

Random Matrix Products

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Spring 2021[†]

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[†]Last updated: May 3, 2021

Contents

Preface

Chapter I	The Multiplicative Ergodic Theorem	
1	Random matrix products and the subadditive ergodic theorem	1
1.1	The Birkhoff ergodic theorem	1
1.2	Random matrix products	1
1.3	Lyapunov exponents and the subadditive ergodic theorem	3
2	Positivity of Lyapunov Exponents	7

Preface

These lecture notes on random matrix products are prepared for the reading group on random matrix products for the [analysis group](#) in Spring 2021. Much of the content is based on Alex Gorodnik's [lecture notes](#) for his course "Random walks on matrix groups". Any errors or omissions can be sent to [the author](#).

I. The Multiplicative Ergodic Theorem

1 RANDOM MATRIX PRODUCTS AND THE SUBADDITIVE ERGODIC THEOREM

1.1 THE BIRKHOFF ERGODIC THEOREM

Let Ω be a separable, second-countable metric space equipped with its Borel σ -algebra \mathcal{B} , and let μ be a Borel probability measure on Ω . Suppose we are given a measurable function $\theta : \Omega \rightarrow \Omega$. We denote the *pushforward* of μ by θ to denote the Borel probability measure defined by the rule

$$\theta_*\mu(E) = \mu(\theta^{-1}(E))$$

for Borel sets $E \subset \Omega$. We say that the function θ is *measure preserving* if $\theta_*\mu = \mu$. In this situation, we call the information (Ω, μ, θ) a *measure-preserving dynamical system*.

Given a Borel set $E \subset \Omega$, we say that E is θ -invariant if $\theta^{-1}(E) = E$, and denote the set of θ -invariant sets by \mathcal{B}_θ . More generally, we say that a measurable function $f : \Omega \rightarrow K$ where K is a topological space is θ -invariant if $f(\omega) = f(\theta(\omega))$ for μ -a.e. ω . One can verify that \mathcal{B}_θ is a Borel σ -subalgebra of \mathcal{B} . In particular, f is θ -invariant if and only if f is \mathcal{B}_θ -measurable. We say that (Ω, μ, θ) is *ergodic* if each θ -invariant set $E \in \mathcal{B}_\theta$ either has $\mu(E) = 0$ or $\mu(E) = 1$.

We will denote by T^n the n -fold composition $T \circ \cdots \circ T$. Given a function f , we write $f = f^+ + f^-$ where $f^+ \geq 0$ and $f^- \leq 0$. A standard result is the following.

1.1 Theorem (Birkhoff Pointwise Ergodic). *Let (Ω, μ, θ) be an ergodic measure-preserving dynamical system and let $f = f^+ + f^-$ satisfy $f_+ \in L^1(\Omega, \mu)$. Then for μ -a.e. $\omega \in \Omega$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(\omega)) = \int_{\Omega} f \, d\mu$$

where the limit may be attained at $-\infty$.

We have written [Theorem 1.1](#) in additive notation, but it can be easily rephrased in multiplicative notation. Denote by $\log^+(x) = \max(0, \log x)$. Write $g = \exp(f)$ and note that $f_+ = \log^+(g)$. Then for μ -a.e. $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} (g(T^{n-1}\omega) \cdots g(\omega))^{1/n} = \exp \left(\int_{\Omega} \log g \, d\mu \right).$$

Of course, here, the group written in product notation is still commutative. In the following section, we consider a more general setting where this is no longer the case.

1.2 RANDOM MATRIX PRODUCTS

The setting of [Theorem 1.1](#) is nice, but in these notes we are interested in a somewhat more general situation. First consider the following example. Let Ω denote the compact product

space $\text{GL}_d(\mathbb{C})^{\mathbb{N}}$ equipped with the left-shift map $\sigma : \Omega \rightarrow \Omega$ given by

$$\sigma((M_n)_{n=1}^{\infty}) = (M_n)_{n=2}^{\infty}$$

for a sequence of matrices $(M_n)_{n=1}^{\infty} \subset \Omega$. Let ν be a probability measure on $\text{GL}_d(\mathbb{C})$ and let $X_i : \Omega \rightarrow \text{GL}_d(\mathbb{C})$ for $i \in \mathbb{N}$ be independent random matrices with distribution ν . Asymptotic behaviour of random products of the form $X_n \cdots X_1$ can be interpreted as a matrix-valued generalization of the law of large numbers.

