Random Matrix Products

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Contents

Preface

These lecture notes on random matrix products are prepared for the reading group on random matrix products for the analysis group in Spring 2021. Much of the content is based on Alex Gorodnik's lecture notes for his course "Random walks on matrix groups". Any errors or omissions can be sent to the author.

I. The Multiplicative Ergodic Theorem

1 Random matrix products and the subadditive ergodic theorem

1.1 The Birkhoff ergodic theorem

Let Ω be a separable, second-countable metric space equipped with its Borel σ-algebra $\mathcal{B}$, and let μ be a Borel probability measure on Ω. Suppose we are given a measurable function $\theta : \Omega \to \Omega$. We denote the *pushforward* of μ by θ to denote the Borel probability measure defined by the rule

$$\theta_*\mu(E) = \mu(\theta^{-1}(E))$$

for Borel sets $E \subset \Omega$. We say that the function θ is *measure preserving* if $\theta_*\mu = \mu$. In this situation, we call the information (Ω, μ, θ) a *measure-preserving dynamical system*.

Given a Borel set $E \subset \Omega$, we say that E is θ-invariant if $\theta^{-1}(E) = E$, and denote the set of θ-invariant sets by $\mathcal{B}_\theta$. More generally, we say that a measurable function $f : \Omega \to K$ where K is a topological space is θ-invariant if $f(\omega) = f \circ \theta(\omega)$ for μ-a.e. ω. One can verify that $\mathcal{B}_\theta$ is a Borel σ-subalgebra of $\mathcal{B}$. In particular, f is θ-invariant if and only if f is $\mathcal{B}_\theta$-measurable. We say that (Ω, μ, θ) is *ergodic* if each θ-invariant set $E \in \mathcal{B}_\theta$ either has $\mu(E) = 0$ or $\mu(E) = 1$.

We will denote by T^n the n-fold composition $T \circ \cdots \circ T$. Given a function f, we write $f = f^+ + f^-$ where $f^+ \geq 0$ and $f^- \leq 0$. A standard result is the following.

1.1 Theorem (Birkhoff Pointwise Ergodic). *Let (Ω, μ, θ) be an ergodic measure-preserving dynamical system and let $f = f^+ + f^-$ satisfy $f_+ \in L^1(\Omega, \mu)$. Then for μ-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(\omega)) = \int_\Omega f \, \mathrm{d}\mu$$

where the limit may be attained at $-\infty$.

We have written Theorem 1.1 in additive notation, but it can be easily rephrased in multiplicative notation. Denote by $\log^+(x) = \max(0, \log x)$. Write $g = \exp(f)$ and note that $f_+ = \log^+(g)$. Then for μ-a.e. $\omega \in \Omega$,

$$\lim_{n\to\infty} (g(T^{n-1}\omega) \cdots g(\omega))^{1/n} = \exp\left(\int_\Omega \log g \, \mathrm{d}\mu\right).$$

Of course, here, the group written in product notation is still commutative. In the following section, we consider a more general setting where this is no longer the case.

1.2 Random matrix products

The setting of Theorem 1.1 is nice, but in these notes we are interested in a somewhat more general situation. First consider the following example. Let Ω denote the compact product space $\mathrm{GL}_d(\mathbb{C})^{\mathbb{N}}$ equipped with the left-shift map $\sigma : \Omega \to \Omega$ given by

$$\sigma((M_n)_{n=1}^{\infty}) = (M_n)_{n=2}^{\infty}$$

for a sequence of matrices $(M_n)_{n=1}^{\infty} \subset \Omega$. Let ν be a probability measure on $\mathrm{GL}_d(\mathbb{C})$ and let $X_i : \Omega \to \mathrm{GL}_d(\mathbb{C})$ for $i \in \mathbb{N}$ be independent random matrices with distribution ν. Asymptotic behaviour of random products of the form $X_n \cdots X_1$ can be interpreted as a matrix-valued generalization of the law of large numbers.

