

Functional Analysis

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I. Analysis in Metric Spaces

1 TOPOLOGY

Let X denote a non-empty set, and $\mathcal{P}(X)$ denote the power set of X .

Definition. A **topology** on a set X is a set τ of subsets of X such that

- (i) $\emptyset, X \in \tau$
- (ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
- (iii) If $n \in \mathbb{N}$ and $U_i \in \tau$ for each $1 \leq i \leq n$, then $\bigcap_{i=1}^n U_i \in \tau$.

The sets $U \in \tau$ are the **open sets** in X , and sets $X \setminus U$ for some open set U are the **closed sets** in X . The pair (X, τ) is called a **topological space**.

Example. (i) *Sorgenfrey line:* Set $X = \mathbb{R}$, and consider

$$\sigma = \{ V \subseteq \mathbb{R} \mid \text{for any } s \in V, \text{ there is } \delta = \delta(s) > 0 \text{ s.t. } [s, s + \delta) \subseteq V \}$$

It is a straightforward exercise to verify that $\tau_{|\cdot|} \subsetneq \sigma$. We say that σ is **finer** than $\tau_{|\cdot|}$.

- (ii) *Relative or subset topology:* let (X, τ) be a topological space and $\emptyset \neq A \subseteq X$. Then we can define a topology $\tau|_A = \{U \cap A : U \in \tau\}$.

1.1 METRIC TOPOLOGY

A metric space (X, d) is naturally a topological space, where the topology is given by

$$\tau_d = \{ U \subseteq X \mid \text{for each } x_0 \in U, \text{ there is } \delta = \delta(x_0) \text{ s.t. } B_\delta(x_0) \subseteq U \}.$$

Given two metrics d, ρ on X , we say that $d \sim \rho$ are **equivalent** if and only if there are $c, C > 0$ such that

$$cd(x, y) \leq \rho(x, y) \leq Cd(x, y) \text{ for any } x, y \in X$$

Note that $d \sim \rho$ implies that $\tau_d = \tau_\rho$, but the reverse implication is not true. An example of this are the metrics on $X = \mathbb{R}$ given by $d(x, y)$ and $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$. Then $d \sim \rho$ but $\tau_d = \tau_\rho$. Let $(X, d), (Y, \rho)$ be metric spaces. A map $f : X \rightarrow Y$ is an **isometry** if for any $x, y \in X$, $d(x, y) = \rho(f(x), f(y))$. By non-degeneracy, f is automatically injective. In particular, when (X, d) is complete, then $(f(X), \rho|_{f(X)})$ is a complete metric space.

Definition. Let (X, τ) and (Y, σ) be topological spaces, and $f : X \rightarrow Y$. We say that f is **$(\tau - \sigma)$ -continuous at x_0** in X if for any $V \in \sigma$ such that $f(x_0) \in V$, then there exists $U \in \tau$ such that $x_0 \in U$ and $f(U) \subseteq V$. We say that f is **$(\tau - \sigma)$ -continuous** if it is continuous at each x_0 in X .

An easy application of definitions yields the following:

1.1 Proposition. Let $(X, \tau), (Y, \sigma)$ be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if for any $U \in \sigma$, $f^{-1}(U) \in \tau$.

1.2 Lemma. *If $x_0 \in X$ where (X, τ) is a topological space, then*

$$\mathcal{I}(x_0) = \{ f \in C_b(X) \mid f(x_0) = 0 \}$$

is closed, hence complete, subspace of $C_b(X)$.

PROOF If $(f_n)_{n=1}^\infty \subseteq \mathcal{I}(x_0)$ and $f = \lim_{n \rightarrow \infty} f_n$ with respect to $\|\cdot\|_\infty$ in $C_b(X)$, then $f(x_0) = \lim_{n \rightarrow \infty} f_n(x_0) = 0$. Thus $f \in \mathcal{I}(x_0)$, and closed subsets of complete spaces are themselves complete. ■

II. Basic Elements of Functional Analysis

2 BANACH SPACES

Throughout, we denote by \mathbb{F} either the field \mathbb{R} or the field \mathbb{C} .

Definition. Let X be a vector space over \mathbb{F} . A **seminorm** is a functional $\|\cdot\| : X \rightarrow \mathbb{R}$ such that it is

- (non-negative) $\|x\| \geq 0$ for any $x \in X$
- (subadditive) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$
- ($|\cdot|$ -homogenous) $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{F}$, $x \in X$.

If in addition, $\|\cdot\|$ satisfies the added requirement

- (non-degenerate) $\|x\| = 0$ if and only if $x = 0$

we call $\|\cdot\|$ a **norm** for X . In this case, the pair $(X, \|\cdot\|)$ a **normed vector space**. We say that $(X, \|\cdot\|)$ is a **Banach space** provided that X is complete with respect to the metric $\rho(x, y) = \|x - y\|$ induced by the norm.

Example. Here are some standard examples of Banach spaces:

- (i) $(\mathbb{F}, |\cdot|)$ is probably the simplest example of a Banach space.
- (ii) *Finite-dimensional space:* denoted $(\mathbb{F}^d, \|\cdot\|_p)$ with points $x = (x_j)_{j=1}^d$ equipped with the p -norm

$$\|x\|_p = \begin{cases} \left(\sum_{j=1}^d |x_j|^p \right)^{1/p} & 1 \leq p < \infty \\ \max_{j=1, \dots, d} |x_j| & p = \infty \end{cases}$$

is a Banach space

- (iii) If you have a background in basic measure theory, the space $L_{p, \mathbb{F}}(\Omega)$, where Ω is a compact domain. For a concrete example, take for example

$$L^p \mathbb{F}([0, 1]) = \left\{ f : [0, 1] \rightarrow \mathbb{F} \mid f \text{ is Lebesgue measurable, } \left(\int_0^1 |f|^p \right)^{1/p} < \infty \right\} \Big/ \sim_{\text{a.e.}}$$

where $1 \leq p < \infty$. To enforce non-degeneracy, we must mod out by equivalence almost everywhere.

- (iv) The space of essentially bounded functions, $L_{\infty}^{\mathbb{F}}[0, 1]$, $\|f\|_{\infty} = \text{ess sup}_{t \in [0, 1]} |f(t)|$.

- (v) *Function spaces:* let (X, d) be a metric space, and define

$$C_b(X, \mathbb{F}) = \{ f : X \rightarrow \mathbb{F} \mid f \text{ is continuous and bounded} \}, \quad \|f\|_{\infty} = \sup_{x \in X} |f(x)|.$$

Here, we define a more involved example.

Example. Let (X, d) be a metric space. We define the space of *Lipschitz functions*

$$\text{Lip}_{\mathbb{F}}(X, d) = \left\{ f : X \rightarrow \mathbb{F} \mid f \text{ is bounded, } L(f) = \sup_{\substack{x, y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\}$$

Note that for any $f : X \rightarrow \mathbb{F}$, $f \in \text{Lip}_{\mathbb{F}}(X, d)$ if and only if there is some $L \geq 0$ such that $|f(x) - f(y)| \leq Ld(x, y)$ for all x, y in X . One may verify that $L(f)$ is the infimum over all values of L for which this inequality holds over X .

It is an easy exercise to see that $\text{Lip}_{\mathbb{F}}(X, d)$ is a vector space and that $L : \text{Lip}_{\mathbb{F}}(X, d) \rightarrow \mathbb{R}$ is a seminorm. However, we do not have non-degeneracy - for example, if f is constant, then $L(f) = 0$. To define a norm on the space of Lipschitz functions, we essentially force non-degeneracy by construction: we define the *Lipschitz norm*

$$\|f\|_{\text{Lip}} = \|f\|_{\infty} + L(f).$$

In this case, we do in fact have what we want:

2.1 Proposition. $(\text{Lip}_{\mathbb{F}}(X, d), \|\cdot\|_{\text{Lip}})$ is a Banach space.

PROOF Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $(\text{Lip}_{\mathbb{F}}(X, d), \|\cdot\|_{\text{Lip}})$. Since $\|\cdot\|_{\infty} \leq \|\cdot\|_{\text{Lip}}$ on $\text{Lip}_{\mathbb{F}}(X, d)$, this sequence is uniformly Cauchy and hence converges to some $f \in C_b(X, \mathbb{F})$ with respect to the uniform norm. Moreover, if $x, y \in X$, then

$$\begin{aligned} |f(x) - f(y)| &= \lim_{n \rightarrow \infty} |f_n(x) - f_n(y)| \leq \sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)| \\ &\leq \sup_{n \in \mathbb{N}} L(f_n) d(x, y) \leq \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Lip}} d(x, y). \end{aligned}$$

Since Cauchy sequences are bounded in norm, we have that $|f(x) - f(y)| \leq Ld(x, y)$ where $L = \sup_{n \in \mathbb{N}} \|f_n\|_{\text{Lip}} < \infty$, so in fact $f \in \text{Lip}_{\mathbb{F}}(X, d)$. It is easy to verify that $\lim_{n \rightarrow \infty} \|f - f_n\|_{\text{Lip}} = 0$. \blacksquare

2.1 SEQUENCE SPACES

Since we do not assume the background of measure theory in this treatment, one of our main basic examples of Banach spaces will be the sequence spaces. Let $\mathbb{F}^{\mathbb{N}}$ denote the set of all sequences in \mathbb{F} , and define

$$\ell^1 = \left\{ x = (x_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_1 = \sum_{j=1}^{\infty} |x_j| < \infty \right\}.$$

It is easy to see that $(\ell^1, \|\cdot\|_1)$ is a normed vector space.

More generally, for $1 < p < \infty$, we may define

$$\ell^p = \left\{ x \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} < \infty \right\}.$$

As always, it is easy to verify that the ℓ^p -spaces, for $1 \leq p < \infty$, are in fact normed vector spaces. The interesting work is in proving that they are Banach spaces.

Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then q is called the **conjugate index** to p . We have a number of standard inequalities on ℓ_p -spaces, the proofs of which can be found in general in [TODO: eventually link measure theory result].

2.2 Proposition. (Inequalities in ℓ^p -spaces) Throughout, let $1 < p, q < \infty$ be conjugate exponents.

- **Young's Inequality:** If $a, b \geq 0$ in \mathbb{R} , then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with equality if and only if $a^p = b^q$.
- **Hölder's Inequality:** If $x \in \ell^p$ and $y \in \ell^q$, then $xy = (x_i y_i)_{i=1}^\infty \in \ell_1$, with

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

Note that equality holds if and only if the following two conditions hold:

- (i) $\text{sgn}(x_i y_i) = \text{sgn}(x_k y_k)$ for all $j, k \in \mathbb{N}$ where $x_i y_i \neq 0 \neq x_k y_k$, and
- (ii) $|x|^p = (|x_j|^p)_{j=1}^\infty$ and $|y|^q$ are linearly dependent in ℓ_1 .
- **Minkowski's Inequality:** If $x, y \in \ell_p$, then $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ with equality exactly when one of x or y is a non-negative scalar combination of the other.

In particular, Minkowski's Inequality [TODO: cite certain labels by name? and also link - would be nice]

2.2 BOUNDED CONTINUOUS FUNCTIONS INTO A NORMED SPACE

Let $(Y, \|\cdot\|)$ be a normed space and $\tau = \tau_{\|\cdot\|}$ the topology induced by $\|\cdot\|$. Let (X, τ) be any topological space. We define the space

$$C_b(X, Y) = \{f : X \rightarrow Y \mid f \text{ is bounded and } \tau - \tau_{\|\cdot\|} \text{ - continuous}\}$$

With pointwise operations, we see that $C_b(X, Y)$ is a vector space. We also define for $f \in C_b(X, Y)$, $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$, making $(C_b(X, Y), \|\cdot\|_\infty)$ a normed vector space.

2.3 Theorem. If $(Y, \|\cdot\|)$ is a Banach space, then $(C_b(X, Y), \|\cdot\|_\infty)$ is a Banach space.

PROOF Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(C_b(X, Y), \|\cdot\|_\infty)$. Then for any $x \in X$, we have that $(f_n(x))_{n=1}^\infty$ is Cauchy in $(Y, \|\cdot\|)$ since $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|_\infty$, and hence admits a limit $f(x)$. This defines a pointwise limit $f : X \rightarrow Y$. Fix $x_0 \in X$: we must show that f is continuous at x_0 . Given $\epsilon > 0$, set

- n_1 so that whenever $n, m \geq n_1$, $\|f_n - f_m\|_\infty < \epsilon/4$.
- n_2 so that whenever $n \geq n_2$, $\|f_n(x_0) - f(x_0)\| < \epsilon/4$.
- $N = \max\{n_1, n_2\}$.
- $U \in \tau$, $x_0 \in U$ such that $f_N(U) \subseteq B_{\epsilon/4}(f(x_0)) \subset Y$.

Then for $x \in U$, we let n_x be so $n_x \geq n_1$ and $n \geq n_x$, so that $\|f_n(x) - f(x)\| < \epsilon/4$. We then have

$$\begin{aligned} \|f(x) - f(x_0)\| &\leq \|f(x) - f_{n_x}(x)\| + \|f_{n_x}(x) - f_N(x)\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &< \frac{\epsilon}{4} + \|f_{n_x} - f_N\|_\infty + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon \end{aligned}$$

in other words that $f(U) \subseteq B_\epsilon(f(x_0))$ so that f is continuous.

Now let us check that $\|f\|_\infty < \infty$. Since $|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$, $(\|f_n\|_\infty)_{n=1}^\infty \subseteq \mathbb{R}$ is Cauchy, hence bounded. If $x \in X$, then

$$\|f(x)\| = \lim_{n \rightarrow \infty} \|f_n(x)\| \leq \sup_{n \in \mathbb{N}} \|f_n(x)\| \leq \sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$$

so $\|f\|_\infty = \sup_{x \in X} \|f(x)\| < \infty$.

Finally, to show that the limit indeed converges appropriately, if ϵ, n_1, x_0, N are as above, we have for $n \geq n_1$

$$\|f_n(x_0) - f(x_0)\| \leq \|f_n(x_0) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\epsilon}{2}$$

so $\|f_n - f\|_\infty = \sup_{x_0 \in X} \|f_n(x_0) - f(x_0)\| \leq \epsilon/2 < \epsilon$. The convergence is uniform since n_1 is chosen uniformly in X . ■

2.4 Corollary. $(C_b(X, \mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

Example. (i) Let T be a non-empty set and let

$$\ell^\infty(T) = \{x = (x_t)_{t \in T} \in \mathbb{F}^T \mid \|x\|_\infty\} < \infty$$

With pointwise operations, $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space. In fact, it is a Banach space, since

$$f \mapsto (f(t))_{t \in T} : C_b(T, \mathcal{P}(T)) \rightarrow \ell_\infty(T)$$

is a surjective linear isometry, and the result follows.

(ii) Let $c = \{x \in \ell_\infty \mid \lim_{n \rightarrow \infty} x_n \text{ exists}\}$. Then $(c, \|\cdot\|_\infty)$ is a Banach space. Consider the topological space given by $\omega = \mathbb{N} \cup \{\infty\}$, with topology

$$\tau_\omega = \mathcal{P}(\mathbb{N}) \cup \bigcup_{n \in \mathbb{N}} \{k \in \mathbb{N} : k \geq n\}$$

The map $f \mapsto (f(n))_{n=1}^\infty : C_b(\omega) \rightarrow c$ is a linear surjective isometry.

(iii) Recall that $\mathcal{I}(\infty)$ is a closed, and hence complete, subspace of c . We may define $c_0 = \{x \in \mathbb{F}^\mathbb{N} \mid \lim_{n \rightarrow \infty} x_n = 0\} \subseteq c \subseteq \ell_\infty$. In this case, $f \mapsto (f(n))_{n=1}^\infty : \mathcal{I}(\infty) \rightarrow c_0$ is a (linear) surjective isometry.

(iv) Consider the Sorgenfrey line (\mathbb{R}, σ) . One may verify that

$$C_b((\mathbb{R}, \sigma), \mathbb{F}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{F} \mid f \text{ is bounded and } \lim_{t \rightarrow t_0^+} f(t) = f(t_0) \text{ for } t \in \mathbb{R} \right\}$$

3 LINEAR FUNCTIONALS AND OPERATORS

Let X, Y be vector spaces. We let $\mathcal{L}(X, Y) = \{S : X \rightarrow Y \mid S \text{ is linear}\}$; this is itself a vector space with pointwise operations. Let $(X, \|\cdot\|)$ be a normed space. We denote

$$D(X) = \{x \in X : \|x\| < 1\}$$

$$S(X) = \{x \in X : \|x\| = 1\}$$

$$B(X) = \{x \in X : \|x\| \leq 1\}$$

(Yes, this notation is confusion. No, I didn't choose it.)

3.1 Proposition. If X, Y are normed spaces and $S \in \mathcal{L}(X, Y)$, then the following are equivalent:

- (i) S is continuous
- (ii) S is continuous at some $x_0 \in X$
- (iii) $\|S\| = \sup_{x \in D(X)} \|Sx\| < \infty$.

Moreover, in this case, we have

$$\begin{aligned} \|S\| &= \min\{L > 0 : \|Sx\| \leq L\|x\| \text{ for } x \in X\} \\ &= \sup_{x \in S(X)} \|Sx\| = \sup_{x \in B(X)} \|Sx\| \end{aligned}$$

PROOF ($i \Rightarrow ii$) By definition.

($ii \Rightarrow iii$) Note that

$$Sx_0 + D(Y) = \{Sx_0 + y : y \in D(Y)\} = \{y \in Y : \|Sx_0 - y\| < 1\}$$

is a neighbourhood of Sx_0 . By the definition of metric continuity, there is $\delta > 0$ such that

$$x_0 + \delta D(X) = \{x_0 + \delta x : x \in D(X)\} = \{x' \in X : \|x_0 - x'\| < \delta\}$$

such that

$$Sx_0 + \delta S(D(X)) = S(x_0 + \delta D(X)) \subseteq Sx_0 + D(Y)$$

which implies that $\delta S(D(X)) \subseteq D(Y)$ and $S(D(X)) \subseteq D(Y)/\delta$, in other words that $\|Sx\| \leq 1/\delta$ for $x \in D(X)$.

($iii \Rightarrow i$) If $x \in X$ and $\epsilon > 0$, then

$$\|Sx\| = (\|x\| + \epsilon) \left\| S \left(\frac{1}{\|x\| + \epsilon} x \right) \right\| \leq (\|x\| + \epsilon) \|S\|$$

Then, letting $\epsilon \rightarrow 0^+$, we see that

$$\|Sx\| \leq \|x\| \|S\| = \|S\| \|x\|$$

If $x, x' \in X$, then $\|Sx - Sx'\| \leq \|S\| \|x - x'\|$ is S is Lipschitz, hence continuous.

To complete the proof, the content of (iii) implies (i) tell us that the Lipschitz constant $L(S) \leq \|S\|$. Furthermore, if $\|x\| = 1$, the preceding proof gives us that $\|S\|_{S(X)}$.

Conversely,

$$\|S\| = \sup_{x \in D(X) \setminus \{0\}} \|Sx\| = \sup_{x \in D(X) \setminus \{0\}} \|x\| \left\| S \left(\frac{1}{\|x\|} x \right) \right\| \leq \sup_{x \in S(X)} \|Sx\|$$

The remaining equivalence is obvious. ■

We now let $\mathcal{B}(X, Y) = \{S \in \mathcal{L}(X, Y) \mid S \text{ is bounded}\}$. We will see that $\|\cdot\|$, above, defines a norm on $\mathcal{B}(X, Y)$.

3.2 Theorem. If X, Y are normed spaces, then $(\mathcal{B}(X, Y), \|\cdot\|)$ is a normed space. Furthermore, if Y is a Banach spaces, then so to is $(\mathcal{B}(X, Y), \|\cdot\|)$.

PROOF Define

$$\Gamma : \mathcal{B}(X, Y) \rightarrow C_b^Y(B(X))$$

given by $\Gamma(S) = S|_{B(X)}$. Then, by definition, Γ is linear, with

$$\|\Gamma(S)\|_\infty = \sup_{x \in B(X)} \|Sx\| = \|S\|$$

Thus $\|\cdot\|$ is a norm: if $S, T \in \mathcal{B}(X, Y)$, $\alpha \in \mathbb{F}$,

$$\begin{aligned} \|S + T\| &= \|\Gamma(S + T)\|_\infty = \|\Gamma(S) + \Gamma(T)\|_\infty \leq \|\Gamma(S)\|_\infty + \|\Gamma(T)\|_\infty = \|S\| + \|T\| \\ \|\alpha S\| &= \|\Gamma(\alpha S)\|_\infty = |\alpha| \|\Gamma(S)\|_\infty = |\alpha| \|S\|. \end{aligned}$$

Furthermore, $\Gamma : \mathcal{B}(X, Y) \rightarrow C_b^Y(B(X))$ is an isometry.

Now suppose that Y is a Banach space. We will show that $\Gamma(\mathcal{B}(X, Y))$ is closed in $C_b^Y(B(X))$, and hence $\mathcal{B}(X, Y) = \Gamma^{-1}(\Gamma(\mathcal{B}(X, Y)))$ is complete. Let $(S_n)_{n=1}^\infty \subset \mathcal{B}(X, Y)$ be $\|\cdot\|$ -Cauchy. Then $(\Gamma(S_n))_{n=1}^\infty$ is $\|\cdot\|_\infty$ -Cauchy in $C_b^Y(B(X))$, and hence there is $f \in C_b^Y(B(X))$ such that $\lim_{n \rightarrow \infty} \|\Gamma(S_n) - f\|_\infty = 0$. Then we let $S : X \rightarrow Y$ be given by

$$Sx = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

If $x, x' \in X$ and $\alpha \in \mathbb{F}$ are all such that $x, x', x + \alpha x' \neq 0$, then

$$\begin{aligned} S(x + \alpha x') &= \|x + \alpha x'\| f\left(\frac{1}{x + \alpha x'}(x + \alpha x')\right) \\ &= \|x + \alpha x'\| \lim_{n \rightarrow \infty} S_n\left(\frac{1}{x + \alpha x'}(x + \alpha x')\right) \\ &= \lim_{n \rightarrow \infty} (S_n x + \alpha S_n x') = \lim_{n \rightarrow \infty} \left[\|x\| S_n\left(\frac{1}{\|x\|}x\right) + \alpha \|x'\| S_n\left(\frac{1}{\|x'\|}x'\right) \right] \\ &= \|x\| f\left(\frac{x}{\|x\|}\right) + \alpha \|x'\| f\left(\frac{x'}{\|x'\|}\right) \\ &= Sx + \alpha Sx' \end{aligned}$$

The above computation is easily performed if any of $x, x', x + \alpha x'$ are 0. Hence $S \in \mathcal{L}(X, Y)$. We see that S is continuous (say, at a point on $S(X)$), so $S \in \mathcal{B}(X, Y)$. Finally, as $S|_{B(X)} = f = \lim_{n \rightarrow \infty} S_n|_{B(X)}$ (with respect to the uniform norm), we have

$$\|S - S_n\| = \sup_{x \in B(X)} \|(S - S_n)x\| = \|f - \Gamma(S_n)\|_\infty$$

goes to 0 as n goes to infinity. ■

Definition. Given a vector space X , let $X' = \mathcal{L}(X, \mathbb{F})$ denote the **algebraic dual**. If further X is a normed space, we let $X^* = \mathcal{B}(X, \mathbb{F})$ denote the (continuous) dual.

3.3 Corollary. If X is a normed spaces, then X^* is always a Banach space.

3.4 Theorem. Let for $x \in \ell_1$, $f_x : c_0 \rightarrow \mathbb{F}$ be given by $f_x(y) = \sum_{j=1}^\infty x_j y_j$. Then $f_x \in c_0^*$ with $\|f_x\| = \|x\|_1$. Furthermore, every element of c_0^* arises as above.

PROOF If $x \in \ell_1$ and $y \in c_0 \subseteq \ell_\infty$, then

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \sum_{j=1}^{\infty} |x_j| \|y\|_\infty = \|x\|_1 \|y\|_\infty < \infty$$

so $f_x(y) = \sum_{j=1}^{\infty} x_j y_j$ is well-defined. It is obvious that f_x is linear: $f_x(y + \alpha y') = f_x(y) + \alpha f(y')$ for $y, y' \in c_0$ and $\alpha \in \mathbb{F}$. Also, $\|f_x\| \leq \|x\|_1$. We let $y^n = (\overline{\text{sgn } x}, \dots, \overline{\text{sgn } x_n}, 0, 0, \dots) \in c_0$, with $\|y^n\| = 1$. Then

$$\|f_x\| \geq |f_x(y^n)| = \sum_{j=1}^n x_j \overline{\text{sgn } x_j} = \sum_{j=1}^n |x_j|$$

so that $\|f_x\| \geq \|x\|_1$, and hence equality holds.

Now let $f \in c_0^*$, and write $e_n = (0, \dots, 0, 1, 0, 0, \dots) \in c_0$, and let $x_n = f(e_n)$. Then, let $y \in c_0$ and $y^n = (y_1, \dots, y_n, 0, 0, \dots)$ and we have

$$\|y - y^n\|_\infty = \sup_{j \geq n+1} |y_j|$$

which goes to 0 as n goes to infinity. Then since f is continuous, we have

$$f(y) = \lim_{n \rightarrow \infty} f(y^n) = \lim_{n \rightarrow \infty} \sum_{j=1}^n y_j x_j = \sum_{j=1}^{\infty} x_j y_j = f_x(y)$$

We use sequence $(y^n)_{n=1}^\infty$ as in $y^n \in c_0$, to see that

$$\sum_{j=1}^n |x_j| = |f(y^n)| \leq \|f\| < \infty$$

so $x \in \ell_1$. Thus $f = f_x$, as desired. ■

3.5 Corollary. $\ell_1 \cong c^*$ isometrically isomorphically.

PROOF For $y \in c$, let $L(y) = \lim_{n \rightarrow \infty} y_n$. Given $y \in c$, let $y^n = (y_1, \dots, y_n, L(y), L(y), \dots) \in c$. Notice that $\|y - y^n\|_\infty \rightarrow 0$ similarly as above.

We let $1 = (1, 1, \dots)$, and $1_n = (0, \dots, 0, 1, 1, \dots)$. If $m < n$, then $1_n - 1_m \in c_0$, so

$$|f(1_n) - f(1_m)| = |f_x(1_n - 1_m)| \leq \sum_{j=m+1}^n |x_j|$$

so that $(f(1_n))_{n=1}^\infty$ is Cauchy in \mathbb{F} . Let $x_0 = \lim_{n \rightarrow \infty} f(1_n)$. Let $\tilde{x} = (x_0, x_1, \dots) \in \ell_1$. Then letting $x_j = f(e_j)$, we see that

$$f(y) = \lim_{n \rightarrow \infty} f(y^n) = \sum_{j=1}^{\infty} x_j y_j + x_0 L(y) \quad \blacksquare$$

Similarly as above, we may show that $\|f\| = \|\tilde{x}\|_1$.

Remark. We write $c_0^* \cong \ell_1$ isometrically.

3.6 Corollary. $(\ell_1, \|\cdot\|_1)$ is complete.

4 AXIOM OF CHOICE AND THE HAHN-BANACH THEOREM

Definition. Let S be a non-empty set. A **partial ordering** is a binary relation \leq on S which satisfies for $s, t, n \in S$,

- (i) (*reflexivity*) $s \leq s$
- (ii) (*transitivity*) $s \leq t, t \leq u$ implies $s \leq u$
- (iii) (*anti-symmetry*) $s \leq t, t \leq s$ implies $s = t$

We call the pair (S, \leq) a **partially ordered set**. We say that (S, \leq) is **totally ordered** if, given $s, t \in S$, at least one of $s \leq t$ or $t \leq s$ holds. We say that (S, \leq) is **well-ordered** if given any $\emptyset \neq S_0 \subseteq S$, there is some $s_0 \in S_0$ such that $s_0 \leq s$ for $s \in S_0$. A **chain** in a poset (S, \leq) is any $\emptyset \neq C \subseteq S$ such that $(S, \leq|_C)$ is totally ordered.

Example. (i) $X \neq \emptyset, (\mathcal{P}(X), \subseteq)$ is a poset
 (ii) (\mathbb{R}, \leq) is a totally ordered set
 (iii) $(\mathbb{N}, \leq), (\omega = \mathbb{N} \cup \{\infty\}, \leq)$, are well-ordered sets.