More generally, we are interested in matrix-valued measurable functions, i.e. functions $X : \Omega \rightarrow \text{GL}_d(\mathbb{C})$ on a measure-preserving space (Ω, μ, θ) . This setting is a generalization of the setting in [Theorem 1.1](#), where we considered a measurable function $f : \Omega \rightarrow \mathbb{R}$ satisfying an integrability criteria. Let $\|\cdot\| : \text{GL}_d(\mathbb{C}) \rightarrow \mathbb{R}$ be a matrix norm. We will assume that $\|\cdot\|$ is *submultiplicative* (i.e. $\|AB\| \leq \|A\| \|B\|$), but we do not lose any generality since all matrix norms are equivalent. We also assume that X satisfies the integrability condition

$$\int_{\Omega} \log^+ \|X(\omega)\| d\omega < \infty.$$

As in the prior section, we are interested in determining statistical information concerning the limit of the random matrix product

$$S_n(\omega) = X(T^{n-1}\omega) \cdots X(\omega).$$

We will investigate various statistical properties of the random products $S_n(\omega)$. Here are three such examples which we will focus on: **TODO: add links here once the sections are written up**

- (i) the growth rate of $\|S_n(\omega)\| = \|X(\theta^{n-1}\omega) \cdots X(\omega)\|$ for large n and “typical” ω .
- (ii) the growth rate from a fixed starting point $\|X(\theta^{n-1}\omega) \cdots X(\omega)v\|$ for some $v \in \mathbb{C}^n$
- (iii) the behaviour of the directions $\|X(\theta^{n-1}\omega) \cdots X(\omega)v\| / \|X(\theta^{n-1}\omega) \cdots X(\omega)v\|$ for some $v \in \mathbb{C}^n$.

Here are some settings where this theory is applicable.

Example. 1. Given fixed matrices $M_1, \dots, M_{\ell} \in \text{GL}_d(\mathbb{C})$, generate a sequence $S_0 = I$ and $S_{n+1} = M_i \cdot S_n$ where we take matrix M_i with probability $1/\ell$. The products S_n can be interpreted as a random walk on $\text{GL}_d(\mathbb{C})$ (or \mathbb{C}^n) where the “steps” are given by multiplication by a matrix M_i .

- 2. If $U \subset \mathbb{R}^d$ is an open set and $F : U \rightarrow U$ is smooth, by the chain rule, the Jacobian of F^n at a point u satisfies

$$D(F^n)_u = (DF)_{F^{n-1}u} \cdots (DF)_u.$$

Here, $DF : U \rightarrow \text{GL}_d(\mathbb{R})$ is a matrix-valued measurable function. The growth rate of DF is related to the entropy of F and the dimension of invariant measures.

- 3. If $T_i(x) = A_i x + t_i$ where $A_1, \dots, A_{\ell} \in \text{GL}_d(\mathbb{R})$ have operator norms $\|A_i\| < 1$ for $i = 1, \dots, \ell$ and $t_i \in \mathbb{R}^n$, then there is a unique *self-affine set* K satisfying

$$K = \bigcup_{i=1}^{\ell} T_i(K)$$

and, given probabilities p_1, \dots, p_ℓ , a unique *self-affine measure*, which is a Borel probability measure ν satisfying

$$\nu = \sum_{i=1}^{\ell} p_i (T_i)_* \mu.$$

Here, dimensional properties of the measure ν are related to properties of random products of the matrices $\{A_1, \dots, A_\ell\}$.

1.3 LYAPUNOV EXPONENTS AND THE SUBADDITIVE ERGODIC THEOREM

A fundamental statistical property associated with the matrix-valued function X is the following.

Definition. With notation as above, we define the *top Lyapunov exponent* $\lambda : \Omega \rightarrow \mathbb{R}$ by

$$\lambda(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|S_n(\omega)\|.$$

We now have the following fundamental result.

1.2 Theorem (Furstenburg-Kesten). *The function λ is θ -invariant and satisfies*

$$\int_{\Omega} \lambda(\omega) d\omega = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|S_n(\omega)\| d\omega.$$

This result can be thought of as interchanging the limit with the integral, i.e. averaging over space is the same as averaging over time.

In fact, we will prove [Theorem 1.2](#) as a consequence of a more general result. We first make some observations about the average $a_n := \int_{\Omega} \log \|S_n(\omega)\| d\omega$. Observe by submultiplicativity of the matrix norm that

$$\begin{aligned} a_{n+m} &:= \int_{\Omega} \log \|S_{n+m}(\omega)\| d\omega \\ &= \int_{\Omega} \log \|X(\theta^{n+m-1}\omega) \cdots X(\omega)\| d\omega \\ &\leq \int_{\Omega} \log \|X(\theta^{n+m-1}\omega) \cdots X(\theta^m\omega)\| d\omega + \int_{\Omega} \log \|X(\theta^{m-1}\omega) \cdots X(\omega)\| d\omega \quad (1.1) \\ &= \int_{\Omega} \log \|S_n(\theta^m\omega)\| d\omega + \int_{\Omega} \log \|S_m(\omega)\| d\omega \\ &= a_n + a_m \end{aligned}$$

where the last line follows by the integrability condition on X along with the fact that θ is measure preserving.

