Functional Analysis

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Contents

I. Fundamentals of Functional Analysis

1 Basic Elements of Functional Analysis

Throughout, we denote by $\mathbb{F}$ either the field $\mathbb{R}$ or the field $\mathbb{C}$.

Banach Spaces

Definition. Let X be a vector space over $\mathbb{F}$. A **norm** is a functional $\|\cdot\| : X \to \mathbb{R}$ such that it is

- *(non-negative)* $\|x\| \geq 0$ for any $x \in X$
- *(non-degenerate)* $\|x\| = 0$ if and only if $x = 0$
- *(subadditivity)* $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$
- *($|\cdot|$−homogeneity)* $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{F}$, $x \in X$.

We call the pair $(X, \|\cdot\|)$ a **normed vector space**. Furthermore, we say that $(X, \|\cdot\|)$ is a **Banach space** provided that X is complete with respect to the metric $\rho(x,y) = \|x-y\|$.

Example. (i) $(\mathbb{F}, |\cdot|)$ is a Banach space.

(ii) $(\mathbb{F}^b, \|\cdot\|_p)$, $x = (x_j)_{j=1}^n$,

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^n |x_j|^p\right)^{1/p} & 1 \leq p < \infty \\ \max_{j=1,\dots,n} |x_j| & p = \infty \end{cases}$$

(iii) Consider the space

$$L_p^{\mathbb{F}} = \left\{ f : [0,1] \to \mathbb{F} \;\middle|\; f \text{ is Lebesgue measurable,} \left(\int_0^1 |f|^p\right)^{1/p} < \infty \right\} \Big/ \sim_{\text{a.e.}}$$

where $1 \leq p < \infty$.

(iv) $L_\infty^{\mathbb{F}}[0,1]$, $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |f(t)|$.

(v) Let (X, d) be a metric space. Then

$$C_b^{\mathbb{F}}(x) = \{ f : X \to \mathbb{F} \mid f \text{ is continuous and bounded} \}, \quad \|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a Banach space.

Here is a more interesting example:

Example. Let (X,d) be a metric space. We define the space of Lipschitz functions

$$\mathrm{Lip}^{\mathbb{F}}(X,d) = \left\{ f : X \to \mathbb{F} \;\middle|\; f \text{ is bounded}, L(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} < \infty \right\}$$

We note that for $f : X \to \mathbb{F}$ that

$$f \in \mathrm{Lip}^{\mathbb{F}}(X,d) \Leftrightarrow \text{there is } L \geq 0 \text{ s.t. } |f(x)-f(x)| \leq Ld(x,y) \text{ for all } x,y \in X \tag{1.1}$$

It is easy to verify that $L(f) = \min\{L \geq 0 : (1.1) \text{ holds for } f\}$. It is an easy exercise to see that $\mathrm{Lip}^{\mathbb{F}}$ is a vector space, and that $L : \mathrm{Lip}^F(X,d) \to \mathbb{R}$ is a **semi-norm** (non-negative, subadditive, $|\cdot|$–homogeneous). However, we do not have non-degeneracy (for example, constants are taken to 0). We define the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + L(f)$$

1.1 Proposition. $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$ *is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathrm{Lip}^{\mathbb{F}}(X,d), \|\cdot\|_{\mathrm{Lip}})$. Since $\|\cdot\|_\infty \leq \|\cdot\|_{\mathrm{Lip}}$ on $\mathrm{Lip}^F(X,d)$, we see that $(f_n)_{n=1}^\infty$ is uniformly Cauchy (and bounded), and hence there is $f = \lim_{n\to\infty} f_n$ in $C_b^{\mathbb{F}}(X)$, where the limit is taken with respect to $\|\cdot\|_\infty$, since $(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space. If $x,y \in X$, then

$$\begin{aligned} |f(x)-f(x)| &= \lim_{n\to\infty} |f_n(x)-f_n(y)| \leq \sup_{n\in\mathbb{N}} |f_n(x)-f_n(y)| \\ &\leq \sup_{n\in\mathbb{N}} L(f_n) d(x,y) \leq \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}}\, d(x,y) \end{aligned}$$

Since Cauchy sequences are bounded, we see that $|f(x)-f(y)| \leq Ld(x,y)$, where $L = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}} < \infty$. Thus by (1.1), $f \in \mathrm{Lip}^{\mathbb{F}}(X,d)$. Exercise: one may verify that $\|f - f_n\|_{\mathrm{Lip}} \to 0$. ■

Another collection of basic examples are given by the sequence spaces. We can define

$$\ell_1^{\mathbb{F}} = \left\{ x = (x_j)_{j=1}^\infty \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_1 = \sum_{j=1}^\infty |x_j| < \infty \right\}$$

It is easy to see that $(\ell_1, \|\cdot\|_1)$ is a normed vector space.

For $1 < p < \infty$, and write

$$\ell_p^{\mathbb{F}} = \left\{ x \in \mathbb{F}^{\mathbb{N}} \;\middle|\; \|x\|_p = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p} < \infty \right\}$$

Note that $0 \in \ell_p$, $\alpha \in \mathbb{F}$, $\alpha x \in \ell_p$ if $x \in \ell_p$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then q is called the **conjugate index**. We have

1.2 Proposition. (Young's Inequality) *If $a,b \geq 0$ in $\mathbb{R}$, then $ab \leq a^p/p + b^q/q$, with equality only if $a^p = b^q$.*

and

1.3 Proposition. (Hölder's Inequality) *If $x \in \ell_p$ and $y \in \ell_q$, then $xy = (x_i y_i)_{i=1}^{\infty} \in \ell_1$, with*

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q$$

with equality exactly when $\operatorname{sgn}(x_i y_i) = \operatorname{sgn}(x_k y_k)$ for all $j,k \in \mathbb{N}$ where $x_i y_i \neq 0 \neq x_k y_k$, and $|x|^p = (|x_j|^p)_{j=1}^{\infty}$ and $|y|^q$ are linearly dependent in ℓ_1.

and finally

1.4 Proposition. (Minkowski's Inequality) *If $x, y \in \ell_p$, then $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ with equality exactly when one of x or y is a non-negative scalar combination of the other.*

Review of Topology

Let X denote a non-empty set, and $\mathcal{P}(X)$ denote the power set of X.

Definition. A **topology** on a set X is a set τ of subsets of X such that
(i) $\emptyset, X \in \tau$
(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} U_i \in \tau$.
The sets $U \in \tau$ are called the **open sets** in X, and sets of the form $X \setminus U$ for some open set U are caled the **closed sets** in X. The pair (X, τ) is called a **topological space**.

The metric topology on a metric space (X, d) is the topology

$$\tau_d = \{\, U \subseteq X \mid \text{ for each } x_0 \in U, \text{ there is } \delta = \delta(x_0) \text{ s.t. } B_\delta(x_0) \subseteq U \,\}$$

Example. (i) Given two metrics d, ρ on X, we say that $d \sim \rho$ if and only if there are $c, C > 0$ such that

$$c d(x,y) \leq \rho(x,y) \leq C d(x,y) \text{ for any } x, y \in X$$

Note that $d \sim \rho$ implies that $\tau_d = \tau_\rho$, but the reverse implication is not true. An example of this are the metrics on $X = \mathbb{R}$ given by $d(x,y)$ and $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$. Then $d \nsim \rho$ but $\tau_d = \tau_\rho$.

(ii) "Sorgenfry line" Set $X = \mathbb{R}$, and consider

$$\sigma = \{\, V \subseteq \mathbb{R} \mid \text{ for any } s \in V, \text{ there is } \delta = \delta(s) > 0 \text{ s.t. } [s, s+\delta) \subseteq V \,\}$$

It is an exercise to verify that $\tau_{|\cdot|} \subsetneq \sigma$. We say that σ is **finer** than $\tau_{|\cdot|}$.

(iii) Relative topology: let (X, τ) be a topological space, and $\emptyset \neq A \subseteq X$. Then we can define a topology $\tau|_A = \{U \cap A : U \in \tau\}$.

Definition. Let (X, τ) and (Y, σ) be topological spaces, and $f : X \to Y$. We say that f is $(\tau - \sigma -)$**continuous** at x_0 in X if,

- given $V \in \sigma$ such that $f(x_0) \in V$, then there exists $U \in \tau$ such that $x_0 \in U$ and $f(U) \subseteq V$.

We say that f is $(\tau - \sigma -)$continuous if it is continuous at each x_0 in X.

Space of bounded continuous functions into a normed space

Let $(Y, \|\cdot\|)$ denote a normed space. We let $\tau_{\|\cdot\|}$ denote the topology given by the metric $\rho(x,y) = \|x-y\|$. Let (X,τ) denote any topological space. Then we write

$$C_b^Y(X) = \left\{ f : X \to Y \;\middle|\; f \text{ is bounded and } \tau - \tau_{\|\cdot\|} - \text{continuous} \right\}$$

With pointwise operations, we see that $C_b^Y(X)$ is a vector space. We also define for $f \in C_b^Y(X)$, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, making $(C_b^Y(X), \|\cdot\|_\infty)$ a normed vector space.

1.5 Theorem. *If $(Y,\|\cdot\|)$ is a Banach space, then $(C_b^Y(X), \|\cdot\|_\infty)$ is a Banach space.*

Proof Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C_b^Y(X), \|\cdot\|_\infty)$. Then for any $x \in X$, we have that $(f_n(x))_{n=1}^\infty$ is Cauchy in $(Y,\|\cdot\|)$ since $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|_\infty$, and hence admis a limit $f(x)$. In particular, $x \mapsto f(x)$ defines a function from X to Y. We shall fix $x_0 \in X$ and show that f is continuous at x_0. Given $\epsilon > 0$, we let
- n_1 be so $n, m \geq n_1$ so that $\|f_n - f_m\|_\infty < \epsilon/4$.
- n_2 be so $n \geq n_2$ so that $\|f_n(x_0) - f(x_0)\| < \epsilon/4$.
- $N = \max\{n_1, n_2\}$.
- $U \in \tau$, $x_0 \in U$ such that $f_N(U) \subseteq B_{\epsilon/4}(f(x_0)) \subset Y$.

Then for $x \in U$, we let n_x be so $n_x \geq n_1$ and $n \geq n_x$, so that $\|f_n(x) - f(x)\| < \epsilon/4$. We then have

$$\begin{aligned}\|f(x) - f(x_0)\| &\leq \left\|f(x) - f_{n_x}(x)\right\| + \left\|f_{n_x}(x) - f_N(x)\right\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &< \frac{\epsilon}{4} + \left\|f_{n_x} - f_N\right\|_\infty + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon\end{aligned}$$

in other words that $f(U) \subseteq B_\epsilon(f(x_0))$.

Now let us check that $\|f\|_\infty < \infty$. Since $|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$, so $(\|f_n\|_\infty)_{n=1}^\infty \subseteq \mathbb{R}$ is Cauchy, hence bounded. If $x \in X$, then

$$\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n(x)\| \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$$

so $\|f\|_\infty = \sup_{x\in X} \|f(x)\| < \infty$.

Notice that if ϵ, n_1 are as above, and further x_0, N are as above, we have for $n \geq n_1$

$$\|f_n(x_0) - f(x_0)\| \leq \|f_n(x_0) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\epsilon}{2}$$

so $\|f_n - f\|_\infty = \sup_{x_0 \in X} \|f_n(x_0) - f(x_0)\| \leq \epsilon/2 < \epsilon$. This is uniform since n_1 is chosen uniformly in X. ∎

1.6 Corollary. *$(C_b^{\mathbb{F}}(X), \|\cdot\|_\infty)$ is a Banach space.*

Let's first note the following general priniple: let (X,d), (Y,ρ) be metric spaces, where (X,d) is complete. If $\psi : X \to Y$ is a $(d-\rho-)$isometry, then $(\psi(X), \rho|_{\psi(X)})$ is a complete metric space.

Example. (i) Let T be a non-empty set and let

$$\ell_\infty(T) = \left\{ x = (x_t)_{t\in T} \in \mathbb{F}^T \;\middle|\; \|x\|_\infty \right\} < \infty$$

With pointwise operations, $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space. In fact, it is a Banach space. Let us note that

$$f \mapsto (f(t))_{t\in T} : C_b(T, \mathcal{P}(T)) \to \ell_\infty(T)$$

is a surjective linear isometry, and the result follows.

(ii) Let $c = \{\, x \in \ell_\infty \mid \lim_{n\to\infty} x_n \text{ exists} \,\}$. Then $(c, \|\cdot\|_\infty)$ is a Banach space. Consider the topological space given by $\omega = \mathbb{N} \cup \{\infty\}$, with topology

$$\tau_\omega = \mathcal{P}(\mathbb{N}) \cup \bigcup_{n\in\mathbb{N}} \{k \in \mathbb{N} : k \geq n\}$$

The map $f \mapsto (f(n))_{n=1}^\infty : C_b(\omega) \to c$ is a linear surjective isometry.

(iii) $c_0 = \left\{ x \in \mathbb{F}^{\mathbb{N}} \,\middle|\, \lim_{n\to\infty} x_n = 0 \right\} \subseteq c \subseteq \ell_\infty$.

1.7 Lemma. *If $x_0 \in X$ where (X, τ) is a topological space, then*

$$\mathcal{I}(x_0) = \{\, f \in C_b(x) \mid f(x_0) = 0 \,\}$$

is closed, hence complete, subspace of $C_b(X)$.

Proof If $(f_n)_{n=1}^\infty \subseteq \mathcal{I}(x_0)$ and $f = \lim_{n\to\infty} f_n$ with respect to $\|\cdot\|_\infty$ in $C_b(X)$, then $f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0$. Thus $f \in \mathcal{I}(x_0)$, and closed subsets of complete spaces are themselves complete. ■

Now, $f \mapsto (f(n))_{n=1}^\infty : \mathcal{I}(\infty) \to c_0$ is a (linear) surjective isometry.

(iv) Consider the Sorgenfty line $(\mathbb{R}, \sigma)$: verify that

$$c_b(\mathbb{R}, \sigma) = \left\{ f : \mathbb{R} \to \mathbb{F} \,\middle|\, f \text{ is bounded and } \lim_{t\to t_0^+} f(t) = f(t_0) \text{ for } t \in \mathbb{R} \right\}$$

2 Linear operators and linear functionals

Let X, Y be vector spaces. We let $\mathcal{L}(X,Y) = \{\, S : X \to Y \mid S \text{ is linear} \,\}$; this is itself a vector space with pointwise operations. Let $(X, \|\cdot\|)$ be a normed space. We denote

$$\begin{aligned} D(X) &= \{x \in X : \|x\| < 1\} \\ S(X) &= \{x \in X : \|x\| = 1\} \\ B(X) &= \{x \in X : \|x\| \leq 1\} \end{aligned}$$

(Yes, this notation is confusion. No, I didn't choose it.)

2.1 Proposition. *If X, Y are normed spaces and $S \in \mathcal{L}(X,Y)$, then the following are equivalent:*

(i) S is continuous

(ii) S is continuous at some $x_0 \in X$

(iii) $\|S\| = \sup_{x\in D(X)} \|Sx\| < \infty$.

Moreover, in this case, we have

$$\begin{aligned} \|S\| &= \min\{L > 0 : \|Sx\| \leq L\|x\| \textit{ for } x \in X\} \\ &= \sup_{x\in S(X)} \|Sx\| = \sup_{x\in B(X)} \|Sx\| \end{aligned}$$

Proof $(i \Rightarrow ii)$ Obvious
$(ii \Rightarrow iii)$ Note that

$$Sx_0 + D(Y) = \{Sx_0 + y : t \in D(Y)\} = \{y \in Y : \left\|Sx_0 - y'\right\| < 1\}$$

is a neighbourhood of Sx_0. By the definition of metric continuity, there is $\delta > 0$ such that

$$x_0 + \delta D(X) = \{x_0 + \delta x : x \in D(x)\} = \{x' \in X : \left\|x_0 - x'\right\| < \delta\}$$

such that

$$Sx_0 + \delta S(D(X)) = S(x_0 + \delta D(x) \subseteq Sx_0 + D(Y)$$

which implies that $\delta S(D(X)) \subseteq D(Y)$ and $S(D(X)) \subseteq D(Y)/\delta$, in other words that $\|Sx\| \leq 1/\delta$ for $x \in D(X)$.

$(iii \Rightarrow i)$ If $x \in X$ and $\epsilon > 0$, then

$$\|Sx\| = (\|x\| + \epsilon)\left\|S\left(\frac{1}{\|x\| + \epsilon}\|x\|\right)\right\| \leq (\|x\| + \epsilon)\|S\|$$

Then, letting $\epsilon \to 0^+$, we see that

$$\|Sx\| \leq \|x\|\|S\| = \|S\|\|X\|$$

If $x, x' \in X$, then $\|Sx - S'x\| \leq \|S\|\|x - x'\|$ is S is Lipschitz, hence continuous.

To complete the proof, the content of (iii) implies (i) tellus us that the Lipschitz constant $L(S) \leq \|S\|$. Furthermore, if $\|x\| = 1$, the preceding proof gives us that $\|S\|_{S(X)}$.

Conversely,

$$\|S\| = \sup_{x \in D(X)\setminus\{0\}} \|Sx\| = \sup_{x \in D(X)\setminus\{0\}} \|x\|\left\|S\left(\frac{1}{\|x\|}x\right)\right\| \leq \sup_{x \in S(X)} \|Sx\|$$

The remaining equivalence is obvious. ■

We now let $\mathcal{B}(X, Y) = \{\, S \in \mathcal{L}(X, Y) \mid S \text{ is bounded} \,\}$. We will see that $\|\cdot\|$, above, defines a norm on $B(X, Y)$.

2.2 Theorem. *If X, Y are normed spaces, then $(\mathcal{B}(X, Y), \|\cdot\|)$ is a normed space. Furthermore, if Y is a Banach spaces, then so to is $(\mathcal{B}(X, Y), \|\cdot\|)$.*

Proof Define

$$\Gamma : \mathcal{B}(X, Y) \to C_b^Y(B(X))$$

given by $\Gamma(S) = S|_{B(X)}$. Then, by definition, Γ is linear, with

$$\|\Gamma(S)\|_\infty = \sup_{x \in B(X)} \|Sx\| = \|S\|$$

Thus $\|\cdot\|$ is a norm: if $S, T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$,

$$\|S + T\| = \|\Gamma(S + T)\|_\infty = \|\Gamma(S) + \Gamma(T)\|_\infty \leq \|\Gamma(S)\|_\infty + \|\Gamma(T)\|_\infty = \|S\| + \|T\|$$
$$\|\alpha S\| = \|\Gamma(\alpha S)\|_\infty = |\alpha|\,\|\Gamma(S)\|_\infty = |\alpha|\,\|S\|.$$

Furthermore, $\Gamma : \mathcal{B}(X,Y) \to C_b^Y(\mathcal{B}(X))$ is an isometry.

Now suppose that Y is a Banach space. We will show that $\Gamma(\mathcal{B}(X,Y))$ is closed in $C_b^Y(B(X))$, and hence $B(X,Y) = \Gamma^{-1}(\Gamma(\mathcal{B}(X,Y)))$ is complete. Let $(S_n)_{n=1}^{\infty} \subset \mathcal{B}(X,Y)$ be $\|\cdot\|$–Cauchy. Then $(\Gamma(S_n))_{n=1}^{\infty}$ is $\|\cdot\|_\infty$–Cauchy in $C_b^Y(B(X))$, and hence there is $f \in C_b^Y(B(X))$ such that $\lim_{n\to\infty} \|\Gamma(S_n) - f\|_\infty = 0$. Then we let $S : X \to Y$ be given by

$$Sx = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

If $x, x' \in X$ and $\alpha \in \mathbb{F}$ are all such that $x, x', x + \alpha x' \neq 0$, then

$$\begin{aligned}
S(x+\alpha x') &= \left\|x + \alpha x'\right\| f\left(\frac{1}{x+\alpha x'}(x+\alpha x')\right) \\
&= \left\|x + \alpha x'\right\| \lim_{n\to\infty} S_n\left(\frac{1}{x+\alpha x'}(x+\alpha x')\right) \\
&= \lim_{n\to\infty} (S_n x + \alpha S_n x') = \lim_{n\to\infty} \left[\|x\| S_n\left(\frac{1}{\|x\|}x\right) + \alpha \left\|x'\right\| S_n\left(\frac{1}{\|x\|}x'\right)\right] \\
&= \|x\| f\left(\frac{x}{\|x\|}\right) + \alpha \left\|x'\right\| f\left(\frac{x'}{\|x\|}\right) \\
&= Sx + \alpha Sx'
\end{aligned}$$

The above computation is easily performed if any of x, x', $x + \alpha x'$ are 0. Hence $S \in \mathcal{L}(X,Y)$. We se that S is continuous (say, at a point on $S(X)$), so $S \in \mathcal{B}(X,Y)$. Finally, as $S|_{B(X)} = f = \lim_{n\to\infty} S_n|_{B(X)}$ (with respect to the uniform norm), we have

$$\|S - S_n\| = \sup_{x \in B(X)} \|(S - S_n)x\| = \|f - \Gamma(S_n)\|_\infty$$

goes to 0 as n goes to infinity. ■

Definition. Given a vector space X, let $X' = \mathcal{L}(X, \mathbb{F})$ denote the **algebraic dual**. If further X is a normed space, we let $X^* = \mathcal{B}(X, \mathbb{F})$ denote the (continuous) dual.