4.1 Theorem. *The following are equivalent:*

- (i) (*Axiom of Choice 1*): For any $X \neq \emptyset$, there is a function $\gamma : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$ such that $\gamma(A) \in A$ for each $A \in \mathcal{P}(X) \setminus \{\emptyset\}$.
- (ii) (*Axiom of Choice 2*): Given any $\{A_\lambda\}_{\lambda \in \Lambda}$ where $A_\lambda \neq \emptyset$ for each λ ,

$$\prod_{\lambda \in \Lambda} A_\lambda = \{(a_\lambda)_{\lambda \in \Lambda} : a_\lambda \in A_\lambda \text{ for each } \lambda\} \neq \emptyset$$

- (iii) (*Zorn's Lemma*): In a poset (S, \leq) , if each chain $C \subseteq S$ admits an upper bound in S , then (S, \leq) admits a maximal element.
- (iv) (*Well-ordering principle*): Any $S \neq \emptyset$ admits a well-ordering

PROOF Exercise. ■

Definition. Let X be a vector space (over k). A subset $S \subseteq X$ is called

- **linearly independent** if for any distinct $x_1, \dots, x_n \in S$, the equation $0 = \alpha_1 x_1 + \dots + \alpha_n x_n = 0$ where $\alpha_i \in k$ implies $\alpha_1 = \dots = \alpha_n = 0$.
- **spanning** if each $x \in X$ admits $x_i \in S$ and $\alpha_i \in k$ such that $x = \alpha_1 x_1 + \dots + \alpha_n x_n$.
- **Hamel basis** if it is both linearly independent and spanning

4.2 Proposition. *Any vector space X admits a Hamel basis.*

PROOF Let $\mathcal{L} = \{L \subseteq X : L \text{ is linearly independent}\}$. Then (\mathcal{L}, \subseteq) is a poset. Verify that for any chain $\mathcal{C} \subseteq \mathcal{L}$, that $U = \bigcup_{L \in \mathcal{C}} L \in \mathcal{L}$ and is an upper bound for \mathcal{C} . Apply Zorn to find a maximal element M in (\mathcal{L}, \subseteq) . Verify that M is spanning for X . ■

4.3 Corollary. *If X is an infinite dimensional normed space, then there exists $f \in X' \setminus X^*$.*

PROOF Our assumption provides $\{e_n\}_{n=1}^\infty$ which is linearly independent. By normalizing each element, we may and will suppose that each $\|e_n\| = 1$. Let

$$\text{span}\{e_n\}_{n=1}^\infty = \left\{ \sum_{j=1}^m \alpha_j e_{n_j} : m \in \mathbb{N}, \alpha_j \in \mathbb{F}, n_1 < \dots < n_m \right\}$$

and let B be any linearly independent set containing $\{e_n\}_{n=1}^\infty$. Define $f : X = \text{span } B \rightarrow \mathbb{F}$ be given for $x = \sum_{b \in B \setminus \{e_n\}_{n=1}^\infty} \alpha_b b + \sum_{j=1}^n \alpha_j e_{n_j}$ by $f(x) = \sum_{j=1}^m \alpha_j n_j$. The point is that $f(e_n) = n$ and $f(e) = 0$ for any other $e \in B$. Notice that

$$\|f\| = \sup_{x \in B(X)} |f(x)| \geq \sup_{n \in \mathbb{N}} |f(e_n)| = \sup_{n \in \mathbb{N}} n = \infty \quad \blacksquare$$

Definition. Let X be a \mathbb{R} -vector space. A **sublinear functional** is any $\rho : X \rightarrow \mathbb{R}$ such that it satisfies

- (non-negative homogeneity) $\rho(tx) = t\rho(x)$ for $t \geq 0, x \in X$.
- (subadditivity) $\rho(x+y) \leq \rho(x) + \rho(y)$ for $x, y \in X$.

4.4 Theorem. (Hahn-Banach) Let X be a \mathbb{R} -vector space, $\rho : X \rightarrow \mathbb{R}$ a sublinear functional, $Y \subseteq X$ a subspace and $f \in Y'$ such that $f \leq \rho|_Y$. Then there exists $F \in X'$ such that $F|_Y = f$ and $F \leq \rho$ on X .

PROOF We first do this for extensions by a single point $x \in X \setminus Y$. We wish to find $c \in \mathbb{R}$ such that

$$f(y) + \alpha c \leq \rho(y + \alpha x)$$

for $y \in Y$ and $\alpha \in \mathbb{R}$. In this case, we let $F : \text{span } Y \cup \{x\} \rightarrow \mathbb{R}$ be given by $F(y + \alpha x) = f(y) + \alpha c$, and we have that F is linear and satisfies $F \leq \rho$ on $\text{span } Y \cup \{x\}$. To do this, let y_+, y_- in Y and observe that $f(y_+) + f(y_-) = f(y_+ + y_-) \leq \rho(y_+ + y_-) \leq \rho(y_+ + x) + \rho(y_- - x)$ so that $f(y_-) - \rho(y_- - x) \leq \rho(y_+ + x) - f(y_+)$. It thus follows that

$$\sup\{f(y) - \rho(y - x) : y \in Y\} \leq \inf\{\rho(y + x) - f(y) : y \in Y\}$$

so we may find $c \in \mathbb{R}$ for which

$$\sup\{f(y) - \rho(y - x) : y \in Y\} \leq c \leq \inf\{\rho(y + x) - f(y) : y \in Y\}$$

If $t > 0$, then for $y \in Y$,

$$c \leq \rho\left(\frac{1}{t}y + x\right) - f\left(\frac{1}{t}y\right) \Rightarrow tc \leq \rho(y + tx) - f(y) \Rightarrow f(y) + tc \leq \rho(y + tx)$$

and if $s > 0$, then for $y \in Y$,

$$f\left(\frac{1}{s}y\right) - \rho\left(\frac{1}{s}y - x\right) \leq c \Rightarrow sc \leq f(y) - \rho(y - sx) \Rightarrow f(y) - sc \leq \rho(y - sx)$$

Clearly, $f(y) + 0 \leq \rho(y + 0x)$. Hence, we have our desired inequality.

We now use Zorn's lemma to lift this result to the whole space. Consider the set of “ p -extensions” of f ,

$$\mathcal{E} = \{(\mathcal{M}, \psi) \mid Y \subseteq \mathcal{M} \subseteq X, \mathcal{M} \text{ is a subspace, } \psi \in \mathcal{M}', \psi|_Y = f, \psi \leq \rho|_{\mathcal{M}}\}$$

Define a partial order on \mathcal{E} by

$$(\mathcal{M}, \psi) \leq (\mathcal{N}, \phi) \text{ iff } \mathcal{M} \subseteq \mathcal{N}, \phi|_{\mathcal{M}} = \psi$$

Suppose $\mathcal{C} \subseteq \mathcal{E}$ is a chain with respect to \leq . We let

- $\mathcal{U} = \bigcup_{(\mathcal{M}, \psi)} \mathcal{M}$ which is a subspace, since \mathcal{C} is a chain.
- and define $\phi : \mathcal{U} \rightarrow \mathbb{R}$ by $\phi(x) = \psi(x)$ whenever $x \in \mathcal{M}$, which is again well-defined since \mathcal{C} is a chain.

Furthermore, we see that $\phi \in \mathcal{U}'$, since if $x, y \in \mathcal{U}$, get $x \in \mathcal{M}$, $y \in \mathcal{N}$ for some $(\mathcal{M}, \psi) \leq (\mathcal{N}, \psi') \in \mathcal{C}$. Then $\phi(x + y) = \psi'(x + y) = \psi'(x) + \psi'(y) = \phi(x) + \phi(y)$, etc. Likewise, $\psi \leq \phi|_{\mathcal{U}}$. Thus by Zorn's lemma, \mathcal{E} admits a maximal element \mathcal{M}, F . Then $\mathcal{M} = X$, for if not, then we would find $x \in X \setminus \mathcal{M}$ and we apply step one to $\text{span } \mathcal{M} \cup \{x\}$ to get F' , a strictly larger element violating maximality. ■

Trivially, any \mathbb{C} -vector space is a \mathbb{R} -vector space.

4.5 Lemma. *Let X be a \mathbb{C} -vector space.*

- (i) *If $f \in X'_{\mathbb{R}}$ into \mathbb{R} , then define $f_{\mathbb{C}}$ given by $f_{\mathbb{C}}(x) = f(x) - if(ix)$ defines an element of $X' = X'_{\mathbb{C}}$.*
- (ii) *If $g \in X'$, then $f = \text{Re } g$ in $X'_{\mathbb{R}}$ satisfies $g = f_{\mathbb{C}}$.*
- (iii) *If X is a normed \mathbb{C} -vector space, then for $f \in X'_{\mathbb{R}}$,*

$$f \in X'_{\mathbb{R}} \text{ if and only if } f_{\mathbb{C}} \in X^* = X^*_{\mathbb{C}} \text{ with } \|f\| = \|f_{\mathbb{C}}\|$$

PROOF (i) and (ii) are straightforward exercises; let's see (iii). We let for $x \in X$, $z = \text{sgn } f_{\mathbb{C}}(x)$. Then

$$\begin{aligned} \mathbb{R} \ni |f_{\mathbb{C}}(x)| &= \bar{z} f_{\mathbb{C}}(x) = f_{\mathbb{C}}(\bar{z}x) = \text{Re } f_{\mathbb{C}}(\bar{z}x) = f(\bar{z}x) = |f(\bar{z}x)| \\ &\leq \|f\| \|\bar{z}x\| = \|f\| |\bar{z}| \|x\| = \|f\| \|x\| \end{aligned}$$

so we see that $\|f_{\mathbb{C}}\| \leq \|f\|$. Conversely,

$$|f(x)| = |\text{Re } f_{\mathbb{C}}(x)| \leq |f_{\mathbb{C}}(x)| \leq \|f_{\mathbb{C}}\| \|x\| \text{ so that } \|f\| \leq \|f_{\mathbb{C}}\| \quad \blacksquare$$

4.6 Corollary. *If X is a normed space, $Y \subseteq X$ a subspace and $f \in Y^*$, then there exists $F \in X^*$ such that $F|_Y = f$ and $\|F\| = \|f\|$.*

PROOF Define $\rho : X \rightarrow \mathbb{R}$ be given by $\rho(x) = \|f\| \cdot \|x\|$, so ρ is sublinear and $\text{Re } f \leq \rho|_Y$. Apply Hahn-banach to this data and get $\tilde{F} \in X^*_{\mathbb{R}}$ such that $\tilde{F}|_Y = \text{Re } f$ and $\tilde{F} \leq \rho$, and let $F = \tilde{F}_{\mathbb{C}}$. ■

4.7 Corollary. *If X is a normed space, $x \in X$, then there is $f \in X^*$ such that*

$$\|x\| = f(x) = |f(x)| \text{ and } \|f\| = 1$$

PROOF Let $f_0 : \mathbb{F}x \rightarrow \mathbb{F}$ be given by $f_0(\alpha x) = \alpha \|x\|$. If $x \neq 0$, then

$$\|f_0\| = \sup_{\|\alpha x\| \leq 1} |f_0(\alpha x)| = \sup_{\|\alpha x\| \leq 1} |\alpha| \|x\| = 1$$

and apply the previous corollary. If $x = 0$, this is trivial. ■

4.8 Theorem. *Let X be a normed space and X^{**} denote the bidual. For $x \in X$, define $\hat{x} : X^* \rightarrow \mathbb{F}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ with $\|\hat{x}\| = \|x\|$, so that $x \mapsto \hat{x} : X \rightarrow X^{**}$ is a linear isometry.*

PROOF Notice that $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$ so $\|\hat{x}\| \leq \|x\|$. The last corollary provides for $x \in X$ an $f_x \in S(X^*)$ with $|f_x(x)| = \|x\|$. Then $\|\hat{x}\| \leq |\hat{x}(f_x)| = \|x\|$. Hence $\|\hat{x}\| = \|x\|$. Clearly $x \mapsto \hat{x}$ is linear. ■

Remark. Since X^{**} , being a dual space, is complete, we have that $\hat{X} = \{\hat{x} : x \in X\}$ satisfies that its closure $\overline{\hat{X}} \subseteq X^{**}$ is complete. Hence $\overline{\hat{X}}$ is a Banach space containing a dense copy of X . Often, we will simply write $\overline{\hat{X}} = \overline{X}$ and call it the **completion** of X .

4.1 GEOMETRIC HAHN-BANACH

If $A, B \subset X$ with $A \cap B = \emptyset$ (and other suitable assumptions), we will find a \mathbb{R} -hyperplane between A and B .

Definition. In a vector space, a **hyperplane** is any set of the form $x_0 + \ker f$ with $x_0 \in X$ and $f \in X'$. Then a **\mathbb{R} -hyperplane** is any set of the form $x_0 + \ker \operatorname{Re} f$.

4.9 Proposition. Let X be a normed space.

(i) If $f \in X^* \setminus \{0\}$, then $\ker f$ is closed and nowhere dense.

(ii) if $f \in X' \setminus X^*$, then $\overline{\ker f} = X$.

Thus a hyperplane in X is either closed and nowhere dense, or it is dense.

PROOF To see (i), $\ker f = f^{-1}(\{0\})$ is a closed set since f is continuous. Furthermore, if $Y \subsetneq X$ is a proper closed subspace, then it is nowhere dense. If not, then there would exist $y_0 \in Y$ and $\delta > 0$ such that $y_0 + \delta D(X) \subseteq Y$. But then $D(X) \subseteq \frac{1}{\delta}(Y - y_0) = Y$, so $X = \operatorname{span} D(X) \subseteq Y$, a contradiction.

To see (ii), suppose that $\ker f$ is not dense in X . Then there would be $x_0 \in X$ and $\delta > 0$ such that $(x_0 + \delta D(X)) \cap \ker f = \emptyset$, so

$$0 \notin f(x_0 + \delta D(X)) = f(x_0) + \delta f(D(X)) \implies \frac{1}{\delta} f(x_0) \notin -f(D(X)) = f(D(X)) \quad (4.1)$$

But then $\|f\| \leq \frac{1}{\delta} f(x_0)$, for if $\|f\| > \frac{1}{\delta} f(x_0)$, there would be $x \in D(X)$ such that $|f(x)| > \frac{1}{\delta} |f(x_0)|$. Thus

$$\left| \frac{f(x_0)}{\delta f(x)} \right| < 1 \implies \frac{f(x_0)}{\delta f(x)} = \frac{1}{\delta} f(x)$$

contradicting the statement in (4.1). ■

Definition. Let $\emptyset \neq A \subseteq X$. We say that A is

- **convex** if for $a, b \in A$ and $0 < \lambda < 1$, $(1 - \lambda)a + \lambda b \in A$.
- **absorbing** at $a_0 \in A$ if for any $x \in X$, there is $\epsilon(a_0, x) > 0$ such that $a_0 + tx \in A$ for $0 \leq t < \epsilon$.

For example, if X is a normed space, then any open set is absorbing around any of its points.

4.10 Lemma. (Minkowski Functional) Let $A \subset X$ be a convex set containing 0 and absorbing at 0. Define $p : X \rightarrow \mathbb{R}$ by $p(x) = \inf\{t > 0 : x \in tA\}$. Then p is a sublinear functional. Moreover, we have that

- (i) $\{x \in X : p(x) < 1\} \subseteq A \subseteq \{x \in X : p(x) \leq 1\}$; and

- (ii) if X is normed and A is a neighbourhood of 0, then there is $N > 0$ such that $p(x) \leq N \|x\|$ for $x \in X$.

PROOF First note, for any $x \in X$, if A is absorbing at 0, there is $s > 0$ such that $sx \in A$, so $x \in \frac{1}{s}A$ and hence $0 \leq p(x) < \infty$.

Let's see non-negative homogeneity. Clearly $p(0) = 0$. If $s > 0$ and $x \in X$, then

$$p(sx) = \inf\{t > 0 : sx \in tA\} = \inf\left\{t > 0 : x \in \frac{t}{s}A\right\} = s \cdot \inf\left\{\frac{t}{s} > 0 : x \in \frac{t}{s}A\right\} = sp(x)$$

We also have subadditivity. First, note that if $s, t > 0$ and $a, b \in A$, then

$$sa + tb = (s+t)\left(\frac{s}{s+t}a + \frac{t}{s+t}b\right) \in (s+t)A \implies sA + tA \subseteq (s+t)A$$

by convexity, and also $(s+t)A = \{(s+t)a : a \in A\} \subseteq \{sa + tb : a, b \in A\} = sA + tA$. Thus $sA + tA = (s+t)A$. Now for $x, y \in X$, we have

$$\begin{aligned} p(x) + p(y) &= \inf\{s > 0 : x \in sA\} + \inf\{t > 0 : y \in tA\} \\ &= \inf\{s+t : s > 0, t > 0, x \in sA, y \in tA\} \\ &\geq \inf\{s+t : s > 0, t > 0, x+y \in sA + tA = (s+t)A\} \\ &= \inf\{r > 0 : x+y \in rA\} = p(x+y) \end{aligned}$$

so that p is a sublinear functional. Then

- (i) If $p(x) < 1$, then there is $0 < t < 1$ so $x \in tA$; i.e. $\frac{1}{t}x \in A$ and $x = (1-t)x + t\frac{1}{t}x \in A$. The second inclusion is obvious.
- (ii) The assumptions provide $\delta > 0$ so $\delta D(X) \subseteq A$. Then for $x \in X$ and $\epsilon > 0$,

$$x \in (\|x\| + \epsilon)D(X) = \frac{\|x\| + \epsilon}{\delta} \delta D(X) \subseteq \frac{\|x\| + \epsilon}{\delta} A$$

so $p(x) \leq \frac{\|x\| + \epsilon}{\delta}$ so $p(x) \leq \frac{1}{\delta} \|x\|$; the result follows with $N = 1/\delta$. ■

4.11 Theorem. (Hyperplane Separation) Let X be an \mathbb{F} -vector space, $A, B \subset X$ be convex with $A \cap B = \emptyset$ and A absorbing at some a_0 . Then there are $f \in X'$ and $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re} f(a) \geq \alpha \geq \operatorname{Re} f(b)$$

for $a \in A$ and $b \in B$. Moreover, if X is normed, then

- If A is a neighbourhood of a_0 , we have $f \in X^*$; and
- if A is absorbing around each of its points (for example if A is open), then we have $\operatorname{Re} f(a) > \alpha \geq \operatorname{Re} f(b)$.

PROOF We first re-centre at 0. Let $A - B = \{a - b : a \in A, b \in B\}$. Then it is easy to verify that

- (i) $A - B$ is absorbing at any $a_0 - b, b \in B$
- (ii) $A - B$ is convex
- (iii) if X is normed and A a neighbourhood of a_0 , then $A - B$ is a neighbourhood of each $a_0 - b, b \in B$; and if A is absorbing around any of its points (resp. open), then A_B is absorbing around any of its points (resp. open).

Let $x_0 = a_0 - b_0$ for some $b_0 \in V$, and set $C = x_0 - (A - B)$, so we have $0 = x_0 - x_0 \in C$. Then by the above points, C is absorbing at 0, convex, and if X is normed and A a neighbourhood of a_0 , then C is a neighbourhood of 0; and if A is absorbing at any of its points (resp. A is open), then C is absorbing at each of its points (resp. open).

Let p be the Minkowski functional of C . Notice that since $A \cap B = \emptyset$, $0 \notin A - B$ so $x_0 \notin C$. Thus by (i) of the lemma, $p(x_0) > 1$.

Let us find f and α . Let $f_0 : \mathbb{R}x_0 \rightarrow \mathbb{R}$, by $f_0(sx) = sp(x_0)$. Hence f_0 is linear and $f_0 \leq p|_{\mathbb{R}x_0}$, so by Hahn-Banach, get $f \in X'_\mathbb{R}$ such that $f \leq p$ on X . If $a \in A$ and $b \in B$, then $x_0 - (a - b) \in C$, so by (i) of the lemma, since $p(x_0) \geq 1$, we have $f(x_0 - (a - b)) \leq p(x_0 - (a - b)) \leq 1$. Thus $f(x_0) + f(b) \leq 1 + f(a)$ so in fact $f(b) \leq f(a)$. Thus there exists some $\alpha \in \mathbb{R}$ such that

$$\sup\{f(b) : b \in B\} \leq \alpha \leq \inf\{f(a) : a \in A\}$$

If $\mathbb{F} = \mathbb{R}$, we are done; otherwise, we shall replace f by f_C

For the remainder of the proof, we suppose X is a normed space, and A is a neighbourhood of a_0 . Then part (ii) of the lemma provides $N > 0$ so that $p(x) \leq N\|x\|$. Then for $x \in X$, $f(x) \leq p(x) \leq N\|x\|$ and $-f(x) = p(-x) \leq N\|-x\| = N\|x\|$ so $|f(x)| \leq N\|x\|$, in other words that $\|f\| \leq N$ and $f \in X^*$. If A is absorbing around any of its points, then $f(a) > \alpha$ for any $a \in A$. Indeed, suppose $f(a) = \alpha$. Then there would be $t > 0$ so $a + t(-x_0) \in A$. But then $\alpha \leq f(a - tx_0) = f(a) - tf(x_0) < \alpha$, a contradiction. ■

Definition. If $\emptyset \neq S \subset X$, then its **convex hull** is given by

$$\text{conv}(S) = \left\{ \sum_{j=1}^n \lambda_j x_j : n \in \mathbb{N}, x_1, \dots, x_n \in S \text{ and } \lambda_1, \dots, \lambda_n \geq 0 \text{ with } \sum_{j=1}^n \lambda_j = 1 \right\}$$

One can verify that $\text{conv}(S)$ is in fact convex, and is the smallest convex set containing S , i.e.

$$\text{conv}(S) = \bigcap \{C : S \subseteq C \subseteq X, C \text{ convex}\}$$

If X is normed, we let $\overline{\text{conv}}(S)$ denote the **closed convex hull**, i.e. the closure of the convex hull.

Definition. A **half-space** of X is any set of the form $H = \{x \in X : \text{Re } f(x) \leq \alpha\}$ for some $f \in X'$, $\alpha \in \mathbb{R}$.

If X is normed, then the last proposition shows H is closed if and only if f is bounded.

4.12 Theorem. If X is a normed vector space and $\emptyset \neq S \subset X$, then $\overline{\text{conv}}(S) = \bigcap \{H : S \subseteq H \subset X, H \text{ a closed half space}\}$.

PROOF It is immediate that $\overline{\text{conv}}(S) \subseteq \bigcap \{H : S \subseteq H \subset X, H \text{ a closed half-space}\}$. Thus suppose $x_0 \notin \overline{\text{conv}}(S)$. Then there is $\delta > 0$ such that $(x_0 + \delta D(X)) \cap \overline{\text{conv}}(S) = \emptyset$. Since $x_0 + \delta D(X)$ is open and convex, hyperplane separation gives provides $f \in X^*$ and $\alpha \in \mathbb{R}$ so $\text{Re } f(a) > \alpha \geq \text{Re } f(b)$ for $a \in x_0 + \delta D(X)$ and $b \in \overline{\text{conv}}(S)$. Then $S \subset H = \{y \in X : \text{Re } f(y) \leq \alpha\}$ but $x_0 \notin H$. ■

5 SOME APPLICATIONS OF BAIRE CATEGORY THEOREM

5.1 Theorem. (Baire Category I) If (X, d) is a complete metric space and $\{U_n\}_{n=1}^\infty$ is a countable collection of dense, open subsets, then $\bigcap_{n=1}^\infty U_n$ is dense in X .

Definition. Let (X, d) be a metric space. A subset $F \subset X$ is **nowhere dense** if $X \setminus F$ is dense in X ; equivalently, \overline{F} contains no non-trivial open subsets. We say that a subset $M \subseteq X$ is **meagre** (1st category) if $M = \bigcup_{n=1}^{\infty} F_n$ and each F_n is nowhere dense; and a set is **non-meagre** (2nd category) otherwise.

5.2 Theorem. (Baire Category II) Let (X, d) be a complete metric space. Then a non-empty open $U \subseteq X$ is non-meagre.

PROOF Suppose not, so $U = \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} \overline{F_n}$, each F_n (hence $\overline{F_n}$) nowhere dense. Then each $V_n = X \setminus \overline{F_n}$ is open and dense, and hence by BCT I, $G = \bigcap_{n=1}^{\infty} V_n$ is dense in X , and hence $U \cap G \neq \emptyset$, violating assumption \blacksquare

5.3 Theorem. (Banach-Steinhaus) Let X, Y be normed spaces, $U \subseteq X$ be non-meagre, and $\mathcal{F} \subset \mathcal{B}(X, Y)$ be such that for each $x \in U$, $\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty$ (pointwise bounded). Then \mathcal{F} is uniformly bounded, i.e. $\sup\{\|T\| : T \in \mathcal{F}\} < \infty$.

PROOF Let for each $n \in \mathbb{N}$

$$F_n = \bigcap_{T \in \mathcal{F}} T^{-1}(nB(Y)) = \{x \in X : \|Tx\| \leq n \text{ for all } T \in \mathcal{F}\}$$

so each F_n is closed and, by the pointwise boundedness assumption, $U \subseteq \bigcup_{n=1}^{\infty} F_n$. By assumption of non-meagreness of U , at least one F_{n_0} admits an interior point: there is $x_0 \in F_{n_0}$ and $\delta > 0$ such that $x_0 + \delta D(X) \subseteq F_{n_0}$. Then if $x \in D(X)$, we have

$$Tx = \frac{1}{\delta} \left[T \left(x_0 + \frac{\delta}{2} x \right) - T \left(x_0 - \frac{\delta}{2} x \right) \right]$$

so $\|Tx\| \leq \frac{2}{\delta} n_0$, in other words

$$\|T\| = \sup_{x \in D(x)} \|Tx\| \leq \frac{2n_0}{\delta} < \infty$$

where the bound is independent of T . \blacksquare

5.4 Theorem. (Open Mapping) Let X, Y be Banach spaces, and $T \in \mathcal{B}(X, Y)$ surjective. Then T is an open map; i.e. $T(U)$ is open in Y whenever U is open in X .

Remark. Given $x \in X$ and $\alpha \in \mathbb{F} \setminus \{0\}$, non-empty $A \subset X$, we have that $\overline{x + \alpha A} = x + \alpha \overline{A}$. Indeed, note that for $(a_k)_{k=1}^{\infty} \subset A$, we have

$$a_k \rightarrow a \in \overline{A} \text{ if and only if } x + \alpha a_k \rightarrow x + \alpha a \in x + \alpha \overline{A}$$

5.5 Lemma. With the assumptions as above, we have that if $\overline{T(D(X))} \supset rB(Y)$ for some $r > 0$, then $T(D(X)) \supseteq rD(Y)$.

PROOF Let $z \in rD(Y)$ and let $0 < \delta < 1$ be so $\|z\| < r(1 - \delta) < r$. Set $y = z/(1 - \delta)$ so $\|y\| < r/(1 - \delta)$. It suffices to show that $y \in \frac{1}{1-\delta} T(D(X))$. To begin, let $A = T(D(X)) \cap rB(Y)$, so $\overline{A} = rB(Y)$. Indeed, if $y \in rB(Y) \subseteq \overline{T(D(X))}$, then there is $(y_k)_{k=1}^{\infty} \subset \overline{T(D(X))}$, so $y = \lim y_k$. But then there is $x_k \in D(X)$ so each $\|y_k - T(x_k)\| < 1/k$ so $y = \lim T(x_k)$ with each $x_k \in D(X)$.

Now we inductively build a sequence $(y_n)_{n=1}^{\infty}$ as follows.

- Since $y \in rD(Y) \subseteq \overline{A}$, there is $y_1 \in A \cap (y + \delta rD(Y))$
- $y \in y_1 + \delta r(D(Y)) \subseteq y_1 + \delta \overline{A} = y_1 + \delta A$, so there is $y_2 \in (y_1 + \delta A) \cap (y + \delta^2 rD(Y))$
- $y \in y_n + \delta^n rD(Y) \subseteq y_n + \delta^n \overline{A}$, so there is $y_{n+1} \in (y_n + \delta^n A) \cap (y + \delta^{n+1} rD(Y))$

By construction, $y_{n+1} - y_n \in \delta^n A$, so $\|y_{n+1} - y_n\| \leq \delta^n r$ and there is $x_n \in \delta^n D(X)$ such that $y_{n+1} - y_n = Tx_n$. Likewise, $y_1 \in A \subseteq T(D(X))$ so $y = T(x_0)$ for some $x_0 \in D(X)$. Notice that each $y_n \in y + \delta^n r \in D(Y)$, so $\|y_n - y\| \leq \delta^n r \rightarrow 0$. Since X is complete, we let $x = \sum_{n=0}^{\infty} x_n$, and by construction

$$\|x\| \leq \sum_{n=0}^{\infty} \|x_n\| < \sum_{n=0}^{\infty} \delta^n = \frac{1}{1-\delta}$$

Then by linearity and continuity of T , we have

$$Tx = \sum_{n=0}^{\infty} Tx_n = y_1 + \sum_{n=1}^{\infty} (y_{n+1} - y_n) = y_N + \sum_{n=N}^{\infty} (y_{n+1} - y_n) \rightarrow y$$

so that indeed $T(x) = y$, as required. ■

Remark. So far, we've only used completeness of X and continuity and linearity of T .

We now proceed with the proof of the open mapping theorem.

