Harmonic Analysis

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Contents

I. Harmonic Analysis

1 Locally Compact Groups

Definition. Let G be a group. A topology τ on G is a **group topology** provided that
- $x \mapsto x^{-1} : G \to G$ is continuous, and
- $(x,y) \mapsto xy : G \times G \to G$ is continuous.

We call (G,τ) a **topological group** where we omit τ when it is clear from context.

Equivalently, we may assert that $(x,y) \mapsto xy^{-1}$ is $\tau \times \tau - \tau$–continuous. Write $L_g(x) = gx$ and $R_g(x) = xg$ to denote the left and right multiplication maps; then it is easy to see that L_g and R_g are homeomorphisms. Similarly, $x \mapsto x^{-1}$ is a homeomorphism.

Definition. We say that a subset $A \subset G$ is **symmetric** if $A^{-1} = A$.

We have the following basic properties:

1.1 Proposition. *Let (G,τ) be a topological group.*
(i) If $\emptyset \neq A \subseteq G$ and U is open, then $AU = \{ay : a \in A, y \in U\}$ and likewise UA are open.
(ii) Given $U \in \tau$ and $x \in U$, then there is a symmetric $V \in \tau$ with $e \in V$ such that $VxV \subseteq U$. In particular, if $e \in U$, then we can find symmetric V so that $V^2 \subseteq U$.
(iii) If H is a subgroup of G, then $\overline{H}$ is also a subgroup.
(iv) An open subgroup is automatically closed.
(v) If $K, L \subseteq G$ are compact, then KL is compact.
(vi) If K is compact and C is closed in G, then KC is closed.

In $(\mathbb{R},+)$, then $\mathbb{Z}+\sqrt{2}\mathbb{Z}$ is not closed, so it is necessary to assume compactness in (vi).

Proof (i) $AU = \bigcup_{a\in A} L_a(U)$ is a union of open sets.

(ii) Consider the continuous map $(y,z) \mapsto yxz$. Since $exe = x \in U$, there is a $\tau \times \tau$–neighbourhood of (e,e) which maps into U have a basic neighbourhood $V_1 \times V_2$. Let $V = V_1 \cap V_2$. Moreover, we may replace V by $V^{-1} \cap V$. to attain symmetry.

(iii) Let $x, y \in \overline{H}$, $U \in \tau$ with $xy \in U$. Then (ii) provides V with $VxyV \subseteq U$. But $Vx \cap H \neq \emptyset$ and $\neq yV \cap H$ so there are $_1 \in Vx \cap H$, $h_2 \in yV \cap H$, and $h_1h_2 \in VxyV \subseteq U$. Thus $U \cap H \neq \emptyset$. Thus $xy \in \overline{H}$.
To use nets for inverses, if $x \in \overline{H}$, then $x = \lim_\alpha x_\alpha$ where $(x_\alpha)_{\alpha\in A} \subset H$ is a net. Then $x^{-1} = \lim_\alpha x_\alpha^{-1} \in \overline{H}$ as each $x_\alpha^{-1} \in H$.

(iv) If H is an open subgroup, then $H = G \setminus \bigcup_{x\in G\setminus H} xH$ is closed.

(v) $K \times L$ is compact, and hence so is its image under multiplication.

(vi) If $x \in \overline{KC}$, then $x = \lim_\alpha k_\alpha x_\alpha$ where $k_\alpha \in H$ and $x_\alpha \in C$. Since K is compact, we may assume (passing to a subnet if necessary) $k = \lim_\alpha k_\alpha$ exists in K. Then

$$k^{-1}x = \lim_\alpha k_\alpha^{-1} \cdot \lim_\alpha k_\alpha x_\alpha = \lim_\alpha k_\alpha^{-1} k_\alpha x_\alpha = \lim x_\alpha \in C$$

so $x = kk^{-1}x \in KC$. ■

1.1 Homogenous Spaces

Let (G,τ) be a topological group, H a subgroup of G, and $G/H = \{xH; x \in G\}$. Let $\pi : G \to G/H$ be given by $\pi(x) = xH$ be the projection map. The **quotient topology** on G/H is $\tau_{G/H} = \{W \in G/H : \pi^{-1}(W) \in \tau\}$. Notice that if $U \in \tau \setminus \{\emptyset\}$, then $UH = \pi^{-1}(\pi(U))$ is open, so $\pi : G \to G/H$ is an open map.

1.2 Proposition. *Let (G,τ), H be as above.*

(i) The map $(x, yH) \mapsto xyH : G \times G/H \to G/H$ is $\tau \times \tau_{G/H} - \tau_{G/H}$ continuous and open.

(ii) If H is normal, then $(G/H, \tau_{G/H})$ is a topological group.

(iii) If H is closed, then $\tau_{G/H}$ is Hausdorff.