2.3 Corollary. *If X is a normed spaces, then X^* is always a Banach space.*

2.4 Theorem. *Let for $x \in \ell_1$, $f_x : c_0 \to \mathbb{F}$ be given by $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$. Then $f_x \in c_0^*$ with $\|f_x\| = \|x\|_1$. Furthermore, every element of c_0^* arises as above.*

Proof If $x \in \ell_1$ and $y \in c_0 \subseteq \ell_\infty$, then

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \sum_{j=1}^{\infty} |x_j| \left\|y\right\|_\infty = \|x\|_1 \left\|y\right\|_\infty < \infty$$

so $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$ is well-defined. It is obvious that f_x is linear: $f_x(y + \alpha y') = f_x(y) + \alpha f(y')$ for $y, yl \in c_0$ and $\alpha \in \mathbb{F}$. Also, $\|f_x\| \leq \|x\|_1$. We let $y^n = (\overline{\operatorname{sgn} x}, \ldots, \overline{\operatorname{sgn} x_n}, 0, 0, \ldots) \in c_0$, with $\left\|y^n\right\| = 1$. Then

$$\|f_x\| \geq |f_x(y^n)| = \sum_{j=1}^{n} x_i \overline{\operatorname{sgn} x_i} = \sum_{j=1}^{n} |x_i|$$

so that $\|f_x\| \geq \|x\|_1$, and hence equality holds.

Now let $f \in c_0^*$, and write $e_n = (0,\ldots,0,1,0,0,\ldots) \in c_0$, and let $x_n = f(e_n)$. Then, let $y \in c_0$ and $y^n = (y_1,\ldots,y_n,0,0,\ldots)$ and we have

$$\left\|y - y^n\right\|_\infty = \sup_{j\geq n+1} |y_j|$$

which goes to 0 as n goes to infinity. Then since f is continuous, we have

$$f(y) = \lim_{n\to\infty} f(y^n) = \lim_{n\to\infty} \sum_{j=1}^{n} y_j x_j = \sum_{j=1}^{\infty} x_j y_j = f_x(y)$$

We use sequence $(y^n)_{n=1}^\infty$ as in $y^n \in c_0$, to see that

$$\sum_{j=1}^{n} |x_i| = |f(y^n)| \leq \|f\| < \infty$$

so $x \in \ell_1$. Thus $f = f_x$, as desired. ■

2.5 Corollary. *$\ell_1 \cong c^*$ isometrically isomorphically.*

Proof For $y \in c$, let $L(y) = \lim_{n\to\infty} y_n$. Given $y \in c$, let $y^n = (y_1,\ldots,y_n,L(y),L(y),\ldots) \in c$. Notice that $\left\|y - y^n\right\|_\infty \to 0$ similarly as above.

We let $1 = (1,1,\ldots)$, and $1_n = (0,\ldots,0,1,1,\ldots)$. If $m < n$, then $1_n - 1_m \in c_0$, so

$$|f(1_n) - f(1_m)| = |f_x(1_n - 1_m)| \leq \sum_{j=m+1}^{n} |x_j|$$

so that $(f(1_n))_{n=1}^\infty$ is Cauchy in $\mathbb{F}$. Let $x_0 = \lim_{n\to\infty} f(1_n)$. Let $\tilde{x} = (x_0,x_1,\ldots) \in \ell_1$. Then letting $x_j = f(e_j)$, we see that

$$f(y) = \lim_{n\to\infty} f(y^n) = \sum_{j=1}^{\infty} x_j y_j + x_0 L(y)$$ ■

Similarly as above, we may show that $\|f\| = \|\tilde{x}\|_1$.

Remark. We write $c_0^* \cong \ell_1$ isometrically.

2.6 Corollary. *$(\ell_1, \|\cdot\|_1)$ is complete.*

3 Axiom of Choice and the Hahn-Banach Theorem

Definition. Let S be a non-empty set. A **partial ordering** is a binary relation $\leq$ on S which satisfies for $s,t,n \in S$,

(i) *(reflexivity)* $s \leq s$
(ii) *(transitivity)* $s \leq t$, $t \leq u$ implies $s \leq u$
(iii) *(anti-symmetry)* $s \leq t$, $t \leq s$ implies $s = t$

We call the pair $(S, \leq)$ a **partially ordered set**. We say that $(S, \leq)$ is **totally ordered** if, given $s, t \in S$, at least one of $s \leq t$ or $t \leq s$ holds. We say that $(S, \leq)$ is **well-ordered** if given any $\emptyset \neq S_0 \subseteq S$, there is some $s_0 \in S_0$ such that $s_0 \leq s$ for $s \in S_0$. A **chain** in a poset $(S, \leq)$ is any $\emptyset \neq C \subseteq S$ such that $(S, \leq|_C)$ is totally ordered.

Example. (i) $X \neq \emptyset$, $(\mathcal{P}(X), \subseteq)$ is a poset
(ii) $(\mathbb{R}, \leq)$ is a totally ordered set
(iii) $(\mathbb{N}, \leq)$, $(\omega = \mathbb{N} \cup \{\infty\}, \leq)$, are well-ordered sets.

3.1 Theorem. *The following are equivalent:*

(i) (Axiom of Choice 1): For any $x \neq \emptyset$, there is a function $\gamma : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ such that $\gamma(A) \in A$ for each $A \in \mathcal{P})X_\setminus\{\emptyset\}$.

(ii) (Axiom of Choice 2): Given any $\{A_\lambda\}_{\lambda \in \Lambda}$ where $A_\lambda \neq \emptyset$ for each λ,

$$\prod_{\lambda \in \Lambda} A_\lambda = \{(a_\lambda)_{\lambda \in \Lambda} : a_\lambda \in A_\lambda \textit{ for each } \lambda\} \neq \emptyset$$

(iii) (Zorn's Lemma): In a poset $(S, \leq)$, if each chain $C \subseteq S$ admits an upper bound in S, then $(S, \leq)$ admis a maximal element.

(iv) (Well-ordering principle): Any $S \neq \emptyset$ admits a well-ordering

PROOF Exercise. ∎

Definition. Let X be a vector space (over k). A subset $S \subseteq X$ is called

- **linearly independent** if for any distinct $x_1, \ldots, x_n \in S$, the equation $0 = \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ where $\alpha_i \in k$ implies $\alpha_1 = \cdots = \alpha_n = 0$.
- **spanning** if each $x \in X$ admits $x_i \in S$ and $\alpha_i \in k$ such that $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$.
- **Hamel basis** if it is both linearly independent and spanning

3.2 Proposition. *Any vector space X admits a Hamel basis.*

PROOF Let $\mathcal{L} = \{L \subseteq X : L \text{ is linearly independent}\}$. Then $(\mathcal{L}, \subseteq)$ is a poset. Verify that for any chain $\mathcal{C} \subseteq \mathcal{L}$, that $U = \bigcup_{L \in \mathcal{C}} L \in \mathcal{L}$ and is an upper bound for $\mathcal{C}$. Apply Zorn to find a maximal element M in $(\mathcal{L}, \subseteq)$. Verify that M is spanning for X. ∎

3.3 Corollary. *If X is an infinite dimensional normed space, then there exists $f \in X' \setminus X^*$.*

PROOF Our assumption provides $\{e_n\}_{n=1}^\infty$ which is linearly independent. By normalizing each element, we may and will suppose that each $\|e_n\| = 1$. Let

$$\operatorname{span}\{e_n\}_{n=1}^\infty = \left\{ \sum_{j=1}^m \alpha_j e_{n_j} : m \in \mathbb{N}, \alpha_i \in \mathbb{F}, n_1 < \cdots < n_m \right\}$$

and let B be any linearly independent set containing $\{e_n\}_{n=1}^\infty$. Define $f : X = \operatorname{span} B \to \mathbb{F}$ be given for $x = \sum_{b \in B \setminus \{e_n\}_{n=1}^\infty} \alpha_b b + \sum_{j=1}^n \alpha_j e_{n_j}$ by $f(x) = \sum_{j=1}^m \alpha_j n_j$. The point is that $f(e_n) = n$ and $f(e) = 0$ for any other $e \in B$. Notice that

$$\|f\| = \sup_{x \in B(X)} |f(x)| \geq \sup_{n \in \mathbb{N}} |f(e_n)| = \sup_{n \in \mathbb{N}} n = \infty$$ ∎

Definition. Let X be a $\mathbb{R}-$vector space. A **sublinear functional** is any $\rho : X \to \mathbb{R}$ such that it satisies

- *(non-negative homogenity)* $\rho(tx) = t\rho(x)$ for $t \geq 0$, $x \in X$.
- *(subadditivity)* $\rho(x+y) \leq \rho(x) + \rho(y)$ for $x, y \in X$.

3.4 Theorem. (Hahn-Banach) *Let X be a $\mathbb{R}-$vector space, $\rho : X \to \mathbb{R}$ a sublinear functional, $Y \subseteq X$ a subspace and $f \in Y'$ such that $f \leq \rho|_Y$. Then there exists $F \in X'$ such that $F|_Y = f$ and $F \leq \rho$ on X.*

Proof We first do this for extensions by a single point $x \in X \setminus Y$. We wish to find $c \in \mathbb{R}$ such that

$$f(y) + \alpha c \leq \rho(y + \alpha x)$$

for $y \in Y$ and $\alpha \in \mathbb{R}$. In this case, we let $F : \operatorname{span} Y \cup \{x\} \to \mathbb{R}$ be given by $F(y+\alpha x) = f(y)+\alpha c$, and we have that F is linear and satisfies $F \leq \rho$ on $\operatorname{span} Y \cup \{s\}$. To do this, let y_+, y_- in Y and observe that $f(y_+) + f(y_-) = f(y_+ + y_-) \leq \rho(y_+ + y_-) \leq \rho(y_+ + x) + \rho(y_- - x)$ so that $f(y_-) - \rho(y_- - x) \leq \rho(y_+ + x) - f(y_+)$. It thus follows that

$$\sup\{f(y) - \rho(y-x) : y \in Y\} \leq \in \{\rho(y+x) - f(y) : y \in Y\}$$

so we may find $c \in \mathbb{R}$ for which

$$\sup\{f(y) - \rho(y-x) : y \in Y\} \leq c \leq \inf\{\rho(y+x) - f(y) : y \in Y\}$$

If $t > 0$, then for $y \in Y$,

$$c \leq \rho\left(\frac{1}{t}y + x\right) - f\left(\frac{1}{t}y\right) \Rightarrow tc \leq \rho(y+tx) - f(y) \Rightarrow f(y) + tc \leq \rho(y+tx)$$

and if $s > 0$, then for $y \in Y$,

$$f\left(\frac{1}{s}y\right) - \rho\left(\frac{1}{s}y - x\right) \leq c \Rightarrow sc \leq f(y) - \rho(y+sx) \Rightarrow f(y) - sc \leq \rho(y-sx)$$

Clearly, $f(y) + 0 \leq \rho(y + 0x)$. Hence, we have our desired inequality.

We now use Zorn's lemma to lift this result to the whole space. Consider the set of "p-extensions" of f,

$$\mathcal{E} = \{\, (\mathcal{M}, \psi) \;\big|\; Y \subseteq \mathcal{M} \subseteq X, \mathcal{M} \text{ is a subspace}, \psi \in \mathcal{M}', \psi|_Y = f, \psi \leq P|_{\mathcal{M}} \,\}$$

Define a partial order on $\mathcal{E}$ by

$$(\mathcal{M}, \psi) \leq (\mathcal{N}, \phi) \text{ iff } \mathcal{M} \subseteq \mathcal{N}, \phi|_{\mathcal{M}} = \psi$$

Suppose $\mathcal{C} \subseteq \mathcal{E}$ is a chain with respect to $\leq$. We let

- $\mathcal{U} = \bigcup_{(\mathcal{M},\varphi)} \mathcal{M}$ which is a subspace, since $\mathcal{C}$ is a chain.
- and define $\phi : \mathcal{U} \to \mathbb{R}$ by $\phi(x) = \psi(x)$ whenever $x \in \mathcal{M}$, which is again well-defined since C is a chain.

Furthermore, we see that $\phi \in U'$, since if $x, y \in \mathcal{U}$, get $x \in \mathcal{M}$, $y \in \mathcal{N}$ for some $(\mathcal{M}, \psi) \leq (\mathcal{N}, \psi') \in \mathcal{C}$. Then $\phi(x+y) = \psi'(x+y) = \psi'(x) + \psi'(y) = \phi(x) + \phi(y)$, etc. Likewise, $\psi \leq p|_{\mathcal{U}}$. Thus by Zorn's lemma, $\mathcal{E}$ admits a maximal element $\mathcal{M}, F$ Then $\mathcal{M} = X$, for if not, then we would find $x \in X \setminus \mathcal{M}$ and we apply step one to $\operatorname{span} \mathcal{M} \cup \{x\}$ to get F', a strictly larger element violating maximality. ■

Trivially, any $\mathbb{C}$–vector siace is a $\mathbb{R}$–vector space.

3.5 Lemma. *Let X be a $\mathbb{C}$–vector space.*

(i) If $f \in X'_{\mathbb{R}}$ into $\mathbb{R}$, then define $f_{\mathbb{C}}$ given by $f_{\mathbb{C}}(x) = f(x) - if(ix)$ defines an element of $X' = X'_{\mathbb{C}}$.

(ii) If $g \in X'$, then $f = \operatorname{Re} g$ in $X'_{\mathbb{R}}$ satisfies $g = f_{\mathbb{C}}$.

(iii) If X is a normed $\mathbb{C}$–vector space, then for $f \in X'_{\mathbb{R}}$,

$$f \in X^*_{\mathbb{R}} \textit{ if and only if} f_{\mathbb{C}} \in X^* = X^*_{\mathbb{C}} \textit{ with } \|f\| = \|f_{\mathbb{C}}\|$$

Proof (i) and (ii) are straightforward exercises; let's see (iii). We let fr $x \in X$, $z = \operatorname{sgn} f_{\mathbb{C}}(x)$. Then

$$\begin{aligned} \mathbb{R} \ni |f_{\mathbb{C}}(x)| &= \overline{z} f_{\mathbb{C}}(x) = f_{\mathbb{C}}(\overline{z}x) = \operatorname{Re} f_{\mathbb{C}}(\overline{z}x) = f(\overline{z}x) = |f(\overline{z}x)| \\ &\leq \|f\| \|\overline{z}x\| = \|f\| |\overline{z}| \|x\| = \|f\| \|x\| \end{aligned}$$

so we see that $\|f_{\mathbb{C}}\| \leq \|f\|$. Conversely,

$$|f(x)| = |\operatorname{Re} f_{\mathbb{C}}(x)| \leq |f_{\mathbb{C}}(x)| \leq \|f_{\mathbb{C}}\| \|x\| \text{ so that } \|f\| \leq \|f_{\mathbb{C}}\| \qquad \blacksquare$$

3.6 Corollary. *If X is a normed space, $Y \subseteq X$ a subspace and $f \in Y^*$, then there exists $F \in X^*$ such that $F|_Y = f$ and $\|F\| = \|f\|$.*

Proof Define $\rho : X \to \mathbb{R}$ be given by $p(x) = \|f\| \cdot \|x\|$, so p is sublinear and $\operatorname{Re} f \leq p|_Y$. Apply Hahn-banach to to this data and get $\tilde{F} \in X^*_{\mathbb{R}}$ such that $\tilde{F}|_Y = \operatorname{Re} f$ and $\tilde{F} \leq p$, and let $F = \tilde{F}_{\mathbb{C}}$. $\blacksquare$

3.7 Corollary. *If X is a normed space, $x \in C$, then there is $f \in X^*$ such that*

$$\|x\| = f(x) = |f(x)| \textit{ and } \|f\| = 1$$

Proof Let $f_0 : \mathbb{F}x \to \mathbb{F}$ be given by $f_0(\alpha x) = \alpha \|x\|$. If $x \neq 0$, then

$$\|f_0\| = \sup_{\|\alpha x\| \leq 1} |f_0(\alpha x)| = \sup_{\|\alpha x\| \leq 1} |\alpha| \|x\| = 1$$

and apply the previous corollary. If $x = 0$, this is trivial. $\blacksquare$

3.8 Theorem. *Let X be a normed space and X^{**} denote the bidual. For $x \in X$, define $\hat{x} : X^* \to \mathbb{F}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ with $\|\hat{x}\| = \|x\|$, so that $x \mapsto \hat{x} : X \to X^{**}$ is a linear isometry.*

Proof Notice that $|\hat{x}(f)| = |f(x) \leq \|f\| \|x\|$ so $\|\hat{x}\| \leq \|x\|$. The last corollary provides for $x \in X$ an $f_x \in S(X^*)$ with $|f_x(x)| = \|x\|$. Then $\|\hat{x}\| \leq |\hat{x}(f_x)| = \|x\|$. Hence $\|\hat{x}\| = \|x\|$. Clearly $x \mapsto \hat{x}$ is linear. $\blacksquare$

Remark. Since X^{**}, being a dual space, is complete, we have that $\hat{X} = \{\hat{x} : x \in X\}$ satisfies that its closure $\overline{\hat{X}} \subseteq X^{**}$ is complete. Hence $\overline{\hat{X}}$ is a Banach space containing a dense copy of X. Often, we will simply write $\overline{\hat{X}} = \overline{X}$ and call it the **completion** of X.

Geometric Hahn-Banach

If $A, B \subset X$ with $A \cap B = \emptyset$ (and other suitable assumptions), we will find a $\mathbb{R}-$hyperplane between A and B.

Definition. In a vector space, a **hyperplane** is any set of the form $x_0 + \ker f$ with $x_0 \in X$ and $f \in X'$. Then a $\mathbb{R}-$**hyperplane** is any set of the form $x_0 + \ker \operatorname{Re} f$.

> ***3.9 Proposition.*** *Let X be a normed space.*
> *(i) If $f \in X^* \setminus \{0\}$, then $\underline{\ker f}$ is closed and nowhere dense.*
> *(ii) if $f \in X' \setminus X^*$, then $\overline{\ker f} = X$.*
> *Thus a hyperplane in X is either closed and nowhere dense, or it is dense.*

Proof To see (i), $\ker f = f^{-1}(\{0\})$ is a closed set since f is continuous. Furthermore, if $Y \subsetneq X$ is a proper closed subspace, then it is nowhere dense. If not, then there would exist $y_0 \in T$ and $\delta > 0$ such that $y_0 + \delta D(X) \subseteq Y$. But then $D(X) \subseteq \frac{1}{\delta}(Y - y_0) = Y$, so $X = \operatorname{span} D(X) \subseteq Y$, a contradiction.

To see (ii), suppose that $\ker f$ is not dense in X. Then there would be $x_0 \in X$ and $\delta > 0$ such that $(x_0 + \delta D(X)) \cap \ker f = \emptyset$, so

$$0 \notin f(x_0 + \delta D(X)) = f(x_0) + \delta f(D(X)) \Longrightarrow \frac{1}{\delta} f(x_0) \notin -f(D(X)) = f(D(X)) \tag{3.1}$$

But then $\|f\| \leq \frac{1}{\delta} f(x_0)$, for if $\|f\| > \frac{1}{\delta} f(x_0)$, there would be $x \in D(X)$ such that $|f(x)| > \frac{1}{\delta}|f(x_0)|$. Thus

$$\left|\frac{f(x_0)}{\delta f(x)}\right| < 1 \Longrightarrow \frac{f(x_0)}{\delta f(x)} = \frac{1}{\delta} f(x)$$

contradicting the statement in (3.1). ■

Definition. Let $\emptyset \neq A \subseteq X$. We say that A is

- **convex** if for $a, b \in A$ and $0 < \lambda < 1$, $(1 - \lambda)a + \lambda b \in A$.
- **absorbing** at $a_0 \in A$ if for any $x \in X$, there is $\epsilon(a_0, x) > 0$ such that $a_0 + tx \in A$ for $0 \leq t < \epsilon$.

For example, if X is a normed space, then any open set is absorbing around any of its points.

> ***3.10 Lemma. (Minkowski Functional)*** *Let $A \subset X$ be a convex set containing 0 and absorbing at 0. Define $p : X \to \mathbb{R}$ by $p(x) = \inf\{t > 0 : x \in tA\}$. Then p is a sublinear functional. Moreover, we have that*
> *(i) $\{x \in X : p(x) < 1\} \subseteq A \subseteq \{x \in X : p(x) \leq 1\}$; and*
> *(ii) if X is normed and A is a neighbourhood of 0, then there is $N > 0$ such that $p(x) \leq N\|x\|$ for $x \in X$.*

Proof First note, for any $x \in X$, if A is absorbing at 0, there is $s > 0$ such that $sx \in A$, so $x \in \frac{1}{s}A$ and hence $0 \leq p(x) < \infty$.

Let's see non-negative homogeneity. Clearly $p(0) = 0$. If $s > 0$ and $x \in X$, then

$$p(sx) = \inf\{t > 0 : sx \in tA\} = \inf\left\{t > 0 : x \in \frac{t}{s}A\right\} = s \cdot \inf\left\{\frac{t}{s} > 0 : x \in \frac{t}{s}\right\} = sp(x)$$

We also have subadditivity. First, note that if $s,t>0$ and $a,b\in A$, then

$$sa+tb=(s+t)\left(\frac{s}{s+t}a+\frac{s}{s+t}b\right)\in(s+t)A\Longrightarrow sA+tA\subseteq(s+t)A$$

by convexity, and also $(s+t)A=\{(s+t)a:a\in A\}\subseteq\{sa+tb:a,b\in A\}=sA+tA$. Thus $sA+tA=(s+t)A$. Now for $x,y\in X$, we have

$$\begin{aligned}p(x)+p(y)&=\inf\{s>0:x\in sA\}+\inf\{t>0:y\in tA\}\\&=\inf\{s+t:s>0,t>0,x\in sA,y\in tA\}\\&\geq\inf\{s+t:s>0,t>0,x+y\in sA+tA=(s+t)A\}\\&=\inf\{r>0:x+y\in rA\}=p(x+y)\end{aligned}$$

so that p is a sublinear functional. Then

(i) If $p(x)<1$, then there is $0<t<1$ so $x\in tA$; i.e. $\frac{1}{t}x\in A$ and $x=(1-t)=+t\frac{1}{t}x\in A$. The second inclusion is obvious.