Definition. We say that the sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ is *subadditive* if $a_{n+m} \leq a_n + a_m$ for each $n, m \in \mathbb{N}$. More generally, we say that a sequence of functions $\varphi_n : \Omega \rightarrow \mathbb{R}$ is *subadditive* if

$$\varphi_{n+m}(\omega) \leq \varphi_n(\theta^m\omega) + \varphi_m(\omega). \quad (1.2)$$

The following lemma is straightforward.

1.3 Lemma. *If $(a_n)_{n=1}^\infty$ is a subadditive sequence, then $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}$.*

In particular, implies that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|S_n(\omega)\| d\omega$$

always exists. Moreover, if we set $\varphi_n(\omega) = \log \|S_n(\omega)\|$, we observed in (1.1) that the sequence of functions φ_n is subadditive. Thus Theorem 1.2 is a consequence of the following more general result.

Throughout the statement and the proof, note that many inequalities implicitly hold for μ -a.e. $\omega \in \Omega$.

1.4 Theorem (Kingman's Subadditive Ergodic). *Let $\varphi_n : \Omega \rightarrow \mathbb{R}$ be a subadditive sequence with $\varphi_1^+ \in L^1(\Omega, \mu)$. Then the limit $\varphi(\omega) := \lim_{n \rightarrow \infty} \frac{\varphi_n(\omega)}{n}$ exists for almost every $\omega \in \Omega$. Moreover, φ is θ -invariant and*

$$\int_{\Omega} \varphi(\omega) d\omega = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \varphi_n(\omega) d\omega =: L.$$

Set

$$\varphi_-(\omega) = \liminf_{n \rightarrow \infty} \frac{\varphi_n(\omega)}{n} \quad \varphi_+(\omega) = \limsup_{n \rightarrow \infty} \frac{\varphi_n(\omega)}{n}.$$

We first observe that φ_- (and by an analogous argument φ_+) is θ -invariant. By the subadditivity assumption (1.2) with $m = 1$,

$$\varphi_-(\omega) \leq \liminf_{n \rightarrow \infty} \frac{\varphi_n(\theta\omega) + \varphi(\omega)}{n+1} = \varphi_-(\theta\omega)$$

so with $X_a = \{\omega \in \Omega : \varphi_-(\omega) \geq a\}$ for any $a \in \overline{\mathbb{R}}$, we have $\theta^{-1}(X_a) \supset X_a$. But θ is measure-preserving, so this can force $\mu(\theta^{-1}(X_a) \setminus X_a) = 0$, i.e. φ_- is θ -invariant.

Our general idea in this proof is to first establish the result for the function φ_- , and then use subadditivity and a repeat application of this result to obtain the result for φ_+ . To subdivide the proof more clearly, we will first prove two intermediate lemmas.

1.5 Lemma. *We have $\int_{\Omega} \varphi_-(\omega) d\omega = L$.*

Proof. Let $\epsilon > 0$ be arbitrary. For $k \in \mathbb{N}$, define

$$E_k = \left\{ \omega \in \Omega : \frac{\varphi_j(\omega)}{j} \leq \varphi_-(\omega) + \epsilon \text{ for some } j = 1, \dots, k \right\}.$$

Note that $E_k \subset E_{k+1}$ and $\bigcup_k E_k = \Omega$. Now set

$$\psi_k(\omega) = \begin{cases} \varphi_-(\omega) + \epsilon & : \omega \in E_k \\ \varphi_1(\omega) & : \omega \in E_k^c \end{cases}$$

Observe that $\psi_k \geq \varphi_-(\omega) + \epsilon$ by definition of E_k .

First, we will prove that for all $n > k$ and almost every $\omega \in \Omega$,

$$\varphi_n(\omega) \leq \sum_{i=0}^{n-k-1} \psi_k(\theta^i \omega) + \sum_{i=n-k}^{n-1} \max\{\psi_k, \varphi_1\}(\theta^i \omega). \quad (1.3)$$

Since φ_- is θ -invariant, we may assume that $\varphi_-(\theta^n \omega) = \varphi_-(\omega)$ for all n .

We will inductively define a sequence $m_0 \leq n_1 < m_1 \leq n_2 < \dots$ as follows. Let $m_0 = 0$. Inductively, let $n_j \geq m_{j-1}$ be the minimal integer such that $\theta^{n_j} \omega \in E_k$ (if it exists). By definition of E_k , there exists m_j such that $1 \leq m_j - n_j \leq k$ and

$$\varphi_{m_j - n_j}(\theta^{n_j} \omega) \leq (m_j - n_j)(\varphi_-(\theta^{n_j} \omega) + \epsilon). \quad (1.4)$$