PROOF It suffices to see that $T(D(X))$ contains a neighbourhood of 0 in Y . Indeed, if $\emptyset \neq U \subseteq X$ is open, $x \in U$, then there is $\delta > 0$ such that $x + \delta D(X) \subseteq U$, so $U - x \supseteq \delta D(X)$. If $T(D(X)) \supseteq rD(Y)$, then $T(U - x) \supseteq \delta T(D(X)) \supseteq r\delta D(Y)$ so that $Tx + r\delta D(Y) \subseteq T(U)$. In other words, $T(U)$ is a neighbourhood of any of its points, and thus open.

Now write $X = \bigcup_{n=1}^{\infty} nD(X)$, and we assume that $T(X) = Y$. Hence $Y = \bigcup_{n=1}^{\infty} nT(D(X))$, so $Y = \bigcup_{n=1}^{\infty} \overline{nT(D(X))}$. But Y is complete, so by Baire category theorem, there is some n so that $\overline{nT(D(X))}$ has non-empty interior. Since $nT(D(X))$ is convex and symmetric, and hence $\overline{nT(D(X))}$ is convex and symmetric as well. Thus if $y \in D(Y)$, then $y_0 \pm \epsilon \in y_0 + \epsilon D(Y)$ so

$$\epsilon y = \frac{1}{2} [y_0 + \epsilon y - (y_0 - \epsilon y)] \in \overline{nT(D(X))}$$

and $\frac{\epsilon}{n} y \in \overline{T(D(X))}$, i.e. $\frac{\epsilon}{n} D(Y) \subseteq \overline{T(D(X))}$. Thus applying the main lemma, $\frac{\epsilon}{n} D(Y) \subseteq T(D(X))$. ■

5.6 Theorem. (Inverse Mapping) If X, Y are Banach spaces and $T \in \mathcal{B}(X, Y)$ is invertible, $T^{-1} \in \mathcal{B}(Y, X)$

PROOF Direct application of the open mapping theorem. ■

Let X, Y be normed spaces. Then we define for $(x, y) \in X \oplus Y$, and we let $\|(x, y)\|_1 = \|x\| + \|y\|$. It is easy to check that $\|\cdot\|_1$ is a norm on $X \oplus Y$, and if X, Y are Banach, then so is $(X \oplus Y, \|\cdot\|_1)$. In this case, we write $X \oplus_1 Y$.

5.7 Theorem. (Closed Graph) Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Then $T \in \mathcal{B}(X, Y)$ if and only if $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \oplus_1 Y$.

PROOF Let $T \in \mathcal{B}(X, Y)$. If $(x_n) \rightarrow x$ in X , then $Tx_n \rightarrow Tx$ in Y . Thus if $(x, y) \in \overline{\Gamma(T)}$, then $(x, y) = \lim(x_n, Tx_n)$ where $(x_n, Tx_n) \in \Gamma(T)$. But then

$$\|y - Tx\| \leq \|y - Tx_n\| + \|Tx_n - Tx\| \leq \|x - x_n\| + \|y - Tx_n\| + \|Tx_n - Tx\| = \|(x - y) - (x_n, Tx_n)\|_1$$

so in fact $y = Tx$ so $(x, y) = (x, Tx)$.

Conversely, if $\Gamma(T)$ is closed in $X \oplus_1 Y$, then $\Gamma(T)$ is a Banach space. Define $S : \Gamma(T) \rightarrow X$ by $S(x, Tx) = x$. Notice that S is linear, and

$$\|S(x, Tx)\| = \|x\| \leq \|(x, Tx)\|_1$$

so $\|S\| \leq 1$, so S is bounded. It is also clear that S is bijective, with $S^{-1} : X \rightarrow \Gamma(T)$ given by $S^{-1}(x) = (x, Tx)$. Thus the inverse mapping theorem gives that S^{-1} is also bounded. Hence for any $x \in X$,

$$\|Tx\| \leq \|(x, Tx)\|_1 = \|S^{-1}x\| \leq \|x\| \|S^{-1}\|$$

so that T is in fact bounded. ■

5.8 Theorem. (Closed graph test) *Given normed spaces and $T \in \mathcal{L}(X, Y)$, we have that $\Gamma(T)$ is closed in $X \oplus_1 Y$ if and only if whenever $x_n \rightarrow 0$ for which we may assume that Tx_n converges in Y , say $y = \lim Tx_n$, then $y = 0$ too.*

PROOF We have $(x_n, Tx_n) \rightarrow (x, z) \in \overline{\Gamma(T)}$ if and only if $(x_n - x, T(x_n - x)) \rightarrow (x, z) - (x, Tx) = (0, z - Tx)$. Set $y = z - Tx$. We have $(x, z) \in \Gamma(T)$ if and only if $z = Tx$ if and only if $y = 0$. ■

5.1 TESTING HYPOTHESIS OF OMT

- (i) Let $1 \leq p < r < \infty$. We have that $\ell_p \subseteq \ell_r$, with $\|x\|_r \leq \|x\|_p$ for $x \in \ell_p$. First, suppose $x \in B(\ell_p)$, so for each k , $|x_k| \leq \|x\|_p \leq 1$ so $|x_k|^{r/p} \leq |x_k|$. Hence

$$\|x\|_r = \left(\sum_{k=1}^{\infty} |x_k|^r \right)^{1/r} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/r} = \|x\|_p^{p/r} \leq 1$$

so if $x \in \ell_p \setminus \{0\}$, then the result follows.

Let $S : (\ell_p, \|\cdot\|_p) \rightarrow (\ell_p, \|\cdot\|_r)$ be the identity map. Then $\|S\| \leq 1$, and furthermore S is bijective. If S were open, then by the proof of inverse mapping theorem, we would see that $\|S^{-1}\| < \infty$. Define $x^{(n)} \in \ell_p$ by

$$x_k^{(n)} = \begin{cases} \frac{1}{ck^{1/p}} & k \leq n \\ 0 & k > n \end{cases}, c = \sum_{k=1}^{\infty} \frac{1}{k^{r/p}}$$

We compute that $\|x^{(n)}\|_r < 1$ while $\|x^{(n)}\|_p = \frac{1}{c} \left(\sum_{k=1}^n \frac{1}{k} \right)^{1/p}$. In other words, $\|S^{-1}x^{(n)}\|_p$ goes to infinity, while $\|x^{(n)}\|_r < 1$, contradicting $\|S^{-1}\| < \infty$. The moral of this is that if the range space is not complete, then OMT may not hold.

- (ii) Take $X = C_b(0, 1)$, $X_0 = \{f \in X : f \text{ is differentiable on } (0, 1), f' \in C_b(0, 1)\}$. We have $X_0 \subseteq X$, and we put the uniform norm $\|\cdot\|_{\infty}$ on both spaces. We let $D : X_0 \rightarrow X$, $Df = f'$. If $h_n(t) = t^n$, then $\|h_n\|_{\infty} = 1$ while $\|Dh_n\|_{\infty} = n$, so D is not bounded. Despite this, we have that $\Gamma(D) = \{(f, f') : f \in X_0\}$ is closed in $X_0 \oplus_1 X$. We apply

the closed graph test: let $(f_n, f'_n) \rightarrow (0, g)$ in $X_0 \oplus_1 X$. Notice that $\|f'_n\|_\infty < \infty$, so f_n is Lipschitz on $(0, 1)$, so f_n is uniformly continuous on $(0, 1)$, so $f_n(0^+) = \lim_{t \rightarrow 0^+} f_n(t)$ exists. Thus by the fundamental theorem of calculus, $f_n(t) = f_n(0^+) + \int_0^t f'_n$ for $t \in (0, 1)$. In particular,

- $f_n \rightarrow 0$ uniformly, so $f_n(0^+) \rightarrow 0$
- $f'_n \rightarrow g$ uniformly, so for each $t \in (0, 1)$,

$$\int_0^t g = \lim_{n \rightarrow \infty} \int_0^t f'_n = \lim_{n \rightarrow \infty} [f_n(t) - f_n(0^+)] = 0$$

and again, by the FT of C, $g(t) = 0$. Thus $g = 0$, so $\Gamma(D)$ is closed. We say that $D : X_0 \rightarrow X$ is a **closed** operator. The moral here is that if the domain is not complete, then closedness of the graph does not imply boundedness of the operator.

Now, let $J : X \rightarrow X_0$ have $Jg(t) = \int_0^t g$ for $t \in (0, 1)$. By the FT of C, $D \circ J(G) = g$, in other words that $D \circ J = I$. We have for $g \in X$,

$$\|Jg\|_\infty = \sup_{t \in (0,1)} \left| \int_0^t g \right| \leq \sup_{t \in (0,1)} t \|g\|_\infty \leq \|g\|_\infty$$

so $\|J\| \leq 1$. Hence $J(D(X)) \subseteq D(X_0)$, and we apply D to see $D(X) \subseteq D(D(X_0))$, in other words, that D is open. As an exercise, show that $C_b(0, 1) = X$ is not separable, while X_0 is separable.

Let $X \subsetneq Y$ be \mathbb{F} -vector spaces. We can always find a subspace $Z \subset Y$ so $X + Z = Y$ and $X \cap Z = \{0\}$. Indeed, let B be a basis for X , and $B' = B \cup B'$ is a basis for Y , and take $Z = \text{span } B'$.

5.9 Theorem. *Let Y be a Banach space and $X \subsetneq Y$ a closed subspace. Then X admits a closed complement Z if and only if there is some $P \in \mathcal{B}(Y)$ such that $P \circ P = P$ and $\text{im } P = P(Y) = X$.*

Remark. We say that $X \subsetneq Y$ is **boundedly complemented** if either of the above conditions hold.

PROOF (\Leftarrow) Let $Z = \ker P$, which is closed. If $y \in Y$, then $y = Py + (I - P)y$ where $Py \in X$ and $P(I - P)y = 0$ so $(I - P)y \in \ker P$. If $z \in Z \cap X$, then $z = Py$ for some $y \in Y$ so $Pz = P^2y = Py = z$, but $z \in \ker P$, so $z = Pz = 0$.

(\Rightarrow) Let $S : X \oplus_1 Z \rightarrow Y$ be given by $S(x, z) = x + z$. Then S is surjective and if $(x, z) \in \ker S$, then $x + z = 0$ so $x = -z \in X \cap Z = \{0\}$, hence S is injective. Furthermore,

$$\|S(x + z)\| = \|x + z\| \leq \|(x, z)\|_1$$

so $\|S\| \leq 1$. Hence S is a bounded bijection between Banach space and hence S^{-1} is bounded by the inverse mapping theorem. Let $P_1 : X \oplus_1 Z \rightarrow X$ be given by $P_1(x, z) = x$; and $J : X \rightarrow Y$ by $Jx = x$. Notice that $\|P_1\| = 1$ and $\|J\| = 1$. Define $P : Y \rightarrow Y$ by $Py = JP_1S^{-1}y$. Then

- $\text{im } J = X$, and each of P_1, S^{-1} are surjective, so $\text{im } P = X$
- If $y \in Y$, $\|Py\| = \|JP_1S^{-1}y\| \leq \|S^{-1}\| \|y\|$ so $\|P\| \leq \|S^{-1}\|$
- Clearly $P^2 = JP_1S^{-1}JP_1S^{-1} = P$ ■

5.10 Theorem. c_0 is not boundedly complemented in ℓ_∞ .

PROOF Let us assume otherwise; hence, there is $P = P^2 \in \mathcal{B}(\ell_\infty)$ such that $\text{im } P = c_0$. Note that $c_0 = \ker(I - P)$. As in A2, we let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a family of infinite subsets such that for $E \neq F$ in \mathcal{F} , $|E \cap F| < \infty$ and $|\mathcal{F}| = \mathfrak{c}$. For each $F \in \mathcal{F}$, we let $y_F = (I_P)\chi_F \neq 0$. If $\alpha_1, \dots, \alpha_n \in F$ are pairwise distinct, $F_1, \dots, F_m \in \mathcal{F}$, then

$$\sum_{i=1}^n \alpha_i \chi_{F_i} = \underbrace{\sum_{i=1}^m \alpha_i \chi_{F_i \setminus \bigcup_{j \in [m] \setminus \{i\}} F_j}}_{:=z} + \underbrace{\sum_{k=2}^m \sum_{1 \leq i < \dots < i_k \leq m} (\alpha_{i_1} + \dots + \alpha_{i_k}) \chi_{F_{i_1} \cap \dots \cap F_{i_k}}}_{\in c_0}$$

where $\|z\|_\infty = \max_{k=1, \dots, m} |\alpha_k|$. Hence

$$\left\| \sum_{i=1}^m \alpha_i y_{F_i} \right\| = \|(I - P)z\| \leq \|I - P\| \|z\| = \|I - P\| \max_{k=1, \dots, m} |\alpha_k| \quad (5.1)$$

Now, let for $n, k \in \mathbb{N}$, $\mathcal{F}_{n,k} = \{F \in \mathcal{F} : |\delta_k(y_F)| \geq \frac{1}{n}\}$ where $\delta_k(x_i)_{i=1}^\infty = x_k$, so $\delta_k \in \ell_\infty^*$ with $\|\delta_k\| \leq 1$. Let F_1, \dots, F_m be pairwise disjoint in $\mathcal{F}_{n,k}$, and $\alpha_i = \text{sgn } \delta_k(y_{F_i})$. Then we have each $|\alpha_i| = 1$, so by (5.1), we find

$$\|I - P\| \geq \left\| \sum_{i=1}^\infty \alpha_i y_{F_i} \right\|_\infty \geq |\delta_k \sum_{i=1}^n \alpha_i y_{F_i}| = \sum_{i=1}^m |\delta_k(y_{F_i})| \geq \frac{m}{n}$$

so $m \leq n\|I - P\|$ and it follows that $\mathcal{F}_{n,k}$ is finite. Since each $y_F \neq 0$ for $F \in \mathcal{F}$, we see that $\mathcal{F} = \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty \mathcal{F}_{n,k}$, which contradicts that $|\mathcal{F}| = \mathfrak{c}$. Hence such a P must not exist. ■

5.11 Theorem. *If X is a finite dimensional vector space over \mathbb{F} , then any two norms are equivalent.*

PROOF Let $\|\cdot\|$ be a norm on X . Fix a basis (e_1, \dots, e_n) for X , and let $x = \sum_{k=1}^n x_k e_k$, $x_k \in \mathbb{F}$, $\|x\|_\infty = \max_{k=1, \dots, n} |x_k|$. This is easily checked to be a norm. Moreover, $B_\infty = \{x \in X : \|x\|_\infty \leq 1\}$ admits a homeomorphic identification

$$B_\infty = \begin{cases} [-1, 1]^n & \mathbb{F} = \mathbb{R} \\ \overline{D}^n & \mathbb{F} = \mathbb{C} \end{cases}$$

and hence is compact. Thus $S_\infty = \{x \in X : \|x\|_\infty = 1\}$ is compact as well. Hence, for $x = \sum_{k=1}^\infty x_k e_k$, we have

$$\|x\| \leq \sum_{k=1}^n |x_k| \|e_k\| \leq \|x\|_\infty \underbrace{\sum_{k=1}^n \|e_k\|}_{:=M}$$

Now for $x, y \in X$, we have $|\|x\| - \|y\|| \leq \|x - y\| \leq M\|x - y\|_\infty$ so $\|\cdot\|$ is Lipschitz with respect to $\|\cdot\|_\infty$, and hence $\tau_{\|\cdot\|_\infty}$ -continuous. Thus the extreme value theorem tells us that $m = \inf_{x \in S_\infty} \|x\| > 0$. Hence for $x \in X \setminus \{0\}$, $\|x\| = \|x\|_\infty \cdot \left\| \frac{1}{\|x\|_\infty} x \right\| \geq \|x\|_\infty m$. In general, $m\|x\|_\infty \leq \|x\| \leq M\|x\|_\infty$. We thus have that $\|\cdot\| \sim \|\cdot\|_\infty$, so any norms are equivalent. ■

5.12 Corollary. *Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Then*

- (i) $K \subseteq X$ is compact if and only if K is closed and bounded.
- (ii) $(X, \|\cdot\|)$ is a Banach space
- (iii) For any normed space Y , we have $\mathcal{L}(X, Y) = \mathcal{B}(X, Y)$
- (iv) We have $X' = X^*$.

PROOF (i) The forward direction is immediate. If K is closed and bounded, it is contained in some scaled copy of B_∞ , which is compact.
 (ii) Cauchy sequences are bounded, and thus contained in some scaled copy of B_∞ , which is compact.
 (iii) Let $T \in \mathcal{L}(X, Y)$, and let $\|x\|_0 = \|x\| + \|Tx\|$. Then the result follows by equivalence of norms.
 (iv) Immediate. ■

5.13 Proposition. *A finite dimensional subspace of normed space is always closed and boundedly complemented.*

PROOF Let $Y \subseteq X$ be so Y is finite dimensional and X a normed space. We can find a basis (e_1, \dots, e_n) for Y . We may assume that each $\|e_k\| = 1$. We define $f_1, \dots, f_n \in Y' = Y^*$ by

$$f_k \left(\sum_{j=1}^n \alpha_j e_j \right) = \alpha_k$$

By Hahn-Banach, get $F_1, \dots, F_n \in X^*$ such that $F_k|_Y = f_k$ and $\|F_k\| = \|f_k\|$. Define $P : X \rightarrow X$ by $Px = \sum_{k=1}^n F_k(x)e_k$. Notice that $\text{im } P \subseteq Y$ and by choice of $F_k|_Y = f_k$, we have $P|_Y = I_Y$. Thus $P^2 = P$. Finally, for $x \in X$, $\|Px\| \leq \sum_{k=1}^n \|F_k\| \|x\|$ so $\|P\| \leq \sum \|F_k\| < \infty$, i.e. P is bounded. Closedness of Y thus follows from the last corollary. Alternatively, $Y = \ker(I - P)$. ■

6 ON COMPACTNESS OF THE UNIT BALL

6.1 Lemma. *Let X be a normed space and $Y \subsetneq X$ a closed subspace. Then given $\epsilon \in (0, 1)$ there is $x_0 \in D(X) \subseteq B(X)$ such that $d(x_0, Y) > 1 - \epsilon$.*

PROOF Let $x \in X \setminus Y$ and let $f : Y + \mathbb{F}x \rightarrow \mathbb{F}$ be given by $f(y + \alpha x) = \alpha$, $y \in Y$, $\alpha \in \mathbb{F}$. Then f is linear and $\ker f = Y$ is closed, $Y \subsetneq Y + \mathbb{F}x$, so f is bounded. Let $F \in X^*$ be any Hahn-Banach extension of f with $\|F\| = \|f\|$.

Now, we find $x_0 \in D(X)$ such that $|F(x_0)| > (1 - \epsilon)\|F\|$. Since $Y \subseteq \ker F$, we have for $y \in Y$ that $\|F\| \|x_0 - y\| \geq |f(x_0 - y)| = |F(x_0)| > (1 - \epsilon)\|F\|$, so $\|x_0 - y\| > 1 - \epsilon$. Hence $d(x_0, Y) = \inf_{y \in Y} \|x_0 - y\| \geq 1 - \epsilon$. ■

6.2 Theorem. *Let X be a normed space. Then $B(X)$ is compact if and only if X is finite dimensional.*

PROOF The reverse implication is standard. Thus suppose X is not finite dimensional. Let $\epsilon \in (0, 1)$ and let $x_1 \in B(X) \setminus \{0\}$. Inductively,

- Find $x_2 \in B(X)$ such that $\text{dist}(x_2, \mathbb{F}x_1) \geq 1 - \epsilon$
- Find $x_3 \in B(X)$ such that $\text{dist}(x_3, \text{span}\{x_1, x_2\}) \geq 1 - \epsilon$
- Find $x_{n+1} \in B(X)$ such that $\text{dist}(x_{n+1}, \text{span}\{x_1, \dots, x_n\}) \geq 1 - \epsilon$

Hence we have $\{x_n\}_{n=1}^\infty \subset B(X)$ such that for $m < n$,

$$\|x_n - x_m\| \geq d(x_n, \text{span}\{x_1, \dots, x_{n-1}\}) \geq 1 - \epsilon$$

so the sequence admits no converging subsequence and $B(X)$ is not compact. \blacksquare

7 MORE TOPOLOGY

Definition. Let (X, τ) be a topological space. A **base** for τ is any family $\beta \subseteq \tau$ such that for any $U \in \tau$ and $x \in U$, there is $B \in \beta$ such that $x \in B \subseteq U$. A **subbase** for τ is any family $\alpha \subseteq \tau$ such that $\{\bigcap_{k=1}^n U_k : n \in \mathbb{N}, U_1, \dots, U_n \in \alpha\}$ is a base for τ .

Note that if $\emptyset \neq X$ and $\beta \subseteq \mathcal{P}(X)$ for which $\bigcup_{B \in \beta} B = X$ and β is closed under finite intersections, then

$$\tau_\beta = \left\{ \bigcup_{i \in I} B_i : \{B_i\}_{i \in I} \subset \beta, I \text{ any index set with } |I| \leq |\beta| \right\}$$

is a topology.

Definition. Let $X \neq \emptyset$. Suppose we are given

- a family $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ of topological spaces, and
- for each $\alpha \in A$, a function $f_\alpha : X \rightarrow X_\alpha$

Then the **initial topology** on X given this data is denoted

$$\sigma = \sigma(X, (f_\alpha)_{\alpha \in A}) = \sigma(X, (f_\alpha, \tau_\alpha)_{\alpha \in A})$$

and is the topology with base

$$\bigcap_{k=1}^n f_{\alpha_k}^{-1}(U_{\alpha_k}), n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in A, \text{ each } U_{\alpha_k} \in \tau_{\alpha_k}$$

In particular, $\{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \tau_\alpha, \alpha \in A\}$ is a subbase for σ .

Remark. The topology is chosen so that each $f_\alpha : X \rightarrow X_\alpha$ is $\sigma - \tau_\alpha$ -continuous. Furthermore, if $\tau \subseteq \mathcal{P}(X)$ is any topology for which every f_α is $\sigma - \tau_\alpha$ -continuous, then $\sigma \subseteq \tau$. We say that σ is the **coarsest** topology so that all the f_α are continuous.

Example. (i) **Metric topology:** If (X, d) is a metric space, for each $x \in X$, let d_x be given by $d_x(x') = d(x, x')$. Then $\sigma(X, (d_x)_{x \in X}) = \tau_d$.

(ii) **Relative topology:** If (Y, τ) -topological space, $\emptyset \neq X \subseteq Y$, and $i : X \rightarrow Y$ is the inclusion map. Then $\tau|_X = \sigma(X, \{i\})$.

(iii) **Product topology:** Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a family of topological spaces. Let $X = \prod_{\alpha \in A} X_\alpha$. Let for $\alpha \in A$, $p_\alpha : X \rightarrow X_\alpha$ denote the projection map onto the component α . Then the product topology $\pi = \sigma(X, \{p_\alpha\}_{\alpha \in A})$. Hence, $V \in \pi$ if and only if for any $x \in V$, there is $\alpha_1, \dots, \alpha_n \in A$ and $U_{\alpha_k} \in \tau_{\alpha_k}$ such that $x_{\alpha_k} = p_{\alpha_k}(x) \in U_{\alpha_k}$ and $x \in \bigcap_{k=1}^n p_{\alpha_k}^{-1}(U_{\alpha_k}) \subseteq V$.

Note that if $X = \prod_{n=1}^\infty X_n$, each (X_n, τ_n) is a topological space, then the basic open sets look like $U_1 \times U_2 \times \dots \times U_m \times X_{m+1} \times X_{m+2} \times \dots$.

- (iv) *Linear topology*: Let X be a vector space and $Z \subseteq X'$ a subspace. Then $\sigma(X, Z)$ is the coarsest topology allowing each $f \in Z$ to be continuous, $f : X \rightarrow \mathbb{F}$. The basic open sets are given as follows: let $x_0 \in X$, $\epsilon > 0$, and $D = D(\mathbb{F})$, and we consider for $f \in Z$

$$f^{-1}(f(x_0) + \epsilon D) = \underbrace{\{x \in X : |f(x) - f(x_0)| < \epsilon\}}_{\text{"affine hypertube"}} = \{x \in X : |\frac{1}{\epsilon}f(x) - \frac{1}{\epsilon}f(x_0)| < 1\}$$

so that

$$\left\{ \bigcap_{k=1}^n \{x \in X : |f_k(x) - f_k(x_0)| < 1\} : f_1, \dots, f_n \in Z, n \in \mathbb{N} \right\}$$

is a base for $\sigma(X, Z)$.

- (v) Now let X be a normed space. Then the **weak topology** on X is $\omega = \sigma(X, X^*)$. Certainly $\omega \subseteq \tau_{\|\cdot\|}$. Similarly, the **weak*-topology** on X^* is $\omega^* = \sigma(X^*, \hat{X})$ (recall for $x \in X$, $\hat{x}(f) = f(x)$). Since $\hat{X} \subseteq X^{**}$, we have $\omega^* \subseteq \omega = \sigma(X^*, X^{**}) \subseteq \tau_{\|\cdot\|}$.

Let (X, τ) be a topological space.

Definition. A subset $K \subseteq X$ is called **compact** if for any collection $\{U_\alpha\}_{\alpha \in A} \subseteq \tau$ with $\bigcup_{\alpha \in A} U_\alpha \supseteq K$, there exists some finite U_1, \dots, U_n covering K . If X itself is τ -compact, we call (X, τ) a compact space.

Definition. A set $F \subseteq X$ is **closed** if $X \setminus F \in \tau$. If $S \subseteq X$, then the **closure** of S is $\bar{S} = \bigcap \{F \subseteq X : S \subseteq F, X \setminus F \in \tau\}$.

Note that $\bar{S} = \{x \in X : \text{for any } U \in \tau \text{ with } x \in U, U \cap S \neq \emptyset\}$.

Definition. A family $\mathcal{F} \subseteq \mathcal{P}(X)$ has the **finite intersection property** if for any $F_1, \dots, F_n \in \mathcal{F}$, $\bigcap_{i=1}^n F_i \neq \emptyset$.

7.1 Proposition. Let (X, τ) be a topological space. Then (X, τ) is compact if and only if any $\mathcal{F} \subseteq \mathcal{P}(X)$ with the finite intersection property has $\bigcap_{F \in \mathcal{F}} \bar{F} \neq \emptyset$.

PROOF Suppose X is compact and $\mathcal{F} \subseteq \mathcal{P}(X)$ has the finite intersection property but with $\bigcap_{F \in \mathcal{F}} \bar{F} = \emptyset$, then $\{X \setminus \bar{F}\}_{F \in \mathcal{F}}$ is an open cover of X with no finite subcover.

Conversely, if $\mathcal{O} \subseteq \tau$ is an open cover of X , then $\mathcal{F} = \{X \setminus U\}_{U \in \mathcal{O}}$ satisfies $\bigcap_{F \in \mathcal{F}} F = \emptyset$, so there is $F_1, \dots, F_n \in \mathcal{F}$ with $\bigcap_{k=1}^n F_k = \emptyset$. Then $\{X \setminus F_i\}_{i=1}^n$ is a finite subcover. ■

Definition. Let X be a non-empty set. An **ultrafilter** is a family $\mathcal{U} \subseteq \mathcal{P}(X)$ such that

- \mathcal{U} has the finite intersection property
- If $A \in \mathcal{P}(X)$, then either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Example. (i) *Principal / trivial ultrafilter*: If $x_0 \in X$, let $U_{x_0} = \{U \subseteq X : x_0 \in U\}$.

7.2 Lemma. (Ultrafilter) If $\mathcal{F} \subseteq \mathcal{P}(X)$ is any set with the finite intersection property, then there is an ultrafilter \mathcal{U} with $\mathcal{F} \subseteq \mathcal{U}$.