(ii) The assumptions provide $\delta>0$ so $\delta D(X)\subseteq A$. Then for $x\in X$ and $\epsilon>0$,

$$x\in(\|x\|+\epsilon)D(X)=\frac{\|x\|+\epsilon}{\delta}\delta D(X)\subseteq\frac{\|x\|+\epsilon}{\delta}A$$

so $p(x)\leq\frac{\|x\|+\epsilon}{\delta}$ so $p(x)\leq\frac{1}{\delta}\|x\|$; the result follows with $N=1/\delta$. ■

3.11 Theorem. (Hyperplane Separation) *Let X be an $\mathbb{F}$−vector space, $A,B\subset X$ be convex with $A\cap B=\emptyset$ and A absorbing at some a_0. Then there are $f\in X'$ and $\alpha\in\mathbb{R}$ such that*

$$\operatorname{Re}f(a)\geq\alpha\geq\operatorname{Re}f(b)$$

for $a\in A$ and $b\in B$. Moreover, if X is normed, then
- *If A is a neighbourhood of a_0, we have $f\in X^*$; and*
- *if A is absorbing around each of its points (for example if A is open), then we have $\operatorname{Re}f(a)>\alpha\geq\operatorname{Re}f(b)$.*

Proof We first re-centre at 0. Let $A-B=\{a-b:a\in A,b\in B\}$. Then it is easy to verify that

(i) $A-B$ is absorbing at any a_0-b, $b\in B$

(ii) $A-B$ is convex

(iii) if X is normed and A a neighbourhood of a_0, then $A-B$ is a neighbourhood of each a_0-b, $b\in B$; and if A is absorbing around any of its points (resp. open), then A_B is absorbing around any of its points (resp. open).

Let $x_0=a_0-b_0$ for some $b_0\in V$, and set $C=x_0-(A-B)$, so we have $0=x_0-x_0\in C$. Then by the above points, C is absorbing at 0, convex, and if X is normed and A a neighbourhood of a_0, then C is a neighbourhood of 0; and if A is absorbing at any of its points (resp. A is open), then C is absorbing at each of its points (resp. open).

Let p be the Minkowski functional of C. Notice that since $A\cap B=\emptyset$, $0\notin A-B$ so $x_0\notin C$. Thus by (i) of the lemma, $p(x_0)>1$.

Let us find f and α. Let $f_0:\mathbb{R}x_0\to\mathbb{R}$, by $f_0(sx)=sp(x_0)$. Hence f_0 is linear and $f_0\leq p|_{Rx_0}$, so by Hahn-Banach, get $f\in X'_{\mathbb{R}}$ such that $f\leq p$ on X. If $a\in A$ and $b\in B$, then

$x_0-(a-b) \in C$, so by (i) of the lemma, since $p(x_0) \geq 1$, we have $f(x_0-(a-b)) \leq p(x_0-(a-b)) \leq 1$. Thus $f(x_0)+f(b) \leq 1+f(a)$ so in fact $f(b) \leq f(a)$. Thus there exists some $\alpha \in \mathbb{R}$ such that

$$\sup\{f(b) : b \in B\} \leq \alpha \leq \inf\{f(a) : a \in A\}$$

If $\mathbb{F} = \mathbb{R}$, we are done; otherwise, we shall replace f by $f_{\mathbb{C}}$

For the remainder of the proof, we suppose X is a normed space, and A is a neighbourhood of a_0. Then part (ii) of the lemma provides $N > 0$ so that $p(x) \leq N\|x\|$. Then for $x \in X$, $f(x) \leq p(x) \leq N\|x\|$ and $-f(x) = p(-x) \leq N\|-x\| = N\|x\|$ so $|f(x)| \leq N\|x\|$, in other words that $\|f\| \leq N$ and $f \in X^*$. If A is absorbing around any of its points, then $f(a) > \alpha$ for any $a \in A$. Indeed, suppose $f(a) = \alpha$. Then there would be $t > 0$ so $a + t(-x_0) \in A$. But then $\alpha \leq f(a - tx_0) = f(a) - tf(x_0) < \alpha$, a contradiction. ■

Definition. If $\emptyset \neq S \subset X$, then its **convex hull** is given by

$$\mathrm{conv}(S) = \{\sum_{i=1}^{n} \lambda_j x_j : n \in \mathbb{N}, x_1, \ldots, x_n \in S \text{ and } \lambda_1, \ldots, \lambda_n \geq 0 \text{ with } \sum_{j=1}^{n} \lambda_j = 1\}$$

One can verify that $\mathrm{conv}(S)$ is in fact convex, and is the smallest convex set containing S, i.e.

$$\mathrm{conv}(S) = \bigcap\{C : S \subseteq C \subseteq X, C \text{ convex}\}$$

If X is normed, we let $\overline{\mathrm{conv}}(S)$ denote the **closed convex hull**, i.e. the closure of the convex hull.

Definition. A **half-space** of X is any set of the form $H = \{x \in X : \mathrm{Re}\, f(x) \leq \alpha\}$ for some $f \in X'$, $\alpha \in \mathbb{R}$.

If X is normed, then the last proposition shows H is closed if and only if f is bounded.

3.12 Theorem. *If X is a normed vector space and $\emptyset \neq S \subset X$, then $\overline{\mathrm{conv}}(S) = \cap\{H : S \subseteq H \subset X, H \text{ a closed half space}\}$.*

Proof It is immediate that $\overline{\mathrm{conv}}(S) \subseteq \cap\{H : S \subseteq H \subset X, H \text{ a closed half-space}\}$. Thus suppose $x_0 \notin \overline{\mathrm{conv}}(S)$. Then there is $\delta > 0$ such that $(x_0 + \delta D(X)) \cap \overline{\mathrm{conv}}(S) = \emptyset$. Since $x_0 + \delta D(X)$ is open and convex, hyperplace separation gives provides $f \in X^*$ and $\alpha \in \mathbb{R}$ so $\mathrm{Re}\, f(a) > \alpha \geq \mathrm{Re}\, f(b)$ for $a \in x_0 + \delta D(X)$ and $b \in \overline{\mathrm{conv}}(S)$. Then $S \subset H = \{y \in X : \mathrm{Re}\, f(x) \leq \alpha\}$ but $x_0 \notin H$. ■

4 Some Applications of Baire Category Theorem

4.1 Theorem. (Baire Category I) *If (X,d) is a complete metric space and $\{U_n\}_{n=1}^{\infty}$ is a countable collection of dense, open subsets, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.*

Definition. Let (X,d) be a metric space. A subset $F \subset X$ is **nowhere dense** if $X \setminus F$ is dense in X; equivalently, $\overline{F}$ contains no non-trivlal open subsets. We say that a subset $M \subseteq X$ is **meagre** (1st category) if $M = \bigcup_{n=1}^{\infty} F_n$ and each F_n is nowehere dense; and a set is **non-meagre** (2nd category) otherwise.

4.2 Theorem. (Baire Category II) *Let (X,d) be a complete metric space. Then a non-empty open $U \subseteq X$ is non-meagre.*

Proof Suppose not, so $U = \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} \overline{F}_n$, each F_n (hence $\overline{F_n}$) nowhere dense. Then each $V_n = X \setminus \overline{F_n}$ is open and dense, and hence by BCT I, $G = \bigcap_{n=1}^{\infty} V_n$ is dense in X, and hence $U \cap G \neq \emptyset$, violating assumption ■

4.3 Theorem. ***(Banach-Steinhaus)*** *Let X, Y be normed spaces, $U \subseteq X$ be non-meagre, and $\mathcal{F} \subset \mathcal{B}(X, Y)$ be such that for each $x \in U$, $\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty$ (pointwise bounded). Then $\mathcal{F}$ is uniformly bounded, i.e. $\sup\{\|T\| : T \in \mathcal{F}\} < \infty$.*

Proof Let for each $n \in \mathbb{N}$

$$F_n = \bigcap_{T \in \mathcal{F}} T^{-1}(nB(Y)) = \{x \in X : \|Tx\| \leq n \text{ for all } T \in \mathcal{F}\}$$

so each F_n is closed and, by the pointwise boundedness assumption, $U \subseteq \bigcup_{n=1}^{\infty} F_n$. By assumption of non-meagreness of U, at least one F_{n_0} admis an interior point: there is $x_0 \in F_{n_0}$ and $\delta > 0$ such that $x_0 + \delta D(X) \subseteq F_{n_0}$. Then if $x \in D(X)$, we have

$$Tx = \frac{1}{\delta}\left[T\left(x_0 + \frac{\delta}{2}x\right) - T\left(x_0 - \frac{\delta}{2}x\right)\right]$$

so $\|Tx\| \leq \frac{2}{\delta} n_0$, in other words

$$\|T\| = \sup_{x \in D(x)} \|Tx\| \leq \frac{2n_0}{\delta} < \infty$$

where the bound is independent of T. ■

4.4 Theorem. ***(Open Mapping)*** *Let X, Y be Banach spaces, and $T \in B(X, Y)$ surjective. Then T is an open map; i.e. $T(U)$ is open in Y whenver U is open in X.*

Remark. Given $x \in X$ and $\alpha \in \mathbb{F} \setminus \{0\}$, non-empty $A \subset X$, we have that $\overline{x + \alpha A} = x + \alpha \overline{A}$. Indeed, note that for $(a_k)_{k=1}^{\infty} \subset A$, we have

$$a_k \to a \in \overline{A} \text{ if and only if } x + \alpha a_k \to x + \alpha a \in x + \alpha \overline{A}$$

4.5 Lemma. *With the assumptions as above, we have that if $\overline{T(D(X)} \supset rB(Y)$ for some $r > 0$, then $T(D(X)) \supseteq rD(Y)$.*

Proof Let $z \in rD(Y)$ and let $0 < \delta < 1$ be so $\|z\| < r(1-\delta) < r$. Set $y = z/(1-\delta)$ so $\left\|y\right\| < r/(1-\delta)$. It suffices to show that $y \in \frac{1}{1-\delta}T(D(X))$. To begin, let $A = T(D(X)) \cap rB(Y)$, so $\overline{A} = rB(Y)$. Indeed, if $y \in rB(Y) \subseteq \overline{T(D(X))}$, then there is $(y_k)_{k=1}^{\infty} \subset \overline{T(D(X))}$, so $y = \lim y_k$. But then there is $x_k \in D(X)$ so each $\left\|y_k - T(x_k)\right\| < 1/k$ so $y = \lim T(x_k)$ with each $x_k \in D(X)$.

Now we inductively build a sequence $(y_n)_{n=1}^{\infty}$ as follows.

- Since $y \in rD(Y) \subseteq \overline{A}$, there is $y_1 \in A \cap (y + \delta rD(Y))$
- $y \in y_1 + \delta r(D(Y)) \subseteq \overline{y_1 + \delta A} = y_1 + \delta\overline{A}$, so there is $y_2 \in (y_1 + \delta A) \cap (y + \delta^2 rD(Y))$
- $y \in y_n + \delta^n rD(Y) \subseteq \overline{y_n + \delta^n A}$, so there is $y_{n+1} \in (y_n + \delta^n A) \cap (y + \delta^{n+1} rD(Y))$

By construction, $y_{n+1}-y_n \in \delta^n A$, so $\left\|y_{n+1}-y_n\right\| \leq \delta^n r$ and there is $x_n \in \delta^n D(X)$ such that $y_{n+1}-y_n = Tx_n$. Likewise, $y_1 \in A \subseteq T(D(X))$ so $y = T(x_0)$ for some $x_0 \in D(X)$. Notice that each $y_n \in y + \delta^n r \in D(Y)$, so $\left\|y_n - y\right\| \leq \delta^n r \to 0$. Since X is complete, we let $x = \sum_{n=0}^{\infty} x_n$, and by construction

$$\|x\| \leq \sum_{n=0}^{\infty} \|x_n\| < \sum_{n=0}^{\infty} \delta^n = \frac{1}{1-\delta}$$

Then by linearity and continuity of T, we have

$$Tx = \sum_{n=0}^{\infty} Tx_n = y_1 + \sum_{n=1}^{\infty}(y_{n+1}-y_n) = y_N + \sum_{n=N}^{\infty}(y_{n+1}-y_n) \to y$$

so that indeed $T(x) = y$, as required. ■

Remark. So far, we've only used completeness of X and continuity and linearity of T.

We now proceed with the proof of the open mapping theorem.

Proof It suffices to see that $T(D(X))$ contains a neighbourhood of 0 in Y. Indeed, if $\emptyset \neq U \subseteq X$ is open, $x \in U$, then there is $\delta > 0$ such that $x + \delta D(X) \subseteq U$, so $U - x \supseteq \delta D(X)$. If $T(D(X)) \supseteq rD(Y)$, then $T(U-x) \supseteq \delta T(D(X)) \supseteq r\delta D(Y)$ so that $Tx + r\delta D(Y) \subseteq T(U)$. In other words, $T(U)$ is a neighbourhood of any of its points, and thus open.

Now write $X = \bigcup_{n=1}^{\infty} nD(X)$, and we assume that $T(X) = Y$. Hence $Y = \bigcup_{n=1}^{\infty} nT(D(X))$, so $Y = \bigcup_{n=1}^{\infty} n\overline{T(D(X))}$. But Y is complete, so by Baire category theorem, there is some n so that $n\overline{T(D(X))}$ has non-empty interior. Since $nT(D(X))$ is convex and symmetric, and hence $n\overline{T(D(X))}$ is convex and symmetric as well. Thus if $y \in D(Y)$, then $y_0 \pm \epsilon \in y_0 + \epsilon D(Y)$ so

$$\epsilon y = \frac{1}{2}\left[y_0 + \epsilon y - (y_0 - \epsilon y)\right] \in n\overline{T(D(X))}$$

and $\frac{\epsilon}{n} y \in \overline{T(D(X))}$, i.e. $\frac{\epsilon}{n} D(Y) \subseteq \overline{T(D(X))}$. Thus applying the main lemma, $\frac{\epsilon}{n} D(Y) \subseteq T(D(X))$. ■

4.6 Theorem. (Inverse Mapping) *If X, Y are Banach spaces and $T \in \mathcal{B}(X,Y)$ is invertible, $T^{-1} \in \mathcal{B}(Y,X)$*

Proof Direct application of the open mapping theorem. ■

Let X, Y be normed spaces. Then we define for $(x,y) \in X \oplus Y$, and we let $\left\|(x,y)\right\|_1 = \|x\| + \left\|y\right\|$. It is easy to check that $\|\cdot\|_1$ is a norm on $X \oplus Y$, and if X, Y are Banach, then so is $(X \oplus Y, \|\cdot\|_1)$. In this case, we write $X \oplus_1 Y$.

4.7 Theorem. (Closed Graph) *Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. Then $T \in \mathcal{B}(X,Y)$ if and only if $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \oplus_1 Y$.*

Proof Let $T \in \mathcal{B}(X,Y)$. If $(x_n) \to x$ in X, then $Tx_n \to Tx$ in Y. Thus if $(x,y) \in \overline{\Gamma(T)}$, then $(x,y) = \lim(x_n, Tx_n)$ where $(x_n, Tx_n) \in \Gamma(T)$. But then

$$\left\|y - Tx\right\| \leq \left\|y - Tx_n\right\| + \|Tx_n - Tx\| \leq \|x - x_n\| + \left\|y - Tx_n\right\| + \|Tx_n - tx\| = \left\|(x-y) - (x_n, Tx_n)\right\|_1$$

so in fact $y = Tx$ so $(x,y) = (x, Tx)$.

Conversely, if $\Gamma(T)$ is closed in $X \oplus_1 Y$, then $\Gamma(T)$ is a Banach space. Define $S : \Gamma(T) \to X$ by $S(x, Tx) = x$. Notice that S is linear, and

$$\|S(x, Tx)\| = \|x\| \le \|(x, Tx)\|_1$$

so $\|S\| \le 1$, so S is bounded. It is also clear that S is bijective, with $S^{-1} : X \to \Gamma(T)$ given by $S^{-1}(x) = (x, Tx)$. Thus the inverse mapping theorem gives that S^{-1} is also bounded. Hence for any $x \in X$,

$$\|Tx\| \le \|(x, Tx)\|_1 = \left\|S^{-1}x\right\| \le \|x\| \left\|S^{-1}\right\|$$

so that T is in fact bounded. ■

4.8 Theorem. (Closed graph test) *Given normed spaces and $T \in \mathcal{L}(X, Y)$, we have that $\Gamma(T)$ is closed in $X \oplus_1 Y$ if and only if whenever $x_n \to 0$ for which we may assume that Tx_n converges in Y, say $y = \lim Tx_n$, then $y = 0$ too.*

Proof We have $(x_n, Tx_n) \to (x, z) \in \overline{\Gamma(T)}$ if and only if $(x_n - x, T(x_n - x)) \to (x, z) - (x, Tx) = (0, z - Tx)$. Set $y = z - Tx$. We have $(x, z) \in \Gamma(T)$ if and only if $z = Tx$ if and only if $y = 0$. ■

Testing hypothesis of OMT

(i) Let $1 \le p < r < \infty$. We have that $\ell_p \subseteq \ell_r$, with $\|x\|_r \le \|x\|_p$ for $x \in \ell_p$. First, suppose $x \in B(\ell_p)$, so for each k, $|x_k| \le \|x\|_p \le 1$ so $|x_k|^{r/p} \le |x_k|$. Hence

$$\|x\|_r = \left(\sum_{k=1}^{\infty} |x_k|^r\right)^{1/r} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/r} = \|x\|_p^{p/r} \le 1$$

so if $x \in \ell_p \setminus \{0\}$, then the result follows.
Let $S : (\ell_p, \|\cdot\|_p) \to (\ell_p, \|\cdot\|_r)$ be the identity map. Then $\|S\| \le 1$, and furthermore S is bijective. If S were open, then by the proof of inverse mapping theorem, we would see that $\left\|S^{-1}\right\| < \infty$. Define $x^{(n)} \in \ell_p$ by

$$x_k^{(n)} = \begin{cases} \frac{1}{ck^{1/p}} & k \le n \\ 0 & k > n \end{cases}, c = \sum_{k=1}^{\infty} \frac{1}{k^{r/p}}$$

We compute that $\left\|x^{(n)}\right\|_r < 1$ while $\left\|x^{(n)}\right\|_p = \frac{1}{c}\left(\sum_{k=1}^{n} \frac{1}{k}\right)^{1/p}$. In other words, $\left\|S^{-1}x^{(n)}\right\|_p$ goes to infinity, while $\left\|x^{(n)}\right\|_r < 1$, contradicting $\left\|S^{-1}\right\| < \infty$. The moral of this is that if the range space is not complete, then OMT may not hold.

(ii) Take $X = C_b(0,1)$, $X_0 = \{f \in X : f \text{ is diferentiable on } (0,1), f' \in C_b(0,1)\}$. We have $X_0 \subseteq X$, and we put the uniform norm $\|\cdot\|_\infty$ on both spaces. We let $D : X_0 \to X$, $Df = f'$. If $h_n(t) = t^n$, then $\|h_n\|_\infty = 1$ while $\|Dh_n\|_\infty = n$, so D is not bounded. Despite this, we have that $\Gamma(D) = \{(f, f') : f \in X_0\}$ is closed in $X_0 \oplus_1 X$. We apply the closed graph test: let $(f_n, f_n') \to (0, g)$ in $X_0 \oplus_1 X$. Notice that $\|f_n'\|_\infty < \infty$, so f_n is Libschitz on $(0,1)$, so f_n is uniformly continuous on $(0,1)$, so $f_n(0^+) = \lim_{t \to 0^+} f(t)$ exists. Thus by the fundamental theorem of calculus, $f_n(t) = f_n(0^+) + \int_0^t f_n'$ for $t \in (0,1)$. In particular,

- $f_n \to 0$ uniformly, so $f_n(0^+) \to 9$

- $f_n' \to g$ uniformly, so for each $t \in (0,1)$,

$$\int_0^t g = \lim_{n\to\infty} \int_0^t f_n' = \lim_{n\to\infty} [f_n(t) - f_n(0^+)] = 0$$

and again, by the FT of C, $g(t) = 0$. Thus $g = 0$, so $\Gamma(D)$ is closed. We say that $D : X_0 \to X$ is a **closed** operator. The moral here is that if the domain is not complete, then closedness of the graph does not imply boundedness of the operator.

Now, let $J : X \to X_0$ have $Jg(t) = \int_0^t g$ for $t \in (0,1)$. By the FT of C, $D \circ J(G) = g$, in other words that $D \circ J = I$. We have for $g \in X$,

$$\|Jg\|_\infty = \sup_{t\in(0,1)} |\int_0^t g| \le \sup_{t\in(0,1)} t\|g\|_\infty \le \|g\|_\infty$$

so $\|J\| \le 1$. Hence $J(D(X)) \subseteq D(X_0)$, and we apply D to see $D(X) \subseteq D(D(X_0))$, in other words, that D is open. As an exercise, show that $C_b(0,1) = X$ is not separable, while X_0 is separable.

