

Harmonic Analysis

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Contents

Chapter I	Harmonic Analysis	
1	Locally Compact Groups	1
1.1	Homogenous Spaces	2
1.2	p -adic Numbers	3
1.3	Haar Integral and Haar Measure	6
1.4	The Modular Function	12
1.5	The Convolution Algebra of Measures	18
1.6	Atomic-Continuous and Lebesgue Decompositions	21
2	Abelian Locally Compact Groups	22
3	Compact Groups	22
4	Introduction to Amenability Theory	22

I. Harmonic Analysis

1 LOCALLY COMPACT GROUPS

Definition. Let G be a group. A topology τ on G is a **group topology** provided that

- $x \mapsto x^{-1} : G \rightarrow G$ is continuous, and
- $(x, y) \mapsto xy : G \times G \rightarrow G$ is continuous.

We call (G, τ) a **topological group** where we omit τ when it is clear from context.

Equivalently, we may assert that $(x, y) \mapsto xy^{-1}$ is $\tau \times \tau - \tau$ -continuous. Write $L_g(x) = gx$ and $R_g(x) = xg$ to denote the left and right multiplication maps; then it is easy to see that L_g and R_g are homeomorphisms. Similarly, $x \mapsto x^{-1}$ is a homeomorphism.

Definition. We say that a subset $A \subset G$ is **symmetric** if $A^{-1} = A$.

We have the following basic properties:

1.1 Proposition. Let (G, τ) be a topological group.

- (i) If $\emptyset \neq A \subseteq G$ and U is open, then $AU = \{ay : a \in A, y \in U\}$ and likewise UA are open.
- (ii) Given $U \in \tau$ and $x \in U$, then there is a symmetric $V \in \tau$ with $e \in V$ such that $VxV \subseteq U$. In particular, if $e \in U$, then we can find symmetric V so that $V^2 \subseteq U$.
- (iii) If H is a subgroup of G , then \overline{H} is also a subgroup.
- (iv) An open subgroup is automatically closed.
- (v) If $K, L \subseteq G$ are compact, then KL is compact.
- (vi) If K is compact and C is closed in G , then KC is closed.

In $(\mathbb{R}, +)$, then $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ is not closed, so it is necessary to assume compactness in (vi).

PROOF (i) $AU = \bigcup_{a \in A} L_a(U)$ is a union of open sets.

- (ii) Consider the continuous map $(y, z) \mapsto yxz$. Since $exe = x \in U$, there is a $\tau \times \tau$ -neighbourhood of (e, e) which maps into U have a basic neighbourhood $V_1 \times V_2$. Let $V = V_1 \cap V_2$. Moreover, we may replace V by $V^{-1} \cap V$ to attain symmetry.

- (iii) Let $x, y \in \overline{H}$, $U \in \tau$ with $xy \in U$. Then (ii) provides V with $VxyV \subseteq U$. But $Vx \cap H \neq \emptyset$ and $yV \cap H \neq \emptyset$ so there are $h_1 \in Vx \cap H$, $h_2 \in yV \cap H$, and $h_1h_2 \in VxyV \subseteq U$. Thus $U \cap H \neq \emptyset$. Thus $xy \in \overline{H}$.

To use nets for inverses, if $x \in \overline{H}$, then $x = \lim_{\alpha} x_{\alpha}$ where $(x_{\alpha})_{\alpha \in A} \subset H$ is a net. Then $x^{-1} = \lim_{\alpha} x_{\alpha}^{-1} \in \overline{H}$ as each $x_{\alpha}^{-1} \in H$.

- (iv) If H is an open subgroup, then $H = G \setminus \bigcup_{x \in G \setminus H} xH$ is closed.
- (v) $K \times L$ is compact, and hence so is its image under multiplication.
- (vi) If $x \in \overline{KC}$, then $x = \lim_{\alpha} k_{\alpha}x_{\alpha}$ where $k_{\alpha} \in K$ and $x_{\alpha} \in C$. Since K is compact, we may assume (passing to a subnet if necessary) $k = \lim_{\alpha} k_{\alpha}$ exists in K . Then

$$k^{-1}x = \lim_{\alpha} k_{\alpha}^{-1} \cdot \lim_{\alpha} k_{\alpha}x_{\alpha} = \lim_{\alpha} k_{\alpha}^{-1}k_{\alpha}x_{\alpha} = \lim_{\alpha} x_{\alpha} \in C$$

so $x = kk^{-1}x \in KC$. ■

1.1 HOMOGENEOUS SPACES

Let (G, τ) be a topological group, H a subgroup of G , and $G/H = \{xH; x \in G\}$. Let $\pi : G \rightarrow G/H$ be given by $\pi(x) = xH$ be the projection map. The **quotient topology** on G/H is $\tau_{G/H} = \{W \in G/H : \pi^{-1}(W) \in \tau\}$. Notice that if $U \in \tau \setminus \{\emptyset\}$, then $UH = \pi^{-1}(\pi(U))$ is open, so $\pi : G \rightarrow G/H$ is an open map.

1.2 Proposition. *Let (G, τ) , H be as above.*

- (i) *The map $(x, yH) \mapsto xyH : G \times G/H \rightarrow G/H$ is $\tau \times \tau_{G/H} - \tau_{G/H}$ continuous and open.*
- (ii) *If H is normal, then $(G/H, \tau_{G/H})$ is a topological group.*
- (iii) *If H is closed, then $\tau_{G/H}$ is Hausdorff.*

PROOF (i) Let $x, y \in G$, $W \in \tau_{G/H}$ satisfy $xyH = \pi(xy) \in W$. Then $xy \in \pi^{-1}(W)$ and by **Proposition 1.1** we have $V \in \tau$ with $e \in V$ such that $VxyV \subseteq \pi^{-1}(W)$. But then $(x, \pi(y)) \in Vx \times \pi(yV) \in \tau \times \tau_{G/H}$ and the latter set maps into $\pi(VxyV) \subseteq W$.

Also, if $U \in \tau \times \tau_{G/H}$, then $U = \bigcup_{(x, yH) \in U} V_x \times W_{yH}$ and

$$\pi(U) = \bigcup_{(x, yH) \in U} \pi(V_x \pi^{-1}(W_{yH}))$$

since π is open.

- (ii) Recall that $(xH)(yH) = xyH$ is our multiplication operation on G/H and π is a group homomorphism. Then the following diagram commutes: We have that $\pi \times \text{id}$ is open and $(x, yH) \mapsto xyH$ is open from (i), so the multiplication from $G/H \times G/H \rightarrow G/H$ must be open and continuous.
- (iii) If $x, y \in G$ with $\pi(x) \neq \pi(y)$, then $e \notin xHy^{-1}$. Now $xHy^{-1} = L_x(R_{y^{-1}}(H))$ so xHy^{-1} is closed. Hence by the last proposition, there is a symmetric open V with $e \in V$ so $V^2 \subseteq G \setminus (xHy^{-1})$. But then $e \notin (VxH)(VyH)^{-1} = VxHy^{-1}V$: if we had $e = vxhy^{-1}u$ with $v, u \in V$ and $h \in H$, then $v^{-1}u^{-1} = xhy^{-1} \in V^2 \cap (xHy^{-1}) = \emptyset$, a contradiction. Hence $VxH \cap VyH = \emptyset$ so $\pi(Vx)$, $\pi(Vy)$ is a pair of separating neighbourhoods of $\pi(x)$, $\pi(y)$. ■

1.3 Corollary. *G is Hausdorff if and only if there exists $x \in G$ so that $\{x\}$ is closed.*

PROOF In a Hausdorff space, points are closed. Conversely, if $\{x\}$ is closed, $\{e\} = L_{x^{-1}}(\{x\})$ is closed and a normal subgroup. Then $G \cong G/\{e\}$ is Hausdorff. ■

If (G, τ) is not Hausdorff, then $\{e\} \subsetneq \overline{\{e\}}$ is the smallest closed subgroup in G . Thus $\overline{\{e\}} \subseteq \bigcap_{x \in G} x\overline{\{e\}}x^{-1} \subseteq \overline{\{x\}}$ so $\overline{\{e\}}$ is normal. In particular, $G/\overline{\{e\}}$ is Hausdorff.

Definition. A **locally compact group** is a Hausdorff topological group (G, τ) which is locally compact.

- (i) If there is any $U \in \tau \setminus \{\emptyset\}$ such that \overline{U} is compact, then for any $x \in U$, we have $e \in x^{-1}U \subseteq L_{x^{-1}}(\overline{U})$ so $\overline{x^{-1}U}$ is compact. If $V \in \tau$ with $e \in V$ and \overline{V} compact, then for any $x \in H$, $x \in xV$ and $\overline{xV} \subseteq L_x(\overline{V})$ and \overline{xV} is compact. In particular, (G, τ) is locally compact if and only if there is some $U \in \tau \setminus \{\emptyset\}$ with \overline{U} compact.

- (ii) If (G, τ) is locally compact and N is a closed normal subgroup, then $(G/N, \tau_{G/N})$ is locally compact. Indeed, $U \in \tau \setminus \{e\}$ with \overline{U} compact, then $\overline{\pi(U)} \subseteq \pi(\overline{U})$ is compact.

Example. (i) If G is any group and τ is the discrete topology, then (G, τ_d) is locally compact.

- (ii) If $((\mathbb{R}, +), \tau_{\|\cdot\|})$ is locally compact.
 (iii) If $\{G_i\}_{i \in I}$ is a family of locally compact groups, then $\prod_{i \in I} G_i$ is a locally compact group if and only if all but finitely many (G_i, τ_i) are compact.
 (iv) $((\mathbb{R}^n, +), \tau_{\|\cdot\|})$ is a locally compact group
 (v) Suppose $\{F_i\}_{i \in I}$ is an infinite family of finite groups (with discrete topologies), then $G = \prod_{i \in I} F_i$ is a compact group. If $F \subset I$ is finite, then $N_F = \{(x_i)_{i \in I} \in G : x_i = e \text{ for } i \in F\}$ is open and a normal subgroup. $\{N_F : F \subset I \text{ finite}\}$ is a base for the topology at e .
 (vi) Let (k, τ) be a locally compact field. Then $\det^{-1}(k \setminus \{0\}) = \text{GL}_n(k) \subseteq M_n(k) \cong k^{n^2}$ is an open subset and multiplicative subgroup, and hence locally compact. Notice that multiplication is governed by linear equations, and hence continuous, while inversion is continuous thanks to Cramer's rule.

