in I$, and $\alpha : I \subseteq \mathbb{R} \rightarrow M$ is a smooth map with $\alpha(s) = p$, we define $\alpha'(s)$ to be the tangent vector $\alpha'(s) \in T_pM$ given by $\phi_*\alpha'(s) = \beta'(s)$ where $\beta(t) = \phi(\alpha(t))$. Note that, by the chain rule, we do have $\phi_*\alpha'(s) = D(\psi\phi^{-1})\phi_*\alpha'(s)$.

When ϕ is a chart on M at p , we often write

$$x = x(p) = \phi(p) = (\phi^1(p), \dots, \phi^n(p)) = (x^1(p), \dots, x^n(p))$$

(so each $x^k = \phi^k$ is a function $x^k, \phi^k : U \subseteq M \rightarrow \mathbb{R}$). When ψ is another chart and we write $y = \psi(p)$, we often write $y = y(x) = (\psi\phi^{-1})(x) = (y^1(x), \dots, y^n(x))$ and we write

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

With this notation, if $u = \phi_*X$ and $v = \psi_*X$, then $v = D(\psi\phi^{-1})u = \frac{\partial y}{\partial x}u$, so $V^k = \sum_{i=1}^n \frac{\partial y^k}{\partial x^i} u^i$.

Definition. Let $f : M \rightarrow N$ be a smooth map of smooth manifolds with $p \in M$. We define the **induced map** or the **pushforward** f_* or the **differential** df to be the map $f_* = df : T_pM \rightarrow T_{f(p)}N$ given as follows. Given $X \in T_pM$, choose $\alpha : (-\epsilon, \epsilon) \rightarrow M$ smooth with $\alpha(0) = p$, $\alpha'(0) = X$, when we let $\beta(t) = f(\alpha(t))$ and define $df(x) = f_*(x) = \beta'(0)$. Given a chart ϕ on M at p and ψ on N at $f(p)$, if $u = \phi_*X$ and $V = \psi_*(f_*X)$, then verify that $v = D(\psi f \phi^{-1})(\phi(p))u$.

1. When ϕ is a chart on M at p and ψ is a chart on N at $f(p)$, $\psi_*f_*X = D(\psi f \phi^{-1})_{\phi(p)}\phi_*X$
2. The map $df = f_*$ is linear
3. If $g : L \rightarrow M$ and $f : M \rightarrow N$ are smooth, then $(f \circ g)_* = f_* \circ g_*$.
4. When $\iota : M \rightarrow M$ is the identity map, $d\iota : T_pM \rightarrow T_pM$ is the identity map
5. If $f : M \rightarrow N$ is a diffeomorphism, then $f_* : T_pM \rightarrow T_pM$ is an isomorphism.
6. For $f : M \rightarrow N$ smooth, f is of rank r at p if and only if f_* is of rank r at p .

When $U \subseteq \mathbb{R}^n$ is open, U is a manifold with atlas $\{\emptyset\}$ where ϕ is the identity map. In this case, we identify $X \in T_pU$ with $\phi_*X \in \mathbb{R}^n$. With this convention, ϕ_*X is equal to ϕ_*X where the second ϕ_* is the pushforward. When $N \subseteq M$ is a submanifold (immersed or embedded), the inclusion map $\iota : N \rightarrow M$ is an injective immersion. Thus, the map $\iota_* : T_pN \rightarrow T_pM$. In this situation, we identify T_pN with the subspace $\iota_*(T_pN) \subseteq T_pM$.

Let X be the vector on \mathbb{S}^2 at p with $\phi_*X = (1, 0)$. Let $\iota : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ be the inclusion map. We have $\phi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ with $u = \phi_*X = (1, 0)$. Then $\iota_*X = D(\iota\phi^{-1})_{\phi(p)}\phi_*X$ where ψ is the identity on \mathbb{R}^3 .

TANGENT VECTORS AS DIFFERENTIAL OPERATORS

Recall that a vector $u \in \mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ acts as a differential operator on smooth maps $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by directional derivative. Choose any smooth map $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$ with

$\alpha(0) = a$ and $\alpha'(0) = u$, and define $u(f) = u_a(f) = D_u f(a) = \beta'(0)$ where $\beta(t) = f(\alpha(t))$. Since $\beta(t) = f(\alpha(t))$, we have $\beta'(t) = Df(\alpha(t)) \cdot \alpha'(t)$ so

$$\begin{aligned} u(f) &= D_u f(a) = \beta'(0) = Df(a) \cdot u \\ &= \left(\frac{\partial f}{\partial x^1}(a), \dots, \frac{\partial f}{\partial x^n}(a) \right) \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) \cdot u^i \end{aligned}$$

or as a differential operator, $u = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}$.

Definition. When M is a smooth manifold, $p \in M$, and $X \in T_p M$, X acts as a differential operator on a smooth function $f : M \rightarrow \mathbb{R}$ as follows: choose a smooth map $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow M$ with $\alpha(0) = p$ and $\alpha'(0) = X$, and define $X(f) = X_p(f) = \beta'(0)$ where $\beta(t) = f(\alpha(t))$.

When ϕ is a chart on M at p , then

$$\begin{aligned} X(f) &= (\phi_* X)(f \circ \phi^{-1}) = D_{\phi_* X}(f \circ \phi^{-1})(\phi(p)) \\ &= D(f \circ \phi^{-1})(\phi(p)) \cdot (\phi_* X) \\ &= \sum_{i=1}^n \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p)) \cdot u^i \end{aligned}$$

where $u = \phi_* X \in \mathbb{R}^n$. So when $u = \phi_* X \in \mathbb{R}^n$, X acts as the differential operator $X = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i} \Big|_p$ where $\frac{\partial}{\partial x^i} \Big|_p(f) = \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p))$. With this notation,

$$T_p M = \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

If ϕ and ψ are two charts at p on M , then $T_p M$ has two representations as differential operators. Let us determine how $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial y^\ell}$ are related. When $X \in T_p M$, $u = \phi_* X \in \mathbb{R}^n$ and $v = \psi_* X \in \mathbb{R}^n$, we have $V = D(\psi \circ \phi^{-1})(\phi(p)) \cdot u = \left(\frac{\partial y}{\partial x} \right)(\phi(p)) \cdot u$. When $u = \frac{\partial}{\partial x^j}$,

$$v = \left(\frac{\partial y}{\partial x} \right) \cdot e_k$$

so $v^\ell = \left(\frac{\partial y}{\partial x} \right)_k^\ell = \frac{\partial y^\ell}{\partial x^k}$ so that

$$v = \sum_{i=1}^n \frac{\partial y^i}{\partial x^k} \frac{\partial}{\partial y^i}$$

Definition. A **derivation** on M at p is a linear map $L : C^\infty(M) \rightarrow \mathbb{R}$ or $L : C_p^\infty(M) \rightarrow \mathbb{R}$ where $C_p^\infty(M)$ is the vector space of **germs** of smooth functions on M at p , which satisfies the product rule at p :

$$L(fg) = L(f) \cdot g(p) + f(p) \cdot L(g)$$

Every $X \in T_p M$ gives a derivation on M at p . Moreover, it can be shown that every derivation on M at p is of this form. Thus allows us to give an alternate definition for $T_p M$ as the space of derivations on M at p .