More generally, we are interested in matrix-valued measurable functions, i.e. functions $X : \Omega \to \mathrm{GL}_d(\mathbb{C})$ on a measure-preserving space (Ω, μ, θ). This setting is a generalization of the setting in Theorem 1.1, where we considered a measurable function $f : \Omega \to \mathbb{R}$ satisfying an integrability criteria. Let $\|\cdot\| : \mathrm{GL}_d(\mathbb{C})$ be a matrix norm. We will assume that $\|\cdot\|$ is *submultiplicative* (i.e. $\|AB\| \leq \|A\| \|B\|$), but we do not lose any generality since all matrix norms are equivalent. We also assume that X satisfies the integrability condition

$$\int_{\Omega} \log^{+} \|X(\omega)\| \, \mathrm{d}\omega < \infty.$$

As in the prior section, we are interested in determining statistical information concerning the limit of the random matrix product

$$S_n(\omega) = X(T^{n-1}\omega) \cdots X(\omega).$$

We will investigate various statistical properties of the random products $S_n(\omega)$. Here are three such examples which we will focus on: **TODO: add links here once the sections are written up**

(i) the growth rate of $\|S_n(\omega)\| = \left\|X(\theta^{n-1}\omega) \cdots X(\omega)\right\|$ for large n and "typical" ω.

(ii) the growth rate from a fixed starting point $\left\|X(\theta^{n-1}\omega) \cdots X(\omega)v\right\|$ for some $v \in \mathbb{C}^n$

(iii) the behaviour of the directions $\left\|X(\theta^{n-1}\omega) \cdots X(\omega)v\right\| / \left\|X(\theta^{n-1}\omega) \cdots X(\omega)v\right\|$ for some $v \in \mathbb{C}^n$.

Here are some settings where this theory is applicable.

Example. 1. Given fixed matrices $M_1, \ldots, M_\ell \in \mathrm{GL}_d(\mathbb{C})$, generate a sequence $S_0 = I$ and $S_{n+1} = M_i \cdot S_n$ where we take matrix M_i with probability $1/\ell$. The products S_n can be interpreted as a random walk on $\mathrm{GL}_d(\mathbb{C})$ (or $\mathbb{C}^n$) where the "steps" are given by multiplication by a matrix M_i.

2. If $U \subset \mathbb{R}^d$ is an open set and $F : U \to U$ is smooth, by the chain rule, the Jacobian of F^n at a point u satisfies

$$D(F^n)_u = (DF)_{F^{n-1}u} \cdots (DF)_u.$$

Here, $DF : U \to \mathrm{GL}_d(\mathbb{R})$ is a matrix-valued measurable function. The growth rate of DF is related to the entropy of F and the dimension of invariant measures.

3. If $T_i(x) = A_i x + t_i$ where $A_1, \ldots, A_\ell \in \mathrm{GL}_d(\mathbb{R})$ have operator norms $\|A_i\| < 1$ for $i = 1, \ldots, \ell$ and $t_i \in \mathbb{R}^n$, then there is a unique *self-affine set* K satisfying

$$K = \bigcup_{i=1}^{\ell} T_i(K)$$

and, given probabilities $p_1, \dots, p_\ell$, a unique *self-affine measure*, which is a Borel probability measure ν satisfying

$$\nu = \sum_{i=1}^{\ell} p_i (T_i)_* \mu.$$

Here, dimensional properties of the measure ν are related to properties of random products of the matrices $\{A_1, \dots, A_\ell\}$.

1.3 LYAPUNOV EXPONENTS AND THE SUBADDITIVE ERGODIC THEOREM

A fundamental statistical property associated with the matrix-valued function X is the following.

Definition. With notation as above, we define the *top Lyapunov exponent* $\lambda : \Omega \to \mathbb{R}$ by

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\| .$$

We now have the following fundamental result.

1.2 Theorem (Furstenberg-Kesten). *The function λ is θ-invariant and satisfies*

$$\int_\Omega \lambda(\omega)\, \mathrm{d}\omega = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|S_n(\omega)\| \, \mathrm{d}\omega.$$

This result can be thought of as interchanging the limit with the integral, i.e. averaging over space is the same as averaging over time.