Let $X \subsetneq Y$ be $\mathbb{F}$ –vector spaces. We can always find a subspace $Z \subset Y$ so $X + Z = Y$ and $X \cap Z = \{0\}$. Indeed, let B be a basis for X, and $B' = B \cup B'$ is a basis for Y, and take $Z = \operatorname{span} B'$.

4.9 Theorem. *Let Y be a Banach space and $X \subsetneq Y$ a closed subspace. Then X admis a closed complement Z if and only if there is some $P \in \mathcal{B}(Y)$ such that $P \circ P = P$ and $\operatorname{im} P = P(Y) = X$.*

Remark. We say that $X \subsetneq Y$ is **boundedly complemented** if either of the above conditions hold.

Proof ($\Leftarrow$) Let $Z = \ker P$, which is closed. If $y \in Y$, then $y = Py + (I - Py$ where $Py \in X$ and $P(I-P)y = 0$ so $(I-P)y \in \ker P$. If $z \in Z \cap X$, then $z = Py$ for some $y \in Y$ so $Pz = P^2 y = Py = z$, but $z \in \ker P$, so $z = Pz = 0$.

($\Rightarrow$) Let $S : X \oplus_1 Z \to Y$ be given by $S(x,z) = x+z$. Then S is surjective and if $(x,z) \in \ker S$, then $x + z = 0$ so $x = -z \in X \cap Z = \{0\}$, hence S is injective. Furthermore,

$$\|S(x+z)\| = \|x+z\| \le \|(x,z)\|_1$$

so $\|S\| \le 1$. Hence S is a bounded bijection between Banach space and hence S^{-1} is bounded by the inverse mapping theorem. Let $P_1 : X \oplus_1 Z \to X$ be given by $P_1(x,z) = x$; and $J : X \to Y$ by $Jx = x$. Notice that $\|P_1\| = 1$ and $\|J\| = 1$. Define $P : Y \to Y$ by $Py = JP_1S^{-1}y$. Then

- $\operatorname{im} J = X$, and each of P_1, S^{-1} are surjective, so $\operatorname{im} P = X$
- If $y \in Y$, $\|Py\| = \|JP_1S^{-1}y\| \le \|S^{-1}\| \|y\|$ so $\|P\| \le \|S^{-1}\|$
- Clearly $P^2 = JP_1S^{-1}JP_1S^{-1} = P$ ∎

4.10 Theorem. *c_0 is not boundedly complemented in ℓ_∞.*

Proof Let us assume otherwise; hence, there is $P = P^2 \in \mathcal{B}(\ell_\infty)$ such that $\operatorname{im} P = c_0$. Note that $c_0 = \ker(I - P)$. As in A2, we let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a family of infinite subsets such that for $E \neq F$ in $\mathcal{F}$, $|E \cap F| < \infty$ and $|\mathcal{F}| = \mathfrak{c}$. For each $F \in \mathcal{F}$, we let $y_F = (I_P)\chi_F \neq 0$. If $\alpha_1, \ldots, \alpha_n \in F$ are pairwise distinct, $F_1, \ldots, F_m \in \mathcal{F}$, then

$$\sum_{i=1}^{n} \alpha_i \chi_{F_i} = \underbrace{\sum_{i=1}^{m} \alpha_i \chi_{F_i \setminus \bigcup_{j \in [m] \setminus \{i\}} F_j}}_{:=z} + \underbrace{\sum_{k=2}^{m} \sum_{1 \leq i < \cdots < i_k \leq m} (\alpha_{i_1} + \cdots + \alpha_{i_k}) \chi_{F_{i_1} \cap \cdots \cap F_{i_k}}}_{\in c_0}$$

where $\|z\|_\infty = \max_{k=1,\ldots,m} |\alpha_k|$. Hence

$$\left\| \sum_{i=1}^{m} \alpha_i y_{F_i} \right\| = \|(I-P)z\| \leq \|I-P\| \|z\| = \|I-P\| \max_{k=1,\ldots,m} |\alpha_k| \tag{4.1}$$

Now, let for $n, k \in \mathbb{N}$, $\mathcal{F}_{n,k} = \{F \in \mathcal{F} : |\delta_k(y_F)| \geq \frac{1}{n}\}$m where $\underline{\delta_k(x_i)_{i=1}^{\infty}} = x_k$, so $\delta_k \in \ell_\infty^*$ with $\|\delta_k\| \leq 1$. Let $F_1, \ldots, F_m$ be pairwise disjoint in $\mathcal{F}_{n,k}$, and $\alpha_i = \overline{\operatorname{sgn} \delta_k(y_{F_i})}$. Then we have each $|\alpha_i| = 1$, so by (4.1), we find

$$\|I-P\| \geq \left\| \sum_{i=1}^{\infty} \alpha_i y_{F_i} \right\|_\infty \geq |\delta_k \sum_{i=1}^{n} \alpha_i y_{F_i}| = \sum_{i=1}^{m} |\delta_k(y_{F_i})| \geq \frac{m}{n}$$

so $m \leq n\|I-P\|$ and it folows that $\mathcal{F}_{n,k}$ is finite. Since each $y_F \neq 0$ for $F \in \mathcal{F}$, we see that $\mathcal{F} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty}$, which contradicts that $|\mathcal{F}| = \mathfrak{c}$. Hence such a P must not exist. ■

4.11 Theorem. *If X is a finite dimensional vector space over $\mathbb{F}$, then any two norms are equivalent.*

Proof Let $\|\cdot\|$ be a norm on X. Fix a basis $(e_1, \ldots, e_n)$ for X, and let $x = \sum_{k=1}^{n} x_k e_k$, $x_i \in \mathbb{F}$, $\|x_k\|_\infty = \max_{k=1,\ldots,n} |x_k|$. This is easily checked to be a norm. Moreover, $B_\infty = \{x \in X : \|x\|_\infty \leq 1\}$ admits a homeomorphic identification

$$B_\infty = \begin{cases} [-1,1]^n & \mathbb{F} = \mathbb{R} \\ \overline{D}^n & \mathbb{F} = \mathbb{C} \end{cases}$$

and hence is compact. Thus $S_\infty = \{x \in X : \|x\|_\infty = 1\}$ is compact as well. Hence, for $x = \sum_{k=1}^{\infty} x_k e_k$, we have

$$\|x\| \leq \sum_{k=1}^{n} |x_k| \|e_k\| \leq \|x\|_\infty \underbrace{\sum_{k=1}^{n} \|e_k\|}_{:=M}$$

Now for $x, y \in X$, we have $|\|x\| - \|y\|| \leq \|x-y\| \leq M\|x-y\|_\infty$ so $\|\cdot\|$ is Lipschitz with respect to $\|\cdot\|_\infty$, and hence $\tau_{\|\cdot\|_\infty}$–continuous. Thus the extreme value theorem tells us that $m = \inf_{x \in S_\infty} \|x\| > 0$. Hence for $x \in X \setminus \{0\}$, $\|x\| = \|x\|_\infty \cdot \left\| \frac{1}{\|x\|_\infty} x \right\| \geq \|x\|_\infty m$. In general, $m\|x\|_\infty \leq \|x\| \leq M\|x\|_\infty$. We thus have that $\|\cdot\| \sim \|\cdot\|_\infty$, so any norms are equivalent. ■

4.12 Corollary. *Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Then*

(i) $K \subseteq X$ is compact if an only if K is closed and bounded.

(ii) $(X, \|\cdot\|)$ is a Banach space

(iii) For any normed space Y, we have $\mathcal{L}(X,Y) = \mathcal{B}(X,Y)$
(iv) We have $X' = X^*$.

Proof (i) The forward direction is immediate. If K is closed and bounded, is contained in some scaled copy of B_∞, which is compact.
(ii) Cauchy sequences are bounded, and thus contained in some scaled copy of B_∞, which is compact.
(iii) Let $T \in \mathcal{L}(X,Y)$, and let $\|x\|_0 = \|x\| + \|Tx\|$. Then the result follows by equivalence of norms.
(iv) Immediate. ■

4.13 Proposition. *A finite dimensional subspace of normed space is always closed and boundedly complemented.*

Proof Let $Y \subseteq X$ be so Y is finite dimensional and X a normed space. We can find a basis $(e_1,\dots,e_n)$ for Y. We may assume that each $\|e_k\| = 1$. We define $f_1,\dots,f_n \in Y' = Y^*$ by

$$f_k\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \alpha_k$$

By Hahn-Banach, get $F_1,\dots,F_n \in X^*$ such that $F_k|_Y = f_k$ and $\|F_k\| = \|f_k\|$. Define $P : X \to X$ by $Px = \sum_{k=1}^{n} F_k(x)e_k$. Notice that $\operatorname{im} P \subseteq Y$ and by choice of $F_k|_Y = f_k$, we have $P|_Y = I_Y$. Thus $P^2 = P$. Finally, for $x \in X$, $\|Px\| \le \sum_{k=1}^{n} \|f_k\|\|x\|$ so $\|P\| \le \sum \|f_k\| < \infty$, i.e. P is bounded. Closedness of Y thus follows from the last corollary. Alternatively, $Y = \ker(I - P)$. ■

5 On Compactness of the Unit Ball

5.1 Lemma. *Let X be a normed space and $Y \subsetneq X$ a closed subspace. Then given $\epsilon \in (0,1)$ there is $x_0 \in D(X) \subseteq B(X)$ such that $d(x_0, Y) > 1 - \epsilon$.*

Proof Let $x \in X \setminus Y$ and let $f : Y + \mathbb{F}x \to \mathbb{F}$ be given by $f(y + \alpha x) = \alpha$, $y \in Y$, $\alpha \in \mathbb{F}$. Then f is linear and $\ker f = Y$ is closed, $Y \subsetneq Y + \mathbb{F}x$, so f is bounded. Let $F \in X^*$ be any Hahn-Banach extension of f with $\|F\| = \|f\|$.

Now, we find $x_0 \in D(X)$ such that $|F(x_0)| > (1-\epsilon)\|F\|$. Since $Y \subseteq \ker F$, we have for $y \in Y$ that $\|F\|\left\|x_0 - y\right\| \ge |f(x_0 - y)| = |F(x_0)| > (1-\epsilon)\|F\|$, so $\left\|x_0 - y\right\| > 1 - \epsilon$. Hence $d(x_0, Y) = \inf_{y \in Y} \left\|x_0 - y\right\| \ge 1 - \epsilon$. ■

5.2 Theorem. *Let X be a normed space. Then $B(X)$ is compact if and only if X is finite dimensional.*

Proof The reverse implication is standard. Thus suppose X is not finite dimensional. Let $\epsilon \in (0,1)$ and let $x_1 \in B(X) \setminus \{0\}$. Inductively,

- Find $x_2 \in B(X)$ such that $\operatorname{dist}(x_2, \mathbb{F}x_1) \ge 1 - \epsilon$
- Find $x_3 \in B(X)$ such that $\operatorname{dist}(x_3, \operatorname{span}\{x_1, x_2\}) \ge 1 - \epsilon$
- Find $x_{n+1} \in B(X)$ such that $\operatorname{dist}(x_{n+1}, \operatorname{span}\{x_1,\dots,x_n\}) \ge 1 - \epsilon$

Hence we have $\{x_n\}_{n=1}^{\infty} \subset B(X)$ such that for $m < n$,

$$\|x_n - x_m\| \ge d(x_n, \operatorname{span}\{x_1,\dots,x_{n-1}\}) \ge 1 - \epsilon$$

so the sequence admis no converging subsequence and $B(X)$ is not compact. ■

6 More Topology

Definition. Let (X,τ) be a topological space. A **base** for τ is any family $\beta \subseteq \tau$ such that for any $U \in \tau$ and $x \in U$, there is $B \in \beta$ such that $x \in B \subseteq U$. A **subbase** for τ is any family $\alpha \subseteq \tau$ such that $\{\bigcap_{k=1}^n U_k : n \in \mathbb{N}, U_1,\ldots,U_n \in \alpha\}$ is a base for τ.

Note that if $\emptyset \neq X$ and $\beta \subseteq \mathcal{P}(X)$ for which $\bigcup_{B\in\beta} B = X$ and β is closed under finite intersections, then

$$\tau_\beta = \{\bigcup_{i\in I} B_i : \{B_i\}_{i\in I} \subset B, I \text{ any index set with } |I| \leq |\beta|\}$$

is a topology.

Definition. Let $X \neq \emptyset$. Suppose we are given

- a family $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in A}$ of topological spaces, and
- for each $\alpha \in A$, a function $f_\alpha : X \to X_\alpha$

Then the **initial topology** on X given this data is denoted

$$\sigma = \sigma(X,(f_\alpha)_{\alpha\in A}) = \sigma(X,(f_\alpha,\tau_\alpha)_{\alpha\in A})$$

and is the topology with base

$$\bigcap_{k=1}^n f_{\alpha_k}^{-1}(U_{\alpha_k}), n \in \mathbb{N}, \alpha_1,\ldots,\alpha_n \in A, \text{ each } U_{\alpha_k} \in \tau_{\alpha_k}$$

In particular, $\{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \tau_\alpha, \alpha \in A\}$ is a subbase for σ.

Remark. The topology is chosen so that each $f_\alpha : X \to X_\alpha$ is $\sigma - \tau_\alpha$–continuous. Furthermore, if $\tau \subseteq \mathcal{P}(X)$ is any topology for which every f_α is $\sigma - \tau_\alpha$–continuous, then $\sigma \subseteq \tau$. We say that σ is the **coarsest** topology so that all the f_α are continuous.

Example. (i) *Metric topology:* If (X,d) is a metric space, for each $x \in X$, let d_x be given by $d_x(x') = d(x,x')$. Then $\sigma(X,(d_x)_{x\in X}) = \tau_d$.

(ii) *Relative topology:* If (Y,τ)–topological space, $\emptyset \neq X \subseteq Y$, and $i : X \to Y$ is the inclusion map. Then $\tau|_X = \sigma(X,\{i\})$.

(iii) *Product topology:* Let $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in A}$ be a family of topological spaces. Let $X = \prod_{\alpha\in A} X_\alpha$. Let for $\alpha \in A$, $p_\alpha : X \to X_\alpha$ denote the projection map onto the component α. Then the product topology $\pi = \sigma(X,\{p_\alpha\}_{\alpha\in A})$. Hence, $V \in \mathcal{P}(X)$, then $V \in \pi$ if and only if for any $x \in V$, there is $\alpha_1,\ldots,\alpha_n \in A$ and $U_{\alpha_k} \in \tau_{\alpha_k}$ such that $x_{\alpha_k} = p_{\alpha_k(x)} \in U_{\alpha_k}$ and $x \in \bigcap_{k=1}^n p_{\alpha_k}^{-1}(U_{\alpha_j}) \subseteq V$.
Note that if $X = \prod_{n=1}^\infty X_n$, each (X_n,τ_n) is a topological space, then the basic open sets look like $U_1 \times U_2 \times \cdots \times U_m \times X_{m+1} \times X_{m+2} \times \cdots$.

(iv) *Linear topology:* Let X be a vector space and $Z \subseteq X'$ a subspace. Then $\sigma(X,Z)$ is the coarsest topology allowing each $f \in Z$ to be continuous, $f : X \to \mathbb{F}$. The basic open sets are given as follows: let $x_0 \in X$, $\epsilon > 0$, and $D = D(\mathbb{F})$, and we consider for $f \in Z$

$$f^{-1}(f(x_0)+\epsilon D) = \underbrace{\{x \in X : |f(x) - f(x_0)| < \epsilon\}}_{\text{"affine hypertube"}} = \{x \in X : |\frac{1}{\epsilon}f(x) - \frac{1}{\epsilon}f(x_0)| < 1\}$$

so that

$$\left\{\bigcap_{k=1}^n \{x \in X : |f_k(x) - f_k(x_0)| < 1\} : f_1,\ldots,f_n \in Z, n \in \mathbb{N}\right\}$$

is a base for $\sigma(X,Z)$.

(v) Now let X be a normed space. Then the **weak topology** on X is $\omega = \sigma(X,X^*)$. Certainly $\omega \subseteq \tau_{\|\cdot\|}$. Similarly, the **weak*-topology** on X^* is $\omega^* = \sigma(X^*,\hat{X})$ (recall for $x \in X$, $\hat{x}(f) = f(x)$). Since $\hat{X} \subseteq X^{**}$, we have $\omega^* \subseteq \omega = \sigma(X^*,X^{**}) \subseteq \tau_{\|\cdot\|}$.

Let (X,τ) be a topological space.

Definition. A subset $K \subseteq X$ is called **compact** if for any collection $\{U_\alpha\}_{\alpha\in A} \subseteq \tau$ with $\bigcup_{\alpha\in A} U_\alpha \supseteq K$, there exists some finite $U_1,\dots,U_n$ covering K. If X itself is τ−compact, we call (X,τ) a compact space.

Definition. A set $F \subseteq X$ is **closed** if $X \setminus F \in \tau$. If $S \subseteq X$, then the **closure** of S is $\overline{S} = \cap\{F \subseteq X : S \subseteq F, X \setminus F \in \tau\}$.

Note that $\overline{S} = \{x \in X : \text{for any } U \in \tau \text{ with } x \in U, U \cap S \neq \emptyset\}$.

Definition. A family $\mathcal{F} \subseteq \mathcal{P}(X)$ has the **finite intersection property** if for any $F_1,\dots,F_n \in \mathcal{F}$, $\bigcap_{l=1}^n F_k \neq \emptyset$.

6.1 Proposition. *Let (X,τ) be a topological space. Then (X,τ) is compact if and only if any $\mathcal{F} \subseteq \mathcal{P}(X)$ with the finite intersection property has $\bigcap_{F\in\mathcal{F}} \overline{F} \neq \emptyset$.*

Proof Suppose X is compact and $\mathcal{F} \subset \mathcal{P}(X)$ has the finite intersection property but with $\bigcap_{F\in\mathcal{F}} \overline{F}$, then $\{X \setminus \overline{F}\}_{F\in\mathcal{F}}$ is an open cover of X with no finite subcover.

Conversely, if $\mathcal{O} \subseteq \tau$ is an open cover of X, then $\mathcal{F} = \{X \setminus U\}_{U\in\mathcal{O}}$ satisfies $\bigcap_{F\in\mathcal{F}} = \emptyset$, so there is $F_1,\dots,F_n \in \mathcal{F}$ with $\bigcap_{k=1}^n F_k = \emptyset$. Then $\{X \setminus F_i\}_{i=1}^k$ is a finite subcover. ■

Definition. Let X be a non-empty set. An **ultrafilter** is a family $\mathcal{U} \subset \mathcal{P}(X)$ such that

- $\mathcal{U}$ has the finite intersection property
- If $A \in \mathcal{P}(X)$, then either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Example. (i) *Principal / trivial ultrafilter:* If $x_0 \in X$, let $U_{x_0} = \{U \subseteq X : x_0 \in U\}$.

6.2 Lemma. (Ultrafilter) *If $\mathcal{F} \subseteq \mathcal{P}(X)$ is any set with the finite intersection property, then there is an ultrafilter $\mathcal{U}$ with $\mathcal{F} \subset \mathcal{U}$.*

Proof Let $\Phi = \{\mathcal{G} \subseteq \mathcal{P}(X) : \mathcal{F} \subseteq \mathcal{G}, \mathcal{G} \text{ has f.i.p.}\}$. Then Φ is partially ordered by inclusion. If $\Gamma \subseteq \Phi$ is a chain, then $\mathcal{G}_\Phi = \bigcup_{\mathcal{G}\in\Gamma} \mathcal{G}$ contains $\mathcal{F}$ and has the finite intersection property. Hence Φ admits a maximal element $\mathcal{U}$. Let $A \in \mathcal{P}(X) \setminus \mathcal{U}$. Then $U \cup \{A\} \supsetneq \mathcal{U}$, so $\mathcal{U} \cup \{A\}$ fails the finite intersection property. Hence get $U_1,\dots,U_n$ so $A \cap \bigcap_{k=1}^n U_k = \emptyset$. Now if $V_1,\dots,V_m \in \mathcal{U}$, then $\bigcap_{j=1}^n V_j \cap \bigcap_{k=1}^n U_j \subseteq \bigcap_{k=1}^n U_k \subseteq X \setminus A$, so $(X \setminus A) \cap \bigcap_{j=1}^m V_j$. Thus $\mathcal{U} \cup \{X \setminus A\}$ has finite intersection property, so $X \setminus A \in \mathcal{U}$ by maximality. ■

6.3 Corollary. *If $U \subseteq \mathcal{P}(X)$ is an ultrafilter, then*

(i) If $A \in \mathcal{P}(X)$, $A \in \mathcal{U}$ if and only if $A \cap U \neq \emptyset$ for each $U \in \mathcal{U}$

(ii) If $A,B \in \mathcal{P}(X)$, then $A \cup B \in \mathcal{U}$ implies at least one of A or B is in $\mathcal{U}$

(iii) If $A \in \mathcal{U}$ and $A \subseteq V$ implies $V \in \mathcal{U}$

Proof The forward implication of (i) follows since $\mathcal{U}$ has finite intersection. Conversely, $X \setminus A \notin \mathcal{U}$, so $A \in \mathcal{U}$. (ii) and (iii) follow consequently. ■

6.4 Corollary. *If X is an infinite set, it admits a non-principle ultrafilter.*

PROOF Let $\mathcal{F} = \{F \in \mathcal{P}(X) : X \setminus F \text{ is finite}\}$. Then $\mathcal{F}$ has the finite intersection property. Apply the lemma. ∎

6.5 Proposition. *There are at least $\mathfrak{c}$ many ultrafilters in $\mathcal{P}(\mathbb{N})$.*

PROOF We let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a collection of infinite sets such that $E \neq F$ in $\mathcal{F}$ implies $|E \cap F| < \infty$, and $|\mathcal{F}| = \mathfrak{c}$. For each $F \in \mathcal{F}$, we let $\mathcal{F}_F = \mathcal{F}_0 \cup \{F\}$, which has the finite intersection property. Moreover, if $E \in \mathcal{F} \setminus \{F\}$, then $\mathcal{F}_F \cup \{E\}$ would fail f.i.p. Hence, for $F \in \mathcal{F}$, let $\mathcal{U}_F$ be any ultrafilter containing $\mathcal{F}_F$, giving $\mathfrak{c}$ many ultrafilters. ∎

Remark. It can be shown (with a lot more work) that $\mathbb{N}$ admits $2^{\mathfrak{c}}$ ultrafilters.