Definition. Let TM be the disjoint union of all the tangent spaces. A **vector field** on M is a function $X : M \rightarrow TM$ such that $X(p) \in T_p M$.

Given a chart $\phi : U \rightarrow \phi(U)$ on M , the restriction of X to U determines and is determined by the vector field $\phi_* X$ on $\phi(U) \subseteq \mathbb{R}^n$ by $(\phi_* X)(\phi(p)) = \phi_*(X(p))$, or $(\phi_* X)(x) = \phi_*(X(\phi^{-1}(x))) \in \mathbb{R}^n$. We say that X is **smooth** at p when for some chart ϕ on M at p , the vector field $\phi_* X$ is smooth at $\phi(p)$. When X is a smooth vector field on M , X acts as a differential operator $X : C^\infty(M) \rightarrow C^\infty(M)$ by $X(f)(p) = X_p(f)$.

The space of smooth vector fields on M is $\Gamma(M, TM) = \Gamma(TM) = \mathcal{X}(M)$.

THE PUSHFORWARD OR DIFFERENTIAL

If X is a smooth vector field on a smooth manifold N and $f : N \rightarrow M$ is a smooth map, for each point $p \in N$, we have the linear map $df = f_* : T_p N \rightarrow T_{f(p)} M$.

Note that f_* does not in general give a map $f_* : \Gamma(TN) \rightarrow \Gamma(TM)$, if f is not surjective, or f is not injective with $p, q \in N$ with $p \neq q$ and $f(p) = f(q)$ and $f_* X_p \neq f_* X_q$.

If $f : N \rightarrow M$ is a diffeomorphism, then f_* does give a well-defined bijective map $f_* : \Gamma(TN) \rightarrow \Gamma(TM)$. If f is an injective immersion, then $f : N \rightarrow f(N)$ is a diffeomorphism.

THE LIE BRACKET OF VECTOR FIELDS

Definition. When X and Y are two smooth vector fields on M , we define the **Lie bracket** of X and Y , denoted by $[X, Y](f)$, by $[X, Y]f = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.

Note that $[X, Y]$ satisfies the product rule since

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X(f \cdot Y(g) + g \cdot Y(f)) - Y(f \cdot X(g) + g \cdot X(f)) \\ &= f \cdot X(Y(g)) + X(f) \cdot Y(g) + g \cdot X(Y(f)) + X(g) \cdot Y(f) - Y(f \cdot X(g)) - Y(g \cdot X(f)) \\ &= g[X, Y](f) + g[X, Y](g) \end{aligned}$$

Given a chart $\phi : U \rightarrow \phi(U)$ on M at p , we can calculate a formula for the Lie bracket: say $u = \phi_* X$ and $v = \phi_* Y$ ($u(x) = \phi_*(X_{\phi^{-1}(x)})$, $v(x) = \phi_*(Y_{\phi^{-1}(x)})$). Then for $f \in C^\infty(M)$,

$$\begin{aligned} [X, Y]_p(f) &= X_p(Y(f)) - Y_p(X(f)) \\ &= \sum_i u^i \frac{\partial}{\partial x^i} \left(\sum_j v^j \frac{\partial g}{\partial x^j} \right) - \sum_i v^i \frac{\partial}{\partial x^i} \left(\sum_j u^j \frac{\partial g}{\partial x^j} \right) \\ &= \sum_{i,j} \left(u^i \frac{\partial v^j}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} + u^i v^j \frac{\partial^2 g}{\partial x^i \partial x^j} - v^i \frac{\partial u^j}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} - v^i u^j \frac{\partial^2 g}{\partial x^i \partial x^j} \right) \\ &= \sum_{i,j} \left(\frac{\partial v^j}{\partial x^i} \cdot u^i - \frac{\partial u^j}{\partial x^i} \cdot v^i \right) \frac{\partial g}{\partial x^j} \end{aligned}$$

Thus $[X, Y]_p$ is a vector in $T_p M$. It is the vector given by $w^j = \sum_i \left(\frac{\partial v^j}{\partial x^i} u^i - \frac{\partial u^j}{\partial x^i} v^i \right)$ and $w = \sum_j w^j \frac{\partial}{\partial x^j} = Dv \cdot u - Du \cdot v$.

INTEGRAL CURVES AND FLOWS

Given a smooth vector field X on a smooth manifold M , and given $p \in M$, the existence and uniqueness theorem for (systems) of ODEs guarantees that there is a unique smooth map (or curve) $\alpha_p : I_p \subseteq \mathbb{R} \rightarrow M$ where I is the (unique) maximal open interval α and $\alpha(0) = p$ and $\alpha'(t) = X_{\alpha(t)}$. A stronger version of the existence and uniqueness theorem also guarantees that $\alpha_p(t)$ varies smoothly with p to give a unique smooth map $\theta : U \subseteq M \times \mathbb{R} \rightarrow M$ where U is the (unique) maximal open connected domain given by $\theta(p, t) = \alpha_p(t)$.

Example. (i) Find a vector field which is a parabola at each point.

(ii) Find a smooth vector field so that the solution curves have vertical asymptote.

When a vector field X on a 2 dimensional manifold M , we define the **index** of X at p as follows. Choose a chart $\phi : C \rightarrow \phi(C) = B(0, 2)$ on M at p . Thus $U = \phi_*X$ is a vector field in \mathbb{R}^2 with no zeros in $B(0, 2)$ except at 0.

When we restrict u to the circle \mathbb{S}^1 and we define the index of X at p to be the winding number of this map $u : \mathbb{S}^1 \rightarrow \mathbb{C} \setminus \{0\}$. When a vector field on X has finitely many isolated zeros, the index of X is the sum of the indices at the zeros of X .

3.1 Theorem. When X is a smooth vector field with isolated zeros on a **compact** 2-dimensional manifold M , $\text{Ind } X = \chi(M)$, the Euler characteristic of M .

4 LIE GROUPS

Definition. A Lie group G is both a smooth manifold and a group such that the group operations $\mu : G \times G \rightarrow G$ and inversion $\nu : G \rightarrow G$ are smooth maps.