In fact, we will prove Theorem 1.2 as a consequence of a more general result. We first make some observations about the average $a_n := \int_\Omega \log \|S_n(\omega)\|$. Observe by submultiplicativity of the matrix norm that

$$\begin{aligned}
a_{n+m} &:= \int_\Omega \log \|S_{n+m}(\omega)\| \, \mathrm{d}\omega \\
&= \int_\Omega \log \left\|X(\theta^{n+m-1}\omega) \cdots X(\omega)\right\| \mathrm{d}\omega \\
&\le \int_\Omega \log \left\|X(\theta^{n+m-1}\omega) \cdots X(\theta^m\omega)\right\| \mathrm{d}\omega + \int_\Omega \log \left\|X(\theta^{m-1}\omega) \cdots X(\omega)\right\| \mathrm{d}\omega \qquad (1.1)\\
&= \int_\Omega \log \|S_n(\theta^m\omega)\| \, \mathrm{d}\omega + \int_\Omega \log \|S_m(\omega)\| \, \mathrm{d}\omega \\
&= a_n + a_m
\end{aligned}$$

where the last line follows by the integrability condition on X along with the fact that θ is measure preserving.

Definition. We say that the sequence $(a_n)_{n=1}^\infty \subset \mathbb{R}$ is *subadditive* if $a_{n+m} \le a_n + a_m$ for each $n, m \in \mathbb{N}$. More generally, we say that a sequence of functions $\varphi_n : \Omega \to \mathbb{R}$ is *subadditive* if

$$\varphi_{n+m}(\omega) \le \varphi_n(\theta^m\omega) + \varphi_m(\omega). \qquad (1.2)$$

The following lemma is straightforward.

1.3 Lemma. *If $(a_n)_{n=1}^\infty$ is a subadditive sequence, then $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}$.*

In particular, implies that the limit

$$\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|S_n(\omega)\| \,\mathrm{d}\omega$$

always exists. Moreover, if we set $\varphi_n(\omega) = \log\|S_n(\omega)\|$, we observed in (1.1) that the sequence of functions φ_n is subadditive. Thus Theorem 1.2 is a consequence of the following more general result.

Throughout the statement and the proof, note that many inequalities implicity hold for μ-a.e. $\omega \in \Omega$.

1.4 Theorem (Kingman's Subadditive Ergodic). *Let $\varphi_n : \Omega \to \mathbb{R}$ be a subadditive sequence with $\varphi_1^+ \in L^1(\Omega, \mu)$. Then the limit $\varphi(\omega) := \lim_{n\to\infty} \frac{\varphi_n(\omega)}{n}$ exists for almost every $\omega \in \Omega$. Moreover, φ is θ-invariant and*

$$\int_\Omega \varphi(\omega)\,\mathrm{d}\omega = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \varphi_n(\omega)\,\mathrm{d}\omega =: L.$$

Set

$$\varphi_-(\omega) = \liminf_{n\to\infty} \frac{\varphi_n(\omega)}{n} \qquad\qquad \varphi_+(\omega) = \limsup_{n\to\infty} \frac{\varphi_n(\omega)}{n}.$$

We first observe that φ_- (and by an analgous argument φ_+) is θ-invariant. By the subadditivity assumption (1.2) with $m = 1$,

$$\varphi_-(\omega) \leq \liminf_{n\to\infty} \frac{\varphi_n(\theta\omega) + \varphi(\omega)}{n+1} = \varphi_-(\theta\omega)$$

so with $X_a = \{\omega \in \Omega : \varphi_-(\omega) \geq a\}$ for any $a \in \overline{\mathbb{R}}$, we have $\theta^{-1}(X_a) \supset X_a$. But θ is measure-preserving, so this can forces $\mu(\theta^{-1}(X_a) \setminus X_a) = 0$, i.e. φ_- is θ-invariant.

Our general idea in this proof is to first establish the result for the function φ_-, and then use subadditivity and a repeat application of this result to obtain the result for φ_+. To subdivide the proof more clearly, we will first prove two intermediate lemmas.