Let ℓ be maximal such that $m_\ell \leq n$. By subadditivity, inductively applying the inequality

$$\varphi_i(\omega) \leq \varphi_1(\theta^i \omega) + \varphi_{i-1}(\omega)$$

if $i \neq m_j$ for some j and the inequality

$$\varphi_{m_j}(\omega) \leq \varphi_{n_j}(\omega) + \varphi_{m_j - n_j}(\theta^{n_j} \omega),$$

we obtain

$$\varphi_n(\omega) \leq \sum_{i \in I} \varphi_1(\theta^i \omega) + \sum_{j=1}^{\ell} \varphi_{m_j - n_j}(\theta^{n_j} \omega) \quad (1.5)$$

where $I = \bigcup_{j=0}^{\ell-1} [m_j, n_{j+1}) \cup [m_\ell, n)$. Now if $i \in I$ with $i < n_{\ell+1}$, we have

$$\varphi_1(\theta^i \omega) = \psi_k(\theta^i \omega)$$

since $\theta^i \omega \notin E_k^c$. Since $\varphi_-(\theta^n \omega) = \varphi_-(\omega)$ and $\psi_k \geq \varphi_- + \epsilon$ by definition, by (1.4),

$$\varphi_{m_j - n_j}(\theta^{n_j} \omega) \leq \sum_{i=n_j}^{m_j-1} (\varphi_-(\theta^i \omega) + \epsilon) \leq \sum_{i=n_j}^{m_j-1} \psi_k(\theta^i \omega).$$

Thus (1.3) follows by (1.5) and the fact that $n - n_\ell < k$.

Now, suppose $\varphi_n/n \geq -C$ for some fixed constant $C > 0$. The upper bound follows by Fatou's Lemma:

$$\int_{\Omega} \varphi_-(\omega) d\omega \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \varphi_n(\omega) d\omega = L.$$

To get the lower bound, by (1.3),

$$\frac{1}{n} \int_{\Omega} \varphi_n(\omega) d\omega \leq \frac{n-k}{n} \int_{\Omega} \psi_k(\omega) d\omega + \frac{k}{n} \int_{\Omega} \max\{\psi_k, \varphi_1\}(\omega) d\omega.$$

Thus taking the limit as n goes to infinity, we have

$$L \leq \int_{\Omega} \psi_k(\omega) d\omega$$

which holds for any $k \in \mathbb{N}$. Moreover, $\lim_{k \rightarrow \infty} \psi_k = \varphi_- + \epsilon$, so that $L \leq \int_{\Omega} \varphi_-(\omega) d\omega + \epsilon$. But $\epsilon > 0$ was arbitrary, giving the desired equality.

More generally, let $\varphi_n^{(C)} = \max\{\varphi_n, -Cn\}$ and $\varphi_-^{(C)} = \max\{\varphi_-, -C\}$. Then by the Monotone Convergence Theorem,

$$\begin{aligned} \int_{\Omega} \varphi_-(\omega) d\omega &= \inf_C \int_{\Omega} \varphi_-^{(C)}(\omega) d\omega = \inf_C \inf_n \int_{\Omega} \frac{\varphi_n^{(C)}(\omega)}{n} d\omega \\ &= \inf_n \int_{\Omega} \frac{\varphi_n(\omega)}{n} d\omega = L \end{aligned}$$

as required. \square

1.6 Lemma. We have $\limsup_{n \rightarrow \infty} \frac{\varphi_{nk}(\omega)}{nk} = \varphi_+(\omega)$ pointwise a.e.

Proof. The upper bound follows since by subadditivity and invariance of φ_+ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\varphi_{nk}(\omega)}{n} &\leq \sum_{j=0}^{k-1} \limsup_{n \rightarrow \infty} \frac{\varphi_n(\theta^{nj}\omega)}{n} \\ &= k\varphi_+(\omega). \end{aligned}$$

Conversely, given $n \in \mathbb{N}$, write $n = kq_n + r_n$ where $r_n \in \{1, \dots, k\}$. By subadditivity,

$$\varphi_n(\omega) \leq \varphi_{kq_n}(\omega) + \varphi_{r_n}(\theta^{kq_n}\omega) \leq \varphi_{kq_n}(\omega) + \psi(\theta^{kq_n}\omega)$$

where $\psi = \max\{\varphi_1^+, \dots, \varphi_k^+\}$. By assumption, $\psi \in L^1$. Below, we will show that

$$\lim_{n \rightarrow \infty} \frac{\psi \circ \theta^{kq_n}}{q_n} = 0 \quad (1.6)$$

pointwise a.e. Assuming this result, we have

$$\limsup_{n \rightarrow \infty} \frac{\varphi_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \varphi_{kq_n} = \frac{1}{k} \limsup_{n \rightarrow \infty} \frac{1}{q_n} \varphi_{kq_n} \leq \frac{1}{k} \limsup_{n \rightarrow \infty} \frac{\varphi_{nk}}{n}.$$

