REPLACE

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Contents

I. REPLACE

1. For $a, b, k \in \mathbb{N}$,
$$\binom{a+b}{k} = \sum_{j=1}^{k} \binom{a}{j} \cdot \binom{b}{k-j} \tag{0.1}$$

   We prove this with a bijection:
$$\mathcal{B}(a+b,k) \leftrightarrows \bigcup_{j=0}^{k} \mathcal{B}(a,j) \times \mathcal{B}(b,k-j)$$

   given by $S \mapsto (S \cap \{1,\ldots,a\}, (S \cap \{a+1,\ldots,a+b\})^{(-a)}$ and $(P,Q) \mapsto P \cup Q^{(a)}$, where $\mathcal{B}(n,i)$ is the set of i–element subsets of $\{1,2,\ldots,n\}$ and for $C \subseteq \mathbb{Z}$ and $q \in \mathbb{Z}$, $C^{(q)} = \{c+q : c \in C\}$. Note that the equation in fact gives the polynomial identity
$$\binom{x+y}{k} = \sum_{j=0}^{k} \binom{x}{j}\binom{y}{k-j}$$

   in $\mathbb{Q}[x,y]$. We denote the falling factorial $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$, which has degree i for each $i \in \mathbb{N}$. In particular, $(x)_i = i!\binom{x}{i}$, so multiplying our identity by $k!$, we get
$$(x+y)_k = \sum_{j=0}^{k} \binom{k}{j} (x)_j (y)_{k-j}$$

   Compare this with the standard binomial theorem
$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

   These are called sequences of binomial type.

2. Here's another identity. For $n \geq 0$ and $s,t \geq 1$,
$$\binom{n+s+t-1}{s+t-1} = \sum_{k=0}^{n} \binom{k+s-1}{s-1}\binom{n-k+t-1}{t-1}$$

   Let $\mathcal{M}(m,r)$ denote a multiset of size m with elements of r types, so that $|\mathcal{M}(m,r)| = \binom{m+r-1}{r-1}$. Let's define a bijection
$$\mathcal{M}(n,s+t) \rightleftharpoons \bigcup_{k=1}^{n} \mathcal{M}(k,s) \times \mathcal{M}(n-k,t) \tag{0.2}$$

$\mu = (m_1, \ldots, m_{s+t}) \mapsto ((m_1, \ldots, m_s), (m_{s+1}, \ldots, m_{s+t}))$ and $(\nu, \theta) \mapsto \nu\theta$. Note that if f, g are polynomials of degree d and e respectively, then $\sum_{k=0}^{n} f(k)g(n-k)$ is a polynomial in n of degree $d+e-1$.

Is there some way to understand (0.2)? It is unclear, with our known techniques, that this corresponds to a polynomial identity since there is a variable n in the exponent. However, we can use generating functions. Define

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{n+s+t-1}{s+t-1} z^n = \sum_{n=0}^{\infty} |\mathcal{M}(n, s+t)| z^n &= \sum_{(m_1, \ldots, m_{s+t})} z^{m_1 + \cdots + m_{s+t}} \\
&= \left(\sum_{m=0}^{\infty} z^m \right)^{s+t} \\
&= \frac{1}{(1-z)^{s+t}} = \frac{1}{(1-z)^s} \frac{1}{(1-z)^t} \\
&= \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} z^k \sum_{\ell=0}^{\infty} \binom{\ell+t-1}{t-1} z^\ell \\
&= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{k+s-1}{s-1} \binom{n-k+t-1}{t-1} \right)
\end{aligned}$$

Similarly, (0.1) is equivalent to saying $(1+z)^{a+b} = (1+z)^a (1+z)^b$. Note that $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ for $n \in \mathbb{N}$.

Can we substitute $\frac{1}{(1-q)^t} = (1+z)^n$ where $z = -q$ and $n = -t$?