1.5 Lemma. *We have $\int_\Omega \varphi_-(\omega)\,\mathrm{d}\omega = L$.*

Proof. Let $\epsilon > 0$ be arbitrary. For $k \in \mathbb{N}$, define

$$E_k = \Big\{\omega \in \Omega : \frac{\varphi_j(\omega)}{j} \leq \varphi_-(\omega) + \epsilon \text{ for some } j = 1, \dots, k\Big\}.$$

Note that $E_k \subset E_{k+1}$ and $\bigcup_k E_k = \omega$. Now set

$$\psi_k(\omega) = \begin{cases} \varphi_-(\omega) + \epsilon & : \omega \in E_k \\ \varphi_1(\omega) & : \omega \in E_k^c \end{cases}$$

Observe that $\psi_k \geq \varphi_-(\omega) + \epsilon$ by definition of E_k.

First, we will prove that for all $n > k$ and almost every $\omega \in \Omega$,

$$\varphi_n(\omega) \leq \sum_{i=0}^{n-k-1} \psi_k(\theta^i\omega) + \sum_{i=n-k}^{n-1} \max\{\psi_k, \varphi_1\}(\theta^i\omega). \tag{1.3}$$

Since φ_- is θ-invariant, we may assume that $\varphi_-(\theta^n\omega) = \varphi_-(\omega)$ for all n.

We will inductively define a sequence $m_0 \le n_1 < m_1 \le n_2 < \cdots$ as follows. Let $m_0 = 0$. Inductively, let $n_j \ge m_{j-1}$ be the minimal integer such that $\theta^{n_j}\omega \in E_k$ (if it exists). By definition of E_k, there exists m_j such that $1 \le m_j - n_j \le k$ and

$$\varphi_{m_j - n_j}(\theta^{n_j}\omega) \le (m_j - n_j)(\varphi_-(\theta^{n_j}\omega) + \epsilon). \tag{1.4}$$

Let ℓ be maximal such that $m_\ell \le n$. By subadditivity, inductively applying the inequality

$$\varphi_i(\omega) \le \varphi_1(\theta^i\omega) + \varphi_{i-1}(\omega)$$

if $i \ne m_j$ for some j and the inequality

$$\varphi_{m_j}(\omega) \le \varphi_{n_j}(\omega) + \varphi_{m_j - n_j}(\theta^{n_j}\omega),$$

we obtain

$$\varphi_n(\omega) \le \sum_{i \in I} \varphi_1(\theta^i\omega) + \sum_{j=1}^{\ell} \varphi_{m_j - n_j}(\theta^{n_j}\omega) \tag{1.5}$$

where $I = \bigcup_{j=0}^{\ell-1}[m_j, n_{j+1}) \cup [m_\ell, n)$. Now if $i \in I$ with $i < n_{\ell+1}$, we have

$$\varphi_1(\theta^i\omega) = \psi_k(\theta^i\omega)$$

since $\theta^i\omega \notin E_k^c$. Since $\varphi_-(\theta^n\omega) = \varphi_-(\omega)$ and $\psi_k \ge \varphi_- + \epsilon$ by definition, by (1.4),

$$\varphi_{m_j - n_j}(\theta^{n_j}\omega) \le \sum_{i=n_j}^{m_j-1} (\varphi_-(\theta^i\omega) + \epsilon) \le \sum_{i=n_j}^{m_j-1} \psi_k(\theta^i\omega).$$

Thus (1.3) follows by (1.5) and the fact that $n - n_\ell < k$.