Let's prove (1.6). Let $\epsilon > 0$ be arbitrary. We first observe that

$$\begin{aligned} \sum_{n=1}^{\infty} \mu(\{\omega \in \Omega : |\psi(\theta^n \omega)| \geq \epsilon n\}) &= \sum_{n=1}^{\infty} \mu(\{\omega \in \Omega : |\psi(\omega)| \geq \epsilon n\}) \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mu(\{\omega \in \Omega : k\epsilon \leq |\psi(\omega)| < (k+1)\epsilon\}) \\ &= \sum_{k=1}^{\infty} k \mu(\{\omega \in \Omega : k\epsilon \leq |\psi(\omega)| < (k+1)\epsilon\}) \\ &\leq \int_{\Omega} \frac{|\psi(\omega)|}{\epsilon} d\omega < \infty. \end{aligned}$$

Thus the result follows by the Borel-Cantelli Lemma. \square

Proof (of Theorem 1.4). We are now in position to complete the proof. As before, we first assume that $\varphi_n/n \geq -C$ for some fixed $C > 0$. Set

$$\phi_k = - \sum_{j=0}^{n-1} \varphi_k \circ \theta^{kj}.$$

By definition, $\phi_{n+m} = \phi_m + \phi_n \circ \theta^{km}$ and $\phi_1 = -\varphi_k \leq Ck$, so $\phi_1^+ \in L^1(\Omega, \mu)$. Let $\phi_- = \liminf_{n \rightarrow \infty} \frac{\phi_n}{n} d\omega$. Then by Lemma 1.5 and the fact that μ is θ -invariant,

$$\int_{\Omega} \phi_-(\omega) d\omega = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\phi_n(\omega)}{n} d\omega = \int_{\Omega} \varphi_k(\omega) d\omega.$$

Now by the subadditivity assumption and [Lemma 1.6](#),

$$-\phi_- = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_k \circ \theta^{kj} \geq \limsup_{n \rightarrow \infty} \frac{\varphi_{kn}}{n} = k\varphi_+.$$

Combining the last two equations, we obtain

$$\int_{\Omega} \varphi_+ d\omega \leq -\frac{1}{k} \int_{\Omega} \phi_- d\omega \leq \frac{1}{k} \int_{\Omega} \varphi_k(\omega) d\omega.$$

But this holds for any $k \in \mathbb{N}$, so that $\int_{\Omega} \varphi_+ d\omega \leq L$.

In general, as in the proof of [Lemma 1.5](#), set $\varphi_n^{(C)} = \max\{\varphi_n, -Cn\}$ and $\varphi_{\pm}^{(C)} = \max\{\varphi_{\pm}, -C\}$. We just showed that $\int_{\Omega} -\varphi_-^{(C)} d\omega = \int_{\Omega} \varphi_+^{(C)}(\omega) d\omega$. But $\varphi_-^{(C)} \leq \varphi_+^{(C)}$, so that $\varphi_-^{(C)} = \varphi_+^{(C)}$. Thus the result follows by the Monotone Convergence Theorem. \square

Remark. This result generalizes [Theorem 1.1](#) since, using the notation from that theorem, the function $\varphi_n(\omega) = \sum_{i=0}^{n-1} f(T^i\omega)$ is subadditive (since it is additive) and by invariance of T ,

$$\int_{\Omega} f(T^i\omega) d\omega = f(T^i\omega).$$

In fact, [Theorem 1.1](#) follows directly from [Lemma 1.5](#) since both $(\varphi_n)_{n=1}^{\infty}$ and $(-\varphi_n)_{n=1}^{\infty}$ are subadditive sequences of functions.

The argument in [Lemma 1.6](#) can be interpreted as a “stability result” for subadditive sequences, which we then use to get control over φ_+ in the general case.

2 POSITIVITY OF LYAPUNOV EXPONENTS

In this section, we specialize slightly to the following setting. Let ν be a probability measure on $\mathrm{GL}_d(\mathbb{C})$. Then we take $\Omega = \mathrm{GL}_d(\mathbb{C})^{\mathbb{N}}$ equipped with the left-shift map σ , and μ is the infinite product $\mu = \nu^{\otimes \mathbb{N}}$. In this setting, the measure-preserving dynamical system (Ω, μ, σ) is ergodic. Since the Lyapunov exponent λ is σ -invariant, λ is constant μ -a.e. Abusing notation, we denote this constant by λ .

What can we say about the almost-everywhere value of λ ? Of course, $\lambda \geq 0$, so we naturally specialize to distinguishing the cases where $\lambda = 0$ or $\lambda > 0$. There are some simple natural settings where $\lambda = 0$. Denote by G_{ν} the closure of the subgroup generated by the matrices in $\mathrm{supp} \nu$.

1. If G_{μ} is compact, then the norms of any random product is uniformly bounded above by a constant, so in fact $\lambda = 0$ everywhere.
2. If G_{μ} is contained in an abelian subgroup, then

$$\lambda = \int_{\Omega} \|M\| d\nu(M)$$

which may be zero depending on the choice of ν .

3. If μ is the atomic measure with support

$$\text{supp } \mu = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

then $\lambda = 0$ almost everywhere. More generally, if μ consists of a uniformly chosen random rational rotation, along with a uniformly chosen contraction or dilation depending on the angle, then $\lambda = 0$ almost everywhere.

