Fractal Geometry

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Contents

I. Topics in Fractal Geometry

1 Dimension Theory

1.1 Constructing Measures in Metric Spaces

[***TODO: fill in proofs and transfer to measure section***] Let X be a metric space.

Definition. Given $A, B \subseteq X$, say $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$. Say A, B have **positive separation** if $d(A,B) > 0$.

If A, B are compact and disjoint, then they have positive separation. We say that an outer measure μ^* is a **metric outer measure** if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ when A, B have positive separation.

Example. The Lebesgue outer measure is a metric outer measure. [***TODO: prove***]

1.1 Theorem. *μ^* is a metric outer measure if and only if every Borel set is μ^*–measurable (in the sense of Caratheodory).*

Proof [***TODO: prove this (homework), and find a proof of the converse? (may not be true)***] ■

Suppose $\mathcal{A} \subseteq \mathcal{B}$ are both covers of X containing $\emptyset$ and $\mathcal{C} : \mathcal{B} \to [0,\infty]$ with $\mathcal{C}(\emptyset)$. Let $\mu^*_{\mathcal{A}}$ and $\mu^*_{\mathcal{B}}$ be the corresponding extensions of $\mathcal{C}$ and $\mathcal{C}|_{\mathcal{A}}$. Then by definition, $\mu^*_{\mathcal{B}}(E) \leq \mu^*_{\mathcal{A}}(E)$ for all $E \in \mathcal{P}(X)$.

Let X be a metric space, $\mathcal{A}$ cover X containing $\emptyset$. Suppose for each $x \in X$ and $\delta > 0$, there exists $A \in \mathcal{A}$ such that $x \in A$ and $A \leq \delta$. Let $\mathcal{C} : \mathcal{A} \to [0,\infty]$ with $\mathcal{C}(\emptyset) = 0$. Set $\mathcal{A}_\epsilon = \{A \in \mathcal{A} : (A) \leq \epsilon\}$, and define μ^*_ϵ by extending $\mathcal{C}|_{\mathcal{A}_\epsilon}$. In particular, as ϵ decreases, μ^*_ϵ increases, and define

$$\mu^*(E) = \sup_\epsilon \mu^*_\epsilon(E) = \lim_{\epsilon \to 0} \mu^*_\epsilon(E)$$

1.2 Theorem. *As defined above, μ^* is a metric outer measure.*

Proof [***TODO: prove this, homework***] ■

Example. The Lebesgue measure arises this way; in fact, the μ^*_ϵ are all the same outer measure.

1.2 Hausdorff Measure and Dimension

For the remainder of this chapter, if X is a metric space and $U \subseteq X$, we denote $|U| = (U)$.

Definition. A **δ–cover** of a set $F \subseteq X$ is any countable collection $\{U_n\}_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty U_n \supseteq F$ and $|U_n| \leq \delta$.

Let $\mathcal{A} = \mathcal{P}(X)$, and $\mathcal{A}_\delta = \{A \subseteq X : |A| \leq \delta\}$. For $\delta \geq 0$, put $\mathcal{C}_s(A) = |A|^s$. Then for $s \geq 0$, $\delta > 0$, and $E \subseteq$, we define

$$\begin{aligned} H^s_\delta(E) &= \inf\left\{\sum_{n=1}^\infty |U_n|^s : \{U_n\} \text{ is a } \delta\text{−cover of } E\right\} \\ &= \inf\left\{\sum_{n=1}^\infty \mathcal{C}_s(U_n) : \bigcup_{n=1}^\infty U_n \supseteq E, U_n \in \mathcal{A}_\delta\right\} \end{aligned}$$

This is the outer measure as constructed in **??** with covering family A_δ and function $\mathcal{C}_s$. In particular, as $\delta \to 0$, H^s_δ increases; in particular, by Theorem 1.2, $H^s(E) = \sup_\delta H^s_\delta(E)$ is a metric outer measure. Then apply Caratheodory (**??**) to get the s−dimensional Hausdorff measure, which is a complete Borel measure.