Here are some common closed subgroups:

$$\det^{-1}(\{1\}) = \text{SL}_n(k)$$

$$O_n(k) = \{x \in \text{GL}_n(k) : x^{-1} = X^T\}$$

As a special case, consider $U_n = \{x \in \text{GL}_n(\mathbb{C}) : x^* = x^{-1}\}$ is governed by continuous equations, and thus closed in $M_n(\mathbb{C})$. Furthermore, U_n is bounded in $M_n(\mathbb{C})$, and hence compact.

1.2 p -ADIC NUMBERS

Let p be a prime in \mathbb{N} . We will establish product structures and topologies on

$$\mathbb{O}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : a_k \in \{0, 1, \dots, p-1\} \right\} \cong \{0, 1, \dots, p-1\}^{\mathbb{N}}$$

$$\mathbb{Q}_p = \left\{ \sum_{k=\mathbb{N}}^{\infty} a_k p^k : N \in \mathbb{Z}, a_k \in \{0, 1, \dots, p-1\} \right\}$$

are topological rings and a topological field respectively. Let $R_p = \prod_{n=0}^{\infty} \mathbb{Z}/p^{n+1}\mathbb{Z}$ which is a ring under pointwise operations.

1.4 Lemma. The map $\rho : R_p \times R_p \rightarrow [0, 1]$ given by

$$\rho(x, y) = \sum_{n \in \mathbb{N}_0} \frac{\rho_n(x_n, y_n)}{p^n} \quad \rho_n(x_n, y_n) = \begin{cases} 1 & : x_n = y_n \\ 0 & : x_n \neq y_n \end{cases}$$

is a metric on R_p which satisfies

- (additively invariant): $\rho(x+z, y+z) = \rho(x, y)$ for $x, y, z \in R_p$
- τ_p is the product topology

PROOF Additive invariance is routine. Notice that if $\frac{1}{p^m} \geq \epsilon > \frac{1}{p^{m+1}}$, then the open ϵ -ball around a point x is $\{x_0\} \times \cdots \times \{x_m\} \times \prod_{n=m+1}^{\infty} \mathbb{Z}/p^{n+1}\mathbb{Z}$ is product-open. Conversely, any product-open set is a finite union of such ϵ -balls. ■

1.5 Corollary. *The function $\|x\|_p = \rho(x, 0)$ in R_p satisfies*

- $\|x\|_p = 0$ if and only if $x = 0$
- $\|x + y\|_p \leq \|x\|_p + \|y\|_p$
- $\|xy\|_p \leq \|x\|_p \|y\|_p$
- $\|-x\|_p = \|x\|_p$

Hence (R_p, τ_ρ) is a compact topological ring.

PROOF The properties follow directly using additive invariance. To see that R_p is a topological ring, if $(x_\alpha), (y_\alpha)$ have $x = \lim x_\alpha$ and $y = \lim y_\alpha$, then, for example,

$$\begin{aligned} \|xy - x_\alpha y_\alpha\|_p &\leq \|xy - x_\alpha y\|_p + \|x_\alpha y - x_\alpha y_\alpha\|_p \\ &\leq \|x - x_\alpha\|_p + \|y - y_\alpha\|_p \end{aligned}$$

as $\|y\|_p, \|x_\alpha\|_p \leq 1$. ■

We now view \mathbb{O}_p as a closed subring of R_p . Define $\alpha : \mathcal{O}_p \rightarrow R_p$ be given on $a = \sum_{k=0}^{\infty} a_k p^k$ by

$$\alpha(a) = \left(\sum_{k=0}^n a_k p^k + p^{n+1} \mathbb{Z} \right)_{n=0}^{\infty}.$$

This map is an injection with range $\alpha(\mathcal{O}_p) = \{(x_n)_{n=0}^{\infty} \in R_p : x_n = \pi_n(x_{n+1}) \text{ for all } n\}$ where $\pi_n : \mathbb{Z}/p^{n+2}\mathbb{Z} \rightarrow \mathbb{Z}/p^{n+1}\mathbb{Z}$ is the canonical quotient map. In fact, this is called an inductive limit with respect to the maps π_n . Hence it is routine to show that

- $\alpha(\mathbb{O}_p)$ is a subring of R_p , and
- $\alpha(\mathbb{O}_p)$ is closed in R_p (just check net limits in product topology)

If $a, b \in \mathbb{O}_p$, define $a + b = \alpha^{-1}(\alpha(a) + \alpha(b))$.

Remark. (i) $1 + \sum_{k=1}^{\infty} 0 \cdot p^k$ is the multiplicative identity in \mathbb{O}_p . Then $-1 = \sum_{k=0}^{\infty} (p-1)p^k$.

- (ii) If $n \in \mathbb{N}$, we can uniquely write $n = \sum_{k=0}^{m(n)} a_k p^k$ with $a_k \in \{0, \dots, p-1\}$. Then $n \cdot 1 = \sum_{k=0}^{m(n)} a_k p^k \in \mathbb{O}_p$. In particular, $n \mapsto n \cdot 1 : \mathbb{N} \rightarrow \mathbb{O}_p$ is an additive semigroup homomorphism with dense range. Hence $n \mapsto n \cdot 1 : \mathbb{Z} \rightarrow \mathbb{Q}_p$ has dense range. We call \mathbb{O}_p the **p -adic integers**.

Let $a = \sum_{k=0}^{\infty} a_k p^k$ in \mathbb{O}_p . Let

$$\begin{aligned} v_p(a) &= \min\{k \in \mathbb{N}_0 : a_k \neq 0\}, \min \emptyset = -\infty \\ |a|_p &= p^{-v_p(a)}, p^{-\infty} = 0 \end{aligned}$$

and notice that $|a|_p = \|\alpha(a)\|_p$. However, $|a|_p$ has even nicer properties:

- (i) $|a|_p = 0$ if and only if $a = 0$
- (ii) $v_p(ab) = v_p(a) + v_p(b)$. Thus $|ab|_p = |a|_p |b|_p$
- (iii) $v_p(a+b) \geq \min\{v_p(a), v_p(b)\}$. Thus $|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$

Notice that (i) and (ii) imply that \mathbb{O}_p is an integral domain.

1.6 Proposition. *The multiplicative unit group of \mathbb{O}_p is $\mathbb{O}_p \setminus p\mathbb{O}_p = \{a \in \mathbb{O}_p : |a|_p = 1\}$. Hence \mathbb{O}_p^\times is open and a topological group.*

PROOF The second equality is trivial. If $a \in \mathbb{O}_p^\times$, then $|a|_p, |a^{-1}|_p \leq 1$ and $1 = |1|_p = |aa^{-1}|_p = |a|_p|a^{-1}|_p$, so $|a|_p = 1$. If $|a|_p = 1$, let

$$x = \alpha(a) = \left(\sum_{k=0}^n a_k p^k + p^{n+1} \mathbb{Z} \right)_{n=0}^\infty \in R_p.$$

Then $x_n \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ since $p \nmid \sum_{k=0}^n a_k p^k$ in \mathbb{Z} . Hence there is $y_n \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ so $x_n y_n = 1 + p^{n+1}\mathbb{Z}$ and thus

$$1 + p^n \mathbb{Z} = \pi_N(1 + p^{n+2} \mathbb{Z}) = \pi_n(x_{n+1} y_{n+1}) = \pi_n(x_{n+1}) \pi_n(y_{n+1}) = x_n \pi_n(y_{n+1})$$

so that $\pi_n(y_{n+1}) = y_n$. Thus if $y \in \alpha(\mathbb{O}_p)$, i.e. $y = \alpha(b)$ with $ab = \alpha^{-1}(\alpha(a)\alpha(b)) = \alpha^{-1}((1 + p^{n+1}\mathbb{Z})_{n=0}^\infty) = 1$ and $a \in \mathbb{O}_p^\times$.

Since $p\mathbb{O}_p$ is closed, we see that \mathbb{O}_p^\times is open in \mathbb{O}_p . Continuity of multiplication follows (ii). Finally, if $a, b \in \mathbb{O}_p$,

$$|a^{-1} - b^{-1}|_p = |a|_p |a^{-1} - b^{-1}|_p |b|_p = |b - a|_p \quad \blacksquare$$

1.7 Corollary. *Each ideal in \mathbb{O}_p is of the form $p^k \mathbb{O}_p$ for $k \in \mathbb{N}_0$.*

PROOF If I is an ideal in \mathbb{O}_p , then let $k(I) = \min\{k \in \mathbb{N}_0 : v_p(a) = k \text{ for some } a \in I\}$. Then there is $a \in I$ with $v_p(a) = k(I)$, so $p^{-k(I)} \in a\mathbb{O}_p^\times \subseteq a\mathbb{O}_p \subseteq I$. Thus $p^{-k(I)}\mathbb{O}_p \subseteq I$. Clearly $I \subseteq p^{-k(I)}\mathbb{O}_p$ as well. \blacksquare

We now extend the valuation and norm to \mathbb{Q}_p . If $a = \sum_{k \in \mathbb{Z}} a_k p^k \in \mathbb{Q}_p$, let $v_p(a) = \min\{k \in \mathbb{Z} : a_k \neq 0\}$ and $|a|_p = p^{-v_p(a)}$. Then for $a \in \mathbb{Q}_p \setminus \{0\}$ admits (formal) factorization

$$a = \sum_{k=v_p(a)}^\infty a_k p^k = p^{v_p(a)} \sum_{k=v_p(a)}^\infty a_k p^{k-v_p(a)} = p^{v_p(a)} \underbrace{\sum_{k=0}^\infty a_{k+v_p(a)} p^k}_{:= a' \in \mathbb{O}_p^\times}$$

Thus, if $a, b \in \mathbb{Q}_p \setminus \{0\}$, we define multiplication and addition by $ab = p^{v_p(a)+v_p(b)} a' b'$ and $a^{-1} = p^{-v_p(a)} (a')^{-1}$. Furthermore, assuming $v_p(a) \leq v_p(b)$, we define addition by

$$a + b = p^{v_p(a)} (a' + p^{v_p(b)-v_p(a)} b')$$

and $0 + a = a$, $0a = 0$. Notice that $|ab|_p = |a|_p |b|_p$, $|1/a|_p = 1/|a|_p$ and if $v_p(a) \leq v_p(b)$, $|a + b|_p = p^{-v_p(a)} |a' + p^{v_p(b)-v_p(a)} b'|_p \leq |a|_p$ so, generally, $|a + b|_p \leq \max\{|a|_p, |b|_p\}$. Also, if $|a|_p = 0$, then $|a| = 0$. Thus $(\mathbb{Q}_p, |\cdot|_p)$ is a “normed field”, and hence a topological field.