Our main theorem in this section is that the three examples above are essentially the only ways in which we can have $\lambda = 0$ almost everywhere. We first state the following definition.

Definition. We say that a subgroup G of $\text{GL}_d(\mathbb{C})$ is *totally irreducible* if there is no finite union of proper subspaces of \mathbb{C}^d which are G -invariant.

We first observe a basic consequence of total irreducibility and non-compactness. Here, $\mathbb{P}(\mathbb{C}^d)$ is $n - 1$ -dimensional projective space, equipped with the projection map $[\cdot] : \mathbb{C}^d \rightarrow \mathbb{P}(\mathbb{C}^d)$ taking $x \in \mathbb{C}^d$ to the equivalence class

$$[x] := \{y \in \mathbb{C}^n : y = \lambda x, \lambda \in \mathbb{C} \setminus \{0\}\}.$$

Of course, $M_d(\mathbb{C})$ acts naturally on $\mathbb{P}(\mathbb{C}^d)$ as well by $M \cdot [x] = [Mx]$ for $M \in M_d(\mathbb{C})$.

2.1 Lemma. Suppose G_ν is totally irreducible and non-compact. Then there is no G_ν -invariant probability measure on $\mathbb{P}(\mathbb{C}^d)$.

Proof. Suppose for contradiction μ is a G_ν -invariant probability measure on $\mathbb{P}(\mathbb{C}^d)$. Since G_ν is unbounded, there exists a sequences of matrices $(g_n)_{n=1}^\infty \subset G_\nu$ such that $\lim_{n \rightarrow \infty} \|g_n\| = \infty$. Let $u_n = g_n / \|g_n\|$, so that $\lim_{n \rightarrow \infty} \det u_n = 0$. Since $\|u_n\| = 1$ for each n , passing to a subsequence if necessary, we may assume

$$\lim_{n \rightarrow \infty} u_n = u \in M_d(\mathbb{C})$$

entry-wise. Write

$$V = [\ker u] \subset \mathbb{P}(\mathbb{C}^n) \text{ and } W = [\text{im } u] \subset \mathbb{P}(\mathbb{C}^n)$$

and since $\|u\| = 1$ so that $u \neq 0$ and $\det u = 0$, V and W are proper projective subspaces of $\mathbb{P}(\mathbb{C}^d)$.

Decompose $\mu = \mu_1 + \mu_2$ where $\mu_1 = \mu|_V$ and $\mu_2 = \mu|_{V^c}$. If $[x] \in V^c$, then $g_n \cdot [x] = u_n \cdot [x]$ so $\lim_{n \rightarrow \infty} g_n \cdot [x] = u \cdot [x]$. Thus

$$\lim_{n \rightarrow \infty} (g_n)_* \mu = \lim_{n \rightarrow \infty} (g_n)_* \mu_1 + u_* \mu_2$$

where we recall $(g_n)_* \mu_1$ denotes the pushforward of μ_1 by g_n (and similarly for $u_* \mu_2$). Now, passing to a subsequence and using compactness of $\mathbb{P}(\mathbb{C}^n)$, we may assume

$$\begin{aligned} \lim_{n \rightarrow \infty} (g_n)_* \mu_1 &= \mu_1^\infty \\ \lim_{n \rightarrow \infty} g_n V &= V^\infty \end{aligned}$$

for some probability measure μ_1^∞ on $\mathbb{P}(\mathbb{C}^d)$ and projective subspace V^∞ .

Since $\text{supp}(g_n)_*\mu_1 \subset g_n V$, we have $\text{supp } \mu_1^\infty \subset V^\infty$, and $\text{supp } u_*\mu_2 \subset W$. Since each $g_n V$ is a proper projective subspace of $\mathbb{P}(\mathbb{C}^n)$, so is V^∞ . But now $\text{supp } \mu \subset V^\infty \cup W$ so that $\mu(V^\infty \cup W) = 1$. Let $F \subset V^\infty \cup W$ be the smallest union of projective subspaces such that $\mu(F) = 1$, and observe that F is a finite union of proper projective subspaces. Thus by invariance of μ under G_ν , we have $gF = F$ for any $g \in G_\nu$, contradicting the assumption of total irreducibility. \square

We now prove our main result on positivity of Lyapunov exponents. For simplicity, we will assume that $G_\nu \subset \text{SL}_d(\mathbb{C})$.

2.2 Theorem (Furstenberg). *Suppose G_ν is totally irreducible and non-compact. Then*

$$\lambda(\omega) > 0$$

for μ -a.e. $\omega \in \Omega$.