Example. (i) H^0 is the counting measure on any metric space.

(ii) Take $X = \mathbb{R}$ and $s = 1$. Then H^1 is the Lebesgue measure (on Borel sets). To see this, we have

$$\begin{aligned} \lambda(E) &= \inf\left\{\sum_{n=1}^\infty |I_n| : \bigcup_{n=1}^\infty I_n \supseteq E, |I_n| \leq \delta\right\} \\ &\geq H^1_\delta(E) \end{aligned}$$

for any $\delta > 0$; and conversely, take any δ−cover of E, say $\{U_n\}_{n=1}^\infty$ and set $I_n = \overline{\operatorname{conv} U_n}$ so $|I_n| = |U_n| \leq \delta$. Thus $\sum_{n=1}^\infty |U_n| = \sum_{n=1}^\infty |I_n| \geq \lambda(E)$ for any such cover, so $\lambda(E) = H^1_\delta(E)$ for any $\delta > 0$. Thus $\lambda(E) = H^1(E)$ for any Borel set E.

(iii) More generally, if $X = \mathbb{R}^n$ and $s = n$, then $\lambda = \pi_n \cdot H^n$ where π_n is the n−dimensional volume of the ball of diameter 1.

We will verify that $H^n \leq m$ where m is n−dimensional Lebesgue measure on $\mathbb{R}^n$; the general result is harder and left as an exercise. To see this, we have

$$\begin{aligned} m(E) &= \inf\left\{\sum_{i=1}^\infty (C_i) : C_i \text{ cube}, \bigcup_{i=1}^\infty C_i \supseteq E, \text{sides } \leq \frac{1}{\sqrt{n}}\delta\right\} \\ &= \inf\left\{\sum_{i=1}^\infty \left(\frac{1}{\sqrt{n}}\right)^n |C_i|^n : \{C_i\} - \delta\text{-cover of cubes of } E\right\} \\ &\geq c_n \inf\left\{\sum_{i=1}^\infty |c_i|^n : \text{all } \delta\text{−covers of } E = c_n H^n_\delta(E)\right\} \end{aligned}$$

where $c_n = (1/\sqrt{n})^n \leq 1$.

(iv) If $s < t$, then $H^s(E) \geq H^t(E)$.

Suppose $s < t$. Clearly $H^s(E) \geq H^t(E)$, but we can in fact make stronger statements. Suppose we have some U_i where $|U_i| \leq \delta$, and

$$\sum_{i=1}^\infty |U_i|^t = \sum_{i=1}^\infty |U_i|^s |U_i|^{t-s} \leq \delta^{t-s} \sum_{i=1}^\infty |U_i|^s$$

so that

$$H^t_\delta(E) \leq \delta^{t-s} \inf\left\{\sum_{i=1}^\infty |U_i|^s : \{U_i\}_{i=1}^\infty \ \delta\text{−cover of } E\right\} = \delta^{t-s} H^s_\delta(E).$$

In particular, as $\delta \to 0$, $H^t_\delta(E) \to H^t(E)$ and $H^s_\delta(E) \to H^s(E)$ and $\delta^{t-s} \to 0$ since $s < t$. Thus if $H^s(E) \neq \infty$, then $H^t(E) = 0$ for all $t > s$. Similarly, if $H^t(E) > 0$, then $H^s(E) = \infty$ for all $s < t$. As a result, there exists some unique number $S_0 := \dim_H(E) \geq 0$ such that for all $s < S_0$, $H^s(E) = \infty$, and for all $t > S_0$, $H^t(E) = 0$. We call this value the **Hausdorff dimension** of E. Note that $H^{S_0}(E) \in [0,\infty]$ and all choices are possible.

Example. (i) Since $1 = m([0,1]) = H^1([0,1])$, $\dim_H[0,1] = 1$

(ii) $\dim_H \mathbb{R} = 1$ but $m(\mathbb{R}) = H^1(\mathbb{R}) = \infty$.