3. Consider

$$(x_1 + x_2)^n = \sum_{i=0}^{n} \binom{n}{i} x_1^i x_2^{n-i}$$

and

$$(x_1 + x_2)^n = \sum_{f : N_n \to \{1,2\}} \prod_{j=1}^{n} x_{f(j)}$$

More generally, we can consider

$$(x_1 + \cdots + x_k)^n = \sum_{f : N_n \to N_k} \prod_{j \in N_n} x_{f(j)}$$

If we set all $x_1 = \cdots = x_k = 1$, then k^n gives the number of functions from N_n to N_k. If we set $x_i = q^i$ for all $i \in N_k$, then we get

$$\left(\frac{q - q^{k+1}}{1-q} \right)^n = (q + q^2 + \cdots + q^k)^n = \sum_{f : N_n \to N_k} q^{f(1) + \cdots + f(k)}$$

Collect all the terms in $(x_1 + \cdots + x_k)^n$ that produce the same monomial. Given a multiset μ with $m_1 + \cdots + m_k = n$, write $x_1^{m_1} \cdots x_k^{m_k} = \underline{x}^\mu$. Then

$$(x_1 + \cdots + x_k)^n = \frac{n!}{m_1! \cdots m_k!} \underline{x}^\mu = \sum_{\mu \in \mathcal{M}(n,t)} \binom{n}{\mu} \underline{x}^\mu$$

4. How can we interpret

$$P_n(q) = \prod_{i=1}^{n}(1 + q + q^2 + \cdots + q^{i-1})$$

In general, if we set $q = 1$, we see that $P_n(1) = n!$. We might hope that there is some weight function on permutations $w : \mathcal{S}_n \to \mathbb{N}$ such that $P_n(q) = \sum_{\sigma\in\mathcal{S}_n} q^{w(\sigma)}$. Recall the bijection $I_n : \mathcal{S}_n \to \mathcal{Q}_n$ from chapter 1. Let's find some weight function $\nu : \mathcal{Q}_n \to \mathbb{N}$ such that $\sum_{\rho\in\mathcal{Q}_n} x^{\nu(\rho)} = P_n(q)$, then "pull back" the definition of $\nu : \mathcal{Q}_n \to \mathbb{N}$ to get a definition for $\omega : \mathcal{S}_n \to \mathbb{N}$. Note that $\sum_{h\in N_r} q^{h-1} = 1 + q + \cdots + q^{r-1}$. Thus

$$\sum_{\rho=(h_1,\dots,h_n)\in\mathcal{Q}_n} q^{(h_1-1)+(h_2-1)+\cdots+(h_n-1)} = \prod_{i=1}^{n}(1 + q + \cdots + q^{i-1}) = P_n(q)$$

so we can define $\nu(\rho) = |\rho| - n$ and $\sum_{q\in\mathcal{Q}_n} q^{|\rho|-n} = P_n(q)$. We also have

$$\sum_{\rho\in\mathcal{Q}_n} q^{(h_1-1)+\cdots+(h_n-1)} = (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2})\cdots(1 + q)(1)$$

For notation, define $[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q}$. Then $[m]_q! = [m]_q[m-1]_q\cdots[1]_q$.

	1	q	q^2	q^3	q^4	
$q[3]_q$	0	1	1	1		
$[2]_q[3]_q$	1	2	2	1		
$-q[2]_q[3]_q$	0	−1	−2	−2	−1	
$q^2[2]_q[3]_q$	0	0	1	2	2	1
$[6]_q$	1	1	1	1	1	1

so that $[6]_q = (1 - q + q^2)[2]_q[3]_q$. An **inversion** in $\sigma = a_1 \dots a_n \in \mathcal{S}_n$ is a pair (i, j) of indices $1 \le i < j \le n$ with $a_i > a)j$. Define $\mathrm{Inv}(\sigma)$ as the set of inversions of σ, and $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$. Notice that if $\sigma = a_1 \dots a_n \mapsto \rho = (h_1, \dots, h_n)$, then for each $1 \le i \le n$, $h_i - 1$ is the number of inversions of σ with i in the first coordinate. Recall

$$\begin{aligned} \mathcal{S}_n &\leftrightarrows \mathcal{B}(n,k) \times \mathcal{S}_k \times \mathcal{S}_{n-k} \\ \sigma = a_1 \dots a_n &\leftrightarrow (A, \beta, \gamma) \\ \mathrm{inv}(\sigma) &= w(A) + \mathrm{inv}(\beta) + \mathrm{inv}(\gamma) \end{aligned}$$