Example. $O_n(\mathbb{R}) = \{A \in \text{GL}_n(\mathbb{R}) : A^T A = I\}$ is a Lie group. Define $F : \text{GL}_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ by $F(X) = X^T X$. Thus $O_n(\mathbb{R}) = F^{-1}(I)$. When $n = 2$, $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ we have $F(X) = \begin{pmatrix} x^2 + y^2 & xz + yw \\ xz + yw & z^2 + w^2 \end{pmatrix}$ so that

$$DF = \begin{pmatrix} 2x & 2y & 0 & 0 \\ z & w & x & y \\ z & w & x & y \\ 0 & 0 & 2z & 2w \end{pmatrix}$$

In general, for $A \in \text{GL}_n(\mathbb{R})$, $F(R_A(X)) = F(XA) = A^T X^T X A = L_{A^T} R_A F(X)$. Thus by the chain rule, $DF(XA) \cdot DR_A(X) = DL_{A^T}(X^T X A) \cdot DR_A(X^T X) \cdot DF(X)$, so we can identify $T_p \text{GL}_n(\mathbb{R})$ or $T_p M_n(\mathbb{R})$ with the vectr space $M_n(\mathbb{R})$. Note that L_{A^T} and R_A are diffeomorphisms of $\text{GL}_n(\mathbb{R})$, so DL_{A^T} and DR_A are invertible. Thus $\text{rank } DF(XA) = \text{rank } DF(X)$. In particular, taking $X = I$, $\text{rank } DF(A) = \text{rank } DF(I)$, so F has constant rank. Let us calculate $\text{rank } DF : T_I \text{GL}_n(\mathbb{R}) \rightarrow T_I M_n(\mathbb{R})$. Let $A \in T_I \text{GL}_n(\mathbb{R})$, so $A \in M_n(\mathbb{R})$, and let $\alpha(t) = I + tA$ so that $\alpha(0) = I$ and $\alpha'(0) = A$. Then $DF(I) \cdot A = \beta'(0)$ where $\beta(t) = F(\alpha(t)) = (I + tA)^T (I + t(A + A^T) + t^2 A^T A)$. Then $\beta'(t) = A + A^T + 2tA^T A$, so $\beta'(0) = A + A^T$ so that $DF(I) \cdot A = A + A^T$. The range of DF at I is the set of matrices B of the form $B = A + A^T$ for some matrix $A \in M_n(\mathbb{R})$, or equivalently, the set of symmetric matrices in $M_n(\mathbb{R})$. Thus the dimension of the range of DF is $(n^2 + n)/2$, so F has constant rank $r = (n^2 + n)/2$ and thus $\dim O_n(\mathbb{R}) = n^2 - r = \frac{n^2 - n}{2}$.

Thus by the constant rank theorem, $O_n(\mathbb{R})$ is a regular embedded submanifold of $\text{GL}_n(\mathbb{R})$. In fact, $T_I O_n(\mathbb{R})$ can be identified with $\ker DF(I) \subseteq T_I \text{GL}_n(\mathbb{R}) = M_n(\mathbb{R})$, which is

$\{A \in M_n(\mathbb{R}^n) : A^T + A = 0\}$. One can do the same for $U_n(\mathbb{C}) = \{A \in GL(\mathbb{C}) : A^*A = I\}$ and $A^* = \overline{A}^T$.

Definition. When $f : M \rightarrow M$ is a diffeomorphism and $X \in \Gamma(M, TM)$, we say that X is **invariant** under f when $f_*X = X$ (where $f_*(X_p) = X_{f(p)}$ for all $p \in M$). When G is a Lie group and $X \in \Gamma(G, TG)$, we say that X is **left-invariant** when X is invariant under the left multiplication map $\ell_a : G \rightarrow G$ where $\ell_a(p) = ap$ for all $a \in G$.

Note that $(\ell_a)_*(X) = X$ for all $a \in G$.

On the other hand, if we define a vector field X on G by the formula $X_a = (\ell_a)_*A$ where $A \in T_eG$, then X is left invariant since for all $a, b \in G$,

$$(\ell_a)_*X_b = ((\ell_a)_* \circ (\ell_b)_*)(X_e) = X_{ab}$$

Definition. A **Lie algebra** is a vector space V with an alternating bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ which satisfies the Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$.

Example. $M_n(\mathbb{R})$ is a Lie algebra using $[A, B] = AB - BA$, as one can verify directly. More generally, when V is a vector space, $\text{End } V$ is a Lie algebra with Lie bracket $[A, B] = AB - BA$. For example, when M is a smooth manifold, $X(M) = \Gamma(M, TM)$ is a vector space with Lie bracket $[X, Y](f) = X(Y(f)) - Y(X(f))$.

Given $A \in T_eG$, there is a unique left invariant vector field X on G with $X_e = A$, and X is given by $X_p = (\ell_p)_*A$. By the assignment if X and Y are left-invariant vector fields on a Lie group G , then $[X, Y]$ is left invariant since $(\ell_a)_*[X, Y] = [(\ell_a)_*X, (\ell_a)_*Y] = [X, Y]$.

Definition. For a Lie group G , the **Lie algebra** of G , denoted by \mathfrak{g} , is the Lie algebra of left-invariant vector fields on G .

Equivalently, we may define $\mathfrak{g} = T_eG$ with the corresponding Lie algebra given by $[A, B] = [X, Y]_e$, where $A, B \in T_eG = \mathfrak{g}$, and X, Y are the left invariant vector fields on G with $X_e = A$ and $Y_e = B$.

Definition. A **Lie subgroup** of a Lie group G is a subgroup $H \subseteq G$ that is also an immersed (or embedded) submanifold.

Let G be a Lie subgroup of $GL_n(\mathbb{R})$. We identify $T_pGL_n(\mathbb{R})$ with $M_n(\mathbb{R})$, and we identify T_pG with a subspace of $M_n(\mathbb{R})$.

Example. 1. Given $A \in T_I G \subseteq M_n(\mathbb{R})$, find a formula for $U_p = U(P)$, where $P \in G \subseteq M_n(\mathbb{R})$ and U is the left-invariant vector field on G with $U_I = A$.

We have $U_p = (L_p)_*A$, where $L_p : G \rightarrow G$ is given by $L_p(X) = PX$. Note that L_p is the restriction of the map $L_p : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$. This map L_p is linear, so DL_p is equal to L_p as a linear map on $M_n(\mathbb{R})$. Thus we have $U_p = (P_p)_*(A) = (DL_p)A = PA$.

2. Given $A, B \in \mathfrak{G} = T_I G \subseteq M_n(\mathbb{R})$, let U and V be given by $U(P) = PA$ and $V(P) = PB$. Note that $U = R_A$, $V = R_B$, so $DU = R_A$ and $DV = R_B$ as linear maps on $M_n(\mathbb{R})$, and we have

$$[A, B] = [U, V]_I = DV(I)U(I) - DU(I)V(I) = R_B(A) - R_A(B) = AB - BA$$

3. Let $A \in \mathfrak{G} = T_I(G) \subseteq M_n(\mathbb{R})$, let $U(P) = PA$. We need to find the integral curve $\alpha : I \subseteq \mathbb{R} \rightarrow G$ with $\alpha(0) = I$. Then we want $\alpha'(t) = U(\alpha(t)) = \alpha(t)A$ for all t . The solution to this DE is given by $\alpha(t) = e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \dots$ so that $\alpha'(t) = (e^{tA})A$. As a consequence of the above formula, note that $\mathfrak{g} = \{A \in M_n(\mathbb{R}) : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$.