Note that

$$\mathbb{O}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\} = \{a \in \mathbb{Q}_p : |a|_p < p\}$$

is a compact open neighbourhood of 0, so \mathbb{Q}_p is locally compact. Moreover, each $p^k \mathbb{Q}_p = \{a \in \mathbb{Q}_p : |a|_p < p^{k-1}\}$ is a closed and open ball about 0.

1.3 HAAR INTEGRAL AND HAAR MEASURE

Let G be a locally compact group. Define for $f : G \rightarrow \mathbb{C}$, $x \in G$, $f \cdot x = f \circ L_x$, and $x \cdot f = f \circ R_x$. Notice that $(f, x) \mapsto f \cdot x$, as an adjoint action, is a right (group) action of G on functions. Likewise, $(x, f) \mapsto x \cdot f$ is a left action.

1.8 Proposition. *If $f \in C_c(G)$, then $\lim_{x \rightarrow e} \|f \cdot x - f\|_\infty = 0 = \lim_{x \rightarrow e} \|x \cdot f - f\|_\infty$.*

PROOF Let $\epsilon > 0$, W a symmetric relatively compact neighbourhood of e , and let $K = \overline{W} \text{supp } f$. Given $y \in K$, $x \mapsto |f(xy) - f(y)|$ is continuous, so there is a neighbourhood U_y of e so $|f(xy) - f(y)| < \epsilon/2$ whenever $x \in U_y$. Let V_y be a symmetric neighbourhood of e so that $V_y^2 \subseteq U_y$. Then $K \subseteq \bigcup_{y \in K} V_y y$ so by compactness get some finite subcover indexed by $\{y_1, \dots, y_n\}$. Set $V = \left(\bigcap_{j=1}^n V_{y_j}\right) \cap W$ and note that V is a symmetric relatively compact neighbourhood of e .

If $x \in V$, then for $y \in K$ we have $y \in V_{y_j} y_j$ for some j , i.e. $y y_j^{-1} \in V_{y_j}$, and hence

$$xy = x y y_j^{-1} y_j \in V V_{y_j} y_j \subseteq V_{y_j}^2 y_j \subseteq U_{y_j} y_j.$$

Note that $xy = x' y_j$ for some $x' \in U_{y_j}$. Similarly, since $y_j y^{-1} \in U_{y_j}$, we $y_j = x'' y$ for some $x'' \in U_{y_j}$. Thus by definition of U_{y_j} , we have

$$|f(xy) - f(y)| \leq |f(x' y_j) - f(y_j)| + |f(x'' y) - f(y)| < \epsilon.$$

Otherwise, if $y \notin K$, then $Wy \cap \text{supp}(f) = \emptyset$: by contrapositive, if $x \in \text{supp}(f) \cap Wy$, then $yx^{-1} \in W$ so $y \in \overline{W} \text{supp } f = K$. Thus for $x \in V \subseteq W$, we have $f(xy) = 0 = f(y)$, and the desired result follows. ■

1.9 Theorem. (Existence of Haar Integral) *There exists a linear functional $I : C_c(G) \rightarrow \mathbb{C}$ satisfying*

- (positivity): $I(f) > 0$ if $f \in C_c^+(G) = \{g \in C_c(G) \setminus \{0\} : g \geq 0\}$.
- (left invariance): $I(f \cdot x) = I(f)$ for $f \in C_c(G)$, $x \in G$.

Let for $f, \phi \in C_c^+(G)$

$$(f : \phi) = \inf \left\{ \sum_{j=1}^n c_j : \text{there are } x_1, \dots, x_n \in G, c_i > 0, n \in \mathbb{N} \text{ s.t. } f \leq \sum_{j=1}^n c_j \phi \cdot x_j \right\}$$

Notive that $0 < \frac{\|f\|_\infty}{\|\phi\|_\infty} \leq (f : \phi)$. Also, if $U = \{x \in G : \phi(x) > \frac{1}{2} \|\phi\|_\infty\}$, then $\text{supp } f$ is covered by finitely many $x^{-1}U$, $x \in G$, and thus $(f : \phi) < \infty$.

CLAIM I *For f, g in $C_c^+(G)$, we have*

- (i) $(f \cdot x : \phi) = (f : \phi)$ for x in G
- (ii) $(cf : \phi) = c(f : \phi) = (f : \frac{1}{c} \phi)$ for $c > 0$
- (iii) $(f + g, \phi) \leq (f : \phi) + (g : \phi)$.
- (iv) $(f : \phi) \leq (f : g)(g : \phi)$

PROOF Note that (i) and (ii) are straightforward. To see (iii) and (iv), consider

$$f \leq \sum_{j=1}^n c_j \phi \cdot x_j \quad g \leq \sum_{j=n+1}^N c_j \phi \cdot x \quad f \leq \sum_{k=1}^m b_k g \cdot y_k$$

so that $f + g \leq \sum_{j=1}^N c_j \phi \cdot x_k$ and $(f + g : \phi) \leq \sum_{j=1}^n c_j + \sum_{j=n+1}^N c_j$, giving (iii). To get (iv), note $f \leq \sum_{k=1}^m b_k \sum_{j=n+1}^N c_j \phi \cdot (x_j y_k)$ so $(f : \phi) \leq \sum_{k=1}^m b_k \sum_{j=k+1}^N c_j$, giving (iv).

We wish to “homogonize” the effect of ϕ . Fix $\psi_0 \in C_c^+(G)$ and let $I_\phi(f) = \frac{(f:\phi)}{(\psi_0:\phi)}$. Then $I_\phi : C_c^+(G) \rightarrow \mathbb{R}_{\geq 0}$ is

(i') left translation invariant

(ii') $\mathbb{R}_{\geq 0}$ -homogenous

(iii') subadditive.

by using the claim above directly. Thus by (iv), $(\psi_0 : \phi) \leq (\psi_0 : f)(f : \phi)$ and $(f : \phi) \leq (f : \psi_0)(\psi_0 : \phi)$, giving

$$\text{iv'} } 0 < \frac{1}{(\psi_0 : f)} \leq I_\phi(f) \leq (f : \psi_0).$$

CLAIM II If $f, g \in C_c^+(G)$, $\epsilon > 0$, there is a neighbourhood V of e such that

$$I_\phi(f) + I_\phi(g) < I_\phi(f + g) + \epsilon$$

whenever $\phi \in C_c^+(G)$ with $\text{supp}(\phi) \subseteq V$.

PROOF Let $k \in C_c^+(G)$ be so $k|_{\text{supp}(f+g)} = 1$ and let $\delta > 0$. Take $h = f + g + \delta k$ and $f' = \frac{f}{h}$, $g' = \frac{g}{h} \in C_c^+(G)$. Uniform continuity of f', g' provides a neighbourhood v of e such that $|f'(x) - f'(y)| < \delta$, $|g'(x) - g'(y)| < \delta$ if $y^{-1}x \in V$. If $\phi \in C_c^+(G)$, $\text{supp}(\phi) \subseteq V$, and x_1, \dots, x_n in G , $c_1, \dots, c_n > 0$ satisfy

$$h \leq \sum_{j=1}^n c_j \phi_j \cdot x_j^{-1}$$

then for x in G

$$\begin{aligned} f(x) &= f'(x)h(x) \leq \sum_{j=1}^n f'(x) c_j \phi(x_j^{-1}x) \\ &\leq \sum_{j=1}^n [f'(x_j) + \delta] c_j \phi(x_j^{-1}x) \end{aligned}$$

by properties of f', g' proven above and $\text{supp}(\phi) \subseteq C$. Likewise,

$$g \leq \sum_{j=1}^n [g'(x_j) + \delta] c_j \phi \cdot x_j^{-1}.$$

Now $f' + g' = (f + g)/h = \frac{f+g}{f+g+\delta k} \leq 1$ so

$$\begin{aligned} (f : \phi) + (g : \phi) &\leq \sum_{j=1}^n [f'(x_j) + \delta] c_j + \sum_{j=1}^n [g'(x_j) + \delta] c_j \\ &\leq \sum_{j=1}^n [1 + 2\delta] c_j \end{aligned}$$

and $(f : \phi) + (g : \phi) \leq (1 + 2\delta)(h : \phi)$. Divide by $(\psi_0 : \phi)$ and (iii') and (iv') above to get

$$I_\phi(f) + I_\phi(g) \leq (1 + 2\delta)I_\phi(h) \leq (1 + 2\delta)[I_\phi(f + h) + \delta I_\phi(k)]$$

Thus with sufficiently small δ , $I_\phi(f) + I_\phi(g) < I_\phi(f + g) + \epsilon$.

We are now in position to complete the proof.