Assuming such a weight fuction $w(A)$ exists, then

$$\begin{aligned} [n]!_q &= \sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{(A,\beta,\gamma)} q^{w(A)+\mathrm{inv}(\beta)+\mathrm{inv}(\gamma)} \\ &= [k]!_q \cdot [n-k]!_q \cdot \sum_{A\in\mathcal{B}(n,k)} q^{w(A)} \end{aligned}$$

so that

$$\sum_{A\in\mathcal{B}(n,k)} q^{w(A)} = \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

$$\sum_{S\in\mathcal{B}(n,k)} q^{\mathrm{sum}(S)} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

0.1 Theorem. *Let V be an n–dimensional vector space over a finite field $\mathbb{F}_q$. Then for $0 \le k \le n$, the number of k–dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

0.2 Lemma. *Let $L : V \to W$ be a linear transformation that is surjective. Then* $\dim V = \dim W + \dim(\ker L)$. *So if this is over a finite field $\mathbb{R}_q$, every $w \in W$ is the image of exactly $q^{\dim(\ker(L)}$ vectors $v \in V$.*

For every $w \in W$, is the image of exactly q^k vectors in V. The number of ordered bases of V is $q^{\binom{n}{2}}(q-1)^n[n]!_q$.

0.3 Theorem. *Let V be an n–dimensional vector space over a finite field $\mathbb{F}_q$. For $0 \le k \le n$, the number of k–dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof Let $\mathrm{OB}(V)$ be the set of ordered bases of V, and let $G(V,k)$ be the set of k–dimensional subspaces of V. Define a function

$$\mathrm{OB}(V) \to \bigcup_{U\in G(V,k)} (\{U\} \times \mathrm{OB}(U) \times \mathrm{OB}(V/U))$$

as follows. Given $(v_1,\ldots,v_n)$ an ordered basis of V, let $U = \mathrm{span}_{\mathbb{F}_q}\{v_1,\ldots,v_k\}$. Then $(v_1,\ldots,v_k) \in \mathrm{OB}(U)$ and $(v_{k+1}+U,\ldots,v_n+U) \in \mathrm{OB}(V/U)$. Consider the map $L : V \to V/U$ given by $L(v) = v + U$, so that every $v + U$ in V/U is the image of q^k vectors in V. Thus $(v_{k-1}+U,\ldots,v_n+U)$ is the image of $q^{k(n-k)}$ sequences $(z_{k+1},\ldots,z_n)$ of vectors in V. Thus the function $(v_1,\ldots,v_n) \mapsto (U,(v_1,\ldots,v_k),(v_{k+1}+U,\ldots,v_n+U))$ is surjective and hits everything on the RHS $q^{k(n-k)}$ times. But then counting both sides,

$$\begin{aligned} q^{\binom{n}{2}}(q-1)^n[n]!_q &= \sum_{U\in G(V,k)} 1 \cdot q^{\binom{k}{2}}(q-1)^k[k]!_q \cdot q^{\binom{n-k}{2}}(q-1)^{n-k}[n-k]!_q \cdot q^{k(n-k)} \\ q^{\binom{n}{2}}[n]!_q &= |G(V,k)| \cdot [k]!_q \cdot [n-k]!_q q^{\binom{k}{2}+\binom{n-k}{2}+k(n-k)} \\ [n]!_q &= |G(V,k)| \cdot [k]!_q \cdot [n-k]!_q \end{aligned}$$

giving our desired result. ∎

A **set partition** π of a set V is a collection of subsets $\pi = \{B_1,\ldots,B_k\}$ of V such that
- Each B_i is not empty
- $B_i \cap B_j = \emptyset$ if $i \ne j$
- $B_1 \cup \cdots \cup B_k = V$

Let $\Pi(n,k)$ be the set of set partitions of N_n with k blocks, and set $S(n,k) = |\Pi(n,k)|$. Certainly $S(0,0) = 1$ for the empty set partition. If $n \ge 1$, then $S(n,0) = 0$, $S(n,n) = 1$, and $S(n,1) = 1$. We can also define a recurrence relation. Let $\Pi'(n,k)$ be those $\pi \in \Pi(n,k)$ in which $\{n\}$ is a block, and $\Pi''(n,k)$ is the set of π in which n is in a block of size at least 2. Note that $\Pi'(n,k) \leftrightarrows \Pi(n-1,k-1)$ by removing or adding the independent element.