(iii) It is possible to have $S_0 = 1$ but $m(E) = 0$.

(iv) There is a Cantor-like set with Hausdorff-dimension 0.

(v) If E is countable and $s > 0$, $H^s_\delta(E) \leq \sum_{x\in E} |\{x\}|^s = 0$. In particular, there exist compact countable sets, and in this case, $\dim_H C = 0$ while $H^0(C) = \infty$.

Here are some basic properties of Hausdorff dimension.

1.3 Proposition. (Properties of Hausdorff Dimension) *(i) If $A \subseteq B$, then $\dim_H A \leq \dim_H B$.*

(ii) If $F \subseteq \mathbb{R}^n$, then $\dim_H F \leq n$.

(iii) If $U \subset \mathbb{R}^n$ is open, then $\dim_H U = n$.

(iv) If $F = \bigcup_{i=1}^\infty F_i$, then $\dim_H(F) = \sup_{i\in\mathbb{N}} \dim_H F_i$.

Proof (i) If $H^s(B) = 0$, then $H^s(A) = 0$ by monotonicity of measures so $\dim_H A \leq \dim_H B$.

(ii) First consider the unit cube $I^n \subset \mathbb{R}^n$. Then

$$H^s_{\sqrt{n}\delta}(I^n) \leq \left(\frac{2}{\delta}\right)^n (\sqrt{n}\delta)^s = 2^n \sqrt{n}^n \delta^{s-n}$$

so if $s > n$, then $\delta^{s-n} \to 0$ as $\delta \to 0$. Thus for all $s > n$, $H^s(I^n) = \lim_{\delta\to 0} H^s_{\sqrt{n}\delta}(I^n) = 0$ so that $\dim_H(I^n) \leq n$. Moreover, $\mathbb{R}^n$ is the countable union of unit cubes, so that $H^s(\mathbb{R}^n) = 0$ and $\dim_H(\mathbb{R}^n) \leq n$. Then appeal to (i).

(iii) Cubes have positive Hausdorff n−measure.

(iv) If $s > \sup\{\dim_H F_i\}$, then $H^s(F_i) = 0$ for all i and by subadditivity $H^s(F) = 0$. Thus $s \geq \dim_H F$. By monotonicity, $\dim_H F \geq \dim_H F_j$ for all j. ■

Suppose $X = \mathbb{R}^n$, $E \subseteq \mathbb{R}^n$, $\lambda > 0$. Set $\lambda E = \{\lambda e : e \in E\}$: then $H^s(\lambda E) = \lambda^s H^s(E)$ since there is a bijection between δ−covers and $\lambda\delta$−covers.

Definition. Let X, Y be metric spaces. A function $f : X \to Y$ is called **Lipschitz** if there exists C such that $d(f(x), f(y)) \leq C d(x,y)$.

Certainly if f is Lipschitz, then f is uniformly continuous. Functions $f : \mathbb{R} \to \mathbb{R}$ with bounded derivative are Lipschitz by the mean value theorem.

Definition. A function $f : X \to Y$ is **Hölder continuous** with exponent α if there exists c such that $d(f(x), f(y)) \leq c d(x,y)^\alpha$.

Example. (i) If $\alpha = 1$, then f is Lipschitz, and if $\alpha = 0$, then f is bounded.

(ii) If $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha > 0$, then f is constant (by considering derivatives). Thus the most interesting cases occur for $0 < \alpha \leq 1$.