CLAIM III *Construction of the functional I .*

PROOF Inequality (iv') tells us that

$$x_\phi = (I_\phi(f))_{f \in C_c^+(G)} \in \prod_{f \in C_c^+(G)} [\frac{1}{(\psi_0 : f)}, (f : \psi_0)] = X$$

which, by Tychonoff, is compact. For $\phi, \phi' \in \Phi = \{\psi \in C_c^+(G) : \psi(e) = 1\}$, $\phi \leq \phi'$ if $\phi \geq \phi'$ pointwise, which is a preorder. Notice that $\phi \phi' \leq \phi \wedge \phi'$ (pointwise minimum), so that (Φ, \leq) is directed. Hence $(x_\phi)_{\phi \in \Phi}$ admits a converging subnet $x = \lim_{\mu \in M} x_{\phi_\mu}$ in X .

Write $x = (I(f))_{f \in C_c^+(G)}$, so $I(f) = \lim_{\mu \in M} I_{\phi_\mu}(f)$ for each $f \in C_c^+(G)$. Then it follows that from (i'), (ii'), and (iii') that for f, g in $C_c^+(G)$, we have

$$I(F \cdot x) = I(f) \quad I(cf) = cI(f) \quad I(f + g) \leq I(f) + I(g)$$

for $x \in G$, $c > 0$. Moreover, by cofinality, if V is a neighbourhood of e , then $\text{supp}(\phi_\mu) \subseteq V$ for μ sufficiently large in M . Hence given $\epsilon > 0$, by **Claim II**, $I_{\phi_\mu}(f) + I_{\phi_\mu}(g) < I_{\phi_\mu}(f + g) + \epsilon$ for μ sufficiently large in M . Since $\epsilon > 0$ as arbitrary, we have $I(f) + I(g) \leq I(f + g)$.

Let $I(0) = 0$. If $f \in C_c^\mathbb{R}(G)$ and $f = f_1 - f_2 = g_1 - g_2$ with $f_1, f_2, g_1, g_2 \geq 0$, then $h = f_1 + g_2 = g_1 + f_2$ satisfies that $I(h) = I(f_1) + I(g_2) = I(g_1) + I(f_2)$ and hence we may define $I(f) = I(f_1) - I(f_2)$, which do not depend on the choice of f_1, f_2 . One may check that $I : C_c^\mathbb{R}(G) \rightarrow \mathbb{R}$ is \mathbb{R} -linear. Finally, if $f \in C_c(G)$, let $I(f) = I(\text{Re } f) + iI(\text{Im } f)$. It is left as an exercise to verify that $I : C_c(G) \rightarrow \mathbb{C}$ is \mathbb{C} -linear.

Finally, the fact that $I(f \cdot x) = I(f)$ for $f \in C_c(G)$ and $x \in G$ follows for f in $C_c^+(G)$ as above. If $f \in C_c^+(G)$, then (iv') tell us that $I(f) \geq \frac{1}{(\psi_0 : f)} > 0$. ■

Remark. (i) In **Claim III**, $I_\phi(\psi_0) = 1$ so $I(\psi_0) = 1$.

(ii) If G is discrete, then $\psi_0 = 1_{\{e\}} = \min \Phi$. Then $I_{\psi_0}(f) = \frac{(f : \psi_0)}{(\psi_0 : \psi_0)} = \sum_{x \in G} f(x)$ for $f \in C_c^+(G)$.

(iii) If $G = \mathbb{R}$, let ψ_0 be the linear function which is 0 on $(-\infty, -1/2 - \delta) \cup (1/2 + \delta, \infty)$, 1 on $(-1/2 + \delta, 1/2 - \delta)$, and continued linearly on the remainder. Notice that $(\psi_0, \phi_n) \approx n$, so $\frac{(f : \phi_n)}{(\psi_0 : \phi_n)}$ is approximately the Riemann-Darboux upper sum.

(iv) Examine \mathbb{O}_p , $\psi_0 = 1_{\mathbb{O}_0}$, $\psi_n = 1_{p^n \mathbb{O}_p}$.

1.10 Theorem. (Harr Measure) Let $\mathcal{B}(G)$ denote the Borel σ -algebra on G . Then there is a Radon measure $m : \mathcal{B}(G) \rightarrow [0, \infty]$ such that

- m is left invariant: $m(xE) = m(E)$ for $x \in G$, $E \in \mathcal{B}(G)$
- $m(U) > 0$ for $U \in \tau \setminus \{\emptyset\}$.

PROOF The Riesz Representation Theorem provides a measure $m : \mathcal{B}(G) \rightarrow [0, \infty]$ for which

$$\int_G f \, dm = I(f)$$

for $f \in C_c(G)$. Notice that

$$\int_G f \cdot x \, dm = I(f \cdot x) = \int_G f$$

for any $x \in G$, $f \in C_c(G)$. Thus if $U \in \tau$, $\text{supp } f \subseteq U$ if and only if $\text{supp}(f \cdot x) \subseteq x^{-1}U$ for $x \in G$ and $f \in C_c(G)$. Thus

$$\begin{aligned} m(U) &= \sup\{I(f) : f \in C_c(G), 0 \leq f \leq 1 \text{ and } \text{supp}(f) \subseteq U\} \\ &= \sup\{I(f \cdot x) : f \in C_c(G), 0 \leq f \leq 1 \text{ and } \text{supp}(f) \subseteq U\} \\ &= \sup\{I(g) : g \in C_c(G), 0 \leq g \leq 1, \text{supp}(g) \subseteq x^{-1}U\} \\ &= m(x^{-1}U). \end{aligned}$$

Therefore, for any $E \in \mathcal{B}(G)$, we have

$$\begin{aligned} m(E) &= \inf\{m(U) : E \subseteq U, U \in \tau\} \\ &= \inf\{m(xU) : E \subseteq U, U \in \tau\} \\ &= \inf\{m(xU) : xE \subseteq xU, U \in \tau\} = m(xE). \end{aligned}$$

Finally, if $U \in \tau \setminus \{\emptyset\}$, there is $f \in C_c^+(G)$ with $0 \leq f \leq 1$ and $\text{supp}(f) \subseteq U$, so $m(U) \geq I(f) > 0$. ■

Remark. If $E \in \mathcal{G}(G)$, $m(E) < \infty$, then $m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$. Inner regularity need not hold on infinite measure sets: taking $G = \mathbb{R}_d \times \mathbb{R}$, then $\mathbb{R}_d \times \{0\}$ is closed, and thus Borel. However, $m(E) = \infty$ while $m(K) = 0$ for each compact $K \subset E$.

1.11 Theorem. (Uniqueness of Haar Measure) Let $m' : \mathcal{B}(G) \rightarrow [0, \infty]$ be any Radon measure such that $m(xE) = m'(E)$ for $x \in G$ and $E \in \mathcal{B}(G)$. Then there is $c \geq 0$ such that $m' = cm$.

PROOF Fix a symmetric neighbourhood $W = W^{-1}$ of e with \overline{W} compact. Given $f \in C_c^+(G)$, $\epsilon > 0$, and U a neighbourhood of e such that $\|f - x \cdot f\|_\infty < \epsilon$. Let $V = U \cap W$. Then let $x \in G$, and for any $x' \in G$ with $x'x^{-1} \in V$, we have

$$\begin{aligned} \left| \int_G f(yx) \, dm'(y) - \int_G f(yx') \, dm'(y) \right| &\leq \|x \cdot f - x' \cdot f\|_\infty m(\text{supp}(f)x^{-1} \cup \text{supp}(f)Vx^{-1}) \\ &< \epsilon m(\text{supp}(f)x^{-1} \cup \text{supp}(f)Wx^{-1}) \end{aligned}$$

and hence $x \mapsto \int_G x \cdot f \, dm'$ is continuous at each point in G . Thus

$$D_f(x) = \frac{\int_G x \cdot f \, dm'}{\int_G f \, dm}$$

defines a continuous function on G .

If $f, g \in C_c^+(G)$, then $(x, y) \mapsto f(x)g(y^{-1})$ is non-negative, continuous, Borel measurable, and compactly supported on $G \times G$. Then by left-invariance and Tonelli's theorem,

$$\begin{aligned}
 \left(\int_G f dm \right) \cdot \left(\int_G g(y^{-1}) dm'(y) \right) &= \int_G \int_G f(x)g(y^{-1}) dm'(y) dm(x) \\
 &= \int_G \int_G f(x)g((x^{-1}y)^{-1}) dm'(y) dm(x) \\
 &= \int_G \int_G f(x)g(y^{-1}x) dm(x) dm'(y) \\
 &= \int_G \int_G f(yx)g(x) dm(x) dm'(y) \\
 &= \int_G g(y) \left(\int_G f(yx) dm'(y) \right) dm(x)
 \end{aligned}$$

Thus,

$$\int_G g(y^{-1}) dm'(y) = \int_G g(x) D_f(x) dm(x).$$

But if we have any other $f' \in C_c^+(G)$, then we would have

$$\int_G g(x) D_f(x) dm(x) = \int_G g(y^{-1}) dm'(y) = \int_G g(x) D_{f'}(x) dm(x)$$

so it follows that $D_f = D_{f'}$ m -a.e. Since $D_f, D_{f'}$ are continuous, we see that $D_f = D_{f'}$ everywhere. In particular, $D_f(e) = D_{f'}(e)$, or that

$$\frac{\int_G f dm'}{\int_G f dm} = D_f(e) = D_{f'}(e) = \frac{\int_G f' dm'}{\int_G f' dm}$$

Let c denote this common value, so $c \int_G f dm = \int_G f dm'$ for $f \in C_c^+(G)$. Hence $m' = cm$. ■

Example. (i) If G is discrete, then $C_c(G)$ is composed of functions with finite support, and $m(E)$ is (a multiple of) the counting measure. In the finite case, we normalize by $|G|$.

(ii) If $G = \mathbb{R}^n$, $I(f) = \int_{\mathbb{R}^n} f$ and m is n -dimensional Lebesgue measure.

(iii) Let $G = \text{GL}_n(\mathbb{R})$.