PROOF Let $\Phi = \{\mathcal{G} \subseteq \mathcal{P}(X) : \mathcal{F} \subseteq \mathcal{G}, \mathcal{G} \text{ has f.i.p.}\}$. Then Φ is partially ordered by inclusion. If $\Gamma \subseteq \Phi$ is a chain, then $\mathcal{G}_\Phi = \bigcup_{\mathcal{G} \in \Gamma} \mathcal{G}$ contains \mathcal{F} and has the finite intersection property. Hence Φ admits a maximal element \mathcal{U} . Let $A \in \mathcal{P}(X) \setminus \mathcal{U}$. Then $U \cup \{A\} \not\supseteq \mathcal{U}$, so $\mathcal{U} \cup \{A\}$ fails the finite intersection property. Hence get U_1, \dots, U_n so $A \cap \bigcap_{k=1}^n U_k = \emptyset$. Now if $V_1, \dots, V_m \in \mathcal{U}$, then $\bigcap_{j=1}^m V_j \cap \bigcap_{k=1}^n U_k \subseteq \bigcap_{k=1}^n U_k \subseteq X \setminus A$, so $(X \setminus A) \cap \bigcap_{j=1}^m V_j \neq \emptyset$. Thus $\mathcal{U} \cup \{X \setminus A\}$ has finite intersection property, so $X \setminus A \in \mathcal{U}$ by maximality. ■

7.3 Corollary. If $\mathcal{U} \subseteq \mathcal{P}(X)$ is an ultrafilter, then

- (i) If $A \in \mathcal{P}(X)$, $A \in \mathcal{U}$ if and only if $A \cap U \neq \emptyset$ for each $U \in \mathcal{U}$
- (ii) If $A, B \in \mathcal{P}(X)$, then $A \cup B \in \mathcal{U}$ implies at least one of A or B is in \mathcal{U}
- (iii) If $A \in \mathcal{U}$ and $A \subseteq V$ implies $V \in \mathcal{U}$

PROOF The forward implication of (i) follows since \mathcal{U} has finite intersection. Conversely, $X \setminus A \notin \mathcal{U}$, so $A \in \mathcal{U}$. (ii) and (iii) follow consequently. ■

7.4 Corollary. If X is an infinite set, it admits a non-principle ultrafilter.

PROOF Let $\mathcal{F} = \{F \in \mathcal{P}(X) : X \setminus F \text{ is finite}\}$. Then \mathcal{F} has the finite intersection property. Apply the lemma. ■

7.5 Proposition. There are at least \mathfrak{c} many ultrafilters in $\mathcal{P}(\mathbb{N})$.

PROOF We let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a collection of infinite sets such that $E \neq F$ in \mathcal{F} implies $|E \cap F| < \infty$, and $|\mathcal{F}| = \mathfrak{c}$. For each $F \in \mathcal{F}$, we let $\mathcal{F}_F = \mathcal{F}_0 \cup \{F\}$, which has the finite intersection property. Moreover, if $E \in \mathcal{F} \setminus \{F\}$, then $\mathcal{F}_F \cup \{E\}$ would fail f.i.p. Hence, for $F \in \mathcal{F}$, let \mathcal{U}_F be any ultrafilter containing \mathcal{F}_F , giving \mathfrak{c} many ultrafilters. ■

Remark. It can be shown (with a lot more work) that \mathbb{N} admits $2^{\mathfrak{c}}$ ultrafilters.

Let $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ be a non-principal ultrafilter. Define $\delta_{\mathcal{U}} : \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\} \subset \mathbb{R}$ by $\delta_{\mathcal{U}}(A) = 1$ if $A \in \mathcal{U}$, and 0 if $X \setminus A \in \mathcal{U}$. Since $\mathbb{N} \in \mathcal{U}$, we see that $\delta_{\mathcal{U}}(\emptyset) = 0$. If $\emptyset \neq A, B \in \mathcal{P}(\mathbb{N})$ with $A \cap B = \emptyset$, then if $A \cup B \in \mathcal{U}$, then exactly one of A or B is in \mathcal{U} . Thus $\delta_{\mathcal{U}}(A \cup B) = \delta_{\mathcal{U}}(A) + \delta_{\mathcal{U}}(B)$. If $E_1, \dots, E_n \subseteq \mathbb{N}$ with $E_j \cap E_k = \emptyset$ for $j \neq k$, then $\sum_{k=1}^n |\delta_{\mathcal{U}}(E_k)| \leq 1$ so $\|\delta_{\mathcal{U}}\|_{\text{var}} \leq 1$. Since $\delta_{\mathcal{U}}(\mathbb{N}) = 1$, we have $\|\delta_{\mathcal{U}}\|_{\text{var}} = 1$. Let $L_{\mathcal{U}} \in \ell_{\infty}^*$ be the linear functional associated to $\delta_{\mathcal{U}}$. We then have (with some verification possibly needed)

- (i) $L_{\mathcal{U}}(1) = 1$, $\|L_{\mathcal{U}}\| = 1$
- (ii) $L_{\mathcal{U}}|_{\mathfrak{c}_0} = 0$, so if $x \in \ell_{\infty}^{\mathbb{R}}$, then $\liminf_{n \rightarrow \infty} x_n \leq L_{\mathcal{U}} \leq \limsup_{n \rightarrow \infty} x_n$
- (iii) Exactly one of $2\mathbb{N}$ and $2\mathbb{N}-1$ is in \mathcal{U} , so $L(\chi_{2\mathbb{N}}) \neq L_{\mathcal{U}}(\chi_{2\mathbb{N}-1})$, so $L_{\mathcal{U}}$ is not translation invariant.
- (iv) Let $S \in \mathcal{B}(\ell_{\infty})$ be given by $Sx = \left(\frac{x_1 + \dots + x_n}{n}\right)_{n=1}^{\infty}$. Then $L_{\mathcal{U}} \circ S$ is a Banach limit.

Definition. If (X, τ) is a topological space, \mathcal{U} an ultrafilter on X , we say that $x_0 \in X$ is a $(\tau-)$ limit point for \mathcal{U} if for each $U \in \tau$ with $x_0 \in U$, we have $U \in \mathcal{U}$.

7.6 Proposition. Let (X, τ) be a topological space. Then (X, τ) is compact if and only if any ultrafilter on X admits a τ -limit point.

PROOF Let us begin with an observation: if $x \in X$ and \mathcal{U} is an ultrafilter on X , then

$$x \in \bigcap_{V \in \mathcal{U}} \overline{V} \Leftrightarrow \text{for any } U \in \tau \text{ with } x \in U, U \cap V \neq \emptyset \text{ for each } V \in \mathcal{U}$$

$$\Leftrightarrow x \text{ is a } \tau\text{-limit point of } \mathcal{U}$$

If (X, τ) is compact, then $\bigcap_{V \in \mathcal{U}} \overline{V} \neq \emptyset$. If $\mathcal{F} \subseteq \mathcal{P}(X)$ has the finite intersection property, then there exists an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$, so $\bigcap_{F \in \mathcal{F}} \overline{F} \supseteq \bigcap_{V \in \mathcal{U}} \overline{V} \neq \emptyset$. ■

7.7 Theorem. (Tychonoff) Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a family of compact spaces, and $X = \prod_{\alpha \in A} X_\alpha$ with the product topology π . Then (X, π) is compact.

PROOF Let \mathcal{U} be an ultrafilter on X ; we will show that it admits a π -limit point. Fix $\alpha \in A$ and let $\mathcal{U}_\alpha = \{p_\alpha(V) : V \in \mathcal{U}\}$, where p_α is the coordinate projection onto α . If $\emptyset \neq S_\alpha \subseteq X_\alpha$, then $S_\alpha = p_\alpha^{-1}(p_\alpha^{-1}(S_\alpha))$, so $S_\alpha \in \mathcal{U}_\alpha$ if and only if $p_\alpha^{-1}(S_\alpha) \in \mathcal{U}$, and since p_α^{-1} commutes with complementation, \mathcal{U}_α is an ultrafilter. The last proposition provides a τ_α -limit point x_α for \mathcal{U}_α . Now let $x = (x_\alpha)_{\alpha \in A}$, where x_α is found as above. If $W \in \pi$ with $x \in W$, then there are $\alpha_1, \dots, \alpha_n$ in A , $U_{\alpha_i} \in \tau_{\alpha_i}$ with $x \in \bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq W$. Since each x_{α_k} is a τ_{α_k} -limit point of \mathcal{U}_{α_k} , we see that each $U_{\alpha_k} \in \mathcal{U}_{\alpha_k}$, so $p_{\alpha_k}^{-1}(U_{\alpha_k}) \in \mathcal{U}$. Thus we see that $W \in \mathcal{U}$, so x is a π -limit point of \mathcal{U} . ■

Remark. (i) Tychonoff's theorem implies the axiom of choice. Given $\{X_\alpha\}_{\alpha \in A}$ be a family of non-empty sets. Find y which is not a member of any X_α , and let $Y_\alpha = X_\alpha \cup \{y\}$ and $\tau_\alpha = \{\emptyset, \{y\}, X_\alpha, Y_\alpha\}$, and (Y_α, τ_α) is compact. The constant element y is an element of Y , so by Tychonoff, (Y, π) is compact. Given $\alpha_1, \dots, \alpha_n \in A$, then $\bigcup_{k=1}^n p_{\alpha_k}^{-1}(\{y\})$. Since $\prod_{k=1}^n X_{\alpha_k} \neq \emptyset$, we see that $Y \subsetneq \bigcup_{k=1}^n p_{\alpha_k}^{-1}(\{y\})$. Hence by compactness, $Y \not\subseteq \bigcup_{\alpha \in A} p_\alpha^{-1}(\{y\})$. Hence $\prod_{\alpha \in A} X_\alpha = Y \setminus \bigcup_{\alpha \in A} p_\alpha^{-1}(\{y\}) \neq \emptyset$.

(ii) If we are given $(X_\alpha, \tau_\alpha)_{\alpha \in A}$ a family of topological spaces, $X = \prod_{\alpha \in A} X_\alpha$, we can define the **box topology**, i.e. the topology with base $\{\prod_{\alpha \in A} U_\alpha : U_\alpha \in \tau_\alpha \setminus \{\emptyset\} \text{ for each } \alpha\}$ Of course, $\pi \subseteq \tau$, and the inclusion is proper on infinite products.

7.8 Proposition. Let (X, τ) be a compact space.

- (i) If $K \subseteq X$ is closed, then K is compact.
- (ii) If (Y, σ) is a topological space and $f : X \rightarrow Y$ is continuous, then $f(X)$ is compact.

PROOF Immediate. ■

Remark. If X is a normed space, $w^* = \sigma(X^*, \hat{X})$, if $x \in X$, $\hat{x} \in X^{**}$, $\hat{x}(f) = f(x)$, $\hat{X} = \{\hat{x} : x \in X\}$. If A, B are non-empty sets, $A^B \cong \{f : B \rightarrow A\}$.

7.9 Theorem. (Alaoglu) Let X be a normed space. Then $B(X^*)$ is $w^* = \sigma(X^*, \hat{X})$ -compact

PROOF Let $\Gamma : X^* \rightarrow \mathbb{F}^X$ be given by $\Gamma(f) = (f(x))_{x \in X}$, so Γ is injective. Let $\pi = \sigma(\mathbb{F}^X, \{p_x\}_{x \in X})$ be the product topology. If $U_1, \dots, U_n \subseteq \mathbb{F}$ are open and $x_1, \dots, x_n \in X$, then

$$\Gamma\left(\bigcap_{k=1}^n \hat{x}_n^{-1}(U_k)\right) = \bigcap_{k=1}^n \Gamma(\hat{x}_n^{-1}(U_k)) = \bigcap_{k=1}^n \hat{x}_n^{-1}(U_k) \cap \Gamma(X^*)$$

so Γ is an open map onto its image in \mathbb{F}^X . Similarly, it is easy to show that Γ^{-1} is also an open map, so in fact Γ is a homeomorphism onto its image.

We now consider $\overline{\Gamma(B(X^*))} \subset \mathbb{F}^X$. Let $g \in \overline{\Gamma(B(X^*))}$ and let $D = D(\mathbb{F})$. Given $x, y \in X$ and $\alpha \in \mathbb{F}$, and then given $\epsilon > 0$, we find $f \in B(X^*)$ such that

$$\Gamma(f) \in p_x^{-1}\left(g(x) + \frac{\epsilon}{3}D\right) \cap p_y^{-1}\left(g(y) + \frac{\epsilon}{3(|\alpha|+1)}D\right) \cap p_{x+\alpha y}^{-1}\left(g(x+\alpha y) + \frac{\epsilon}{3}D\right)$$

We have that f is linear with $\Gamma(f)(x) = f(x)$, etc. so we have

$$|g(x) + \alpha g(y) - g(x + \alpha y)| \leq |g(x) - f(x)| + |\alpha| |g(y) - f(y)| + |g(x + \alpha y) - f(x + \alpha y)| < \epsilon$$

and since $\|f\| \leq 1$, we have $|g(x)| \leq |g(x) - f(x)| + |f(x)| < \epsilon/3 + \|x\|$. Then since $\epsilon > 0$ is arbitrary, get $g \in X'$ and $|g(x)| \geq \|x\|$, i.e. $g \in B(X^*)$. Hence we have that $g = \Gamma(g)$.

Thus $\Gamma(B(X^*)) \subseteq \prod_{x \in X} \|x\| \overline{D} \subseteq \mathbb{F}^X$ is a closed subset of a compact subset of \mathbb{F}^X . Thus $B(X^*)$ is the continuous image of a compact set and hence compact. ■

Remark. If $r > 0$, then we may replace $B(X^*)$ with $rB(X^*)$ in the proof above, with trivial modifications. Thus any ball is w^* -compact. Hence bounded w^* -closed sets in X^* are automatically w^* -compact.

Definition. A topological space (X, τ) is Hausdorff if given $x \neq y$ in X , there are $U_x, V_y \in \tau$ such that $x \in U_x$ and $y \in V_y$ and $U_x \cap V_y = \emptyset$.

Example. (i) A metric space is Hausdorff.

(ii) X a normed space, $w = \sigma(X, X^*)$ is Hausdorff (by Hahn-Banach and A2Q1).

(iii) If X is a normed space, then $w^* = \sigma(X^*, \hat{X})$ on X^* is Hausdorff.

(iv) $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ family of topological spaces, $X = \prod_{\alpha \in A} X_\alpha$ with π the product topology. Then (X, π) is Hausdorff if and only if all (X_α, τ_α) are Hausdorff. (Straightforward exercise).

7.10 Proposition. Let (X, τ) be a Hausdorff space, $K \subseteq X$ τ -compact. Then K is τ -closed.

PROOF Straightforward exercise. ■

7.11 Proposition. Let (X, τ) be a compact space.

(i) If (Y, σ) is a Hausdorff space and $\phi : X \rightarrow Y$ is continuous and bijective, then $\phi^{-1} : Y \rightarrow X$ is continuous.

(ii) If $\tau' \subseteq \tau$ is a Hausdorff topology on X , so $\tau' = \tau$.

PROOF (i) If $F \subseteq X$ is τ -closed, then it is τ -compact. Hence $(\phi^{-1})^{-1}(F) = \phi(F)$ is σ -closed, so by A1Q1, ϕ^{-1} is continuous.

(ii) $\text{id} : X \rightarrow X$ is continuous, so if $U \in \tau'$, then $\text{id}^{-1}(U) = U \in \tau$, so id is continuous. Hence by (1) id^{-1} is continuous so $\tau \subseteq \tau'$. ■

7.12 Theorem. (Metrization) If X is a separable normed space, then $B(X^*)$ is w^* -metrizable, i.e. there exists a metric ρ on $B(X^*)$ such that $w^*|_{B(X^*)} = \tau_\rho$.

PROOF Let $\{x_n\}_{n=1}^\infty \subset B(X)$ be any set which is separating for X^* , i.e. if $f \in X^* \setminus \{0\}$, then $f(x_n) \neq 0$ for some n (for example, take any dense subset of $D(X) \setminus \{0\}$). Let ρ be given by

$$\rho(f, g) = \sum_{k=1}^{\infty} \frac{|(f - g)(x_k)|}{2^k} \leq 2$$

It is easy to see that this is a metric.

Given $f_0 \in B(X^*)$, take $\epsilon > 0$ and let

- n be so $\sum_{k=n+1}^{\infty} \frac{2}{2^k} < \frac{\epsilon}{2}$, and

• $V = \bigcap_{k=1}^n \{f \in B(X^*) : |\hat{x}_k(f) - \hat{x}_k(f_0)| < \epsilon/2\} \in w^*|_{B(X^*)}, f_0 \in V$.
Then if $f \in V$,

$$g(f, f_0) = \sum_{k=1}^n \frac{|f(x_k) - f_0(x_k)|}{2^k} + \sum_{k=n+1}^{\infty} \frac{|f(x_k) - f_0(x_k)|}{2^k} < \epsilon$$

so $f_0 \in V \subset B_{\rho, \epsilon}^\circ(f_0)$. Since f_0 is arbitrary, we have $\tau_\rho \subseteq w^*|_{B(X^*)}$, but since w^* is compact and τ_ρ is Hausdorff, these must be equal. ■

- (i) Note that different separating families from $B(X)$ may produce different metrics, but always the same topology.
- (ii) The definition of ρ above extends to all of $X^* \times X^*$. However, X^* with the weak* topology is not metrizable if X is infinite dimensional.
- (iii) $X^* = \bigcup_{n=1}^{\infty} nB(X^*)$, so each $nB(X^*)$ is metrizable and compact, and thus w^* -separable. Thus if X is separable, then X^* is itself separable.

8 NETS

Definition. A pair (N, \leq) is a **preorder** on N if

- $v \leq v$ for $v \in N$
- $v_1 \leq v_2$ and $v_2 \leq v_3$ implies $v_1 \leq v_3$.

This pair is **cofinal** if for any $v_1, v_2 \in N$, there is $v_3 \in N$ so $v_1 \leq v_3$ and $v_2 \leq v_3$. Then (N, \leq) is a **directed set** if \leq is a cofinal preorder. Given a non-empty set X , a **net** is a function $x : N \rightarrow X$.

Definition. If $(x_\nu)_{\nu \in N}$ is a net in X , $A \subseteq X$, we say that $(x_\nu)_{\nu \in N}$ is

- **eventually** in A if there is $v_A \in N$ so $x_\nu \in A$ whenever $\nu \geq v_A$
- **frequently** in A if for any $v \in N$, there is $v' \in N$ with $v' \geq v$ so $x_{v'} \in A$.

Definition. Now, let (M, \leq) be another directed set. A map $\phi : M \rightarrow N$ is **eventually cofinal** if for any $v \in N$, there is $\mu_v \in M$ s $\phi(\mu) \geq v$ whenever $\mu \geq \mu_v$. Given a net $(x_\nu)_{\nu \in N}$ and an eventually cofinal $\phi : M \rightarrow N$, we call $(x_{\phi(\mu)})_{\mu \in M}$ a **subnet**.

Definition. We call $\phi : M \rightarrow N$ a **directed map** if

- (i) $\mu \leq \mu'$ in M implies $\phi(\mu) \leq \phi(\mu')$ in N
- (ii) For any $v \in N$, there is $\mu \in M$ s $v \leq \phi(\mu)$.

Directed maps are always cofinal. Different sources use directed maps over eventually cofinal maps.

Example. (i) (\mathbb{N}, \leq) is directed, and subsequences are special types of subnets.

(ii) (\mathbb{R}, \leq) is directed

(iii) (*Riemann sums*) Let $a < b$ in \mathbb{R} . We let

$$N = \{(P, P^*) : P = \{a = t_0 < t_1 < \dots < t_n = b\}, P^* = \{t_1^*, \dots, t_n^*\}, t_k^* \in [t_{k-1}, t_k]\}$$

and say $(P, P^*) \leq (Q, Q^*)$ if $P \subseteq Q$. One can verify that this is a net (the Riemann sum net).

(iv) (*Nets from filtering families*). We say that $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ is a **filtering family** if for each $F_1, F_2 \in \mathcal{F}$, there is $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$. For example, an ultrafilter is a filtering family. Let

$$N_{\mathcal{F}} = \{(x, F) : x \in F, F \in \mathcal{F}\}$$

equipped with the pre-order $(x, F) \leq (x', F')$ if and only if $F \supseteq F'$. Since \mathcal{F} is a filtering family, $(N_{\mathcal{F}}, \leq)$ is directed. Let $x_{(x,F)} = x$, so $(x)_{(x,F) \in N_{\mathcal{F}}}$ is the net built from \mathcal{F} . Note that if $F \in \mathcal{F}$, then $(x)_{(x,F) \in \mathcal{F}}$ is eventually in F .

An **ultranet** $(x_v)_{v \in N} \subset X$ is a net for which any $A \in \mathcal{P}(X)$, $(x_v)_{v \in N}$ is either eventually in A or eventually in $X \setminus A$. If \mathcal{F} is an ultrafilter, then $(x)_{(x,F) \in N_{\mathcal{F}}}$ is an **ultranet**.

8.1 NETS AND TOPOLOGY

Now, suppose (X, τ) is a topological space.

Definition. We say that $x_0 \in X$ is

- Some $x_0 \in X$ is a **limit point** if for any $U \in \tau$ with $x_0 \in U$, $(x_v)_{v \in N}$ is eventually in U . That is, there is v_U such that $x_v \in U$ whenever $v \geq v_U$. We write $x_0 = \lim_{v \in N} x_v$, the τ -limit of $(x_v)_{v \in N}$. Note that this is an abuse of notation, since limit points need not be unique (when (X, τ) is not Hausdorff).
- Some $x_0 \in X$ is a **cluster point** of $(x_v)_{v \in N}$ if for any $U \in \tau$ with $x_0 \in U$, $(x_v)_{v \in N}$ is frequently in U .

8.1 Proposition. If $(x_v)_{v \in N}$ is a net in (X, τ) and $x_0 \in X$, then x_0 is a cluster point for $(x_v)_{v \in N}$ if and only if x_0 is a τ -limit point of x_{v_μ} for some subnet $(x_{v_\mu})_{\mu \in M}$ of $(x_v)_{v \in N}$.

PROOF (\implies) Suppose x_0 is a cluster point for $(x_v)_{v \in N}$. Then for each $v \in N$ and $U \in \tau$ containing x_0 , define

$$F_{v,U} = \{v' \in N : v' \geq v, x_{v'} \in U\}$$

which is non-empty since x_0 is a cluster point. Then set

$$\mathcal{F} = \{F_{v,U} : v \in N, U \in \tau, x_0 \in U\} \subset \mathcal{P}(N)$$

Let's see that \mathcal{F} is filtering: suppose $F_{v,U}$ and $F_{v',U'}$ are in \mathcal{F} . Get $\mu \geq v$ and $\mu \geq v'$ by definition of a net and set $V = U \cap U'$, which is open and contains x_0 . Then since x_0 is a cluster point, get some $\mu' \geq \mu$ such that $x_{\mu'} \in V$, so $F_{\mu',V} \subseteq F_{v,U} \cap F_{v',U'}$. We then let $M = N_{\mathcal{F}}$ be the net construction from the filtering family and set $v_{(v,F)} = V$.

Now set $N_{\mathcal{F}} = \{(v, F) : v \in F, F \in \mathcal{F}\}$ with the standard preorder and $v_{(v,F)} = v$. Then the map $(v, F) \mapsto v$ from $N_{\mathcal{F}} \rightarrow N$ is eventually cofinal: if $v_0 \in N$ is arbitrary, take any $F_0 = F_{v_0, U} \in \mathcal{F}$. Then $F_0 = \{v \in N : v \geq v_0, x_v \in U\}$, so if $F_{\mu, V} \in \mathcal{F}$ with $F_{\mu, V} \subseteq F_0$, we let $M = N_{\mathcal{F}}$ as in (iv) above, and $v_{v, \mathcal{F}} = v$. Check that $(x_v)_{(v,F) \in N_{\mathcal{F}}}$ is eventually in U for any $U \in \tau$ with $x_0 \in U$. [Check: $(v, F) \mapsto v : N_{\mathcal{F}} \rightarrow N$ is cofinal, but is not evidently directed]

(\impliedby) If for some subnet $(x_{v_\mu})_{\mu \in M}$ is eventually in U for any $U \in \tau$ with $x_0 \in U$, then $(x_v)_{v \in N}$ is frequently in U for such U by definition of a subnet. ■

8.2 Proposition. If (Y, σ) is another topological space, then $f : X \rightarrow Y$ is continuous if and only if for any $x_0 \in X$ and net $(x_v)_{v \in N}$ with having x_0 as a limit, $f(x_0) = \lim_{v \in N} f(x_v)$.

PROOF If $V \in \sigma$ with $f(x_0) \in V$, then $f^{-1}(V) \in \tau$ with $x_0 \in f^{-1}(V)$. Since $(x_v)_{v \in N}$ is eventually in $f^{-1}(V)$, so $(f(x_v))_{v \in N}$ is eventually in V .

Conversely, let $\tau_{x_0} = \{U \in \tau : x_0 \in U\}$, which is filtering on X . Let $N_{\tau_{x_0}} = \{(x, U) : x \in U, U \in \tau_{x_0}\}$ be directed by $(x, U) \leq (x', U')$ if and only if $U \supseteq U'$ as in (iv) above. Then $x_0 = \lim_{(x,U) \in N_{\tau_{x_0}}} x$. Now, let $V \in \sigma$ with $f(x_0) \in V$. The assumptions on f tell us there is $v - V \in N_{\tau_{x_0}}$ such that for $v \geq v_V$, we have $f(x_0) \in V$. We have $v_V = (x, U)$ for some

$U \in \tau_{x_0}$ and $x \in U$, so for any $x' \in U$, $(x', U) \geq (x, U)$ and $f(x') = f(x_{x', U}) \in V$, so that $x_0 \in U = \bigcup_{x' \in U} \{x'\} \subseteq f^{-1}(V)$, so f is continuous at x_0 . But $x_0 \in X$ was arbitrary. ■

Remark. We get the following consequences of this result:

- (i) Given topologies τ, τ' on X , $\tau' \subseteq \tau$ if and only if $\tau' - \lim_{v \in N} x_v = x_0$ whenever $\tau - \lim_{v \in N} x_v = x_0$ for any $x_0 \in X$.
- (ii) (limits in product topology) $\{(x_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be topological space and $X = \prod_{\alpha \in A} X_\alpha$ equipped with the product topology π . If $(x^{(v)})_{v \in N}$ is a net in X and $x^{(0)} \in X$, then $\pi - \lim_{v \in N} x^{(v)} = x^{(0)}$ if and only if for every $\alpha \in A$, $\tau_\alpha - \lim_{v \in N} x_\alpha^{(v)} = x_\alpha^{(0)}$. Recall that π is the coarsest topology making each μ_α continuous.
- (iii) If X is a normed space and $(f_v)_{v \in N} \subset X^*$, $f_0 \in X^*$, then $w^* - \lim_{v \in N} f_v = f_0$ if and only if $\lim_{v \in N} f_v(x) = f_0(x)$ for each $x \in X$.

8.2 ROLES OF WEAK AND WEAK* TOPOLOGIES IN CONVEXITY

We begin with a useful technical lemma:

8.3 Lemma. *In an \mathbb{F} -vector space, if $f_0, f_1, \dots, f_n \in X'$ with $\ker f_0 \supseteq \bigcap_{i=1}^n \ker f_i$, then $f \in \text{span}\{f_1, \dots, f_n\}$.*

PROOF Define $T : X \rightarrow \mathbb{F}^n$ by $Tx = (f_1(x), \dots, f_n(x))$. Then $\ker T = \bigcap_{i=1}^n \ker f_i$. Let $\mathcal{R} = \text{im } T \subseteq \mathbb{F}^n$ and $g_0 \in \mathcal{R}'$ by $g_0(Tx) = f_0(x)$. Then g_0 is well-defined: if $Tx = Ty$, then $x - y \in \ker T \subseteq \ker f_0$, so $f_0(x - y) = 0$ so $f_0(x) = f_0(y)$. Also g_0 is linear. Let $g \in (\mathbb{F}^n)'$ such that $g|_{\mathcal{R}} = g_0$. Hence there are $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $g(y_1, \dots, y_n) = \sum_{j=1}^n \alpha_j y_j$. Hence for $x \in X$,

$$f_0(x) = g_0(Tx) = g(Tx) = g(f_1(x), \dots, f_n(x)) = \sum_{j=1}^n \alpha_j f_j(x)$$

so that $f_0 = \sum_{j=1}^n \alpha_j f_j$. ■

We now show that the w^* -continuous functions are actually sufficient for the separation theorems:

8.4 Theorem. (w^* -Separation) *Let X be a normed space, $A, B \subset X^*$ each be non-empty and convex, with $A \cap B = \emptyset$ and B w^* -open. Then there is $x \in X$ and $\alpha \in \mathbb{R}$ such that*

$$\text{Re } f(x) \leq \alpha < \text{Re } g(x)$$

for $f \in A$ and $g \in B$.

PROOF Since $w^* \subseteq \tau_{\|\cdot\|}$, the norm separation theorem gives $F \in X^{**}$ and $\alpha \in \mathbb{R}$ such that $\text{Re } F(f) \leq \alpha < \text{Re } F(g)$ for $f \in A$, $g \in B$. It suffices to show that $F = \hat{x}$ for some $x \in X$: assuming this, we have $\text{Re } f(x) = \text{Re } \hat{x}(f) \leq \alpha < \text{Re } \hat{x}(g) = \text{Re } g(x)$, which is the desired result.