Let $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ be a non-principal ultrafilter. Define $\delta_{\mathcal{U}} : \mathcal{P}(\mathbb{N}) \to \{0,1\} \subset \mathbb{R}$ by $\delta_{\mathcal{U}}(A) = 1$ if $A \in \mathcal{U}$, and 0 if $X \setminus A \in \mathcal{U}$. Since $\mathbb{N} \in \mathcal{U}$, we see that $\delta_{\mathcal{U}}(\emptyset) = 0$. If $\emptyset \neq A, B \in \mathcal{P}(\mathbb{N})$ with $A \cap B = \emptyset$, then if $A \cup B \in \mathcal{U}$, then exactly one of A or B is in $\mathcal{U}$. Thus $\delta_U(A \cup B) = \delta_U(A) + \delta_U(B)$. If $E_1, \ldots, E_n \subseteq \mathbb{N}$ with $E_j \cap E_k = \emptyset$ for $j \neq k$, then $\sum_{k=1}^n |\delta_{\mathcal{U}}(E_k)| \leq 1$ so $\|\delta_{\mathcal{U}}\|_{\text{var}} \leq 1$. Since $\delta_{\mathcal{U}}(\mathbb{N}) = 1$, we have $\|\delta_{\mathcal{U}}\|_{\text{var}} = 1$. Let $L_{\mathcal{U}} \in \ell_\infty^*$ be the linear functional associated to $\delta_{\mathcal{U}}$. We then have (with some verification possibly needed)

(i) $L_{\mathcal{U}}(1) = 1$, $\|L_{\mathcal{U}}\| = 1$
(ii) $L_{\mathcal{U}}|_{\mathbf{c}_0} = 0$, so if $x \in \ell_\infty^{\mathbb{R}}$, then $\liminf_{n\to\infty} x_n \leq L_{\mathcal{U}} \leq \limsup_{n\to\infty} x_n$
(iii) Exactly one of $2\mathbb{N}$ and $2\mathbb{N}-1$ is in $\mathcal{U}$, so $L(\chi_{2\mathbb{N}}) \neq L_{\mathcal{U}}(\chi_{2\mathbb{N}-1})$, so $L_{\mathcal{U}}$ is not translation invariant.
(iv) Let $S \in \mathcal{B}(\ell_\infty)$ be given by $Sx = \left(\frac{x_1 + \cdots + x_n}{n}\right)_{n=1}^\infty$. Then $L_{\mathcal{U}} \circ S$ is a Banach limit.

Definition. If (X, τ) is a topological space, $\mathcal{U}$ an ultrafilter on X, we say that $x_0 \in X$ is a $(\tau-)$limit point for $\mathcal{U}$ if for each $U \in \tau$ with $x_0 \in U$, we have $U \in \mathcal{U}$.

6.6 Proposition. *Let (X, τ) be a topological space. Then (X, τ) is compact if and only if any ultrafilter on X admits a τ–limit point.*

PROOF Let us begin with an observation: if $x \in X$ and $\mathcal{U}$ is an ultrafilter on X, then

$$x \in \bigcap_{V \in \mathcal{U}} \overline{V} \Leftrightarrow \text{for any } U \in \tau \text{ with } x \in U, U \cap V \neq \emptyset \text{ for each } V \in \mathcal{U}$$
$$\Leftrightarrow x \text{ is a } \tau\text{–limit point of } \mathcal{U}$$
∎

If (X, τ) is compact, then $\bigcap_{V \in \mathcal{U}} \overline{V} \neq \emptyset$. If $\mathcal{F} \subseteq \mathcal{P}(X)$ has the finite intersection property, then there exists an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$, so $\bigcap_{F \in \mathcal{F}} \overline{F} \supseteq \bigcap_{V \in \mathcal{U}} \overline{V} \neq \emptyset$.

6.7 Theorem. (Tychonoff) *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a family of compact spaces, and $X = \prod_{\alpha \in A} X_\alpha$ with the product topology π. Then (X, π) is compact.*

PROOF Let $\mathcal{U}$ be an ultrafilter on X; we will show that it admits a π–limit point. Fix $\alpha \in A$ and let $\mathcal{U}_\alpha = \{p_\alpha(V) : V \in \mathcal{U}\}$, where p_α is the coordinate projection onto α. If $\emptyset \neq S_\alpha \subseteq X_\alpha$, then $S_\alpha = p_\alpha^{-1}(p_\alpha^{-1}(S_\alpha))$, so $S_\alpha \in \mathcal{U}_\alpha$ if and only if $p^{-1}(S_\alpha) \in \mathcal{U}$, and since p^{-1} commutes with complementation, $\mathcal{U}_\alpha$ is an ultrafilter. The last proposition provides a τ_α–limit point x_α for $\mathcal{U}_\alpha$. Now let $x = (x_\alpha)_{\alpha \in A}$, where x_α is found as above. If $W \in \pi$ with $x \in W$, then there are $\alpha_1, \ldots, \alpha_n$ in A, $U_{\alpha_i} \in \tau_{\alpha_i}$ with $x \in \bigcap_{k=1}^n p_{\alpha_k}^{-1}(U_{\alpha_k}) \subseteq W$. Since each x_{α_k} is a τ_{α_k}–limit point of $\mathcal{U}_{\alpha_k}$, we see that each $U_{\alpha_k} \in \mathcal{U}_{\alpha_k}$, so $p_{\alpha_k}^{-1}(U_{\alpha_k}) \in \mathcal{U}$. Thus we see that $W \in \mathcal{U}$, so x is a π–limit point of $\mathcal{U}$. ∎

Remark. (i) Tychonoff's theorem implies the axiom of choice. Given $\{X_\alpha\}_{\alpha\in A}$ be a family of non-empty sets. Find y which is not a member of any X_α, and let $Y_\alpha = X_\alpha \cup \{y\}$ and $\tau_\alpha = \{\emptyset, \{y\}, X_\alpha, Y_\alpha\}$, and $(Y_\alpha . \tau_\alpha)$ is compact. The constant element y is an element of Y, so by Tychonoff, (Y,π) is compact. Given $\alpha_1,\dots,\alpha_n \in A$, then $\bigcup_{k=1}^n p_{\alpha_k}^{-1}(\{y\})$. Since $\prod_{k=1}^n X_{\alpha_k} \neq 0$, we see that $Y \subsetneq \bigcup_{k=1}^n p_{\alpha_k}^{-1}(\{y\})$. Hence by compactness, $Y \not\subseteq \bigcup_{\alpha\in A} p_\alpha^{-1}(\{y\})$. Hence $\prod_{x\in A} X_\alpha = Y \setminus \bigcup_{\alpha\in A} p_\alpha^{-1}(\{y\}) \neq 0$.

(ii) If we are given $(X_\alpha,\tau_\alpha)_{\alpha\in A}$ a family of topological spaces, $X = \prod_{\alpha\in A} X_\alpha$, we can define the **box topology**, i.e. the topology with base $\{\prod_{\alpha\in A} U_\alpha : U_\alpha \in \tau_\alpha \setminus \{\emptyset\}$ for each $\alpha\}$ Of course, $\pi \subseteq \tau$, and the inclusion is proper on infinite products.

6.8 Proposition. *Let (X,τ) be a compact space.*
(i) If $K \subseteq X$ is closed, then K is compact.
(ii) If (Y,σ) is a topological space and $f : X \to Y$ is continuous, then $f(X)$ is compact.

Proof Immediate. ■

Remark. If X is a normed space, $w^* = \sigma(X^*, \hat{X})$, if $x \in X$, $\hat{x} \in X^{**}$, $\hat{x}(f) = f(x)$, $\hat{X} = \{\hat{x} : x \in X\}$. If A, B are non-empty sets, $A^B \cong \{f : B \to A\}$.

6.9 Theorem. (Alaoglu) *Let X be a normed space. Then $B(X^*)$ is $w^* = \sigma(X^*,\hat{X})$–compact*

Proof Let $\Gamma : X^* \to \mathbb{F}^X$ be given by $\Gamma(f) = (f(x))_{x\in X}$, so Γ is injective. Let $\pi = \sigma(\mathbb{F}^X, \{p_x\}_{x\in X})$ be the product topology. If $U_1,\dots,U_n \subseteq \mathbb{F}$ are open and $x_1,\dots,x_n \in X$, then

$$\Gamma\left(\bigcap_{k=1}^n \hat{x}_n^{-1}(U_k)\right) = \bigcap_{k=1}^n \Gamma\left(\hat{x}_n^{-1}(U_k)\right) = \bigcap_{k=1}^n \hat{x}_n^{-1}(U_k) \cap \Gamma(X^*)$$

so Γ is an open map onto its image in $\mathbb{F}^X$. Similarly, it is easy to show that Γ^{-1} is also an open map, so in fact Γ is a homeomorphism onto its image.

We now consider $\overline{\Gamma(B(X^*))} \subset \mathbb{F}^X$. Let $g \in \overline{\Gamma(B(X^*))}$ and let $D = D(\mathbb{F})$. Given $x, y \in X$ and $\alpha \in \mathbb{F}$, and then given $\epsilon > 0$, we find $f \in B(X^*)$ such that

$$\Gamma(f) \in p_x^{-1}\left(g(x) + \frac{\epsilon}{3}D\right) \cap p_y^{-1}\left(g(y) + \frac{\epsilon}{3(|\alpha|+1)}D\right) \cap p_{x+\alpha y}^{-1}\left(g(x+\alpha y) + \frac{\epsilon}{3}D\right)$$

We have that f is linear with $\Gamma(f)(x) = f(x)$, etc. so we have

$$|g(x) + \alpha g(y) - g(x+\alpha y)| \leq |g(x) - f(x)| + |\alpha||g(y) - f(y)| + |g(x+\alpha y) - f(x+\alpha y)| < \epsilon$$

and since $\|f\| \leq 1$, we have $|g(x)| \leq |g(x) - f(x)| + |f(x)| < \epsilon/3 + \|x\|$. Then since $\epsilon > 0$ is arbitrary, get $g \in X'$ and $|g(x)| \geq \|x\|$, i.e. $g \in B(X^*)$. Hence we have that $g = \Gamma(g)$.

Thus $\Gamma(B(X^*)) \subseteq \prod_{x\in X} \|x\| \overline{D} \subseteq \mathbb{F}^X$ is a closed subset of a compact subset of $\mathbb{F}^X$. Thus $B(X^*)$ is the continuous image of a compact set and hence compact. ■

Remark. If $r > 0$, then we may replace $B(X^*)$ with $rB(X^*)$ in the proof above, with trivial modifications. Thus any ball is w^*–compact. Hence bounded w^*–closed sets in X^* are automatically w^*–compact.

Definition. A topological space (X,τ) is Hausdorff if given $x \neq y$ in X, there are $U_x, V_y \in \tau$ such that $x \in U_x$ and $y \in V_y$ and $U_x \cap U_y = \emptyset$.

Example. (i) A metric space is Hausdorff.
(ii) X a normed space, $w = \sigma(X, X^*)$ is Hausdorff (by Hahn-Banach and A2Q1).
(iii) If X is a normed space, then $w^* = \sigma(X^*, \hat{X})$ on X^* is Hausdorff.
(iv) $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in A}$ family of topological spaces, $X = \prod_{\alpha\in A} X_\alpha$ with π the product topology. Then (X, π) is Hausdorff if and only if all (X_α, τ_α) are Hausdorff. (Straightfoward exercise).

6.10 Proposition. *Let (X, τ) be a Hausdorff space, $K \subseteq X$ τ–compact. Then K is τ–closed.*

Proof Straightforward exercise. ∎

6.11 Proposition. *Let (X, τ) be a compact space.*
(i) If (Y, σ) is a Hausdorff space and $\phi : X \to Y$ is continuous and bijective, then $\phi^{-1} : Y \to X$ is continuous.
(ii) If $\tau' \subseteq \tau$ is a Hausdorff topology on X, so $\tau' = \tau$.

Proof (i) If $F \subseteq X$ is τ–closed, then it is τ–compact. Hence $(\phi^{-1})^{-1}(F) = \phi(F)$ is σ–closed, so by A1Q1, ϕ^{-1} is continuous.
(ii) id : $X \to X$ is continuous, so if $U \in \tau'$, then $\mathrm{id}^{-1}(U) = U \in \tau$, so id is continuous. Hence by (1) id^{-1} is continuous so $\tau \subseteq \tau'$. ∎

6.12 Theorem. (Metrization) *If X is a separable normed space, then $B(X^*)$ is w^*–metrizable, i.e. there exists a metric ρ on $B(X^*)$ such that $w^*|_{B(X^*)} = \tau_\rho$.*

Proof Let $\{x_n\}_{n=1}^\infty \subset B(X)$ be any set which is separating for X^*, i.e. if $f \in X^* \setminus \{0\}$, then $f(x_n) \neq 0$ for some n (for example, take any dense subset of $D(X) \setminus \{0\}$). Let ρ be given by

$$\rho(f, g) = \sum_{k=1}^{\infty} \frac{|(f - g)(x_k)|}{2^k} \leq 2$$

It is easy to see that this is a metric.

Given $f_0 \in B(X^*)$, take $\epsilon > 0$ and let
- n be so $\sum_{k=n+1}^\infty \frac{2}{2^k} < \frac{\epsilon}{2}$, and
- $V = \bigcap_{k=1}^n \{f \in B(X^*) : |\hat{x}_k(f) - \hat{x}_k(f_0)| < \epsilon/2\} \in w^*|_{B(X^*)}$, $f_0 \in V$.

Then if $f \in V$,

$$g(f, f_0) = \sum_{k=1}^{n} \frac{|f(x_k) - f_0(x_k)|}{2^k} + \sum_{k=n+1}^{\infty} \frac{|f(x_k) - f_0(x_k)|}{2^k} < \epsilon$$

so $f_0 \in V \subset B^\circ_{\rho,\epsilon}(f_0)$. Since f_0 is arbitrary, we have $\tau_\rho \subseteq w^*|_{B(X^*)}$, but since w^* is compact and τ_ρ is Hausdorff, these must be equal. ∎

(i) Note that different separating families from $B(X)$ may produce different metrics, but always the same topology.
(ii) The definition of ρ above extends to all of $X^* \times X^*$. However, X^* with the weak* topology is not in metrizable if X is infinite dimensional.
(iii) $X^* = \bigcup_{=1}^\infty nB(X^*)$, so each $nB(X^*)$ is metrizable and compact, and thus w^*–separable. Thus if X is separable, then X^* is itself separable.

7 Nets

Definition. A pair $(N, \leq)$ is a **preorder** on N if
- $v \leq v$ for $v \in N$
- $v_1 \leq v_2$ and $v_2 \leq v_3$ implies $v_1 \leq v_3$.

This pair is **cofinal** if for any $v_1, v_2 \in N$, there is $v_3 \in N$ so $v_1 \leq v_3$ and $v_2 \leq v_3$. Then $(N, \leq)$ is a **directed set** if $\leq$ is a cofinal preorder. Given a non-empty set X, a **net** is a function $x : N \to X$.

Definition. If $(x_0)_{v \in N}$ is a net in X, $A \subseteq X$, we say that $(x_0)_{n \in \mathbb{N}}$ is
- **eventually** in A if there is $v_A \in N$ so $x_v \in A$ whenever $v \geq v_A$
- **frequently** in A if for any $v \in N$, there is $v' \in N$ with $v' \geq g$ so $x_{v'} \in A$.

Definition. Now, let $(M, \leq)$ be anther directed set A map $\phi : M \to N$ is **eventually cofinal** if for any $v \in N$, there is $\mu_v \in N$ s $\phi(u) \geq v$ whenever $\mu \geq \mu_0$. Given a net $(x_v)_{v \in N}$ and an eventually cofinal $\phi : M \to N$, we call $(x_{\phi(\mu)})_{\mu \in M}$ a **subnet**.

Definition. We call $\phi : M \to N$ a **directed map** if
(i) $\mu \leq \mu'$ in M implies $\phi(\mu) \leq \phi(\mu')$ in N
(ii) For any $v \in N$, there is $\mu \in M$ s $v \leq \phi(\mu)$.

Directed maps are always cofinal. Different sources use directed maps over eventually cofinal maps.

Example. (i) $(\mathbb{N}, \leq)$ is directed, and subsequences are special types of subnets.
(ii) $(\mathbb{R}, \leq)$ is directed
(iii) (*Riemann sums*) Let $a < b$ in $\mathbb{R}$. We let

$$N = \{(P, P^*) : P = \{a = t_0 < t_1 < \cdots < t_n = b\}, P^* = \{t_1^*, \ldots, t_n^*\}, t_k^* \in [t_{k-1}, t_k]\}$$

and say $(P, P^*) \leq (Q, Q^*)$ if $P \subseteq Q$. One can verify that this is a net (the Riemann sum net).
(iv) (*Nets from filtering families*). We say that $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ is a **filtering family** if for each $F_1, F_2 \in \mathcal{F}$, there is $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$. For example, an ultrafilter is a filtering family. Let

$$N_{\mathcal{F}} = \{(x, F) : x \in F, F \in \mathcal{F}\}$$

equipped with the pre-order $(x, F) \leq (x', F')$ if and only if $F \supseteq F'$. Since $\mathcal{F}$ is a filtering family, $(N_{\mathcal{F}}, \leq)$ is directed. Let $x_{(x,F)} = x$, so $(x)_{(x,F) \in N_{\mathcal{F}}}$ is the net built from $\mathcal{F}$. Note that if $F \in \mathcal{F}$, then $(x)_{(x,F) \in \mathcal{F}}$ is eventually in F.
An **ultranet** $(x_v)_{v \in N} \subset X$ is a net for which any $A \in \mathcal{P}(X)$, $(x_v)_{v \in N}$ is either eventually in A or eventually in $X \setminus A$. If $\mathcal{F}$ is an ultrafilter, then $(x)_{(x,F) \in N_{\mathcal{F}}}$ is an **ultranet**.

Nets and Topology

Now, suppose (X, τ) is a topological space.

Definition. We say that $x_0 \in X$ is
- Some $x_0 \in X$ is a **limit point** if for any $U \in \tau$ with $x_0 \in U$, $(x_v)_{v \in N}$ is eventually in U. That is, there is ν_U such that $x_\nu \in U$ whenever $\nu \geq \nu_U$. We write $x_0 = \lim_{v \in N} x_v$, the τ–limit of $(x_v)_{v \in N}$. Note that this is an abuse of notation, since limit points need not be unique (when (X, τ) is not Hausdorff).

- Some $x_0 \in X$ is a **cluster point** of $(x_v)_{v\in N}$ if for any $U \in \tau$ with $x_0 \in U$, $(x_v)_{v\in N}$ is frequently in U.

7.1 Proposition. *If $(x_v)_{v\in N}$ is a net in (X,τ) and $x_0 \in X$, then x_0 is a cluster point for $(x_v)_{v\in N}$ if and only if x_0 is a τ–limit point of x_{v_μ} for some subnet $(x_{v_\mu})_{\mu\in M}$ of $(x_v)_{v\in N}$.*

Proof ($\Longrightarrow$) Suppose x_0 is a cluster point for $(x_v)_{v\in N}$. Then for each $v \in N$ and $U \in \tau$ containing x_0, define

$$F_{\nu,U} = \{\nu' \in N : \nu' \geq \nu, x_{\nu'} \in U\}$$

which is non-empty since x_0 is a cluster point. Then set

$$\mathcal{F} = \{F_{\nu,U} : \nu \in N, U \in \tau, x_0 \in U\} \subset \mathcal{P}(N)$$

Let's see that $\mathcal{F}$ is filtering: suppose $F_{\nu,U}$ and $F_{\nu',U'}$ are in $\mathcal{F}$. Get $\mu \geq \nu$ and $\mu \geq \nu'$ by definition of a net and set $V = U \cap U'$, which is open and contains x_0. Then since x_0 is a cluster point, get some $\mu' \geq \mu$ such that $x_{\mu'} \in V$, so $F_{\mu',V} \subseteq F_{\nu,U} \cap F_{\nu',U'}$ We then let $M = N_{\mathcal{F}}$ be the net construction from the filtering family and set $v_{(\nu,F)} = V$.