(a) If $t \in \text{GL}_n(\mathbb{R})$, then for $f \in C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f \circ t(y) dy = \frac{1}{|\det t|} \int_{\mathbb{R}^n} f(y) dy.$$

Indeed, show that this holds for an elementary matrix t , and $\text{GL}_n(\mathbb{R})$ is the algebra generated by these elements.

(b) If $X \in \text{GL}_n(\mathbb{R})$, then $L_X : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ is isomorphic to

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} xy_1 \\ \vdots \\ xy_n \end{pmatrix} : (\mathbb{R}^n)^n \rightarrow (\mathbb{R}^n)^n$$

and hence $\det L_X = \det X^n$. Thus if $f \in C_c(M_n(\mathbb{R}))$, we have that

$$\int_{M_n(\mathbb{R})} f(xy) dy = \int_{M_n(\mathbb{R})} f \circ L_X(y) dy = \frac{1}{|\det X|^n} \int_{M_n(\mathbb{R})} f(y) dy.$$

Now since $\mathrm{GL}_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$, so $C_c(\mathrm{GL}_n(\mathbb{R})) \subset C_c(M_n(\mathbb{R}))$, and we define for $f \in C_c(\mathrm{GL}_n(\mathbb{R}))$

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{R})} f(y) \frac{1}{|\det y|^n} dy.$$

Then for $x \in \mathrm{GL}_n(\mathbb{R})$, we have

$$\begin{aligned} I(f \cdot x) &= \int_{\mathrm{GL}_n(\mathbb{R})} f(xy) \frac{1}{|\det xy|^n} \cdot |\det x|^n dy \\ &= \int_{\mathrm{GL}_n(\mathbb{R})} f(y) \frac{1}{|\det y|^n} \cdot \frac{|\det x|^n}{|\det x|^n} dy = I(f) \end{aligned}$$

If $M_{\mathrm{GL}_n(\mathbb{R})}$ is the measure associated with I , then with m the Lebesgue measure on $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we have

$$\frac{dM_{\mathrm{GL}_n(\mathbb{R})}}{dm}(y) = \frac{1}{|\det y|^n}$$

(c) If we take $\mathbb{R}^\times \cong \mathrm{GL}_1(\mathbb{R})$, then

$$I(f) = \int_{\mathbb{R}^\times} f(y) \frac{dy}{|y|}$$

(d) Consider $\mathbb{C}^\times = \mathbb{C} \setminus \{0\} \subseteq \mathbb{C} \cong \mathbb{R}^2$. Then $[L_{x+iy}]_{(1,i)} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ so that $\det L_{x+iy} = |x + iy|^2$. Thus we get an integral on $G = \mathbb{C}^\times$ by

$$I(f) = \int_{\mathbb{C}^\times} f(z) \frac{dz}{|z|^2}$$

(e) On $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{R})$, we likewise find Haar integral

$$I(f) = \int_{\mathrm{GL}_n(\mathbb{C})} f(y) \frac{1}{|\det y|^{2n}} dy.$$

(iv) Suppose G admits an open (hence closed) subgroup H . If m is a Haar measure on G , then $m_H = m|_{\mathcal{B}(H)}$ is a Haar measure on H . Let T be a transversal for left cosets of H (A of C), so $G = \bigcup_{t \in T} tH$. If $U \subset G$ is open with $m(U) < \infty$, then

$$\begin{aligned} \{t \in T : U \cap tH \neq \emptyset\} &= \{t \in T : m(U \cap tH) > 0\} \\ &= \bigcup_{n=1}^{\infty} \{t \in T : m(U \cap tH) < \frac{1}{n}\} \end{aligned}$$

is countable, so if $E \in \mathcal{B}(G)$, $m(E) < \infty$, $E \subseteq \bigcup_{j=1}^{\infty} t_j H$ and then

$$\begin{aligned} m(E) &= \sum_{j=1}^{\infty} m(E \cap t_j H) = \sum_{j=1}^{\infty} m(t_j^{-1}(E \cap t_j H)) \\ &= \sum_{j=1}^{\infty} m_H((t_j^{-1}E) \cap H) \end{aligned}$$

(v) Suppose H is a closed, non-open subgroup of G . We wish to see that for compact $K \subseteq H$, $m(K) = 0$.

(a) Given open U with $K \subseteq U$, then there is open V with $e \in V$ so $VK \subseteq U$. Indeed, for $x \in K$, find open U_x with $e \in U_x$, so $U_x x \subseteq U$. Find open V_x , $e \in V_x$, so $V_x^2 \subseteq U_x$, then $K \subseteq \bigcup_{j=1}^n V_{x_j} x_j$ where $x_1, \dots, x_n \in K$. Let $V = \bigcap_{j=1}^n V_{x_j}$. If $x \in K$, $x \in V_{x_j} x_j$ for some j so $Vx \subseteq V V_{x_j} x_j \subseteq V_{x_j}^2 x_j \subseteq U_{x_j} x_j \subseteq U$, i.e. $VK = \bigcup_{x \in K} Vx \subseteq U$.

(b) Suppose we had compact $K \subseteq H$ such that $m(K) > 0$. We may find open U so $K \subseteq U$ and $m(U) < 2m(K)$ (by outer regularity). Take V as above. Since H is non-open, there is $x \in V \setminus H$. Then

- $K \cap xK = \emptyset$ as $K \subseteq H$, while
- $K \cup xK \subseteq U$.

. Thus $2m(K) = m(K \cup xK) \leq m(U) < 2m(K)$, a contradiction.

Thus, a closed non-open subgroup H of G is always m -locally null. Hence, if G is σ -compact, then closed non-open H are m -null. However, if $G = \mathbb{R} \times \mathbb{R}_d$, $H = \{0\} \times \mathbb{R}_d$ is closed, m -locally null, but not m -null.

(vi) The measure on $(\mathbb{Q}_p, +)$ is determined by $(\mathbb{Q}_p, +)$. Likewise, the measure $\text{GL}_n(\mathbb{Q}_p)$ is determined by $\text{GL}_n(\mathbb{Q}_p) = \{a \in M_n(\mathbb{Q}_p) : \det a \in \mathbb{Q}_p^\times\}$

(vii) On $\text{GL}_n(\mathbb{Q}_p)$, we have Haar integral

$$I(f) = \int_{\text{GL}_n(\mathbb{Q}_p)} f(y) \frac{1}{|\det y|_p^n} dy$$

(viii) G is compact if and only if $m(G) < \infty$. The forward is clear since m is Radon. If G is not compact, let U be a open neighbourhood of e so \overline{U} is compact, so $0 < m(U) < \infty$. For any compact set K , $KU \subseteq \overline{KU}$ is compact, hence $KU \subsetneq G$. Inductively find $(x_n)_{n=1}^\infty \subset G$ so $x_{n+1} \notin \{x_1, \dots, x_n\}U$. Notice that $x_j V \cap x_k V = \emptyset$ for $j \neq k$ for V a neighbourhood of e with $V = V^{-1}$, $V^2 \subset U$. Then $m(G) \geq nm(V)$ for any $n \in \mathbb{N}$, so $m(G) = \infty$.

Let G be a locally compact group equipped with Haar measure m . Then

$$L^1(G) = \{f : G \rightarrow \mathbb{C} : f \text{ measurable}, \|f\|_1 = \int_G |f| dm < \infty\} / \sim_m \quad a.e.$$

This is a Banach space. Recall that by definition of the Lebesgue integral

$$S^1(G) = \text{span}\{\chi_E : E \in \mathcal{B}(G), m(E) < \infty\} / \sim_m \quad a.e.$$

If $0 < m(E) < \infty$, then, given $\epsilon > 0$, there are compact $K \subseteq E$ and open $U \supseteq E$. Hence Urysohn's lemma provides $f \in C_c^+(G)$ such that $f|_K = 1$, $\text{supp}(f) \subseteq U$, and $0 \leq f \leq 1$. Hence $\|f - 1_E\|_1 < \epsilon$. Note that if $f, g \in C_c(G)$, then $f = g$ m a.e. if and only if $f = g$. Thus $C_c(G) \subseteq L^1(G)$ is dense.

1.4 THE MODULAR FUNCTION

If $x \in G$, define $m_x : \mathcal{B}(G) \rightarrow [0, \infty]$ by $m_x(E) = m(Ex)$. Then since R_x is a homeomorphism, one may verify that

- m_x is left invariant,
- $m_x(U) = m(Ux) > 0$ if U is non-empty and open, and

- $m_x(K) = m(Kx) < \infty$ if K is compact.

Hence by uniqueness of Haar measure there exists some function $\Delta : G \rightarrow \mathbb{R}^\times$ such that $m_x = \Delta(x)m$. In fact, Δ is a group homomorphism. To see this, if $E \in \mathcal{B}(G)$ with $0 < m(E) < \infty$ and $x, y \in G$, then

$$\Delta(xy)m(E) = m(Exy) = \Delta(y)m(Ex) = \Delta(x)\Delta(y)m(E).$$

Denote by $x \cdot f$ the function $x \cdot f(y) = f(yx)$. We then have the following result:

1.12 Proposition. (i) For any $f \in L^1(G)$ and $x \in G$, $x \cdot f \in L^1(G)$ with

$$\int_G f \, dm = \Delta(x) \int_G x \cdot f \, dm.$$

(ii) Δ is a continuous function.

PROOF (i) Let $E \in \mathcal{B}(G)$ with $m(E) < \infty$. Then

$$\Delta(x) \int \mathbf{1}_E \, dm = \Delta(x)m(E) = m(Ex) = \int \mathbf{1}_{Ex} \, dm = \int_G x^{-1} \cdot \mathbf{1}_E \, dm$$

since $\mathbf{1}_{Ex} = x^{-1} \cdot \mathbf{1}_E$. Thus replacing x by x^{-1} , we have

$$\int \mathbf{1}_E \, dm = \Delta(x) \int x \cdot \mathbf{1}_E \, dm$$

so that the desired result holds for characteristic functions. Then by density of simple functions in L^1 and applying dominated convergence, the result holds for any $f \in L^1$.

