

Random Matrix Products

Alex Rutar^{*}
University of St Andrews

Spring 2021[†]

^{*}*alex@rutar.org*

[†]Last updated: April 12, 2021

Contents

Preface

Chapter I	The Multiplicative Ergodic Theorem	
1	Random matrix products and the subadditive ergodic theorem	1
1.1	The Birkhoff ergodic theorem	1
1.2	Random matrix products	1
1.3	Lyapunov exponents and the subadditive ergodic theorem	2

Preface

These lecture notes on random matrix products are prepared for the reading group on random matrix products for the [analysis group](#) in Spring 2021. Much of the content is based on Alex Gorodnik's [lecture notes](#) for his course "Random walks on matrix groups". Any errors or omissions can be sent to [the author](#).

I. The Multiplicative Ergodic Theorem

1 RANDOM MATRIX PRODUCTS AND THE SUBADDITIVE ERGODIC THEOREM

1.1 THE BIRKHOFF ERGODIC THEOREM

Let Ω be a separable, second-countable metric space equipped with its Borel σ -algebra \mathcal{B} , and let μ be a Borel probability measure on Ω . Suppose we are given a measurable function $\theta : \Omega \rightarrow \Omega$. We denote the *pushforward* of μ by θ to denote the Borel probability measure defined by the rule

$$\theta_*\mu(E) = \mu(\theta^{-1}(E))$$

for Borel sets $E \subset \Omega$. We say that the function θ is *measure preserving* if $\theta_*\mu = \mu$. In this situation, we call the information (Ω, μ, θ) a *measure-preserving dynamical system*.

Given a Borel set $E \subset \Omega$, we say that E is θ -invariant if $\theta^{-1}(E) = E$, and denote the set of θ -invariant sets by \mathcal{B}_θ . More generally, we say that a measurable function $f : \Omega \rightarrow K$ where K is a topological space is θ -invariant if $f = f \circ \theta$. One can verify that \mathcal{B}_θ is a Borel σ -subalgebra of \mathcal{B} . In particular, f is θ -invariant if and only if f is \mathcal{B}_θ -measurable. We say that (Ω, μ, θ) is *ergodic* if each θ -invariant set $E \in \mathcal{B}_\theta$ either has $\mu(E) = 0$ or $\mu(E) = 1$.

We will denote by T^n the n -fold composition $T \circ \cdots \circ T$. Given a function f , we write $f = f^+ + f^-$ where $f^+ \geq 0$ and $f^- \leq 0$. A standard result is the following.

1.1 Theorem. (Birkhoff Pointwise Ergodic) *Let (Ω, μ, θ) be an ergodic measure-preserving dynamical system and let $f = f^+ + f^-$ satisfy $f_+ \in L^1(\Omega, \mu)$. Then for μ -a.e. $\omega \in \Omega$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(\omega)) = \int_{\Omega} f \, d\mu$$

where the limit may be attained at $-\infty$.

We have written [Theorem 1.1](#) in additive notation, but it can be easily rephrased in multiplicative notation. Denote by $\log^+(x) = \max(0, \log x)$. Write $g = \exp(f)$ and note that $f_+ = \log^+(g)$. Then for μ -a.e. $\omega \in \Omega$,

$$\lim_{N \rightarrow \infty} (g(T^{N-1}\omega) \cdots g(\omega))^{1/N} = \exp \left(\int_{\Omega} \log g \, d\mu \right).$$

Of course, here, the group written in product notation is still commutative. In the following section, we consider a more general setting where this is no longer the case.

1.2 RANDOM MATRIX PRODUCTS

The setting of [Theorem 1.1](#) is nice, but in these notes we are interested in a somewhat more general situation. First consider the following example. Let Ω denote the compact product space $\text{GL}_n(\mathbb{C})^{\mathbb{N}}$ equipped with the left-shift map $\sigma : \Omega \rightarrow \Omega$ given by

$$\sigma((M_n)_{n=1}^{\infty}) = (M_n)_{n=2}^{\infty}$$

for a sequence of matrices $(M_n)_{n=1}^\infty \in \Omega$. Let ν be a probability measure on $\text{GL}_n(\mathbb{C})$ and let $X_i : \Omega \rightarrow \text{GL}_n(\mathbb{C})$ for $i \in \mathbb{N}$ be independent random matrices with distribution ν . Asymptotic behaviour of random products of the form $X_n \cdots X_1$ can be interpreted as a matrix-valued generalization of the law of large numbers.

Example. 1. Given fixed matrices $M_1, \dots, M_\ell \in \text{GL}_n(\mathbb{C})$, generate a sequence $S_0 = I$ and $S_{n+1} = M_i \cdot S_n$ where we take matrix M_i with probability $1/\ell$. The products S_n can be interpreted as a random walk on $\text{GL}_n(\mathbb{C})$ (or \mathbb{C}^n) where the “steps” are given by multiplication by a matrix M_i .