Now, suppose $\varphi_n/n \ge -C$ for some fixed constant $C > 0$. The upper bound follows by Fatou's Lemma:

$$\int_\Omega \varphi_-(\omega)\,\mathrm{d}\omega \le \liminf_{n\to\infty} \frac{1}{n} \int_\Omega \varphi_n(\omega)\,\mathrm{d}\omega = L.$$

To get the lower bound, by (1.3),

$$\frac{1}{n}\int_\Omega \varphi_n(\omega)\,\mathrm{d}\omega \le \frac{n-k}{n}\int_\Omega \psi_k(\omega)\,\mathrm{d}\omega + \frac{k}{n}\int_\Omega \max\{\psi_k, \varphi_1\}(\omega)\,\mathrm{d}\omega.$$

Thus taking the limit as n goes to infinity, we have

$$L \le \int_\Omega \psi_k(\omega)\,\mathrm{d}\omega$$

which holds for any $k \in \mathbb{N}$. Moreover, $\lim_{k\to\infty}\psi_k = \varphi_- + \epsilon$, so that $L \le \int_\Omega \varphi_-(\omega)\,\mathrm{d}\omega + \epsilon$. But $\epsilon > 0$ was arbitrary, giving the desired equality.

More generally, let $\varphi_n^{(C)} = \max\{\varphi_n, -Cn\}$ and $\varphi_-^{(C)} = \max\{\varphi_-, -C\}$. Then by the Monotone Convergence Theorem,

$$\begin{aligned}\int_\Omega \varphi_-(\omega)\,\mathrm{d}\omega &= \inf_C \int_\Omega \varphi_-^{(C)}(\omega)\,\mathrm{d}\omega = \inf_C \inf_n \int_\Omega \frac{\varphi_n^{(C)}(\omega)}{n}\,\mathrm{d}\omega \\ &= \inf_n \int_\Omega \frac{\varphi_n(\omega)}{n}\,\mathrm{d}\omega = L\end{aligned}$$

as required. $\square$

1.6 Lemma. *We have* $\limsup_{n\to\infty} \frac{\varphi_{nk}(\omega)}{nk} = \varphi_+(\omega)$ *pointwise a.e.*

Proof. The upper bound follows since by subadditivity and invariance of φ_+,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\varphi_{nk}(\omega)}{n} &\le \sum_{j=0}^{k-1} \limsup_{n\to\infty} \frac{\varphi_n(\theta^{nj}\omega)}{n} \\ &= k\varphi_+(\omega).\end{aligned}$$

Conversely, given $n \in \mathbb{N}$, write $n = kq_n + r_n$ where $r_n \in \{1, \dots, k\}$. By subadditivity,

$$\varphi_n(\omega) \le \varphi_{kq_n}(\omega) + \varphi_{r_n}(\theta^{kq_n}\omega) \le \varphi_{kq_n}(\omega) + \psi(\theta^{kq_n}\omega)$$

where $\psi = \max\{\varphi_1^+, \dots, \varphi_k^+\}$. By assumption, $\psi \in L^1$. Below, we will show that

$$\lim_{n\to\infty} \frac{\psi \circ \theta^{kq_n}}{q_n} = 0 \tag{1.6}$$

pointwise a.e. Assuming this result, we have

$$\limsup_{n\to\infty} \frac{\varphi_n}{n} \le \limsup_{n\to\infty} \frac{1}{n}\varphi_{kq_n} = \frac{1}{k} \limsup_{n\to\infty} \frac{1}{q_n}\varphi_{kq_n} \le \frac{1}{k} \limsup_{n\to\infty} \frac{\varphi_{nk}}{n}.$$

Let's prove (1.6). Let $\epsilon > 0$ be arbitrary. We first observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \mu\big(\{\omega \in \Omega : |\psi(\theta^n\omega)| \ge \epsilon n\}\big) &= \sum_{n=1}^{\infty} \mu\big(\{\omega \in \Omega : |\psi(\omega)| \ge \epsilon n\}\big) \\ &= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mu\big(\{\omega \in \Omega : k\epsilon \le |\psi(\omega)| < (k+1)\epsilon\}\big) \\ &= \sum_{k=1}^{\infty} k\mu\big(\{\omega \in \Omega : k\epsilon \le |\psi(\omega)| < (k+1)\omega\}\big) \\ &\le \int_\Omega \frac{|\psi(\omega)|}{\epsilon}\,\mathrm{d}\omega < \infty.\end{aligned}$$