Fix some $f_0 \in B$, and by w^* -continuity get x_1, \dots, x_n in X such that

$$f_0 \in U = \bigcap_{i=1}^n \hat{x}_i^{-1}(V_i) \subseteq B$$

where each $V_i \subset \mathbb{F}$ is an open neighbourhood of $f_0(x_i)$. Let $Y = \bigcap_{i=1}^n \ker \hat{x}_i \subseteq X^*$. Then for each $i = 1, \dots, n$,

$$\hat{x}_i(f_0 + Y) = \{f_0(x_i)\} \subset V_i,$$

so that $f_0 + Y \subseteq U \subseteq B$. Thus if $f \in Y$ is arbitrary, then $f_0 + f \in B$ so that $\operatorname{Re} F(f_0 + f) > \alpha$, or equivalently $\operatorname{Re} F(f) > \alpha - \operatorname{Re} F(f_0) =: c$ where c is a fixed quantity. But Y is a vector subspace, and hence closed under scalar multiplication; thus $\alpha \operatorname{Re} F(f) > c$ for any $\alpha \in \mathbb{F}$, forcing $F(f) = 0$. Thus $Y \subseteq \ker F$, and apply [Lemma 8.3](#) so that $F \in \operatorname{span}\{\hat{x}_1, \dots, \hat{x}_n\} \subseteq \hat{X}$. Thus $F = \hat{x}$ for some $x \in X$. ■

8.5 Theorem. (w^* -Closed Convex Hull) If $S \subset X^*$, then

$$\overline{\operatorname{co}}^{w^*} S = \bigcap \{f \in X^* : \operatorname{Re} f(x) \leq \alpha\} \supseteq S : x \in X, \alpha \in \mathbb{R}$$

PROOF The set on the right is w^* -closed and convex being the intersection of such. Conversely, if $f \in X^* \setminus \overline{\operatorname{co}}^{w^*} S$, which is open, then there is a basic w^* -open neighbourhood

$$B = \bigcap_{j=1}^n \hat{x}_j^{-1}(f(x_j) + \mathbb{D}) \subseteq X^* \setminus \overline{\operatorname{co}}^{w^*} S$$

so that $B \cap \overline{\operatorname{co}}^{w^*} S = \emptyset$. Also, B is convex. ■

Remark. If X is a normed space, a closed half space $H = \{x \in X : \operatorname{Re} f(x) \leq \alpha\}$ for some f in X^* , $\alpha \in \mathbb{R}$. Hence, H is weakly closed $(\operatorname{Re} f)^{-1}([\alpha, \infty)) = f^{-1}(\{z \in \mathbb{C} : \operatorname{Re} z \geq \alpha\})$ is w -closed. Thus if $S \subset X$, we have $\overline{\operatorname{co}} S \in w = \sigma(X, X^*) \subseteq \tau_{\|\cdot\|}$, so $\overline{\operatorname{co}} S$ is automatically weakly closed. Hence if $C \subseteq X$ is convex, then C is norm closed if and only if C is w -closed.

Definition. Let X be a normed space. If $E \subseteq X$ (non-empty), the **polar** of E is given by

$$\begin{aligned} E^\circ &= \{f \in X^* : \operatorname{Re} f(x) \leq 1 \text{ for all } x \text{ in } E\} \subseteq X^* \\ &= \bigcap_{x \in E} \{f \in X^* : \operatorname{Re} \hat{x}(f) \leq 1\} \end{aligned}$$

so E° is convex and w^* -closed in X^* , and $0 \in E^\circ$.

If $F \subseteq X^*$ (non-empty), let the **pre-polar** of F be given by

$$F_\circ = \{x \in X : \operatorname{Re} f(x) \leq 1 \text{ for all } f \text{ in } F\}$$

so, like above, F_\circ is convex, $(w-)$ closed, and $0 \in F_\circ$.

8.6 Theorem. (Bipolar) (i) If $\emptyset \neq E \subseteq X$, then $(E^\circ)_\circ = \overline{\operatorname{co}}(E \cup \{0\})$.

(ii) If $\emptyset \neq F \subseteq X^*$, then $(F_\circ)^\circ = \overline{\operatorname{co}}^{w^*}(F \cup \{0\})$.

PROOF (i) Note that $E \cup \{0\} \subseteq (E^\circ)_\circ$, so $\overline{\operatorname{co}}(E \cup \{0\}) \subseteq (E^\circ)_\circ$. If $x_0 \in X \setminus \overline{\operatorname{co}}(E \cup \{0\})$, then the separation theorem provides $f \in X^*$, $\alpha \in \mathbb{R}$ so $\operatorname{Re} f(x_0) > \alpha \geq \operatorname{Re} f(x)$ for $x \in E \cup \{0\}$. Notice that $\alpha \geq \operatorname{Re} f(0) = 0$, and we let $\beta = \frac{1}{2}[\operatorname{Re} f(x_0) + \alpha] > 0$, so $\operatorname{Re} f(x_0) > \beta \geq \operatorname{Re} f(x)$ for $x \in E \cup \{0\}$, $\beta > 0$. Let $g = \frac{1}{\beta}f$ and we see that $g \in E^\circ$ and $\operatorname{Re} g(x_0) > 1$, $x_0 \notin (E^\circ)_\circ$.

(ii) Similar, use w^* -separation. ■

Remark. Let $Y \subseteq X$ be a subspace. If $f \in Y^0$, then $\operatorname{Re} f(y) \leq 1$ for $y \in Y$ implies that $f(y) = 0$ for all $y \in Y$. We write $Y^a = Y^0$, and $Y^a = \{f \in X^* : f|_Y = 0\}$ is called the **annihilator** of Y . Likewise, if $Z \subseteq X^*$ is a subspace, then $Z_a = Z_0$ where $Z_a = \{x \in X : f(x) = 0 \text{ for each } f \in Z\}$ is called the **pre-annihilator**. Notice that Y^a and Z_a are subspaces.

8.7 Corollary. (i) If $Y \subseteq X$ is a subspace, then $(X^a)_a = \overline{X}$.

(ii) If $Z \subseteq X^*$ is a subspace, then $(Z_a)^a = \overline{Z}^{w^*}$.

8.8 Lemma. If X is a normed space, then $B(X)^0 = B(X^*)$ and $B(X^*)_0 = B(X)$.

PROOF If $f \in B(X)^0$, then $\operatorname{Re} f(x) \leq 1$ for $x \in B(X)$. Thus for $x \in B(X)$, $|f(x)| = \operatorname{sgn} f(x) f(x) = f(\operatorname{sgn} f(x) x) \leq 1$, so $\|f\| \leq 1$ and $f \in B(X^*)$. Conversely, if $f \in B(X^*)$, $x \in B(X)$, then $\operatorname{Re} f(x) \leq |f(x)| \leq 1$ so $f \in B(X)^0$. Then use the Bipolar theorem. ■

8.9 Theorem. (Goldstine) If X is a normed space, then $\overline{B(\hat{X})}^{w^*} = B(X^{**})$. Note that $w^* = \sigma(X^{**}, \hat{X}^*)$.

PROOF The Bipolar theorem provides $\overline{B(\hat{X})}^{w^*} = \overline{B(\hat{X})}^{w^*} B(\hat{x}) = (B(\hat{X})_0)^\circ$. But, in X^* ,

$$\begin{aligned} B(X)^\circ &= \{f \in X^* : \operatorname{Re} f(x) \leq 1 \text{ for } x \in B(X)\} \\ &= \{f \in \hat{X}^* : \operatorname{Re} \hat{x}(f) \leq 1 \text{ for } x \in B(X)\} \\ &= B(\hat{X})_0. \end{aligned}$$

Hence we have, using the lemma,

$$\overline{B(\hat{X})}^{w^*} = (B(\hat{X})_0)^\circ = (B(X)^\circ)^\circ = B(X^*)^\circ = B(X^{**})$$

Example. (i) Recall that $c_0^* \cong \ell_1$ and $\ell_1^* \cong \ell_\infty$, where $c_0 \subseteq \ell_\infty$. Thus by Goldstine, $\overline{B(c_0)}^{w^*} = B(\ell_\infty)$, so $w^* = \sigma(\ell_\infty, \ell_1)$. Since ℓ_1 is separable, we have that $(B(\ell_\infty), w^*)$ is metrizable. In fact, if $x \in \ell_\infty$, then if $x^{(n)} = (x_1, \dots, x_n, 0, 0, \dots) \in c_0$, we have $x = w^* - \lim_{n \rightarrow \infty} x^{(n)}$.
(ii) $\ell_\infty^* \cong \operatorname{FA}(\mathbb{N})$. But $B(\operatorname{FA}(\mathbb{N}), w^*)$ is not metrizable. Since $\ell_1^* \cong \ell_\infty$, there is a natural isometric embedding $\ell_1 \hookrightarrow \operatorname{FA}(\mathbb{N})$. Then $y^{(n)} = \frac{1}{n}(1, 1, \dots) \in B(\ell_1)$, and w^* -cluster point of $(y^{(n)})_{n=1}^\infty \subset B(\operatorname{FA}(\mathbb{N}))$ is a Banach limit.

8.10 Corollary. If $F \in X^{**}$, there always exists a net $(x_\nu)_{\nu \in N} \subset X$ such that

$$F = w^* - \lim_{\nu \in N} \hat{x}_\nu \text{ and } \|x_\nu\| \leq \|F\|$$

PROOF If $F \neq 0$, $\frac{1}{\|F\|}F \in B(X^{**}) = \overline{B(\hat{X})}^{w^*}$, and we may find $(y_\nu)_{\nu \in N} \subset B(X)$ such that $(\hat{y}_\nu)_{\nu \in N} \subset B(\hat{X})$ and $\frac{1}{\|F\|}F = w^* - \lim_{\nu \in N} \hat{y}_\nu$. Let $x_\nu = \|F\|y_\nu$. ■

Consider $\mathcal{F} = w^*_{\frac{1}{\|F\|}F} = \{U \in w^*|_{B(X^{**})} : F \in U\}$ is a filtering family. Each $U \in w^*_{\frac{1}{\|F\|}F}$ has $U \cap B(\hat{X}) \neq \emptyset$ by Goldstine. Let $N_{\mathcal{F}} = \{(x, U) : x \in B(X), \hat{x} \in U, U \in \mathcal{F}\}$. Then $(x_\nu)_{\nu \in N_{\mathcal{F}}} = (x)_{(x, U) \in N_{\mathcal{F}}}$ works.

Definition. A normed space X is **reflexive** if $\hat{X} = X^{**}$.

Notice that $X^{**} = (X^*)^*$ is complete, and $x \mapsto \hat{x}$ is an isometry, so a reflexive space is always complete.

8.11 Theorem. Let X be a Banach space. The following are equivalent:

- (i) X is reflexive
- (ii) $B(X)$ is w -compact
- (iii) $w^* = w$ on X^*
- (iv) X^* is reflexive.

PROOF The map $\iota : x \mapsto \hat{x}$ is a $w - w^*|_{\hat{X}}$ -homeomorphism. Recall $w^* = \sigma(X^{**}, \hat{X}^*)$, and $w^*|_{\hat{X}} = \sigma(\hat{X}, (\hat{X}^*)^*|_{\hat{X}})$ and we have for $x_0 \in X$, net $(x_\nu)_{\nu \in N}$ in X ,

$$\begin{aligned} w - \lim_{\nu \in N} x_\nu = x_0 &\iff \lim_{\nu \in N} f(x_\nu) = f(x_0) \forall f \in X^* \\ &\iff \lim_{\nu \in N} \hat{x}_\nu(f) = \hat{x}_0(f) \forall f \in X^* \\ &\iff \lim_{\nu \in N} \hat{f}(\hat{x}_\nu) = \hat{f}(\hat{x}_0) \end{aligned}$$

and having the same convergent nets means that the topologies are the same.

(i \Rightarrow ii) By assumption, $\widehat{B(X)} = B(\hat{X}) = B(X^{**})$. Since $B(X^{**})$ is w^* -compact, and hence $\iota^{-1}(B(X^{**})) = B(X)$ is w -compact

(ii \Rightarrow i) If $B(X)$ is w -compact, then since $x \mapsto \hat{x} : X \rightarrow X^{**}$ is continuous, we see that $B(\hat{X}) = \widehat{B(X)}$ is w^* -compact.

(i \Rightarrow iii) We have $\hat{X} = X^{**}$ so on X^* , we have $w = \sigma(X^*, X^{**}) = \sigma(X^*, \hat{X}) = w^*$.

(iii \Rightarrow iv) $B(X^*)$ is compact, hence w -compact, so by (ii) implies (i) applied to X^* , we have that X^* is reflexive.

(iv \Rightarrow i) We assume $\widehat{X^*} = X^{***}$. Thus on X^{***} , we have $w = \sigma(X^{**}, X^{***}) = \sigma(X^{**}, \widehat{X^*}) = w^*$. Now $B(\hat{X}) = B(X^{**}) \cap \hat{X}$ is norm-closed and convex, hence w -closed, by Closed Convex Hull theorem. Thus from above, $B(\hat{X})$ is w^* -closed, so $B(\hat{X}) = \overline{B(\hat{X})}^{w^*} = B(X^{**})$ by Goldstine, so $\hat{X} = X^{**}$. ■

8.12 Corollary. (i) Any finite dimensional normed space is reflexive.

(ii) Any closed subspace Y of a normed space X is reflexive.

PROOF (i) A finite dimensional normed space is complete, and its closed ball is compact, and thus w -compact as $\tau_{\|\cdot\|} \supseteq w$.

(ii) By Hahn-Banach, $Y^* = X^*|_Y$, so $\sigma(Y, Y^*) = \sigma(Y, X^*|_Y) = \sigma(X, X^*)|_Y$. Now $B(Y) = B(X) \cap Y$ is norm-closed and convex, hence w -closed in $B(X)$. But $B(X)$ is w -compact, so $B(Y)$ is a w -closed subset of a w -compact space and thus w -compact. ■

8.3 EXTREME POINTS AND THE KREIN-MILMAN THEOREM

Definition. Let X be a vector space and $C \subset X$ convex. A **face** F of C is any non-empty subset such that if $x \in F$, $x = (1-t)y + tz$, $t \in (0, 1)$, $y, z \in C$ implies that $y, z \in F$. A **extreme point** of C is a singleton face, i.e. $\text{ext } C = \{x \in C : \{x\} \text{ is a face of } C\}$. Hence $x \in \text{ext } C$ if for any $t \in (0, 1)$ and $y, z \in C$, if $x = (1-t)y + tz$ then $x = y = z$.

- Remark.* (i) Faces of C are not necessarily convex.
 (ii) A face F' of a convex face F of C is itself a face of C .
 (iii) $\text{ext } F \subseteq \text{ext } C$.
 (iv) If $f \in X'$ and $\text{Re } f(C) = [a, b]$, then $(\text{Re } f)^{-1}(\{b\})$ is itself a face of C .

8.13 Theorem. (Krein-Milman) *Let X be a normed space and $C \subset X^*$ convex and w^* -compact. Then $C = \overline{\text{co}}^{w^*} \text{ext } C$.*

PROOF We first verify that any w^* -closed face of C admits an extreme point. We let $\mathcal{F} = \{F : F \text{ is a convex } w^*\text{-closed face of } C\}$, which is partially ordered by reverse inclusion. If \mathcal{C} is a chain in \mathcal{F} with $F_1, \dots, F_n \in \mathcal{C}$, we may assume $F_1 \supseteq \dots \supseteq F_n$ so that \mathcal{C} has the finite intersection property. Thus $\emptyset \neq F_0 = \bigcap_{F \in \mathcal{C}} F$. If $x \in F_0$, $t \in (0, 1)$, $y, z \in C$ and $x = (1 - t)y + tz$, then $x \in F$ for any $F \in \mathcal{C}$ so $y, z \in F$ for any $F \in \mathcal{C}$. Thus $y, z \in \bigcap_{F \in \mathcal{C}} F = F_0$. Also F_0 is closed, so $F_0 \in \mathcal{F}$. Thus F_0 is an upper bound in \mathcal{F} for \mathcal{C} , so by Zorn, get some maximal element M .

Let M be a minimal w^* -closed convex face of F . Then given $x \in X$, $\text{Re } \hat{x} : X^* \rightarrow \mathbb{R}$ is w^* -continuous, and hence $\text{Re } \hat{x}(M) = [a_x, b_x]$ since the only compact convex subsets of \mathbb{R} are compact intervals. But then $F_x = (\text{Re } \hat{x})^{-1}(\{b_x\}) \cap M$ is a w^* -closed convex face in M , so that $F_x = M$. If $f, g \in M$, then $\text{Re } f(x) = \text{Re } \hat{x}(f) = b_x = \text{Re } \hat{x}(g) = \text{Re } g(x)$, so $f = g$ and hence $M = \{f\}$ and $f \in \text{ext } F$.

Now let $f_0 \in X^* \setminus \overline{\text{co}}^{w^*} \text{ext } C$. Since C is w^* -compact and convex, $\text{Re } \hat{x}(C) = [a_x, b_x]$, so $C_x = (\text{Re } \hat{x})^{-1}(\{b_x\}) \cap C$ is a w^* -closed convex face of C . Hence by above, there is $f \in \text{ext } C_x \subseteq \text{ext } C$ with $\text{Re } \hat{x}(f) = b_x$. But then $\text{Re } \hat{x}(f_0) > a \geq \text{Re } \hat{x}(f) = b_x$, so $\text{Re } \hat{x}(f_0) \notin [a_x, b_x] = \text{Re } \hat{x}(C)$, so $f_0 \notin C$. Thus $C \subseteq \overline{\text{co}}^{w^*} \text{ext } C$, where the converse inclusion is obvious. ■

- 8.14 Corollary.** (i) *If $C \subset X$ is a w -compact convex set, then $C = \overline{\text{co}} \text{ext } C$.*
 (ii) *If $C \subset X$ is a norm-compact convex set, then $C = \overline{\text{co}} \text{ext } C$.*

PROOF (i) We have that $x \mapsto \hat{x} : X \rightarrow \hat{X} \subseteq X^{**}$ is continuous. Hence \hat{C} is w^* -compact in X^{**} , so $x \mapsto \hat{x} : C \rightarrow \hat{C}$ is a homeomorphism. In \hat{C} , we have

$$\widehat{\overline{\text{co}}^w \text{ext } C} = \overline{\text{co}}^{w^*} \text{ext } \hat{C} = \hat{C}$$

so that $C = \overline{\text{co}}^w \text{ext } C = \overline{\text{co}} \text{ext } C$ by the closed convex hull theorem.

- (ii) Since $w \subseteq \tau_{\|\cdot\|}$, any norm-compact is w -compact. ■

Remark. Let X be a normed space. Then $\text{ext } B(X) \subseteq S(X)$.

8.15 Proposition. *Let $1 < p < \infty$. Then $\text{ext } B(\ell_p) = S(\ell_p)$.*

PROOF Let $x \in S(\ell_p)$, so $x = (1 - t)y + tz$. Then

$$1 = \|x\|_p \leq (1 - t)\|y\|_p + t\|z\|_p \leq 1$$

so that $\|y\|_p = \|z\|_p = 1$ and $\|x\|_p = (1 - t)\|y\|_p + t\|z\|_p$. Thus by the equality case for Minkowski, there is $s \geq 0$ so $s(1 - t)y = tz$. Taking norms, we have $y = z$. ■

8.16 Proposition. *We have $\text{ext } B(c_0) = \emptyset$.*

PROOF Let $x = (x_1, x_2, \dots) \in B(C_0)$. Since $\lim x_n = 0$, get n_0 so $|x_{n_0}| \leq 1/2$. If $x_{n_0} \neq 0$, let $y = (x_1, \dots, x_{n_0-1}, 2x_{n_0}, x_{n_0+1}, \dots)$ and $z = (x_1, \dots, x_{n_0-1}, 0, x_{n_0+1}, \dots)$, and similarly for $x_{n_0} = 0$. Thus we have in each case that $y, z \in B(C_0)$ and $x = y/2 + z/2$. ■

8.17 Corollary. *There exists no normed space X for which $c_0 \cong X^*$.*

PROOF If there were such X , then $B(c_0)$ would be w^* -compact, and hence Krein-Milman would imply $\text{ext } B(c_0) \neq \emptyset$. ■

Definition. Let (X, τ) be a compact Hausdorff space, and let

$$P(X) = \{\mu \in B(C^{\mathbb{R}}(X, \tau)^*) : \mu(1) = 1\}$$

8.18 Theorem. $\text{ext } P(X) = \{\hat{x} : x \in X\}$, where $\hat{x}(f) = f(x)$. Furthermore, $\overline{\text{co}}^{w^*} \text{ext } P(X) = P(X)$.

PROOF Write $C = C^{\mathbb{R}}(X, \tau)$. Note that $P(X) = B(C^*) \cap \hat{1}^{-1}(\{1\})$ is w^* -compact and convex. Hence by Krein-Milman, we have that $\overline{\text{co}}^{w^*} \text{ext } P(X) = P(X)$. It remains to describe $\text{ext } P(X)$.

(I) Some inequalities. Fix $\mu \in P(X)$. If $0 \leq f \leq 1$ in C , then $0 \leq 1 - f \leq 1$ so $\|f\|_{\infty}, \|1 - f\|_{\infty} \leq 1$. Thus $|\mu(f)| \leq 1$ and $|1 - \mu(f)| = |\mu(1 - f)| \leq 1$. Thus $0 \leq \mu(f) \leq 1$. Then if $g \neq 0$ and $g \geq 0$ in C , then we have $\mu(g/\|g\|_{\infty}) \geq 0$, so $\mu(g) > 0$; if $g \leq h$ in C , then $h - g \geq 0$ and $\mu(h) \geq \mu(g)$.

If $g \in C$, $g^+ = \max\{g, 0\}$, $g^- = \max\{-g, 0\} \in C$, and $g = g^+ - g^-$ while $|g| = g^+ + g^-$. Hence if $0 \leq f \leq 1$ in C and let $\mu_f(g) = \mu(fg)$ for $g \in C$, we have

$$\begin{aligned} |\mu_f(g)| &= |\mu_f(g^+ - g^-)| = |\mu(fg^+) - \mu(fg^-)| \leq \mu(fg^+) + \mu(fg^-) = \mu(fg) \\ &= \mu(f\|g\|_{\infty}) = \mu(f)\|g\|_{\infty} \end{aligned} \quad (8.1)$$

and, with $f = 1$, we have

$$|\mu(g)| \leq \mu(|g|) \quad (8.2)$$

(II) Let $\delta \in \text{ext } P(X)$. We first show for h, g in C that $\delta(hg) = \delta(h)\delta(g)$. To see this, since $\delta \neq 0$, we may find $0 \leq f \leq 1$ such that $0 < \delta(f) < 1$. Now let $\mu = \frac{1}{\delta(f)}\delta_f$ so, for $g \in C$, (8.1) provides

$$|\mu(g)| = \frac{1}{\delta(f)}|\delta_f(g)| \leq \frac{1}{\delta(f)}\delta(f)\|g\|_{\infty} = \|g\|_{\infty}$$

so that $\mu \in B(C^*)$. We also know that $\mu(1) = 1$. Hence $\mu \in P(X)$. Likewise, $\nu = \frac{1}{1-\delta(f)}\delta_{1-f} \in P(X)$. We have that

$$\delta(f)\mu + (1 - \delta(f))\nu = \delta$$

so by assumption on δ , $\mu = \delta$. Thus $\frac{1}{\delta(f)}\delta(fg) = \mu(g) = \delta(g)$, so that $\delta(fg) = \delta(f)\delta(g)$. Note that $C = \text{span}\{f \in C : 0 \leq f \leq 1\}$, so we get multiplicativity of δ .

Suppose now for each $x \in X$, there exists some $f_x \in \ker \delta$ so that $f_x(x) \neq 0$. Let $U_x = f_x^{-1}(\mathbb{R} \setminus \{0\})$, so $X = \bigcup_{x \in X} \{x\} = \bigcup_{x \in X} U_x$ so there are x_1, \dots, x_n in X so $X = \bigcup_{j=1}^n U_{x_j}$. Then $f = \sum_{j=1}^n f_{x_j}^2 > 0$ on X (by definition of each U_{x_j}), so $1/f \in C$. Then

$$1 = \delta(1) = \delta\left(\frac{1}{f}\right)\delta(f) = \delta\left(\frac{1}{f}\right)\sum_{j=1}^n \delta(f_{x_j})^2 = 0$$

since each $f_{x_i} \in \ker \delta$, which is absurd. Hence there is $x \in X$ so $f(x) = 0$ whenever $f \in \ker \delta$, so $\ker \delta \supsetneq \ker \hat{x}$, so $\delta \in \mathbb{R} \hat{x}$ and since $\delta(1) = 1 = \hat{x}(1)$, so $\delta = \hat{x}$.

(III) If $\hat{x} = (1 - t)\mu + tv$ and $t \in (0, 1)$, $\mu, v \in P(X)$, then by (8.2),

$$t|\nu(f)| \leq tv(|f|) \leq \hat{x}(|f|) = |f(x)|$$

so $\ker \nu \supseteq \ker \hat{x}$ and as above, $\nu = \hat{x}$. Then $\mu = \hat{x}$. ■

Remark. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, it is similar to show that $\text{ext } B(C^{\mathbb{F}}(X, \tau)^*) = \{z\hat{x} : z \in \mathbb{F}, |z| = 1, x \in X^*\}$.

Let $PA(\mathbb{N}) = \{\mu \in \text{FA}(\mathbb{N}) : \|\mu\|_{\text{var}} \leq 1, \mu(\mathbb{N}) = 1\}$ so, as above, $PA(\mathbb{N})$ is a $w^* = \sigma(\text{FA}(\mathbb{N}), \ell_{\infty})$ -compact set.

8.19 Proposition. $\text{ext } PA(\mathbb{N}) = \{\delta_{\mathcal{U}} : \mathcal{U} \text{ is an ultrafilter on } \mathbb{N}\}$

PROOF If $\delta \in \text{ext } PA(\mathbb{N})$, let $f_{\delta} \in \ell_{\infty}^*$ be as in A1. As above, we compute that $f_{\delta}(\chi_E \chi_F) = f_{\delta}(\chi_E) f_{\delta}(\chi_F)$, and we have $\chi_E \chi_F = \chi_{E \cap F}$ and hence $\delta(E \cap F) = \delta(E) \delta(F)$. Hence

$$\mathcal{U} = \{E \subseteq \mathbb{N} : \delta(E) \neq 0\} = \{E \subseteq \mathbb{N} : \delta(E) = 1\}$$

is an ultrafilter. The converse is easy. ■

9 EUCLIDEAN AND HILBERT SPACES

Definition. Let X be a vector space over \mathbb{F} (\mathbb{R} or \mathbb{C}). A form $[\cdot, \cdot] : X \rightarrow \mathbb{F}$ is called **Hermitian** if for x, x', y in X , $\alpha \in \mathbb{F}$, we have

- (i) $[x + \alpha x', y] = [x, y] + \alpha [x', y]$
- (ii) $[\overline{[y, x]}, x] = [x, y]$

and furthermore **positive** if

- 3. $[x, x] \geq 0$ for all $x \in X$

and **non-degenerate** if

- 4. $[x, y] = 0$ for all $y \in X$ implies $x = 0$.

9.1 Proposition. Let $[\cdot, \cdot]$ be a positive Hermitian form. Let $p(x) = [x, x]^{1/2}$, so $p : X \rightarrow [0, \infty)$. Then for $x, y \in X$ and $\alpha \in \mathbb{F}$, we have

- (i) $p(\alpha x) = |\alpha| p(x)$
- (ii) $[x, y] \leq p(x) p(y)$
- (iii) $p(x + y) \leq p(x) + p(y)$
- (iv) $[\cdot, \cdot]$ is non-degenerate if and only if $[x, x] > 0$ for $x \in X \setminus \{0\}$.

Furthermore, in this case, we have

- Equality in (ii) if and only if x, y are linearly dependent
- $[x, y] = p(x) p(y)$ if and only if there is $s \geq 0$ such that $x = sy$ or $y = sx$ if and only if equality holds in (iii).

PROOF (i) $p(\alpha x) = (\alpha \overline{\alpha} [x, x])^{1/2} = |\alpha| p(x)$

(ii) If $\alpha \in F$, then

$$\begin{aligned} 0 \leq [x - \alpha y, x - \alpha y] &= [x, x] - \bar{\alpha}[x, y] - \overline{\alpha[x, y]} + |\alpha|^2[y, y] \\ &= p(x)^2 - 2\operatorname{Re} \bar{\alpha}[x, y] + |\alpha|p(y)^2 \end{aligned}$$

Set $\alpha = \operatorname{sgn}[x, y]$ so that $\bar{\alpha}[x, y] = |[x, y]|$ so

$$|[x, y]| \leq \frac{1}{2}(p(x)^2 + p(y)^2)$$

Then if $t > 0$, by (i),

$$|[x, y]| = \left| \left[tx, \frac{1}{t}y \right] \right| \leq \frac{1}{2}(t^2 p(x)^2 + \frac{1}{t^2} p(y)^2)$$

If $p(x) = 0$, we let $t \rightarrow \infty$ so that $[x, y] = 0$; if $p(y) = 0$, we let $t \rightarrow 0^+$ and again that $[x, y] = 0$. If $[x, y] \neq 0$, set $t = p(y)/p(x)$ and we are done.