Now set $N_{\mathcal{F}} = \{(\nu, F) : \nu \in F, F \in \mathcal{F}\}$ with the standard preorder and $\nu_{(\nu,F)} = \nu$. Then the map $(\nu, F) \mapsto \nu$ from $N_{\mathcal{F}} \to N$ is eventually cofinal: if $\nu_0 \in N$ is arbitrary, take any $F_0 = F_{\nu_0,U} \in \mathcal{F}$. Then $F_0 = \{\nu \in N : \nu \geq \nu_0, x_\nu \in U\}$, so if $F_{\mu,V} \in \mathcal{F}$ with $F_{\mu,V} \subseteq$ We let $M = N_{\mathcal{F}}$ as in (iv) above, and $v_{v,\mathcal{F}} = v$. Check that $(x_v)_{(v,F)\in N_{\mathcal{F}}}$ is eventually in U for any $U \in \tau$ with $x_0 \in U$. [Check: $(v,F) \mapsto v : N_{\mathcal{F}} \to N$ is cofinal, but is not evidently directed]

($\Longleftarrow$) If for some subnet $(x_{v_\mu})_{\mu\in M}$ is eventually in U for any $U \in \tau$ with $x_0 \in U$, then $(x_v)_{v\in N}$ is frequently in U for such U by definition of a subnet. ∎

7.2 Proposition. *If (Y,σ) is another topological space, then $f : X \to Y$ is continuous if and only if for any $x_0 \in X$ and net $(x_v)_{v\in N}$ with having x_0 as a limit, $f(x_0) = \lim_{v\in N} f(x_v)$.*

Proof If $V \in \sigma$ with $f(x_0) \in V$, then $f^{-1}(V) \in \tau$ with $x_0 \in f^{-1}(V)$. Since $(x_v)_{v\in N}$ is eventually in $f^{-1}(V)$, so $(f(x_v))_{v\in N}$ is eventually in V.

Conversely, let $\tau_{x_0} = \{U \in \tau : x_0 \in U\}$, which is filtering on X. Let $N_{\tau_{x_0}} = \{(x,U) : x \in U, U \in \tau_{x_0}\}$ be directed by $(x,U) \leq (x',U')$ if and only if $U \supseteq U'$ as in (iv) above. Then $x_0 = \lim_{(x,U)\in N_{\tau_{x_0}}} x$. Now, let $V \in \sigma$ with $f(x_0)$. The assumptions on f tell us there is $v - V \in N_{\tau_{x_0}}$ such that for $v \geq v_V$, we have $f(x_0) \in V$ We have $v_V = (x,U)$ for some $U \in \tau_{x_0}$ and $x \in U$, so for any $x' \in U$, $(x',U) \geq (x,U)$ and $f(x') = f(x_{x',U}) \in V$, so that $x_0 \in U = \bigcup_{x'\in U}\{x'\} \subseteq f^{-1}(V)$, so f is continuous at x_0. But $x_0 \in X$ was arbitrary. ∎

Remark. We get the following consequences of this result:

(i) Given topologies τ, τ' on X, $\tau' \subseteq \tau$ if and only if $\tau' - \lim_{v\in N} x_v = x_0$ whenever $\tau - \lim_{v\in N} x_v = x_0$ for any $x_0 \in X$.

(ii) (limits in product topology) $\{(x_\alpha, \tau_\alpha)\}_{\alpha\in A}$ be topological space and $X = \prod_{\alpha\in A} X_\alpha$ equipped with the product topology π. If $(x^{(v)})_{v\in N}$ is a net in X and $x^{(0)} \in X$, then $\pi - \lim_{v\in N} x^{(v)} = x^{(0)}$ if and only if for every $\alpha \in A$, $\tau_\alpha - \lim_{v\in N} x_\alpha^{(v)} = x_\alpha^{(0)}$. Recall that π is the coarsest topology making each μ_α continuous.

(iii) If X is a normed space and $(f_v)_{v\in N} \subset X^*$, $f_0 \in X^*$, then $w^* - \lim_{v\in N} f_v = f_0$ if and only if $\lim_{v\in N} f_v(x) = f_0(x)$ for each $x \in X$.

Roles of weak and weak* topologies in convexity

***7.3 Theorem.** **(w^*–Separation)** Let X be a normed space, $A, B \subset X^*$ each be non-empty and convex, with $A \cap B = \emptyset$ and B w^*–open. Then there is $x \in X$ and $\alpha \in \mathbb{R}$ such that*

$$\operatorname{Re} f(x) \leq \alpha < \operatorname{Re} g(x)$$

for $f \in A$ and $g \in B$.

Proof The separation theorem and the fact that B is $\|\cdot\|$–open (i.e. $w^* \subseteq \tau_{\|\cdot\|}$) provides $F \in X^{**}$ and $\alpha \in \mathbb{R}$ such that $\operatorname{Re} F(f) \leq \alpha \operatorname{Re} F(g)$ for $f \in A$, $g \in B$. Since $B \in w^* = \sigma(X^*, \hat{X})$, if $f_0 \in B$, then there are $x_1, \ldots, x_n$ in X such that

$$f_0 \in U = \bigcap_{i=1}^{n} \hat{x}_i^{-1}(f_0(x_i) + \mathbb{D}) \subseteq B$$

Let $Y = \bigcap_{i=1}^{n} \ker \hat{x}_i \subseteq X^*$. Then for $i = 1, \ldots, n$, $\hat{x}_i(f_0 + Y) = \{f_0(x_i)\} \subset f_0(x_i) + \mathbb{D}$, so that $f_0 + Y \subseteq U \subseteq B$. Thus if $f \in Y$, then $\operatorname{Re} F(f_0 + f) > \alpha$ and hence $\operatorname{Re} F(f) > \alpha - \operatorname{Re} F(f_0)$ which implies that $f \in \ker \operatorname{Re} F$, so $f \in \ker F$. That is, $Y \subseteq \ker F$. The next lemma shows that $F \in \operatorname{span}\{\hat{x}_1, \ldots, \hat{x}_n\} \subseteq \hat{X}$, i.e. $F = \hat{x}$ for some $x \in X$. ■

***7.4 Lemma.** In an $\mathbb{F}$–vector space, if $f_0, f_1, \ldots, f_{\in} X'$ with $\ker f_0 \supseteq \bigcap_{i=1}^{n} \ker f_i$, then $f \in \operatorname{span}\{f_1, \ldots, f_n\}$.*

Proof Define $T : X \to \mathbb{F}^n$ by $Tx = (f_1(x), \ldots, f_n(x))$. Then $\ker T = \bigcap_{i=1}^{n} \ker f_i$. Let $\mathcal{R} = \operatorname{im} T \subseteq \mathbb{F}$ and $g_0 \in \mathcal{R}'$ by $g_0(Tx) = f_0(x)$. Then g_0 is well-defined: if $Tx = Ty$, then $x - y \in \ker T \subseteq \ker f_0$, so $f_0(x - y) = 0$ so $f_0(x) = f_0(y)$. Also g_0 is linear. Let $g \in (\mathbb{F}^n)'$ such that $g|_{\mathcal{R}} = g_0$. Hence there are $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $g(y_1, \ldots, y_n) = \sum_{j=1}^{n} \alpha_j y_j$. Hence for $x \in X$,

$$f_0(x) = g_0(Tx) = g(Tx) = g(f_1(x), \ldots, f_n(x)) = \sum_{j=1}^{n} \alpha_j f_j(x)$$

so that $f_0 = \sum_{j=1}^{n} \alpha_j f_j$. ■

***7.5 Theorem.** **(w^*–Closed Convex Hull)** If $S \subset X^*$, then*

$$\overline{\operatorname{co}}^{w^*} S = \cap\{\{f \in X^* : \operatorname{Re} f(x) \leq \alpha\} \supseteq S : x \in X, \alpha \in \mathbb{R}\}$$

Proof The set on the right is w^*–closed and convex being the intersection of such. Conversely, if $f \in X^* \setminus \overline{\operatorname{co}}^{w^*} S$, which is open, then there is a basic w^*–open neighbourhood

$$B = \bigcap_{j=1}^{n} \hat{x}_j^{-1}(f(x_j) + \mathbb{D}) \subseteq X^* \setminus \overline{\operatorname{co}}^{w^*} S$$

so that $B \cap \overline{\operatorname{co}}^{w^*} S = \emptyset$. Also, B is convex. ■

Remark. If X is a normed spacee, a closed half space $H = \{x \in X : \operatorname{Re} f(x) \leq \alpha\}$ for some f in X^*, $\alpha \in \mathbb{R}$. Hence, H is weakly closed $(\operatorname{Re} f)^{-1}([\alpha, \infty)) = f^{-1}(\{z \in \mathbb{C} : \operatorname{Re} z \geq \alpha\})$ is w–closed. Thus if $S \subset X$, we have $\overline{\operatorname{co}}\, S \in w = \sigma(X, X^*) \subseteq \tau_{\|\cdot\|}$, so $\overline{\operatorname{co}}\, S$ is automatically weakly closed. Hence if $C \subseteq X$ is convex, then C is norm closed if and only if C is w–closed.

Definition. Let X be a normed space. If $E \subseteq X$ (non-empty), the **polar** of E is given by

$$E^\circ = \{f \in X^* : \operatorname{Re} f(x) \leq 1 \text{ for all } x \text{ in } E\} \subseteq X^*$$
$$= \bigcap_{x \in E} \{f \in X^* : \operatorname{Re} \hat{x}(f) \leq 1\}$$

so E° is convex and w^*–closed in X^*, and $0 \in E^\circ$.

If $F \subseteq X^*$ (non-empty), let the **pre-polar** of F be given by

$$F_\circ = \{x \in X : \operatorname{Re} f(x) \leq 1 \text{ for all } f \text{ in } F\}$$

so, like above, F_0 is convex, (w–)closed, and $0 \in F_0$.

7.6 Theorem. (Bipolar) *(i) If $\emptyset \neq E \subseteq X$, then $(E^\circ)_\circ = \overline{\text{co}}(E \cup \{0\})$.*
(ii) If $\emptyset \neq F \subseteq X^*$, then $(F_\circ)^\circ = \overline{\text{co}}^{w^*}(F \cup \{0\})$.

PROOF (i) Note that $E \cup \{0\} \subseteq (E^\circ)_\circ$, so $\overline{\text{co}}(E \cup \{0\}) \subseteq (E^\circ)_\circ$. If $x_0 \in X \setminus \overline{\text{co}}(E \cup \{0\})$, then the separation theorem provides $f \in X^*$, $\alpha \in \mathbb{R}$ so $\operatorname{Re} f(x_0) > \alpha \geq \operatorname{Re} f(x)$ for $x \in E \cup \{0\}$. Notice that $\alpha \geq \operatorname{Re} f(0) = 0$, and we let $\beta = \frac{1}{2}[\operatorname{Re} f(x_0) + \alpha] > 0$, so $\operatorname{Re} f(x_0) > \beta \geq \operatorname{Re} f(x)$ for $x \in E \cup \{0\}$, $\beta > 0$. Let $g = \frac{1}{\beta} f$ and we see that $g \in E^\circ$ and as $\operatorname{Re} g(x_0) > 1$, $x_0 \notin (E^\circ)_\circ$.

(ii) Similar, use w^*–separation. ■

Remark. Let $Y \subseteq X$ be a subspace. If $f \in Y^0$, then $\operatorname{Re} f(y) \leq 1$ for $y \in Y$ implies that $f(y) = 0$ for all $y \in Y$. We write $Y^a = Y^0$, and $Y^a = \{f \in X^* : f|_Y = 0\}$ is called the **annhilator** of Y. Likewise, if $Z \subseteq X^*$ is a subspace, then $Z_a = Z_0$ where $Z_a = \{x \in X : f(x) = 0 \text{ for each } f \in Z\}$ is called the **pre-annhilator**. Notice that Y^a and Z_a are subspaces.

7.7 Corollary. *(i) If $Y \subseteq$ is a subspace, then $(X^a)_a = \overline{X}$.*
(ii) If $Z \subseteq X^*$ is a subspace, then $(Z_a)^a = \overline{Z}^{w^*}$.

7.8 Lemma. *If X is a normed space, then $B(X)^0 = B(X^*)$ and $B(X^*)_0 = B(X)$.*

PROOF If $f \in B(X^0)$, then $\operatorname{Re} f(x) \leq 1$ for $x \in B(X)$. Thus for $x \in B(X)$, $|f(x)| = \overline{\operatorname{sgn} f(x)} f(x) = f(\overline{\operatorname{sgn} f(x)} x) \leq 1$, so $\|f\| \leq 1$ and $f \in B(X^*)$. Conversely, if $f \in B(X^*)$, $x \in B(X)$, then $\operatorname{Re} f(x) \leq |f(x)| \leq 1$ so $f \in B(X)^\circ$. Then use the Bipolar theorem. ■

7.9 Theorem. (Goldstine) *If X is a normed space, then $\overline{B(\hat{X})}^{w^*} = B(X^{**})$. Note that $w^* = \sigma(X^{**}, \hat{X^*})$.*

PROOF The Bipolar theorem provides $\overline{B(\hat{X})}^{w^*} = \overline{\text{co}}^{w^*} B(\hat{x}) = (B(\hat{X})_\circ)^\circ$. But, in X^*,

$$B(X)^\circ = \{f \in X^* : \operatorname{Re} f(x) \leq 1 \text{ for } x \text{ in } B(X)\}$$
$$= \{f \in \hat{X^*} : \operatorname{Re} \hat{x}(f) \leq 1 \text{ for } x \text{ in } B(X)\}$$
$$= B(\hat{X})_\circ$$

Hence we have, using the lemma,

$$\overline{B(\hat{X})}^{w^*} = (B(\hat{X})_\circ)^\circ = (B(X)^\circ)^\circ = B(X^*)^\circ = B(X^{**})$$ ■

Example. (i) Recall that $c_0^* \cong \ell_1$ and $\ell_1^* \cong \ell_\infty$, wheren $c_0 \subseteq \ell_\infty$. Thus by Goldstine, $\overline{B(c_0)}^{w^*} = B(\ell_\infty)$, so $w^* = \sigma(\ell_\infty, \ell_1)$. Since ℓ_1 is separable, we have that $(B(\ell_\infty), w^*)$ is metrizable. In fact, if $x \in \ell_\infty$, then if $x^{(n)} = (x_1, \ldots, x_n, 0, 0, \ldots) \in c_0$, we have $x = w^* - \lim_{n\to\infty} x^{(n)}$.

(ii) $\ell_\infty^* \cong \mathrm{FA}(\mathbb{N})$. But $B(\mathrm{FA}(\mathbb{N}), w^*)$ is not metrizable. Since $\ell_1^* \cong \ell_\infty$, there is a natural isometric embedding $\ell_1 \hookrightarrow \mathrm{FA}(\mathbb{N})$. Then $y^{(n)} = \frac{1}{n}(1, 1, \ldots) \in B(\ell_1)$, and w^*–cluster point of $(y^{(n)})_{n=1}^\infty \subset B(\mathrm{FA}(\mathbb{N}))$ is a Banach limit.

7.10 Corollary. *If $F \in X^{**}$, there always exists a net $(x_\nu)_{\nu \in N} \subset X$ such that*

$$F = w^* - \lim_{\nu \in N} \hat{x}_\nu \text{ and } \|x_\nu\| \leq \|F\|$$

Proof If $F \neq 0$, $\frac{1}{\|F\|} F \in B(X^{**}) = \overline{B(\hat{X})}^{w^*}$, and we may find $(y_\nu)_{\nu \in N} \subset B(X)$ such that $(\hat{y}_\nu)_{\nu \in N} \subset B(\hat{X})$ and $\frac{1}{\|F\|} F = w^* - \lim_{\nu \in N} \hat{y}_\nu$. Let $x_\nu = \|F\| y_\nu$. ■

Consider $\mathcal{F} = w^*_{\frac{1}{\|F\|}F} = \{U \in w^*|_{B(X^{**})} : F \in U\}$ is a filtering family. Each $U \in w^*_{\frac{1}{\|F\|}F}$ has $U \cap B(\hat{X}) \neq \emptyset$ by Goldstine. Let $N_{\mathcal{F}} = \{(x, U) : x \in B(X), \hat{x} \in U, U \in \mathcal{F}\}$. Then $(x_\nu)_{\nu \in N_{\mathcal{F}}} = (x)_{(x,U) \in N_{\mathcal{F}}}$ works.

Definition. A normed space X is **reflexive** if $\hat{X} = X^{**}$.

Notice that $X^{**} = (X^*)^*$ is complete, and $x \mapsto \hat{x}$ is an isometry, so a reflexive space is always complete.

7.11 Theorem. *Let X be a Banach space. The following are equivalent:*

(i) X is reflexive

(ii) $B(X)$ is w–compact

(iii) $w^* = w$ on X^*

(iv) X^* is reflexive.

Proof The map $\iota : x \mapsto \hat{x}$ is a $w - w^*|_{\hat{X}}$–homeomorphism. Recall $w^* = \sigma(X^{**}, \hat{X}^*)$, and $w^*|_{\hat{X}} = \sigma(\hat{X}, (\hat{X})^*|_{\hat{X}})$ and we have for $x_0 \in X$, net $(x_\nu)_{\nu \in N}$ in X,

$$\begin{aligned} w - \lim_{\nu \in N} x_\nu = x_0 &\iff \lim_{\nu \in N} f(x_\nu) = f(x_0) \forall f \in X^* \\ &\iff \lim_{\nu \in N} \hat{x}_\nu(f) = \hat{x}_0(f) \forall f \in X^* \\ &\iff \lim_{\nu \in N} \hat{f}(\hat{x}_\nu) = \hat{f}(\hat{x}_0) \end{aligned}$$

and having the same convergent nets means that the topologies are the same.

$(i \Rightarrow ii)$ By assumption, $\widehat{B(X)} = B(\hat{X}) = B(X^{**})$. Since $B(X^{**})$ is w^*–compact, and hence $\iota^{-1}(B(X^{**})) = B(X)$ is w–compact

$(ii \Rightarrow i)$ If $B(X)$ is w–compact, then since $x \mapsto \hat{x} : X \to X^{**}$ is continuous, we see that $B(\hat{X}) = \widehat{B(X)}$ is w^*–compact.

$(i \Rightarrow iii)$ We have $\hat{X} = X^{**}$ so on X^*, we have $w = \sigma(X^*, X^{**}) = \sigma(X^*, \hat{X}) = w^*$.

$(iii \Rightarrow iv)$ $B(X^*)$ is compact, hence w–compct, so by (ii) implies (i) applied to X^*, we have that X^* is reflexive.

$(iv \Rightarrow i)$ We assume $\widehat{X^*} = X^{***}$. Thus on X^{***}, we have $w = \sigma(X^{**}, X^{***}) = \sigma(X^{**}, \widehat{X^*}) = w^*$. Now $B(\hat{X}) = B(X^{**}) \cap \hat{X}$ is norm-closed and convex, hence w–closed, by Closed Convex Hull theorem. Thus from above, $B(\hat{X})$ is w^*–closed, so $B(\hat{X}) = \overline{B(\hat{X})}^{w^*} = B(X^{**})$ by Goldstine, so $\hat{X} = X^{**}$. ■

7.12 Corollary. *(i) Any finite dimensional normed space is reflexive.*
(ii) Any closed subspace Y of a normed space X is reflexive.

Proof (i) A finite dimensional normed space is complete, and its closed ball is compact, and thus w–compact as $\tau_{\|\cdot\|} \supseteq w$.

(ii) By Hahn-Banach, $Y^* = X^*|_Y$, so $\sigma(Y, Y^*) = \sigma(Y, X^*|_Y) = \sigma(X, X^*)|_Y$. Now $B(Y) = B(X) \cap Y$ is norm-closed and convex, hence w–closed in $B(X)$. But $B(X)$ is w–compact, so $B(Y)$ is a w–closed subset of a w–compact space and thus w–compact. ■

Extreme Points and the Krein-Milman Theorem

Definition. Let X be a vector space and $C \subset X$ convex. A **face** F of C is any non-empty subset such that if $x \in F$, $x = (1-t)y + tz$, $t \in (0,1)$, $y, z \in C$ implies that $y, z \in F$. A **extreme point** of C is a singleton face, i.e. $\operatorname{ext} C = \{x \in C : \{x\} \text{ is a face of } C\}$. Hence $x \in \operatorname{ext} C$ if for any $t \in (0,1)$ and $y, z \in C$, if $x = (1-t)y + tz$ then $x = y = z$.

Remark. (i) Faces of C are not necessarily convex.
(ii) A face F' of a convex face F of C is itself a face of C.
(iii) $\operatorname{ext} F \subseteq \operatorname{ext} C$.
(iv) If $f \in X'$ and $\operatorname{Re} f(C) = [a, b]$, then $(\operatorname{Re} f)^{-1}(\{b\})$ is itself a face of C.

7.13 Theorem. (Krein-Milman) *Let X be a normed space and $C \subset X^*$ convex and w^*–compact. Then $C = \overline{co}^{w^*} \operatorname{ext} C$.*

Proof We first verify that any w^*–closed face of C admits an extreme point. We let $\mathcal{F} = \{F : F \text{ is a convex } w^*\text{–closed face of } C\}$, which is partially ordered by reverse inclusion. If $\mathcal{C}$ is a chain in $\mathcal{F}$ with $F_1, \ldots, F_n \in \mathcal{C}$, we may assume $F_1 \supseteq \cdots \supseteq F_n$ so that $\mathcal{C}$ has the finite intersection property. Thus $\emptyset \neq F_0 = \bigcap_{F \in \mathcal{C}} F$. If $x \in F_0$, $t \in (0,1), y, z \in C$ and $x = (1-t)y + tz$, then $x \in F$ for any $F \in \mathcal{C}$ so $y, z \in F$ for any $f \in \mathcal{C}$. Thus $y, z \in \bigcap_{F \in \mathcal{C}} F = F_0$. Also F_0 is closed, so $F_0 \in \mathcal{F}$. Thus F_0 is an upper bound in $\mathcal{F}$ for $\mathcal{C}$, so by Zorn, get some maximal element M.