1.4 Proposition. *If $f : X \to Y$ is Hölder continuous with exponent α. Then $H^{s/\alpha}(f(E)) \leq cH^s(E)$ for some constant c.*

Proof If $\{U_i\}$ are a δ-cover of E, then $\{f(U_i)\}$ cover $f(E)$. Then $f(U_i) = \sup\{d(f(x), f(y)) : x, y \in U_i\} \leq c\sup\{d(x,y)^\alpha : x, y \in U_i\} = C \cdot (U_i)^\alpha$. Thus if $\{U_i\}$ is a δ–cover of E, then $\{f(U_i)\}$ is a $c\delta^\alpha$–cover of $f(E)$. Passing through the definition, we get $H^{s/\alpha} \leq c^{s/\alpha} H^s(E)$. ■

We then have the easy corollaries

1.5 Corollary. $\dim_H f(X) \leq \frac{1}{\alpha} \dim_H X$.

1.6 Corollary. *If f is an isometry, then $H^s(f(X)) = H^s(X)$.*

1.7 Corollary. *If $f : X \to Y$ are bi-Lipschitz, then* $\dim_H X = \dim_H Y$.

Example. Let C denote the Cantor set. Let's show that $\frac{1}{2} \leq H^s(C) \leq 1$ for $s = \frac{\log 2}{\log 3}$. In particular, this implies that $\dim_H C = \frac{\log 2}{\log 3}$.

Let $\delta = 3^{-n}$ and cover C with a δ–covering with generation n Cantor intervals. Then $H^s_\delta(C) \leq \sum_{I \in C_n} |I|^s = 2^n 3^{-ns} = 1$ by choice of s. Thus $\lim_{\delta\to 0} H^s_\delta(C) = \lim_{n\to\infty} H^s_{3^{-n}}(C) \leq 1$.

For the lower bound, take any δ–cover $\{U_i\}$ of C. Without loss of generality, we may assume that the U_i are open intervals. Since C is compact, get some finite subcover $U_1, \ldots, U_N$. For each i, get $k_i \in \mathbb{N}$ so that $3^{-(k_i+1)} \leq |U_i| < 3^{-k_i}$; set $k = \max\{k_1, \ldots, k_N\}$. Since U_i intersects at most 1 interval in C_{k_i}, U_i intersects at most 2^{k-k_i} intervals of C_k. Thus $2^k \leq \sum_{i=1}^N 2^{k-k_i}$ where $2^{k-k_i} = 2^k 3^{-sk_i} = 2^k 3^{-s(k_i+1)} \leq 2^k |U_i|^s 3^s$. Thus

$$2^k \leq \sum_{i=1}^N 2^k |U_i|^s 3^s$$

so $\frac{1}{2} = 3^{-s} \leq \sum_{i=1}^N |U_i|^s \leq \sum_{i=1}^\infty |U_i|^s$ so $H^s_\delta(C) \geq \frac{1}{2}$ so $H^s(C) \geq \frac{1}{2}$.

1.8 Proposition. *Let (X,d) be a metric space. If* $\dim_H X < 1$, *then X is totally disconnected.*

Proof Let $x \in X$ and define $f : X \to [0,\infty)$ by $f(z) = d(z,x)$. Then f is Lipschitz with constant 1 so $\dim_H f(X) \leq \dim_H X < 1$ so $m(f(X)) = 0$. Then if $y \neq x$, $d(y,x) = f(y) > 0$ while $f(x) = 0$. In particular, $(0, f(y)) \not\subset f(X)$ so there exists $0 < r < f(y)$ such that $r \notin f(X)$. Then $U_1 = \{z \in X : f(z) < r\}$ and $U_2 = \{z \in X : f(z) > r\}$ are disconnecting sets for X separating x and y.

1.3 Box Dimensions

Definition. Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set, and for each $\delta > 0$, let $N_\delta(E)$ be the least number of closed balls of diameter δ. We then define the **upper box dimension** of E

$$\overline{\dim}_B E = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{|\log \delta|}$$

and similarly $\underline{\dim}_B E$ (the **lower box dimension**) with a lim inf in place of lim sup. If $\underline{\dim}_B E = \overline{\dim}_B E$, then we define the **box dimension** to be this shared quantity.

If I is any interval, it is easy to see that $\dim_B I = 1$. Note that if $N_\delta(E) \sim \delta^{-s}$, then $\dim_B E = S$.