It is meaningful to obtain the following operator-theoretic formulation of [Theorem 2.2](#); this perspective will also reappear in **TODO: cite Furstenberg measures section**. Consider the Hilbert space

$$\mathcal{H} = L^2(\mathbb{C}^d) = \left\{ f : \mathbb{C}^d \rightarrow \mathbb{C} : \int_{\mathbb{C}^d} |f(x)|^2 dm(x) < \infty \right\}.$$

Then a matrix $g \in \text{SL}_d(\mathbb{C})$ acting on \mathbb{C}^d induces a natural action $\pi(g) : \mathcal{H} \rightarrow \mathcal{H}$ by $\pi(g)f(x) = f(g^{-1}x)$, so we may define the operator $P_\nu : \mathcal{H} \rightarrow \mathcal{H}$ given by the rule

$$P_\nu f(x) = \int_{\text{GL}_d(\mathbb{C})} \pi(g)f(x) d\nu(g).$$

One can interpret the operator P_ν as applying a random transformation of f by a matrix g chosen according to the probability measure ν . We first list some basic properties of the action π and the operator P_ν .

2.3 Lemma. (i) $\|\pi(g)f\|_2 = \|f\|_2$ for any $g \in \text{SL}_d(\mathbb{C})$

(ii) $\|P_\nu\| \leq 1$

(iii) $P_{\nu_1}P_{\nu_2} = P_{\nu_1*\nu_2}$

(iv) $P_\nu^* = P_{\nu^*}$ where $d\nu^*(g) = d\nu(g^{-1})$

Proof. Part (i) follows by a change of variables since $|\det g| = 1$, and parts (iii) and (iv) follow directly from the definition of P_ν .

It remains to see (ii). By Jensen's inequality and an application of Fubini's Theorem,

$$\begin{aligned} \|P_\nu f\|_2^2 &= \int_{\mathbb{C}^d} \left| \int_{\text{GL}_d(\mathbb{C})} \pi(g)f(x) d\nu(g) \right|^2 dm(x) \\ &\leq \int_{\mathbb{C}^d} \int_{\text{GL}_d(\mathbb{C})} |\pi(g)f(x)|^2 d\nu(g) dm(x) \\ &= \int_{\text{GL}_d(\mathbb{C})} \int_{\mathbb{C}^d} |\pi(g)f(x)|^2 dm d\nu(g) \\ &= \int_{\text{GL}_d(\mathbb{C})} \|\pi(g)f\|_2^2 d\nu(g) \\ &= \|f\|_2^2 \end{aligned}$$

where the last line follows by (i) and the fact that ν is a probability measure. \square

Our proof approach is bound $\|P_\nu\|$ and then relate [Theorem 2.2](#) to the operator P_ν . We first need a standard result from analysis in Hilbert spaces, which we include for completeness.

2.4 Lemma. *Let P be a self-adjoint operator on a Hilbert space \mathcal{H} . Then*

$$\|P\| = \sup_{\|f\|=1} |\langle Pf, f \rangle| =: \alpha.$$

Proof. Set

$$\sup_{\|f\|=1} |\langle Pf, f \rangle| =: \alpha$$

Of course, we always have $\alpha \leq \|P\|$ by the Cauchy-Schwarz inequality. Conversely, it suffices to show that $|\langle Pf, g \rangle| \leq \alpha$ for any f, g with $\|f\| = \|g\| = 1$ (since taking $g = Pf / \|Pf\|$, $|\langle Pf, g \rangle| = \|P\|$). It suffices to prove the case where $\langle Pf, g \rangle \in \mathbb{R}$. Then since P is self-adjoint,

$$\langle Pf, g \rangle = \frac{\langle P(f+g), f+g \rangle - \langle P(f-g), f-g \rangle}{4}$$

so that

$$|\langle Pf, g \rangle| \leq \alpha \cdot \frac{\|f+g\|^2 + \|f-g\|^2}{4} = \alpha$$

by the parallelogram identity. \square

2.5 Proposition. *Suppose G_ν is totally irreducible and non-compact. Then $\|P_\nu\| < 1$.*

Proof. We have that $P_\nu P_{\nu^*} = P_{\nu^* \nu^*}$ is self adjoint, and $\|P_\nu P_{\nu^*}\| = \|P_\nu\|^2$ (this is just the C^* identity). Thus $\|P_\nu\| < 1$ if and only if $\|P_{\nu^* \nu^*}\| < 1$, so without loss of generality, we may assume that P_ν is self-adjoint.

Suppose for contradiction $\|P_\nu\| = 1$. By [Lemma 2.4](#), get $(f_n)_{n=1}^\infty \subset \mathcal{H}$ with $\|f_n\|_2 = 1$ and $\lim_{n \rightarrow \infty} |\langle Pf_n, f_n \rangle| = 1$. Since $|\langle Pf_n, f_n \rangle| \leq \langle P|f_n|, |f_n| \rangle \leq 1$, we may assume $f_n \geq 0$. Now,

$$\lim_{n \rightarrow \infty} \int_{G_\nu} \langle \pi(g) f_n, f_n \rangle d\nu(g) = \lim_{n \rightarrow \infty} \langle P_\nu f_n, f_n \rangle.$$

Since $\langle \pi(g) f_n, f_n \rangle \leq 1$, we have $\lim_{n \rightarrow \infty} \langle \pi(g) f_n, f_n \rangle = 1$ for ν -a.e. $g \in G_\nu$.