Thus formula allows us to give an explicit description of the Lie algebras of many Lie subgroups of $GL_n(\mathbb{R})$.

Given $A \in M_n(\mathbb{R})$, $\det e^A = e^{\text{tr} A}$. By Schur's Theorem or the Jordan Normal Form, there is a matrix $P \in GL_n(\mathbb{C})$ so that $P^{-1}AP = T$ where T is upper triangular, so that

$$\det e^A = \det(Pe^T P^{-1}) = \det e^T = e^{\text{tr} A}$$

Recall when G is a Lie subgroup of $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ and if $J = T_I G \subseteq T_I GL_n(\mathbb{R})$, the left invariant vector field U on G with $U(I) = A \in J$ is given by $U(P) = PA$. The Lie bracket on J is given by $[A, B] = AB - BA$, and the integral curve of $U(P) = PA$ is given by $\alpha : \mathbb{R} \rightarrow G$ where $\alpha(t) = e^{tA}$, and hence

$$J = \{A \in M_n(\mathbb{R}) : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$$

For example, the Lie algebra of $SL_n(\mathbb{R})$ is

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : e^{tA} \in SL_n(\mathbb{R}) \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \det e^{tA} = 1 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : e^{\text{tr} tA} = 1 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \text{tr}(tA) = 0 \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : \text{tr}(A) = 0\} \end{aligned}$$

The Lie algebra of $O_n(\mathbb{R})$ is

$$\begin{aligned} \mathfrak{o}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}) : e^{tA} \in O_n(\mathbb{R}) \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : (e^{tA})^T (e^{tA}) = I \forall t\} \\ &= \{A \in M_n(\mathbb{R}) : (e^{tA^T})(e^{tA}) = I \forall t\} \end{aligned}$$

If $(e^{tA^T})(e^{tA}) = I$ for all $t \in \mathbb{R}$, then $\frac{d}{dt}(e^{tA^T})(e^{tA}) = \frac{d}{dt}I$ so that

$$(e^{tA^T} A^T (e^{tA}) + (e^{tA^T})(e^{tA}) \cdot A = 0$$

and taking $t = 0$ gives $A^T + A = 0$. Then $A^T = -A$ so $tA^T = -tA$ so $e^{tA^T} = e^{-tA} = (e^{tA})^{-1}$ for all t , so $e^{tA^T} \cdot e^{tA} = I$ for all t . Thus $\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$.

Table of Lie algebras:

G	\mathfrak{g}
$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$GL_n^+(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A > 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$	$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \text{tr} A = 0\}$
$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T A = I\}$	$\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$
$SO_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T A = I, \det A = 1\}$	$\mathfrak{so}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\}$
$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A \neq 0\}$	$\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$
$SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A = 1\}$	$\mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \text{tr} A = 0\}$
$O_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : A^T A = I\}$	$\mathfrak{o}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A + A^T = 0\}$
$SO_n(\mathbb{C}) = \{A \in SL_n(\mathbb{C}) : A^T A = I, \det A = 1\}$	$\mathfrak{so}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \text{tr} A = 0, A + A^T = 0\}$
$U_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : A^* A = I\}$	$\mathfrak{u}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A^* + A = 0\}$
$SU_n(\mathbb{C}) = \{A \in SL_n(\mathbb{C}) : A^* A = I\}$	$\mathfrak{su}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \text{tr} A = 0, A^* + A = 0\}$

5 SMOOTH k -FORMS

Suppose $\alpha : I \subseteq \mathbb{R} \rightarrow U \subseteq \mathbb{R}^3$ and let $f : U \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$, then the length of α is

$$\int_C dL = \int_\alpha dL = \int_{t \in I} |\alpha'(t)| dt$$

and

$$\int_C f dL = \int_\alpha f dL = \int_{t \in I} f(\alpha(t)) |\alpha'(t)| dt$$

Given $\sigma : R \subseteq \mathbb{R}^2 \rightarrow U \subseteq \mathbb{R}^3$, $f : U \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$, the area of $\text{im } \sigma$ is given by

$$\begin{aligned} \sigma(s, t) &= (x(s, t), y(s, t), z(s, t)) \\ D\sigma &= \begin{pmatrix} \frac{\partial}{\partial s} x(s, t) & \frac{\partial}{\partial t} x(s, t) \\ \frac{\partial}{\partial s} y(s, t) & \frac{\partial}{\partial t} y(s, t) \\ \frac{\partial}{\partial s} z(s, t) & \frac{\partial}{\partial t} z(s, t) \end{pmatrix} \end{aligned}$$

and denote σ_s, σ_t as the respective columns, so

$$A = \int_S dA = \int_\sigma dA = \iint_{(s,t) \in R} |\sigma_s(s, t) \times \sigma_t(s, t)| ds dt$$

and

$$\int_S f dA = \int_\sigma f dA = \iint_{(s,t) \in R} f(\sigma(s, t)) |\sigma_s \times \sigma_t| ds dt$$

For $\alpha : I \subseteq \mathbb{R} \rightarrow U \subseteq \mathbb{R}^3$, $F : U \rightarrow \mathbb{R}^3$, say $F = (P, Q, R)$, then

$$\begin{aligned} W &= \int_C F \cdot T dL = \int_\alpha F \cdot T dL \\ &= \int_{t \in I} F(\alpha(t)) \cdot \frac{\alpha'(t)}{|\alpha'(t)|} |\alpha'(t)| dt \\ &= \int_{t \in I} (P(\alpha(t))x'(t) + Q(\alpha(t))y'(t) + R(\alpha(t))z'(t)) dt \\ &= \int_\alpha P dx + Q dy + R dz \end{aligned}$$

Definition. A **smooth k -form** in $U \subseteq \mathbb{R}^n$ is an expression of the form $a(x) = \sum_I a_I(x) dx^I$ where the sum is taken over multi-indices $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \dots < i_k \leq n$ and each $a_I : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth map and $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

For a smooth map $\sigma : R \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^n$ and $s = \text{im } \sigma$ and for $a(x) = \sum_I a_I(x) dx^I$, we define

$$\int_S a = \int_\sigma a := \sum_I \int_R a_I(\sigma(t)) \det \left(\frac{\partial x^I}{\partial t} \right) dt^{i_1} \dots dt^{i_k}$$

where

$$\frac{\partial x^I}{\partial x} = \begin{pmatrix} \frac{\partial x^{i_1}}{\partial t^1} & \dots & \frac{\partial x^{i_1}}{\partial t^k} \\ \vdots & & \vdots \\ \frac{\partial x^{i_k}}{\partial t^1} & \dots & \frac{\partial x^{i_k}}{\partial t^k} \end{pmatrix}$$

For $a(x) = \sum_I a_I(x) dx^I$, we define $da = \sum_I \sum_j \frac{\partial a_I}{\partial x_j} dx^j \wedge dx^I$, using the rule $dx^j \wedge dx^i = -dx^i \wedge dx^j$. With this notation, Gauss' Theorem and Stoke's Theorem become

$$\int_S d\alpha = \int_{\partial S} \alpha$$

where $S = \text{im } \sigma$, $\sigma : R \subseteq \mathbb{R}^{k+1} \rightarrow \mathbb{R}^n$, $\alpha = \sum a_I dx^I$ is a k -form, and $d\alpha$ is a $(k+1)$ -form.