Furthemore, the function which removes the element n from a block in $\Pi''(n,k)$ is a surjective function onto $\Pi(n-1,k)$ which hits every element of $\Pi(n-1,k)$ k times. Thus combing these observations, $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$. Thus we can compute

$S(n,k)$	0	1	2	3	4	5	6
0	1	X	X	X	X	X	X
1	0	1	X	X	X	X	X
2	0	1	1	X	X	X	X
3	0	1	3	1	X	X	X
4	0	1	7	6	1	X	X
5	0	1	15	25	10	1	X
6	0	1	31		1		

From homework 2, we have that

$$x^n = \sum_{k=0}^{n} k! S(n,k) \binom{n}{k}$$

Invert this using Binomial Inversion.

0.4 Theorem. (Binomial Inversion) *Let $a_0, a_1, \ldots$ be a sequence.*

Proof For $h \in \mathbb{N}$, let $b_h = \sum_{i=0}^{h} \binom{h}{i} a_i$. Let $A(t) = \sum_{i=0}^{\infty} a_i t^i$ and $B(t) = \sum_{h=0}^{\infty} b_h t^h$. Then

$$\begin{aligned}
B(t) &= \sum_{h=0}^{\infty} t^h \sum_{i=0}^{h} \binom{h}{i} a_i \\
&= \sum_{i=0}^{\infty} a_i t^i \sum_{h=i}^{\infty} \binom{h}{h-i} t^{h-i} \\
&= \sum_{i=0}^{\infty} a_i t^i \sum_{j=0}^{\infty} \binom{i+j}{j} t^j = \sum_{i=0}^{\infty} \frac{a_i t^i}{(1-t)^{i+1})} \\
&= \frac{1}{1-t} \sum_{i=0}^{\infty} a_i \left(\frac{t}{1-t}\right)^i = \frac{1}{1-t} A\left(\frac{t}{1-t}\right)
\end{aligned}$$

Let $z = t/(1-t)$, so that $t = z/(1+z)$. Thus

$$B\left(\frac{z}{1+z}\right) = (1+z)A(z)$$

so that

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i z^i = \frac{1}{1+z} B\left(\frac{z}{1+z}\right) &= \sum_{h=0}^{\infty} b_h \frac{z^h}{(1+z)^{h+1}} \\
&= \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\infty} \binom{j+h}{h} z^h (-z)^j \\
&= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} b_j
\end{aligned}$$

Thus for all $m \in \mathbb{N}$, $a_m = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} b_j$. ■

In particular, applying this to Stirling numbers of the second kind, for all $n \in \mathbb{N}$ in $\mathbb{R}[x]$, we have

$$x^n = \sum_{k=0}^{n} S(n,k)\binom{x}{k}k!$$

Let $b_i = i^n$ for $i = 0, 1, 2, \ldots$. If $k > n$ or $k > i$, then $S(n,k)\binom{i}{k} = 0$; thus,

$$\begin{aligned} i^n &= \sum_{k=0}^{n} S(,nk)\binom{i}{k}k! = \sum_{k=0}^{\min(n,i)} S(n,k)\binom{i}{k}k! = \sum_{k=0}^{i} S(n,k)\binom{i}{k}k! \\ &= \sum_{k=0}^{i}\binom{i}{k}a_k \end{aligned}$$

where $a_k = k!S(n,k)$ for all $k \in \mathbb{N}$. But then apply binomial inversion to get

$$a_k = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}b_j$$

Suppose $m^n = \sum_{k=0}^{n} S(n,k)\binom{m}{k}k!$. Then $[m]_q^n = \sum_{k=0}^{\infty} S[n,k]_q \begin{bmatrix} m \\ k \end{bmatrix}_q [k]!_q$, where $S[n,k]_q = \sum_{\pi\in\Pi(n,k)} q^{w(\pi)}$. Is there some function $w : \Pi(n,k) \to \mathbb{N}$ that makes this work?