(iii)

$$\begin{aligned} p(x+y)^2 &= [x+y, x+y] = p(x)^2 + 2\operatorname{Re}[x, y] + p(y)^2 \\ &\leq p(x)^2 + 2|[x, y]| + p(y)^2 \\ &\leq p(x)^2 + 2p(x)p(y) + p(y)^2 = (p(x) + p(y))^2 \end{aligned}$$

(iv) We see, by (iii), if $p(x)^2 = [x, x] = 0$, then $[x, y] = 0$ for all y . Hence $[\cdot, \cdot]$ is non-degenerate if and only if $[x, x] > 0$ for $x \in X \setminus \{0\}$. If x, y are linearly dependant, then equality holds in (ii) by direct computation. If x, y are not linearly dependent, then the choice of $\alpha = \operatorname{sgn}[x, y]$ in (ii) gives strict inequality. The condition $[x, y] = p(x)p(y)$ requires non-negativity of $[x, y]$, showing one is a $R_{\geq 0}$ multiple of the other. This is equivalent to having equality in (iii). ■

Definition. A non-degenerate positive Hermitian form on a vector space \mathcal{E} is called an **inner product**. The pair $(\mathcal{E}, (\cdot, \cdot))$ is called a Euclidean space. If, further, \mathcal{E} is complete with respect to the induced norm $\|x\| = (x, x)^{1/2}$, then we call $(\mathcal{E}, (\cdot, \cdot))$ a **Hilbert space**.

Example. (i) (Euclidean Space) $(C[0, 1], \langle \cdot, \cdot \rangle)$ given by $(f, g) = \int_0^1 f \bar{g}$

(ii) (Euclidean Space) Recall $\ell = \{x \in \mathbb{F}^{\mathbb{N}} : x_n = 0 \text{ for all but finitely many } n\}$, and $(\ell, \langle \cdot, \cdot \rangle)$ with $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j \bar{y}_j$

(iii) (Hilbert Space) $(L_2[0, 1], (\cdot, \cdot))$, $(f, g) = \int_{[0, 1]} f \bar{g}$.

(iv) (Hilbert Space) $(\ell_2, (\cdot, \cdot))$, $(x, y) = \sum_{j=1}^{\infty} x_j \bar{y}_j$ (convergence by Hölder's inequality)

(v) (Non-separable Hilbert Space) Let Γ be an uncountable set. If $a = (a_\gamma)_{\gamma \in \Gamma} \in [0, \infty)^\Gamma$, we let $\mathcal{F} = \{F \subset \Gamma : |F| < \infty\}$. We define $\sum_{\gamma \in \Gamma} a_\gamma = \sup_{F \in \mathcal{F}} \sum_{\gamma \in F} a_\gamma = \lim_{F \in \mathcal{F}} \sum_{\gamma \in F} a_\gamma$ where \mathcal{F} is pre-ordered by inclusion. Suppose that $\sum_{\gamma \in \Gamma} a_\gamma < \infty$. Let $\Gamma_n = \{\gamma \in \Gamma : a_\gamma \geq 1/n\}$ and we have

$$\infty > \sum_{\gamma \in \Gamma} a_\gamma \geq \sup_{F \in \mathcal{F}} \sum_{\gamma \in F \cap \Gamma_n} a_\gamma \geq \sum_{F \in \mathcal{F}} \frac{|F \cap \Gamma_n|}{n}$$

so that $|\Gamma_n| < \infty$. Thus $\Gamma_a = \{\gamma \in \Gamma : a_\gamma > 0\} = \bigcup_{n=1}^{\infty} \Gamma_n$ is countable.

Now, define $\ell_2(\Gamma) = \{x = (x_\gamma) \in \mathbb{F}^\Gamma : \sum_{\gamma \in \Gamma} |x_\gamma|^2 < \infty\}$. If $x, y \in \ell_2(\Gamma)$, then we may let $\Gamma_{|x|^2} \cup \Gamma_{|y|^2} \subseteq \{\gamma_k\}_{k=1}^{\infty}$ so Hölder's inequality for ℓ_2 says that

$$\sum_{k=1}^{\infty} |x_{\gamma_k} \bar{y}_{\gamma_k}| \leq \left(\sum_{k=1}^{\infty} |x_{\gamma_k}|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |y_{\gamma_k}|^2 \right)^{1/2} < \infty.$$

Thus, $\sum_{k=1}^{\infty} x_{\gamma_k} \bar{y}_{\gamma_k}$ is absolutely converging. Write $(x, y) = \sum_{\gamma \in \Gamma} x_\gamma \bar{y}_\gamma = \sum_{k=1}^{\infty} x_{\gamma_k} \bar{y}_{\gamma_k}$. Now if $(x^{(n)})_{n=1}^{\infty} \subset \ell_2(\Gamma)$ is $\|\cdot\|_2$ -Cauchy, then $\Gamma' = \bigcup_{n=1}^{\infty} \Gamma_{|x^{(n)}|^2}$ is countable. Then since $\ell_2(\Gamma') \cong \ell_2$ (up to counting Γ'), so the Cauchy sequence has a limit. Thus $\ell_2(\Gamma)$ is a Hilbert space. It is immediate that $(\ell_2(\Gamma), \|\cdot\|_2)$ is non-separable.

(vi) Let $w : \mathbb{N} \rightarrow (0, \infty)$. Let $\ell_2^w = \{x \in \mathbb{F}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^2 w(k) < \infty\}$. Notice that if $x, y \in \ell_2^w$, then $(x_k w(k)^{1/2})_{k=1}^{\infty}, (y_k w(k)^{1/2})_{k=1}^{\infty} \in \ell_2$, so it follows that

$$(x, y)_w = \sum_{k=1}^{\infty} x_k \bar{y}_k w(k)$$

defines an inner product, and $W : \ell_2^w \rightarrow \ell_2$ by $W(x_k)_{k=1}^{\infty} = (x_k w(k)^{1/2})_{k=1}^{\infty}$ is a surjective linear isometry, so ℓ_2^w is a Hilbert space.

9.1 VARIOUS IDENTITIES

Let $[\cdot, \cdot]$ be a Hermitian form on X . Then we have the *polarization identity*: then over \mathbb{R} , $4[x, y] = [x + y, x + y] - [x - y, x - y]$, and over \mathbb{C} , $4[x, y] = \sum_{k=0}^3 i^k [x + i^k y, x + i^k y]$.

Now suppose $(\mathcal{E}, (\cdot, \cdot))$ is a Euclidean space. We say that $x, y \in \mathcal{E}$ are **orthogonal** if $(x, y) = 0$ and write $x \perp y$. We call a subset $E \subset \mathcal{E}$ **orthogonal** if $x \neq y \in E$ implies $x \perp y$. We write $x \perp E$ if $x \perp y$ for each $y \in E$. We have

- *Pythagoreans' identity*: if $\{x_1, \dots, x_n\} \subset \mathcal{E}$ orthogonal, then $\left\| \sum_{j=1}^n x_j \right\|^2 = \sum_{j=1}^n \|x_j\|^2$.
- *Parallelogram law*: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Note that if $\mathbb{F} = \mathbb{C}$, $(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2$ defines an inner product, for any norm satisfying the parallelogram law.

9.2 Proposition. If $y \in \mathcal{E}$ with $(\mathcal{E}, (\cdot, \cdot))$ a Euclidean space, then $f_y : \mathcal{E} \rightarrow \mathbb{F}$ by $f_y(x) = (x, y)$ is linear with $\|f_y\| = \|y\|$. Furthermore, $|f_y(x)| = \|y\|$ for $y \neq 0$, $x \in B(\mathcal{E})$ if and only if $x = \frac{\bar{y}}{\|y\|} y$ where $|\zeta| = 1$.

PROOF Linearity is from an assumption on (\cdot, \cdot) . Furthermore, Cauchy-Schwarz tells us that

$$|f_y(x)| = |(x, y)| \leq \|x\| \|y\| \Rightarrow \|f_y\| \leq \|y\|$$

so the equality case for Cauchy-Schwarz provides the last statement of the proposition, and supplies $\|f_y\| \geq \|y\|$. ■

Definition. In a Euclidean space $(\mathcal{E}, (\cdot, \cdot))$, a set $E \subset \mathcal{E}$ is called **orthonormal** provided that for $e, e' \in E$,

$$(e, e') = \begin{cases} 1 & : e = e' \\ 0 & : e \neq e' \end{cases}$$

9.3 Lemma. (Closest Approximation to Finite) Let $\{e_1, \dots, e_n\}$ be orthonormal in a Euclidean space $(\mathcal{E}, (\cdot, \cdot))$ and $\mathcal{M} = \text{span}\{e_1, \dots, e_n\}$. Then for $x \in \mathcal{E}$ we have that

- (i) $P_{\mathcal{M}}x = \sum_{j=1}^n (x, e_j) e_j$ satisfies that $x - P_{\mathcal{M}}x \perp \mathcal{M}$ and hence $\|x\|^2 = \|P_{\mathcal{M}}x\|^2 + \|x - P_{\mathcal{M}}x\|^2$
- (ii) $d(x, \mathcal{M}) = \left\| x - \sum_{j=1}^n (x, e_j) e_j \right\|^{1/2}$

PROOF (i) If $1 \leq k \leq n$, we have

$$(x - P_{\mathcal{M}}x, e_k) = (x, e_k) - \sum_{j=1}^n (x, e_j) (e_j, e_k) = (x, e_k) - (x, e_k) = 0$$

and it follows that $x - P_{\mathcal{M}}x \perp \mathcal{M}$. Pythagoras' law provides the second formula.

- (ii) Endow \mathbb{F}^n with the usual inner product $\|\cdot\|_2$. By Cauchy-Schwarz, for $x \in \mathcal{E}$ and $\alpha \in \mathbb{F}^n$,

$$\left| \left(\left((x, e_j) \right)_{j=1}^n, \alpha \right) \right| = \left| \sum_{j=1}^n (x, e_j) \bar{\alpha}_j \right| \leq \left(\sum_{j=1}^n |(x, e_j)|^2 \right)^{1/2} \|\alpha\|_2$$

so that

$$\begin{aligned} \left\| x - \sum_{j=1}^n \alpha_j e_j \right\|^2 &= \|x\|^2 - 2 \operatorname{Re} \sum_{j=1}^n (x, e_j) \bar{\alpha}_j + \sum_{j=1}^n |\alpha_j|^2 \\ &\geq \|x\|^2 - 2 \left| \left(\left((x, e_j) \right)_{j=1}^n, \alpha \right) \right| + \|\alpha\|_2^2 \\ &\geq \|x\|^2 - 2 \|P_{\mathcal{M}}x\| \|\alpha\|_2 + \|\alpha\|_2^2 \\ &= \|x - P_{\mathcal{M}}x\|^2 + (\|P_{\mathcal{M}}x\| - \|\alpha\|_2)^2 \end{aligned}$$

We get equality above if $x \perp \mathcal{M}$ or otherwise there is some $s \geq 0$ so $\alpha_j = s(x, e_j)$ for $j = 1, \dots, n$. Hence, in this case,

$$\left\| x - \sum_{j=1}^n s(x, e_j) e_j \right\|^2 = \|x - P_{\mathcal{M}}x\|^2 + \|P_{\mathcal{M}}x\|^2 (1 - s)^2$$

which is minimized when $s = 1$. ■

- Remark.* (i) The proof above shows that $P_{\mathcal{M}}x$ is the unique element of \mathcal{M} satisfying $\text{dist}(x, \mathcal{M}) = \|x - P_{\mathcal{M}}x\|$.
- (ii) It may be shown that $P_{\mathcal{M}} : \mathcal{E} \rightarrow \mathcal{E}$ is linear with $\text{im } P_{\mathcal{M}} = \mathcal{M}$, $P_{\mathcal{M}}^2 = P_{\mathcal{M}}$, and $\|P_{\mathcal{M}}\| = 1$ (in other words, this map is actually a projection operator)

9.4 Theorem. (Orthonormal Basis)

let $(\mathcal{E}, (\cdot, \cdot))$ be a Euclidean space, $E \subset \mathcal{E}$ an orthonormal set. Then the following are equivalent:

- (i) $\overline{\text{span}} E = \mathcal{E}$
- (ii) for $x \in \mathcal{E}$ $x = \sum_{e \in E} (x, e) e = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) e$, where $\mathcal{F} = \{F \subseteq E : |F| < \infty\}$, directed by inclusion (Bessel's identity)

(iii) For $x, y \in \mathcal{E}$, $(x, y) = \sum_{e \in E} (x, e)(e, y)$ (Parseval's identity).

PROOF ($i \Rightarrow ii$) For $F \in \mathcal{F}$, let $\mathcal{E}_F = \text{span } F$, so that $\mathcal{E}_F \subseteq \mathcal{E}_{F'}$ if $F \subseteq F'$ in \mathcal{F} and $\text{span } E = \bigcup_{F \in \mathcal{F}} \mathcal{E}_F$. Hence for $x \in \mathcal{E}$, we have

$$0 = \text{dist}(x, \text{span } E) = \text{dist}\left(x, \bigcup_{F \in \mathcal{F}} \mathcal{E}_F\right) = \inf_{F \in \mathcal{F}} \text{dist}(x, \mathcal{E}_F) = \lim_{F \in \mathcal{F}} \text{dist}(x, \mathcal{E}_F)$$

Thus by the f.d. approximation lemma, we have

$$0 = \lim_{F \in \mathcal{F}} \text{dist}(x, \mathcal{E}_F) = \lim_{F \in \mathcal{F}} \left\| x - \sum_{e \in F} (x, e)e \right\|$$

(ii \Leftrightarrow iii) We have

$$\begin{aligned} 0 &= \lim_{F \in \mathcal{F}} \left\| x - \sum_{e \in F} (x, e)e \right\|^2 \\ &= \lim_{F \in \mathcal{F}} \left(\|x\|^2 - 2 \operatorname{Re} \sum_{e \in F} \overline{(x, e)}(x, e) + \sum_{e \in F} \|(x, e)\|^2 \right) \\ &= \lim_{F \in \mathcal{F}} \left(\|x\|^2 - \sum_{e \in F} |(x, e)|^2 \right) \\ &= \|x\|^2 - \sum_{e \in E} |(x, e)|^2 \end{aligned}$$

(ii \Rightarrow iv) Recall that $f_y = (\cdot, y) \in \mathcal{E}^*$ so that

$$(x, y) = f_y \left(\lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e)e \right) = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) f_y(e) = \sum_{e \in E} (x, e)(e, y)$$

(iv \Rightarrow ii) Take $x = y$.

(iii \Rightarrow i) Obvious; $x = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e)e \in \overline{\text{span } E}$, i.e. $\mathcal{E} \subseteq \overline{\text{span } E} \subseteq \mathcal{E}$. ■

Definition. Any set $E \subset \mathcal{E}$ satisfying the above conditions is called a **orthonormal basis** for \mathcal{E} .

9.5 Theorem. (Gram-Schmidt) Let (x_1, x_2, \dots) be a linearly independent sequence in a euclidean space $(\mathcal{E}, (\cdot, \cdot))$. There exists an orthogonal sequence (z_1, z_2, \dots) which satisfies $\text{span}\{z_1, \dots, z_n\} = \text{span}\{x_1, \dots, x_n\}$ for $n = 1, 2, \dots$ so that $\text{span}\{z_1, z_2, \dots\} = \text{span}\{x_1, x_2, \dots\}$.

PROOF Let $\mathcal{E}_n = \text{span}\{x_1, \dots, x_n\}$. We set

$$\begin{aligned} z_1 &= x_1 & e_1 &= \frac{z_1}{\|z_1\|} \\ z_2 &= x_2 - P_{\mathcal{E}_1} x_2 & e_2 &= \frac{z_2}{\|z_2\|} \\ &\vdots & & \\ z_{n+1} &= x_{n+1} - P_{\mathcal{E}_n} x_{n+1} & e_{n+1} &= \frac{z_{n+1}}{\|z_{n+1}\|} \end{aligned}$$

where $P_{\mathcal{E}_n} x = \sum_{j=1}^n (x, e_j) e_j$. Inductively, $z_n \in \mathcal{E}_n$ and $z_n \perp \mathcal{E}_k$ for $k = 1, \dots, n-1$. Hence each set $\{z_1, \dots, z_n\}$ is orthonormal and $\text{span}\{z_1, \dots, z_n\} \subseteq \text{span}\{x_1, \dots, x_n\}$ is of full dimension and hence equal. ■

9.6 Corollary. *Any separable Euclidean space admits an orthonormal basis.*

PROOF Let $\{x_n\}_{n=1}^\infty$ be dense in \mathcal{E} . Let $n_1 = \min\{n : x_n \neq 0\}$, and $n_{k+1} = \min\{n : x_n \notin \text{span}\{x_{n_1}, \dots, x_{n_k}\}\}$. Then $\{x_{n_1}, x_{n_2}, \dots\}$ and normalize to get an orthonormal set $E = \{e_1, e_2, \dots\}$ which satisfies $\overline{\text{span } E} = \overline{\text{span}\{x_n\}_{n=1}^\infty} = \mathcal{E}$. ■

9.7 Theorem. (Riesz Fischer) *Let $(\mathcal{E}, (\cdot, \cdot))$ be a Euclidean space. Then \mathcal{E} is a Hilbert space if and only if for any orthonormal set E and an $\alpha = (\alpha_e)_{e \in E} \in \ell_2(E)$, we have that $\sum_{e \in E} \alpha_e e \in \mathcal{E}$.*

PROOF (\implies) If $\alpha \in \ell_2(E)$ then $E_\alpha = \{e \in E : \alpha_e \neq 0\}$ is countable, and write $E_\alpha = \{e_1, e_2, \dots\}$. If $m < n$, we have

$$\left\| \sum_{k=1}^n \alpha_{e_k} e_k - \sum_{k=1}^m \alpha_{e_k} e_k \right\|^2 = \sum_{k=m+1}^n |\alpha_{e_k}|^2 \leq \sum_{k=n+1}^\infty |\alpha_{e_k}|^2 \rightarrow 0$$

so $x_\alpha = \sum_{k=1}^\infty \alpha_{e_k} e_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_{e_k} e_k$ converges. If $F \in \mathcal{F}$, $F \supseteq \{e_1, \dots, e_n\}$, then

$$\left\| x_\alpha - \sum_{e \in F} \alpha_e e \right\|^2 = \sum_{e \in \{e_1, e_2, \dots\} \setminus F} |\alpha_e|^2 \leq \sum_{k=n+1}^\infty |\alpha_{e_k}|^2 \rightarrow 0$$

so $x_\alpha = \sum_{e \in E} \alpha_e e = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) e$.

(\impliedby) Let $(x^{(n)})_{n=1}^\infty$ be Cauchy in \mathcal{E} . Let $\mathcal{M} = \overline{\text{span}\{x^{(n)}\}_{n=1}^\infty} \subset \mathcal{E}$ so \mathcal{M} is separable and admits a countable orthonormal basis $E = \{e_1, e_2, \dots\}$. Then we appeal to orthonormal basis to see that for any $x \in \mathcal{M}$, $\sum_{k=1}^\infty |(x, e_k)|^2 = \|x\|^2 < \infty$ and $x = \sum_{k=1}^\infty (x, e_k) e_k$.

Our present assumption show that $U : \ell_2(E) \rightarrow \mathcal{M}$ given by $U_\alpha = \sum_{k=1}^\infty \alpha_k e_k$ always converges in $\mathcal{M} \subseteq \mathcal{E}$. Then orthonormal basis theorem gives $\|U_\alpha\| = \|\alpha\|_2$ so U is a surjective isometry. We let $\alpha^{(n)} = ((x^{(n)}, e_k))_{k=1}^\infty \in \ell_2(E)$, then $\|\alpha^{(n)} - \alpha^{(m)}\|_2 = \|U_\alpha^{(n)} - U_\alpha^{(m)}\| = \|x^{(n)} - x^{(m)}\|$ so $(\alpha^{(n)})_{n=1}^\infty$ is Cauchy and in $\ell_2(E)$ and hence admits a limit α . Furthermore,

$$\left\| \sum_{k=1}^\infty \alpha_k e_k - x^{(n)} \right\| = \|U_\alpha - U_\alpha^{(n)}\| = \|\alpha - \alpha^{(n)}\| \rightarrow 0$$

as required. ■

Definition. If $\emptyset \neq S \subset \mathcal{E}$, $(\mathcal{E}, (\cdot, \cdot))$ a Euclidean space, we define its **perpendicular** by $S^\perp = \{y \in \mathcal{E} : (x, y) = 0 \text{ for any } x \in S\}$.

Remark. (i) $S \subseteq T$ implies $T^\perp \subseteq S^\perp$

(ii) $S^\perp = \bigcap_{x \in S} \ker f_x$ and is thus closed.

(iii) $\overline{S}^\perp = S^\perp$, since $\overline{S}^\perp \subseteq \overline{S}^\perp$, and if $y \in S^\perp$ and $x \in \overline{S}$, then $x = \lim x_n$ with $x_n \in S$ so $(x, y) = f_y(x) = f_y(\lim x_n) = \lim f_y(x_n) = \lim (x_n, y) = 0$.

(iv) $(\overline{\text{span } S})^\perp = S^\perp$. Notice that $(\text{span } S)^\perp = S^\perp$ (easy test) and use (iii)

(v) $\overline{\text{span} S} \cap S^\perp = \{0\}$.

9.8 Theorem. (Existence of Orthonormal Basis) Let $(H, (\cdot, \cdot))$ be a Hilbert space.

(i) Given an orthonormal set $E \subset H$, $P_E : H \rightarrow H$, $P_E x = \sum_{e \in E} (x, e) e$ satisfies

$$\text{im } P_E \subseteq \overline{\text{span} E} \text{ for } x \in H, x - P_E x \in E^\perp$$

(ii) H admits an orthonormal basis, i.e. an orthonormal set M such that $\overline{\text{span} M} = H$.

PROOF (i) Let $\mathcal{F} = \{F \subseteq E : |F| < \infty\}$ be directed by inclusion, and for $F \in \mathcal{F}$, $\mathcal{E}_F = \text{span} F$. Then as in the proof of OMBT, we have for $x \in H$

$$0 \leq \text{dist}(x, \text{span} E)^2 = \lim_{F \in \mathcal{F}} \text{dist}(x, \mathcal{E}_F)^2 = \|x\|^2 - \sum_{e \in E} |(x, e)|^2$$

so $\sum_{e \in E} |(x, e)|^2 \leq \|x\|^2 < \infty$. Thus appealing to Riesz-Fischer, $P_E x = \sum_{e \in E} (x, e) e$ converges in H . Since $P_E x = \lim_{F \in \mathcal{F}} \sum_{e \in F} (x, e) e$, we see that $P_E x \in \overline{\text{span} E}$, so $\text{im } P_E \subseteq \overline{\text{span} E}$. Moreover, if $e' \in E$, $f_{e'} = (\cdot, e') \in H^*$ so

$$(x - P_E x, e') = (x, e') - f_{e'} \left(\sum_{e \in E} (x, e) e \right) = (x, e') - \sum_{e \in E} (x, e) f_{e'}(e) = -$$

so $x - P_E x \in E^\perp$.

(ii) Let $\mathcal{O} = \{E \subseteq H : E \text{ is orthonormal}\}$, which is partially ordered by inclusion. Note that $\emptyset \in \mathcal{O}$ vacuously. If $\mathcal{C} \subseteq \mathcal{O}$ is a chain, then $\bigcup_{E \in \mathcal{C}} E \in \mathcal{O}$ is an upper bound for \mathcal{C} . By Zorn' get a maximal element M .

Suppose $\overline{\text{span} M} \subsetneq H$, and get $x \in H \setminus \overline{\text{span} M}$ and $y = x - P_M x \in (\overline{\text{span} M})^\perp \setminus \{0\}$. But then $M \subsetneq M \cup \left\{ \frac{1}{\|y\|} y \right\}$, violating maximality. ■

9.9 Corollary. If H is a Hilbert space with orthonormal basis E , then the map

$$U : H \rightarrow \ell_2(E), Ux = ((x, e))_{e \in E}$$

is a surjective isometry which respects inner products.

PROOF We know $\|x\|^2 = \sum_{e \in E} |(x, e)|^2 = \|Ux\|_2^2$ from ONBT. It is evident that U is linear and $\text{im } U$ is dense in $\ell_2(E)$ so that U is surjective. Finally, if $x, y \in H$, then

$$(x, y)_H = \sum_{e \in E} (x, e) (y, e) = \sum_{e \in E} (x, e) \overline{(y, e)} = (Ux, Uy)_{\ell_2(E)}$$

as required. ■

Remark. If each of E, E' is an orthonormal basis for a Euclidean space $(\mathcal{E}, (\cdot, \cdot))$, then $|E| = |E'|$. We let \mathbb{k} be any countable dense subfield of \mathbb{F} . Then $D = \text{span}_{\mathbb{k}} E$, so $|D| = \aleph_0 |E| = |E|$ when $|E|$ is infinite. Since for e', e'' in E' , $\|e' - e''\| = \sqrt{2}$, we have that any ball $e' + \frac{1}{\sqrt{2}} D(\mathcal{E})$ contains at least one element of D , and $d_{e'} \neq d_{e''}$ if $e' \neq e''$ in E' . This shows that $|E| \geq |E'|$. Likewise $|E'| \leq |E|$.

9.10 Corollary. (Orthogonal complementation) Let $(\mathcal{E}, \|\cdot\|)$ be a Euclidean space and $\mathcal{M} \subseteq \mathcal{E}$ a subspace which is complete with respect to the norm induced from (\cdot, \cdot) , i.e. $(\mathcal{M}, (\cdot, \cdot))$ is a Hilbert space. Then there is a unique operator $P_{\mathcal{M}} = P : \mathcal{E} \rightarrow \mathcal{E}$ such that $\text{im } P = \mathcal{M}$ and $\text{im}(I - P) = \mathcal{M}^\perp$. Moreover,

- P is linear
- $\|P\| \leq 1$, $\|P\| = 1$ if $\mathcal{M} \neq \{0\}$
- $P^2 = P$
- for $x, y \in \mathcal{E}$, $(Px, y) = (Px, Py) = (x, Py)$.

Such an operator is called the **orthogonal projection**.

PROOF The theorem above provides an orthonormal basis E for \mathcal{M} . Then P_E , as defined above, satisfies $\text{im } P = \mathcal{M}$ and $\text{im}(I - P) = \mathcal{M}^\perp$. Moreover, if P satisfies those conditions, then for $x \in \mathcal{E}$,

$$Px + x - Px = x = P_E x + x - P_E x$$

so that

$$Px - P_E x = [x - P_E x] - [x - Px] \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$$

so $Px = P_E x$. Then if $x, y \in \mathcal{E}$ and $\alpha \in \mathbb{F}$,

$$P(x + \alpha y) + x + \alpha y - P(x + \alpha y) = x + \alpha y = Px + x - Px + \alpha[Py + y - Py]$$

so

$$P(x + \alpha y) - [Px + \alpha Py] = x - Px + \alpha[y - Py] - [x + \alpha y - P(x + \alpha y)] \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$$

and we have linearity.

If $x \in \mathcal{E}$, Pythagoras tells us that $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$ so $\|Px\| \leq \|x\|$, i.e. $\|P\| \leq 1$. If $e' \in E$, $Pe' = P_E e' = \sum_{e \in E} (e', e)e = e'$, so $P|_{\text{span } E} = \text{id}$ and by uniqueness of extension of bounded linear functionals (uniformly continuous), we see that $P|_{\mathcal{M}} = \text{id}_{\mathcal{M}}$. This shows that if $\mathcal{M} \neq \{0\}$, $\|P\| = 1$ and $P = P^2$. Furthermore, this also shows that $\text{im } P = \mathcal{M}$. Finally,

$$(Px, y) = (Px, Py + y - Py) = (Px, Py)$$

and likewise $(x, Py) = (Px, Py)$. ■

9.11 Corollary. Let H be a Hilbert space.

- (i) If \mathcal{M} is a closed subspace, then $(\mathcal{M}^\perp)^\perp = \mathcal{M}$.
- (ii) If $\emptyset \neq S \subset H$, then $(S^\perp)^\perp = \overline{\text{span}} S$.

PROOF (i) We have $\mathcal{M} \subseteq \mathcal{M}^{\perp\perp}$ and \mathcal{M} is complete and thus admits an orthogonal projection $P_{\mathcal{M}} : H \rightarrow H$ with $\text{im } P_{\mathcal{M}} = \mathcal{M}$ and $\text{im}(I - P_{\mathcal{M}}) = \mathcal{M}^\perp$. Now if $x \in \mathcal{M}^{\perp\perp}$, $P_{\mathcal{M}} x \in \mathcal{M}$ so that $x - P_{\mathcal{M}} x \in \mathcal{M}^\perp + \mathcal{M} = \mathcal{M}^{\perp\perp}$ so that $x - P_{\mathcal{M}} x \in \mathcal{M}^\perp$. Thus

$$x - P_{\mathcal{M}} x \in \mathcal{M}^{\perp\perp} \cap \mathcal{M}^\perp = \{0\}$$

so that $x = P_{\mathcal{M}} x \in \mathcal{M}$. Hence $\mathcal{M}^{\perp\perp} \subseteq \mathcal{M}$.

