# Random Matrix Products

Alex Rutar*
University of St Andrews

Spring 2021†

*alex@rutar.org
†Last updated: April 12, 2021

# Contents

# Preface

These lecture notes on random matrix products are prepared for the reading group on random matrix products for the analysis group in Spring 2021. Much of the content is based on Alex Gorodnik's lecture notes for his course "Random walks on matrix groups". Any errors or omissions can be sent to the author.

# I. The Multiplicative Ergodic Theorem

## 1 Random matrix products and the subadditive ergodic theorem

### 1.1 The Birkhoff ergodic theorem

Let $\Omega$ be a separable, second-countable metric space equipped with its Borel $\sigma$-algebra $\mathcal{B}$, and let $\mu$ be a Borel probability measure on $\Omega$. Suppose we are given a measurable function $\theta : \Omega \to \Omega$. We denote the *pushforward* of $\mu$ by $\theta$ to denote the Borel probability measure defined by the rule

$$\theta_*\mu(E) = \mu(\theta^{-1}(E))$$

for Borel sets $E \subset \Omega$. We say that the function $\theta$ is *measure preserving* if $\theta_*\mu = \mu$. In this situation, we call the information $(\Omega, \mu, \theta)$ a *measure-preserving dynamical system.*

Given a Borel set $E \subset \Omega$, we say that $E$ is $\theta$-invariant if $\theta^{-1}(E) = E$, and denote the set of $\theta$-invariant sets by $\mathcal{B}_\theta$. More generally, we say that a measurable function $f : \Omega \to K$ where $K$ is a topological space is $\theta$-invariant if $f = f \circ \theta$. One can verify that $\mathcal{B}_\theta$ is a Borel $\sigma$-subalgebra of $\mathcal{B}$. In particular, $f$ is $\theta$-invariant if and only if $f$ is $\mathcal{B}_\theta$-measurable. We say that $(\Omega, \mu, \theta)$ is *ergodic* if each $\theta$-invariant set $E \in \mathcal{B}_\theta$ either has $\mu(E) = 0$ or $\mu(E) = 1$.

We will denote by $T^n$ the $n$-fold composition $T \circ \cdots \circ T$. Given a function $f$, we write $f = f^+ + f^-$ where $f^+ \geq 0$ and $f^- \leq 0$. A standard result is the following.

***1.1 Theorem. (Birkhoff Pointwise Ergodic)*** *Let $(\Omega, \mu, \theta)$ be an ergodic measure-preserving dynamical system and let $f = f^+ + f^-$ satisfy $f_+ \in L^1(\Omega, \mu)$. Then for $\mu$-a.e. $\omega \in \Omega$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(\omega)) = \int_\Omega f \,\mathrm{d}\mu$$

*where the limit may be attained at $-\infty$.*

We have written Theorem 1.1 in additive notation, but it can be easily rephrased in multiplicative notation. Denote by $\log^+(x) = \max(0, \log x)$. Write $g = \exp(f)$ and note that $f_+ = \log^+(g)$. Then for $\mu$-a.e. $\omega \in \Omega$,

$$\lim_{n\to\infty} (g(T^{n-1}\omega) \cdots g(\omega))^{1/n} = \exp\left(\int_\Omega \log g \,\mathrm{d}\mu\right).$$

Of course, here, the group written in product notation is still commutative. In the following section, we consider a more general setting where this is no longer the case.

### 1.2 Random matrix products

The setting of Theorem 1.1 is nice, but in these notes we are interested in a somewhat more general situation. First consider the following example. Let $\Omega$ denote the compact product space $\mathrm{GL}_n(\mathbb{C})^{\mathbb{N}}$ equipped with the left-shift map $\sigma : \Omega \to \Omega$ given by

$$\sigma((M_n)_{n=1}^\infty) = (M_n)_{n=2}^\infty$$

for a sequence of matrices $(M_n)_{n=1}^{\infty} \in \Omega$. Let $\nu$ be a probability measure on $\mathrm{GL}_n(\mathbb{C})$ and let $X_i : \Omega \to \mathrm{GL}_n(\mathbb{C})$ for $i \in \mathbb{N}$ be independent random matrices with distribution $\nu$. Asymptotic behaviour of random products of the form $X_n \cdots X_1$ can be interpreted as a matrix-valued generalization of the law of large numbers.