B.5 For $\mathcal{S}$ a set of BTs, let $\mathcal{R}$ be the trees in $\mathcal{S}$ with a red root and $\mathcal{B}$ be the trees in $\mathcal{S}$ with a blue root, so $\mathcal{S} = \mathcal{R} \cup \mathcal{B}$ disjointly. Let $r(T)$ count the number of red notes, and $b(T)$ count the number of blue nodes, and let $S(x,y) = \sum_{T\in\mathcal{S}} x^{r(T)}y^{b(T)}$. In particular, $S(t,t) = \sum_{T\in\mathcal{S}} t^{n(T)}$ where $n(T) = r(T) + b(T)$ is the total number of notes.

We have bijections

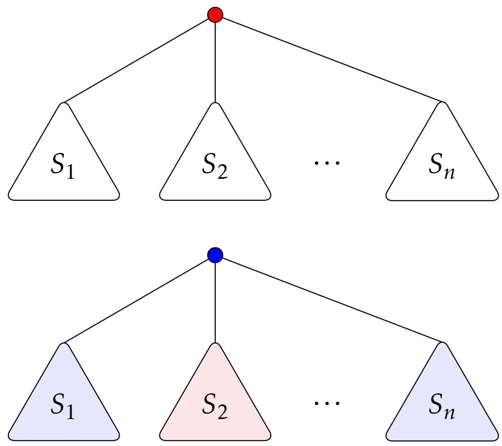
$$\begin{aligned} \mathcal{R} &\leftrightharpoons \{\color{red}{\bullet}\} \times \bigcup_{k=0}^{\infty} \mathcal{S}^k \\ \mathcal{B} &\leftrightharpoons \{\color{blue}{\bullet}\} \times \Big((\epsilon \cup \mathcal{R})(\mathcal{B}\mathcal{R})^*(\epsilon \cup \mathcal{B})\Big) \\ \mathcal{S} &\leftrightharpoons \mathcal{R} \cup \mathcal{B} \end{aligned}$$

so that

$$\begin{aligned} S &= R + B \\ R &= \frac{x}{1-S} \\ B &= y(1+R)\frac{1}{1-BR}(1+B) \end{aligned}$$

Substituting R and B using the first two equations, we get

$$S - \frac{x}{1-S} = \frac{y\left(1+\frac{x}{1-S}\right)\left(1+S-\frac{x}{1-S}\right)}{1-\frac{x}{1-S}\left(S-\frac{x}{1-S}\right)}$$


S_1
S_2
$\cdots$
S_n
S_1
S_2
$\cdots$
S_n

II. Power Series Identities

(i) $\dfrac{1}{(1-z)^h} = \sum_{k=0}^{\infty} \binom{k+h-1}{h-1} z^k$

(ii) Let $a_0, a_1, \ldots$ be a sequence, and $b_h = \sum_{i=0}^{h} \binom{h}{i} a_i$. Then $a_m = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} b_i$.

(iii) General Binomial Series. For $k \in \mathbb{N}$, let $\binom{y}{k} = \frac{y(y-1)\cdots(y-k+1)}{k!} \in \mathbb{Q}[y]$. Then we define

$$(1+x)^y = \sum_{k=0}^{\infty} \binom{y}{k} x^k$$

which is a power series in x. Each coefficient of $[x^n]$ is in $\mathbb{Q}[y]$. Then by Vandermone convolution,

$$\begin{aligned}(1+x)^y(1+x)^z &= \sum_{i=0}^{\infty} \binom{y}{i} x^i \sum_{j=0}^{\infty} \binom{z}{j} x^j \\ &= \sum_{n=0}^{\infty} x^n \left(\sum_{i=0}^{n} \binom{y}{i} \binom{z}{n-i} \right) \\ &= \sum_{n=0}^{\infty} \binom{y+z}{n} x^n = (1+x)^{y+z}\end{aligned}$$

Furthermore, if $y = -p < 0$ is an integer, then

$$\begin{aligned}(1+x)^{-p} &= \sum_{k=0}^{\infty} \binom{-p}{k} x^k \\ &= \sum_{k=0}^{\infty} \binom{k+p-1}{p-1}\end{aligned}$$

For $\alpha \in \mathbb{C}$, $f(x) = (1+x)^\alpha$ is analytic for $|x| < 1$. In particular, by Taylor's theorem,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} c_k x^k$$

where $c_k = \frac{1}{k!} \frac{d^k}{dx^k} (1+x)^\alpha |_{x=0}$.