Thus the result follows by the Borel-Cantelli Lemma. □

Proof (of Theorem 1.4). We are now in position to complete the proof. As before, we first assume that $\varphi_n/n \ge -C$ for some fixed $C > 0$. Set

$$\phi_k = -\sum_{j=0}^{n-1} \varphi_k \circ \theta^{kj}.$$

By definition, $\phi_{n+m} = \phi_m + \phi_n \circ \theta^{km}$ and $\phi_1 = -\varphi_k \le Ck$, so $\phi_1^+ \in L^1(\Omega, \mu)$. Let $\phi_- = \liminf_{n\to\infty} \frac{\phi_n}{n}\,\mathrm{d}\omega$. Then by Lemma 1.5 and the fact that μ is θ-invariant,

$$\int_\Omega \phi_-(\omega)\,\mathrm{d}\omega = \lim_{n\to\infty} \int_\Omega \frac{\phi_n(\omega)}{n}\,\mathrm{d}\omega = \int_\Omega \varphi_k(\omega)\,\mathrm{d}\omega.$$

Now by the subadditivity assumption and Lemma 1.6,

$$-\phi_- = \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi_k \circ \theta^{kj} \geq \limsup_{n\to\infty} \frac{\varphi_{kn}}{n} = k\varphi_+.$$

Combining the last two equations, we obtain

$$\int_\Omega \varphi_+ \,\mathrm{d}\omega \leq -\frac{1}{k}\int_\Omega \phi_- \,\mathrm{d}\omega \leq \frac{1}{k}\int_\Omega \varphi_k(\omega)\,\mathrm{d}\omega.$$

But this holds for any $k \in \mathbb{N}$, so that $\int_\Omega \varphi_+ \,\mathrm{d}\omega \leq L$.

In general, as in the proof of Lemma 1.5, set $\varphi_n^{(C)} = \max\{\varphi_n, -Cn\}$ and $\varphi_\pm^{(C)} = \max\{\varphi_\pm, -C\}$. We just showed that $\int_\Omega -\varphi_-^{(C)}\,\mathrm{d}\omega = \int_\Omega \varphi_+^{(C)}(\omega)\,\mathrm{d}\omega$. But $\varphi_-^{(C)} \leq \varphi_+^{(C)}$, so that $\varphi_-^{(C)} = \varphi_+^{(C)}$. Thus the result follows by the Monotone Convergence Theorem. □

Remark. This result generalizes Theorem 1.1 since, using the notation from that theorem, the function $\varphi_n(\omega) = \sum_{i=0}^{n-1} f(T^i\omega)$ is subadditive (since it is additive) and by invariance of T,

$$\int_\Omega f(T^i\omega)\,\mathrm{d}\omega = f(T^i\omega).$$

In fact, Theorem 1.1 follows directly from Lemma 1.5 since both $(\varphi_n)_{n=1}^\infty$ and $(-\varphi_n)_{n=1}^\infty$ are subadditive sequences of functions.

The argument in Lemma 1.6 can be interpreted as a "stability result" for subadditive sequences, which we then use to get control over φ_+ in the general case.

2 POSITIVITY OF LYAPUNOV EXPONENTS

In this section, we specialize slightly to the following setting. Let ν be a probability measure on $\mathrm{GL}_d(\mathbb{C})$. Then we take $\Omega = \mathrm{GL}_d(\mathbb{C})^\mathbb{N}$ equipped with the left-shift map σ, and μ is the infinite product $\mu = \nu^{\otimes\mathbb{N}}$. In this setting, the measure-preserving dynamical system (Ω, μ, σ) is ergodic. Since the Lyapunov exponent λ is σ-invariant, λ is constant μ-a.e. Abusing notation, we denote this constant by λ.

What can we say about the almost-everywhere value of λ? Of course, $\lambda \geq 0$, so we naturally specialize to distinguishing the cases where $\lambda = 0$ or $\lambda > 0$. There are some simple natural settings where $\lambda = 0$. Denote by G_ν the closure of the subgroup generated by the matrices in $\operatorname{supp}\nu$.

1. If G_μ is compact, then the norms of any random product is uniformly bounded above by a constant, so in fact $\lambda = 0$ everywhere.