(ii) Let $f \in C_c^+(G)$, $\epsilon > 0$, and $V = V^{-1}$ be a relatively compact neighbourhood of e so that $\|x \cdot f - f\|_\infty < \epsilon$ for any $x \in V$. Then for $x \in V$, applying (i) above,

$$\begin{aligned} |\Delta(x) - 1| &= \frac{|\Delta(x) \int f \, dm - \int f \, dm|}{\int f \, dm} \\ &\leq \frac{1}{\int f \, dm} \int |x^{-1} \cdot f - f| \, dm < \epsilon \frac{m(\text{supp}(f)V)}{\int f \, dm}. \end{aligned}$$

where $\text{supp}(x^{-1} \cdot f - f) \subseteq \text{supp}(f)V$ and $\text{supp}(f)V$ has compact closure so that $m(\text{supp}(f)V) < \infty$. Moreover, as $\epsilon \rightarrow 0$, we may arrange for V to be decreasing, yielding continuity of Δ at e . Now if $x, y \in G$ are arbitrary, we have

$$|\Delta(xy) - \Delta(y)| = |\Delta(x) - 1|\Delta(y)$$

which shows that Δ is continuous at y . ■

1.13 Proposition. (i) The integral $f \mapsto \int_G f(x) \frac{1}{\Delta(x)} \, dx$ on $C_c(G)$ is right invariant.

(ii) For $f \in L^1(G)$,

$$\int_G f(x^{-1}) \frac{1}{\Delta(x)} \, dx = \int_G f(x) \, dx$$

PROOF (i) If $y \in G$ and $f \in C_c(G)$, then

$$\begin{aligned} \int_G y \cdot f(x) \frac{1}{\Delta(x)} dx &= \int_G f(xy) \frac{1}{\Delta(x)} dx = \int_G f(xy) \frac{1}{\Delta(xy)} \Delta(y) dx \\ &= \int_G f(x) \frac{1}{\Delta(x)} dx \end{aligned}$$

(ii) If $f \in C_c^+(G)$, then for any $y \in G$,

$$\begin{aligned} \int_G f \cdot y(x^{-1}) \frac{1}{\Delta(x)} dx &= \int_G f((xy^{-1})^{-1}) \frac{1}{\Delta(x)} dx \\ &= \int_G f(x^{-1}) \frac{1}{\Delta(x)} dx \end{aligned}$$

by the proof above. Hence by uniqueness of left Haar integral, there is $c > 0$ so that

$$\int_G f(x^{-1}) \frac{1}{\Delta(x)} dx = c \int_G f(x) dx$$

for $f \in C_c(G)$. Notice, by continuity of $f \mapsto \int_G f dm$ on $L^1(G)$, this holds for $f \in L^1(G)$. Now, if $c \neq 1$, there is a relatively compact neighbourhood $U = U^{-1}$ of e such that $|\Delta(x) - 1| < \frac{1}{2}|c - 1|$ for $x \in U$. Then we have

$$\begin{aligned} 0 &= \left| \int_G \mathbf{1}_U(x^{-1}) \frac{1}{\Delta(x)} dx - c \int_G \mathbf{1}_U(x) dx \right| \\ &= \left| \int_U \left(\frac{1}{\Delta(x)} - c \right) dx \right| \\ &= \left| \int_U \left(\frac{1}{\Delta(x)} - 1 + 1 - c \right) dx \right| \\ &\geq m(U) \left| 1 - c - \frac{1}{2}|c - 1| \right| = \frac{m(U)}{2} |1 - c| > 0 \end{aligned}$$

which is a contradiction. ■

For $f \in L^1(G)$, $x \in G$, we let

$$\begin{aligned} x * f(y) &= f(x^{-1}y) \\ f * x(y) &= f(yx^{-1}) \frac{1}{\Delta(x)} \\ f^*(x) &= \overline{f(x^{-1})} \frac{1}{\Delta(x)} \end{aligned}$$

Notice that $\|f\|_1 = \|x * f\|_1 = \|f * x\|_1 = \|f^*\|_1$. Moreover,

- $x' * (x * f) = (x'x) * f$ and $(f * x) * x' = f * (xx')$
- $f^{**} = f$.

- $(x * f)^* = f^* * x^{-1}$. Indeed, for m -a.e. y , we have

$$\begin{aligned} (x * f)^*(y) &= \overline{[x * f](y^{-1})} \frac{1}{\Delta(y)} \\ &= \overline{f(x^{-1}y^{-1})} \frac{1}{\Delta(y)} \\ &= \overline{f((yx)^{-1})} \frac{1}{\Delta(yx)} \frac{1}{\Delta(x^{-1})} \\ &= f^* * x^{-1}(y) \end{aligned}$$

1.14 Proposition. For $f \in L^1(G)$, $\lim_{x \rightarrow e} \|x * f - f\|_1 = 0 = \lim_{x \rightarrow e} \|f * x - f\|_1$.

PROOF First, consider $g \in C_c(G)$ and $\epsilon > 0$, and let $V = V^{-1}$ be a relatively compact neighbourhood of e so $\|g \cdot x - g\|_\infty < \epsilon$ for $x \in V$. Then

$$\|x * g - g\|_1 = \int_G |g \cdot x^{-1}| dm \leq \|g \cdot x^{-1} - g\|_\infty m(V \text{ supp}(g)) < \epsilon m(V \text{ supp}(g))$$

so $\lim_{x \rightarrow e} \|x * g - g\|_1 = 0$. If $f \in L^1(G)$, $\epsilon > 0$, find $g \in C_c(G)$ such that $\|f - g\|_1 < \epsilon$. Then

$$\begin{aligned} \|x * f - f\|_1 &\leq \|x * f - x * g\|_1 + \|x * g - g\|_1 + \|g - f\|_1 \\ &< 2\epsilon + \|x * g - g\|_1 \end{aligned}$$

where $\|x * g - g\|_1 \rightarrow 0$ as $x \rightarrow e$. Since $\epsilon > 0$ was arbitrary, we are done.

Now, for f, x as above,

$$\|f_x - f\|_1 = \|(f * x - f)^*\|_1 = \|x^{-1} * f^* - f^*\|_1$$

where $x^{-1} \rightarrow e$ as $x \rightarrow e$. ■

1.15 Theorem. (Weil Integral Formula) Let N be a closed normal subgroup of G .

- (i) If $f \in C_c(G)$, then $x \mapsto \int_N f(xn) dn : G \rightarrow \mathbb{C}$ is constant on cosets and hence defines a function $T_n f : G/N \rightarrow \mathbb{C}$. Furthermore, $T_N(C_c^*(G)) \subseteq C_c^+(G/N)$ and the operator $T_N : C_c(G) \rightarrow C_c(G/N)$ is linear and covariant:

$$(T_N f) \cdot (yN) = T_N(f \cdot y)$$

for $f \in C_c(G)$ and $y \in G$.

- (ii) The functional on $C_c(G)$ given by $f \mapsto \int_{G/N} T_n f(xN) dxN$ is hence a Haar integral on G , so we may write

$$\int_{G/N} \int_N f(xn) dn dxN = \int_G f(x) dx$$

PROOF (ii) is a direct consequence of (i); let's see the proof of (i).

The N -invariance of the first function is evident. Let $f \in C_c(G)$. We inspect the continuity of $T_N f$ at x in G . Given $\epsilon > 0$, let $V = V^{-1}$ be a real compact neighbourhood

of e , so $\|f \cdot y - f\|_\infty < \epsilon$ for $y \in V$. Let $g \in C_c^+(G)$ satisfy that $0 \leq g \leq 1$ and $g|_{Vx^{-1}\text{supp}(f)} = 1$. For $y \in V$, $yN = q_N(y) \in q_N(V)$ so

$$\begin{aligned} |T_N f(yxN) - T_N f(xN)| &\leq \int_N |f(yx) - f(xn)|g(n) \, dn \\ &< \epsilon m_N(\text{supp}(g) \cap N) \end{aligned}$$

Notice as $\epsilon \rightarrow 0$, we may shrink V and hence $\text{supp}(g)$. Hence $T_N f$ is continuous at xN . Now, $\text{supp}(T_N f) \subseteq q_N(\text{supp } f)$ so $T_N f \in C_c(G/N)$.

If $g \in C_c^+(G)$, $x \in G$ is such that $f(x) > 0$, let U be a neighbourhood of e , $f(xy) > \frac{1}{2}f(x)$ for $y \in U$. Then

$$T_N f(xN) = \int_N f(xn) \, dn \geq \frac{1}{2}f(x)m_N(U \cap N) > 0 \quad \blacksquare$$

1.16 Corollary. *If N is closed and normal in G , Then $\Delta_G|_N = \Delta_N$.*

PROOF Let $n' \in N$ and $f \in C_c^+(G)$. Then

$$\begin{aligned} \int_G n' \cdot f(x) \, dx &= \int_{G/N} \int_N f(xnn') \, dn \, dxN \\ &= \int_{G/N} \frac{1}{\Delta_N(n')} \int_N f(xn) \, dn \, dxN = \frac{1}{\Delta_N(n')} \int_G f(x) \, dx \end{aligned}$$

so that $\Delta_G(n') = \Delta_N(n')$. ■

Definition. We say that G is **unimodular** if $\Delta = 1$ on G .

1.17 Proposition. *G is unimodular in any of the following cases:*

- (i) G is abelian, discrete, or compact
- (ii) G is perfect: $G = \overline{[G, G]}$
- (iii) $G/Z(G)$ is unimodular.
- (iv) There is a closed, unimodular normal subgroup N such that G/N is compact.

PROOF (i) This is (nearly) obvious for G abelian or discrete. If G is compact, then $\log \circ \Delta : G \rightarrow (\mathbb{R}, +)$ is a continuous homomorphism whose range is a compact subgroup.

(ii) Any commutator $[x, y] \in xyx^{-1}y^{-1} \in \ker \Delta$.