Consider the class $\mathcal{Q}$ of (unrooted) tres in which every vertex has odd degree. We identify $\mathcal{Q}^\bullet \equiv \chi * \xi_{\text{odd}}[\mathcal{N}]$ for some class $\mathcal{N}$ describing the components of $T \setminus \{v\}$. A structure in $\mathcal{N}$ is a rooted tree in which every vertex has an even number of children. Moreover, $\mathcal{N} \equiv \chi * \xi_{\text{even}}[\mathcal{N}]$. Note that the exponential generating function for ξ_{odd} is $\sum_{j=0}^{\infty} \frac{e^{2j+1}}{(2j+1)!}$, and similarly for the even components. This give

$$\mathcal{Q}^\bullet = x \cdot E_{\text{odd}}(N(x)) = x \cdot \sinh(N(x))$$
$$N(x) = x \cdot E_{\text{even}}(N(x)) = x \cdot \cosh(N(x))$$

Now apply LIFT with $K = \mathbb{Q}$, $G(u) = \cosh(u)$, and $F(u) = \sinh(u)$, so $F'(u) = \cosh(u)$. Now for $n \geq 2$,

$$\begin{aligned}
|\mathcal{Q}_n| &= \frac{1}{n}|\mathcal{Q}_n^\bullet| = \frac{1}{n} \cdot n![x^n]Q^\bullet(x) \\
&= (n-1)![x^n]x\sinh(N(x)) \\
&= (n-1)![x^{n-1}]\sinh(N(x)) \\
&= (n-1)! \cdot \frac{1}{n-1}[u^{n-2}]F'(u)G(u)^{n-1} \\
&= (n-2)! \cdot [u^{n-2}]\cosh(u)^n \\
&= (n-2)![u^{n-2}]\left(\frac{e^u + e^{-u}}{2}\right)^n \\
&= \frac{(n-2)!}{2^n}[u^n]\sum_{j=0}^{n}\binom{n}{j}(e^u)^j(e^{-u})^{n-j} \\
&= \frac{(n-2)!}{2^n}[u^{n-2}]\sum_{j=0}^{n}\binom{n}{j}e^{(2j-n)u} \\
&= \frac{(n-2)!}{2^n}\sum_{j=0}^{n}\binom{n}{j}[u^{n-2}]\sum_{i=0}^{\infty}\frac{((2j-n)u)^i}{i!} \\
&= \frac{(n-2)!}{2^n}\sum_{j=0}^{n}\binom{n}{j}\frac{(2j-n)^{n-2}}{(n-2)!} \\
&= \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}(2j-n)^{n-2}
\end{aligned}$$

If n is odd, then this summation is zero, as expected.

Endofunctions

An **endofunction** is any function $\phi : X \to X$. If $|X| = n$, then there are n^n endofunctions $\phi : X \to X$. Call this class $\mathcal{F}$. We can define the **functional directed graph** of $\phi : X \to X$ with vertices X and directed edges $v \to \phi(v)$ for $v \in X$. When we say ϕ is connected, we mean the underlying undirected graph is connected. Call this class $\mathfrak{C}$. Certainly $\mathcal{F} \equiv \xi[\mathfrak{C}]$.

What is the expected number of components among all n^n endofunctions on $\{1, 2, \ldots, n\}$? Certainly $F(x) = \exp(C(x))$ for the EGFs $F(x)$ and $C(x)$ for $\mathcal{F}$ and $\mathfrak{C}$ respectively. Let $c(\phi)$ be the number of connected components of $\phi \in \mathcal{F}_X$. Then

$$F(x, y) = \sum_{n=0}^{\infty}\left(\sum_{\phi \in \mathcal{F}_n} y^{c(\phi)}\right)\frac{x^n}{n!}$$