Let M be a minimal w^*–closed convex face of F. Then given $x \in X$, $\operatorname{Re} \hat{x} : X^* \to \mathbb{R}$ is w^*–continuous, and hence $\operatorname{Re} \hat{x}(M) = [a_x, b_x]$ since the only compact convex subsets of $\mathbb{R}$ are compact intervals. But then $F_x = (\operatorname{Re} \hat{x})^{-1}(\{b_x\}) \cap M$ is a w^*–closed convex face in M, so that $F_x = M$. If $f, g \in M$, then $\operatorname{Re} f(x) = \operatorname{Re} \hat{x}(f) = b_x = \operatorname{Re} \hat{x}(g) = \operatorname{Re} g(x)$, so $f = g$ and hence $M = \{f\}$ and $f \in \operatorname{ext} F$.

Now let $f_0 \in X^* \setminus \overline{\text{co}}^{w^*} \operatorname{ext} C$. Since C is w^*–compact and convex, $\operatorname{Re} \hat{x}(C) = [a_x, b_x]$, so $C_x = (\operatorname{Re} \hat{x})^{-1}(\{b_x\}) \cap C$ is a w^*–closed convex face of C. Hence by above, there is $f \in \operatorname{ext} C_x \subseteq \operatorname{ext} C$ with $\operatorname{Re} \hat{x}(f) = b_x$. But then $\operatorname{Re} \hat{x}(f_0) > \alpha \geq \operatorname{Re} \hat{x}(f) = b_x$, so $\operatorname{Re} \hat{x}(f_0) \notin [a_x, b_x] = \operatorname{Re} \hat{x}(C)$, so $f_0 \notin C$. Thus $C \subseteq \overline{\text{co}}^{w^*} \operatorname{ext} C$, where the converse inclusion is obvious. ■

7.14 Corollary. *(i) If $C \subset X$ is a w–compact convex set, then $C = \overline{\text{co}} \operatorname{ext} C$.*
(ii) If $C \subset X$ is a norm-compact convex set, then $C = \overline{\text{co}} \operatorname{ext} C$.

Proof (i) We have that $x \mapsto \hat{x} : X \to \hat{X} \subseteq X^{**}$ is continuous. Hence $\hat{C}$ is w^*–compact in X^{**}, so $x \mapsto \hat{x} : C \to \hat{C}$ is a homeomorphism. In $\hat{C}$, we have

$$\widehat{\overline{\mathrm{co}}^{w} \operatorname{ext} C} = \overline{\mathrm{co}}^{w^*} \operatorname{ext} \hat{C} = \hat{C}$$

so that $C = \overline{\mathrm{co}}^{w} \operatorname{ext} C = \overline{co} \operatorname{ext} C$ by the closed convex hull theorem.
(ii) Since $w \subseteq \tau_{\|\cdot\|}$, any norm-compact is w–compact. ■

Remark. Let X be a normed space. Then $\operatorname{ext} B(X) \subseteq S(X)$.

7.15 Proposition. *Let* $1 < p < \infty$. *Then* $\operatorname{ext} B(\ell_p) = S(\ell_p)$.

Proof Let $x \in S(\ell_p)$, so $x = (1-t)y + tz$. Then

$$1 = \|x\|_p \leq (1-t)\left\|y\right\|_p + t\|z\|_p \leq 1$$

so that $\left\|y\right\|_p = \|z\|_p = 1$ and $\|x\|_p = (1-t)\left\|y\right\|_p + t\|z\|_p$. Thus by the equality case for Minkowski, there is $s \geq 0$ so $s(1-t)y = tz$. Taking norms, we have $y = z$. ■

7.16 Proposition. *We have* $\operatorname{ext} B(c_0) = \emptyset$.

Proof Let $x = (x_1, x_2, \ldots) \in B(C_0)$. Since $\lim x_n = 0$, get n_0 so $|x_{n_0}| \leq 1/2$. If $x_{n_0} \neq 0$, let $y = (x_1, \ldots, x_{n_0-1}, 2x_{n_0}, x_{n_0+1}, \ldots)$ and $z = (x_1, \ldots, x_{n_0-1}, 0, x_{n_0+1}, \ldots)$, and similarly for $x_{n_0} = 0$. Thus we have in each case that $y, z \in B(c_0)$ and $x = y/2 + z/2$. ■

7.17 Corollary. *There exists no normed space X for which* $c_0 \cong X^*$.

Proof If there were such X, then $B(c_0)$ would be w^*–compact, and hence Krein-Milman would imply $\operatorname{ext} B(c_0) \neq \emptyset$. ■

Definition. Let (X, τ) be a compact Hausdorff space, and let

$$P(X) = \{\mu \in B(C^{\mathbb{R}}(X,\tau)^*) : \mu(1) = 1\}$$

7.18 Theorem. $\operatorname{ext} P(X) = \{\hat{x} : x \in X\}$, *where* $\hat{x}(f) = f(x)$. *Furthermore,* $\overline{\mathrm{co}}^{w^*} \operatorname{ext} P(X) = P(X)$.

Proof Write $C = C^{\mathbb{R}}(X,\tau)$. Note that $P(X) = B(C^*) \cap \hat{\mathbf{1}}^{-1}(\{1\})$ is w^*–compact and convex. Hence by Krein-Milman, we have that $\overline{\mathrm{co}}^{w^*} \operatorname{ext} P(X) = P(X)$. It remains to describe $\operatorname{ext} P(X)$.

(I) Some inequalities. Fix $\mu \in P(X)$. If $0 \leq f \leq 1$ in C, then $0 \leq \mathbf{1} - f \leq 1$ so $\|f\|_\infty$, $\|\mathbf{1} - f\|_\infty \leq 1$. Thus $|\mu(f)| \leq 1$ and $|\mathbf{1} - \mu(f)| = |\mu(\mathbf{1} - f)| \leq 1$. Thus $0 \leq \mu(f) \leq 1$. Then if $g \neq 0$ and $g \geq 0$ in C, then we have $\mu(g/\|g\|_\infty) \geq 0$, so $\mu(g) > 0$; if $g \leq h$ in C, then $h - g \geq 0$ and $\mu(h) \geq \mu(g)$.

If $g \in C$, $g^+ = \max\{g, 0\}$, $g^- = \max\{-g, 0\} \in C$, and $g = g^+ - g^-$ while $|g| = g^+ + g^-$. Hence if $0 \leq f \leq 1$ in C and let $\mu_f(g) = \mu(fg)$ for $g \in C$, we have

$$\begin{aligned} |\mu_f(g)| = |\mu_f(g^+ - g^-)| = |\mu(fg^+) + \mu(fg^-)| \leq \mu(fg^+) + \mu(fg^-) = \mu(f(g)) \\ \leq \mu(f\|g\|_\infty) = \mu(f)\|g\|_\infty \end{aligned} \tag{7.1}$$

and, with $f = \mathbf{1}$, we have

$$|\mu(g)| \leq \mu(|g|) \tag{7.2}$$

(II) Let $\delta \in \operatorname{ext} P(X)$. We first show for h, g in C that $\delta(hg) = \delta(h)\delta(g)$. To see this, since $\delta \neq 0$, we may find $0 \leq f \leq 1$ such that $0 < \delta(f) < 1$. Now let $\mu = \frac{1}{\delta(f)}\delta_f$ so, for $g \in C$, (7.1) provides

$$|\mu(g)| = \frac{1}{\delta(f)}|\delta_f(g)| \leq \frac{1}{\delta(f)}\delta(f)\left\|y\right\|_\infty = \left\|y\right\|_\infty$$

so that $\mu \in B(C^*)$. We also know that $\mu(\mathbf{1}) = 1$. Hence $\mu \in P(X)$. Likewise, $\nu = \frac{1}{1-\delta(f)}\delta_{\mathbf{1}-f} \in P(X)$. We have that

$$\delta(f)\mu + (1-\delta(f))\nu = \delta$$

so by assumption on δ, $\mu = \delta$. Thus $\frac{1}{\delta(f)}\delta(fg) = \mu(g) = \delta(g)$, so that $\delta(fg) = \delta(f)\delta(g)$. Note that $C = \operatorname{span}\{f \in C : 0 \leq f \leq 1\}$, so we get multiplicativity of δ.

Suppose now for each $x \in X$, there exists some $f_x \in \ker\delta$ so that $f_x(x) \neq 0$. Let $U_x = f_x^{-1}(\mathbb{R}\setminus\{0\})$, so $X = \bigcup_{x\in X}\{x\} = \bigcup_{x\in X} U_x$ so there are $x_1, \ldots, x_n$ in X so $X = \bigcup_{j=1}^n U_{x_j}$. Then $f = \sum_{j=1}^n f_{x_j}^2 > 0$ on X (by definition of each U_{x_j}), so $1/f \in C$. Then

$$1 = \delta(\mathbf{1}) = \delta\left(\frac{1}{f}\right)\delta(f) = \delta\left(\frac{1}{f}\right)\sum_{j=1}^n \delta(f_{x_j})^2 = 0$$

since each $f_{x_i} \in \ker\delta$, which is absurd. Hence there is $x \in X$ so $f(x) = 0$ whenever $f \in \ker\delta$, so $\ker\delta \supsetneq \ker\hat{x}$, so $\delta \in \mathbb{R}\hat{x}$ and since $\delta(\mathbf{1}) = 1 = \hat{x}(\mathbf{1})$, so $\delta = \hat{x}$.

(III) If $\hat{x} = (1-t)\mu + t\nu$ and $t \in (0,1)$, $\mu, \nu \in P(X)$, then by (7.2),

$$t|\nu(f)| \leq t\nu(|f|) \leq \hat{x}(|f|) = |f(x)|$$

so $\ker\nu \supseteq \ker\hat{x}$ and as above, $\nu = \hat{x}$. Then $\mu = \hat{x}$. ■

Remark. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, it is similar to show that $\operatorname{ext} B(C^{\mathbb{F}}(X,\tau)^*) = \{z\hat{x} : z \in \mathbb{F}, |z| = 1, x \in X^*\}$.

Let $PA(\mathbb{N}) = \{\mu \in \mathrm{FA}(\mathbb{N}) : \left\|\mu\right\|_{\mathrm{var}} \leq 1, \mu(\mathbb{N}) = 1\}$ so, as above, $PA(\mathbb{N})$ is a $w^* = \sigma(\mathrm{FA}(\mathbb{N}), \ell_\infty)$–compact set.

7.19 Proposition. $\operatorname{ext} PA(\mathbb{N}) = \{\delta_{\mathcal{U}} : \mathcal{U}$ *is an ultrafilter on* $\mathbb{N}\}$

Proof If $\delta \in \operatorname{ext} PA(\mathbb{N})$, let $f_\delta \in \ell_\infty^*$ be as in $A1$. As above, we compute that $f_\delta(\chi_E\chi_F) = f_\delta(\chi_E)f_\delta(\chi_F)$, and we have $\chi_E\chi_F = \chi_{E\cap F}$ and hence $\delta(E\cap F) = \delta(E)\delta(F)$. Hence

$$\mathcal{U} = \{E \subseteq \mathbb{N} : \delta(E) \neq 0\} = \{E \subseteq \mathbb{N} : \delta(E) = 1\}$$

is an ultrafilter. The converse is easy. ■

8 Euclidean and Hilbert Spaces

Definition. Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A form $[\cdot,\cdot] : X \to \mathbb{F}$ is called **Hermitian** if for x, x', y in X, $\alpha \in \mathbb{F}$, we have

(i) $[x + \alpha x', y] = [x,y] + \alpha[x',y]$

(ii) $\overline{[y,x]} = [x,y]$
and furthermore **positive** if

3. $[x,x] \geq 0$ for all $x \in X$

and **non-degenerate** if

4. $[x,y] = 0$ for all $y \in X$ implies $x = 0$.

8.1 Proposition. *Let $[\cdot,\cdot]$ be a positive Hermitian form. Let $p(x) = [x,x]^{1/2}$, so $p : X \to [0,\infty)$. Then for $x, y \in X$ and $\alpha \in \mathbb{F}$, we have*
(i) $p(\alpha x) = |\alpha| p(x)$
(ii) $[x,y]| \leq p(x)p(y)$
(iii) $p(x+y) \leq p(x) + p(y)$
(iv) $[\cdot,\cdot]$ is non-degenerate if and only if $[x,x] > 0$ for $x \in X \setminus \{0\}$.
Furthermore, in this case, we have
- *Equality in (ii) if and only if x, y are linearly dependent*
- *$[x,y] = p(x)p(y)$ if and only if there is $s \geq 0$ such that $x = sy$ or $y = sx$ if and only if equality holds in (iii).*

Proof (i) $p(\alpha x) = (\alpha\overline{\alpha}[x,x])^{1,2} = |\alpha| p(x)$
(ii) If $\alpha \in F$, then

$$0 \leq [x - \alpha y, x - \alpha y] = [x,x] - \overline{\alpha}[x,y] - \overline{\overline{\alpha}[x,y]} + |\alpha|^2 [y,y]$$
$$= p(x)^2 - 2\operatorname{Re}\overline{\alpha}[x,y] + |\alpha| p(y)^2$$

Set $\alpha = \operatorname{sgn}[x,y]$ so that $\overline{\alpha}[x,y] = |[x,y]|$ so

$$|[x,y]| \leq \frac{1}{2}\left(p(x)^2 + p(y)^2\right)$$

Then if $t > 0$, by (i),

$$|[x,y]| = |\left[tx, \frac{1}{t}y\right] \leq \frac{1}{2}(t^2 p(x)^2 + \frac{1}{t^2} p(y)^2)$$

If $p(x) = 0$, we let $t \to \infty$ so that $[x,y] = 0$; if $p(y) = 0$, we let $t \to 0^+$ and again that $[x,y] = 0$. If $[x,y] \neq 0$, set $t = p(y)/p(x)$ and we are done.
(iii)

$$\begin{aligned} p(x+y)^2 = [x+y, x+y] &= p(x)^2 + 2\operatorname{Re}[x,y] + p(y)^2 \\ &\leq p(x)^2 + 2|[x,y]| + p(y)^2 \\ &\leq p(x)^2 + 2p(x)p(y) + p(y)^2 = (p(x) + p(y))^2 \end{aligned}$$

(iv) We see, by (iii), if $p(x)^2 = [x,x] = 0$, then $[x,y] = 0$ for all y. Hence $[\cdot,\cdot]$ is non-degenerate if and only if $[x,x] > 0$ for $x \in X \setminus \{0\}$. If x, y are linearly dependant, then equality holds in (ii) by direct computation. If x, y are not linearly dependent, then the choice of $\alpha = \operatorname{sgn}[x,y]$ in (ii) gives strict inequality. The condition $[x,y] = p(x)p(y)$ requires non-negativity of $[x,y]$, showing one is a $R_{\geq 0}$ multiple of the other. This is equivalent to having equality in (iii). ■

Definition. A non-degenerate positive Hermitian form on a vector space $\mathcal{E}$ is called an **inner product**. The pair $(\mathcal{E},(\cdot,\cdot))$ is called a Euclidean space. If, further, $\mathcal{E}$ is complete with respect to the induced norm $\|x\| = (x,x)^{1/2}$, then we call $(\mathcal{E},(\cdot,\cdot))$ a **Hilbert space**.

Example. (i) (Euclidean Space) $(C[0,1],\langle\cdot,\cdot\rangle)$ given by $(f,g) = \int_0^1 f\overline{g}$

(ii) (Euclidean Space) Recall $\ell = \{x \in \mathbb{F}^{\mathbb{N}} : x_n = 0 \text{ for all but finitely many } n\}$, and $(\ell,\langle\cdot,\cdot\rangle)$ with $\langle x,y\rangle = \sum_{j=1}^{\infty} x_j\overline{y}_j$

(iii) (Hilbert Space) $(L_2[0,1],(\cdot,\cdot))$, $(f,g) = \int_{[0,1]} f\overline{g}$.

(iv) (Hilbert Space) $(\ell_2,(\cdot,\cdot))$, $(x,y) = \sum_{j=1}^{\infty} x_j\overline{y}_j$ (convergence by Hölder's inequality)

(v) (Non-separable Hilbert Space) Let Γ be an uncountable set. If $a = (a_\gamma)_{\gamma\in\Gamma} \in [0,\infty)^\Gamma$, we let $\mathcal{F} = \{F \subset \Gamma : |F| < \infty\}$. We define $\sum_{\gamma\in\Gamma} a_\gamma = \sup_{F\in\mathcal{F}} \sum_{\gamma\in F} a_\gamma = \lim_{F\in\mathcal{F}} \sum_{\gamma\in F} a_\gamma$ where $\mathcal{F}$ is pre-ordered by inclusion. Suppose that $\sum_{\gamma\in\Gamma} a_\gamma < \infty$. Let $\Gamma_n = \{\gamma \in \Gamma : a_\gamma \geq 1/n\}$ and we have

$$\infty > \sum_{\gamma\in\Gamma} a_\gamma \geq \sup_{F\in\mathcal{F}} \sum_{\gamma\in F\cap\Gamma_n} a_\gamma \geq \sum_{F\in\mathcal{F}} \frac{F\cap\Gamma_n}{n}$$

so that $|\Gamma_n| < \infty$. Thus $\Gamma_a = \{\gamma \in \Gamma : a_\gamma > 0\} = \bigcup_{n=1}^{\infty} \Gamma_n$ is countable.

Now, define $\ell_2(\Gamma) = \{x = (x_\gamma) \in \mathbb{F}^\Gamma : \sum_{\gamma\in\Gamma} |x_\gamma|^2 < \infty\}$. If $x,y \in \ell_2(\Gamma)$, then we may let $\Gamma_{|x|^2} \cup \Gamma_{|y|^2} \subseteq \{\gamma_k\}_{k=1}^{\infty}$ so Hölder's inequality for ℓ_2 says that

$$\sum_{k=1}^{\infty} |x_{\gamma_k}\overline{y}_{\gamma_k}| \leq \left(\sum_{k=1}^{\infty} |x_{\gamma_k}|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |y_{\gamma_k}|^2\right)^{1/2} < \infty.$$

Thus, $\sum_{k=1}^{\infty} x_{\gamma_k}\overline{y_{\gamma_k}}$ is absolutely converging. Write $(x,y) = \sum_{\gamma\in\Gamma} x_\gamma\overline{y_\gamma} = \sum_{k=1}^{\infty} x_{\gamma_k}\overline{y_{\gamma_k}}$. Now if $(x^{(n)})_{n=1}^{\infty} \subset \ell_2(\Gamma)$ is $\|\cdot\|_2$–Cauchy, then $\Gamma' = \bigcup_{n=1}^{\infty} \Gamma_{|x^{(n)}|^2}$ is countable. Then since $\ell_2(\Gamma') \cong \ell_2$ (up to counting Γ'), so the Cauchy sequence has a limit. Thus $\ell_2(\Gamma)$ is a Hilbert space. It is immediate that $(\ell_2(\Gamma),\|\cdot\|_2)$ is non-separable.

(vi) Let $w : \mathbb{N} \to (0,\infty)$. Let $\ell_2^w = \{x \in \mathbb{F}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^2 w(k) < \infty\}$. Notice that if $x,y \in \ell_2^w$, then $(x_k w(k)^{1/2})_{k=1}^{\infty}, (y_k w(k)^{1/2})_{k=1}^{\infty} \in \ell_2$, so it follows that

$$(x,y)_w = \sum_{k=1}^{\infty} x_k\overline{y_k}w(k)$$

defines an inner product, and $W : \ell_2^2 \to \ell_2$ by $W(x_k)_{k=1}^{\infty} = (x_k w(k)^{1/2})_{k=1}^{\infty}$ is a surjective linear isometry, so ℓ_2^w is a hilbert space.

Various Identities

Let $[\cdot,\cdot]$ be a Hermitian form on X. Then we have the *polarization identitiy*: then over $\mathbb{R}$, $4[x,y] = [x+y,x+y] - [x-y,x-y]$, and over $\mathbb{C}$, $4[x,y] = \sum_{k=0}^{3} i^k[x+i^k y, x+i^k y]$.

Now suppose $(\mathcal{E},(\cdot,\cdot))$ is a Euclidean space. We say that $x,y \in \mathcal{E}$ are **orthogonal** if $(x,y) = 0$ and write $x \perp y$. We call a subset $E \subset \mathcal{E}$ **orthogonal** if $x \neq y \in E$ implies $x \perp y$. We write $x \perp E$ if $x \perp y$ for each $y \in E$. We have

- *Pythagoreans' identity*: if $\{x_1,\ldots,x_n\} \subset \mathcal{E}$ orthogonal, then $\left\|\sum_{j=1}^n x_j\right\|^2 = \sum_{j=1}^n \left\|x_j\right\|^2$.
- *Parallelogram law*: $\left\|x+y\right\|^2 + \left\|x-y\right\|^2 = 2\left\|x\right\|^2 + 2\left\|y\right\|^2$.

Note that if $\mathbb{F} = \mathbb{C}$, $(x,y) = \frac{1}{4}\sum_{k=0}^3 i^k \left\|x + i^k y\right\|^2$ defines an inner product, for any norm satisfying the parallelogram law.