(iii) Let $Z = Z(G)$. If $y \in G$ and $f \in C_c(G)$, we have by Weyl's integral formula

$$\begin{aligned} \int_G y \cdot f(x) dx &= \int_{G/Z} \int_Z f(xz) dz dx Z \\ &= \int_{G/Z} \int_Z f(xyz) dz dx Z \\ &= \int_{G/Z} T_Z f(xyZ) dx Z \\ &= \int_{G/Z} T_Z f(xZYZ) dx Z \\ &= \int_{G/Z} T_Z f(xZ) dx Z = \int_G f(x) dx \end{aligned}$$

(iv) We have $\Delta_{\underline{G}}|_N = \Delta_N = 1$ by assumption, i.e. $N \in \ker \Delta_G$, so $\Delta_{\underline{G}}$ induces a homomorphism $\bar{\Delta} : G/N \rightarrow (0, \infty)$ where $\Delta_G = \bar{\Delta} \circ \pi_N$. Verify that $\bar{\Delta}$ is continuous, so $\log \circ \bar{\Delta} : G/N \rightarrow \mathbb{R}^+$ is a continuous homomorphism, whose range is a closed subgroup. It follows that $\Delta_G = 1$ on G . ■

Example. Here are some examples of unimodular groups.

(i) Let \mathbb{k} be a locally compact field with $|\mathbb{k}| > 3$. Then $\mathrm{SL}_n(\mathbb{k})$ is perfect.

Let $\{E_{ij}\}_{i,j=1}^n$ be a matrix unit for $M_n(\mathbb{k})$, so $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$.

If $\lambda \in \mathbb{k}$, i, j, k distinct (i.e. $n \geq 3$), then

$$[e + \lambda E_{ik}, e + E_{kj}] = (e + \lambda E_{ik})(e + E_{kj})(e - \lambda E_{ik})(e - E_{kj}) = e + \lambda E_{ij}.$$

If $n = 2$, then

$$\left[\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right] = e + \lambda E_{12}$$

where $\lambda = (1 - \alpha^2)\beta$.

If $S = \langle e + \lambda E_{ij} : \lambda \in \mathbb{k}, i \neq j \rangle$. Using only elementary operations induced by multiplying by elements of S on the left, and element a of $\mathrm{SL}_n(\mathbb{k})$ satisfies (see pic)

By an evident induction, we see that there are $s_1, s_2 \in S$ so $s_1 s a s_2 = e$. Thus $a = s^{-1} s_1^{-1} s_2^{-1} \in S$.

(ii) Let $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . Consider $G = \mathrm{GL}_n(\mathbb{k})$. Notice that $Z = Z(G) = \mathbb{k}^\times e$. From (i), $\mathrm{SL}_n(\mathbb{k})$ is perfect.

Let $H = Z \cdot \mathrm{SL}_n(\mathbb{k})$, which is closed (check!) and $H/Z \cong \mathrm{SL}_n(\mathbb{k})/Z \cap \mathrm{SL}_n(\mathbb{k})$ is perfect, being the quotient of a perfect group, hence unimodular.

If $\mathbb{k} = \mathbb{C}$ or $\mathbb{k} = \mathbb{R}$ and n is odd, $H = G$. Else if $\mathbb{k} = \mathbb{R}$ and n is even, then $H = \mathrm{GL}_n(\mathbb{R})_e = \det^{-1}((0, \infty))$ (connected component of e) and $G = \mathrm{GL}_n(\mathbb{R})_e \cup (-e) \mathrm{GL}_n(\mathbb{R})_e$, so $G/H \cong \{-1, 1\}$ is compact.

(iii) $E(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$. Since $N = \mathbb{R}^n \rtimes \{e\}$ is closed, normal, and abelian, and $G/N = \mathrm{SO}(n)$ is compact.

(iv) Consider

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

then

$$Z(\mathbb{H}) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

has $\mathbb{H}/Z \cong \mathbb{R}^2$.

Remark. In A1, a “Braconnier” modular function $\delta : \text{Aut}(G) \rightarrow (0, \infty)$ is defined.

- (i) If $\gamma : G \rightarrow \text{Aut}(G)$ has $\gamma(x)(y) = xyx^{-1}$, so γ is a homomorphism. Then $\delta(\gamma(x)) = \frac{1}{\Delta(x)}$.
- (ii) If G is compact and $\alpha \in \text{Aut}(G)$, then $\alpha(G) = G$ so $1 = m(G) = m(\alpha(G))$ and $\delta(\alpha) = 1$.
- (iii) If G is discrete and $\alpha \in \text{Aut}(G)$, then for any non-empty finite $F \subseteq G$, we have $|F| = |\alpha(F)|$, and it follows that $\delta(\alpha) = 1$.
- (iv) If G is unimodular and H an open subgroup of G , then H is unimodular. However, there is some subtlety here:

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

is closed in $\text{SL}_2(\mathbb{R})$ and $H \cong \mathbb{R} \rtimes (0, \infty)$ is not unimodular. Moreover,

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}, a > 0 \right\}$$

is open, normal and abelian in $G = \left\{ \begin{pmatrix} 2^n & r \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}, r \in \mathbb{R} \right\}$ and G is closed in $\text{GL}_2(\mathbb{R})$.

1.5 THE CONVOLUTION ALGEBRA OF MEASURES

Let G be a locally compact group. Let

$$\begin{aligned} M(G) &= \{\mu : \mathcal{B}(G) \rightarrow \mathbb{C} : \mu \text{ Radon measure}\} = \text{span } M_+(G) \\ M_+(G) &= \{\mu : \mathcal{B}(G) \rightarrow [0, \infty) : \mu \text{ Radon}\} \end{aligned}$$

If $\mu \in M_+(G)$ with $\mu(G) < \infty$ so μ is finite.

Recall the Hahn-Jordan Decomposition: each μ in $M(G)$ admits a decomposition $\mu = \sum_{k=0}^3 i^k \mu_k$ where each $\mu_i \in M_+(G)$, $\mu_0 \perp \mu_2$, and $\mu_1 \perp \mu_3$. Any measures satisfying this decomposition are unique.

Definition. If $\mu \in M(G)$, we define the $|\mu| : \mathcal{B}(G) \rightarrow [0, \infty)$ by

$$|\mu|(E) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(E_k)| : E = \bigcup_{k=1}^{\infty} E_k, E_k \in \mathcal{B}(G) \right\}$$

and $|\mu| \in M_+(G)$. If $\mu = \sum_{k=0}^3 i^k \mu_k$ as in Hahn-Jordan, then $|\mu_0 - \mu_2| = \mu_0 + \mu_2$ and $|\mu_1 - \mu_3| = \mu_1 + \mu_3$. Furthermore,

$$|\mu| \leq |\mu_0 - \mu_2| + |\mu_1 - \mu_3| \text{ and } |\mu_0 - \mu_2| |\mu_1 - \mu_3| \leq |\mu|$$

1.18 Theorem. (Riesz-Markov Duality) Let $C_0(G) = \overline{C_c(G)} \subseteq C_b(G)$ with the uniform topology. Then $C_0(G)^* \cong M(G)$ through the map $\mu \mapsto \langle \mu, \cdot \rangle$ where $\langle \mu, f \rangle = \int_G f d\mu$. Moreover, $\|\langle \mu, \cdot \rangle\|_{op} = |\mu|(G)$.

Remark. Let $\mathcal{B}^\infty(G) = \{f : G \rightarrow \mathbb{C} : f \text{ bounded and Borel measurable}\}$, which is a Banach space under the uniform norm. Note that $\mathcal{B}^\infty(G) = \overline{\text{span}\{1_E : E \in \mathcal{B}(G)\}}$. We have

$$\left| \int_G f d\mu \right| \leq \int_G |f| d|\mu| \leq \|f\|_\infty \|\mu\|_1$$

If $\mu \in M(G)$ and $\epsilon > 0$, then inner regularity provides compact $K \subseteq G$ such that $|\mu|(K) > |\mu|(G) - \epsilon$. Hence $|\mu|(G \setminus K) < \epsilon$. Then $\|\mu - \mu_K\|_1 = \|\mu_{G \setminus K}\|_1 = |\mu|(G \setminus K) < \epsilon$.

1.19 Lemma. (Continuous Fubini) Let X, Y be locally compact spaces, $\mu \in M(X)$, $\nu \in M(Y)$. Then there is a measure $\mu \times \nu \in M(X \times Y)$ such that

$$\int_{X \times Y} f d(\mu \times \nu) + \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu$$

for any $f \in C_b(X \times Y)$.

PROOF Let $\mathcal{A} = \text{span}\{\phi \times \psi : \phi \in C_0(X), \psi \in C_0(Y)\}$ where $\phi \times \psi(x, y) = \phi(x)\psi(y)$. Then $\overline{\mathcal{A}} = C_0(X \times Y)$ in the uniform topology. We define for $f \in \mathcal{A}$

$$J(f) = \int_X \int_Y f d\nu d\mu \int_Y \int_X f d\mu d\nu$$

so that J is linear on \mathcal{A} and

$$|J(f)| \leq \|f\|_\infty |\mu|(X) |\nu|(Y)$$

so J is bounded. Thus J is uniformly continuous and hence extends uniquely to $C_0(X \times Y)$ as a bounded linear functional with $\|J\| \leq \|\mu\|_1 \|\nu\|_1$. By Riesz-Markov, there is $\mu \times \nu \in M(X \times Y)$ such that $J(f) = \int_{X \times Y} f d(\mu \times \nu)$. Uniform limits are pointwise limits, so LDCT tells us that we have Fubini for $f \in C_0(X \times Y)$.