Recall $F(x) = \sum_{k=0}^{\infty}\frac{C(x)^k}{k!}$, where $C(x)^k$ is the generating function for a graph with k connected components. Thus

$$F(x, y) = \sum_{k=0}^{\infty}\frac{(C(x)y)^k}{k!} = \exp(yC(x))$$

Let's determine the structure of a connected endofunction $\phi \in \mathfrak{C}_X$. By following arrows, the graph must contain a directed cycle; in fact, this directed cycle must be unique. The same argument allows use to decompose the graph into a set of components, one for each vertex in the directed cycle. But then each component is in fact a rooted tree. We can thus identify $\mathfrak{C} \equiv \mathcal{C}[\mathcal{R}]$ where $\mathcal{R} = \mathcal{T}^\bullet$ is the class of rooted trees and $\mathcal{C}$ is the class of cyclic permutations. Passing to EGFs, we have

$$\begin{aligned} F(x,y) &= \exp(yC(x)) \\ C(x) &= \log\left(\frac{1}{1-R(x)}\right) \\ R(x) &= x\exp(R(x)) \end{aligned}$$

Thus,

$$F(x,y) = \exp\left(\log\left(\left(\frac{1}{(1-R(x))}\right)^y\right)\right) = \left(\frac{1}{1-R(x)}\right)^y$$

Apply LIFT with $R(x) = x\exp(R(x))$, $G(u) = \exp(u)$, $F(u) = \frac{1}{(1-u)^y}$. Then $F'(u) = uF(u) = \frac{y}{(1-u)^{y+1}}$ Thus the total number of components among all n^n $\phi \in \mathcal{F}_n$ is

$$\begin{aligned} n![x^n]yF(x,y)|_{y=1} &= n!y\frac{1}{n}[u^{n-1}]\frac{y}{(1-u)^{y+1}}\exp(u)^n\Bigg|_{y=1} \\ &= (n-1)![u^{n-1}]\exp(nu)\left[\frac{(1-u)^{y+1} - y(y+1)(1-u)^y}{(1-u)^{2y+2}}\right]_{y=1} \\ &= (n-1)![u^{n-1}]\exp(nu)\left[\frac{(1-u)^2 - 2(1-u)}{(1-u)^4}\right] \end{aligned}$$

For each $j \geq 1$, let $M_i \subseteq \mathbb{N}$ be a set of **allowed multiplicites** (for parts of size j)

$$\lambda \mapsto \underline{m}(\lambda) = \langle m_1, m_2, m_3, \ldots \rangle$$

We require that only finitely many $j \geq 0$ have $0 \notin m_j$. Consider $\mathcal{Z} \subseteq \mathcal{Y}$ given by

$$\mathcal{Z} = \{\lambda \in \mathcal{Y} : m_j(\lambda) \in M_j \text{ for all } j \geq 1\}$$

0.5 Theorem.

$$\Phi_{\mathcal{Z}}(x,y) = \sum_{\lambda \in \mathcal{Z}} x^{n(\lambda)}y^{k(\lambda)} = \prod_{j=1}^{\infty}\left(\sum_{m \in M_j} x^{jm}y^m\right)$$

Proof Let $\hat{\mathcal{Z}} = \{\underline{m}(\lambda) : \lambda \in \mathcal{Z}\}$. For $\ell \geq 1$, let $\mathcal{M}(\ell) = \{\rho \in \mathcal{M} : r_j = 0 \text{ if } j > \ell\}$. Then $\bigcup_{\ell=1}^{\infty} \mathcal{M}(\ell) = \mathcal{M}$. Consider partitions in $\hat{\mathcal{Z}} \cap \mathcal{M}(\ell)$ when ℓ is bigger than the greatest index i such that $0 \notin M_i$. Then $\lambda \in \hat{\mathcal{Z}} \cap \mathcal{M}(\ell)$ if and only if $\underline{m}(\lambda) = \langle m_1, m_2, \ldots \rangle$ with $m_j \in M_j$ and

$m_j = 0$ if $j > \ell$. Thus $\hat{\mathcal{Z}} \cap \mathcal{M}(\ell) \leftrightharpoons M_1 \times \cdots \times M_\ell$. Thus

$$\begin{aligned}\sum_{\substack{\lambda\in\mathcal{Z}\\ \underline{m}(\lambda)\in\mathcal{M}(\ell)}} x^{n(\lambda)}y^{k(\lambda)} &= \sum_{\rho\in M_1\times\cdots\times M_\ell} x^{r_1+2r_2+\cdots+\ell r_\ell}y^{r_1+\cdots+r_\ell}\\ &= \sum_{r\in M_1} x^r y^r \sum_{r\in M_2} x^{2r}y^r \cdots \sum_{j\in M_\ell} x^{\ell r}y^r\\ &= \prod_{j=1}^{\ell}\left(\sum_{r\in M_j} x^{jr}y^r\right)\end{aligned}$$