THE EXTERIOR ALGEBRA

If V is a vector space with basis $\{u_1, \dots, u_n\}$, then the dual space $V^* = \{\text{linear maps } g : V \rightarrow \mathbb{R}\}$ has dual basis $\{f^1, \dots, f^n\}$ where each $f^k : V \rightarrow \mathbb{R}$ and $f^k(u_\ell) = \delta_\ell^k$.

We have a canonical evaluation map $E : V \rightarrow V^{**}$ given by $E(v)(g) = g(v)$, which is an isomorphism.

The space $\Lambda^k V = \{\text{alternating } k\text{-linear maps } L : V^* \times \dots \times V^* \rightarrow \mathbb{R}\}$ has a basis

$$\{U_i = U_{i_1} \wedge \dots \wedge U_{i_k} : I \text{ is an increasing multi-index}\}$$

where for $v^i \in V$ and $g^i \in V^*$,

$$(v_1 \wedge \dots \wedge v_k)(g^1, \dots, g^k) = \det \begin{pmatrix} g^1(v_1) & \dots & g^1(v_k) \\ \vdots & & \vdots \\ g^k(v_1) & \dots & g^k(v_k) \end{pmatrix}$$

Also $\Lambda^k V^*$ has basis given similarly.

\mathbb{R}^n has standard basis $\{e_1, \dots, e_n\}$, which we can consider as differential operators $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$. The dual basis for $(\mathbb{R}^n)^*$ is denoted by $\{dx^1, \dots, dx^n\}$ where $dx^k(\frac{\partial}{\partial x^\ell}) = \delta_\ell^k$. So for example,

$$(dx^{i_1} \wedge \dots \wedge dx^{i_k}) \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \begin{cases} (-1)^\sigma & : J = \sigma(I) \\ 0 & : \text{otherwise} \end{cases}$$

A smooth k -form on $U \subseteq \mathbb{R}^n$ is a smooth map $\alpha : U \subseteq \mathbb{R}^n \rightarrow \Lambda^k(\mathbb{R}^n)^*$. Note that $dx^I : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$ has

$$\begin{aligned} dx^I(u_1, \dots, u_k) &= dx^I \left(\sum_{j_1=1}^n u_1^{j_1} \frac{\partial}{\partial x^{j_1}}, \dots, \sum_{j_k=1}^n u_k^{j_k} \frac{\partial}{\partial x^{j_k}} \right) \\ &= \sum_{\text{all } J} u_1^{j_1} \dots u_k^{j_k} \underbrace{dx^I \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right)}_{(-1)^\sigma \text{ if } J=\sigma(I); 0 \text{ otherwise}} \\ &= \sum_{\sigma \in S_n} (-1)^\sigma u_1^{i_{\sigma(1)}} \dots u_k^{i_{\sigma(k)}} \\ &= \det(A^I) \end{aligned}$$

where A^I consists of the rows i_1, \dots, i_k of the matrix $A = (u_1, \dots, u_k)$.

k -FORMS AT A POINT ON A MANIFOLD

Let M be a smooth manifold and fix a point $p \in M$. Given a chart ϕ on M at p , $T_p M$ has a basis $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$. We denote the dual basis for $T_p^* M = (T_p M)^*$ by $\{dx^1, \dots, dx^n\}$. Then $\Lambda^k T_p^* M$ has basis $\{dx^I : I \text{ increasing}\}$ (it is $\binom{n}{k}$ dimensional). If $X_j = \sum_{i=1}^n u_j^i \frac{\partial}{\partial x^i} \in T_p M$, then

$$dx^I(X_1, \dots, X_k) = \det \begin{pmatrix} u_1^{i_1} & \cdots & u_k^{i_1} \\ \vdots & & \vdots \\ u_1^{i_k} & \cdots & u_k^{i_k} \end{pmatrix}$$

An element $\alpha \in \Lambda^k T_p^* M$ can be written uniquely as $\alpha = \sum_{I \text{ increasing}} a_I dx^I$ with $a_I \in \mathbb{R}$, and α is called a k -form on M at p .

Change of Coordinates

Suppose that ϕ and ψ are two charts on M at p , so that $T_p M$ has bases $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ and $\left\{ \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \right\}$, and $T_p^* M$ has dual bases $\{dx^1, \dots, dx^n\}$ and $\{dy^1, \dots, dy^n\}$ with corresponding bases for $\Lambda^k T_p^* M$. Let $\alpha \in \Lambda^k T_p^* M$. Say $\alpha = \sum_I a_I dx^I = \sum_J b_J dy^J$. If $X = \sum_i u^i \frac{\partial}{\partial x^i} = \sum_j v^j \frac{\partial}{\partial y^j}$, then $v = D(\psi \phi^{-1})$ is

$$v^j = \sum_i \frac{\partial y^j}{\partial x^i} u^i$$

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

where $y(x) = \psi \circ \phi^{-1}(x)$. Then for each increasing multi-index I ,

$$\begin{aligned} a_I &= \alpha \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \\ &= \left(\sum_J b_J dy^J \right) \left(\sum_{\ell_1} \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \frac{\partial}{\partial y^{\ell_1}}, \dots, \sum_{\ell_k} \frac{\partial y^{\ell_k}}{\partial x^{i_k}} \frac{\partial}{\partial y^{\ell_k}} \right) \\ &= \sum_{\text{incr } J} \sum_{\text{all } L} b_J \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\ell_k}}{\partial x^{i_k}} dy^L \left(\frac{\partial}{\partial y^{\ell_1}}, \dots, \frac{\partial}{\partial y^{\ell_k}} \right) \\ &= \sum_{\text{incr } J} \sum_{\sigma \in S_n} (-1)^\sigma b_J \frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_{\sigma(k)}}}{\partial x^{i_k}} \\ &= \sum_{\text{incr } J} b_J \det \left(\frac{\partial y^J}{\partial x^I} \right) \end{aligned}$$

where

$$\frac{\partial y^J}{\partial x^I} = \begin{pmatrix} \frac{\partial y^{j_1}}{\partial x^{i_1}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{i_1}} \\ \vdots & & \vdots \\ \frac{\partial y^{j_1}}{\partial x^{i_k}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{i_k}} \end{pmatrix}$$