By inner regularity, find $(K_n)_{n=1}^\infty$ and $(L_n)_{n=1}^\infty$ so that $\lim_{n \rightarrow \infty} \|\mu - \mu_{K_n}\|_1 = 0 = \lim_{n \rightarrow \infty} \|\nu - \nu_{L_n}\|_1$. For each n , let $f_n \in C_c(X \times Y)$ be such that $f|_{K_n \times L_n} = f_n|_{K_n \times L_n}$ (Urysohn). We also notice that $(\mu \times \nu)_{K_n \times L_n} = \mu_{K_n} \times \nu_{L_n}$. Then check that $\lim_{n \rightarrow \infty} \|(\mu \times \nu)_{K_n \times L_n} - \mu \times \nu\|_1 = 0$. Then for $f \in C_b(X \times Y)$,

$$\begin{aligned} \int_{X \times Y} f d(\mu \times \nu) &= \lim_{n \rightarrow \infty} \int_{X \times Y} f d(\mu \times \nu)_{K_n \times L_n} \\ &= \lim_{n \rightarrow \infty} \int_{X \times Y} f_n d(\mu \times \nu)_{K_n \times L_n} \\ &= \lim_{n \rightarrow \infty} \int_X \int_Y f_n d\nu_{L_n} d\mu_{K_n} \\ &= \lim_{n \rightarrow \infty} \int_X \int_Y f d\nu_{L_n} d\mu_{K_n} = \int_X \int_Y f d\nu d\mu \end{aligned} \quad \blacksquare$$

1.20 Theorem. Given μ, ν in $M(G)$, there is a unique measure $\mu * \nu$ in $M(G)$ such that

$$\int_G f d\mu * \nu = \int_G \int_G f(xy) d\mu(x) d\nu(y)$$

for $f \in C_0(G)$. The map $(\mu, \nu) \mapsto \mu * \nu$ is bilinear, associative, and satisfies $\|\mu * \nu\|_1 \leq \|\mu\|_1 \|\nu\|_1$. Hence $(M(G), *)$ is a Banach algebra.

CLAIM I Given $\mu \in M(G)$ and $f \in C_0(G)$, define $f \cdot \mu, \mu \cdot f : G \rightarrow \mathbb{C}$ by

$$\begin{aligned} f \cdot \mu(x) &= \langle \mu, x \cdot f \rangle = \int_G f(yx) d\mu(y) \\ \mu \cdot f(x) &= \langle \mu, f \cdot x \rangle = \int_G f(xy) d\mu(y) \end{aligned}$$

Notice that $\mu \cdot f(e) = f \cdot \mu(e)$. Then $f \cdot \mu, \mu \cdot f$ in $C_0(G)$.

PROOF Indeed, let us check continuity of $\mu \cdot f$. Let $\epsilon > 0$ and V be a neighbourhood of e such that $|f(x) - f(x')| < \epsilon$ for $x'x^{-1} \in V$ (uniform continuity, $\overline{C_c(G)} = C_0(G)$, uniform limit of uniformly cts is uniformly cts). Then

$$|\mu \cdot f(x) - \mu \cdot f(x')| \leq \int_G |f(xy) - f(x'y)| d|\mu|(y) \leq \epsilon |\mu|(G)$$

is uniformly continuous, hence continuous. Notice that $|\mu \cdot f(x)| \leq \int_G |f(xy)| d|\mu|(Y) \leq \|f\|_\infty \|\mu\|_1$. Now, given $\epsilon > 0$, let $K \subseteq G$ be compact so $\|\mu - \mu_K\|_1 < \epsilon$ and $f' \in C_c(G)$ be so $\|f - f'\|_\infty < \epsilon$. Then

$$\begin{aligned} \|\mu \cdot f - \mu_K \cdot f'\|_\infty &\leq \|\mu \cdot f - \mu_K \cdot f\|_\infty + \|\mu_K \cdot f - \mu_K \cdot f'\|_\infty \\ &\leq \|\mu - \mu_K\|_1 \|f\|_\infty + \|\mu_K\|_1 \|f - f'\|_\infty \\ &< (\|\mu\|_1 + \|f\|_\infty) \epsilon \end{aligned}$$

and $\text{supp}(\mu_K \cdot f') \subseteq \text{supp}(f')K^{-1}$, so $\mu_K \cdot f' \in C_c(G)$. Thus $\mu \cdot f \in C_0(G)$. Similarly, $f \cdot \mu \in C_0(G)$. It is evident that $(\mu, f) \mapsto \mu \cdot f$ and $(f, \mu) \mapsto f \cdot \mu$ are bilinear.

CLAIM II If $\mu, \nu \in M(G)$, then $\mu \cdot (f \cdot \nu) = (\mu \cdot f) \cdot \nu$ and $\langle \mu, f \cdot \mu \rangle = \langle \nu, \mu \cdot f \rangle$.

PROOF Use Fubini for continuous functions: for x in G ,

$$\begin{aligned} \mu \cdot (f \cdot \nu)(x) &= \int_G (f \cdot \nu)(xy) d\mu(y) = \int_G \int_G f(zxy) d\nu(z) d\mu(y) \\ &= \int_G \int_G f(zxy) d\mu(y) d\nu(z) \\ &= \int_G (\mu \cdot f)(zx) d\nu(z) = (\mu \cdot f) \cdot \nu(x) \end{aligned}$$

CLAIM III For μ, ν in $M(G)$, define for f in $C_0(G)$

$$\langle \mu * \nu, f \rangle = \langle \mu, \nu \cdot f \rangle = \langle \nu, f \cdot \mu \rangle$$

Then $\mu * \nu$ is unique and satisfies the required properties.

PROOF Uniqueness follows by Riesz-Markov (...?)
 Moreover, $(\mu, \nu) \mapsto \mu * \nu$ is evidently bilinear and

$$|\langle \mu * \nu, f \rangle| = |\langle \mu, \nu \cdot f \rangle| \leq \|\mu\|_1 \|\nu \cdot f\|_\infty \leq \|\mu\|_1 \|\nu\|_1 \|f\|_\infty.$$

This shows that

- $f \mapsto \langle \mu * \nu, f \rangle$ is bounded, and hence $\mu * \nu$ is an element of $M(G)$ (by Riesz-Markov)
- $\|\mu * \nu\|_1 \leq \|\mu\|_1 \|\nu\|_1$

It remains to see associativity. If $\rho \in M(G)$, then for $f \in C_0(G)$,

$$\begin{aligned} \langle \mu * (\nu * \rho), f \rangle &= \langle \nu * \rho, f \cdot \mu \rangle = \langle \nu, \rho \cdot (f \cdot \mu) \rangle \\ &= \langle \mu * \nu, \rho \cdot f \rangle \end{aligned}$$

■

Remark. (i) If $\mu \in M(G)$, let $R_\mu, L_\mu : C_0(G) \rightarrow C_0(G)$ be given by $L_\mu f = \mu \cdot f$, $R_\mu f = f \cdot \mu$. Then for $\nu \in M(G)$, $\mu * \nu = R_\mu^*(\nu) = L_\nu^*(\mu)$ which shows $\nu \mapsto \mu * \nu$ or $\nu \mapsto \nu * \mu$ are each $w^* - w^*$ -continuous. Note that $(\mu, \nu) \mapsto \mu * \nu$ may not be $w^* - w^*$ -continuous.

(ii) Let for x in G $\delta_x(E)$ be the point mass measure at x . Then $\langle \delta_x, f \rangle = f(x)$ for $f \in C_0(G)$. Then $\delta_x * \delta_y = \delta_{xy}$.

(iii) If $\mu, \nu \in M_+(G)$, then $\mu * \nu \in M_+(G)$.

1.6 ATOMIC-CONTINUOUS AND LEBESGUE DECOMPOSITIONS

Let $\mu \in M(G)$ and set

$$A(\mu) = \{x \in G : |\mu|(\{x\}) > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in G : |\mu|(\{x\}) > \frac{1}{n}\right\}$$

so $A(\mu)$ is countable. For any $x \in G$, $|\mu|(\{x\}) = |\mu(\{x\})|$ by definition of $|\mu|$ and hence

$$\sum_{x \in A(\mu)} |\mu|(\{x\}) = \sum_{x \in A(\mu)} |\mu(\{x\})| = |\mu|(A(\mu)) < \infty.$$

We then define the **atomic** or **discrete** part of μ by

$$\mu_d = \sum_{x \in A(\mu)} \mu(\{x\}) \delta_x \in M(G)$$

and the **continuous** part by

$$\mu_c = \mu - \mu_d$$

Then $\mu_c \perp \mu_d$ so

$$\|\mu\|_1 = |\mu|(G) = |\mu_c|(G) + |\mu_d|(G) = \|\mu_c\|_1 + \|\mu_d\|_1$$

The set $M_d(G) = \overline{\text{span}}\{\delta_x : x \in G\}$ is a subspace of $M(G)$, and $M_d(G) \cong \ell^1(G)$ isometrically. Thus $M_c(G) = \text{im } P_c$ is a closed subspace.

If G is discrete, then $|\mu_c|(G) = 0$ so $M(G) = M_d(G) \cong \ell^1(G)$.

If G is not discrete, then $\{e\}$ is a closed, non-open subgroup, so for $x \in G$, $m(\{x\}) = 0$ where m is the Haar measure. Hence the measures absolutely continuous with respect to m satisfy $M_d(G) \subseteq M_c(G)$. We can employ the Lebesgue decomposition to write $\mu_c = \mu_a + \mu_{cs}$

where $\mu_a \ll m$ and $\mu_a \perp \mu_{cs}$. Moreover, the Radon-Nikodym derivative has $\frac{d\mu}{dm} \in L^1(G)$. To conclude,

$$\begin{aligned} M(G) &= M_c(G) \oplus_1 M_d(G) \\ &= M_a(G) \oplus_1 M_{cs}(G) \oplus_1 M_d(G) \\ &\cong L^1(G) \oplus_1 M_{cs}(G) \oplus_1 \ell^1(G) \end{aligned}$$

Certainly $\ell^1(G)$ is a subalgebra. We will show that the remaining components are also subalgebras.

2 ABELIAN LOCALLY COMPACT GROUPS

3 COMPACT GROUPS

4 INTRODUCTION TO AMENABILITY THEORY