Since $\mathcal{M}(1) \subseteq \mathcal{M}(2) \subseteq \cdots \subseteq \mathcal{M}$ and $\bigcup_{\ell=1}^\infty \mathcal{M}(\ell) = \mathcal{M}$, by taking limits,

$$\begin{aligned}\Phi_{\mathcal{Z}}(x,y) &= \lim_{\ell\to\infty} \Phi_{\hat{\mathcal{Z}}\cap\mathcal{M}(\ell)}(x,y)\\ &= \lim_{\ell\to\infty}\prod_{j=1}^{\infty}\left(\sum_{r\in\mathcal{M}_j} x^{jr}y^r\right)\\ &= \prod_{j=1}^{\infty}\left(\sum_{r\in M_j} x^{jr}y^r\right)\end{aligned}$$

■

Example. Partitions with distinct parts $\mathcal{D}$. Then $M_j = \{0,1\}$ for all $j \geq 1$. Then $\phi_{\mathcal{D}}(x,y) = \prod_{j=1}^\infty (1 + x^j y)$

Example. Partitions with only odd parts $\mathcal{O}$. Then $M_j = \{0\}$ if $j = 2i$ is even, and $M_j = \mathbb{N}$ if $j = 2i+1$ is odd. Then

$$\Phi_{\mathcal{O}}(x,y) = \prod_{i=0}^{\infty}\left(\sum_{r\in\mathbb{N}} x^{(2i+1)r}y^r\right) = \prod_{i=0}^{\infty}\frac{1}{1-x^{2i+1}y}$$

Set $y = 1$ in $\Phi_{\mathcal{D}}(x,y)$ and $\Phi_{\mathcal{O}}(x,y)$. Then

$$\begin{aligned}\Phi_{\mathcal{D}}(x,1) &= \prod_{j=1}^{\infty}(1+x^j) = \prod_{j=1}^{\infty}\frac{(1+x^j)(1-x^j)}{(1-x)^j}\\ &= \prod_{j=1}^{\infty}\frac{(1-x^{2j})}{(1-x^j)}\\ &= \prod_{i=0}^{\infty}\frac{1}{1-x^{2i+1}} = \Phi_{\mathcal{O}}(x,1)\end{aligned}$$

We thus have a bijection $\mathcal{D} \leftrightharpoons \mathcal{O}$ where if $\lambda \leftrightarrow \mu$, then $n(\lambda) = n(\mu)$ (but in general, lengths are not preserved).

Example. Let $\mathcal{A}$ denote the set of partitions in which each part occurs 0,1,4, or 5 times. Then

$$\Phi_{\mathcal{A}}(x,1) = \prod_{j=1}^{\infty}(1 + x^j + x^{4j} + x^{5j})$$

Let $\mathcal{B}$ denote the set of partitions with no parts congrudent to 2 mod 4, and parts divisible by 4 are distinct. Then

$$M_j = \begin{cases} \mathbb{N} & : j \text{ odd} \\ \{0\} & : j \equiv 2 \pmod 4 \\ \{0,1\} & : 4 \mid j \end{cases}$$

So

$$\Phi_{\mathcal{B}}(x) = \prod_{i=1}^{\infty} \left(\frac{1}{1 - x^{2i-1}} \right) (1 + x^{4i})$$

Now,

$$\begin{aligned} \Phi_{\mathcal{A}}(x) &= \prod_{j=1}^{\infty} (1 + x^j + x^{4j} + x^{5j}) \\ &= \prod_{j=1}^{\infty} (1 + x^j)(1 + x^{4j}) \\ &= \prod_{j=1}^{\infty} (1 + x^{4j}) \frac{(1 + x^j)(1 - x^j)}{(1 - x^j)} \\ &= \prod_{j=1}^{\infty} (1 + x^{4j}) \left(\frac{1 - x^{2j}}{1 - x^j} \right) \\ &= \prod_{j=1}^{\infty} \frac{(1 + x^{4j}}{1 - x^{2j-1}} = \Phi_{\mathcal{B}}(x) \end{aligned}$$