More generally, we are interested in matrix-valued measurable functions, i.e. functions $X : \Omega \to \mathrm{GL}_n(\mathbb{C})$ on a measure-preserving space $(\Omega, \mu, \theta)$. This setting is a generalization of the setting in Theorem 1.1, where we considered a measurable function $f : \Omega \to \mathbb{R}$ satisfying an integrability criteria. Let $\|\cdot\| : \mathrm{GL}_n(\mathbb{C})$ be a matrix norm. We will assume that $\|\cdot\|$ is *submultiplicative* (i.e. $\|AB\| \leq \|A\| \|B\|$), but we do not lose any generality since all matrix norms are equivalent. We also assume that $X$ satisfies the integrability condition

$$\int_{\Omega} \log^{+} \|X(\omega)\| \, \mathrm{d}\omega < \infty.$$

As in the prior section, we are interested in determining statistical information concerning the limit of the random matrix product

$$S_n(\omega) = X(T^{n-1}\omega) \cdots X(\omega).$$

We will investigate various statistical properties of the random products $S_n(\omega)$. Here are three such examples which we will focus on: **TODO: add links here once the sections are written up**

(i) the growth rate of $\|S_n(\omega)\| = \left\|X(\theta^{n-1}\omega) \cdots X(\omega)\right\|$ for large $n$ and "typical" $\omega$.

(ii) the growth rate from a fixed starting point $\left\|X(\theta^{n-1}\omega) \cdots X(\omega)v\right\|$ for some $v \in \mathbb{C}^n$

(iii) the behaviour of the directions $\left\|X(\theta^{n-1}\omega) \cdots X(\omega)v\right\| / \left\|X(\theta^{n-1}\omega) \cdots X(\omega)v\right\|$ for some $v \in \mathbb{C}^n$.

Here are a couple examples of settings

*Example.* 1. Given fixed matrices $M_1, \ldots, M_\ell \in \mathrm{GL}_n(\mathbb{C})$, generate a sequence $S_0 = I$ and $S_{n+1} = M_i \cdot S_n$ where we take matrix $M_i$ with probability $1/\ell$. The products $S_n$ can be interpreted as a random walk on $\mathrm{GL}_n(\mathbb{C})$ (or $\mathbb{C}^n$) where the "steps" are given by multiplication by a matrix $M_i$.

2. If $U \subset \mathbb{R}^d$ is an open set and $F : U \to U$ is smooth, by the chain rule, the Jacobian of $F^n$ at a point $u$ satisfies

$$D(F^n)_u = (DF)_{F^{n-1}u} \cdots (DF)_u.$$

Here, $DF : U \to \mathrm{GL}_d(\mathbb{R})$ is a matrix-valued measurable function. The growth rate of $DF$ is related to the entropy of $F$ and the dimension of invariant measures.

3. If $T_i(x) = A_i x + t_i$ where $A_1, \ldots, A_\ell \in \mathrm{GL}_n(\mathbb{R})$ have operator norms $\|A_i\| < 1$ for $i = 1, \ldots, \ell$ and $t_i \in \mathbb{R}^n$, then there is a unique *self-affine set* $K$ satisfying

$$K = \bigcup_{i=1}^{\ell} T_i(K)$$

and, given probabilities $p_1, \ldots, p_\ell$, a unique *self-affine measure*, which is a Borel probability measure $\nu$ satisfying

$$\nu = \sum_{i=1}^{\ell} p_i (T_i)_* \mu.$$

Here, dimensional properties of the measure $\nu$ are related to properties of random products of the matrices $\{A_1, \ldots, A_\ell\}$.