Proof (i) Let $x, y \in G$, $W \in \tau_{G/H}$ satisfy $xyH = \pi(xy) \in W$. Then $xy \in \pi^{-1}(W)$ and by Proposition 1.1 we have $V \in \tau$ with $e \in V$ such that $VxyV \subseteq \pi^{-1}(W)$. But then $(x, \pi(y)) \in Vx \times \pi(yV) \in \tau \times \tau_{G/H}$ and the latter set maps into $\pi(VxyV) \subseteq W$. Also, if $U \in \tau \times \tau_{G/H}$, then $U = \bigcup_{(x,yH)\in U} V_x \times W_{yH}$ and

$$\pi(U) = \bigcup_{(x,yH)\in U} \pi(V_x \pi^{-1}(W_{yH}))$$

since π is open.

(ii) Recall that $(xH)(yH) = xyH$ is our multiplication operation on G/H and π is a group homomorphism. Then the following diagram commutes: We have that $\pi \times \mathrm{id}$ is open and $(x, yH) \mapsto xyH$ is open from (i), so the multiplication from $G/H \times G/H \to G/H$ must be open and continuous.

(iii) If $x, y \in G$ with $\pi(x) \neq \pi(y)$, then $e \notin xHy^{-1}$. Now $xHy^{-1} = L_x(R_{y^{-1}}(H))$ so xHy^{-1} is closed. Hence by the last proposition, there is a symmetric open V with $e \in V$ so $V^2 \subseteq G \setminus (xHy^{-1})$. But then $e \notin (VxH)(VyH)^{-1} = VxHy^{-1}V$: if we had $e = vxhy^{-1}u$ with $v, u \in V$ and $h \in H$, then $v^{-1}u^{-1} = xhy^{-1} \in V^2 \cap (xHy^{-1}) = \emptyset$, a contradiction. Hence $VxH \cap VyH = \emptyset$ so $\pi(Vx)$, $\pi(Vy)$ is a pair of separating neighbourhoods of $\pi(x)$, $\pi(y)$. ■

1.3 Corollary. *G is Hausdorff if and only if there exists $x \in G$ so that $\{x\}$ is closed.*

Proof In a Hausdorff space, points are closed. Conversely, if $\{x\}$ is closed, $\{e\} = L_{x^{-1}}(\{x\})$ is closed and a normal subgroup. Then $G \cong G/\{e\}$ is Hausdorff. ■

If (G,τ) is not Hausdorff, then $\{e\} \subsetneq \overline{\{e\}}$ is the smallest closed subgrup in G. Thus $\overline{\{e\}} \subseteq \bigcap_{x\in G} x\overline{\{e\}}x^{-1} \subseteq \overline{\{x\}}$ so $\overline{\{e\}}$ is normal. In particular, $G/\overline{\{e\}}$ is Hausdorff.

Definition. A **locally compact group** is a Hausdorff topological group (G,τ) which is locally compact.

(i) If there is any $U \in \tau \setminus \{\emptyset\}$ such that $\overline{U}$ is compact, then for any $x \in U$, we have $e \in x^{-1}U \subseteq L_{x^{-1}}(\overline{U})$ so $\overline{x^{-1}U}$ is compact. If $V \in \tau$ with $e \in V$ and $\overline{V}$ compact, then for any $x \in H$, $x \in xV$ and $\overline{xV} \subseteq L_x(\overline{V})$ and $\overline{xV}$ is compact. In particular, (G,τ) is locally compact if and only if there is some $U \in \tau \setminus \{\emptyset\}$ with $\overline{U}$ compact.

(ii) If (G, τ) is locally cmpact and N is a closed normal subgroup, then $({}^{G}\!/_{N}, \tau_{{}^{G}\!/_{N}})$ is locally compact. Indeed, $U \in \tau \setminus \{e\}$ with $\overline{U}$ compact, then $\overline{\pi(U)} \subseteq \pi(\overline{U})$ is compact.

Example. (i) If G is any group and τ is the discrete topology, then (G, τ_d) is locally compact.

(ii) If $((\mathbb{R}, +), \tau_{\|\cdot\|})$ is locally compact.

(iii) If $\{G_i\}_{i \in I}$ is a family of locally compact groups, then $\prod_{i \in I} G_i$ is a locally compact group if and only if all but finitely many (G_i, τ_i) are compact.

(iv) $((\mathbb{R}^n, +), \tau_{\|\cdot\|})$ is a locally compact group

(v) Suppose $\{F_i\}_{i \in I}$ is an infinite family of finite groups (with discrete topologies). then $G = \prod_{i \in I} F_i$ is a compact group.

2 Abelian Locally Compact Groups

3 Compact Groups

4 Introduction to Amenability Theory