When $\alpha = \sum_I a_I dx^I = \sum_J b_J dy^J$,

$$a_I = \sum_J b_J \det \left(\frac{\partial y^J}{\partial x^I} \right)$$

For an increasing multi-index L , taking $b_K = 1$ and $b_J = 0$ for $J \neq L$, we obtain

$$dy^L = \sum_I a_I dx^I, a_I = 1 \cdot \det \left(\frac{\partial y^L}{\partial x^I} \right)$$

so

$$dy^L = \sum_I \det \left(\frac{\partial y^L}{\partial x^I} \right) dx^I$$

THE WEDGE PRODUCT OR EXTERIOR PRODUCT

When ϕ is a chart on M at p and $\alpha = \sum_I a_I dx^I \in \Lambda^k T_p^* M$ and $\beta = \sum_J b_J dx^J \in \Lambda^\ell T_p^* M$, we would like to define $\alpha \wedge \beta \in \Lambda^{k+\ell} T_p^* M$ by $\alpha \wedge \beta = \sum_{I,J} a_I b_J dx^I \wedge dx^J$ (where we can use the rule $dx^i \wedge dx^j = -dx^j \wedge dx^i$ to put the multi-index in increasing order). We need to make sure that the definition does not depend on the choice of the chart ϕ . Note that for vectors $X_1, \dots, X_{k+\ell} \in T_p M$, given (in the chart ϕ) by $X_j = \sum_{i=1}^n u_j^i \frac{\partial}{\partial x^i}$ we have

$$\begin{aligned} (dx^I \wedge dx^J)(X_1, \dots, X_{k+\ell}) &= \det \begin{pmatrix} u_1^{i_1} & \cdots & u_{k+\ell}^{i_1} \\ \vdots & & \vdots \\ u_1^{i_k} & \cdots & u_{k+\ell}^{i_k} \\ u_1^{j_1} & \cdots & u_{k+\ell}^{j_1} \\ \vdots & & \vdots \\ u_1^{j_\ell} & \cdots & u_{k+\ell}^{j_\ell} \end{pmatrix} \\ &= \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma u^{i_1}_{\sigma(1)} \cdots u^{i_k}_{\sigma(k)} u^{j_1}_{\sigma(k+1)} \cdots u^{j_\ell}_{\sigma(k+\ell)} \\ &= \sum_{\tau} \sum_{\mu} \sum_{\nu} (-1)^\tau (-1)^\mu (-1)^\nu u^{i_1}_{\mu(\tau(1))} \cdots u^{i_k}_{\mu(\tau(k))} u^{j_1}_{\nu(\tau(k+1))} \cdots u^{j_\ell}_{\nu(\tau(k+\ell))} \end{aligned}$$

where the sums are over τ a permutation of $\{1, \dots, k+\ell\}$ so that $\tau(1) < \cdots < \tau(k)$ and $\tau(k+1) < \cdots < \tau(k+\ell)$, μ is a permutation of $\{\tau(1), \dots, \tau(k)\}$ and ν is a permutation of $\{\tau(k+1), \dots, \tau(k+\ell)\}$, so that

$$\begin{aligned} &= \sum_{\tau} (-1)^\tau \det \begin{pmatrix} u_{\tau(1)}^{i_1} & \cdots & u_{\tau(1)}^{i_k} \\ \vdots & & \vdots \\ u_{\tau(k)}^{i_1} & \cdots & u_{\tau(k)}^{i_k} \end{pmatrix} \det \begin{pmatrix} u_{\tau(k+1)}^{j_1} & \cdots & u_{\tau(k+1)}^{j_\ell} \\ \vdots & & \vdots \\ u_{\tau(k+\ell)}^{j_1} & \cdots & u_{\tau(k+\ell)}^{j_\ell} \end{pmatrix} \\ &= \sum_{\tau} (-1)^\tau dx^I(X_{\tau(1)}, \dots, X_{\tau(k)}) dx^J(X_{\tau(k+1)}, \dots, X_{\tau(k+\ell)}) \end{aligned}$$

Thus for $\alpha = \sum_I a_I dx^I$, $\beta = \sum_J b_J dx^J$, $\gamma = \sum_{I,J} a_I b_J dx^I \wedge dx^J$ we have

$$\gamma(X_1, \dots, X_{k+\ell}) = \sum_{\tau \in T_{k,\ell}} (-1)^\tau \alpha(X_{\tau(1)}, \dots, X_{\tau(k)}) \cdot \beta(X_{\tau(k+1)}, \dots, X_{\tau(k+\ell)})$$

where $T_{k,l}$ is the set of permutations τ of $\{1, \dots, k + \ell\}$ such that $\tau(1) < \dots < \tau(k)$ and $\tau(k+1) < \dots < \tau(k+\ell)$.

Definition. When $f : M \rightarrow N$ is a smooth map of smooth manifolds and $p \in M$, we define the **pullback**

$$f^* = f^*(p) : \Lambda^k T_{f(p)}^* N \rightarrow \Lambda^k T_p^* M$$

by $f^*(\beta)(X_1, \dots, X_k) = \beta(f_* X_1, \dots, f_* X_k)$ where $\beta \in \Lambda^k T_{f(p)}^* N$ and each $X_j \in T_p M$ so that $f_* X_j \in T_{f(p)} N$.

Let M be a chart on M at p and ψ a chart on N at $f(p)$. Let $\beta = \sum b_J dx^J$, write $X_j = \sum_i u_j^i \frac{\partial}{\partial x^i}$ and say $\alpha = f^* \beta = \sum_I a_I dx^I$.

$$\begin{aligned} a_I &= \alpha \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \\ &= \beta \left(f_* \frac{\partial}{\partial x^{i_1}}, \dots, f_* \frac{\partial}{\partial x^{i_k}} \right) \\ &= \left(\sum_J b_J dy^J \right) \left(\sum_{\ell_1} \frac{\partial y^{\ell_1}}{\partial x^{i_1}}, \dots, \sum_{\ell_k} \frac{\partial y^{\ell_k}}{\partial x^{i_k}} \right) \\ &= \sum_J \sum_{\text{incr all } L} B_J \frac{\partial y^{\ell_1}}{\partial x^{i_1}} \dots \frac{\partial y^{\ell_k}}{\partial x^{i_k}} dy^J \left(\frac{\partial}{\partial y^{\ell_1}}, \dots, \frac{\partial}{\partial y^{\ell_k}} \right) \\ &= \sum_J \sum_{\sigma \in S_k} (-1)^\sigma b_J \frac{\partial y^{j_{\sigma(1)}}}{\partial x^{i_1}} \dots \frac{\partial y^{j_{\sigma(k)}}}{\partial x^{i_k}} \\ &= \sum_J b_J \det \left(\frac{\partial y^J}{\partial x^I} \right) \end{aligned}$$

where $y = y(x) = (\psi f \phi^{-1})(x)$.