2. If G_μ is contained in an abelian subgroup, then

$$\lambda = \int_\Omega \|M\|\,\mathrm{d}\nu(M)$$

which may be zero depending on the choice of ν.

3. If μ is the atomic measure with support

$$\operatorname{supp}\mu = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

   then $\lambda = 0$ almost everywhere. More generally, if μ consists of a uniformly chosen random rational rotation, along with a uniformly chosen contraction or dilation depending on the angle, then $\lambda = 0$ almost everywhere.

Our main theorem in this section is that the three examples above are essentially the only ways in which we can have $\lambda = 0$ almost everywhere. We first state the following definition.

Definition. We say that a subgroup G of $\mathrm{GL}_d(\mathbb{C})$ is *totally irreducible* if there is no finite union of proper subspaces of $\mathbb{C}^d$ which are G-invariant.

For simplicity, we will assume that $G_\nu \subset \mathrm{SL}_d(\mathbb{C})$.

2.1 Theorem (Furstenberg). *Suppose G_ν is totally irreducible and non-compact. Then*

$$\lambda(\omega) > 0$$

for μ-a.e. $\omega \in \Omega$.

It is meaningful to obtain the following operator-theoretic formulation of Theorem 2.1; this perspective will also reappear in **TODO: cite Furstenberg measures section**. Consider the Hilbert space

$$\mathcal{H} = L^2(\mathbb{C}^d) = \left\{ f : \mathbb{C}^d \to \mathbb{C} : \int_{\mathbb{C}^d} |f(x)|^2 \, \mathrm{d}m(x) < \infty \right\}.$$

Then a matrix $g \in \mathrm{SL}_d(\mathbb{C})$ acting on $\mathbb{C}^d$ induces a natural action $\pi(g) : \mathcal{H} \to \mathcal{H}$ by $\pi(g)f(x) = f(g^{-1}x)$, so we may define the operator $P_\nu : \mathcal{H} \to \mathcal{H}$ given by the rule

$$P_\nu f(x) = \int_{\mathrm{GL}_d(\mathbb{C})} \pi(g) f(x) \, \mathrm{d}\nu(g).$$

One can interpret the operator P_ν as applying a random transformation of f by a matrix g chosen according to the probability measure ν. We first list some basic properties of the action π and the operator P_ν.

2.2 Lemma. *(i)* $\|\pi(g)f\|_2 = \|f\|_2$ *for any* $g \in \mathrm{SL}_d(\mathbb{C})$
(ii) $\|P_\nu\| \le 1$
(iii) $P_{\nu_1} P_{\nu_2} = P_{\nu_1 * \nu_2}$
(iv) $P_\nu^* = P_{\nu^*}$ *where* $\mathrm{d}\nu^*(g) = \mathrm{d}\nu(g^{-1})$

Proof. Part (i) follows by a change of variables since $|\det g| = 1$, and parts (iii) and (iv) follow directly from the definition of P_ν.

It remains to see (ii). By Jensen's inequality and an application of Fubini's Theorem,

$$\begin{aligned}
\|P_\nu f\|_2^2 &= \int_{\mathbb{C}^d} \left| \int_{\mathrm{GL}_d(\mathbb{C})} \pi(g) f(x) \, \mathrm{d}\nu(g) \right|^2 \mathrm{d}m(x) \\
&\leq \int_{\mathbb{C}^d} \int_{\mathrm{GL}_d(\mathbb{C})} |\pi(g) f(x)|^2 \, \mathrm{d}\nu(g) \, \mathrm{d}m(x) \\
&= \int_{\mathrm{GL}_d(\mathbb{C})} \int_{\mathbb{C}^d} |\pi(g) f(x)|^2 \, \mathrm{d}m \, \mathrm{d}\nu(g) \\
&= \int_{\mathrm{GL}_d(\mathbb{C})} \|\pi(g) f\|_2 \, \mathrm{d}\nu(g) \\
&= \|f\|_2
\end{aligned}$$

where the last line follows by (i) and the fact that ν is a probability measure. □

Our proof approach is bound $\|P_\nu\|$ and then relate Theorem 2.1 to the operator P_ν.