## 1.3 LYAPUNOV EXPONENTS AND THE SUBADDITIVE ERGODIC THEOREM

A fundamental statistical property associated with the matrix-valued function $X$ is the following.

***Definition.*** With notation as above, we define the *top Lyapunov exponent* $\lambda : \Omega \to \mathbb{R}$ by

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\| .$$

We now have the following fundamental result.

***1.2 Theorem. (Furstenburg-Kesten)*** *The function $\lambda$ is $\theta$-invariant and satisfies*

$$\int_\Omega \lambda(\omega)\, \mathrm{d}\omega = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|S_n(\omega)\|\, \mathrm{d}\omega.$$

This result can be thought of as interchanging the limit with the integral, i.e. averaging over space is the same as

In fact, we will prove Theorem 1.2 as a consequence of a more general result. We first make some observations about the average $a_n := \int_\Omega \log \|S_n(\omega)\|$. Observe by submultiplicative of the matrix norm that

$$\begin{aligned}
a_{n+m} &:= \int_\Omega \log \|S_{n+m}(\omega)\|\, \mathrm{d}\omega \\
&= \int_\Omega \log \left\|X(\theta^{n+m-1}\omega) \cdots X(\omega)\right\| \mathrm{d}\omega \\
&\leq \int_\Omega \log \left\|X(\theta^{n+m-1}\omega) \cdots X(\theta^m\omega)\right\| \mathrm{d}\omega + \int_\Omega \log \left\|X(\theta^{m-1}\omega) \cdots X(\omega)\right\| \mathrm{d}\omega \\
&= \int_\Omega \log \|S_n(\theta^m\omega)\|\, \mathrm{d}\omega + \int_\Omega \log \|S_m(\omega)\|\, \mathrm{d}\omega \\
&= a_n + a_m
\end{aligned} \tag{1.1}$$

where the last line follows by the integrability condition on $X$ along with the fact that $\theta$ is measure preserving.

***Definition.*** We say that the sequence $(a_n)_{n=1}^\infty \subset \mathbb{R}$ is *subadditive* if $a_{n+m} \leq a_n + a_m$ for each $n, m \in \mathbb{N}$. More generally, we say that a sequence of functions $\varphi_n : \Omega \to \mathbb{R}$ is *subadditive* if

$$\varphi_{n+m}(\omega) \leq \varphi_n(\theta^m\omega) + \varphi_m(\omega).$$

The following lemma is straightforward.

***1.3 Lemma.*** *If* $(a_n)_{n=1}^{\infty}$ *is a subadditive sequence, then* $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}$.

In particular, implies that the limit

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Omega} \log \|S_n(\omega)\| \,\mathrm{d}\omega$$

always exists. Moreover, if we set $\varphi_n(\omega) = \log \|S_n(\omega)\|$, we observed in (1.1) that the sequence of functions $\varphi_n$ is subadditive. Thus Theorem 1.2 is a consequence of the following more general result.

***1.4 Theorem. (Kingman's Subadditive Ergodic)*** *Let* $\varphi_n : \Omega \to \mathbb{R}$ *be a subadditive sequence with* $\varphi_1^+ \in L^1(\Omega, \mu)$. *Then the limit* $\varphi(\omega) := \lim_{n\to\infty} \frac{\varphi_n(\omega)}{n}$ *exists a.e., and*

$$\int_{\Omega} \varphi(\omega) \,\mathrm{d}\omega = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega} \varphi_n(\omega) \,\mathrm{d}\omega.$$

PROOF **TODO: write** ■

*Remark.* This result generalizes Theorem 1.1 since, using the notation from that theorem, the function $\varphi_n = \sum_{i=0}^{n-1} f(T^i\omega)$ is subadditive and by invariance of $T$,

$$\int_{\Omega} f(T^i\omega) \,\mathrm{d}\omega = f(T^i\omega).$$