2. If $U \subset \mathbb{R}^d$ is an open set and $F : U \rightarrow U$ is smooth, by the chain rule, the Jacobian of F^n at a point u satisfies

$$D(F^n)_u = (DF)_{F^{n-1}u} \cdots (DF)_u.$$

Here, $DF : U \rightarrow \text{GL}_d(\mathbb{R})$ is a matrix-valued measurable function. The growth rate of DF is related to the *entropy* of F and the *dimension* of invariant measures. **TODO: elaborate?**

More generally, we are interested in matrix-valued measurable functions, i.e. functions $X : \Omega \rightarrow \text{GL}_n(\mathbb{C})$ on a measure-preserving space (Ω, μ, θ) . This setting is a generalization of the setting in [Theorem 1.1](#), where we considered a measurable function $f : \Omega \rightarrow \mathbb{R}$ satisfying an integrability criteria. Let $\|\cdot\| : \text{GL}_n(\mathbb{C}) \rightarrow \mathbb{R}$ be a submultiplicative matrix norm (i.e. $\|MN\| \leq \|M\| \|N\|$): we assume that X satisfies the integrability condition

$$\int_{\Omega} \log^+ \|X(\omega)\| d\omega < \infty.$$

As in the prior section, we are interested in determining statistical information concerning the limit of the random matrix product

$$S_n(\omega) = X(T^{n-1}\omega) \cdots X(\omega).$$

We will investigate various properties of the random products $S_n(\omega)$.

1.3 LYAPUNOV EXPONENTS AND THE SUBADDITIVE ERGODIC THEOREM

A fundamental statistical property associated with the matrix-valued function X is the following.

Definition. With notation as above, we define the *top Lyapunov exponent* $\lambda : \Omega \rightarrow \overline{\mathbb{R}}$ by

$$\lambda(\omega) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \|S_N(\omega)\|.$$

We now have the following fundamental result.

1.2 Theorem. (Furstenberg-Kesten) *The function λ is θ -invariant and satisfies*

$$\int_{\Omega} \lambda(\omega) d\omega = \lim_{n \rightarrow \infty} \int_{\Omega} \log \|S_n(\omega)\| d\omega.$$

In fact, we will prove [Theorem 1.2](#) as a consequence of a more general result. We first make some observations about the average $a_n := \int_{\Omega} \log \|S_n(\omega)\|$. Observe by submultiplicative of the matrix norm that

$$\begin{aligned}
 a_{n+m} &:= \int_{\Omega} \log \|S_{n+m}(\omega)\| d\omega \\
 &= \int_{\Omega} \log \|X(\theta^{n+m-1}\omega) \cdots X(\omega)\| d\omega \\
 &\leq \int_{\Omega} \log \|X(\theta^{n+m-1}\omega) \cdots X(\theta^m\omega)\| d\omega + \int_{\Omega} \log \|X(\theta^{m-1}\omega) \cdots X(\omega)\| d\omega \quad (1.1) \\
 &= \int_{\Omega} \log \|S_n(\theta^m\omega)\| d\omega + \int_{\Omega} \log \|S_m(\omega)\| d\omega \\
 &= a_n + a_m
 \end{aligned}$$

where the last line follows by the integrability condition on X along with the fact that θ is measure preserving.

Definition. We say that the sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ is *subadditive* if $a_{n+m} \leq a_n + a_m$ for each $n, m \in \mathbb{N}$. More generally, we say that a sequence of functions $\varphi_n : \Omega \rightarrow \mathbb{R}$ is *subadditive* if

$$\varphi_{n+m}(\omega) \leq \varphi_n(\theta^m\omega) + \varphi_m(\omega).$$

The following lemma is straightforward.

1.3 Lemma. *If $(a_n)_{n=1}^{\infty}$ is a subadditive sequence, then $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}$.*

In particular, implies that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|S_n(\omega)\| d\omega$$

always exists. Moreover, if we set $\varphi_n(\omega) = \log \|S_n(\omega)\|$, we observed in (1.1) that the sequence of functions φ_n is subadditive. Thus [Theorem 1.2](#) is a consequence of the following more general result.

1.4 Theorem. (Kingman's Subadditive Ergodic) *Let $\varphi_n : \Omega \rightarrow \mathbb{R}$ be a subadditive sequence with $\varphi_1^+ \in L^1(\Omega, \mu)$. Then the limit $\varphi(\omega) := \lim_{n \rightarrow \infty} \frac{\varphi_n(\omega)}{n}$ exists a.e., and*

$$\int_{\Omega} \varphi(\omega) d\omega = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \varphi_n(\omega) d\omega.$$

PROOF **TODO: write** ■

Remark. This result generalizes [Theorem 1.1](#) since, using the notation from that theorem, the function $\varphi_N = \sum_{n=0}^{N-1} f(T^n\omega)$ is subadditive.