2.3 *Proposition.* *Suppose G_ν is totally irreducible and non-compact. Then $\|P_\nu\| < 1$.*

Proof. **TODO: write**

We can now obtain our desired result.

Proof (of Theorem 2.1). By Theorem 1.2, it suffices to show that

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \int \log \|S_n(\omega)\| \, \mathrm{d}\mu(\omega) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{GL}_d(\mathbb{C})} \log \|g\| \, \mathrm{d}\nu^{*n}(g) > 0$$

for μ-a.e. $\omega \in \Omega$.

Let

$$\begin{aligned}
f(x) &= \min\{C, |x|^{-\alpha}\} \\
K &= \{x : 1 \leq |x| \leq 2\}
\end{aligned}$$

where α is chosen so that $f \in L^2(\mathbb{C}^d)$ and $C > 0$ is a constant to be determined below. Set $\gamma = \|P_\nu\| < 1$. We then have

$$\begin{aligned}
\limsup_{n\to\infty} |\langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle|^{1/n} &= \limsup_{n\to\infty} |\langle P_\nu^n f, \mathbf{1}_k \rangle|^{1/n} \\
&\leq \limsup_{n\to\infty} \|P_\nu^n\|^{1/n} \cdot \|f\|_2^{1/n} \cdot \|\mathbf{1}_K\|_2^{1/n} \leq \gamma.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle P_{\nu^{*n}} f, \mathbf{1}_K \rangle &= \int_{1\leq|x|\leq 2} \int_{\mathrm{GL}_d(\mathbb{C})} \min\{C, \left\|g^{-1}x\right\|^{-\alpha} \, \mathrm{d}\nu^{*n}(g) \, \mathrm{d}m(x) \\
&\geq \int_{1\leq|x|\leq 2} \int_{\mathrm{GL}_d(\mathbb{C})} \min\{C, \left\|g^{-1}\right\|^{-\alpha} \cdot \|x\|^{-\alpha}\} \, \mathrm{d}\nu^{*n}(g) \, \mathrm{d}m(x) \\
&\geq C_0 \int_{\mathrm{GL}_d(\mathbb{C})} \min\{C, \left\|g^{-1}\right\|^{-\alpha}\} \, \mathrm{d}\nu^{*n}(g)
\end{aligned}$$

for some constant C_0 depending only on α. Since $\inf_{g\in \mathrm{SL}_d(\mathbb{C})} \|g\| > 0$, we can take C sufficiently large so that $\min\{C, \|g^{-1}\|^{-\alpha}\} = \|g^{-1}\|^{-\alpha}$ for any $g \in \mathrm{SL}_d(\mathbb{C})$ We also use the fact that $\|g^{-1}\| \le C_0' \|g\|^{d-1}$, which follows by the adjoint formula for the matrix (since the entries in the adjoint are degree $d-1$ polynomial functions of the entries of g, and $|\deg g| = 1$). Thus there is some constant $C_1 > 0$ such that

$$\langle P_{\nu^{*n}} f, \mathbf{1}_K\rangle \ge C_1 \int_{\mathrm{GL}_d(\mathbb{C})} \|g\|^{-\alpha(d-1)} \,\mathrm{d}\nu^{*n}(g).$$

Thus taking logarithms, applying Jensen's inequality, and rearranging, we have

$$\int_{\mathrm{GL}_d(\mathbb{C})} \log \|g\| \,\mathrm{d}\nu^{*n}(g) \ge \frac{\log C_1}{\alpha(d-1)} - \frac{1}{\alpha(d-1)} \log\langle P_{\nu^*} f, \mathbf{1}_K\rangle$$

and therefore

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{GL}_d(\mathbb{C})} \log \|g\| \,\mathrm{d}\nu^{*n}(g) &= -\frac{1}{\alpha(d-1)} \limsup_{n\to\infty} \frac{1}{n} \log\langle P_{\nu^{*n}} f, \mathbf{1}_K\rangle \\ &\ge -\frac{1}{\alpha(d-1)} \log \gamma > 0\end{aligned}$$

as required. $\square$