Definition. A k -form at each point p on a smooth manifold M is given by a map $\alpha : M \rightarrow \bigcup_{p \in M} \Lambda^k T_p^* M$, where $\alpha(p) \in \Lambda^k T_p^* M$ for all $p \in M$. We say that such a map α is **smooth** at $p \in M$ when for some (hence for every) chart $\phi : U \subseteq M \rightarrow \phi(U) \subseteq \mathbb{R}^m$ on M at p , when we write the restriction of α to U as $\alpha(p) = \sum_I \alpha_I(p) dx^I$, the coefficient functions $\alpha_I : U \subseteq M \rightarrow \mathbb{R}$ are smooth. Such a map $\alpha : M \rightarrow \bigcup_{p \in M} \Lambda^k T_p^* M$ is called smooth (on M) when it is smooth at every point $p \in M$.

Another way to think about smooth k -forms is as follows. Consider $\alpha : M \rightarrow \bigcup_{p \in M} \Lambda^k T_p^* M$ with $\alpha(p) \in \Lambda^k T_p^* M$ for all $p \in M$. Let $\Lambda^k T^* M = \bigcup_{p \in M} \Lambda^k T_p^* M$ and define the **projection map** $\pi : \Lambda^k T^* M \rightarrow M$ by $\pi(\alpha_p) = p$, when $\alpha_p \in \Lambda^k T_p^* M$. We give $\Lambda^k T^* M$ the structure of a smooth vector bundle of rank $\binom{n}{k}$ on M as follows. For each chart $\phi : U \rightarrow \phi(U)$ on M , we define a chart

$$\Phi : \pi^{-1}(U) = \bigcup_{p \in U} \Lambda^k T_p^* M \rightarrow \phi(U) \times \Lambda^k(\mathbb{R}^n)^* \equiv \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$

by $\Phi(\alpha_p) = (\phi(p), \sum_I \alpha_I(p) dx^I)$, where the restriction of α to U is given by $\alpha(p) = \sum_I \alpha_I(p) dx^I$.

With this definition, a smooth k -form on M is a smooth map $\alpha : M \rightarrow \Lambda^k T^* M$ such that $\pi(\alpha(p)) = p$. We denote the space of all k -forms on M by $\Omega^k(M)$ or $\Gamma(M, \Lambda^k T^* M)$ (sections) or $\Gamma(\Lambda^k T^* M)$.

When $U \subseteq \mathbb{R}^n$ is open and $p \in U$, we often identify

$$\begin{aligned} T_p U &= T_p \mathbb{R}^n = \mathbb{R}^n = M_{n \times 1}(\mathbb{R}) \\ T_p^* U &= T_p^* \mathbb{R}^n = \mathbb{R}^n = M_{1 \times n}(\mathbb{R}) \\ \Lambda^k T_p^* U &= \Lambda^k T_p^* \mathbb{R}^n = \Lambda^k(\mathbb{R}^n)^* = \text{span}\{dx^I : I \text{ incr}\} \\ TU &= \bigcup_{p \in U} T_p U = \bigcup_{p \in U} \mathbb{R}^n = U \times \mathbb{R}^n \\ T^* U &= U \times (\mathbb{R}^n)^* \\ \Lambda^k T^* U &= U \Lambda^k(\mathbb{R}^n)^* \\ \mathfrak{X}(U) &= \Gamma(TU) = C^\infty(U, \mathbb{R}^n) \\ \Omega^k(U) &= \Gamma(\Lambda^k T^* U) = \Gamma(U \times \Lambda^k(\mathbb{R}^n)^*) = C^\infty(U, \Lambda^k(\mathbb{R}^n)^*) \end{aligned}$$

where we identify $\Gamma(TU) = C^\infty(U, \mathbb{R}^n)$ by $X : U \rightarrow U \times \mathbb{R}^n$ and $X(p) = (p, u(p))$ with $u : U \rightarrow \mathbb{R}^n$.

When $U \subseteq M$ is open and $p \in U$, $T_p U = T_p M$, $T_p^* U = T_p^* M$, $\Lambda^k T_p^* U = \Lambda^k T_p^* M$. When $\phi : U \rightarrow \phi(U)$ is a chart on M and if $X \in \mathfrak{X}(M) = \Gamma(TM)$ is given locally by $X(p) = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x^i}$ then

$$(\phi_* X)(x) = \sum_{i=1}^n u^i(x) \frac{\partial}{\partial x^i}$$

for $x \in \phi(U)$ where $u^i(x) = X^i(\phi^{-1}(x))$, $X^i(p) = u^i(\phi(p))$. If $\alpha \in \Omega^k(M) = \Gamma(\Lambda^k T^* M)$ and the restriction of α to U is given by

$$\alpha(p) = \sum_{I \text{ incr}} \alpha_I(p) dx^I$$

then

$$(\phi^{-1})^*(\alpha)(x) = \sum_{I \text{ incr}} a_I(x) dx^I$$

where

$$a_I(x) = \alpha_I(\phi^{-1}(x)) \quad \alpha_I(p) = a_I(\phi(p))$$

Note that when $f \in C^\infty(M)$, we have $df = f_* \in \Omega^1(M)$. Indeed, we have $f : M \rightarrow \mathbb{R}$, $df(p) = f_*(p) : T_p M \rightarrow T_p \mathbb{R} = \mathbb{R}$, so $df(p) = f_*(p) \in T_p^* M$ so that $df = f_* : M \rightarrow T^* M$ with $df(p) = f_*(p) \in T_p^* M$ for all $p \in M$.

Locally, $df = f_*$ is given by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

To be precise, if $\phi : U \rightarrow \phi(U)$ is a chart on M then the restriction of f to U is given by

$$df(p) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i$$

so that

$$(\phi^{-1})^*(df)(x) = \sum_{i=1}^n \frac{\partial f \circ \phi^{-1}}{\partial x^i} dx^i$$

Definition. When M is a smooth manifold, we define a smooth 0-form on M to be a smooth function $f : M \rightarrow \mathbb{R}$.

We define

$$\begin{aligned}\Omega^0(M) &= C^\infty(M) = C^\infty(M, \mathbb{R}) \\ \Lambda^0 T_p^* M &= \mathbb{R} \\ \Lambda^0 T^* M &= \bigsqcup_{p \in M} \mathbb{R} = M \times \mathbb{R}\end{aligned}$$

so that $\Omega^0(M) = \Gamma(\Lambda^0 T^* \mathbb{R}) = C^\infty(M)$.