8.2 Proposition. *If $y \in \mathcal{E}$ with $(\mathcal{E},(\cdot,\cdot))$ a Euclidean space, then $f_y : \mathcal{E} \to \mathbb{F}$ by $f_y(x) = (x,y)$ is linear with $\left\|f_y\right\| = \left\|y\right\|$. Furthermore, $|f_y(x)| = \left\|y\right\|$ for $y \neq 0$, $x \in B(\mathcal{E})$ if and only if $x = \frac{\zeta}{\|y\|}y$ where $|\zeta| = 1$.*

Proof Linearity isfrom an assumption on $(\cdot,\cdot)$. Furthermore, Cauchy-Schwarz tells us that

$$|f_y(x)| = |(x,y)| \leq \|x\|\left\|y\right\| \Rightarrow \left\|f_y\right\| \leq \left\|y\right\|$$

so the equality case for Cauchy-Schwarz provides the last statement of the proposition, and supplies $\left\|f_y\right\| \geq \left\|y\right\|$. ∎

Definition. In a Euclidean space $(\mathcal{E},(\cdot,\cdot))$, a set $E \subset \mathcal{E}$ is called **orthonormal** provided that for $e, e' \in E$,

$$(e,e') = \begin{cases} 1 & : e = e' \\ 0 & : e \neq e' \end{cases}$$

8.3 Lemma. (Closest Approximation to Finite) *Let $\{e_1,\ldots,e_n\}$ be orthonormal in a Euclidean space $(\mathcal{E},(\cdot,\cdot))$ and $\mathcal{M} = \mathrm{span}\{e_1,\ldots,e_n\}$. Then for $x \in \mathcal{E}$ we have that*

(i) $P_{\mathcal{M}}x = \sum_{j=1}^n \left(x,e_j\right)e_j$ satisfies that $x - P_{\mathcal{M}}x \perp \mathcal{M}$ and hence $\|x\|^2 = \|P_{\mathcal{M}}\|^2 + \|x - P_{\mathcal{M}}x\|^2$

(ii) $d(x,\mathcal{M}) = \left\|x - \sum_{j=1}^n (x,e_j)e_j\right\|^{1/2}$

Proof (i) If $1 \leq k \leq n$, we have

$$(x - P_{\mathcal{M}}x, e_k) = (x,e_k) - \sum_{j=1}^n \left(e,e_j\right)\left(e_j,e_k\right) = (x,e_k) - (x,e_k) = 0$$

and it follows that $x - P_{\mathcal{M}}x \perp \mathcal{M}$. Pythagoras' law provides the second formula.

(ii) Endow $\mathbb{F}^n$ with the usual inner product $\|\cdot\|_2$. By Cauchy-Schwarz, for $x \in \mathcal{E}$ and $\alpha \in \mathbb{F}^n$,

$$\left|\left(((x,e_j))_{j=1}^n, \alpha\right)\right| = \left|\sum_{j=1}^n (x,e_j)\overline{\alpha}_j\right| \leq \left(\sum_{j=1}^n |\left(x,e_j\right)|^2\right)^{1/2} = \|P_{\mathcal{M}}x\|\,\|\alpha\|_2$$

so that

$$\begin{aligned}\left\|x - \sum_{j=1}^n \alpha_j e_j\right\|^2 &= \|x\|^2 - 2\,\mathrm{Re}\sum_{i=1}^n \left(x,e_j\right)\overline{\alpha}_j + \sum_{j=1}^n |\alpha_j|^2 \\ &\geq \|x\|^2 - 2\left|\left(\left(\left(x,e_j\right)\right)_{j=1}^n, \alpha\right)\right| + \|\alpha\|_2^2 \\ &\geq \|x\|^2 - 2\|P_{\mathcal{M}}x\|\,\|\alpha\|_2 + \|\alpha\|_2^2 \\ &= \|x - P_{\mathcal{M}}x\|^2 + (\|P_{\mathcal{M}}x\| - \|\alpha\|_2)\end{aligned}$$

We get equality above if $x \perp \mathcal{M}$ or otherwise there is some $s \geq 0$ so $\alpha_j = s\left(x, e_j\right)$ for $j = 1, \ldots, n$. Hence, in this case,

$$\left\| x - \sum_{j=1}^{n} s\left(x, e_j\right) e_j \right\|^2 = \|x - P_{\mathcal{M}} x\|^2 + \|P_{\mathcal{M}} x\|^2 (1-s)^2$$

which is minimized when $s = 1$. ■

Remark. (i) The proof above shows that $P_{\mathcal{M}} x$ is the unique elemet of $\mathcal{M}$ satisfying $\operatorname{dist}(x, \mathcal{M}) = \|x - P_{\mathcal{M}} x\|$.

(ii) It may be shown that $P_{\mathcal{M}} : \mathcal{E} \to \mathcal{E}$ is linear with $\operatorname{im} P_{\mathcal{M}} = \mathcal{M}$, $P_{\mathcal{M}}^2 = P_{\mathcal{M}}$, and $\|P_{\mathcal{M}}\| = 1$ (in other words, this map is actually a projection operator)

8.4 Theorem. (Orthonormal Basis)

let $(\mathcal{E}, (\cdot, \cdot))$ be a Euclidean space, $E \subset \mathcal{E}$ an orthonormal set. Then the following are equivalent:

(i) $\overline{\operatorname{span}} E = \mathcal{E}$

(ii) for $x \in \mathcal{E} = x = \sum_{e \in E} (x, e) e = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) e$, where $\mathcal{F} = \{F \subseteq E : |f| < \infty\}$, directed by inclusion (Bessel's identity)

(iii) For $x, y \in \mathcal{E}$, $(x, y) = \sum_{e \in E} (x, e)(e, y)$ (Parseval's identity).

Proof $(i \Rightarrow ii)$ For $F \in \mathcal{F}$, let $\mathcal{E}_F = \operatorname{span} F$, so that $\mathcal{E}_F \subseteq \mathcal{E}_{F'}$ if $F \subseteq F'$ in $\mathcal{F}$ and $\operatorname{span} E = \bigcup_{F \in \mathcal{F}} \mathcal{E}_F$. Hence for $x \in \mathcal{E}$, we have

$$0 = \operatorname{dist}(x, \operatorname{span} E) = \operatorname{dist}\left(x, \bigcup_{F \in \mathcal{F}} \mathcal{E}_F\right) = \inf_{F \in \mathcal{F}} \operatorname{dist}(x, \mathcal{E}_F) = \lim_{F \in \mathcal{F}} \operatorname{dist}(x, \mathcal{E}_F)$$

Thus by the f.d. approximation lemma, we have

$$0 = \lim_{F \in \mathcal{F}} \operatorname{dist}(x, \mathcal{E}_F) = \lim_{F \in \mathcal{F}} \left\| x - \sum_{e \in F} (x, e) e \right\|$$

$(ii \Leftrightarrow iii)$ We have

$$\begin{aligned} 0 &= \lim_{F \in \mathcal{F}} \left\| x - \sum_{e \in F} (x, e) e \right\|^2 \\ &= \lim_{F \in \mathcal{F}} \left(\|x\|^2 - 2 \operatorname{Re} \sum_{e \in F} \overline{(x, e)} (x, e) + \sum_{e \in F} \|(x, e)\|^2 \right) \\ &= \lim_{F \in \mathcal{F}} \left(\|x\|^2 - \sum_{e \in F} |(x, e)^2| \right) \\ &= \|x\|^2 - \sum_{e \in E} |(x, e)|^2 \end{aligned}$$

$(ii \Rightarrow iv)$ Recall that $f_y = (\cdot, y) \in \mathcal{E}^*$ so that

$$(x, y) = f_y\left(\lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) e\right) = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) f_y(e) = \sum_{e \in E} (x, e)(e, y)$$

$(iv \Rightarrow ii)$ Take $x = y$.
$(iii \Rightarrow i)$ Obvious; $x = \lim_{F\in\mathcal{F}} \sum_{e\in F} (x,e)\, e \in \overline{\operatorname{span} E}$, i.e. $\mathcal{E} \subseteq \overline{\operatorname{span} E} \subseteq \mathcal{E}$. ■

Definition. Any set $E \subset \mathcal{E}$ satisfying the above conditions is called a **orthonormal basis** for $\mathcal{E}$.

8.5 Theorem. *Let $(x_1, x_2, \ldots)$ be a linearly independent sequence in a euclidean space $(\mathcal{E}, (\cdot,\cdot))$. There exists an orthogonal sequence $(z_1, z_2, \ldots)$ which satisfies* $\operatorname{span}\{z_1, \ldots, z_n\} = \operatorname{span}\{x_1, \ldots, x_n\}$ *for $n = 1, 2, \ldots$ so that* $\operatorname{span}\{z_1, z_2, \ldots\} = \operatorname{span}\{x_1, x_2, \ldots\}$.

Proof Let $\mathcal{E}_n = \operatorname{span}\{x_1, \ldots, x_n\}$. We set

$$\begin{aligned} z_1 &= x_1 & e_1 &= \frac{z_1}{\|z_1\|} \\ z_2 &= x_2 - P_{e_1} x_2 & e_2 &= \frac{z_2}{\|z_2\|} \\ &\vdots \\ z_{n+1} &= x_{n+1} - P_{\mathcal{E}_n} x_{n+1} & e_{n+1} &= \frac{z_{n+1}}{\|z_{n+1}\|} \end{aligned}$$

where $P_{\mathcal{E}_n} x = \sum_{j=1}^{n} \left(x, e_j\right) e_j$. Inductively, $z_n \in \mathcal{E}_n$ and $z_n \perp \mathcal{E}_k$ for $k = 1, \ldots, n-1$. Hence each set $\{z_1, \ldots, z_n\}$is orthonormal and $\operatorname{span}\{z_1, \ldots, z_n\} \subseteq \operatorname{span}\{x_1, \ldots, x_n\}$ is of full dimension and hence equal. ■

8.6 Corollary. *Any separable Euclidean space admits an orthonormal basis.*

Proof Let $\{x_n\}_{n=1}^{\infty}$ be dense in $\mathcal{E}$. Let $n_1 = \min\{n : x_n \neq 0\}$, and $n_{k+1} = \min\{n : x_n \notin \operatorname{span}\{x_{n_1}, \ldots, x_{n_k}\}\}$. Then $\{x_{n_1}, x_{n_2}, \ldots\}$ and normalize to get an orthonormal set $E = \{e_1, e_2, \ldots\}$ which satisfies $\overline{\operatorname{span} E} = \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty} = \mathcal{E}$. ■

8.7 Theorem. (Riesz Fischer) *Let $(\mathcal{E}, (\cdot,\cdot))$ be a Euclidean space. Then $\mathcal{E}$ is a Hilbert space if and only if for any orthonrmal set E and an $\alpha = (\alpha_e)_{e\in E} \in \ell_2(E)$, we have that $\sum_{e\in E} \alpha_e e \in \mathcal{E}$.*

Proof ($\Longrightarrow$) If $\alpha \in \ell_2(E)$ then $E_\alpha = \{e \in E : \alpha_e \neq 0\}$ is countable, and write $E_\alpha = (e_1, e_2, \ldots)$. If $m < n$, we have

$$\left\| \sum_{k=1}^{n} \alpha_{e_k} e_k - \sum_{k=1}^{m} \alpha_{e_k} e_k \right\|^2 = \sum_{k=m+1}^{n} |\alpha_{e_k}|^2 \leq \sum_{k=n+1}^{\infty} |\alpha_{e_k}|^2 \to 0$$

so $x_\alpha = \sum_{k=1}^{\infty} \alpha_{e_k} e_k = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{e_k} e_k$ converges. If $F \in \mathcal{F}$, $F \supseteq \{e_1, \ldots, e_n\}$, then

$$\left\| x_\alpha - \sum_{e\in F} \alpha_e e \right\|^2 = \sum_{e=\{e_1,e_2,\ldots\}\setminus F} |\alpha_e|^2 \leq \sum_{k=n+1}^{\infty} |\alpha_{e_k}|^2 \to 0$$

so $x_\alpha = \sum_{e\in E} \alpha_e e = \lim_{F\in\mathcal{F}} \sum_{e\in F} (x,e)\, e$.

($\Longleftarrow$) Let $(x^{(n)})_{n=1}^{\infty}$ be Cauchy in $\mathcal{E}$. Let $\mathcal{M} = \overline{\operatorname{span}\{x^{(n)}\}_{n=1}^{\infty}} \subset \mathcal{E}$ so $\mathcal{M}$ is separable and admits a countable orthonormal basis $E = (e_1, e_2, \ldots)$. Then we appeal to orthonormal basis to see that for any $x \in \mathcal{M}$, $\sum_{k=1}^{\infty} |(x, e_k)|^2 = \|x\|^2 < \infty$ and $x = \sum_{k=1}^{\infty} (x, e_k)\, e_k$.

Our present assumption show that $U : \ell_2(E) \to \mathcal{M}$ given by $U_\alpha = \sum_{k=1}^\infty \alpha_k e_k$ always converges in $\mathcal{M} \subseteq \mathcal{E}$. Then orthonormal basis theorem gives $\|U_\alpha\| = \|\alpha\|_2$ so U is a surjective isometry. We let $\alpha^{(n)} = ((x^{(n)}, e_k))_{k=1}^\infty \in \ell_2(E)$, then $\left\|\alpha^{(n)} - \alpha^{(m)}\right\|_2 = \left\|U_\alpha^{(n)} - U_\alpha^{(m)}\right\| = \left\|x^{(n)} - x^{(m)}\right\|$ so $(\alpha^{(n)})_{n=1}^\infty$ is Cauchy and in $\ell_2(E)$ and hence admits a limit α. Furthermore,

$$\left\|\sum_{k=1}^\infty \alpha_k e_k - x^{(n)}\right\| = \left\|U_\alpha - U_\alpha^{(n)}\right\| = \left\|\alpha - \alpha^{(n)}\right\| \to 0$$

as required. ■

Definition. If $\emptyset \neq S \subset \mathcal{E}$, $(\mathcal{E}, (\cdot,\cdot))$ a Euclidean space, we define its **perpindicular** by $S^\perp = \{y \in \mathcal{E} : ((x,y) = 0 \text{ for any } x \in S)\}$.

Remark. (i) $S \subseteq T$ implies $T^\perp \subseteq S^\perp$
(ii) $S^\perp = \bigcap_{x\in S} \ker f_x$ and is thus closed.
(iii) $\overline{S}^\perp = S^\perp$, since $\overline{S}^\perp \subseteq \overline{S}^\perp$, and if $y \in S^\perp$ and $x \in \overline{S}$, then $x = \lim x_n$ with $x_n \in S$ so $(x,y) = f_y(x) = f_y \lim x_n = \lim f_y(x_n) = \lim (x_n, y) = 0$.
(iv) $(\overline{\operatorname{span}} S)^\perp = S^\perp$. Notice that $(\operatorname{span} S)^\perp = S^\perp$ (easy test) and use (iii)
(v) $\overline{\operatorname{span}} S \cap S^\perp = \{0\}$.

8.8 Theorem. (Existence of Orthonormal Basis) *Let $(H, (\cdot,\cdot))$ be a Hilbert space.*
(i) Given an orthonormal set $E \subset H$, $P_E : H \to H$, $P_E x = \sum_{e\in E} (x,e)\, e$ satisfies

$$\operatorname{im} P_E \subseteq \overline{\operatorname{span} E} \text{ for } x \in H, x - P_E x \in E^\perp$$

(ii) H admits an orthonormal basis, i.e. an orthonormal set M such that $\overline{\operatorname{span} M} = H$.

Proof (i) Let $\mathcal{F} = \{F \subseteq E : |F| < \infty\}$ be directed by inclusion, and for $F \in \mathcal{F}$, $\mathcal{E}_F = \operatorname{span} F$. Then as in the proof of OMBT, we have for $x \in H$

$$0 \le \operatorname{dist}(x, \operatorname{span} E)^2 = \lim_{F\in\mathcal{F}} \operatorname{dist}(x, \mathcal{E}_F)^2 = \|x\|^2 - \sum_{e\in E} |(x,e)|^2$$

so $\sum_{e\in E} |(x,e)|^2 \le \|x\|^2 < \infty$. Thus appealing to Riesz-Fischer, $P_E x = \sum_{e\in E} (x,e)\, e$ converges in H. Since $P_E x = \lim_{F\in\mathcal{F}} \sum_{e\in F} (x,e), e$, we see that $P_E x \in \overline{\operatorname{span}} E$, so $\operatorname{im} P_E \subseteq \overline{\operatorname{span}} E$. Moreover, if $e' \in E$, $f_{e'} = (\cdot, e') \in H^*$ so

$$(x - P_E x, e') = (x, e') - f_{e'}\left(\sum_{e\in E} (x,e)\, e\right) = (x, e') - \sum_{e\in E} (x,e)\, f_{e'}(e) = -$$

so $x - P_E x \in E^\perp$.

(ii) Let $\mathcal{O} = \{E \subseteq H : E \text{ is orthonormal}\}$, which is partially ordered by inclusion. Note that $\emptyset \in \mathcal{O}$ vacuously. If $\mathcal{C} \subseteq \mathcal{O}$ is a chain, then $\bigcup_{E\in\mathcal{C}} \in \mathcal{O}$ is an upper bound for $\mathcal{C}$. By Zorn' get a maximal element M.

Suppose $\overline{\operatorname{span}} M \subsetneq H$, and get $x \in H \setminus \overline{\operatorname{span}} M$ and $y = x - P_M x \in (\overline{\operatorname{span}} M)^\perp \setminus \{0\}$. But then $M \subsetneq M \cup \{\frac{1}{\|y\|} y\}$, violating maximality. ■

8.9 Corollary. *If H is a Hilber space with orthonormal basis E, then the map*

$$U : H \to \ell_2(E), Ux = ((x,e))_{e\in E}$$

is a surjective isometry which respects inner products.

Proof We know $\|x\|^2 = \sum_{e\in E} |(x,e)|^2 = \|Ux\|_2$ from ONBT. It is evident that U is linear and $\operatorname{im} U$ is dense in $\ell_2(E)$ so that U is surjective. Finally, if $x, y \in H$, then

$$(x,y)_H = \sum_{e\in E}(x,e)(e,y) = \sum_{e\in E}(x,e)\overline{(y,e)} = (Ux, Uy)_{\ell_2(E)}$$

as required. ■

Remark. If each of E, E' is an orthonormal basis for a Euclidean space $(\mathcal{E},(\cdot,\cdot))$, then $|E| = |E'|$. We let $\Bbbk$ be any countable dense subfield of $\mathbb{F}$. Then $D = \operatorname{span}_{\Bbbk}$, so $|D| = \aleph_0|E| = |E|$ when $|E|$ is infinite. Since for e', e'' in E', $\|e' - e''\| = \sqrt{2}$, we have that any ball $e' + \frac{1}{\sqrt{2}}D(\mathcal{E})$ contains at least one element of D, and $d_{e'} \neq d_{e''}$ if $e' \neq e''$ in E'. This shows that $|E| \geq |E'|$. Likewise $|E'| \leq |E|$.

8.10 Corollary. (Orthogonal complementation) *Let $(\mathcal{E}, \|\cdot,\cdot\|)$ be a Euclidean space and $\mathcal{M} \subseteq \mathcal{E}$ a subspace which is complete with respect to the norm induced from $(\cdot,\cdot)$, i.e. $(\mathcal{M},(\cdot,\cdot))$ is a Hilbert space. Then there is a unique operator $P_{\mathcal{M}} = P : \mathcal{E} \to \mathcal{E}$ such that $\operatorname{im} P = \mathcal{M}$ and $\operatorname{im}(I-P) = \mathcal{M}^\perp$. Moreover,*

- *P is linear*
- *$\|P\| \leq 1$, $\|P\| = 1$ if $\mathcal{M} \neq \{0\}$*
- *$P^2 = P$*
- *for $x, y \in \mathcal{E}$, $(Px, y) = (Px, Py) = (x, Py)$.*

Such an operator is called the ***orthogonal projection****.*

Proof The theorem above prvides an orthonormal basis E for $\mathcal{M}$. Then P_E, as defined above, satisfies $\operatorname{im} P = \mathcal{M}$ and $\operatorname{im}(I-P) = \mathcal{M}^\perp$. Moreover, if P satisfies those conditions, then for $x \in \mathcal{E}$,

$$Px + x - Px = x = P_E x + x - P_e X$$

so that

$$Px - P_e X = [x - P_E x] - [x - Px] \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$$

so $Px = P_e x$. Then if $x, y \in \mathcal{E}$ and $\alpha \in \mathbb{F}$,

$$P(x+\alpha y) + x + \alpha y - P(x+\alpha y) = x + \alpha y = Px + x - Px + \alpha[Py + y - Py]$$

so

$$P(x+\alpha y) - [Px + \alpha py] = x - Px + \alpha[y - Py] - [x + \alpha y - P(x+\alpha y)] \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$$

and we have linearity.

If $x \in \mathcal{E}$, Pythagoras tells us that $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$ so $\|Px\| \leq \|x\|$, i.e. $\|P\| \leq 1$. If $e' \in E$, $Pe' = P_E e' = \sum_{e\in E}(e',e)e = e'$, so $P|_{\operatorname{span} E} = \mathrm{id}$ and by uniqueness of extension of bounded linear functionals (uniformly continuous), we see that $P|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$. This shows that if $\mathcal{M} \neq \{0\}$, $\|P\| = 1$ and $P = P^2$. Furthermore, this also shows that $\operatorname{im} P = \mathcal{M}$. Finally,

$$(Px, y) = (Px, Py + y - Py) = (Px, Py)$$

and likewise $(x, Py) = (Px, Py)$. ■

8.11 Corollary. *Let H be a Hilbert space.*

(i) If $\mathcal{M}$ is a closed subspace, then $(\mathcal{M}^{\perp})^{\perp} = \mathcal{M}$.

(ii) If $\emptyset \neq S \subset H$, then $(S^{\perp})^{\perp} = \overline{\mathrm{span}} S$.