(ii) We have $(\overline{\text{span}} S)^\perp = S^\perp$ and apply (i). ■

9.12 Theorem. (Riesz-Fréchet) If H is a Hilbert space and $f \in H^*$, then there is a unique $x_0 \in H$ such that $f = f_{x_0}$; i.e. $f(x) = (x, x_0)$ for all $x \in H$.

PROOF If $f = 0$, let $x_0 = 0$. If $f \neq 0$, $\ker f \subsetneq H$ so $(\ker f)^{\perp\perp} = f$, so $(\ker f)^{\perp} \neq \{0\}$. If $x_1, x_2 \in (\ker f)^{\perp}$, then $f(x_2)x_1 - f(x_1)x_2 \in (\ker f)^{\perp} \cap \ker f = \{0\}$, so that $\dim(\ker f)^{\perp} = 1$ and $(\ker f)^{\perp} = \mathbb{F}x_1$. But then $f_{x_1} = (\cdot, x_1)$ has $\ker f_{x_1} = (\mathbb{F}x_1)^{\perp} = (\ker f)^{\perp\perp} = \ker f$, so there is $\alpha \in \mathbb{F}$ so $f = \alpha f_{x_1} = f_{\bar{\alpha}x_1}$. Set $x_0 = \bar{\alpha}x_1$.

Uniqueness holds since $x \mapsto f_x : H \rightarrow H^*$ is an isometry and thus injective. \blacksquare

- Remark.* (i) Many results above may fail in a non-complete Euclidean space. Consider $(\ell, (\cdot, \cdot))$ where ℓ is the space of finitely supported sequences. Define $f : \ell \rightarrow \mathbb{F}$ by $f(x) = \sum_{k=1}^{\infty} \frac{1}{k} x_k$. By Hölder, $|f(x)| \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) \|x\|_2$ so that $f \in \ell^*$. If there were $x^{(0)} \in \ell$ so that $f = f_{x^{(0)}}$ for some $x^{(0)} \in (\ker f)^{\perp} \setminus \{0\}$, we would then have $x_k^{(0)} = (e_k, x^{(0)}) = \frac{1}{k}$, which is non-zero for infinitely many k , giving a contradiction. In fact, $(\ker f)^{\perp} = \{0\}$ so that $(\ker f)^{\perp\perp} = \ell$.
- (ii) Let $(\mathcal{E}, (\cdot, \cdot))$ be a Euclidean space. Let $H = \bar{\mathcal{E}}$ be the metrical completion with respect to $\|x\|_2$. If $x, y \in H$, then $x = \lim x_n = \lim x'_n$ with $x_n, x'_n \in \mathcal{E}$, and $y = \lim y_n = \lim y'_n$ similarly. Then

$$\begin{aligned} |(x_n, y_n) - (x_m, y_m)| &\leq |(x_n, y_n) - (x_n, y_m)| + |(x_n, y_m) - (x_m, y_m)| \\ &\leq \|x_n\| \|y_n - y_m\| + \|x_n - x_m\| \|y_m\| \end{aligned}$$

so that $((x_n, y_n))_{n=1}^{\infty} \subset \mathbb{F}$ is Cauchy, and thus admits a limit. Moreover, $|(x_n, y_n) - (x'_n, y'_n)| \leq \|x_n\| \|y_n - y'_n\| + \|x_n - x'_n\| \|y'_n\|$. Thus, $(x, y) = \lim_{n \rightarrow \infty} (x_n, y_n) = \lim_{n \rightarrow \infty} (x'_n, y'_n)$ is well-defined on $H \times H$. It is straightforward to verify that this is an inner product, and $\|x\| = \lim_{n \rightarrow \infty} \|x_n\| = (x, x)^{1/2}$. Thus the completion of a Euclidean space is a Hilbert space.

- (iii) As a consequence of (ii), we have $\mathcal{E}^* = \{f_x : x \in H\}$ where $H = \bar{\mathcal{E}}$, as above. Furthermore, $\bar{\mathcal{E}} \cong H^{**}$.
- (iv) If H is a Hilbert space, the map $f \mapsto f_x$ from $H \rightarrow H^*$ is
- a conjugate linear map: $f_{x+\alpha y} = f_x + \bar{\alpha} f_y$
 - an isometry: $\|f_x\| = \|x\|$

10 ADJOINT OPERATORS

Definition. Let X, Y be vector spaces over \mathbb{F} , and $T \in \mathcal{L}(X, Y)$. Define the **adjoint** of T , $T^* : Y' \rightarrow X'$ by $T^*f = f \circ T$.

Notice that $T^* \in \mathcal{L}(Y', X')$.

10.1 Proposition. Let X, Y, Z be normed spaces, $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$. Then

- (i) $T^* \in \mathcal{B}(Y^*, X^*)$ with $\|T^*\| = \|T\|$
- (ii) $T \mapsto T^* : \mathcal{B}(X, Y) \rightarrow \mathcal{B}(Y^*, X^*)$ is linear
- (iii) $T^{**} := (T^*)^*$ satisfies $T^{**} \in \mathcal{B}(X^{**}, Y^{**})$ and $T^{**}\hat{x} = \widehat{Tx}$.
- (iv) $(S \circ T)^* = T^* \circ S^* \in \mathcal{B}(Z^*, X^*)$.

PROOF (i), (iii) If $f \in Y^*$, then

$$\|T^*f\| = \sup\{|T^*f(x)| : x \in B(X)\} \leq \sup\{\|f\| \|Tx\| : x \in B(X)\} \leq \|f\| \|T\|$$

so $\|T^*\| \leq \|T\|$. If $x \in X$ and $f \in Y^*$, then

$$T^{**}\hat{x}(f) = \hat{x}(T^*f) = T^*f(x) = f(Tx) = \widehat{Tx}(f)$$

so that $\|T\| = \|T^{**}|_X\| \leq \|T^{**}\| \leq \|T^*\|$.

(ii) Immediate.

(iv) Immediate. ■

Remark. If H, K are Hilbert spaces and $T \in \mathcal{B}(H, K)$, then we define for $x \in K$, T^*x by $f_{T^*x} = T^*f_x$. Notice that (i) and (iv) hold in this setting. However, (ii) is replaced by $T \mapsto T^*$ is conjugate linear. Notice that T^* satisfies $(Tx, y) = (x, T^*y)$ for $x, y \in H$.

10.2 Theorem. (Kernel-Annihilator) *If X and Y are Banach spaces, $T \in \mathcal{B}(X, Y)$, then $\ker T = [\text{im}(T^*)]_a$ and $\ker(T^*) = (\text{im } T)^a$.*

PROOF We have

$$\ker T = \{x \in X : Tx = 0\} = \{x \in X : T^*g(x) = g(Tx) = 0 \text{ for all } x \in X\} = [\text{im}(T^*)]_a$$

and

$$\ker(T^*) = \{g \in Y^* : T^*g = 0\} = \{g \in Y^* : g(Tx) = T^*g(x) = 0 \text{ for all } x \in X\} = [\text{im}(T)]^a \quad \blacksquare$$

Remark. If $T \in \mathcal{B}(H, K)$ where H, K are Hilbert spaces, then $\ker T = (\text{im } T^*)^\perp$, identifying $T^{**} = T$ since Hilbert spaces are reflexive.

10.3 Theorem. (Characterization of Invertibility) *Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then TFAE:*

- (i) T is invertible
- (ii) T^* is invertible
- (iii) $\overline{\text{im } T} = Y$ and $\inf\{\|Tx\| : x \in S(X)\} > 0$, we say that T is **bounded below**, and
- (iv) both T and T^* are bounded below.

PROOF $(i \Rightarrow ii)$ Let $T^{-1} \in \mathcal{B}(Y, X)$, so $I_Y = TT^{-1}$, $I_X = T^{-1}T$. Then $(T^{-1})^*T^* = (TT^{-1})^* = I_Y^* = I_{Y^*}$ and likewise for the reverse.

$(ii \Rightarrow iii)$ By the kernel-annihilator theorem, we have $(\text{im } T)^a = \ker(T^*) = \{0\}$ in Y^* , so by annihilator-preannihilator, $\overline{\text{im } T} = (\text{im } T)_a^a = \{0\}_a = Y$. Now if $x \in S(X)$, find $f \in X^*$ so $f(x) = \|x\| = 1 = \|f\|$ (by Hahn-Banach). Then

$$1 = f(x) = [T^*(T^*-1)f](x) = [(T^*)^{-1}f](Tx) \leq \|(T^*)^{-1}f\| \|Tx\| \leq \|(T^*)^{-1}\| \|Tx\|$$

so that $\|Tx\| \geq \frac{1}{\|(T^*)^{-1}\|} > 0$ and T is bounded below.

$(iii \Rightarrow i)$ Let T be bounded below, and set $c = \inf\{\|Tx\| : x \in S(X)\} > 0$, then for $x \in X \setminus \{0\}$, $\|Tx\| = \|x\| \left\| T \left(\frac{1}{\|x\|} x \right) \right\| \geq c \|x\|$. If $y \in \overline{\text{im } T}$, then $y = \lim y_n$, each $y_n = Tx_n \in \text{im } T$. Then

$$\|x_n - x_m\| \leq \frac{1}{c} \|Tx_n - Tx_m\|$$

so $(x_n)_{n=1}^\infty$ is Cauchy as $(Tx_n)_{n=1}^\infty$ converges. Then $x = \lim x_n \in X$ and by continuity of T , $y = Tx \in \text{im } T$. Notice as well that bounded below implies $\ker T = \{0\}$.

We assume that T is bounded below and $\text{im } T = \overline{\text{im } T} = Y$, so T is bijective, hence invertible.

$(i, ii \Rightarrow iv)$ Use (iii)

$(iv \Rightarrow iii)$ We suppose that T is bounded below, and so is T^* . Then $\{0\} = \ker(T^*)$ in Y^* , so $Y = \{0\}_a = \ker(T^*)_a = \overline{\text{im } T}$ and T is bounded below provides $\text{im } T = \overline{\text{im } T} = Y$, so $\ker T = \{0\}$. ■

Remark. Reasons why $T \in \mathcal{B}(X, Y)$ is not intervible: $\ker T \supsetneq \{0\}$, $\operatorname{im} T \subsetneq Y$, T is not bounded below.

Example. Let $T : \ell_p \rightarrow \ell_p$ be given by $T(x_n)_{n=1}^\infty = \left(\frac{1}{n}x_n\right)_{n=1}^\infty$, so $\|T\| = 1$. Notice that $\ker T = \{0\}$ and $\overline{\operatorname{im} T} = \ell_p$. However, T is not bounded below.

11 SPECTRAL THEORY FOR BOUNDED OPERATORS

Let X be a \mathbb{C} -Banach space, and $\mathcal{B}(X) = \mathcal{B}(X, X)$.

Definition. If $T \in \mathcal{V}(X)$, we define the **resolvent** of T by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$. Then the **spectrum** of T , $\sigma(T)$, is given by $\sigma(T) = \mathbb{C} \setminus \rho(T)$. We define the **point spectrum** $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \supsetneq \{0\}\}$, so $\sigma_p(T) \subseteq \sigma(T)$.

Example. (i) If X is finite dimensional, then $\sigma(T) = \sigma_p(T)$.

(ii) Let $1 \leq p < \infty$ and define $S : \ell_p \rightarrow \ell_p$ by $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Notice that S is linear and $\|Sx\|_p = \|x\|_p$, so $\|S\| = 1$. Also, $\ker S = \{0\}$. Suppose $\lambda \in \sigma_p$, so there is $x \in \ker(\lambda I - S) \setminus \{0\}$. We let $k = \min\{n \in \mathbb{N} : x_n \neq 0\}$, we see $0 = (Sx)_k = \lambda x_k$, but $|S\rangle = \{0\}$, so $0 \notin \sigma_p(T)$, but hence no λ as above exists, so $\sigma_p(X) = \emptyset$.

For any $T \in \mathcal{B}(X)$, is $\sigma(T) \neq \emptyset$?

Let

$$\mathcal{G}(X) = \{T \in \mathcal{B}(X) : T \text{ is invertible}\}$$

Notice that if $S, T \in \mathcal{G}(X)$, then $(ST)^{-1} = T^{-1}S^{-1}$, so $\mathcal{G}(X)$ is a group in $\mathcal{B}(X)$ with identity I . Note that $\mathcal{B}(X)$ is complete, and if $S, T \in \mathcal{B}(X)$, then $\|ST\| \leq \|S\|\|T\|$, so that $S \mapsto ST$ and $S \mapsto TS$ for some $T \in \mathcal{B}(X)$ are continuous.

11.1 Theorem. (Inversion) (i) If $T \in \mathcal{D}(X)$, then $\sum_{k=0}^\infty T^k$ converges in $\mathcal{B}(X)$, and $\sum_{k=0}^\infty T^k = (I - T)^{-1}$

(ii) If $S, T \in \mathcal{B}(X)$ such that $S \in \mathcal{G}(X)$ and $\|T - S\| < \frac{1}{\|S^{-1}\|}$, then $T \in \mathcal{G}(X)$ with $T^{-1} = S^{-1} + \sum_{k=1}^\infty [S^{-1}(S - T)]^k S$.

Thus, we find that $\mathcal{G}(X)$ is open in $\mathcal{B}(X)$ and $T \mapsto T^{-1}$ on $\mathcal{G}(X)$ is continuous.

PROOF (i) Let $S_n = \sum_{k=0}^n T^k$, so for $m < n$, we have

$$\|S_n - S_m\| \leq \sum_{k=m+1}^n \|T^k\| \leq \sum_{k=m+1}^n \|T\|^k = \frac{\|T\|^{m+1}}{1 - \|T\|} \rightarrow 0$$

since $\|T\| < 1$, so $(S_n)_{n=1}^\infty$ is Cauchy, and thus convergent in $\mathcal{B}(X)$. Also,

$$(I - T)S_n = I - T^{n+1} \rightarrow I \text{ as } T^{n+1} \rightarrow 0$$

since $\|T\| < 1$. Similarly, $S_n(I - T) \rightarrow I$, so that $S = \sum_{k=0}^\infty T^k$ has $S(I - T) = I = (I - T)S$.

(ii) We have $\|S^{-1}(S - T)\| \leq \|S^{-1}\|\|S - T\| < 1$ so by (i)

$$T = S - (S - T) = S[I - S^{-1}(S - T)] \in \mathcal{G}(X)$$

Furthermore,

$$T^{-1} = [I - S^{-1}(S - T)]^{-1} S^{-1} = \sum_{k=0}^\infty [S^{-1}(S - T)]^k S^{-1}$$

In particular, we see that for $S \in \mathcal{G}(X)$, $S + \frac{1}{\|S^{-1}\|}D(X) \subseteq \mathcal{G}(X)$, so (a) holds. From (ii), we see that

$$\|T^{-1} - S^{-1}\| \leq \sum_{k=1}^{\infty} \|[S^{-1}(T - S)]^k S\| \leq \sum_{k=1}^{\infty} \|S^{-1}\|^k \|T - S\|^k \|S^{-1}\| = \frac{\|S^{-1}\|^2 \|T - S\|}{1 - \|S^{-1}\| \|T - S\|}$$

so that $\lim_{TS} \|T^{-1} - S^{-1}\| = 0$. ■

Definition. Suppose \mathcal{B} is a \mathbb{C} -Banach space, $U \subseteq \mathbb{C}$ and $F : U \rightarrow \mathcal{B}$. We say that F is **holomorphic** if for any $z_0 \in U$,

$$F'(z_0) = \lim_{z \rightarrow z_0} \frac{1}{z - z_0} [F(z) - F(z_0)]$$

Remark. Just as in calculus, a holomorphic function is continuous on its domain.

11.2 Proposition. Let $T \in \mathcal{B}(X)$. Then

- (i) $\rho(T)$ is open in \mathbb{C}
- (ii) $R(z) = R_T(z) = (zI - T)^{-1}$ defines a holomorphic function on $\rho(T)$, called the **resolvent function**, and
- (iii) $\sigma(T) \subseteq \|T\|\overline{\mathbb{D}}$, and for $|z| > \|T\|$, $R(z) \leq \frac{1}{|z| - \|T\|}$.

PROOF (i) Define $F : \mathbb{C} \rightarrow \mathcal{B}(X)$ by $F(z) = zI - T$. Then F is continuous and $\rho(T) = F^{-1}(\mathcal{G}(X))$.

(ii) If $z, z_0 \in \rho(T)$, then

$$\begin{aligned} R(z) - R(z_0) &= (zI - T)^{-1} - (z_0I - T)^{-1} = (zI - T)^{-1} [(z_0I - T) - (zI - T)](z_0I - T)^{-1} \\ &= (z_0 - z)(zI - T)^{-1}(z_0I - T)^{-1} \end{aligned}$$

Hence

$$\frac{1}{z - z_0} [R(z) - R(z_0)] = -(zI - T)^{-1}(z_0I - T)^{-1} \rightarrow -(z_0I - T)^{-2}$$

by continuity of inversion.

- (iii) If $|z| > \|T\|$, then $\left\|\frac{1}{z}T\right\| < 1$ so $zI - T = z(I - \frac{1}{z}T) \in \mathcal{G}(X)$, so $\sigma(T) \subseteq \|T\|\overline{\mathbb{D}}$. Furthermore, for $|z| > \|T\|$, we have

$$R(z) = (zI - T)^{-1} = \frac{1}{z} \left(I - \frac{1}{z}T\right)^{-1} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} T^k \quad \blacksquare$$

11.3 Theorem. (Liouville) If $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and bounded, then f is constant.

PROOF Apply Cauchy integral formula. ■

11.4 Theorem. (Liouville for Banach Spaces) If $F : \mathbb{C} \rightarrow \mathcal{B}$ is holomorphic and bounded, then F is constant.

PROOF Let $f \in \mathcal{B}^*$ and let $F_f = f \circ F : \mathbb{C} \rightarrow \mathbb{C}$. Notice that for $z, z_0 \in \mathbb{C}$,

$$\frac{F_f(z) - F_f(z_0)}{z - z_0} = f\left(\frac{1}{z - z_0}[F(z) - F(z_0)]\right) \rightarrow f(F^1(z_0))$$

by linearity and continuity of f , and hence $F'_f = f \circ F'$. Also, if F is bounded, then for $z \in \mathbb{C}$, $|F_f(z)| = |f(F(z))| \leq \|f\| \|F(z)\|$ shows that F_f is bounded, so by Liouville's theorem, is constant. In particular, if $z, z' \in \mathbb{C}$, $f(F(z) - F(z')) = F_f(z) - F_f(z') = 0$. Thus by Hahn-Banach, we have $F(z) = F(z')$ for any $z, z' \in \mathbb{C}$, so F is constant. ■

11.5 Theorem. *If $T \in \mathcal{B}(X)$, then $\sigma(T) \neq \emptyset$ and compact.*

PROOF If $\sigma(T) = \emptyset$, then $R : \mathbb{C} \rightarrow \mathcal{B}(X)$ is holomorphic. Hence, as $\|T\|\overline{\mathbb{D}}$ is compact in \mathbb{C} , R is bounded on $\|T\|\overline{\mathbb{D}}$; and if $|z| > \|T\|$, we have

$$\|R(z)\| \leq \frac{1}{|z| - \|T\|} \rightarrow 0$$

It follows that R is bounded on $\mathcal{B}(X)$, and hence constant, and thus $R = 0$. But $R(z)(zI - T) = I$, a contradiction.

Moreover, $\rho(T) = \mathbb{C} \setminus \sigma(T)$ is open, and $\sigma(T) \subseteq \|T\|\overline{\mathbb{D}} \subset \mathbb{C}$. Thus $\sigma(T)$ is a non-empty compact set. ■

11.6 Corollary. (Joke) *\mathbb{C} is algebraically closed.*

PROOF Let $p(x) \in \mathbb{C}[x]$ be an arbitrary irreducible polynomial with $p(x) = (x - r_1) \cdots (x - r_n)$ for some $r_i \in \overline{\mathbb{C}}$. Consider the operator $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with diagonal r_1, \dots, r_n and hence characteristic polynomial $p(x)$. Then $\emptyset \neq \sigma(T) = \sigma_p(T) = \{x \in \mathbb{C} : p(x) = 0\}$, so p has some root in \mathbb{C} , so that $\deg p = 1$. ■

11.7 Proposition. (i) *If X is a (non-Hilbert) Banach space, then $\sigma(T^*) = \sigma(T)$.*
 (ii) *If H is a Hilbert space, $T \in \mathcal{B}(H)$, then $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.*

PROOF (i) $(\lambda I_X - T)^* = \lambda I_{X^*} - T^*$ and is invertible if and only if $\lambda I_X - T$ is invertible
 (ii) Same. ■

Definition. We define the **point spectrum** $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\}$, the **approximate point spectrum** $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$, and the **compression spectrum** $\sigma_{com}(T) = \{\lambda \in \mathbb{C} : \overline{\text{im}(\lambda I - T)} \subsetneq X\}$.

Remark. (i) $\sigma_p(T) \subseteq \sigma_{ap}(T)$.

(ii) We have $[(\lambda I - T)]^a = \ker(\lambda I - T^*)$ by kernel-annihilator so $\overline{\text{im}(\lambda I - T)} = \ker(\lambda I - T^*)$ by annihilator-preannihilator, so that $\sigma_{com}(T) = \sigma_p(T^*)$.

11.8 Lemma. *If $(T_n)_{n=1}^\infty \subset \mathcal{G}(X)$ satisfies that*

- $T = \lim_{n \rightarrow \infty} T_n$
- $M = \sup_{n \in \mathbb{N}} \|T_n^{-1}\| < \infty$

then $T \in \mathcal{G}(X)$.

PROOF Since $M > 0$, for sufficiently large n , we have $\|T - T_n\| \leq \frac{1}{M} \leq \frac{1}{\|T_n^{-1}\|}$, so $T \in \mathcal{G}(X)$ by inversion theorem. ■

11.9 Proposition. (i) $\partial\sigma(T) \subseteq \sigma_{ap}(T)$

(ii) $\sigma_{ap}(T)$ is closed

Hence $\sigma_{ap}(T)$ is always a non-empty closed subset of \mathbb{C} .

PROOF (i) Let $\lambda \in \partial\sigma(T)$, so there is $(\lambda_n)_{n=1}^\infty \subset \rho(T) = \mathbb{C} \setminus \sigma(T)$ such that $\lambda = \lim_{n \rightarrow \infty} \lambda_n$.

Then $\|(\lambda_n I - T) - (\lambda I - T)\| = |\lambda_n - \lambda| \rightarrow 0$, but $\lambda I - T \notin \mathcal{G}(X)$, so by the lemma, $\sup_{n \in \mathbb{N}} \|(\lambda_n I - T)^{-1}\| = \infty$. Passing to a subsequence if necessary, we may suppose $\lim_{n \rightarrow \infty} \|(\lambda_n I - T)^{-1}\| = \infty$.

For each index n , let $x_n \in S(X)$ so $\alpha_n = \|(\lambda_n I - T)^{-1} x_n\| > \|(\lambda_n I - T)^{-1}\| - \frac{1}{n}$ so $\lim_{n \rightarrow \infty} \alpha_n = \infty$. Then $y_n = \frac{1}{\alpha_n} (\lambda_n I - T)^{-1} x_n$, so $y_n \in S(X)$ and

$$\begin{aligned} (\lambda I - T)y_n &= (\lambda_n I - T)y_n + (\lambda - \lambda_n)y_n \\ &= \frac{1}{\alpha_n} x_n + (\lambda - \lambda_n)y_n \rightarrow 0 \end{aligned}$$

so $\lambda I - T$ is not bounded below.

(ii) If $\lambda = \lim_{n \rightarrow \infty} \lambda_n$, each $\lambda_n \in \sigma_{ap}(T)$, for each n find $x_n \in S(X)$ so $\|(\lambda_n I - T)x_n\| < \frac{1}{n}$. Then

$$\|(\lambda I - T)x_n\| \leq \|(\lambda_n I - T)x_n\| + \|(\lambda - \lambda_n)x_n\| < \frac{1}{n} + |\lambda - \lambda_n| \rightarrow 0$$

so $\lambda I - T$ is not bounded below. ■

Example. Let $S \in B(\ell_p)$, $1 < p < \infty$, where $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$, the *unilateral shift* map. It is immediate that $\|Sx\|_p = \|x\|_p$ for $x \in \ell_p$, so $\|S\| = 1$. Recall that $\ell_p^* \cong \ell_q$ where p, q are conjugate. Define a bilinear form on $\ell_p \times \ell_q$ by $\langle x, y \rangle = \sum_{k=1}^\infty x_k y_k$. We compute $\langle x, S^* y \rangle = \langle Sx, y \rangle = \sum_{k=1}^\infty x_k y_{k+1} = \langle x, (y_2, y_3, \dots) \rangle$ so $S^*(y_1, y_2, \dots) = (y_2, y_3, \dots)$. Recall that $\sigma_p(S) = \emptyset$. However, if $\lambda \in \mathbb{D}$, let $y_\lambda = (1, \lambda, \lambda^2, \dots) \in \ell_q$. Then $S^*(y_\lambda) = \lambda \cdot y_\lambda$. Hence $\sigma_p(S^*) \supseteq \mathbb{D}$. Furthermore, if $\lambda \in \sigma_p(S^*)$ and $y \in \ker(\lambda I - S^*)$, then $\lambda^n y = (S^*)^n y \rightarrow 0$ so $\lambda^n \rightarrow 0$, forcing $|\lambda| < 1$. Thus

$$\mathbb{D} = \sigma_p(S^*) \subseteq \sigma(S^*) = \sigma(S) \subseteq \overline{\mathbb{D}}$$

since $\|S\| = 1$, and since $\sigma(S)$ is compact, $\sigma(S) = \overline{\mathbb{D}}$.

We know that $\sigma_{ap}(S) \supseteq \partial\sigma(S) = \mathbb{S}$. If $\lambda \in \mathbb{D}$, then for $x \in \ell_p$, $\|(S - \lambda I)x\|_p \geq \|Sx\|_p - \|\lambda x\|_p = (1 - |\lambda|)\|x\|_p$, so $S - \lambda I$ is bounded below. Thus $\sigma_{ap}(S) \cap \mathbb{S} = \emptyset$, so $\sigma_{ap}(S) = \mathbb{S}$. In conclusion,

$$\begin{array}{ll} \sigma(S) = \mathbb{D} & \sigma_p(S) = \emptyset \\ \sigma_{ap}(S) = \mathbb{S} = \partial\sigma(S) & \sigma_{com}(S) = \sigma_p(S^*) = \mathbb{D} \\ \sigma(S^*) = \overline{\mathbb{D}} & \sigma_p(S^*) = \mathbb{D} \\ \sigma_{ap}(S^*) = \partial\sigma(S^*) \cup \sigma_p(S^*) = \overline{\mathbb{D}} & \sigma_{com}(S^*) = \emptyset \end{array}$$

Remark. Let $\sigma_p(T)$, $\sigma_{com}(T)$ may be empty, and if non-empty need not be closed.

Remark. If $p = 1$ and $S \in B(\ell_1)$ is the unilateral shift, as above, and $L \in \ell_\infty^*$ be a Banach limit. Then

$$S^{**}L = L \circ S^L$$

so $\sigma_p(S^{**}) \ni 1$. Thus $\sigma_{com}(S^*) = \sigma_p(S^{**}) \neq \emptyset$.

11.10 Theorem. (Spectral Mapping) Let $T \in \mathcal{B}(X)$, $p \in \mathbb{C}[x]$, then $\sigma(p(T)) = p(\sigma(T))$.

PROOF We may assume that $p \neq 0$. Let $\lambda_0 \in \mathbb{C}$ and write $p(t) - \lambda_0 = \alpha \prod_{k=1}^n (t - \lambda_k)$. Then

$$p(T) - \lambda_0 I = \alpha \prod_{k=1}^n (T - \lambda_k I)$$

Thus $p(T) - \lambda_0 I \notin \mathcal{G}(X)$ if and only if some $T - \lambda_k I \notin \mathcal{G}(X)$, so $\lambda_0 \in \sigma(p(T))$ if and only if at least one $\lambda_k \in \sigma(T)$ if and only if $p(\lambda) - \lambda_0 = 0$ for some $\lambda \in \sigma(T)$, i.e. $\lambda_0 = p(\lambda) \in p(\sigma(T))$. ■

11.11 Theorem. (Spectral Radius Formula) If $T \in \mathcal{B}(X)$, let $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Then $r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$.

PROOF By the spectral mapping theorem, $r(T^n) = r(T)^n$. Moreover, $r(T^n) \leq \|T^n\|$ since $\sigma(T^n) \subseteq \|T^n\| \overline{\mathbb{D}}$. Thus $r(T) = r(T^n)^{1/n} \leq \|T^n\|^{1/n}$ so that $r(T) \leq \liminf_{n \rightarrow \infty} \|T^n\|^{1/n}$.