Now, we want to define $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ by the local definition

$$d\left(\sum_I a_I x^*\right) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

To be precise, given $\alpha \in \Omega^k(M)$, if $\phi : U \rightarrow \phi(U)$ is a chart and the restriction of α to U is given by $\alpha(p) = \sum \alpha_I(p) dx^I$, so

$$(\phi^{-1})^*(\alpha)(x) = \sum_I a_I(x) dx^I$$

then we want $d\alpha$ restricted to U to be given by

$$d\alpha(p) = \sum d\alpha_I(p) \wedge dx^I$$

that is

$$(\phi^{-1})(d\alpha)(x) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

However, it is not immediate that this definition does not depend on the chosen chart ϕ .

5.1 Theorem. (Exterior Derivative) Let M be a smooth manifold. Then there exists a unique \mathbb{R} -linear map $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ for all $k \geq 0$ such that

1. $d : \Omega^0(M) \rightarrow \Omega^1(M)$ is given by $df = f_*$ (as previously defined)
2. If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
3. $d^2 = 0$ (where $d^2 = d \circ d$).

Such a map d is given locally by $d\left(\sum_I a_I dx^I\right) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$.

PROOF We claim that if such a map d exists, then d is determined locally: for $\alpha, \beta \in \Omega^k(M)$, if $\alpha(p) = \beta(p)$ for all $p \in U$ where $U \subseteq \mathbb{R}^n$ is open, then $d\alpha(p) = d\beta(p)$ for all $p \in U$. Suppose $\alpha = \beta$ in U . Let $\gamma = \beta - \alpha$, so $\gamma = 0$ in U . Let $p \in U$.

Choose a smooth bump function $S : M \rightarrow \mathbb{R}$ with $S = 1$ in a neighbourhood of p and $\text{supp}(S) \subseteq U$. Then $s\gamma = 0 \in \Omega^k(M)$, so that

$$\begin{aligned}0 &= d(s\gamma) \\ &= ds \wedge \gamma + s \wedge d\gamma \\ &= 0 \wedge \gamma + 1 \wedge d\gamma\end{aligned}$$

in a neighbourhood of p . Thus $d\gamma = 0$ in a neighbourhood of p , so that $d\gamma(p) = 0$. In particular, $p \in U$ was arbitrary, so $\gamma = 0$ in U .

We now claim that if such a map d exists, then d must be given locally by the formula $d(\sum_I a_I dx^I) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$. To be precise, if $\alpha \in \Omega^k(M)$ and $\phi : U \rightarrow \phi(U)$ is a chart, and if the restriction of α to U is given by $\alpha(p) = \sum_I \alpha_I(p) dx^I$ so that $(\phi^{-1})^*(\alpha)(x) = \sum_I a_I(x) dx^I$ then $d\alpha$, restricted to U , is given by

$$(d\alpha)(p) = \sum_I d\alpha_I(p) dx^I$$

so that

$$(\phi^{-1})^*(d\alpha)(x) = \sum_I \sum_j \frac{\partial a_I}{\partial x^j}(x) dx^j \wedge dx^I$$

Suppose d exists. Let $\alpha \in \Omega^k(M)$ and $\phi : U \rightarrow \phi(U)$ be a chart on M . Say the restriction of α to U is given by $\alpha(p) = \sum_I \alpha_I(p) dx^I$ for all $p \in U$. Fix $p \in U$. Use a smooth step function which is equal to 1 in an open neighbourhood of p and supported in U to construct smooth functions $\beta_I : M \rightarrow \mathbb{R}$ and $\gamma^i : M \rightarrow \mathbb{R}$ such that $\beta_I = \alpha_I$ and $\gamma^i = \phi^i$ in an open neighbourhood of p (where $\phi : U \rightarrow \phi(U)$ is given by $\phi = (\phi^1, \dots, \phi^n)$). Define $\beta \in \Omega^k(M)$ by $\beta = \sum_I \beta_I d\gamma^I$. Then $\beta = \alpha$ in an open neighbourhood of p . Then in this neighbourhood $d\alpha = d\beta = d(\sum_I \beta_I d\gamma^I) = \sum_I d\beta_I \wedge d\gamma^I$ by linearity and rules (2) and (3). Also, by our knowledge of $d : \Omega^0(M) \rightarrow \Omega^1(M)$ in the open neighbourhood, we have $d\beta_I = d\alpha_I$ and $d\gamma^i = d\phi^i = dx^i$.

On the other hand, given a chart $\phi : U \rightarrow \phi(U)$, we can define $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$ by the local formula

$$d(\sum_I \alpha_I dx^I) = \sum_I d\alpha_I \wedge dx^I$$

and then d is linear and satisfies the required properties since

$$\begin{aligned} d\left(\sum_I a_I dx^I \wedge \sum_J b_J dx^J\right) &= d\left(\sum_{I,J} a_I b_J dx^I \wedge dx^J\right) \\ &= \sum_{I,J} \sum_j \left(\frac{\partial a_I}{\partial x^j} b_J + a_I \frac{\partial b_J}{\partial x^i}\right) dx^j \wedge dx^I \wedge dx^J \\ &= \sum_{I,J} \sum_j \left(\frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I\right) \wedge b_J dx^J + (-1)^k (a_I dx^I) \wedge \left(\frac{\partial b_J}{\partial x^j} dx^j \wedge dx^I\right) \end{aligned}$$

and

$$\begin{aligned} d^2(\sum_I a_I dx^I) &= d\left(\sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I\right) \\ &= \sum_I \sum_{i,j} \frac{\partial^2 a_I}{\partial x^i \partial x^j} dx^i \wedge dx^j \wedge dx^I \\ &= 0 \end{aligned}$$

Finally, note that we can define $d : \Omega^k \rightarrow \Omega^{k+1}$ globally by using the local formula in the domains of all charts, and the definitions agree on intersections of charts. ■

As an exercise, we have the following formulas:

- (i) For $\alpha \in \Omega^k(M)$, $\beta \in \Omega^\ell(M)$, $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$.
- (ii) For $f : L \rightarrow M$, $g : M \rightarrow N$, $(g \circ f)^* = f^* \circ g^*$. Also, if $f : M \rightarrow M$ is the identity, then $f^* : \Omega^k(M) \rightarrow \Omega^k(M)$ is the identity.
- (iii) $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$
- (iv) $d \circ f^* = f^* \circ d$
- (v) For $\alpha \in \Omega^k(M)$ and each $X_i \in \mathfrak{X}(M)$, then

$$\begin{aligned} d\alpha(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\alpha(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \end{aligned}$$

In particular, when $\alpha = f \in \Omega^0(M)$, $df(X) = X(f)$ and when $\alpha \in \Omega^1$,

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$