Example. Let's show that the box dimension of $C_{1/3}$ exists, and compute it. Given some $\delta > 0$, let n be so that $3^{-n} \leq \delta < 3^{-(n-1)}$. Certainly we can cover $C_{1/3}$ by Cantor intervals of level n, so that $N_\delta(C_{1/3}) \leq 2^n$. Moreover, the endpoints of Cantor inverals of level $n-1$ are distance at least $3^{-(n-1)} > \delta$ apart. Thus $N_\delta(C_{1/3})$ is at least the number of endpoints of level $n-1$, i.e. $N_\delta(C_{1/3}) \geq 2^n$. Thus $N_\delta(C_{1/3}) = 2^n$, so that

$$\frac{\log 2}{\log 3} = \frac{\log 2^n}{\log 3^n} \leq \frac{\log N_\delta(C_{1/3})}{|\log \delta|} \leq \frac{\log 2^n}{\log 3^{n-1}} = \frac{n}{n-1} \cdot \frac{\log 2}{\log 3}$$

and, as $\delta \to 0$, $n \to \infty$ so that the $C_{1/3} = \frac{\log 2}{\log 3}$.

More generally, using the same technique, we may compute $C_r = \frac{\log 2}{\log 1/r}$.

However, the box dimension has poor properties: for example, we may verify $\dim_B\{0, 1, 1/2, 1/3, \ldots\} = \frac{1}{2}$. But this is very concerning from a measure theoretic perspective, since this is a countable set with larger "dimension" than some uncountable sets (e.g. C_r for small r).

1.9 Theorem. *The value of the various box dimensions are equal for all following definitions of $N_\delta(E)$:*

1. *least number of open balls of radius δ that cover E*
2. *least number of cubes of side length δ*
3. *the number of δ–mesh cubes that intersect E: $[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$ for $(m_1, \ldots, m_n) \in \mathbb{Z}^n$.*
4. *the largest number of disjoint closed balls of radius δ with centers in E.*

Proof Throughout, from the logarithms in the definition, it suffices to bound $N_\delta^{(i)}(E)$ with respect to $N_\delta(E)$ up to some constant factor either with respect to δ or with respect to N_δ.

1. Exercise.
2. Exercise.
3. In general, the diameter of a δ–cube in $\mathbb{R}^n$ is $\sqrt{n}\delta$. Let $N_\delta^{(3)}(E)$ denote the number of δ–mesh cubes intersecting E. Then the cubes which intersect E cover E and these have diameter $\sqrt{n}\delta$, so $N_{\sqrt{n}\delta}(E) \leq N_\delta^{(3)}(E)$.
   Conversely, any set with diameter at most δ is contained in at most 3^n δ–mesh cubes. Thus $N_\delta^{(3)}(E) \leq 3^n N_\delta(E)$.
4. Let $N_\delta^{(4)}$ denote the largest number of disjoint balls of radius δ centred in E. Say $B_1, \ldots, B_{N_\delta^{(4)}(F)}$ are such balls. If $x \in F$, then $d(x, B_i) \leq \delta$ for some i, else $B(x,\delta)$ would be disjoint from all B_i, contradicting maximality. Thus the balls $B_1^1, \ldots, B^1_{N_\delta^{(4)}(E)}$ cover E and have diameter 4δ, so $N_{4\delta}(E) \leq N_\delta^{(4)}(E)$.
   Conversely, let $U_1, \ldots, U_{N_\delta(E)}$ be any collection of sets of diameter at most δ that cover E. Let $B_1, \ldots, B_m$ be any disjoint balls with radius δ and centres $x_i \in E$. Since the U_j cover E, each $x_i \in U_{j(i)}$ for some $j(i)$ so $U_{j(i)} \subseteq B_i$ and $U_{j(i)} \cap B_k = \emptyset$ for $k \neq i$. Thus $N_\delta(E) \geq N_\delta^{(4)}(E)$. ■