Now, let $f \in \mathcal{B}(X)^*$. We recall that for $|z| > \|T\|$,

$$\begin{aligned} R(z) &= (zI - T)^{-1} = \frac{1}{z} (I - \frac{1}{z} T)^{-1} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} T^k \\ &= \sum_{k=1}^{\infty} \frac{1}{z^k} T^{k-1}. \end{aligned}$$

Thus the Holomorphic function $F_f = f \circ R : \rho(T) = \mathbb{C} \setminus \sigma(T) \rightarrow \mathbb{C}$ satisfies $F_f(z) = \sum_{k=1}^{\infty} f(T^{k-1}) \frac{1}{z^k}$. From \mathbb{C} -analysis, the holomorphic function admits a Laurent series for all z with $|z| > r(T)$, and by uniqueness of Laurent series, $F_f(z) = \sum_{n=1}^{\infty} f(T^{n-1}) \frac{1}{z^n}$ for $|z| > r(T)$. Hence if $|z_0| > r(T)$, we have that $\left\{ f\left(\frac{1}{z_0^k} T^k\right) \right\}_{k=1}^{\infty}$ is bounded in \mathbb{C} . Doing this for any $f \in \mathcal{B}(X)^*$, we may apply Banach-Steinhaus to see that $\left\{ \frac{1}{z_0^k} T^k \right\}_{k=1}^{\infty}$ is bounded in $\mathcal{B}(X)$, so

$$\sup_{n \in \mathbb{N}} \left\| \frac{1}{z_0^n} T^n \right\| \leq M < \infty$$

Then $\|T^n\| \leq M |z_0|^n$ so $\|T^n\|^{1/n} \leq M^{1/n} |z_0|$, so that

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq |z_0|.$$

This applies for any $|z_0| > r(T)$, so the result follows. ■

12 COMPACT OPERATORS

Definition. Let X, Y be Banach spaces over \mathbb{F} . A linear operator $K : X \rightarrow Y$ is called **compact** if $\overline{K(B(X))}$ is compact in Y . We let

$$\mathcal{K}(X, Y) = \{K \in \mathcal{L}(X, Y) : K \text{ is compact}\}.$$

Since compact sets in Y are bounded, we immediately see that $\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y)$.

12.1 Proposition. Let X, Y be Banach spaces. Then

- (i) $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$, and
- (ii) If W, Z are also Banach spaces, $S \in \mathcal{B}(Y, Z)$, $T \in \mathcal{B}(W, X)$, $K \in \mathcal{K}(X, Y)$, then $SKT \in \mathcal{K}(W, Z)$.

PROOF (i) Let $K, L \in \mathcal{K}(X, Y)$, $\alpha \in \mathbb{F}$, and $(x_n)_{n=1}^\infty \subset B(X)$. Then $\overline{K(B(X))} \times \overline{L(B(X))}$ is compact in $Y \times Y \cong Y \oplus Y$, so we have converging

$$\left((Kx_{n_j}, Lx_{n_j}) \right)_{j=1}^\infty \subset \overline{K(B(X))} \times \overline{L(B(X))}$$

Now, $(K + \alpha L)(x_{n_j}) = Kx_{n_j} + \alpha Lx_{n_j} \rightarrow y + \alpha y'$, so $((K + \alpha L)x_n)_{n=1}^\infty$ admits a converging subsequence in Y .

Now let $K \in \overline{\mathcal{K}(X, Y)} \subseteq \mathcal{B}(X, Y)$, so $K = \lim_{n \rightarrow \infty} K_n$ with each $K_n \in \mathcal{K}(X, Y)$. Let $\epsilon > 0$. Let $n_0 \in \mathbb{N}$ be so $n \geq n_0$ implies $\|K - K_n\| < \epsilon/3$. Since K_{n_0} is compact, there are $\{x_1, \dots, x_m\} \subset B(X)$ such that

$$K_{n_0}(B(X)) \subseteq \bigcup_{j=1}^m (K_{n_0}x_j + \frac{\epsilon}{3}B(Y))$$

Then for $x \in B(X)$, find x_j so $K_{n_0}x \in K_{n_0}x_j + \frac{\epsilon}{3}B(X)$ and hence

$$\begin{aligned} \|Kx - Kx_j\| &\leq \|Kx - K_{n_0}x\| + \|K_{n_0}x - K_{n_0}x_j\| + \|K_{n_0}x_j - Kx_j\| \\ &\leq \|K - K_{n_0}\|\|x\| + \frac{\epsilon}{3} + \|K - K_{n_0}\|\|x_j\| < \end{aligned}$$

and we see that $K(B(X)) \subseteq \bigcup_{j=1}^m (Kx_j + \epsilon B(Y))$, and hence is totally bounded.

(ii) We have

$$\begin{aligned} \overline{SKT(B(W))} &\subseteq \overline{SK(\|T\|B(X))} = \|T\| \overline{SK(B(X))} \\ &= \|T\| \overline{S(K(B(X)))} \subseteq \|T\| \overline{S(\overline{K(B(X))})} = \|T\| S(\overline{K(B(X))}) \end{aligned}$$

is a continuous image of a compact set and hence compact. ■

Example. (i) Given Banach spaces X, Y , let $\mathcal{F}(X, Y) = \{F \in \mathcal{B}(X, Y) : \text{rank } F < \infty\}$. Note that $\text{rank } F = \dim(\text{im } F)$. If $F \in \mathcal{F}(X, Y)$, then $F(B(X)) \subseteq \|F\|B(\text{im } F)$ is subset of a compact set, so F is compact. Thus $\overline{\mathcal{F}(X, Y)} \subseteq \mathcal{K}(X, Y)$.

(ii) Also W, Z Banach spaces, $S \in \mathcal{B}(Y, Z)$, $T \in \mathcal{B}(W, X)$, then $SFT \in \mathcal{F}(W, Z)$ for $F \in \mathcal{F}(X, Y)$.

(iii) Let $I = [0, 1]$ and $k \in C(I^2)$. Then for $f \in C(I)$, $x \in I$, define

$$Kf(x) = \int_0^1 k(x, y)f(y)dy$$

This defines a compact operator in $\mathcal{K}(C(I), C(I))$.

To see this, let

$$\mathcal{A} = \text{span}\{(x, y) \mapsto \phi(x)\psi(y), \phi, \psi \in C(I)\} \subseteq C(I^2).$$

Then \mathcal{A} is an algebra of functions, $1 \in \mathcal{A}$, \mathcal{A} is point separating, and if $H \in \mathcal{A}$, then $\bar{H} \in \mathcal{A}$. Thus by Stone-Weierstrass, the uniform closure $\overline{\mathcal{A}} = C(I^2)$. Hence, $k = \lim k_n$ where each $k_n(x, y) = \sum_{j=1}^{m_n} \phi_{n_j}(x)\psi_{n_j}(y)$. Let

$$\begin{aligned} K_n f(x) &= \int_0^1 k_n(x, y)f(y)dy = \int_0^1 \sum_{j=1}^{m_n} \phi_{n_j}(x)\psi_{n_j}(y)f(y)dy \\ &= \sum_{j=1}^{m_n} \left[\int_0^1 \psi_{n_j}(y)f(y)dy \right] \phi_{n_j}(x) \end{aligned}$$

so that

$$K_n f = \sum_{j=1}^{m_n} \left[\int_0^1 \psi_{n_j} f \right] \phi_{n_j} \in \text{span}\{\phi_{n_1}, \dots, \phi_{n_{m_n}}\} \subset C(I)$$

so K has finite rank, and furthermore,

$$|K_n f(x)| \leq \int_0^1 |k_n(x, y)| \|f(y)\| dy \leq \int_0^1 \|k_n\|_\infty \|f\|_\infty dy = \|k_n\|_\infty \|f\|_\infty$$

so that $\|K_n f\|_\infty \leq \|k_n\|_\infty \|f\|_\infty$, so K_n is bounded. Thus each $K_n \in \mathcal{F}(C(I))$. Furthermore, for $f \in C(I)$,

$$\|(K - K_n)f\|_\infty \leq \|k - k_n\|_\infty \|f\|_\infty$$

so $\|K - K_n\| \leq \|k - k_n\|_\infty \xrightarrow{n \rightarrow \infty} 0$ so $K \in \overline{\mathcal{F}(C(I))} \subseteq \mathcal{K}(C(I))$.

Exercise: if $1 \leq p < \infty$, then K , as above, defines an operator in $\mathcal{K}(L_p(I))$. Notice that $\|Kf\|_p \leq \|k\|_\infty \|f\|_p$, $J : C(I) \rightarrow L_p(I)$ “identity” is bounded.

12.2 Theorem. Let X, Y be Banach spaces, $K \in \mathcal{B}(X, Y)$. Then the following are equivalent:

- (i) $K \in \mathcal{K}(X, Y)$
- (ii) $K^*|_{B(Y^*)} : B(Y^*) \rightarrow X^*$ is w^* -norm continuous
- (iii) $K^* \in \mathcal{K}(Y^*, X^*)$

PROOF ($i \Rightarrow ii$) Let $f_0 = w^* - \lim_{v \in N} f_v$ in $B(Y^*)$. Given $\epsilon > 0$, let $\{x_1, \dots, x_n\} \subset B(X)$ such that $K(B(X)) \subseteq \bigcup_{j=1}^n (Kx_j + \frac{\epsilon}{3} B(X))$. Hence if $x \in B(X)$, there is x_j such that $\|Kx - Kx_j\| < \frac{\epsilon}{3}$. Then for each $v \in N$, we have

$$\begin{aligned} |f_v(Kx) - f_0(Kx)| &\leq |f_v(Kx) - f_v(Kx_j)| + |f_v(Kx_j) - f_0(Kx_j)| + |f_0(Kx_j) - f_0(Kx)| \\ &\leq \|f_v\| \|Kx - Kx_j\| + |f_v(Kx_j) - f_0(Kx_j)| + \|f_0\| \|Kx_j - Kx\| \\ &< \frac{2\epsilon}{3} + |f_v(Kx_j) - f_0(Kx_j)| \end{aligned}$$

so

$$\|K^* f_\nu - K^* f_0\| = \sup_{x \in B(X)} |f_\nu(Kx) - f_0(Kx)| \leq \frac{2\epsilon}{3} + \max_{j=1, \dots, n} |f_\nu(Kx_j) - f_0(Kx_j)|$$

and hence there is $\nu_0 \in N$ so $\nu \geq \nu_0$ implies that $|f_\nu(Kx_j) - f_0(Kx_j)| < \epsilon/3$ for $j = 1, \dots, n$, so $\nu \geq \nu_0$ implies that $\|K^* f_\nu - K^* f_0\| \leq \epsilon$.

(ii \Rightarrow iii) We have that $B(Y^*)$ is w^* -compact, so $K^*(B(Y^*))$ is norm compact.

(iii \Rightarrow i) By the proof above, K^{**} is compact. Hence $K(\widehat{B(X)}) = K^{**}(B(\hat{X})) \subseteq K^{**}(B(X^{**}))$ is compact, and $\overline{K(B(X))} \cong K(\widehat{B(X)})$ is compact. ■

12.3 Corollary. If X and Y are reflexive, $K \in \mathcal{B}(X, Y)$, then the following are equivalent:

- (i) $K \in \mathcal{K}(X, Y)$
- (ii) $K(B(X))$ is compact
- (iii) $K|_{B(X)} : B(X) \rightarrow Y$ is w -norm continuous.

PROOF In a reflexive space, $w = w^*$ and $K = K^{**}$. ■

Remark. Let X, Y be reflexive, $K \in \mathcal{B}(X, Y)$. Then K is w -norm continuous if and only if $K \in \mathcal{F}(X, Y)$.

13 SPECTRAL THEORY FOR COMPACT OPERATORS

13.1 Lemma. Let $K \in \mathcal{K}(X)$. Suppose there are sequences

- of closed subspaces $Y_1 \subsetneq Y_2 \subsetneq \dots$ of X
- scalars $(\alpha_j)_{j=1}^\infty$ such that $(K - \alpha_j I)Y_j \subseteq Y_{j-1}$ for $j = 1, 2, 3, \dots$

Then $\lim_{j \rightarrow \infty} \alpha_j = 0$.

PROOF Suppose not, i.e. $\limsup_{j \rightarrow \infty} |\alpha_j| > 0$. Passing to a subsequence if necessary, we may assume $|\alpha_j| \geq \epsilon > 0$ for all j . By the Riesz lemma, find $x_j \in B(Y_j)$ so $\text{dist}(x_j, Y_{j-1}) > 1/2$. Then $y_j = (K - \alpha_j I)x_j \in Y_{j-1}$, implying that $Kx_j = y_j + \alpha_j x_j \in Y_j$. If $i < j$, we have

$$\|Kx_j - Kx_i\| = \|y_j + \alpha_j x_j - Kx_i\| = |\alpha_j| \left\| x_j + \frac{1}{\alpha_j}(y_j - Kx_i) \right\| > \frac{|\alpha_j|}{2} \geq \frac{\epsilon}{2}$$

so that $(Kx_j)_{j=1}^\infty$ admits no Cauchy sequence, a contradiction. ■

Remark. If $T \in \mathcal{B}(X)$, $\{\lambda_1, \dots, \lambda_n\} \subseteq \sigma_p(T)$ distinct. Then $\{\ker(\lambda_i I - T)\}_{i=1}^n$ is a linearly independent sequence of subspace. This is true for $n = 1$, so suppose it holds for $n - 1$. Let $x_j \in \ker(\lambda_j I - T) \setminus \{0\}$ for each $1 \leq j \leq n$, and suppose $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ satisfy $0 = \alpha_1 x_1 + \dots + \alpha_n x_n$. Suppose $\alpha_{j_0} \neq 0$. Write $x_{j_0} = \sum_{j=1, j \neq j_0}^n \alpha'_j x_j$ where $\alpha'_j = \alpha_j / \alpha_{j_0}$. Then

$$\sum_{\substack{j=1 \\ j \neq j_0}}^n \alpha'_j \lambda_{j_0} x_{j_0} = \lambda_{j_0} x_{j_0} = T x_{j_0} = \sum_{\substack{j=1 \\ j \neq j_0}}^n \alpha'_j \lambda_j x_j$$

so that $0 = \sum_{j=1, j \neq j_0}^n \alpha'_j (\lambda_j - \lambda_{j_0}) x_j$, a contradiction.

13.2 Theorem. (Spectral Theorem for Compact Operators) Let X be an infinite dimensional \mathbb{C} -Banach space and $K \in \mathcal{K}(X)$. Then

(i) $\lambda \in \mathbb{C} \setminus \{0\}$, then each “generalized eigenspace” $\ker[(\lambda I - K)^n]$ is finite dimensional.

Furthermore, there is $n_\lambda \in \mathbb{N}$ such that $\ker[(\lambda I - K)^n] = \ker[(\lambda I - K)^{n_\lambda}]$ for $n \geq n_\lambda$.

(ii) $\sigma(K) = \sigma_p(K) \cup \{0\}$, where $\sigma_p(K) = \{\lambda_1, \lambda_2, \dots\}$ with $\lim_{n \rightarrow \infty} \lambda_n = 0$ if infinite.

PROOF (i) We have $\ker(\lambda I - K) \subseteq \ker[(\lambda I - K)^2] \subseteq \dots$. If n_λ as above does not exist, let $n_1 = 1$ and $n_{k+1} = \min\{n : \ker[(\lambda I - K)^{n_k}] \subsetneq \ker[(\lambda I - K)^n]\}$, and set $Y_0 = \{0\}$ and $Y_k = \ker[(\lambda I - K)^{n_k}]$. Then $Y_0 \subsetneq Y_1 \subsetneq \dots$ and $(\lambda I - K)Y_k \subseteq Y_{k-1}$. But the preceding lemma implies that $\lim_{n \rightarrow \infty} \lambda = 0$, a contradiction.

Now suppose $\dim \ker[(\lambda I - K)^n] = \infty$ for some $n = n_\lambda$. Then let $k \in \mathbb{N}$ be so Y_k is infinite dimensional, while Y_{k-1} is finite dimensional. Thus we can find a linearly independent $\{x_j\}_{j=1}^\infty \subset Y_k \setminus Y_{k-1}$. We let $V_0 = Y_{k-1}$ and $V_n = \overline{\text{span}\{V_0, x_1, \dots, x_n\}}$ for each $N \in \mathbb{N}$. Then $V_0 \subsetneq V_1 \subsetneq \dots$ and $(\lambda I - K)V_n \subseteq (\lambda I - K)Y_k \subseteq Y_{k-1} = V_0 \subseteq V_{n-1}$ for $N \in \mathbb{N}$. Again by the preceding lemma, we have our contradiction.

(ii) We first show that $\sigma_{ap}(K) \subseteq \sigma_p(K) \cup \{0\}$. If $\lambda \in \sigma_{ap}(K) \setminus \{0\}$, then there is $(x_n)_{n=1}^\infty \subset S(X)$ such that $(\lambda I - K)x_n \xrightarrow{n \rightarrow \infty} 0$. Since K is compact, passing to a subsequence if necessary, we may assume $y = \lim Kx_n$ exists. Then

$$\|\lambda x_n - y\| \leq \|(\lambda I - K)x_n\| + \|Kx_n - y\| \xrightarrow{n \rightarrow \infty} 0$$

where each $\|\lambda x_n\| = |\lambda| \neq 0$ so $y \neq 0$. Furthermore, $Ky = \lim_{n \rightarrow \infty} K(\lambda x_n) = \lambda \lim_{n \rightarrow \infty} Kx_n = \lambda y$, so $\lambda \in \sigma_p(K)$.

Now suppose that $\{\lambda_n\}_{n=1}^\infty \subseteq \partial\sigma(K) \setminus \{0\} \subseteq \sigma_{ap}(K) \setminus \{0\} \subseteq \sigma_p(K)$. Let $Y_0 = \{0\}$, $Y_n = \sum_{j=1}^n \ker(\lambda_j I - K)$. By linear independence of eigenspaces, $Y_0 \subsetneq Y_1 \subsetneq \dots$ and $(\lambda_n I - K)Y_n \subseteq Y_{n-1}$ for $n \in \mathbb{N}$. Then the previous lemma shows that $\lim_{n \rightarrow \infty} \lambda_n = 0$. Hence for each $n \in \mathbb{N}$ we have that $\partial\sigma(K) \setminus \frac{1}{n}\mathbb{D}$ is finite, and hence $\partial\sigma(K) \subseteq \{0\} \cup \bigcup_{n=1}^\infty (\partial\sigma(K) \setminus \frac{1}{n}\mathbb{D})$ is countable. Thus $\partial\sigma(K)$ is countable, so $\sigma(K) \setminus \{0\}$ is countable. Moreover, note that $0 \in \sigma(K)$, for if not, then $K \in \mathcal{G}(X)$ and $I = KK^{-1} \in \mathcal{K}(X)$, contradicting infinite dimensionality of X . ■

Example. We may have $\sigma(K) = \{0\}$, $\sigma_p(K) = \emptyset$. Let $K : \ell_p \rightarrow \ell_p$ be given by $K(x_1, x_2, \dots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. It is easy to see that $K \in \mathcal{K}(\ell_p)$ with $\|K\| = 1$. A straightforward induction shows $K^n x = (0, \dots, 0, \frac{1}{n!}x_1, \frac{1}{(n+1)!}x_2, \dots)$ so $\|K^n\| = \frac{1}{n!}$. The spectral radius formula shows that $r(K) = \lim_{n \rightarrow \infty} \frac{1}{(n!)^{1/n}} = 0$, so $\sigma(K) = \{0\}$. However, as for the unilateral shift, $\ker K = \{0\}$ but $0 \notin \sigma_p(K)$, so $\sigma_p(K) = \emptyset$.

Let \mathcal{H} be a complex Hilbert space. Recall that $\mathcal{H}^* = \{f_y : y \in \mathcal{H}\}$ where $f_y(x) = (x, y)$. If $T \in \mathcal{B}(X)$, implicitly define T^*y as $f_{T^*y} = T^*f_y$. Note that $(\alpha T)^* = \overline{\alpha}T^*$ and $(Tx, y) = f_y(Tx) = T^*f_y(x) = f_{T^*y}(x) = (x, T^*y)$.

Definition. Let $H \in \mathcal{B}(\mathcal{H})$. Then H is called **Hermitian** or **self-adjoint** if $H = H^*$. Let $N \in \mathcal{B}(H)$ then N is **normal** if $N^*N = NN^*$.

Certainly Hermitian operators are normal.

13.3 Proposition. (i) If $T \in \mathcal{B}(\mathcal{H})$, there are unique Hermitian operators $\text{Re } T$, $\text{Im } T$, such that $T = \text{Re } T + i \text{Im } T$.

(ii) T is normal if and only if $(\text{Re } T)(\text{Im } T) = (\text{Im } T)(\text{Re } T)$.

PROOF (i) Let $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ and $\operatorname{Im} T = \frac{1}{2i}(T - T^*)$. If we had $T = H_1 + iH_2$, H_1, H_2 each Hermitian, then

$$H_1 + iH_2 = T = \operatorname{Re} T + i \operatorname{Im} T \implies H_2 - \operatorname{Re} T = i(\operatorname{Im} T - H_2) := S$$

where $S^* = S = -S^*$ so that $S^* = 0$ and $S = 0$.

(ii) We have

$$TT^* = (\operatorname{Re} T + i \operatorname{Im} T)(\operatorname{Re} T - i \operatorname{Im} T) = (\operatorname{Re} T)^2 + i(\operatorname{Im} T \operatorname{Re} T + \operatorname{Re} T \operatorname{Im} T) + (\operatorname{Im} T)^2$$

and

$$T^*T = (\operatorname{Re} T - i \operatorname{Im} T)(\operatorname{Re} T + i \operatorname{Im} T) = (\operatorname{Re} T)^2 - i(\operatorname{Im} T \operatorname{Re} T - \operatorname{Re} T \operatorname{Im} T) + (\operatorname{Im} T)^2$$

so that $TT^* = T^*T$ if and only if $2 \operatorname{Im} T \operatorname{Re} T = 2 \operatorname{Re} T \operatorname{Im} T$. ■

13.4 Proposition. (C*-identity) If $T \in \mathcal{B}(\mathcal{H})$, then $\|T^*T\| = \|T\|^2$.

PROOF We have

$$\begin{aligned} \|T\|^2 &= \sup_{x \in \mathcal{B}(\mathcal{H})} \|Tx\|^2 = \sup_{x \in \mathcal{B}(\mathcal{H})} (Tx, Tx) \\ &= \sup_{x \in \mathcal{B}(\mathcal{H})} \|T^*Tx, x\| \leq \sup_{x \in \mathcal{B}(\mathcal{H})} \|T^*Tx\| \|x\| = \|T^*T\| \end{aligned}$$

while $\|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$. ■

13.5 Proposition. If $N \in \mathcal{B}(\mathcal{H})$ is normal, then $r(N) = \|N\|$.

PROOF Suppose first that $H = H^*$ in $\mathcal{B}(\mathcal{H})$. The C*-identity gives $\|H^2\| = \|H^*H\| = \|H\|^2$ and hence inductively for $k \in \mathbb{N}$ $\|H^{2k}\| = \|H\|^{2k}$ since H^k is Hermitian for any k . Now since $N^*N = NN^*$, we see that $(N^*N)^n = (N^*)^n N^n$ for $n \in \mathbb{N}$. Hence if $k \in \mathbb{N}$, we use the C*-identity to see

$$\|N\|^{2^{k+1}} = \|N^*N\|^{2^k} = \|(N^*N)^{2^k}\| = \|(N^{2^k})^* N^{2^k}\| = \|N^{2^k}\|^2$$

and hence $\|N\|^{2^k} = \|N^{2^k}\|$. Now, by the spectral radius formula, $r(N) = \lim_{n \rightarrow \infty} \|N^n\|^{1/n} = \lim_{k \rightarrow \infty} \|N^{2^k}\|^{1/2^k} = \|N\|$. ■

Remark. If $T \in \mathcal{B}(X)$, then $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{com}(T)$. Indeed, if $\lambda \in \sigma(T)$, then $\lambda I - T \notin \mathcal{G}(X)$, and hence at least one of the following holds:

- $\lambda I - T$ is not bounded below, i.e. $\lambda \in \sigma_{ap}(T)$
- $\operatorname{im}(\lambda I - T)$ is not dense, i.e. $\lambda \in \sigma_{com}(T)$.

13.6 Proposition. If $H = H^*$ in $\mathcal{B}(\mathcal{H})$, then $\sigma(H) = \sigma_{ap}(H) \subseteq \mathbb{R}$.

PROOF If $\lambda \in \sigma_{ap}(H)$, then there is $(x_n)_{n=1}^\infty \subset S(\mathcal{H})$ such that $(\lambda I - H)x_n \rightarrow 0$. Then

$$\lambda - (Hx_n, x_n) = \lambda(x_n, x_n) - (Hx_n, x_n) = ((\lambda I - H)x_n, x_n) \rightarrow 0$$

and $(Hx_n, x_n) = \overline{(x_n, Hx_n)} = \overline{(Hx_n, x_n)} \in \mathbb{R}$ for each n , so $\lambda \in \mathbb{R}$. Now, $\sigma_{com}(H) = \{\bar{\lambda} : \lambda \in \sigma_p(H^*) = \sigma_p(H)\}$, so as $\sigma_p(H) \subseteq \sigma_{ap}(H) \subseteq \mathbb{R}$, we see that $\sigma_{com}(H) \subseteq \sigma_p(H) \subseteq \sigma_{ap}(H)$ and use the remark above. ■

Remark. (i) $(Hx, x) \in \mathbb{R}$ if $H = H^*$, $x \in \mathcal{H}$. If we define $[\cdot, \cdot] : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ by $[x, y] = (Hx, y)$, then $[\cdot, \cdot]$ is Hermitian by the Polarization identity.

(ii) If, further, $(Hx, x) \geq 0$ for any $x \in \mathcal{H}$, then $\sigma(H) \subseteq [0, \infty)$. We call such H **positive**

(iii) If $T \in \mathcal{B}(\mathcal{H})$, then $\sigma(T^*T) \subseteq [0, \infty)$ as T^*T is Hermitian, $(T^*Tx, x) = \|Tx\|^2 \geq 0$.

13.7 Theorem. (Functional Calculus for Bounded Hermitian Operators) Let $H \in \mathcal{B}(\mathcal{H})$ be Hermitian. Then there is a unique bounded operator $\Phi : C(\sigma(H)) \rightarrow \mathcal{B}(\mathcal{H})$ satisfying

- $\Phi(1) = I$, $\Phi(\iota) = H$, where ι is the identity function on $\sigma(H)$
- $\Phi(fg) = \Phi(f)\Phi(g)$ for f, g in $C(\sigma(H))$.

Furthermore, Φ

- Φ is an isometry
- $\Phi(f)T = T\Phi(f)$ for any $f \in C(\sigma(H))$, provided that $HT = TH$.
- $\Phi(f) = \Phi(f)^*$ for $f \in C(\sigma(H))$

PROOF Let $\text{Pol}(\sigma(H))$ denote the polynomial functions restricted to $\sigma(H)$. Then we have

- (i) $\sigma(p(H)) = p(\sigma(H))$
- (ii) $\|p(H)\| = r(p(H))$ as $p(H)$ is normal.
- (iii) $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$

We let $\Phi_0 : \text{Pol}(\sigma(H)) \rightarrow \mathcal{B}(\mathcal{H})$ by $\Phi_0(p) = p(H)$. Then Φ_0 is the unique linear operator such that $\Phi_0(1) = I$, $\Phi_0(\iota) = H$, and $\Phi_0(pq) = \Phi_0(p)\Phi_0(q)$ for $p, q \in \text{Pol}(\sigma(H))$. Moreover, (ii) shows that $\|p\|_\infty = r(p(H))$ so that $r(p(H)) = \|p(H)\|$ so that $\Phi_0 : \text{Pol}(\sigma(H)) \rightarrow \mathcal{B}(\mathcal{H})$ is an isometry. Thus since $\sigma(H)$ is compact, by Stone-Weierstrass, $\overline{\text{Pol}(\sigma(H))}^{\|\cdot\|_\infty} = C(\sigma(H))$, where we are conjugacy closed since $\sigma(H) \subseteq \mathbb{R}$. Hence Φ_0 admits a unique extension $\Phi : C(\sigma(H)) \rightarrow \mathcal{B}(\mathcal{H})$. This is clearly isometric and satisfies the properties of Φ_0 . If $f, g \in C(\sigma(H))$, say $f = \lim p_n$, $g = \lim q_n$, then $\|fg - p_nq_n\|_\infty \rightarrow 0$, and hence

$$\|\Phi(fg) - \Phi(f)\Phi(g)\| \leq \|\Phi(fg) - \Phi_0(p_nq_n)\| + \|\Phi_0(p_n)\Phi_0(q_n) - \Phi(f)\Phi(g)\| \rightarrow 0$$

so $\Phi(fg) = \Phi(f)\Phi(g)$. The other properties follow similarly. ■