In particular, for ν -a.e. $g \in G_\nu$, we have

$$\lim_{n \rightarrow \infty} \|\pi(g) f_n - f_n\|_2^2 = 2 - 2 \lim_{n \rightarrow \infty} \langle \pi(g) f_n, f_n \rangle = 0$$

so by Cauchy-Schwarz,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\pi(g) f_n^2 - f_n^2\|_2 &\leq \lim_{n \rightarrow \infty} \|\pi(g) f_n - f_n\|_2 \cdot \|\pi(g) f_n + f_n\|_2 \\ &\leq 2 \lim_{n \rightarrow \infty} \|\pi(g) f_n - f_n\|_2 = 0. \end{aligned} \tag{2.1}$$

Now, consider the probability measures $d\mu_n = f_n^2 d\mu$ on \mathbb{C}^d , and let $\bar{\mu}_n$ denote the pushforward onto the projective space $\mathbb{P}(\mathbb{C}^d)$. Since $\mathbb{P}(\mathbb{C}^d)$ is compact, $\{\bar{\mu}_n\}_{n=1}^\infty$ has a weak*-accumulation point $\bar{\mu}$, and by [\(2.1\)](#), $\bar{\mu}$ is g -invariant for ν -a.e. $g \in G_\nu$. This contradicts [Lemma 2.1](#). \square

We now finish the proof by relating the operators P_ν with Lyapunov exponents.

Proof (of Theorem 2.2). By Theorem 1.2, it suffices to show that

$$\lambda(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|S_n(\omega)\| d\mu(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathrm{GL}_d(\mathbb{C})} \log \|g\| d\nu^{*n}(g) > 0$$

for μ -a.e. $\omega \in \Omega$.

Let

$$\begin{aligned} f(x) &= \min\{C, |x|^{-\alpha}\} \\ K &= \{x : 1 \leq |x| \leq 2\} \end{aligned}$$

where α is chosen so that $f \in L^2(\mathbb{C}^d)$ and $C > 0$ is a constant to be determined below. Set $\gamma = \|P_\nu\| < 1$. We then have

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle|^{1/n} &= \limsup_{n \rightarrow \infty} |\langle P_\nu^n f, \mathbf{1}_K \rangle|^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} \|P_\nu^n\|^{1/n} \cdot \|f\|_2^{1/n} \cdot \|\mathbf{1}_K\|_2^{1/n} \leq \gamma. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle &= \int_{1 \leq |x| \leq 2} \int_{\mathrm{GL}_d(\mathbb{C})} \min\{C, \|g^{-1}x\|^{-\alpha}\} d\nu^{*n}(g) dm(x) \\ &\geq \int_{1 \leq |x| \leq 2} \int_{\mathrm{GL}_d(\mathbb{C})} \min\{C, \|g^{-1}\|^{-\alpha} \cdot \|x\|^{-\alpha}\} d\nu^{*n}(g) dm(x) \\ &\geq C_0 \int_{\mathrm{GL}_d(\mathbb{C})} \min\{C, \|g^{-1}\|^{-\alpha}\} d\nu^{*n}(g) \end{aligned}$$

for some constant C_0 depending only on α . Since $\inf_{g \in \mathrm{SL}_d(\mathbb{C})} \|g\| > 0$, we can take C sufficiently large so that $\min\{C, \|g^{-1}\|^{-\alpha}\} = \|g^{-1}\|^{-\alpha}$ for any $g \in \mathrm{SL}_d(\mathbb{C})$. We also use the fact that $\|g^{-1}\| \leq C'_0 \|g\|^{d-1}$, which follows by the adjoint formula for the matrix (since the entries in the adjoint are degree $d-1$ polynomial functions of the entries of g , and $|\deg g| = 1$). Thus there is some constant $C_1 > 0$ such that

$$\langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle \geq C_1 \int_{\mathrm{GL}_d(\mathbb{C})} \|g\|^{-\alpha(d-1)} d\nu^{*n}(g).$$

Thus taking logarithms, applying Jensen's inequality, and rearranging, we have

$$\int_{\mathrm{GL}_d(\mathbb{C})} \log \|g\| d\nu^{*n}(g) \geq \frac{\log C_1}{\alpha(d-1)} - \frac{1}{\alpha(d-1)} \log \langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle$$

and therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathrm{GL}_d(\mathbb{C})} \log \|g\| d\nu^{*n}(g) &= -\frac{1}{\alpha(d-1)} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle \\ &\geq -\frac{1}{\alpha(d-1)} \log \gamma > 0 \end{aligned}$$

as required. \square